CS 2

Introduction to Programming Methods

Last Time: Sorting

Introduced ways to sort, and complexity

- no optimal way, though

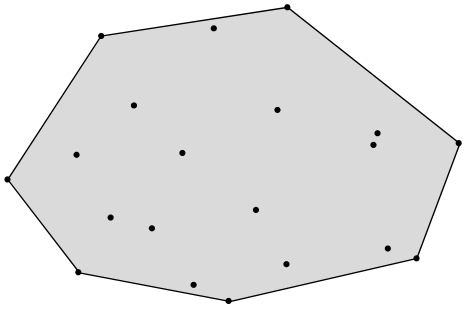
Before then...

Assuming a comparison-based method... what is the best complexity we can get?

- for n numbers, how many permutations?
  - $n \cdot (n-1) \cdot (n-2) \cdots \cdot 2 \cdot 1 = n!$
- the sorted list is only ***one*** of these n! combos
- each comparison kills half the permutations
  - e.g., for 1,2,3: (123), (132), (213), (231), (312), (321)
  - if A[1]<A[2], then we are left with (123), (132), (231)
  - like the "20 questions" game... (can find one out of 2^{20})
- so sorting is $\Omega(\log_2 n!) = \Omega(n \log n)$ (Stirling's)

Introducing Convex Hull

Given a set S = $\{p_1, p_2, ..., p_N\}$ of points in 2D, convex hull H(S) is the smallest convex polygon in the plane that contains all of S.



Q: find the ordered list of points from S defining the convex hull H(S)

Why is CH Interesting

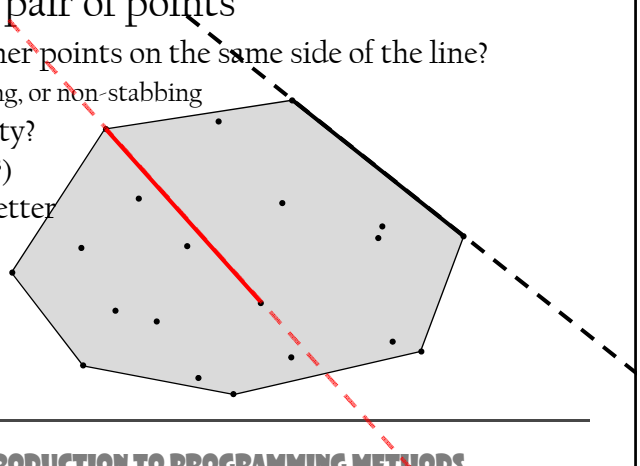
Many reasons to look at it a bit

- extends sorting, in a way
  - in fact, easy to prove in 1D that the two problems share the same lower bound
    - from $\{x_i\}$, compute CH of $\{x_i, x_i^2\}$
- neat in its own right in Comp. Geo.
- results you can see

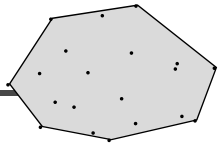
Convex Hull Construction

A first algorithm

- test every pair of points
  - are all other points on the same side of the line?
    - stabbing, or non-stabbing
  - complexity?
    - in $O(n^3)$
  - let's do better

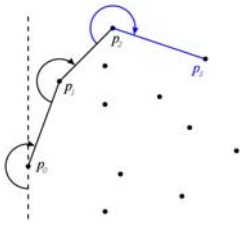


Gift Wrapping



Improving on previous algorithm

- find a point on convex hull
  - by going thru points and finding leftmost one (O(n))
- then find non-stabbing line (linear time)
- then repeat from new point
- Worst case?
  - n^2
- More precisely, O(nh)
  - output sensitive!



Can We Do Even Better

Yes, if we can sort in O(n log n)…

So, let's revisit the kooky idea of "dividing" up the problem in smaller problems…

- actually pretty convenient in many cases
  - example: factorial
    - n! = n·(n-1)· … ·2·1
    - n! = n·(n-1)!
  - or Fibonacci numbers
    - Fib(n) = Fib(n-1)+Fib(n-2)
    - Fib(0)=0, Fib(1)=1.

```
static int factorial(int k)
{ If (k == 0) return 1;    // 0! = 1
  else return k*factorial(k-1);    }

static int Fib(int k)
{ If (k < 2) return k;    //Fib(0/1)
  else return Fib(k-1)+Fib(k-2); }
```

Recursive Sorting?

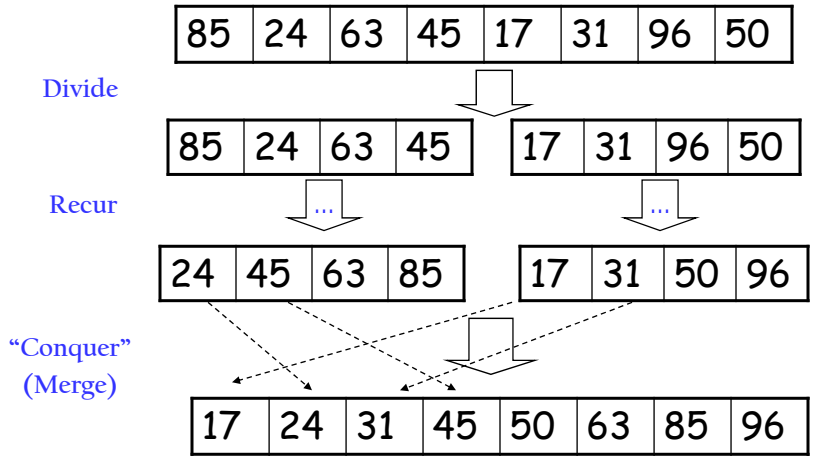
Could we:

- Divide input in half (trivial)
- Recursively sort each half (just two calls)
- Merge two halves together
  - this last step is less trivial, but…
  - how hard is it to merge two sorted lists?
    - compare "head elements" repeatedly → O(n)

Base case:

- Sort({k}) = … {k}.

Execution



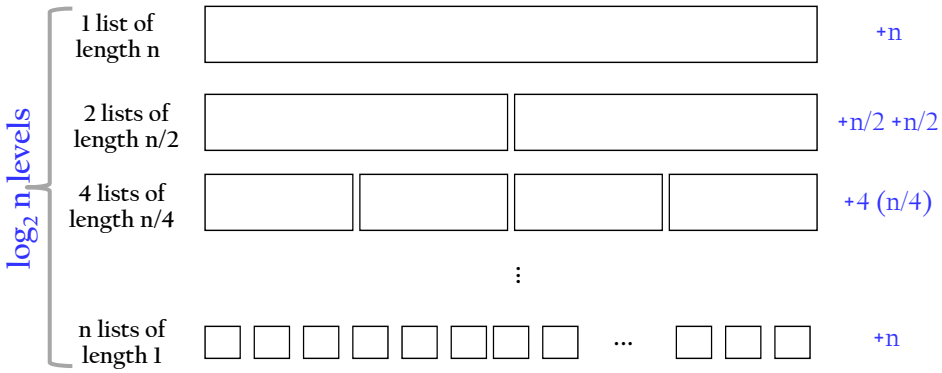
Complexity (I)

So, did we gain anything in doing so?

- let's call T(n) the computational complexity
- T(n) = T(n/2) + T(n/2) + n
  - 2 recursive calls, and one O(n) merge
  - and T(1) = 1

Complexity (II)

Let's trace a full execution



Complexity (III)

So, did we gain anything in doing so?

- let's all T(n) the computational complexity
- T(n) = T(n/2) + T(n/2) + n
  - 2 recursive calls, and one O(n) merge
  - and T(1) = 0
- T(n) = O(n log n)

Code

+n more than I said
but no change in complexity

```
public class MergeSorter
{
   private static int[] a, b;    // auxiliary array b

   public static void sort(int[] input)
   {
      a=input;
      int n=a.length; b=new int[n];
      mergesort(0, n-1);
   }

   private static void mergesort(int lo, int hi)
   {
      if (lo<hi)
      {
         int m=(lo+hi)/2;
         mergesort(lo, m);
         mergesort(m+1, hi);
         merge(lo, m, hi);
      }
   }
```

```
   private static void merge(int lo, int m, int hi)
   {
      int i, j, k;

      // copy both halves of a to auxiliary array b
      for (i=lo; i<=hi; i++)
         b[i]=a[i];

      i=lo; j=m+1; k=lo;
      // copy the next-greatest elmt, each time
      while (i<=m && j<=hi)
        if (b[i]<=b[j])
          a[k++]=b[i++];
        else
          a[k++]=b[j++];

      // copy remaining elmts of first half (if any)
      while (i<=m)
         a[k++]=b[i++];
   }
}  // end class MergeSorter
```

Recursion, More Generally

Common recursive strategies

- Handle first and/or last, recur on rest
- Divide in half, recur on halves
- process, recur on updated state

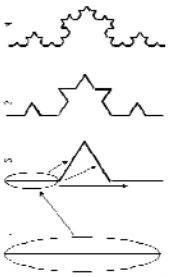
Complexity of divide-and-conquer methods

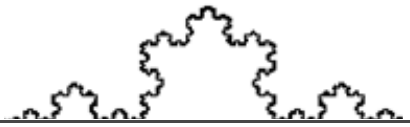
- use of master theorem [CLRS]
  - $T(n) = a\,T(n/b) + f(n)$
- and variants

Recursion & Fractals

Snowflakes in "turtle graphics"

- F means "draw forward", + means "turn left by α", and - means "turn right by α"
- set α = 60°, and start with F
- then apply rule: F → F+F--F+F





Using Recursion on Printers?

```
%!PS
/N 7 def                    ← less than 20, or you'll wait  a while…
/X {  dup 0 ne
      {1 sub 4 {dup} repeat - F X + + F Y -}
      if pop } def
/Y {  dup 0 ne
      {1 sub 4 {dup} repeat + F X - - F Y +}
      if pop } def
/F {  0 eq { 10 0 rlineto } if } bind def
/- { -45 rotate } bind def
/+ { 45 rotate } bind def
1 setlinejoin 1 setlinecap
newpath
220 180 moveto
50 N { 2 sqrt div } repeat dup scale
90 rotate N X stroke
showpage
```

X → -FX++FY-
Y → +FX--FY+
F →

angle 22.5
X→ F
F → FF-[-F+F+F]+[+F-F-F]