XIII. BOSE GAS

Bosons are particles of the quiet type - they don't fuss when they are put in the same state - in fact, they hardly see eachother. Whenever you add a boson to a system, the system's entropy goes up, unlike the fussy fermions which may actually reduce the entropy (positive chemical potential). Nevertheless they do a really remarkable thing - they bose condense. This gives rise to all those beatiful picture i'm passing. macroscopic ground state participation, superfluidity, vortices. In this lecture we will derive the properties of a bose gas. We will start with a pedestrian determination of their pressure as a function of the density, and from there, move to the celebrated BEC, and explain some of these photos.

A. Low density expansion

The BE distribution is

$$f(\epsilon) = \frac{1}{e^{\epsilon\beta}e^{-\mu\beta} - 1} \tag{406}$$

A common notation introduces a quantity which i think you encountered before - the fugacity:

$$\zeta = e^{\mu\beta} \tag{407}$$

this is just a popular reparametrization of μ. With the fugacity we can write the BE distribution as:

$$f(\epsilon) = \frac{1}{e^{\epsilon\beta}/\zeta - 1} \tag{408}$$

Now, μ is always negative:

$$\mu < 0 \tag{409}$$

and therefore, ζ is always smaller than 1:

$$\zeta \leq 1 \tag{410}$$

the more negative μ is, the smaller ζ is, and the lower the density is.

Now it is time to meet the menace of the BE distirubiton head on as we calculate the pressure as a function of density.

The number density of a bose gas in d dimensions is:

$$\frac{N}{V} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\epsilon\beta}/\zeta - 1} \tag{411}$$

Also, we should keep an eye on the pressure. We know that:

$$\omega = -pV = V \int \frac{d^d k}{(2\pi)^d} \ln\left(1 - e^{-\epsilon\beta}\zeta\right) \tag{412}$$

The ideal gas approximation of low density - which translates to:

$$\zeta \ll 1 \tag{413}$$

gave us the following thing:

$$\frac{N}{V} = \int \frac{d^d k}{(2\pi)^d} e^{-\epsilon\beta}\zeta = \left(\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-\hbar^2 k^2/2mT}\right)^d \zeta = \left(\frac{2\pi m T}{h^2}\right)^{d/2} \zeta \tag{414}$$

The thing in the brackets we also saw before - it is the thermal wavelength:

$$\lambda_T = \frac{h}{\sqrt{2\pi m T}} \tag{415}$$

the reason we think about this as the thermal wavelength is that the denominator is a characteristic momentum of a gas particule: mT is mass times energy, which is also momentum square. The de-Broglie law says that h over momentum is the effective wavelength. Notice how quantum mechanics creeps in.

Thus we can write the number density as:

$$\frac{N}{V} = \zeta \frac{1}{\lambda_T^d} \tag{416}$$

The condition of this derivation was that $\zeta \ll 1$. Indeed, if the density is lower than the 'quantum' density, λ_T^{-d}, then our assumptions are validated. Similarly we can also say:

$$p \to T \cdot \zeta \frac{1}{\lambda_T^d} \tag{417}$$

These expression though are just the first in a low density expansion. For bosons it easy to write the higher order expressions. Think of ζ as small, but don't throw all of its dependece away except for the ζ. Instead carry out the following expansion:

$$\frac{1}{e^{\epsilon\beta}/\zeta - 1} = e^{-\epsilon\beta}\zeta \frac{1}{1 - e^{-\epsilon\beta}\zeta} = e^{-\epsilon\beta}\zeta \left(1 + \zeta e^{-\epsilon\beta} + \zeta^2 e^{-2\epsilon\beta} \ldots\right) \tag{418}$$

Now we can get an expansion in terms of powers of ζ. Just plug this into the expression for N:

$$\frac{N}{V} = \int \frac{d^d k}{(2\pi)^d} \sum_{n=1}^{\infty} e^{-n \cdot \hbar^2 k^2 / 2mT} \zeta^n \tag{419}$$

The only difference in each term from the large density first term is the power in the exponent. ah - that's easy to deal with. We can almost redily write:

$$\to \frac{1}{\lambda_T^d} \overbrace{\sum_{n=1}^{\infty} \frac{\zeta^n}{n^{d/2}}}^{g_{d/2}(\zeta)} \tag{420}$$

neat. We also defined a new function - $g_p(\zeta)$, this function is very useful for bosons. We can also do the same thing to the pressure:

$$-\ln(1 - e^{-\epsilon\beta}\zeta) = \sum_n \frac{e^{-n\epsilon\beta}\zeta^n}{n} \tag{421}$$

This means that

$$-p \to T \int \frac{d^d k}{(2\pi)^d} (-) \sum_n \frac{e^{-n\epsilon\beta}\zeta^n}{n} = T \frac{1}{\lambda_T^d} \overbrace{\sum_n \frac{\zeta^n}{n^{d/2+1}}}^{g_{d/2+1}(\zeta)} \tag{422}$$

Here we define a sister function for $g_{d/2}$ - $g_{d/2+1}(\zeta)$. indeed - if you take only the first term in this expansion and the one for N, you get the ideal gas law. But including higher terms gives you deviations.

B. Virial expansion for a bose gas

The ideal gas law is obtained as the first order in the above expressions. but now, we would like to consider higher order corrections to the equation of state. For low densisties these would take the form of a power-law. The small paramter as we have seen is roughly:

$$\frac{N}{V} \lambda_T^d \ll 1 \tag{423}$$

and therefore we expect:

$$p = \frac{N}{V}T\left(1 + B_1\frac{N}{V}\lambda_T^d + B_2\left(\frac{N}{V}\right)^2\lambda_T^{2d}\dots\right) \tag{424}$$

This expansion is called the Virial expansion. How would we obtain this? (ASK) from elimination of ζ between the equations for N and p.

Let is carry it out to obtain B_1 in this case. We start with the equation for N. For simplicity, and to preserve the perfect health of my hands, let's redefine

$$\frac{N}{V}\lambda_T^d = \nu \tag{425}$$

kinda like a filling factor. And then, the equation for N becomes:

$$\nu = \zeta + \zeta^2\frac{1}{2^{d/2}} + \mathcal{O}\left(\zeta^3\right) \tag{426}$$

We are interested in things only upto second order in ν. To first order we know that:

$$\nu = \zeta, \tag{427}$$

So we truncate a power series for ζ in the second term:

$$\zeta = \nu + A_2\nu^2 + \mathcal{O}(\nu^3). \tag{428}$$

Plugging this into the equation, we see:

$$\nu = \nu + A_2\nu^2 + \frac{1}{2^{d/2}}\nu^2 + \mathcal{O}(\nu^3). \tag{429}$$

and:

$$A_2 = -\frac{1}{2^{d/2}} \tag{430}$$

By this we solved for $1/\zeta$ as much as we wanted from the N equation. Now, we put this in the pressure equation:

$$p = T\frac{1}{\lambda_T^d}\left(\zeta + \frac{\zeta^2}{2^{d/2+1}} + \mathcal{O}\left(\zeta^3\right)\right) = T\frac{1}{\lambda_T^d}\left(\nu - \frac{1}{2^{d/2}}\nu^2 + \frac{1}{2^{d/2+1}}\nu^2 + \mathcal{O}\left(\frac{1}{\nu^3}\right)\right) = T\frac{N}{V}\left(1 - \frac{1}{2^{d/2+1}}\frac{N}{V}\lambda_T^2 + \mathcal{O}(\nu^3)\right) \tag{431}$$

And we could do this to any order we want. This will be used in the problem set to derive the van-der-Waals equation.

C. Bose-Einstein Condensation

From the expansion of N and p in a power series in ζ we can glean some very important information about the limit in which the gas is *dense*, i.e., the filling factor is of order 1. But there are questions we need to ask:

1. Is there a solution of ζ for any N/V?

2. can the system attain any pressure?

Since the power series for both p and N consists of all positive terms, we can answer these questions qualitatively fairly easily.

As for question 1, let's reconsider the equation fro N:

$$\frac{N}{V} = \frac{1}{\lambda_T^d}\sum_{n=1}^{\infty}\frac{\zeta^n}{n^{d/2}} = \frac{1}{\lambda_T^d}g_{d/2}(\zeta) \tag{432}$$

In order to find a solution we need to effectively invert $g_{d/2}(\zeta)$. Graphically this looks like this: g starts at zero when $\zeta = 0$, and keeps growing until ζ reaches 1. To find zeta we just look at the intercept of $1/\lambda_T^d g_{d/2}(\zeta)$ with the density

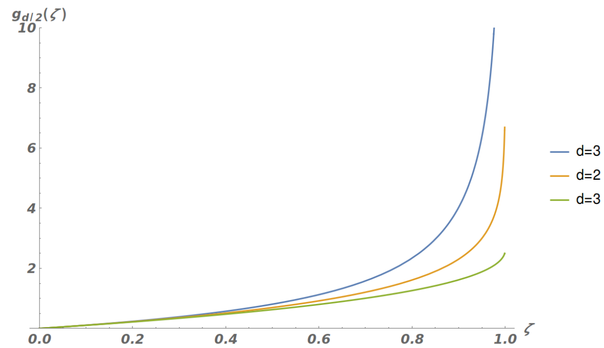

FIG. 6. The function $g_{d/2}(\zeta)$ for 1, 2, and 3 dimensions.

n. In order for there to be a solution for ζ we need g to diverge at $\zeta = 1$; the highest that ζ could be is 1, since the chemical potential cannot go positive. Let's see if it indeed diverges: at:

$$\zeta = 1 \tag{433}$$

We have:

$$\frac{N}{V} \leq \frac{1}{\lambda_T^d} \sum_{n=1}^{\infty} \frac{1}{n^{d/2}} \tag{434}$$

Now, note that for:

- $d = 1$, the sum over $1/\sqrt{n}$ definitely diverges.
- $d = 2$, the sum diverges more slowly. If the previous sum diverged as $\sqrt{\infty}$, the diverges here is more like $\ln \infty$. Nevertheless it diverges.
- $d > 2$ the sum converges. $d = 3$ we have $g_{3/2}(1) = 2.612$.

When the sum converges. It means that somehow there aren't enough states to house all bosons... In three-d the maximum density is:

$$n_c = \frac{2.612}{\lambda_T^d} \tag{435}$$

Which is roughly two particles in a thrmal wavelength cube...

As the density increases, ζ increases towards 1, and achieves it at a finite density, since $g(1)$ is convergent.

After that, our math seems to getjammed.

So what do the other bosons do when pushed into the system? Let's also look at the pressure expression. The maximum pressure that we can enforce is:

$$p \leq \frac{T}{\lambda_T^d} \sum_{n=1}^{\infty} \frac{1}{n^{d/2+1}} = \frac{1}{\lambda_T^d} \overbrace{g_{d/2+1}(1)}^{d=3:\ 1.341} \tag{436}$$

This sum converges for *any* dimension. This looks a bit more intriguing than the sum used for the density.

What is the issue? In our derivations we made one crucial assumption - we took the continuum limit while evaluating the number of particle in a box. We assumed that:

$$g = \sum_{\vec{k}} = V \int \frac{d^d k}{(2\pi)^d} \tag{437}$$

This should be true for large volumes, since:

$$dk = \frac{2\pi}{L} \tag{438}$$

was our guiding principle. But here we see that in 3-d the continuum limit and higher dimensions the continuum limit doesn't give us the goods - it doesn't provide enough states for our bosons. When this happens we need to look very carefully at divergences.

Let's take a step back and rewrite the integrals as sum:

$$\frac{N}{V} = \frac{1}{V}\sum_{\vec{k}} \frac{1}{e^{\epsilon_k \beta}/\zeta - 1} \tag{439}$$

The risky thing happens when we have a diverging term in the sum, that we then multiply by a dk which makes it vanish. If this happens - we have to be very careful, since there might be an order of limits that we are doing wrong.

The only term that does diverge in the sum is the $k = 0$ term. All others have a finite energy, and therefore the denominator can never vanish. In oredre to rectify whatever it is we are doing wrong, let's seperate this term from the rest, and carry out the continuum approximation for the rest:

$$\frac{N}{V} = \frac{1}{V}\frac{\zeta}{1-\zeta} + \int_{\vec{k}} \frac{k^2 dk}{2\pi^2}\frac{1}{e^{\epsilon_k \beta}/\zeta - 1} \tag{440}$$

With the help of our previous analysis we obtain:

$$\to \frac{1}{V}\frac{\zeta}{1-\zeta} + \frac{1}{\lambda_T^d} g_{3/2}(\zeta) \tag{441}$$

Now we have to be very careful. Let's assume that

$$V \to \infty \tag{442}$$

when the density reaches the maximum value from below:

$$n_c = \frac{1}{\lambda_T^3} g_{d/2}(1) = \left(\frac{2\pi m}{h^2}\right)^{3/2} T^{3/2} \overbrace{g_{3/2}(1)}^{2.612} \tag{443}$$

ζ reaches one, and the first term is undetermined, the second term saturates at n_c. Say that the indeterminate term is n_0, and we see that it is determined by the remainder of bosons, not hosted by the nonzero energy states:

$$n = n_0 + n_c \tag{444}$$

or just

$$n_0 = n - n_c = n - \frac{2.612}{h^3}(2\pi m)^{3/2} T^{3/2} \tag{445}$$

This is Bose-Einstein condensation.

n_0 is zero for all densities below the critical density, but then, once the critical density is reached, all extra particles join in to the ground state.

If we look at this phenomena as a function of T, we see that for any density, there is a low enough temperature such that the density exceeds the critical density. This temperature is obtained from:

$$\frac{N}{V} = \frac{2.612}{h^3}(\sqrt{2\pi m T})^3 \tag{446}$$

then as long as:

$$T > T_{BEC} = \frac{h^2}{2\pi m}\left(\frac{n}{2.612}\right)^{2/3} \tag{447}$$

which is obtained as the temperature in which $n = n_c$, we have: $n_0 = 0$. But as soon as the temperature drops below, we have:

$$n_0 = n - \left(\frac{2\pi m}{h^2}\right)^{3/2} T^{3/2} g_{3/2}(1) \tag{448}$$

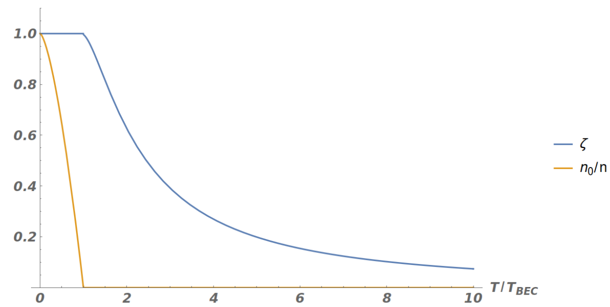

FIG. 7. ζ and the fraction of the density in the $\vec{k}=0$ mode as a function of the temperature ratio T/T_{BEC}

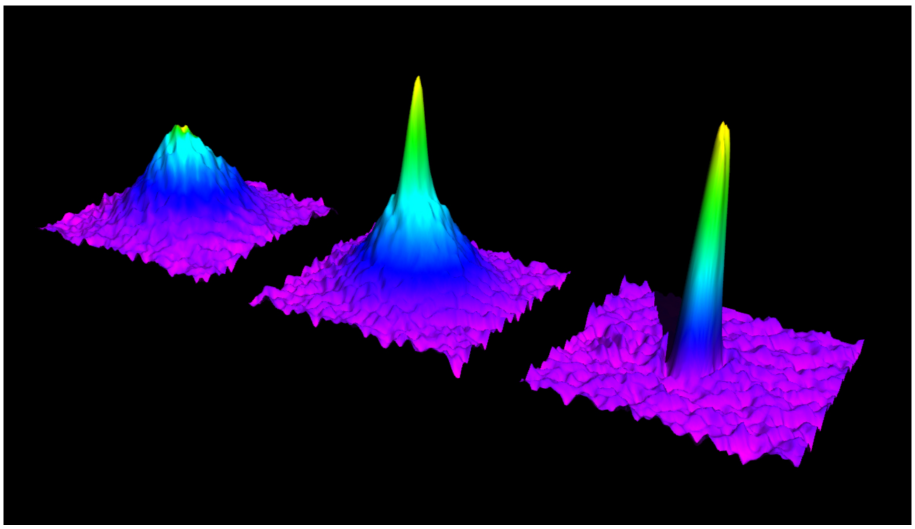

FIG. 8. The momentum distribution of a gas above, near, and below the BEC temperature. The large peak that develops is the hallmark of a macroscopic occupation of the $\vec{k}=0$ state. Taken from the LMU (Munich) website.

which can also be written as:

$$n_0 = \left(\frac{2\pi m}{h^2}\right)^{3/2} 2.612\left(T_{BEC}^{3/2} - T^{3/2}\right) \tag{449}$$

When we have a finite volume, there is no need for ζ to go all the way to 1, because the zero-enbergy occupation is always finite:

$$\frac{1}{V}\frac{\zeta}{1-\zeta} > 0 \tag{450}$$

but ζ will come close to V, and at $T=0$ will be, roughly:

$$\zeta \sim 1 - \frac{\lambda_{T_{BEC}}^3}{V} \tag{451}$$

This was indeed seen in several experiments by now. Using a technique called 'time of flight', cold atoms experimentalists are able to map out the momentum distribution of a gas. Fig. 7 shows the momentum distribution of a Rb gas (I believe) above, near, and below T_{BEC}.