(1) G542 Problem set 1

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1.6

a) when two variables are independent

$E(x.y) = E(x).E(y)$

$\operatorname{cov}(x,y) = E(x.y) - E(x).E(y)$

$= E(x).E(y) - E(x).E(y)$

$\boxed{\operatorname{cov}(x,y) = 0}$

b) i. $P(+ \mid \text{has HIV}) = \boxed{\frac{72}{75}}$

ii $P(- \mid \text{with HIV}) = \boxed{\frac{3}{75}}$

iii $P(\text{has HIV} \mid +) = \frac{P(+ \mid \text{has HIV}) \cdot P(\text{has HIV})}{P(+)}$

$P(+) = P(+ \mid HIV) \cdot P(HIV) + P(+ \mid \text{No HIV}) \times P(\text{No HIV}) = 0.2424$

$P(\text{has HIV} \mid +) = \frac{P(HIV)\, P(+ \mid HIV)}{P(+)}$

$= \frac{0.12 \times 0.86}{0.2424}$

$= 0.475$

$\boxed{P(\text{has HIV} \mid +) = 47.5\%}$

$$P(-) = P(-\mid HIV)\cdot P(HIV) + P(-\mid No\,HIV)\cdot P(No\,HIV)$$

$$= 0.7576$$

$$\boxed{P(HIV \mid -) = 0.0063}$$

2. Bayes Theorem

(a) $P(C) + \frac{1}{2}P(C) + \frac{1}{2}P(C) + \frac{1}{2}P(C) = 1$

$$\boxed{\begin{aligned} P(A\mid B) &= \tfrac{3}{5}\big|2 = 0.3 \\ P(D\mid B) &= \tfrac{3}{5}\big|2 = 0.3 \\ P(C\mid B) &= \tfrac{2}{5} = 0.4 \end{aligned}}$$

$\left(1 + \frac{3}{2}\right)P(C) = 1$

$P(C) = \frac{2}{5}$

$P(A) = \frac{1}{5}$ $P(B) = \frac{1}{5}$

$P(D) = \frac{1}{5}$

$P(A) + P(B) + P(D) = \frac{3}{5}$

by opening the door B $P(B)$ and $P(A)\cdot P(D) \Rightarrow$ $P(A)$ and $P(D)$ $\Rightarrow \frac{3}{5}$

b)

	Suspected	Not suspected	Total
Cheating	1.8	0.2	2
Not cheating	14.6	58.4	73
Total	16.4	58.6	75

$P(\text{cheating} \mid \text{Suspected}) = \frac{1.8}{16.4} = \boxed{0.1097 = 10.97\%}$

3. Linear Algebra (2)

(a)

$$\begin{bmatrix}1&2\\2&1\end{bmatrix}^T \Rightarrow \begin{bmatrix}1&2\\2&1\end{bmatrix}$$

$$\begin{bmatrix}1&2\\2&1\end{bmatrix}^{-1} \Rightarrow \frac{1}{1-4}\begin{bmatrix}1&-2\\-2&1\end{bmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix}1&-2\\-2&1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\frac{1}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{1}{3}\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{3}&-\frac{2}{3}\\-\frac{2}{3}&1\end{bmatrix}^T \Rightarrow \begin{bmatrix}\frac{1}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{1}{3}\end{bmatrix}$$

True, the inverse of a symmetric matrix is itself symmetric.

$$A = \begin{bmatrix}3&4&-1\\-1&-2&1\\3&9&0\end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 4 & -1 \\ -1 & -2-\lambda & 1 \\ 0 & 9 & -\lambda \end{bmatrix}$$

$$3-\lambda \begin{bmatrix} -2-\lambda & 1 \\ 9 & -\lambda \end{bmatrix} - 4 \begin{bmatrix} -1 & 1 \\ 0 & -\lambda \end{bmatrix} - 1 \begin{bmatrix} -1 & -2-\lambda \\ 0 & 9 \end{bmatrix}$$

$$3-\lambda(-2\lambda+\lambda^2-9) - 4(\lambda-0) - 1[-9-0)$$

$$6\lambda + 3\lambda^2 - 27 - 2\lambda^2 - \lambda^3 + 9\lambda - 4\lambda + 9 + 12 - 6 - 3\lambda$$

$$\cancel{-\lambda^3 + \lambda^2 + 12\lambda}$$

$$8\lambda + \lambda^2 - \lambda^3 - 12$$

$$\cancel{-1(x} \cdot (\lambda^2 + 4 - 4\lambda)(\lambda + 3)$$

$$\boxed{\lambda = 2 \quad \text{and } \lambda = -3}$$ eigenvalues.

$\lambda = 2$ Eigenvectors

$$\left(\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ -1 & -4 & 1 & 0 \\ 3 & 9 & -2 & 0 \end{array}\right) \Rightarrow \text{vector} \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

$\lambda = -3$

$$\begin{pmatrix} 6 & 4 & -1 \\ -1 & 1 & 1 \\ 3 & 9 & +3 \end{pmatrix} \Rightarrow \text{vector} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$\lambda = 2$ $\lambda = -3$

Not all the eigenvalues are positive. so

A is not positive definite

2. Probability distributions. (3)

Bishop.

2.2 Normalization

$$P(x=+1|\mu)+P(x=-1|\mu)=\left(\frac{1+\mu}{2}\right)+\left(\frac{1-\mu}{2}\right)=1$$

The mean is given by

$$\boxed{E(x)=\left(\frac{1+\mu}{2}\right)-\left(\frac{1-\mu}{2}\right)=\mu.}$$

Variance.

$$E(x^2)=\left(\frac{1-\mu}{2}\right)+\left(\frac{1+\mu}{2}\right)=1.$$

$$\boxed{Var[x]=E[x^2]-E[x]^2=1-\mu^2}$$

Entropy.

$$H[x]=-\sum_{x=-1}^{x=+1}P(x|\mu)\ln P(x|\mu)$$

$$\boxed{H[x]=-\left(\frac{1-\mu}{2}\right)\ln\left(\frac{1-\mu}{2}\right)-\left(\frac{1+\mu}{2}\right)\ln\left(\frac{1+\mu}{2}\right).}$$

2.10

$$\int \prod_{k=1}^{M} \mu_k^{\alpha_k - 1} d\mu = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_M)}{\Gamma(\alpha_0)} \qquad (88)$$

$$E[\mu_j] = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_M)} \int \mu_j \prod_{k=1}^{M} \mu_k^{\alpha_k - 1} d\mu$$

$$= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_M)} \cdot \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_j + 1) \cdots \Gamma(\alpha_M)}{\Gamma(\alpha_0 + 1)}$$

$$\boxed{E[\mu_j] = \frac{\alpha_j}{\alpha_0}}$$

$$\mathrm{Var}[\mu_j] = E[\mu_j^2] - E[\mu_j]^2 = \frac{\alpha_j(\alpha_j + 1)}{\alpha_0(\alpha_0 + 1)} - \frac{\alpha_j^2}{\alpha_0^2}$$

$$\mathrm{Var}[\mu_j] = \frac{\alpha_j(\alpha_0 - \alpha_j)}{\alpha_0^2(\alpha_0 + 1)}$$

$$\mathrm{cov}[\mu_j \mu_l] = E[\mu_j \mu_l] - E[\mu_j] E[\mu_l]$$

$$\mathrm{cov}[\mu_j \mu_l] = \frac{\alpha_j \alpha_l}{\alpha_0(\alpha_0 + 1)} - \frac{\alpha_j \alpha_l}{\alpha_0 \alpha_0} = -\frac{\alpha_j \alpha_l}{\alpha_0^2(\alpha_0 + 1)}$$

(4)

2.12.

$$\int_a^b \frac{1}{b-a}\,dx = \frac{b-a}{b-a} = 1 \qquad \text{Normalization}$$

$$E(x) = \int_a^b \frac{1}{b-a}\,x\,dx = \left[\frac{x^2}{2(b-a)}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}.$$

$$E[x^2] = \int_a^b \frac{1}{b-a}\,x^2\,dx = \left[\frac{x^3}{3(b-a)}\right]_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

$$\mathrm{Var}[X] = E[x^2] - E[x]^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)^2}{12}.$$

2.15.

$$H[x] = -\int N(x|\mu,\Sigma)\ln N(x|\mu,\Sigma)\,dx$$

$$= \int N(x|\mu,\Sigma)\,\frac{1}{2}\left(D\ln(2\pi) + \ln|\Sigma| + (x-\mu)^T\Sigma^{-1}(x-\mu)\right)dx$$

$$= \frac{1}{2}\left(D\ln(2\pi) + \ln|\Sigma| + \mathrm{Tr}\left[\Sigma^{-1}\Sigma\right]\right)$$

$$H[x] = \frac{1}{2}\left(D\ln(2\pi) + \ln|\Sigma| + D\right)$$