

The Spanning Tree Problem with One Quadratic Term

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1 Introduction

A common approach to quadratic optimization over binary variables is to linearize the quadratic terms and to develop an appropriate polyhedral description of the corresponding set of feasible solutions. A straightforward idea is to linearize each product in the objective function independently and simply combine the result with the given linear side constraints. This approach yields a correct integer programming model of the problem, but the resulting LP-relaxations lead to very weak bounds in general, so that branch-and-cut algorithms based on this simple linearization idea perform very poorly in general. For this reason, one usually searches for stronger or even facet defining inequalities to tighten the description; see, e.g., [2].

In this paper, we consider another approach that, to the best of our knowledge, has not been investigated yet: we examine the problem version with only one product term in the objective function, but with all linear side constraints taken into account. Any valid cutting plane for this problem will remain valid for the original problem as well and potentially improves over the straightforward model, since the chosen product is considered together with all side constraints. The advantage of our approach lies in the fact that there exists a polynomial time separation algorithm for the one-product problem whenever the underlying linear problem is tractable: in this case, the corresponding optimization problem is polynomially solvable by solving the underlying linear problem four times, with different fixings of the two variables appearing in the chosen product.

From a practical point of view, this indirect separation approach does not pay off within a branch-and-cut algorithm; the effort for computing a single cutting plane is too high. Therefore, we apply this idea to an important specific problem in the following, namely the quadratic minimum spanning tree (QMST) problem. The linear spanning tree problem is well studied and solvable in polynomial time, while additional costs for pairs of edges render the problem NP-hard [1]. We introduce two new classes of facet defining inequalities for QMST with only one quadratic term in the objective function. These inequalities arise from certain subtour elimination constraints by adding the new product variable to the left hand side. We actually conjecture that the addition of these classes of inequalities suffices to obtain a complete polyhedral description of the linearized polytope in the one-product case. The new inequalities can be separated in polynomial time by adapting the separation routines for subtour elimination constraints. Our experimental results show a significant improvement of both bounds and running times when adding the new inequalities to a branch-and-cut algorithm for solving the original problem with all quadratic terms, with respect to the straightforward linearization.

2 Preliminaries

We assume that $G = (V, E)$ is a complete undirected graph. The *quadratic minimum spanning tree problem* can be formulated as an integer program with linear constraints and a quadratic objective function:

$$(\text{QIP}_{\text{QMST}}) \quad \min \sum_{e \in E} c_e x_e + \sum_{e, f \in E, e \neq f} c_{ef} x_e x_f \quad (1)$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = |V| - 1 \quad (2)$$

$$\sum_{e \in E(G[S])} x_e \leq |S| - 1 \quad \forall \emptyset \neq S \subsetneq V \quad (3)$$

$$x_e \in \{0, 1\} \quad \forall e \in E.$$

Here $G[S]$ denotes the subgraph of G induced by the vertices in S and $E(G[S])$ denotes its edge set. The subtour elimination constraints (3) ensure that no subgraph induced by S contains a cycle; combined with Equation (2), this also guarantees connectivity.

To get rid of the quadratic terms in the objective function, we linearize all products $x_e x_f$ by introducing artificial binary variables y_{ef} and link them to the original variables using the following additional linear inequalities:

$$y_{ef} \leq x_e, x_f \quad \forall e, f \in E \quad (4)$$

$$y_{ef} \geq x_e + x_f - 1 \quad \forall e, f \in E. \quad (5)$$

The x -entries of all feasible solutions of the linearized problem (QIP_{QMST}) model exactly the incidence vectors of all spanning trees, and due to the binary constraints, the value of every y_{ef} is exactly the product of x_e and x_f by (4) and (5). The latter does not remain true, however, after relaxing integrality.

When considering a quadratic objective function with a single product term, we have to distinguish between two cases. In the first case, the product term consists of variables of two adjacent edges. We denote these edges by $e_1 := \{u, v\}$ and $e_2 := \{v, w\}$ and the product of their variables is called a *connected monomial*. The corresponding problem is denoted by QMST^c in the following. In the second case, the edges of the product variables are non-adjacent in the graph, therefore, the edges are $e_1 := \{u, v\}$ and $e_2 := \{w, z\}$ with pairwise distinct vertices $u, v, w, z \in V$. We refer to a *disconnected monomial* and denote the problem by QMST^d . As the context leads to the correct association, we shortly denote the linearization variables $y_{\{u,v\}\{v,w\}}$ and $y_{\{u,v\}\{w,z\}}$ by y .

Our aim is thus to investigate the polytope corresponding to QMST^c , defined as

$$P^c := \text{conv} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \{0, 1\}^{|E|+1} \mid x \text{ satisfies (2) to (3) and } y = x_{\{u,v\}} x_{\{v,w\}} \right\}$$

and the polytope corresponding to QMST^d , defined as

$$P^d := \text{conv} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \{0, 1\}^{|E|+1} \mid x \text{ satisfies (2) to (3) and } y = x_{\{u,v\}} x_{\{w,z\}} \right\}.$$

3 Polyhedral Results

In the following we assume $|V| \geq 4$. The dimension of the (linear) spanning tree polytope P^{lin} is $|E| - 1$. Clearly, the additional linearization variable y increases the dimension by at most one. In fact, we have

Theorem 3.1.

$$\dim(P^c) = \dim(P^d) = \dim(P^{\text{lin}}) + 1 = |E|.$$

The following results introduce two classes of facet defining inequalities for the polytopes P^c and P^d , respectively. They strengthen the subtour elimination constraints (3); we call them *quadratic subtour elimination constraints* in the following.

Theorem 3.2. *Let $S \subset V$ be a set of vertices with $u, w \in S$ and $v \notin S$. Then the inequality*

$$\sum_{e \in E(G[S])} x_e + y \leq |S| - 1 \quad (6)$$

induces a facet of P^c .

Theorem 3.3. *Let $S_1, S_2 \subset V$ be disjoint subsets of vertices such that both edges $\{u, v\}$ and $\{w, z\}$ have exactly one end node in S_1 and one end node in S_2 . Then the inequality*

$$\sum_{e \in E(G[S_1])} x_e + \sum_{e \in E(G[S_2])} x_e + y \leq |S_1| + |S_2| - 2 \quad (7)$$

induces a facet of P^d .

In fact, we conjecture that the classes of facets described in Theorems 3.2 and 3.3 yield a complete polyhedral description of P^c and P^d , respectively.

Conjecture 3.4.

$$P^c = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in [0, 1]^{|E|+1} \mid \begin{pmatrix} x \\ y \end{pmatrix} \text{ satisfies (2), (3), (4), (5), and (6)} \right\}$$

$$P^d = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in [0, 1]^{|E|+1} \mid \begin{pmatrix} x \\ y \end{pmatrix} \text{ satisfies (2), (3), (4), (5), and (7)} \right\}.$$

All three classes of subtour elimination constraints (3), (6) and (7) are of exponential size, so that these inequalities cannot be separated by enumeration. Therefore, a polynomial time separation routine is required. For (3), a well-known separation algorithm is based on a minimal cut algorithm. Using an appropriate adaption of this algorithm, we can show the following result:

Theorem 3.5. *The separation problems for the classes of inequalities (6) and (7) are polynomial time solvable.*

The problems can be reduced to solving one maximum s - t -flow problem for (6) and eight such problems for (7), corresponding to different fixings.

4 Experimental Results

Our aim is to determine the impact of the new inequalities for instances with more or all quadratic terms in the objective function. We implemented the separation algorithm of Theorem 3.5 and embedded it into the branch-and-cut software SCIL [4]. We considered the basic problem formulation using Constraints (2) to (5) where the subtour elimination constraints were separated dynamically and no further reformulation was applied (*stdlin*). For comparison we additionally separated the quadratic subtour elimination constraints (6) or (7) for each of the appearing products (*+qsec*). We tested graphs with edge densities of 33 %, 67 % and 100 %. The absolute weights are chosen randomly between 1 and 10, similar to the instances used in [3].

We discovered that, in the case of positive product weights, the separation of quadratic subtour elimination constraints does not improve the bounds at all, so we consider the case of negative product weights in the following. The separation for both connected and disconnected monomials leads to slightly longer running times but has a positive influence on the bounds and numbers of subproblems. In terms of running times the best approach turns out to be the separation for only connected monomials even if other monomials exist.

Actually, in most applications, the objective function contains only connected monomials. In this case using quadratic subtour elimination constraints (6) leads to significant improvements over *stdlin*, see the table below. Each line corresponds to the average of four tested instances.

edges	nodes	sep	# subs	# LPs	cputime	septime	rootgap
10	33 %	stdlin	5.00	8.50	0.01	0.00	1.63 %
		+qsec	1.00	3.25	0.00	0.00	0.00 %
	67 %	stdlin	38.50	45.50	0.14	0.00	23.65 %
		+qsec	27.50	32.75	0.21	0.05	21.91 %
	100 %	stdlin	3869.00	3635.25	36.39	0.42	72.31 %
		+qsec	1694.50	1788.25	24.91	3.42	71.46 %
15	33 %	stdlin	1434.00	1260.00	3.04	0.24	42.58 %
		+qsec	393.00	452.75	2.13	0.49	33.67 %
	67 %	stdlin	790.00	689.50	17.70	0.23	26.11 %
		+qsec	329.50	332.00	12.02	2.39	24.20 %

One can see that the separation time (*septime*) increases slightly but the bounds are improved significantly, leading to better root gaps (*rootgap*) and numbers of subproblems (*# subs*). Altogether, the running times (*cputime*) decrease considerably, showing the potential of our approach of considering only single product terms.

References

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