EAS 502, Fall 2018, Final Exam, Part II
Take Home, Due Midnight EST Monday Dec. 17, 2018, submitted to UBLearns
Total Possible Points: 75

Work **all** problems. Please read all directions carefully. You may use course notes, books, internet search, and tools such as Matlab or Python. If you use external references such as journal papers, be sure to cite them accordingly. **Consultation, either in-person or virtually, with anyone other than Professor Bauman or the TAs is STRICTLY FORBIDDEN.** All work submitted must be your own! Submit all code written and one PDF document with your answers to each problem. Your PDF document should be well organized and your code well commented. Any copying of work or consultation with others will result in a grade of **ZERO** for the **ENTIRE** midterm exam.

1. (Initial Value Problems, Numerical Integration, 15 pts.)

   We often speak of integrating an initial value problem to find the value of the solution at a particular time. Indeed we can think of time-stepping methods as an alternative algorithm for integration. Consider the following initial value problem:

$$\dot{y} = \begin{cases} y\left(-2t + \frac{1}{t}\right) & t \neq 0 \\ 1 & t = 0 \end{cases}$$
$$y(0) = 0$$

   We wish to get an approximation for the solution at $t = 1$.

   (a) Approximate the solution of the initial value problem using a fourth order Runge-Kunga method, such as `ode45` in Matlab. (5 pts.)

   (b) Rewrite the initial value problem as an integral problem by integrating both sides of the equation above. When determining the integration constant consider the time derivative of the analytic solution. Be sure to think carefully about the end points of integration. (2 pts.)

   (c) Compute the exact solution $y(1)$ by computing the integrals from the previous part. (3 pts.)

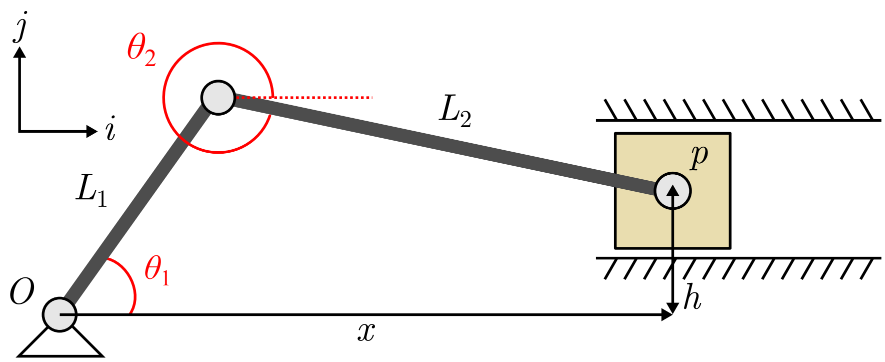
   (d) Estimate the value of the solution at $t = 1$ using an appropriate quadrature rule. How many points do you need to get within 1% accuracy of the exact solution? How does this number compare to the number of timesteps taken in your Runge-Kutta method? (5 pts.)

2. (Nonlinear Systems, Numerical Differentiation, Numerical Integration, 35 pts.) The goal of this problem is to study the behavior of a simple mechanical system. Consider the system shown in Figure 1. The bars are rigid and connected to a piston. The angle θ_1 will be given a function of time, but it will have noise. You will use the input $\theta_1(t)$ to compute the resulting $\theta_2(t)$ and $x(t)$. Then, using this information, you will compute the work done by the piston. Because the input $\theta_1(t)$ has noise, you will solve the problem two different ways: using the data directly and developing a curve fit for $\theta_1(t)$ and then use the curve fit to solve the problem.

   (a) The position of the point p is given by the equations

$$x = (L_1 \cos\theta_1 + L_2 \cos\theta_2) \tag{1a}$$
$$h = (L_1 \sin\theta_1 + L_2 \sin\theta_2) \tag{1b}$$

   The unknowns are θ_2 and x; you may assume θ_1, L_1, L_2, and h are given. Cast this problem as a root problem. (5 pts.)



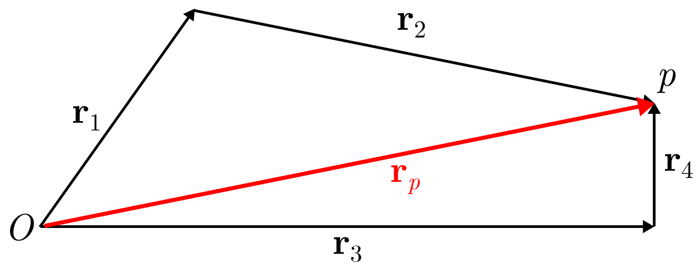


Figure 1: A schematic of a simple piston mechanism. The vectors correspond to the location of the piston p is illustrated.

(b) For the remainder of the problem, take $L_1 = 0.5$ m, $L_2 = 0.75$ m, $h = 0$ m, and the mass of the piston as 10 kg. Now, write a MATLAB script that does the following:

i. Load the time series of $\theta_1(t)$ given in `theta1.dat` (use the command `load('theta1.dat')`). The data is a two-dimensional array containing the value of time t in the first row and the corresponding values of θ_1 in the second row. (2 pt.)

ii. For each value of $\theta_1(t)$, use a Newton solver to solve the nonlinear system of equations for θ_2 and x. For example, you can use the one posted with the homework solutions. (5 pts.)

iii. Plot θ_1 vs. t, θ_2 vs. t, and x vs. t. (2 pts.)

iv. Compute the velocity and acceleration of the piston based on your $x(t)$. Use a central difference method when applicable and a forward/backward difference method when central difference is not applicable. (5 pts.)

v. Plot the computed piston force vs. x. (1 pts.)

vi. Compute the work done by the piston. You may use the Matlab function, `trapz`, for example. Hint: total work is $\int_{x_0}^{x_1} \mathbf{F} \cdot d\mathbf{x}$. (5 pts.)

(c) Now, assume that $\theta_1(t)$ should obey a function of the form

$$f(t) = c_1 \frac{t}{t + c_2}$$

Write a MATLAB script that uses a least square regression analysis to find coefficients c_1 and c_2 that best fit the data. Plot the data and your best curve fit on the same plot. (5 pts.)

(d) Now repeat steps 2(b)ii–2(b)vi using your curve fit for $\theta_1(t)$. (5 pts.)

3. (Nonlinear minimzation, 25 pts) In this problem you will use non-linear minimization techniques to determine the closest point on a parametric interface. Consider a parametric interface in two-dimensional space given by $(x(s), y(s))$ for $s \in [0, s_{max}]$. For example, a unit circle is given by $x(s) = \cos(s)$ and $y(s) = \sin(s)$ for $s \in [0, 2\pi]$. Let $\vec{p}$ be any point in the two-dimensional space. You will be finding value of s which minimizes the distance between the point $\vec{p}$ and the interface, which we call $\vec{q}$.

(a) Write down the function which is at a minimum when $\vec{q}$ is the closest point to $\vec{p}$. (2 pt.)

(b) Let $x(s) = \cos(s)$ and $y(s) = \sin(s)$ for $s \in [0, 2\pi]$. Develop the analytical expression for the closest point $\vec{q}$ for any point in space $\vec{p}$. (3 pt.)

(c) Analytically show that the result you developed is indeed the minimum distance to the interface. (3 pt.)

(d) Using the Matlab function `fminbnd` determine the closest point $\vec{q}$ to the point $\vec{p} = (4, 2)$. Compare the result to the analytic solution determined above. (3 pt.)

(e) Using Newton's method for minimization, write a function which takes as an input $x(s)$, $y(s)$, the bounds on s, an initial guess for s_{min}, and the point $\vec{p}$ and returns the closest point on the interface, $\vec{q}$. Use this to determine the closest point $\vec{q}$ to the point $\vec{p} = (4, 2)$ for the unit circle. (4 pt.)

(f) Using your Newton's method for minimization, determine the closest point $\vec{q}$ to the point $\vec{p} = (1, 1)$ for $x(s) = (1 + 0.5\cos(4s))\cos(s)$ and $y(s) = (1 + 0.5\cos(4s))\sin(s)$ for $s \in [0, 2\pi]$ using an initial guess of $s = 0$, $s = \pi/2$, and $s = 5\pi/4$. Do you get the same result? Explain your results using a figure with the interface, the point p and the solutions. (10 pt.)