Iterations of Line Graphs

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What is a Line Graph?

- G is a finite graph.
- $L(G)$ generates G', the line graph of G.
- $L(G)$ maps each edge of G to a corresponding node in G'. An edge between two nodes in G' exists if their corresponding edges have a common node in G.

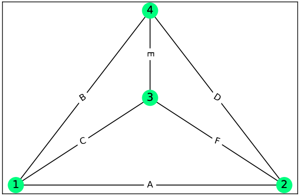


Figure 1: Original Graph G

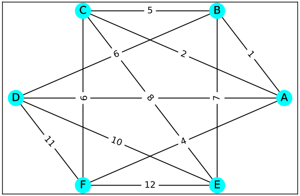


Figure 2: G'

Families of Graphs

Supposing that G has no self-loops or multiple edges. We can define four families of graphs that will have different forms after multiple iterations of $L(G)$. [1]

Definitions by Rooij and Wilf

Where $(a \in G)$ and $\mathrm{Q}(a)$ is the local degree of node a.

- $\mathrm{Q}(a) = 2$
- $\mathrm{Q}(a) \leq 2$ and some $\mathrm{Q}(a) < 2$
- Special case $K_{1,3}$
- At least one $\mathrm{Q}(a) \geq 3$, $G \neq K_{1,3}$

Henceforth we will respectively refer to the defined families as:

- Cycle Graphs
- Path Graphs
- Claw Graphs
- All Other Graphs

[1] A. C. Rooij and H. S. Wilf. "The interchange graph of a finite graph". In: *Acta Mathematica Academiae Scientiarum Hungaricae* 16.3-4 (1965), 263–269. DOI: 10.1007/bf01904834. (Visited on 11/04/2022)

Cycle Graphs

The line graph of a cycle graph is isomorphic and thus its iterations are of the form $G, G, G, \ldots$

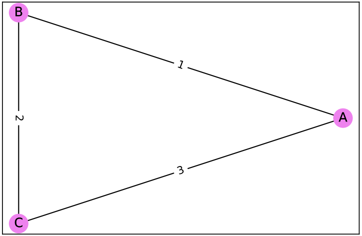


Figure 3: G

$\xRightarrow{L(G)}$

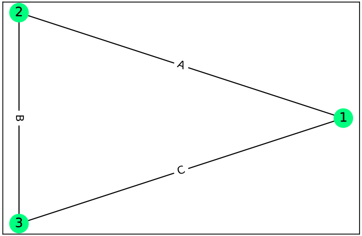


Figure 4: G'

Path Graphs

A path graph of size n generates another path graph of size n-1 until it reaches the null graph. Its iterations are of the form: $G, G', G'', ..., K_0$

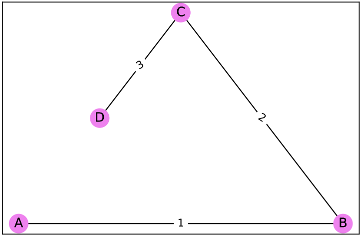


Figure 5: G

$\xRightarrow{L(G)}$

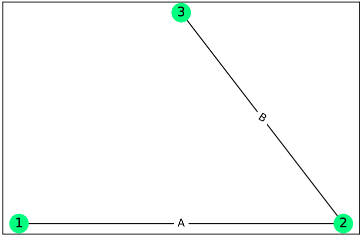


Figure 6: G'

Claw Graphs

Claw graphs are a special case as after one application of $L(G)$ their line graph is K_3, a cycle graph. Clearly after the first iteration they follow the same pattern as a cycle graph: $G, G', G', G', \ldots$

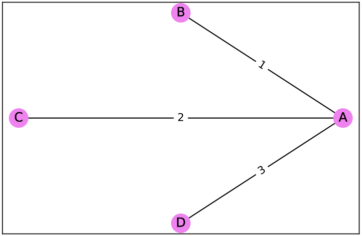


Figure 7: G

$\xRightarrow{L(G)}$

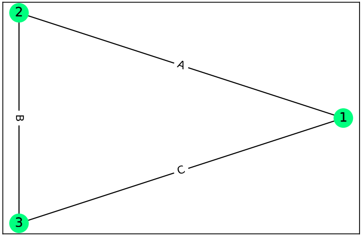


Figure 8: G'

All Other Graphs

The growth in the size of all other graphs is unbounded:
$G, G', G'', G''', \ldots$

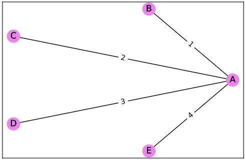


Figure 9: G

$\xRightarrow{L(G)}$

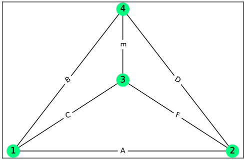


Figure 10: G'

$\xRightarrow{L(G')}$

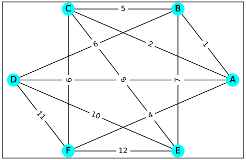


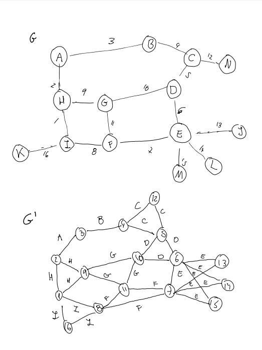
Figure 11: G''

Applications

Whitney Theorem

The Whitney Graph Isomorphism Theorem states that any two connected graphs $G_1, G_2 \neq K_{1,3}$ are not isomorphic unless their line graphs G_1', G_2' are also isomorphic. [2] This has been extended to Hypergraphs by Vertigan. [3]

Possible Interpretations?



[2] Hassler Whitney. "Congruent Graphs and the Connectivity of Graphs". In: *American Journal of Mathematics* 54.1 (1932), pp. 150–168. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2371086 (visited on 11/04/2022)

[3] Dirk Vertigan and Geoff Whittle. "A 2-Isomorphism Theorem for Hypergraphs". In: *Journal of Combinatorial Theory, Series B* 71.2 (1997), pp. 215–230. ISSN: 0095-8956. DOI: https://doi.org/10.1006/jctb.1997.1789. URL: https://www.sciencedirect.com/science/article/pii/S0095895697917895 (visited on 11/04/2022)

Reference List

A. C. Rooij and H. S. Wilf. "The interchange graph of a finite graph". In: *Acta Mathematica Academiae Scientiarum Hungaricae* 16.3-4 (1965), 263–269. DOI: 10.1007/bf01904834. (Visited on 11/04/2022).

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Thank you for your attention!

All graphs used in this project were generated using Python code which we have made accessible through the GitHub repository below, along with the LaTeX code and PDF of this presentation:

https://github.com/paulncassidy/MP305-Project.git