EAS596- Take Home Final

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Question 1

Part A

$$\dot{y} = y(t), \qquad y(0) = 0$$

$$y(t) = \begin{cases} y\left[-2t + \frac{1}{t}\right], & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Find solution $y(1)$

Using RK4 to approximate the solution:

$$y(1) = 0.4330$$

Part B

$$y\dot{(t)} = y(t)$$

For t ≠ 0

$$\frac{\dot{y}(t)}{y} = -2t + \frac{1}{t}$$

$$\frac{1}{y}\frac{dy}{dt} = -2t + \frac{1}{t}$$

$$\int \frac{1}{y} dy = \int -2t + \frac{1}{t} dt$$

$$\log(y(t)) = -t^2 + \log(t) + c$$

$$y(t) = te^{c-t^2}$$

Simplify the constant

$$y(t) = c_1 t e^{-t^2}$$

Part C

Now using the initial condition $y'(0) = 1$

$$y'(t) = c_1 e^{-t^2} - 2c_1 t^2 e^{-t^2}$$

$$c_1 = 1$$

$$y(t) = te^{-t^2}$$

Now evaluating the analytical solution for $y(1)$

$$y(1) = 1e^{-(1)^2} = \frac{1}{e} = 0.3679$$

Part D

Rewriting the equation for Gaussian Quadrature the equivalent integral becomes:

$$y(t) = \int_0^1 te^{-t^2}.dt = \int_{-0.5}^{0.5} \left(\frac{x+1}{2}\right) e^{-\left(\frac{x+1}{2}\right)^2}.dx$$

Table – Comparison of the Accuracy of the Numerical solutions to y(1)

Method	Estimate	Error (%)
RK4 (n=1000)	0.4330	+ 17.70
Simpson's Composite (n=1000)	0.3161	- 14.09 %
Gaussian Quadrature (n = 2)	0.3600	+ 0.4149 %
Gaussian Quadrature (n = 3)	0.3696	+ 0.4763 %
Gaussian Quadrature (n = 4)	0.3696	+ 0.4759 %
Gaussian Quadrature (n = 5)	0.3712	+ 0.9033 %

Using Gaussian Quadrature only 2 points are needed to get within 1% accuracy of the solution. Comparing the accuracy, we can see that Gaussian quadrature is an order of magnitude more accurate on this problem compared with RK4 and Simpson's Composite. Not only is Gaussian Quadrature more accurate on this problem but it is less computationally expensive than the brute force approach of RK4. This shows the power and elegance of quadrature in selecting optimal abscissas at which to evaluate, and weighting those evaluations accordingly.

Question 2

Part A

From equation 1

$$x = L_1 \cos\theta_1 + L_2 \cos\theta_2$$

$$0 = x - \ L_1 \cos\theta_1 + L_2 \cos\theta_2$$

From equation 2

$$h = L_1 \sin\theta_1 + L_2 \sin\theta_2$$

$$0 = h - \ L_1 \sin\theta_1 + L_2 \sin\theta_2$$

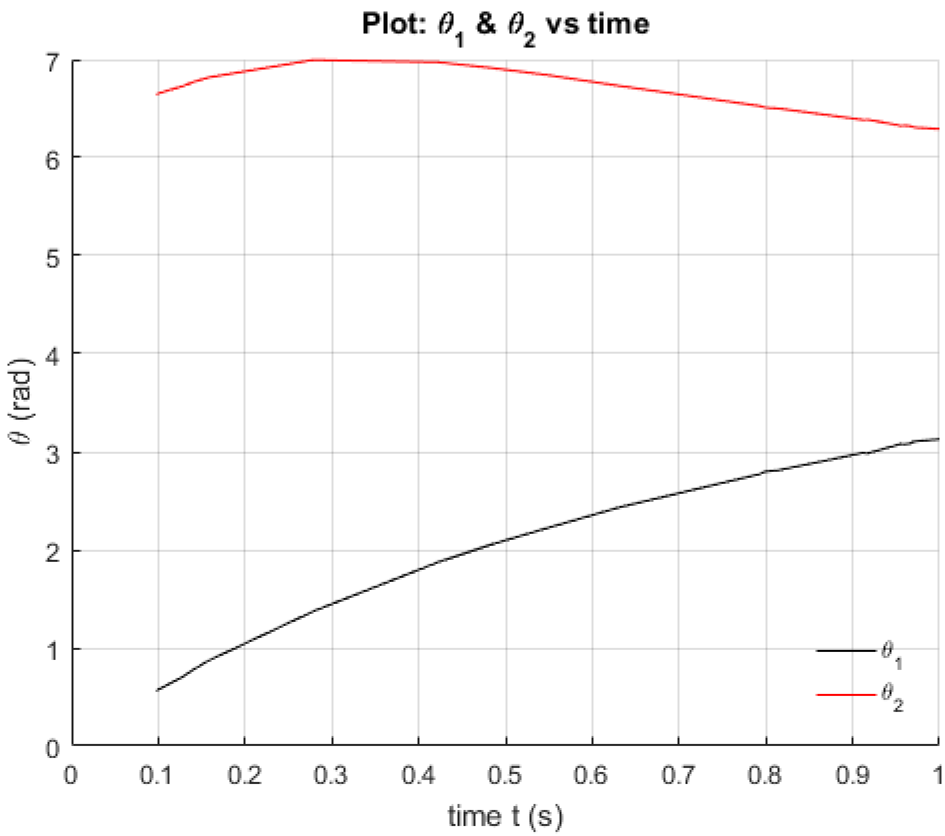
$$x - \ L_1 \cos\theta_1 + L_2 \cos\theta_2 = \ h - \ L_1 \sin\theta_1 + L_2 \sin\theta_2$$

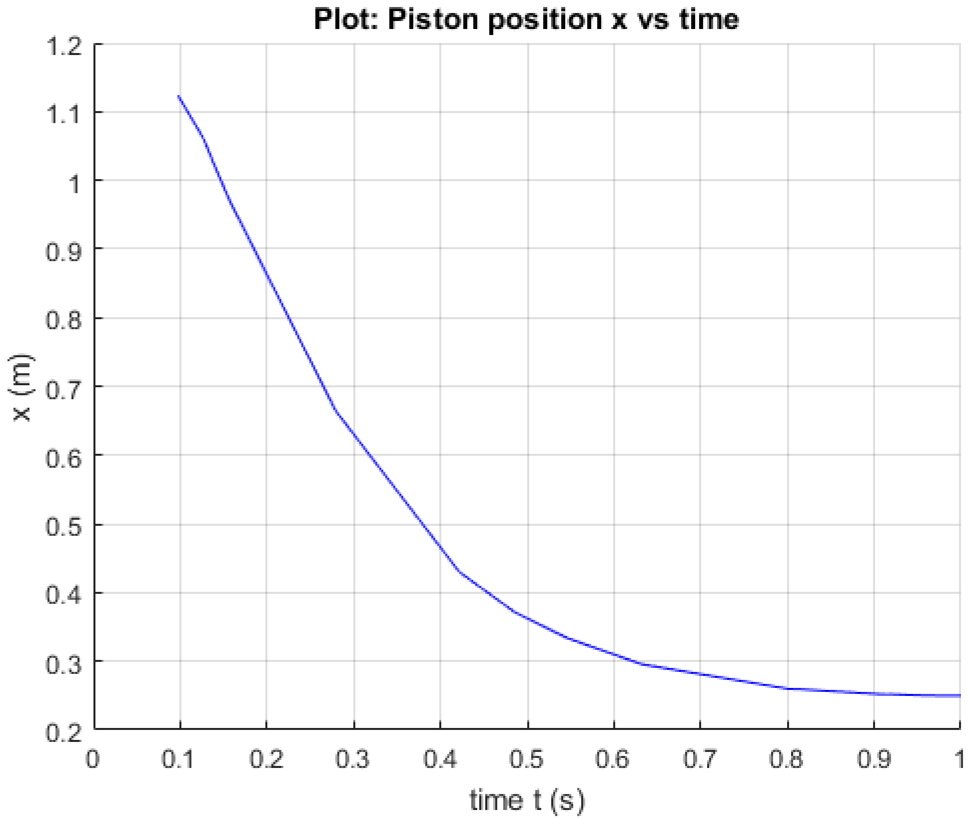
$$f(x, \theta_2) = \ x - h + \ \ L_1(\sin\theta_1 - \ \cos\theta_1) + L_2(\sin\theta_2 - \ \cos\theta_2) = 0$$

This is now a root finding problem with x, θ_2 unknowns, and θ_1 L_1 L_2 and h given.

Part B

Section (iii)





Section (iii)

$$f(x, \theta_2) = x - h + L_1(\sin\theta_1 - \cos\theta_1) + L_2(\sin\theta_2 - \cos\theta_2) = 0$$

Substituting in given values

$$f(x, \theta_2) = x + 0.5(\sin\theta_1 - \cos\theta_1) + 0.75(\sin\theta_2 - \cos\theta_2) = 0$$

NOTE:

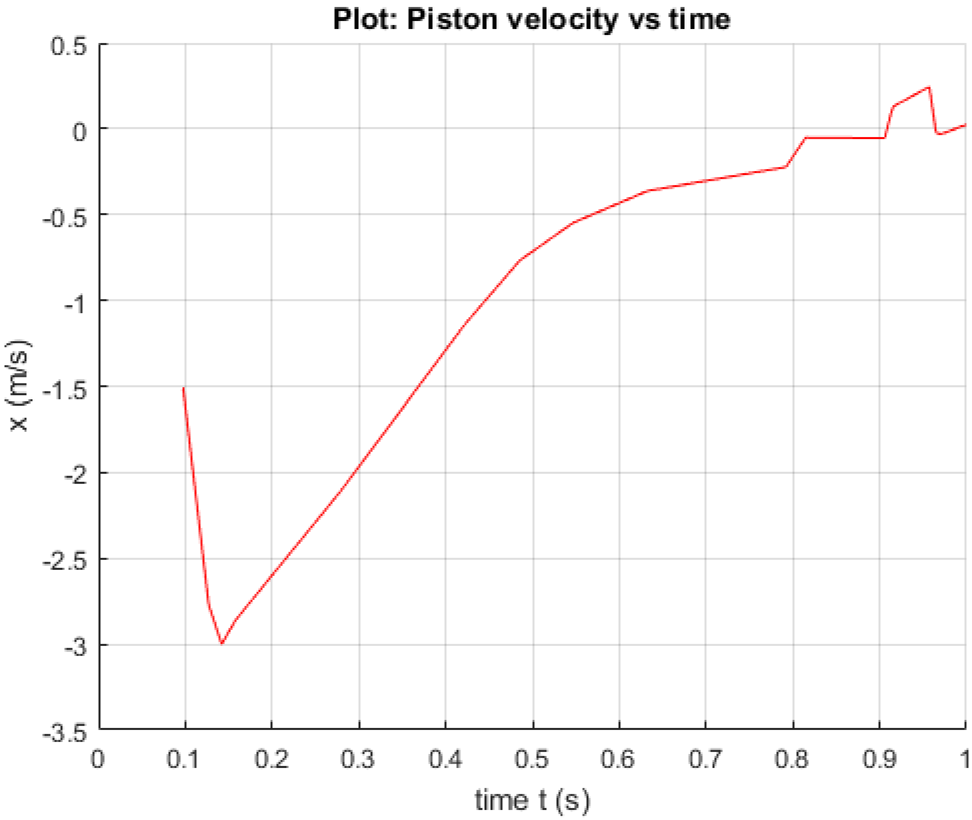
$$\sin\theta - \cos\theta = -\sqrt{2}\sin(\frac{\pi}{4} - \theta)$$

Section (iv)

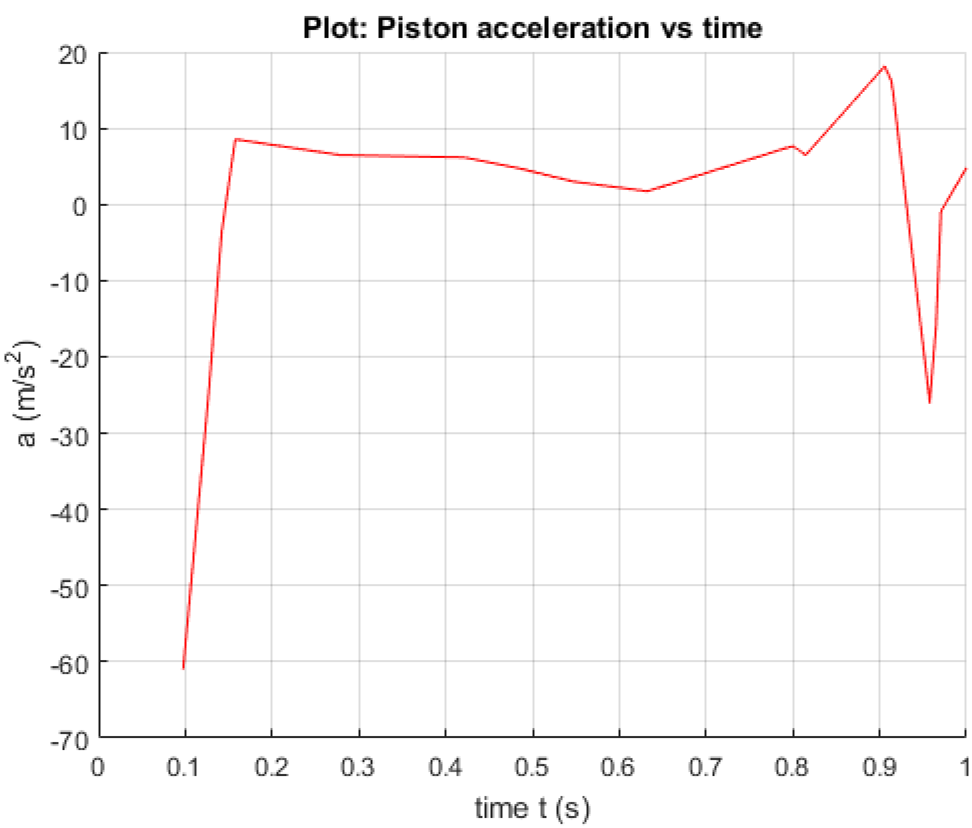
Velocity and acceleration were computed using the following scheme:

- Forward difference at the left end;
- Backward difference at the right end;
- Central difference in the interior.

Since the data was said to have noise, the h were evaluated at each point. This was done in case the time-steps also had noise, and was uneven.

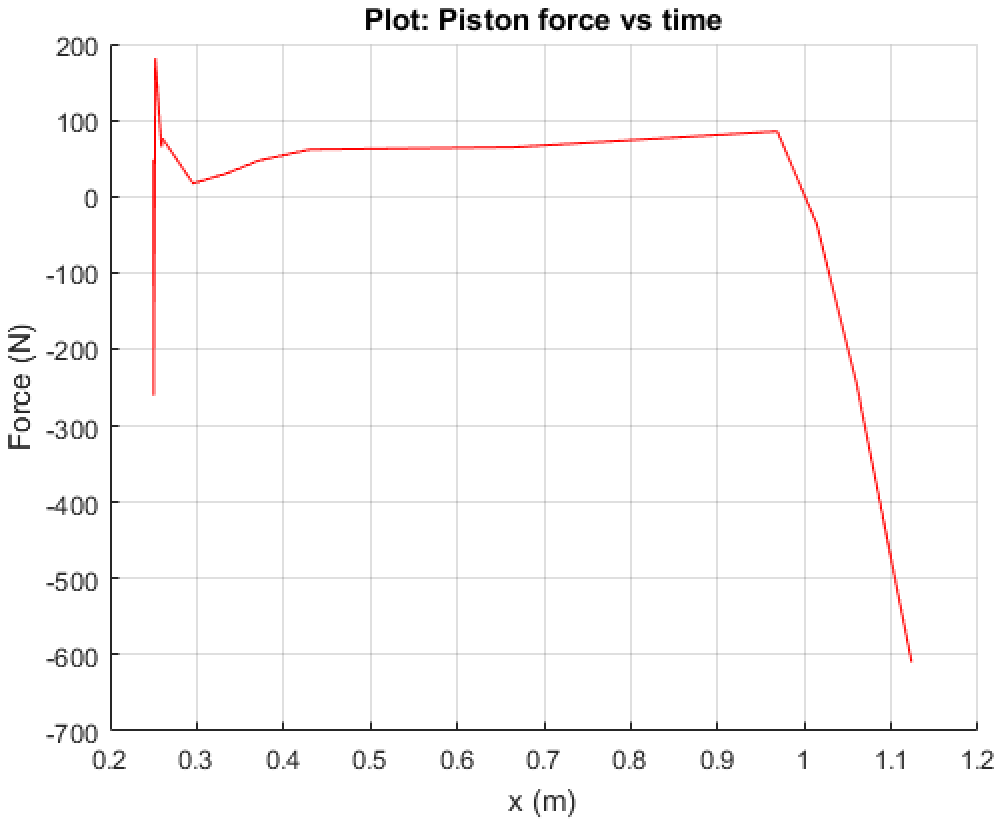


Plot: Piston velocity vs time
x (m/s)
time t (s)
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Plot: Piston acceleration vs time
a (m/s^2)
time t (s)
20
10
0
-10
-20
-30
-40
-50
-60
-70
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Section (v)



Section (vi)

$$W = m \int_{x_o}^{x_1} a.\,dx$$

With the integral being the evaluation of the area under the acceleration vs x curve using the trapezoidal rule.

W = -13.2220 J

Part C

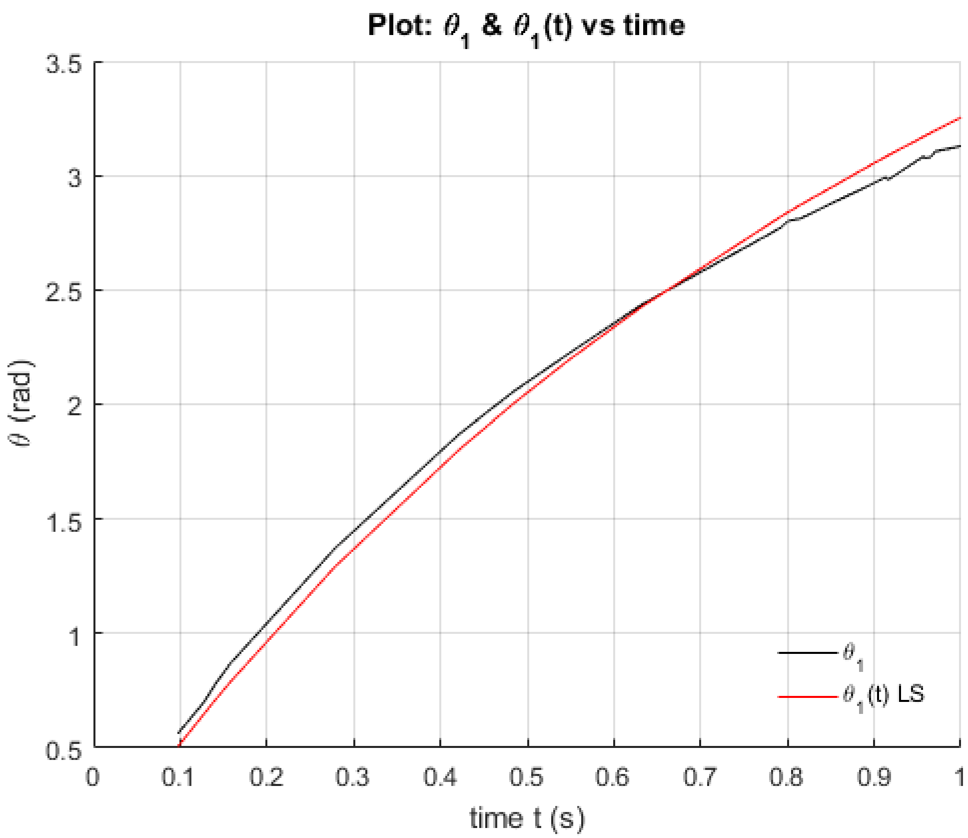
$$\theta_1(t) = \frac{c_1 t}{t + c_2}$$

Evaluated

C1 = 7.8540

C2 = 1.4142

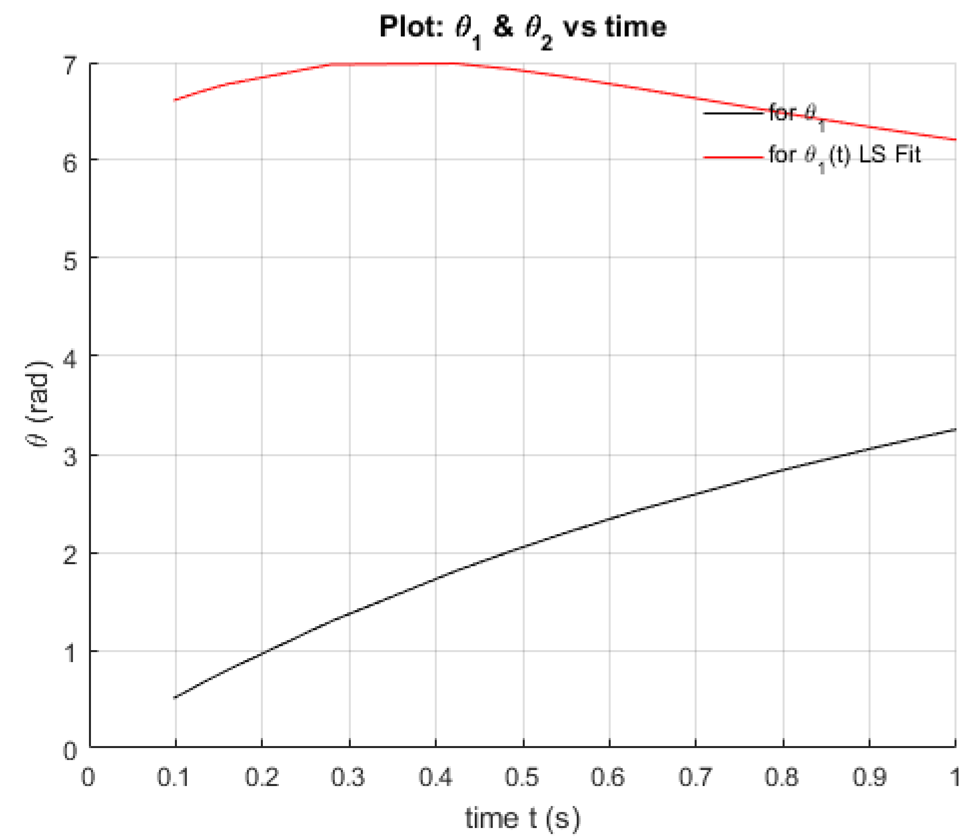
Examining the fit below we can see the function is a good approximation.

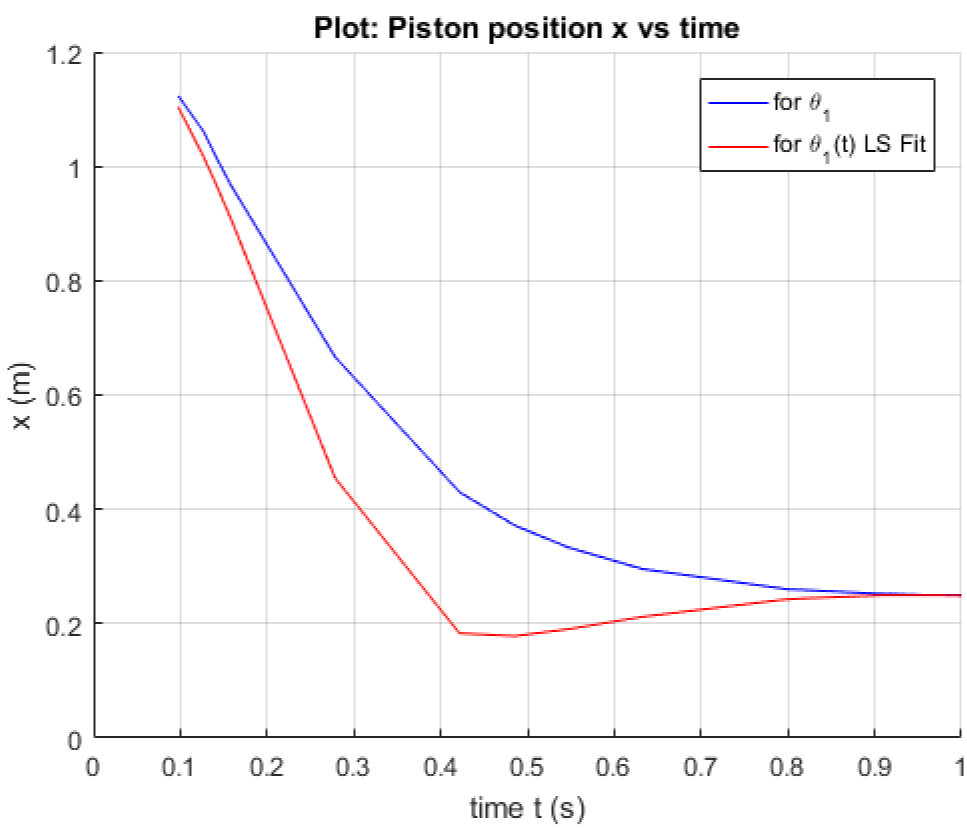


Part D

Repeating the previous steps with curve fit for $\theta_1(t)$.

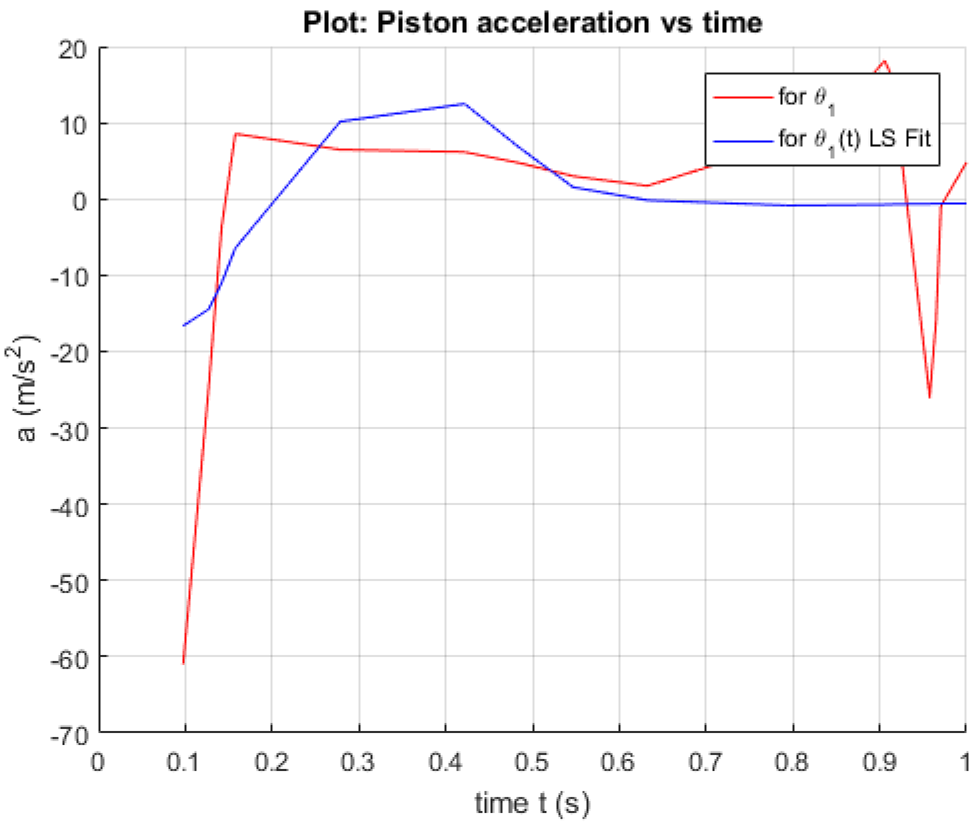
See graphs below of non-linear solution of x and θ_1


Plot: θ_1 & θ_2 vs time
for θ_1
for θ_1(t) LS Fit
θ (rad)
time t (s)
0
1
2
3
4
5
6
7
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

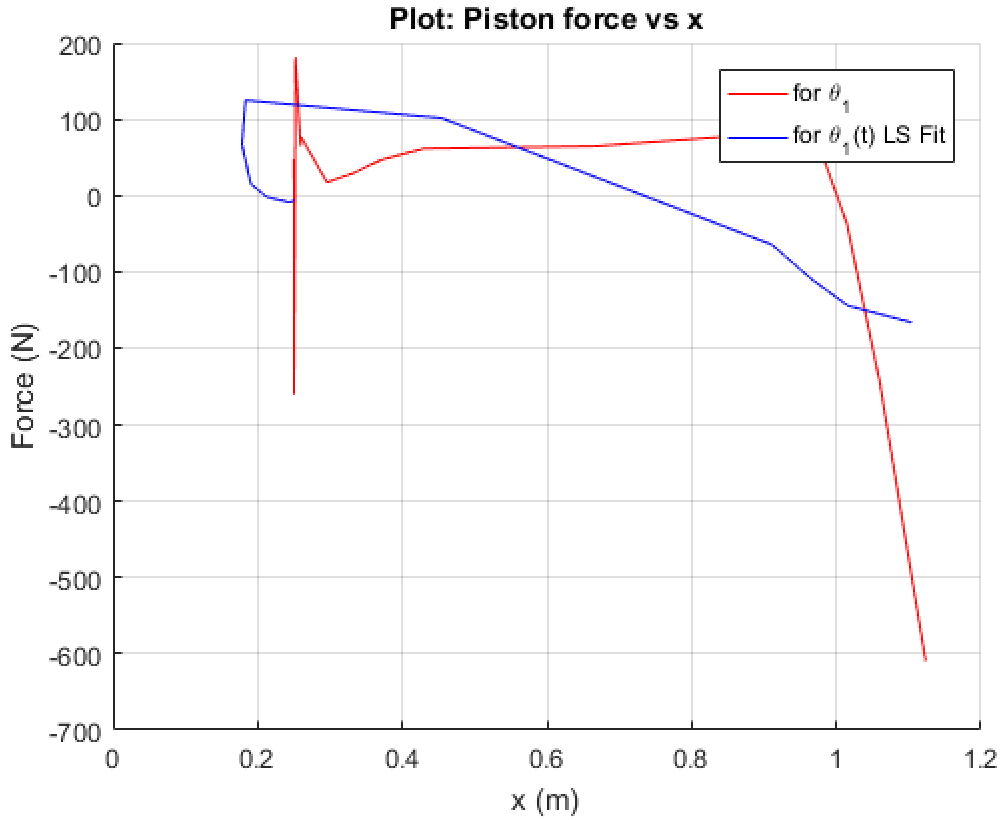

Plot: Piston position x vs time
for θ_1
for θ_1(t) LS Fit
x (m)
time t (s)
0
0.2
0.4
0.6
0.8
1
1.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

As previously, the velocity and acceleration were computed using the following scheme:

- Forward difference at the left end;
- Backward difference at the right end;
- Central difference in the interior.



As we can see above the piston acceleration derived from the $\theta_1(t)$ least squares approximation is smoother, especially at t ≈ 0.15, and t ≈ 0.95. These discontinuities in the original plot may have arisen due to the switch between differentiation methods at the left and right ends. The least squares approximation does not suffer as much from this issue.



As we can see above the piston force derived from the $\theta_1(t)$ least squares approximation is smoother.

$$W = m\int_{x_o}^{x_1} a.dx$$

With the integral being the evaluation of the area under the acceleration vs x curve using the trapezoidal rule.

W = - 14.2496 J

Case	Calculate Work	Difference
θ_1	-13.2220 J	-3.74 %
$\theta_1(t)$	- 14.2496 J	3.74 %
Ave	-13.7358 J	N/A

Comparing the calculated solutions above against the average of the two solutions we can see that each solution varies from each other by about ± 3.74 %.

Question 3

Part A

$$Given\ points\ p(x,y), and\ q(x(s),y(s))$$

The distance between the two points is given by the equation below:

$$f\big(x(s),y(s)\big) = \sqrt{\big(x - x(s)\big)^2 + \big(y - y(s)\big)^2}$$

The points p and q are closest when this distance is minimized.

Part B

$$\begin{gathered} x(s) = \cos(s) \\ y(s) = \sin(s) \\ for\ s \in [0, 2\pi] \end{gathered}$$

$$f(s) = \sqrt{(x - \cos(s))^2 + (y - \sin(s))^2}$$

Part C

Now, this specifies a unit circle.

By geometry the angle s which minimizes the distance is.

Therefore

$$s_{min} = \begin{cases} 0, & x > 0, \quad y = 0 \\ \tan^{-1}\left(\frac{y}{x}\right), & x > 0, \quad y > 0 \\ \frac{\pi}{2}, & x = 0, \quad y > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2}, & x < 0, \quad y > 0 \\ \pi, & x < 0, \quad y = 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi, & x < 0, \quad y < 0 \\ \frac{3\pi}{2}, & x = 0, \quad y < 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \frac{3\pi}{2}, & x > 0, \quad y < 0 \end{cases}$$

The minimum will also occur at the root of $f'(s)$

Therefore point the closest point q will be $(\cos(s_{min}), \sin(s_{min}))$

Part D

$$p(4,2)$$

Analytical solution:

$$s_{min} = \tan^{-1}\left(\frac{2}{4}\right) = 0.4636$$

$$q(\cos(s_{min}), \sin(s_{min}))$$
$$q(0.8944, 0.4472)$$

Numerical Solution:

Minimize f(s) using fminbnd

$$f(s) = \sqrt{(x-\cos(s))^2 + (y-\sin(s))^2}$$

$$q(0.8944, 0.4472)$$

Solution	q(x,y)	Smin Absolute Rel Error
Analytical	(0.8944,0.4472)	N/A
Numerical (fminbnd)	(0.8944,0.4472)	6.2794e-11

Part E

Using Newton's method for minimization requires finding the root of the equation $f'(s)$

The function diff(f, s) was used to get the first and second derivatives of $f(s)$.

$$q(0.8944, 0.4472)$$

Solution	q(x,y)	Smin Absolute Rel Error
Analytical	(0.8944,0.4472)	N/A
Numerical (fminbnd)	(0.8944,0.4472)	6.2794e-11
Numerical (Newton)	(0.8944,0.4472)	4.9651e-17

Newton's method had an absolute relative error of 4.9651e-17, which was better than the solution using fminbnd which only yielded an absolute relative error of 6.2794e-11.

Part F

$$Given\ points\ p(x,y), and\ q(x(s),y(s))$$

The distance between the two points is given by the equation below:

$$f(x(s),y(s)) = \sqrt{(x - x(s))^2 + (y - y(s))^2}$$

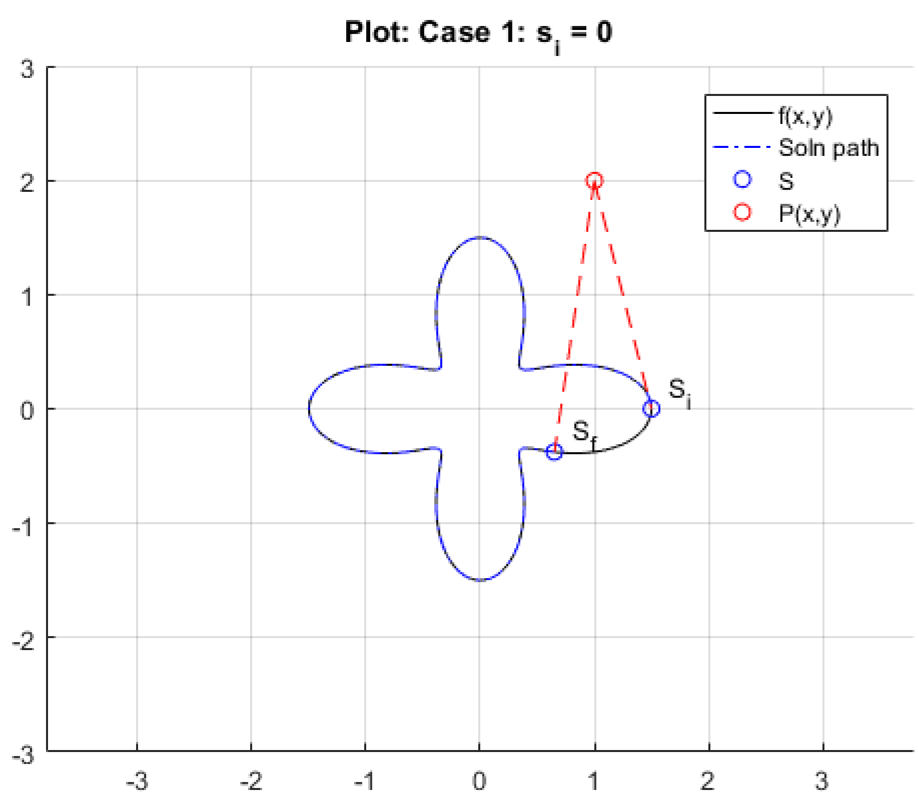
The points p and q are closest when this distance is minimized.

$$x(s) = 1 + 0.5\cos(4s)$$
$$y(s) = 1 + 0.5\sin(4s)$$
$$for\ s \in [0, 2\pi]$$

Once again using Newton's method for minimization requires finding the root of the equation $f'(s)$.

Case	q(x,y)	smin
$s = 0$	(0.6584, -0.3760)	5.7643
$s = \frac{\pi}{2}$	(-0.3760, 1.3750)	1.4051
$s = \frac{5\pi}{2}$	(-0.3760, 1.3750)	1.4051

Here the solutions are plotted below (including the solution path).



Plot: Case 2 & 3: $s_i = \pi/2,\ 5\pi/2$