Linear instability of Reissner-Nordström's Cauchy horizon under scalar perturbations

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Abstract

I motivate the strong cosmic censorship conjecture from a physical point of view. I introduce some central concepts and present a proof of an essential step of the main theorem established by Luk and Oh [2017]. This was part of an undergraduate research project supervised by Professor Sung-Jin Oh at UC Berkeley.

1 Introduction

Physicists mostly think of their classical (non-quantum) theories as being deterministic. This is because physics is associated with predictability, and its capacity to describe the world. What is meant by determinism can be captured in this following quote by Laplace and Dale [1998].

> We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

General relativity admits an initial value formulation, where the present state of the universe is a Cauchy hypersurface with initial data specified on it. Those initial data have to satify some constraint equations that are implied by Einstein field equations. And within this setting we have the following celebrated result.

Theorem 1.1 (Choquet-Bruhat Geroch)**.** *For any initial data there exists a unique (up to diffeomorphism) GHD, called the **maximal globally hyperbolic development** (MGHD), such that all GHD of that data is are isometrically embedded into the MGHD.*

But some MGHD are future extendible in an non unique way. For example this is the case of Reissner-Nordström and Kerr spacetimes. This would mean that an observer travelling through spacetime would not know its fate despite full information on the Cauchy initial hypersurface. But this is conjectured to not be generic. In this spirit Penrose conjectured the strong cosmic censorship (SCC), which modernly reads

Conjecture 1. For 'generic' initial data for the vacuum equations or for suitable Einstein–matter systems, the maximal Cauchy development is 'appropriately' inextendible.

This conjecture has been keeping mathematical minds busy since its first formulation and no real consensus has been established, only partial results. Einstein field equation are complicated non-linear coupled partial differential equations that lead to many developments of PDE techniques that not only apply to general relativity but to other fields. A way to attack the conjecture is to only consider initial data that lead to solutions that obey some symmetries. An interesting class of spacetimes are the ones that are spherically symmetric, they exhibit behavior that are at the center of what makes the SCC hard to prove or disprove.

The latest results for spherically symmetric spacetimes concern the Einstein-Maxwell-(real)-scalar system. The SCC in this scenario is thought to be true because of Penrose blue-shift argument, which was formalized by Chandrasekhar and Hartle [1982]. In short it says that there exist arbitrarily small perturbations of Reissner-Nordström initial data will create wave solution which will lead to infinite energy near to Cauchy horizon. It has been proven in this case that for all 2-ended asymptotically flat initial data on $\mathbb{R} \times \mathbb{S}^2$ the MHD is C^0-extendible by Dafermos [2003], Dafermos and Rodnianski [2005]. This is a negative result for the SCC but C^0-extensions might not be sufficient to make sense of Einstein field equations. For this reason, it has been recently proven under the same assumptions that

Theorem 1.2 (Luk and Oh [2019b,a])**.** *For generic initial of the Einstein–Maxwell–(real)–scalar field system with spherical symmetry and 2-ended asymptotically flat initial data on $\mathbb{R} \times \mathbb{S}^2$ the MHD are C^2-inextendible.*

In this discussion, we will not examine this theorem but rather a linearized version which states that

Theorem 1.3 (Luk and Oh [2017])**.** *Generic smooth and compactly supported initial data to* (2.1) *on a Cauchy surface of Reissner-Nordström have solution that are not in $W^{1,2}_{loc}$.*

In section 2, I will be describing the Reissner-Nordsröm spacetime and explain what $W^{1,2}_{\text{loc}}$ means in this context. In section 3, the main steps of the proof will be outlined and finally in section 4, a proof of a central theorem required to prove theorem 1.3 will be given.

2 Reissner-Nordström

The Reissner-Nordström is a solution to the Einstein-Maxwell system. Further more this solution turns out to be unique in the following sense.

Theorem 2.1 (Birkhoff)**.** *All solutions of the globally hyperbolic spherically symmetric Einstein-Maxwell system with a regular Cauchy hypersurface with asymptotically flat ends are characterized by two parameters M and e ranging for the following values:*

1. *$M = 0$ and $e = 0$ (Minkowski)*
2. *$M > 0$ an $e = 0$ (Schwarzschild)*
3. *$M > 0$ and $M^2 > e^2 > 0$ (Subextremal Reissner-Nordström)*

Subextremal Reissner-Nordström ,is a spacetime diffeomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$ with metric locally given by

$$g = -(1 - \frac{2M}{r} + \frac{e^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{e^2}{r^2})^{-1}dr^2 + r^2 d\sigma_{\mathbb{S}^2},$$

where $d\sigma_{\mathbb{S}^2}$ is the 2-sphere metric with radius 1, $t \in \mathbb{R}$ and $r > r_-$, for $r_\pm := M \pm \sqrt{M^2 - e^2}$.

There is no such theorem when adding the scalar field into the system. For this reason, it is interesting to approximate the Einstein-Maxwell-scalar system by neglecting the back reaction of the scalar field into the Einstein field equations. In this sense the Reissner-Nordström spacetime is taken as a background on which we will study the spherically symmetric wave equation.

There are 4 different regions of the spacetime that can be covered by this local coordinate charts, exterior region, parallel exterior region, black hole and white hole. In this discussion, we will only work with the subspacetime of the exterior region, the black hole region (which we will call interior region) and the event horizon $\mathscr{H}^+ = \{r = r_+\}$ which is the region that divides those two regions. We will mostly work with null coordinate (u, v).

2.1 Coordinate in the exterior region

In the exterior region, consider the following change of coordinate given by

$$v = \frac{1}{2}(t + r^*) \text{ and } u = \frac{1}{2}(t - r^*),$$

where

$$r^* := r + \left(M + \frac{2M^2 - e^2}{2\sqrt{M^2 - e^2}}\right)\log(r - r_+) + \left(M - \frac{2M^2 - e^2}{2\sqrt{M^2 - e^2}}\right)\log(r - r_-).$$

So we have

$$t = v + u, \quad r^* = v - u, \quad \partial_v = \partial_{r^*} + \partial_t \quad \text{and} \quad \partial_u = \partial_t - \partial_{r^*}.$$

The metric takes the form

$$g = -4\Omega^2 du dv + r^2 d\sigma_{\mathbb{S}^2}.$$

where $\Omega^2 = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)$. We get that $\partial_v r = -\partial_u r = \Omega^2$. The event horizon $\mathscr{H}^+$ formally corresponds to the limit $\{u = \infty\}$.

2.2 Coordinate in the interior region

In the interior region, in the same way consider the follwing change of coordinate

$$v = \frac{1}{2}(t + r^*) \text{ and } u = \frac{1}{2}(r^* - t),$$

where

$$r^* := r + \left(M + \frac{2M^2 - e^2}{2\sqrt{M^2 - e^2}}\right)\log|r - r_+| + \left(M - \frac{2M^2 - e^2}{2\sqrt{M^2 - e^2}}\right)\log(r - r_-).$$

We get

$$\partial_v = \partial_{r^*} + \partial_t, \quad \partial_u = \partial_{r^*} - \partial_t.$$

The metric takes the form

$$g = -4\Omega^2 du dv + r^2 d\sigma_{\mathbb{S}^2},$$

where $\Omega^2 = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)$. We get that $\partial_v r = -\partial_u r = -\Omega^2$. The event horizon $\mathscr{H}^+$ formally corresponds to the limit $\{u = -\infty\}$ and the Cauchy horizon $\mathscr{CH}^+$ to the limit $\{v = \infty\}$.

2.3 The wave equation

The wave equation is $\Box_g \phi = 0$, where $\Box_g$ is the Laplace-Beltrami operator with respect to Reissner-Nordström metric. We will only be considering spherically symmetric solutions, which takes the form

$$\partial_u \partial_v \phi = -\frac{\partial_v r \partial_u \phi}{r} - \frac{\partial_u r \partial_v \phi}{r}. \tag{2.1}$$

So in the exterior it takes the form

$$\partial_u \partial_v \phi = \frac{\Omega^2}{r}(\partial_v \phi - \partial_u \phi), \quad \partial_u \partial_v (r\phi) = -\frac{2\left(M - \frac{e^2}{r}\right)\Omega^2 \phi}{r^2}, \tag{2.2}$$

and the interior it reads

$$\partial_u \partial_v \phi = \frac{\Omega^2}{r}(\partial_v \phi + \partial_u \phi). \tag{2.3}$$

We say that a function is in $W^{1,2}_{\text{loc}}$ when its $W^{1,2}$-norm is finite for all open set $\mathscr{U}$ with compact closure. The $W^{1,2}$-norm for $\mathscr{U}$ is

$$\int_{\mathscr{U}} \left(\Omega^{-2}\left(\partial_v \phi\right)^2 + \Omega^2\left(\left(\partial_u \phi\right)^2 + \phi^2\right)\right) du\, dv.$$

The reason taking $\frac{\partial_v}{\Omega^2}$ is that this vector is smoothly extendible across the Cauchy horizon, so that we can capture the blue-shift effect.

3 The main steps of the proof

Since we will only be considering spherically symmetric scalar fields on Reissner-Nordström, it will be more convenient to work with Reissner-Nordström quotiented by $SO(3)$ which is a (1+1)-dimensional Lorentzian manifold. We will introduce some notation that will be practical to prove theorem 1.3.

$$\begin{aligned}
C_{u'} &:= \{u = u'\}, \\
\underline{C}_{v'} &:= \{v = v'\}, \\
\underline{C}^{\text{ext}}_{v'} &:= \underline{C}_{v'} \cap \{r \geq r_+\}, \\
\underline{C}^{\text{int}}_{v'} &:= \underline{C}_{v'} \cap \{r \leq r_+\}, \\
\gamma_R &:= \{r = R\}.
\end{aligned}$$

Integrals will always be taken with respect to the measure $dudv$ unless otherwise stated.

Instead of directly working with initial data on the Cauchy surface it is easier to work with initial data on $\left(\left(\underline{C}^{\text{ext}} \cap \{u \geq U_0\}\right) \cup \left(\underline{C}^{\text{int}} \cap \{u \leq -1\}\right)\right) \subset \underline{C}_1$ and $C_{-U_0} \cap \{v \geq 1\}$, which take the following form.

(C1) For some $D > 0$, ϕ is C^1 on $C_{-U_0} \cap \{v \geq 1\}$ and

$$\sup_{C_{-U_0}\cap\{v\geq 1\}} r^2 \left|\frac{\partial_v \phi}{\partial_v r}\right| \leq D, \qquad \sup_{C_{-U_0}\cap\{v\geq 1\}} r^2 \left|\frac{\partial_v (r\phi)}{\partial_v r}\right| \leq D.$$

(C2) For some $D > 0$, ϕ is C^1 on $C_{-U_0} \cap \{v \geq 1\}$ and

$$\sup_{\left(\underline{C}^{\text{ext}}\cap\{u\geq U_0\}\right)\cup\left(\underline{C}^{\text{int}}\cap\{u\leq -1\}\right)} \left|\frac{\partial_v \phi}{\partial_v r}\right| \leq D, \qquad \sup_{\left(\underline{C}^{\text{ext}}\cap\{u\geq U_0\}\right)\cup\left(\underline{C}^{\text{int}}\cap\{u\leq -1\}\right)} |\phi| \leq D$$

(C3)

$$\lim_{v\to\infty} r^3 \partial_v (r\phi)(-U_0, v) \text{ exists.}$$

Smooth and compactly supported initial data on a Cauchy hypersurface can be shown to have solution that satisfy (C1), (C2) and (C3). Let

$$\mathscr{L} := \lim_{v\to\infty} r^3 \partial_v (r\phi)(-U_0, v) - \int_{-U_0}^{\infty} 2M\Phi(u)du$$

where $\Phi(u) := \lim_{v\to\infty} r\phi(u,v)$. This quantity will be important for the following theorem.

Theorem 3.1 (Theorem 1.3 of Luk and Oh [2017]). *Let ϕ satisfy* (2.1)*, (C1), (C2) and (C3). If $\mathscr{L} \neq 0$, then in the interior region, for all $u \in \mathbb{R}$ and integer $\alpha_0 > 7$ we have*

$$\int_1^{\infty} \log^{\alpha_0}\left(\frac{1}{\Omega}\right) (\partial_v \phi)^2 (u,v) dv = \infty. \tag{3.1}$$

Observe that if (3.1) is satisfied, then near the Cauchy horizon the $W^{1,2}_{\text{loc}}$-norm is infinite. Now all is left is to show that such solution exists and is generic.

Theorem 3.2 (Theorem 1.5 of Luk and Oh [2017]). *For $U_0 > 1$, there exists ϕ solution to* (2.1) *with smooth and compactly supported initial data on $\underline{C}_1$ and vanishing on C_{-U_0} such that $\mathscr{L} \neq 0$.*

To summarize, with the constructed solution satisfying $\mathscr{L} \neq 0$ in theorem 3.2, and from theorem 3.1 we can conclude theorem 1.3 by showing that such solution is generic, in the sense that for any perturbation $\epsilon\phi$ added to a solution by linearity it will have $\mathscr{L} \neq 0$. For more details see proof of Corollary 1.6 from Luk and Oh [2017].

In order to prove theorem 3.1 some intermediate results need to be shown. In the rest of this section, I will show how those intermediate results imply theorem 3.1. The following two theorems allow us to derive a condition on the event horizon for solutions that satisfy $\mathscr{L} \neq 0$.

Theorem 3.3 (Theorem 1.7 of Luk and Oh [2017]). *There exits a $R_1(M) > 2M$ such that for all ϕ that satisfy* (2.1)*, (C1), (C2), (C3), if $\mathscr{L} \neq 0$ and there exists $A' > 0$ such that*

$$\sup_{\{r=R_1\}\cap\{u\geq 1\}} u^3 |\phi| \leq A', \tag{3.2}$$

then there exist $R(\mathscr{L}, A', D, U_0, R_1) \geq R_1$ and $U(\phi)$ such that

$$|\partial_v(r\phi)|(u,v) \geq \frac{|\mathscr{L}|}{8} v^{-3} on\ \{(u,v) : u \geq U and\ r(u,v) = R\} \tag{3.3}$$

Theorem 3.4 (Theorem 1.8 of Luk and Oh [2017]). *For all ϕ that satisfy* (2.1)*, (C1), (C2), if there exist $0 < A < \infty$ and $\epsilon > 0$ such that*

$$\int_{\mathscr{H}^+\{v\geq 1\}} v^{7+\epsilon}(\partial_v\phi)^2 = A, \tag{3.4}$$

then for all $R \geq r_+$ there exists $C > 0$ such that,

$$\sup_{\{r_+\leq r\leq R\}\cap\{v\geq 1\}} v^{3+\frac{\epsilon}{2}} |\phi| \leq C$$

and,

$$\int_{\{r=R,v\geq 1\}} v^5(\partial_v(r\phi))^2 \leq C$$

Observe that under the assumption of theorem 3.4, equation (3.2) holds true. This is because for a fixed R_1 there exists C such that $u \leq Cv$. But, it can also be seen that (3.3) will not hold. This implies the following.

Corollary 3.1 (Corollary 1.9 of Luk and Oh [2017]). *Let ϕ satisfy* (2.1)*, (C1), (C2), (C3) and $\mathscr{L} \neq 0$, then for all $\epsilon > 0$*

$$\int_{\mathscr{H}^+\cap\{v\geq 1\}} v^{7+\epsilon} (\partial_v\phi)^2 = \infty$$

Finally consider a theorem that relates the event horizon to the interior region.

Theorem 3.5 (Theorem 1.10 of Luk and Oh [2017]). *Let ϕ satisfy* (2.1)*, (C1), (C2), (C3). If in the interior region there exist $u \leq -1$ and $\alpha_0 \in \mathbb{N}^\star$ such that*

$$\int_1^\infty \log^{\alpha_0}\left(\frac{1}{\Omega}\right)(\partial_v\phi)^2(u,v)dv < \infty$$

then

$$\int_{\mathscr{H}^+\cap\{v\geq 1\}} \frac{v^{\alpha_0}}{\log^2(1+v)} (\partial_v\phi)^2 < \infty$$

Under the assumption of corollary 3.1, theorem 3.5 directly implies (3.1), which ends our proof of theorem 3.1. Theorem 3.2 and all theorems to prove theorem 3.1 are proven in Luk and Oh [2017] in their respective sections. In the last section of this discussion, I will give a proof of theorem 3.4.

4 Proof of theorem 3.4

In this section, we will be proving theorem 3.4, the proof doesn't rely directly on ϕ satisfying (C1) and (C2), but rather on

(C4) $\phi \to 0$ along $\mathscr{H}^+$ as $v \to \infty$.

It is known from Dafermos and Rodnianski [2005] that (C4) is an implication of (C1) and (C2). This proof will only involve the exterior region, so u, v, in this section, will always refer to the coordinate in the exterior region.

Proposition 4.1. *Let ϕ satisfy* (2.2)*, (C4) and* (3.4)*, then there exists a $C = C(A) > 0$ such that*

$$|\phi| \leq Cv^{-3-\frac{\epsilon}{2}} \text{ on } \mathscr{H}^+ \cap \{v \geq 1\}. \tag{4.1}$$

Proof. In (u, v) coordinate

$$\begin{aligned}
&|\phi(\infty, v)| \\
&= |\phi(\infty, \infty) - \phi(\infty, v)| && \text{from (C4)} \\
&= \left| \int_v^\infty (\partial_v \phi)(\infty, v') dv' \right| \\
&\leq \int_v^\infty |(\partial_v \phi)|(\infty, v') dv' \\
&= \int_v^\infty (v'^{\frac{-7-\epsilon}{2}})(v'^{\frac{7+\epsilon}{2}} |\partial_v \phi|) dv' && \text{because } v' \geq 1 \\
&\leq \left(\int_v^\infty v'^{-7-\epsilon} dv' \right)^{\frac{1}{2}} \left(\int_v^\infty v'^{7+\epsilon} (\partial_v \phi)^2 (\infty, v') dv' \right)^{\frac{1}{2}} && \text{by Cauchy-Schwarz} \\
&\leq Cv^{-3-\frac{\epsilon}{2}}
\end{aligned}$$

□

Proposition 4.2. *Let ϕ satisfy* (2.1) *and (C4). If there exist $0 < A < \infty$ and $\epsilon > 0$ such that*

$$\int_{\mathscr{H}^+ \{v \geq 1\}} v^{7+\epsilon} (\partial_v \phi)^2 = A,$$

Then for $R_2 > r+$ such that $(R_2 - r_+)$ is sufficiently small, there exists $C = C(A, \epsilon, R_2, D) > 0$ such that,

$$\sup_{(u,v) \in \{r_+ \leq r \leq R_2\} \cap \{v \geq 1\}} v^{\frac{7+\epsilon}{2}} \left| \frac{\partial_u \phi}{\partial_u r} \right| (u, v) \leq C \tag{4.2}$$

$$\int_{C_u \cap \{r_+ \leq r \leq R_2\} \cap \{v \geq 1\}} v^{7+\epsilon} (\partial_v \phi)^2 \leq C \tag{4.3}$$

$$\sup_{(u,v) \in \{r_+ \leq r \leq R_2\}} v^{3+\frac{\epsilon}{2}} |\phi| \leq C \tag{4.4}$$

$$\sup_{r_+ \leq r \leq R_2} \int_{\gamma_r \cap \{v \geq 1\}} v^{7+\epsilon} (\partial_v \phi)^2 \leq C \tag{4.5}$$

Proof. Let $\epsilon > 0$ and $\alpha = 7 + \epsilon$ and observe that,

$$\begin{aligned}\frac{\partial_v \Omega^2}{\Omega^2} &= \frac{1}{\Omega^2}\left(\partial_v r \partial_r + \partial_v t \partial_t\right)\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) \\ &= \frac{2M}{r^2} - \frac{2e^2}{r^3},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial_v \Omega^2}{\Omega^2}(r_+) &= \frac{1}{r_+}\left(\frac{2M}{r_+} - \frac{e^2}{{r_+}^2} - \frac{e^2}{{r_+}^2}\right) \\ &= \frac{1}{r_+}\left(1 - \frac{e^2}{{r_+}^2}\right) \\ &= \frac{r_+^2 - e^2}{r_+^3} \\ &= \frac{M^2 + 2M\sqrt{M^2 - e^2} - e^2}{r_+^3} \\ &> 0. && \text{since } M^2 > e^2\end{aligned}$$

So by continuity of $\frac{\partial_v \Omega^2}{\Omega^2}$, there exist $R_2' > r_+$ and $c > 0$ such that

$$\frac{\partial_v \Omega^2}{\Omega^2} \geq c \quad \text{for } r \in [r_+, R_2'].$$

Furthermore, there exists $V = V(R_2', \alpha, c) > 0$ such that

$$\frac{\partial_v \Omega^2}{\Omega^2} - \frac{\alpha}{2(v+V)} \geq \frac{c}{2} \quad \text{for } r \in [r_+, R_2'] \text{ and } v \geq 1.$$

Consider the following equalities.

$$\begin{aligned}\partial_v\left(\frac{\partial_u \phi}{\Omega^2}\right) &= -\frac{\partial_u \phi \partial_v \Omega^2}{\Omega^4} + \frac{\partial_v \partial_u \phi}{\Omega^2} \\ &= -\frac{\partial_u \phi \partial_v \Omega^2}{\Omega^4} + \frac{1}{r}(\partial_v \phi - \partial_u \phi) && \text{by (2.2)} \\ &= -\left(\frac{\partial_v \Omega^2}{\Omega^2} + \frac{\Omega^2}{r}\right)\frac{\partial_u \phi}{\Omega^2} + \frac{1}{r}\partial_v \phi\end{aligned}$$

Then we observe the following.

$$\begin{aligned}&\frac{1}{2}\partial_v\left((v+V)^\alpha\left(\frac{\partial_u \phi}{\Omega^2}\right)^2\right) + \left(\frac{\partial_v \Omega^2}{\Omega^2} + \frac{\Omega^2}{r} - \frac{\alpha}{2(v+V)}\right)(v+V)^\alpha\left(\frac{\partial_u \phi}{\Omega^2}\right)^2 \\ &= \frac{1}{2}\alpha(v+V)^{\alpha-1}\left(\frac{\partial_u \phi}{\Omega^2}\right)^2 + (v+V)^\alpha\left(\frac{\partial_u \phi}{\Omega^2}\right)\partial_v\left(\frac{\partial_u \phi}{\Omega^2}\right) + \left(\frac{\partial_v \Omega^2}{\Omega^2} + \frac{\Omega^2}{r} - \frac{\alpha}{2(v+V)}\right)(v+V)^\alpha\left(\frac{\partial_u \phi}{\Omega^2}\right)^2 \\ &= (v+V)^\alpha\left(\frac{\partial_u \phi}{\Omega^2}\right)\left(\partial_v\left(\frac{\partial_u \phi}{\Omega^2}\right) + \left(\frac{\partial_v \Omega^2}{\Omega^2} + \frac{\Omega^2}{r}\right)\left(\frac{\partial_u \phi}{\Omega^2}\right)\right) \\ &= (v+V)^\alpha\left(\frac{\partial_u \phi}{\Omega^2}\right)\left(\frac{\partial_v \phi}{r}\right) && (4.6)\end{aligned}$$

Now integrating (4.6) on $\{u\} \times [v_1, v_2]$ such that $r(u,v) \leq R_2'$ and $v_1 \geq 1$. We have for the first term on the LHS

$$\int_{v_1}^{v_2} \frac{1}{2}\partial_v (v+V)^\alpha \left(\frac{\partial_u \phi}{\Omega^2}\right)^2 (u,v)\, dv$$

$$
\begin{aligned}
&= \frac{1}{2}(v_2+V)^\alpha \left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v_2) - \frac{1}{2}(v_1+V)^\alpha \left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v_1)\\
&\geq K \sup_{v\in[v_1,v_2]} v^\alpha \left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v) - \frac{1}{2}(v_1+V)^\alpha \left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v_1) \text{ for some } K>0.
\end{aligned}
$$

Second term on the LHS

$$
\begin{aligned}
\int_{v_1}^{v_2}\left(\frac{\partial_v\Omega^2}{\Omega^2}+\frac{\Omega^2}{r}-\frac{\alpha}{2(v+V)}\right)(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v)\,dv &\geq \int_{v_1}^{v_2}\left(\frac{c}{2}+\frac{\Omega^2}{r}\right)(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v)\,dv\\
&\geq \int_{v_1}^{v_2}\left(\frac{c}{2}\right)(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v)\,dv.
\end{aligned}
$$

And the RHS, we have

$$
\begin{aligned}
\int_{v_1}^{v_2}(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)\left(\frac{\partial_v\phi}{r}\right)dv &\leq \frac{1}{2}\frac{c}{2}\int_{v_1}^{v_2}(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2dv+\frac{1}{2}\frac{2}{c}\int_{v_1}^{v_2}(v+V)^\alpha\left(\frac{\partial_v\phi}{r}\right)^2dv\\
&\leq \int_{v_1}^{v_2}\frac{c}{4}(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2dv+\frac{1}{cr_+}\int_{v_1}^{v_2}(v+V)^\alpha(\partial_v\phi)^2\,dv.
\end{aligned}
$$

So all together in equation (4.6), we have that for some $C=C(R_2',\alpha,c)>0$

$$
\sup_{v\in[v_1,v_2]} v^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v)+\int_{v_1}^{v_2}v^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v)dv \leq C\left(v_1^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v_1)+\int_{v_1}^{v_2}v^\alpha(\partial_v\phi)^2\,dv\right). \tag{4.7}
$$

In the same way

$$
\frac{\partial_v\Omega}{\Omega}=\frac{1}{2}\left(\frac{2M}{r^2}-\frac{2e^2}{r^3}\right)\geq\frac{c}{2}.
$$

And

$$
\begin{aligned}
\partial_v\left(\frac{\partial_u\phi}{\Omega}\right) &= \frac{\partial_v\partial_u\phi}{\Omega}-\frac{\partial_u\phi}{\Omega}\frac{\partial_v\Omega}{\Omega}\\
&= \frac{\Omega}{r}(\partial_v\phi-\partial_u\phi)-\frac{\partial_u\phi}{\Omega}\frac{\partial_v\Omega}{\Omega} && \text{by (2.2)}\\
&= -\left(\frac{\partial_v\Omega}{\Omega}+\frac{\Omega^2}{r}\right)\frac{\partial_u\phi}{\Omega}+\frac{\Omega\partial_v\phi}{r}
\end{aligned}
$$

Which gives

$$
\begin{aligned}
&\frac{1}{2}\partial_v\left((v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2\right)+\left(\frac{\partial_v\Omega}{\Omega}+\frac{\Omega^2}{r}-\frac{\alpha}{2\,(v+V)}\right)(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2\\
&=\frac{1}{2}\alpha\,(v+V)^{\alpha-1}\left(\frac{\partial_u\phi}{\Omega}\right)^2+(v+V)^\alpha\frac{\partial_u\phi}{\Omega}\partial_v\left(\frac{\partial_u\phi}{\Omega}\right)+\left(\frac{\partial_v\Omega}{\Omega}+\frac{\Omega^2}{r}-\frac{\alpha}{2\,(v+V)}\right)(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2\\
&=(v+V)^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)\left(\partial_v\left(\frac{\partial_u\phi}{\Omega}\right)+\left(\frac{\partial_v\Omega}{\Omega}+\frac{\Omega^2}{r}\right)\frac{\partial_u\phi}{\Omega}\right)\\
&=(v+V)^\alpha\left(\frac{\partial_u\phi\partial_v\phi}{r}\right).
\end{aligned}
$$

Now let $R_2\leq R_2'$. Since $v\geq 1$, let $u_{R_2}(v)$ be the unique u such that $r(u_{R_2}(v),v)=R_2$, similarly for $v_{R_2}(u)$. In the same way as equation (4.7), integrate previous equation on $\{(u,v):u\in[u_{R_2}(v),u_2],v\in[v_1,v_2]\}$ we get

$$
\begin{aligned}
&\int_{u_{R_2}(v_1)}^{u_2} \sup_{v\in[v_1,v_2],r(u,v)\le R_2} v^\alpha \left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v)\,du + \int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v)\,du\,dv\\
&\le C\left(\int_{u_{R_2}(v_1)}^{u_2} v_1^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v_1)\,du + \int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^\alpha\Omega^2(\partial_v\phi)^2(u,v)\,du\,dv\right)\\
&\le C\left(\sup_{u\in[u_1,u_2]} v_1^\alpha\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v_1) + \int_{v_1}^{v_2}\sup_{u\in[u_{R_2}(v),u_2]} v^\alpha(\partial_v\phi)^2(u,v)\,dv\right),
\end{aligned}
\tag{4.8}
$$

because $\sup_{v\in[v_1,v_2]}\int_{u_{R_2}(v)}^{u_2}\Omega^2(u,v)\,du\le C$. Now consider

$$
\begin{aligned}
\frac{1}{2}v^\alpha\partial_u\left((r\partial_v\phi)^2\right) &= v^\alpha r\partial_v\phi\left(\partial_u r\partial_v\phi + r\partial_u\partial_v\phi\right)\\
&= -v^\alpha(\partial_v r\partial_u\phi)(r\partial_v\phi). \qquad\qquad \text{because } \partial_u r = -\partial_v r
\end{aligned}
$$

Integrating it over $u\in[u',u_2]$, then taking the supremum over $u'\in[u_{R_2}(v),u_2]$ and integrating over $v\in[v_1,v_2]$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{v_1}^{v_2}\sup_{u'\in[u_{R_2}(v),u_2]} r^2v^\alpha\left(\partial_v\phi\right)^2(u',v)\,dv\\
&\le \frac{1}{2}\int_{v_1}^{v_2} r^2v^\alpha\left(\partial_v\phi\right)^2(u_2,v)\,dv + \int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^\alpha\left|(\partial_v r\partial_u\phi)(r\partial_v\phi)\right|(u,v)\,du\,dv.
\end{aligned}
$$

We also have that

$$
\begin{aligned}
&\int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^\alpha\left|(\partial_v r\partial_u\phi)(r\partial_v\phi)\right|(u,v)\,du\,dv\\
&\le C\left(\int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v)\,du\,dv\right)^{\frac{1}{2}}\left(\int_{v_1}^{v_2}\sup_{u\in[u_{R_2}(v),u_2]} r^2v^\alpha(\partial_v\phi)^2(u,v)\,dv\right)^{\frac{1}{2}}\\
&\qquad\times\left(\sup_{v\in[v_1,v_2]}\left(\int_{u_{R_2}(v)}^{u_2}\partial_u r)^2\,du\right)^{\frac{1}{2}}\right)\\
&\le C\left(\int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v)\,du\,dv\right)^{\frac{1}{2}}\left(\int_{v_1}^{v_2}\sup_{u\in[u_{R_2}(v),u_2]} r^2v^\alpha(\partial_v\phi)^2(u,v)\,dv\right)^{\frac{1}{2}}(R_2-r_+)^{\frac{1}{2}},
\end{aligned}
$$

where last inequality is because $0\le\partial_v r\le 1$ so $(\partial_v r)^2\le\partial_v r=\Omega^2$. Combining last two steps we get that

$$
\begin{aligned}
&\int_{v_1}^{v_2}\sup_{u'\in[u_{R_2}(v),u_2]} r^2v^\alpha\left(\partial_v\phi\right)^2(u',v)\,dv\\
&\le 2\int_{v_1}^{v_2} r^2v^\alpha\left(\partial_v\phi\right)^2(u_2,v)\,dv + C(R_2-r_+)\left(\int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v)\,du\,dv\right).
\end{aligned}
\tag{4.9}
$$

Therefore for R_2-r_+ sufficiently small, which depends on R_2', r_+, α and c. From equations (4.7), (4.8) and (4.9) we have

$$
\int_{u_{R_2}(v_1}^{u_2}\sup_{v\in[v_1,v_2],r(u,v)\le R_2} v^\alpha\left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v)\,du
$$

$$
\begin{aligned}
&+\int_{v_1}^{v_2}\int_{u_{R_2}(v)}^{u_2} v^{\alpha}\left(\frac{\partial_u\phi}{\Omega}\right)^2(u,v)\,du\,dv\\
&+\sup_{u\in[u_{R_2}(v),u_2],v\in[v_1,v_2]} v^{\alpha}\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v)\\
&+\int_{v_1}^{v_2}\sup_{u\in[u_{R_2}(v),u_2]} v^{\alpha}(\partial_v\phi)^2(u,v)\,dv\\
&\leq C\left(\sup_{u\in[u_1,u_2]} v_1^{\alpha}\left(\frac{\partial_u\phi}{\Omega^2}\right)^2(u,v_1)+\int_{v_1}^{v_2} v^{\alpha}(\partial_v\phi)^2(u_2,v)\,dv\right).
\end{aligned}
$$

So for $u_2=\infty$ $v_1=1$, we can use equation (3.4). We get equations (4.2), (4.3) and (4.5) by observing the independence of v_2. And by integrating over $[u,\infty)\times\{v\}$ equation (4.2) using equation (4.1) we get (4.4). □

It can be observed that for the case where $R\leq R_2$ theorem 3.4 is proven. So now we prove a generalized version of equations (4.4) and (4.5).

Proposition 4.3. *Let $R>R_2$, where R_2 is from Proposition 4.2. Furthermore let ϕ satisfy* (2.1)*, (C4) and* (3.4)*, then there exists $C=C(A,\epsilon,R,R_2)>0$ such that*

$$\sup_{\{r\leq R\}\cap\{v\geq 1\}} v^{3+\frac{\epsilon}{2}}\,|\phi|\leq C, \tag{4.10}$$

and

$$\int_{\gamma_R\cap\{v\geq1\}}\frac{v^{6+\epsilon}}{\log^2(1+v)}\left(\partial_v\phi\right)^2\leq C. \tag{4.11}$$

Proof. Let R^* be that value of R in the in the r^* coordinate, same for R_2^*. Denote $u_{r'^*}(v)=v-r'^*$, $v_{r'^*}(u)=u+r'^*$ and $\gamma_{r'^*}:=\{r^*=r'^*\}$. Furthermore,

$$
\begin{aligned}
\underline{C}_{v'}(r_1^*,r_2^*)&:=\{(u,v'):u_{r_1^*}(v')\leq u\leq u_{r_2^*}(v')\},\\
C_{u'}(r_1^*,r_2^*)&:=\{(u',v):v_{r_1^*}(u')\leq v\leq v_{r_2^*}(u')\}.
\end{aligned}
$$

And given $v_0\geq1$, let

$$\gamma_{r'^*}^{(v_0)}:=\gamma_{r'^*}\cap\{v_{r'^*}(u_{R^*}(v_0))\leq v\leq 2v_0\}.$$

Let $\mathscr{D}^{(v_0)}(r_1^*,r_2^*)$ be the region bounded by $\gamma_{r_1^*}^{(v_0)}$, $\gamma_{r_2^*}^{(v_0)}$, $\underline{C}_{2v_0}(r_1^*,r_2^*)$ and $C_{u_{R^*}(v_0)}(r_1^*,r_2^*)$. Observe that,

$$
\begin{aligned}
-\frac{1}{2}\partial_v\left(\partial_u\phi\right)^2+\frac{1}{2}\partial_u\left(\partial_v\phi\right)^2&=\partial_u\partial_v\phi\left(\partial_v-\partial_u\right)\phi\\
&=\left(-\frac{\Omega^2}{r}\partial_u\phi+\frac{\Omega^2}{r}\partial_v\phi\right)\left(\partial_v-\partial_u\right)\phi. \qquad \text{by (2.2)}
\end{aligned}
$$

Integration on $\mathscr{D}^{(v_0)}(R_2^*,r*)\cap\{v\leq v_1\}$, where $r^*\in[R_2^*,R^*]$ and $v_1\in[v_0,2v_0]$. So for some $C=C(R_2^*,R^*)>0$ we get

$$
\begin{aligned}
&\sup_{r^*\in[R_2^*,R^*]}\left(\int_{\gamma_{r^*}^{(v_0)}}\left(\partial_v\phi\right)^2+\left(\partial_u\phi\right)^2\right)+\sup_{v\in[v_0,2v_0]}\int_{\underline{C}_v(R_2^*,R^*)}\left(\partial_u\phi\right)^2\\
&\leq C\left(\int_{\gamma_{R_2^*}^{(v_0)}}\left(\partial_v\phi\right)^2+\left(\partial_u\phi\right)^2+\int_{R_2^*}^{R^*}\left(\int_{\gamma_{r^*}^{(v_0)}}\left(\partial_v\phi\right)^2+\left(\partial_u\phi\right)^2\right)dr^*\right).
\end{aligned}
$$

If we let $f(r'^*) = \int_{\gamma_{r'^*}^{(v_0)}} (\partial_v\phi)^2 + (\partial_u\phi)^2$. We have

$$f(r'^*) \leq Cf(R_2^*) + \int_{R_2^*}^{R*} f(r^*)\, dr^*.$$

And so by Grownwall's inequality we get

$$f(R^*) \leq Cf(R_2^*)\exp{(R^* - R_2^*)} = Cf(R_2^*).$$

So for all $v_0 \geq 1$, there exists $C = C(R_2^*, R^*) > 0$ such that

$$\sup_{r^*\in[R_2^*,R^*]} \left(\int_{\gamma_{r^*}^{(v_0)}} (\partial_v\phi)^2 + (\partial_u\phi)^2 \right) + \sup_{v\in[v_0,2v_0]} \int_{\underline{C}_v(R_2^*,R^*)} (\partial_u\phi)^2 \leq C \left(\int_{\gamma_{R_2^*}^{(v_0)}} (\partial_v\phi)^2 + (\partial_u\phi)^2 \right). \quad (4.12)$$

It can be observed that for $v_0 \geq 2(R^* - R_2^*)$

$$\gamma_{R_2^*}^{(v_0)} \subseteq \gamma_{R_2^*} \cap \{\frac{1}{2}v_0 \leq v \leq 2v_0\}.$$

From (4.2) and (4.5) we get that for some $C = C(A, \epsilon, R_2, D) > 0$

$$\int_{\gamma_{R_2^*}\cap\{\frac{1}{2}v_0\leq v\leq 2v_0\}} v^{6+\epsilon} \left((\partial_v\phi)^2 + (\partial_u\phi)^2 \right) \leq C. \quad (4.13)$$

By summing over all $v_0 \in 2^{\mathbb{N}}$ we will cover all $v \geq 1$. So if we sum up equation (4.12) and (4.13) we get equation (4.11), where $\log^{-2}(1+v)$ is multiplied for convergence of sum. We also get

$$\sup_{\{v\geq 1\}} \int_{\underline{C}_v(R_2^*,R^*)} v^{6+\epsilon} (\partial_u\phi)^2 = \sup_{\{v\geq 1\}} v^{6+\epsilon} \int_{\underline{C}_v(R_2^*,R^*)} (\partial_u\phi)^2 \leq C.$$

From this previous inequality we get a bound for equation (4.10) but only for $R_2^* \leq R^*$, we get the rest of the bound between $r_+ \leq r \leq R_2^*$ with equation (4.4) which implies (4.10).This is the first bound of theorem 3.4. $\square$

Proof of Theorem 3.4. All that is rest to do is to bound the second term of theorem 3.4.

$$\begin{aligned}
&\int_{\gamma_R\cap\{v\geq 1\}} v^5 (\partial_v(r\phi))^2 \\
&= \int_{\gamma_R\cap\{v\geq 1\}} v^5 (r\partial_v\phi)^2 + 2\int_{\gamma_R\cap\{v\geq 1\}} v^5 r\Omega^2\partial_v\phi + \int_{\gamma_R\cap\{v\geq 1\}} v^5\Omega^2\phi^2 \\
&\leq 2R^2 \int_{\gamma_R\cap\{v\geq 1\}} v^5 (\partial_v\phi)^2 + 2\Omega_R^2 \int_{\gamma_R\cap\{v\geq 1\}} v^5\phi^2 && \text{by Cauchy Schwarz} \\
&\leq C. && \text{by equation (4.10) and (4.11),} \\
& && \text{since } \frac{v}{\log^2(1+v)} > 1 \text{ for } v \geq 1
\end{aligned}$$

$\square$

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