Determinism and Strong Cosmic Censorship

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Abstract

This discussion addresses the question of determinism in the general relativity setting, where it can be observed that the debate is still not settled. This is tightly related to what is consider as physical solutions. This question of what is a physical solution could be answered with the help of the strong cosmic censorship conjecture, where we will be giving mathematical results related to this conjecture. We then argue that if a theory is not deterministic, it could still be considered scientific as long as it satisfies some conditions. This work was the outcome of a undergraduate summer research project supervised by Professor Christopher Smeenk at the Western University, Ontario.

1 Introduction

Most physicist think that classical theories are deterministic, in Laplacian sense. But whether a theory is deterministic is not as obvious as commonly thought. In fact, even in Newton gravitation theory, one can build thought experiments that exhibit unwanted behaviors that challenge the belief that the theory is deterministic. Such examples are addressed in Norton [2008], Xia [1992]. Let me start with the definition of determinism.

Definition 1.1 (Determinism)**. A theory is deterministic** when for any 'appropriate' initial data there exists a unique 'physical' model of that theory that 'admits' the initial data.

'Appropriate', 'physical' and 'admits' are words that will depend on the specific theory under study. Since this discussion will be focused on General relativity, We will start by defining what are the mathematical representations of the theory. We will then explain what 'appropriate' initial data are, from this one can then define what 'admits' means in this context. Then then all is left is to define what physical is, which will be a subclass of the mathematical representation. This is the hardest part and we will be exploring its complications.

Definition 1.2. (M, g, D) is a **mathematical representation of general relativity (or spacetime)** when M is a smooth connected 4-dimensional manifold, g is a Lorentzian metric, and D is a finite sequence of matter fields. The metric g and the matter fields D have to satisfy equation of motion obtained by extremizing the action

$$S(g, D) = \int_M \sqrt{-g}\left(\frac{1}{4}R + L_{matter}(g, D)\right)$$

with respect to all the variables, where R is the Ricci scalar with respect to g.

An example of a matter fields is the massless (real)-scalar matter field ϕ without self interaction, where

$$L_{sf} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

Another matter field is the Maxwell vector field A where

$$L_{Maxwell} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We will refer often to solutions of $L_{matter} = L_{Maxwell} + L_{sf}$ as Einstein-Maxwell-scalar solutions. Similarly, when $L_{matter} = L_{sf}$ we call them Einstein-scalar solutions, when $L_{matter} = L_{Maxwell}$ we call them Einstein-Maxwell and when $L_{matter} = 0$ we call them Einstein-vacuum.

Definition 1.3. $(S, \bar{g}, \bar{k}, \bar{D}, \bar{F})$ is **appropriate initial data in general relativity** when $(S, \bar{g})$ is geodesically complete 3-dimensional Riemannian Manifold, $\bar{k}$ is a rank 2 tensor, $\bar{D}$ are matter fields and $\bar{F}$ a tensors of the same rank as the matter fields. $\bar{g}, \bar{k}, \bar{D}$ and $\bar{F}$ are required to satisfy the constraint equations[1].

Definition 1.4. A mathematical representation (M, g, D) **admits the initial data** $(S, \bar{g}, \bar{k}, \bar{D}, \bar{F})$ when there exists and embedding $i : S \to M$ such that

1. $i^\star(g) = \bar{g}$
2. $i^\star(k) = \bar{k}$ where k is the second fundamental form of g
3. $i^\star(D_i) = \bar{D}_i$ + optional additional condition on F that are related to D. (For example, derivative wrt to time of the wave equation.)

Now that we have all those notion, we could start discussion what are the physical models of the theory. First observe that given a mathematical representation (M, g, D), let $d : M' \to M$ be a diffeomorphism. Then, $(M', d^\star g, d^\star D)$ is also a mathematical representation. This is called the Hole argument. For this reason we will say that any two mathematical representation that are linked in this way by a diffeomorphism are the same physical model. So all uniqueness means uniqueness up to diffeomorphism. This is just a necessary condition for a model to be physical but we will see that it covers too many models that we will not consider physical. Consider the following three examples.

First example is called Gödel spacetime which is $\mathbb{R}^4$ as a smooth manifold. This spacetime has the particularity that through every point there is a closed timelike curve (CTC). This implies that it is impossible to choose a global spacelike slice. For this reason, it is not clear how to make sense of determinism.

Another example is Minkowski spacetime but with particularity that we identify a every slice at t_0 with a slice at $t_0 + 1$. This spacetime is a vacuum solution that has the same initial condition as Minkowski but has not the same topology as Minkowski. So those two spacetimes are not diffeomorphic, but admit the same initial data. In this case determinism will fail.

As a last example, consider any spacetime which admits some initial data. All submanifolds which contain the initial data are also mathematical representation that admit the same initial. They are not diffeomorphic, so we have a failure of determinism.

If we want our theory to be deterministic, the first example tells us that physical models should at least admit a global spacelike slice. The second tells us that that we should not consider weird causal structures as physical. And the last example tells us that we should always consider the largest spacetimes. In the following section we will consider spacetimes that get rid of the first two examples. Those are called globally hyperbolic spacetimes. We will investigate mathematical results related to those spacetimes.

2 Globally hyperbolic spacetimes and strong cosmic censorship

Lets start by assuming that all physical models are globally hyperbolic, refer to appendix A for a precise definition and some properties. For our purposes it enough to know that a **globally hyperbolic spacetime** admits a spacelike surface (Cauchy surface) S and the spacetime is homeomorphic to $\mathbb{R} \times S$. A **global hyperbolic development** (GHD) of an initial data S is a globally hyperbolic spacetime that admits S as a Cauchy surface. The GHD is unique in the following sense.

[1] Ringström [2009] chapter 13 for discussion on the constraint equations.

Theorem 2.1 (Choquet-Bruhat and Geroch [1969], Sbierski [2015]). *For any initial data there exists a unique (up to diffeomorphism) GHD, called the* ***maximal globally hyperbolic development*** *(MGHD), such that all GHD of that data is are isometrically embedded into the MGHD.*

So if we only consider spacetimes that are globally hyperbolic to be physical then it is a win for determinism.The problem with assuming globally hyperbolic is that an observer living in a globally hyperbolic spacetime could not know from its local observation that the spacetime is globally hyperbolic. Another critic raised by Hawking and Penrose [2010] is,

> ...[G]lobal hyperbolicity may be a physical necessity. But my view point is that one shouldn't assume it because that may be ruling out something that gravity is trying to tell us. Rather one should deduce that certain regions of spacetime are globally hyperbolic from other physically reasonable assumptions.

He brings to our attention that there exist certain 'reasonable' spacetimes that are not globally hyperbolic but contain globally hyperbolic regions. This motivates the following definition.

Definition 2.1. We say that a connected Lorentzian manifold (M, g) is **P-extendible or extendible with P** if there exists a connected Lorentzian manifold $(\tilde{M}, \tilde{g})$ with the same dimension as M and $\tilde{g}$ being P so that (1) (M, g) isometrically embeds into $(\tilde{M}, \tilde{g})$, and (2) $M \neq \tilde{M}$ (here, M denotes the image of the isometric embedding). Such an $(\tilde{M}, \tilde{g})$ is called an **extension** of (M, g) and it is said to be **maximal** if it is not extendable.

For example, the maximal extended Reissner-Nordström spacetime is a case of what Hawking pointed out. But should we consider it physical, or only consider its hyperbolic region to be the physical one? Furthermore, Reissner-Nordström is a particular case of the following theorem.

Theorem 2.2. *If (M, g) is extendible MGH spacetime and N is an extension. If there exists a spacelike hypersurface S that intersects both M and $N \setminus \overline{M}$, then there exits non unique solution to the EFE.*

This applies to the globally hyperbolic submanifold of Reissner-Nordström so we have non unique extensions. In this sense, when giving initial data that develops to Reissner-Nordström, we have non-unique solutions. In order to save determinism, Penrose conjectured the strong cosmic censorship (SCC), which, in its modern version, reads as.

Conjecture 2.1 (Strong cosmic censorship). For generic initial data for the vacuum equations or[2] for suitable Einstein–matter systems, the maximal Cauchy development is appropriately inextendible.

To define genericity we need to equip the class of initial data with a topology so that it has the structure of a manifold. A subclass of the initial data is generic means that its complement has positive codimension. It could also mean that it is open and dense with respect to some norm topology[3]. There are still debates regarding the notion of genericity but we will not address them or state which exact definition is used for mathematical results. This genericity has to be intuitively understood as, given an initial data that is extendible, if we perturb, we will always end up with a solution that have no extensions to their MGHD. If this conjecture is true then it will be physically reasonable to assume globally hyperbolicity. This will give us a "cosmic" law that justifies our choice of global hyperbolicity.

Concerning the meaning of suitable matter fields, it is up to debate. Most mathematical results are for Maxwell and real-scalar matter fields. There is no clear answer regarding what we should consider a suitable matter fields. Earman [1986] as the following conditions.

(i) local conservation of energy, namely $div(T) = 0$.

[2]"or" is in the sense that we can consider both scenario, not in the logical sense.
[3]For more details refer to Ringström [2009].

(ii) Dominant energy condition. For every timelike vector V, $T^{ij}V_iV_j \geq 0$ and $T^{ij}V_j$ non-spacelike. Which means energy density is non-negative and energy flow is non-spacelike.

(iii) T vanishes on an open neighborhood $U \subset M$ iff all matter fields vanish on U.

Those conditons imply that given a matter field on an archronal surface S, it is uniquely determined on $D(S)$. So for a Cauchy surface it will uniquely be determined on all MGHD, see Malament [2012] propositon 2.5.1. Another way to address the question could be using Weinberg–Witten theorem by consider considering only massless matter fields with spin smaller or equal to 1, but this does not cover massive matter fields.

The meaning of appropriately inextendible is subject to discussion and will further analysed in the following sections. In the following section we will see the latest mathematical results regarding the SCC.

3 SCC formulations and results

Mathematicians agree that it is not realistic to prove this conjecture in all generality directly. Rather we should try to study the SCC in some class of spacetimes with symmetries. A generic set in a class of initial data that follow some symmetries might not be generic in the class of all initial data. Being dense in a space does not imply it being dense in a larger space. So positive or negative result in a class of initial data that has some symmetry does not prove or disprove anything, but they give us some insight and techniques for a more general proof.

3.1 No symmetries

It has been shown that

Theorem 3.1 (Moncrief and Isenberg [2019]). *All globally hyperbolic vacuum spacetimes that do not posses any symmetries, do not admit Cauchy horizons which are compact, non-degenerate and generated by closed null geodesics.*

This result is a partial positive result for the SCC but imposes many condition on the Cauchy horizon. Nevertheless this result implies that the Cauchy horizon in Taub-Nut spacetimes is unstable under perturbations. Ideally one would have to relax the condition on this theorem, but no recent result are known. On the negative side we have the following result.

Theorem 3.2 (Dafermos and Luk [2017]). *For generic compact or asymptotically flat vacuum initial data, the maximal Cauchy development is C^0-extendible.*

To prove this theorem, they integrate twice the tidal force in the direction of the Cauchy horizon of Kerr, if this quantity is finite and non-degenerate as you approach the Cauchy horizon then it implies that the metric is C^0-extendible. They showed that this characteristic of Kerr Cauchy horizon is present for generic compact or asymptotically flat vacuum initial data. If we think about a test matter field such as pressureless dust, this quantity will be the "tidal" displacement. This can be thought as the observer not being torn apart.

Here we are assuming compactness of asymptotically flat initial data, which does not represent all initial data possible. But most of the mathematical formulation of the SCC impose the asymptotically flat initial data. This is because there exists counter example to the conjecture when the initial data in no asymptotically flat, see Cardoso et al. [2018]. So if we consider appropriate extension to be as low regularity as C^0 then the SCC fails for the vacuum case. But such low regularity on the metric might not be sufficient to make sense of Einstein equations. Having locally square integrable christoffel symbols is the weakest known condition to makes sense of Einstein equations Demetrios [2009]. So it is not a complete loss for the SCC. All following results are within a class of initial data that has symmetries.

3.2 Spherically symmetric

Lets first start with Einstein-Maxwell system in the spherically symmetric class.

Theorem 3.3 (Birkhoff). *All solutions of the globally hyperbolic spherically symmetric Einstein-Maxwell system with a regular Cauchy hypersurface with asymptotically flat ends are characterized by two parameters m and e ranging for the following values:*

1. *$m = 0$ and $e = 0$ (Minkowski),*
2. *$m > 0$ and $e = 0$ (Schwarzschild),*
3. *$m > 0$ and $m^2 > e^2 > 0$ (Subextremal Reissner-Nordström).*

Theorem 3.4 (Sbierski [2018]). *Minkowski and Schwarzschild spacetimes are C^0-inextendible.*

Since both those solution are vacuum solution (Maxwell matter vanishing) we have a positive result for the SCC in the best way possible, because the initial data that is C^0-inextendible are all the initial data, so it is trivially generic. But now if we consider a larger class of spherically symmetric initial data that includes Maxwell matter field, we get a failure of the SCC since Reissner-Nordström is C^∞-extendible, so we cannot save the SCC in any way.

Now if we consider Einstein-scalar system we have the following theorem in the spherically symmetric case.

Theorem 3.5 (Christodoulou [1991, 1999]). *For generic initial data of the spherically symmetric Einstein–scalar field system with either 1-ended asymptotically flat initial data on $\mathbb{R}^3$ or 2-ended asymptotically flat initial data on $\mathbb{R} \times \mathbb{S}^2$ the MGHD is C^0-inextendible.*

Theorem 3.6 (Dafermos [2003], Dafermos and Rodnianski [2005]). *For generic initial data of the spherically symmetric Einstein–Maxwell–scalar field system with 2-ended asymptotically flat initial data on $\mathbb{R}\times\mathbb{S}^2$ the MGHD is C^0-extendible.*

Theorem 3.7 (Luk and Oh [2019b,a]). *For generic initial data of the spherically symmetric Einstein–Maxwell–scalar field system with 2-ended asymptotically flat initial data on $\mathbb{R} \times \mathbb{S}^2$ the MGHD is C^2-inextendible.*

We have seen if we just consider Maxwell matter field in the class of spherically symmetric solution by Birkoff theorem we have Reissner-Nordström which is C^∞-extendible. But if we consider Maxwell-Scalar matter field in the class of spherically symmetric solutions we have positive results for the SCC, because the subclass with vanishing scalar is "small" enough to be non-generic. We see that the questions of which matter field and what regularity we should consider are important for the resolution of the conjecture.

4 Why determinism

Why determinism? The question seems to be attached to what it means to do physics. The most humble physicist would say that physics has just the goal to describe the material world into some language. But some more ambitious physicist would say that physics has to predict. The question boils down to what it means to be a scientific theory. Some might say that a scientific theory is not complete if it is not deterministic. Determinism can be seen as something we always go for.

Lets say that we have a theory that is not deterministic. So given some initial data we have multiple models of the theory that admits it. Some might say lets do an experiment to decide which one is the physical one. But then lets say we get different outcomes when repeating the experiment. The requirement of a theory being deterministic would not be rationally justified. Determinism is not something we should take a priori as a requirement for our theory, but rather something a theory could posses or not. But then if a theory is not deterministic does it qualify as a scientific theory? A theory that only has descriptive power would be incomplete as being scientific, since there is no way of gaining confidence in it. In this way, I define that *a scientific theory should necessarily have the capability of updating our confidence about it* (in a bayesianism sense for example). In this sense, determinism gives us a way to update our confidence about it by setting up initial data and doing experiments.

It seems that determinism only makes sense if time could be a variable in the theory, some dynamical equations. But what if time is not a variable of the equation but something emergent, as it is with thermal time. Could we still talk about determinism if our equation are not dynamical? It could be observed that the definition of determinism stated in the introduction does not make reference to time or initial data on a time slice. In this way, determinism could be applied to theories that do not have time as a variable. If a theory is deterministic with respect to some initial condition that could be set up for observation, then the theory would give us a way to update our confidence about it.

Going back to general relativity, if we impose it to be deterministic, such as by choosing only globally hyperbolic spacetimes to be physical then it would qualify as a scientific theory. But what about if we let all mathematical models (up to diffeomorphism) be physical? Would we still have a way to gain confidence about the theory? It doesn't seem plausible, for the same initial data we would have multiple evolution and by observation we only see one solution. By allowing all those possible models we have no way of updating our confidence about the theory, since the theory does not tell us what should be observed, not even in a probabilistic sense. This would not make general relativity, in all its generality, a scientific theory. Imposing globally hyperbolicity makes it scientific, there is no a priori principle that tells that the restriction of our mathematical models by Einstein equation or by global hyperbolicity is more fundamental. None of them are implication of the other. It might be argued the the SCC is Einstein equation imply global hyperbolicity, but this is not true. The SCC only tells us that assuming Einstein equations [4], the space of non-globally hyperbolic spacetimes is 'small' with respect to the globally hyperbolic ones. Einstein equation and globally hyperbolicity should be seen on the same level and would be both postulates of a complete scientific theory.

One can argue that determinism is not the only way to satisfy the condition that a scientific theory has to have the capability of updating our confidence about it. There might be other ways of doing so. Since all theories seem to be deterministic in their own way, I cannot think of any example that would satisfy this condition but not be deterministic. I didn't not find any such examples in the field and would be curious to study further this question.

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[4] Additionally, we have to impose the possibility of embedding a complete Riemannian 3-manifold into the solution and many other conditions.

Appendix A Causal structure

Here a couple of definitions and theorem about the causal structures of spacetime.

Definition A.1. A vector $V \in T_pM$ is:

causal	when $g(V, V) \leq 0$,
timelike	when $g(V, V) < 0$,
lightlike	when $g(V, V) = 0$ and $v \neq 0$,
null	when $g(V, V) = 0$,
spacelike	when $g(V, V) > 0$.

Definition A.2. $D(S)$ is the **domain of dependence of** S **in** (M, g) when

$$D(S) = \{p \in M : \text{Every inextendible causal curves through } p \text{ intersect } S\}$$

Definition A.3. A set S is **acausal** when there is no causal curve which starts and ends at S

Definition A.4. A set S is **Cauchy hypersurface** when it is acausal, closed and $D(S) = M$.

Definition A.5. A spacetime (M, g) is **globally hyperbolic** when it admits a Cauchy hypersurface S

Theorem A.1. *All Cauchy hypersurfaces are homeomorphic.*

Theorem A.2. *If (M, g) is globally hyperbolic there is a homeomorphism* $\phi : M \to R \times S$.

Definition A.6. A manifold is **geodesically complete** when the exponential map domain is the TM.

Theorem A.3. *If (M,g) is geodesically complete then it is* C^2*-inextendible.*

Theorem A.4 (Galloway et al. [2017])**.** *If (M,g) is geodesically complete and hyperbolic then it is* C^0*-inextendible.*

For a more in-dept treatment of causal structure refer to Minguzzi [2019].

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