Slaying the Mythical p

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April 12, 2017

Binomial distribution:

$$P(F_h|p_h, F) = \binom{F}{F_h} p_h^{F_h} (1 - p_h)^{F - F_h} \tag{1}$$

Statistician:

$$F = 250, F_h = 140, p_h = \frac{1}{2}, F - F_h = F_t = 110 \tag{2}$$

$$P(F_h = 140|p_h = \frac{1}{2}, F = 250) = \binom{250}{140} \left(\frac{1}{2}\right)^{140} \left(1 - \frac{1}{2}\right)^{110} \simeq 10^{-21} \tag{3}$$

$$P(F_h \geq 140|p_h, F) = \sum_{F_h=140}^{250} \binom{250}{F_h} \left(\frac{1}{2}\right)^{F_h} \left(1 - \frac{1}{2}\right)^{F - F_h} \tag{4}$$

$$P(F_h \geq 140|p_h, F) = \left(\frac{1}{2}\right)^{250} \sum_{F_h=140}^{250} \binom{250}{F_h} \tag{5}$$

$$P(F_h \geq 140|p_h, F) \simeq 0.0332 \tag{6}$$

$\mathcal{H}_0$: Coin is fair
$\mathcal{H}_1$: Coin is biased
D : Data of 140 heads in 250 spins
F_h : 140 (frequency of heads)
F_t : 110 (frequency of tails)
F : 250 (number of coin spins)
p_h : Probability of heads
Getting model comparison:

$$P(\mathcal{H}_0|D) = \frac{P(D|\mathcal{H}_0)P(\mathcal{H}_0)}{P(D)} \tag{7}$$

$$P(\mathcal{H}_1|D) = \frac{P(D|\mathcal{H}_1)P(\mathcal{H}_1)}{P(D)} \tag{8}$$

$$P(\mathcal{H}_0) = P(\mathcal{H}_1) = \frac{1}{2} \tag{9}$$

$$\frac{P(\mathcal{H}_1|D)}{P(\mathcal{H}_0|D)} = \frac{P(D|\mathcal{H}_1)}{P(D|\mathcal{H}_0)} \tag{10}$$

Done getting model comparison.
Getting $P(D|\mathcal{H}_1)$:

$$P(D|\mathcal{H}_1) = \int_0^1 \mathrm{d}p_h P(D|p_h, \mathcal{H}_1)P(p_h|\mathcal{H}_1) \tag{11}$$

$$P(D|p_h, \mathcal{H}_1) = p_h^{F_h}(1-p_h)^{F_t} \tag{12}$$

$$P(p_h|\mathcal{H}_1) = 1 \tag{13}$$

$$P(D|\mathcal{H}_1) = \int_0^1 \mathrm{d}p_h p_h^{F_h}(1-p_h)^{F_t} \tag{14}$$

$$B(x,y) = \int_0^1 \mathrm{d}t t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{15}$$

$$P(D|\mathcal{H}_1) = B(F_h+1, F_t+1) = \frac{\Gamma(F_h+1)\Gamma(F_t+1)}{\Gamma(F_h+F_t+2)} \tag{16}$$

$$\Gamma(x) = (x-1)! \tag{17}$$

$$P(D|\mathcal{H}_1) = \frac{(F_h+1)!(F_t+1)!}{(F_h+F_t+1)!} = \frac{F_h!F_t!}{(F+1)!} \tag{18}$$

End of getting $P(D|\mathcal{H}_1)$.
Getting $P(D|\mathcal{H}_0)$:

$$p_f = \frac{1}{2} \tag{19}$$

$$P(D|\mathcal{H}_0) = \left(\frac{1}{2}\right)^{F_h} \left(1 - \left(\frac{1}{2}\right)\right)^{F_t} = \left(\frac{1}{2}\right)^{F} \tag{20}$$

Done getting $P(D|\mathcal{H}_0)$.
Computing model comparison:

$$\frac{P(\mathcal{H}_1|D)}{P(\mathcal{H}_0|D)} = \frac{\frac{F_h!F_t!}{(F+1)!}}{\left(\frac{1}{2}\right)^{F}} = \frac{\frac{140!110!}{251!}}{\left(\frac{1}{2}\right)^{250}} \simeq 0.48 \tag{21}$$

End computing model comparison.
Getting Beta prior:

$$P(p_h|\mathcal{H}_1, \alpha) = \frac{1}{Z(\alpha)} p_h^{\alpha-1}(1-p_h)^{\alpha-1} \tag{22}$$

$$Z(\alpha) = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} \tag{23}$$

$$P(D|\mathcal{H}_1) = \int_0^1 \mathrm{d}p_h P(D|p_h, \mathcal{H}_1) P(p_h|\mathcal{H}_1) \tag{24}$$

$$P(D|\mathcal{H}_1) = \int_0^1 \mathrm{d}p_h p_h^{F_h}(1-p_h)^{F_t} \frac{1}{Z(\alpha)} p_h^{\alpha-1}(1-p_h)^{\alpha-1} \tag{25}$$

$$P(D|\mathcal{H}_1) = \frac{1}{Z(\alpha)} \int_0^1 \mathrm{d}p_h p_h^{F_h+\alpha-1}(1-p_h)^{F_t+\alpha-1} \tag{26}$$

$$B(F_h+\alpha-1+1, F_t+\alpha-1+1) = \frac{\Gamma(F_h+\alpha)\Gamma(F_t+\alpha)}{\Gamma(F+2\alpha)} \tag{27}$$

$$P(D|\mathcal{H}_1) = \frac{1}{Z(\alpha)} \frac{\Gamma(F_h+\alpha)\Gamma(F_t+\alpha)}{\Gamma(F+2\alpha)} \tag{28}$$

$$P(D|\mathcal{H}_1) = \frac{\Gamma(2\alpha)\Gamma(F_h+\alpha)\Gamma(F_t+\alpha)}{\Gamma(\alpha)^2\Gamma(F+2\alpha)} \tag{29}$$

$$\frac{P(\mathcal{H}_1|D)}{P(\mathcal{H}_0|D)} = \frac{\Gamma(2\alpha)\Gamma(F_h+\alpha)\Gamma(F_t+\alpha)2^{250}}{\Gamma(\alpha)^2\Gamma(F+2\alpha)} \tag{30}$$

End getting Beta prior.
Getting best possible prior:

$$\frac{P(\mathcal{H}_1|D)}{P(\mathcal{H}_0|D)} = f^{F_h}(1-f)^{F_t}2^F \simeq 6.1 \tag{31}$$