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Fuzzy quadratic minimum spanning tree problem

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Abstract

In this paper, a fuzzy quadratic minimum spanning tree problem is formulated as expected value model, chance-constrained programming and dependent-chance programming according to different decision criteria. Then the crisp equivalents are derived when the fuzzy costs are characterized by trapezoidal fuzzy numbers. Furthermore, a simulation-based genetic algorithm using Prüfer number representation is designed for solving the proposed fuzzy programming models as well as their crisp equivalents. Finally, a numerical example is provided for illustrating the effectiveness of the genetic algorithm.

Keywords: Minimum spanning tree; Fuzzy programming; Genetic algorithm; Credibility measure

1. Introduction

The minimum spanning tree (MST) problem is to find a least cost spanning tree in a weighted graph. The MST problem is one of the most important

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network optimization problems, and has important applications in transportation, communications, distribution systems, etc.

The MST problem has been well studied and many efficient algorithms have been developed by Dijkstra [6], Kruskal [12], Prim [18], Gabow et al. [8], Nondy and Murty [1], and Christofides [5]. As an extension of MST problem, the quadratic MST problem introduced by Xu [20] take into account the interaction effects between a pair of edges when considering the minimum cost spanning tree. Then the additional interactive cost makes the objective function no longer linear but quadratic. Xu also proved that the quadratic MST problem is NP-hard and gave two heuristic algorithm which can only get some local optima. Then Zhou and Gen [24] showed that genetic algorithm could find much better solutions than Xu's heuristic algorithm.

In practice, different uncertainties must be considered for different problems. It is well-known that the randomness of a stochastic network can be dealt with by probability theory, and the stochastic network optimization problem can be formulated as stochastic models. However, in many real applications, the problem parameters are vague or in a subjective nature. Then the problem parameters may be specialized as fuzzy variables by an experts system. In [11], Itoh and Ishii formulated a MST problem with fuzzy cost as chance-constrained programming based on the necessity measure. In [3], Chang and Lee defined three means based on the Overall Existence Ranking Index [2] for ranking fuzzy costs of spanning trees. Recently, Liu [16,17] developed a credibility theory including credibility measure, pessimistic value and expected value as fuzzy ranking methods. In this paper, based on the credibility theory, we propose the concepts of expected minimum spanning tree (EMST), α-pessimistic minimum spanning tree (α-PMST) and most minimum spanning tree (MMST) in a fuzzy quadratic minimum spanning tree (FQMST) problem. Then in order to find the EMST, α-PMST and MMST, we formulate the FQMST problem as expected value model, chance-constrained programming, and dependent-chance programming, respectively. The crisp equivalents are also discussed when the fuzzy costs are characterized by trapezoidal fuzzy numbers. Furthermore, a genetic algorithm using Prüfer number representation is designed for solving the proposed fuzzy programming models as well as their deterministic equivalents.

This paper is arranged as follows. After recalling some preliminaries on credibility theory, Section 3 introduces the concepts of EMST, α-PMST and MMST. Section 4 proposes three types of fuzzy programming models for a FQMST problem. Then the crisp equivalents of the three models are discussed. Nextly, a simulation-based genetic algorithm is designed for solving the proposed fuzzy programming models as well as their crisp equivalents. Lastly, a numerical example is provided for illustrating the effectiveness of the genetic algorithm.

2. A brief introduction to credibility theory

Possibility theory was proposed by Zadeh [23], and developed by many researchers such as Dubois and Prade [7]. Let Θ be a nonempty set, $\mathscr{P}(\Theta)$ be the power set of Θ, and Pos a possibility measure. The triplet $(\Theta, \mathscr{P}(\Theta), \mathrm{Pos})$ is called a possibility space. Then for each fuzzy event $A \in \mathscr{P}(\Theta)$, its possibility to occur is $\mathrm{Pos}\{A\}$, while its necessity to occur is defined by $\mathrm{Nec}\{A\} = 1 - \mathrm{Pos}\{A^c\}$. It is obvious that a fuzzy event may fail even though its possibility achieves 1, and hold even though its necessity is 0.

In the fuzzy optimization area, the most important fuzzy ranking methods are the possibility and necessity measures. Many other fuzzy ranking methods have also proposed in literature (e.g., [2,10,21,22]). Here we introduce the credibility theory developed by Liu [17], which includes credibility measure, expected value and pessimistic value as fuzzy ranking methods.

The credibility measure Cr is an average of possibility measure and necessity measure, i.e.,

$$\mathrm{Cr}\{\xi \leqslant r\} = \frac{1}{2}(\mathrm{Pos}\{\xi \leqslant r\} + \mathrm{Nec}\{\xi \leqslant r\}). \tag{1}$$

It is easy to verify that Cr has the following properties:

(i) $\mathrm{Cr}\{\emptyset\} = 0$, $\mathrm{Cr}\{\Theta\} = 1$;
(ii) $\mathrm{Cr}\{A\} \leqslant \mathrm{Cr}\{B\}$ whenever $A, B \in \mathscr{P}(\Theta)$, and $A \subset B$;
(iii) $\mathrm{Pos}\{A\} \geqslant \mathrm{Cr}\{A\} \geqslant \mathrm{Nec}\{A\}$ for all $A \in \mathscr{P}(\Theta)$;
(iv) $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ for all $A \in \mathscr{P}(\Theta)$.

Directly from these basic properties, we can see that the credibility measure may play the role of probability measure more appropriately than the possibility and necessity measures.

Based on the credibility measure, we have the expected value operator as follows.

Definition 1 (Liu and Liu [15]). Let ξ be a fuzzy variable. The expected value of ξ is defined as

$$E[\xi] = \int_0^{\infty} \mathrm{Cr}\{\xi \geqslant r\}\,\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leqslant r\}\,\mathrm{d}r \tag{2}$$

provided that at least one of the two integrals is finite.

Lemma 1 (Liu and Liu [16]). *Let ξ and η be independent fuzzy variables. Then for any real numbers a and b, we have $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$.*

Definition 2 (Liu and Liu [16]). Let ξ be a fuzzy variable defined on the possibility space $(\Theta, \mathscr{P}(\Theta), \text{Pos})$. Then the pessimistic value of ξ is defined as

$$\xi_{\inf}(\alpha) = \inf\{r | \text{Cr}\{\xi \leqslant r\} \geqslant \alpha\}, \quad \alpha \in (0, 1]. \tag{3}$$

3. Fuzzy quadratic minimum spanning tree

Let $G = (V, E)$ be an undirected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. A spanning tree $T = (V, S)$ is a subgraph of G such that $S \subseteq E$, $|S| = n - 1$ (where $|S|$ denotes the cardinality of set S) and T is connected. In real world, the transportation, communication, and distribution networks may be characterized by spanning trees. The direct cost ξ_i associated with edge e_i may represent the construction or running cost, $i = 1, 2, \ldots, m$. The interactive cost η_{ij}, which is caused by edge e_i and e_j's being selected simultaneously within a tree, $i, j = 1, 2, \ldots, m$, may explain two interactional activities. Let x be a binary decision variable defined as

$$x_i = \begin{cases} 1 & \text{if edge } e_i \text{ is selected,} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then the cost of a spanning tree $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ is denoted by

$$C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{i=1}^{m} \xi_i x_i + \sum_{i=1}^{m} \sum_{k=1}^{m} \eta_{ik} x_i x_k. \tag{5}$$

Let Γ be the set of all spanning trees corresponding to the graph G. Then a spanning tree $\boldsymbol{x}^*$ is called a minimum spanning tree if

$$C(\boldsymbol{x}^*, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) \tag{6}$$

for all $\boldsymbol{x} \in \Gamma$.

In real-world decision systems, the decision-maker often faces with some uncertain situations like insufficient information about the construction or running costs. For these cases, the direct costs ξ_i and interactive costs η_{ij}, $i, j = 1, 2, \ldots, m$ may be specified as fuzzy variables according to the expert system. Then the cost function $C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta})$ becomes fuzzy too. In order to rank spanning trees or the fuzzy costs, different decision makers may have different ideas. Suppose that the decision maker want to minimize the expected value of the fuzzy cost, we present the concept of expected minimum spanning tree (EMST).

Definition 3. A spanning tree $\boldsymbol{x}^*$ is called the expected minimum spanning tree if

$$E[C(\boldsymbol{x}^*, \boldsymbol{\xi}, \boldsymbol{\eta})] \leqslant E[C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta})]$$

for all spanning tree $\boldsymbol{x} \in \Gamma$, where $\mathrm{E}[C(x^*, \boldsymbol{\xi}, \boldsymbol{\eta})]$ is called the expected minimum cost.

In many cases, the decision maker sets a confidence level α as an appropriate safety margin, and hopes to minimize a critical value $\overline{C}$ with $\mathrm{Cr}\{C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant \overline{C}\} \geqslant \alpha$. For this case, we propose the concept of α-pessimistic minimum spanning tree (α-PMST) as follows:

Definition 4. A spanning tree $\boldsymbol{x}^*$ is called the α-pessimistic minimum spanning tree if

$$\min\{\overline{C} \mid \mathrm{Cr}\{C(\boldsymbol{x}^*, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant \overline{C}\} \geqslant \alpha\} \leqslant \min\{\overline{C} \mid \mathrm{Cr}\{C(\boldsymbol{x}, \xi, \boldsymbol{\eta}) \leqslant \overline{C}\} \geqslant \alpha\} \tag{7}$$

for all spanning tree $\boldsymbol{x} \in \Gamma$, where α is predetermined confidence level and $\min\{\overline{C} \mid \mathrm{Cr}\{C(\boldsymbol{x}^*, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant \overline{C}\} \geqslant \alpha\}$ is called the α-pessimistic minimum cost.

Sometimes, the decision maker may provide a cost supremum $\overline{C}$ and hope that the credibility of the cost's not exceeding $\overline{C}$ will be maximized as possible. For this case, we propose the concept of most minimum spanning tree (MMST) as follows:

Definition 5. A spanning tree $\boldsymbol{x}^*$ is called the most minimum spanning tree if

$$\mathrm{Cr}\{C(\boldsymbol{x}^*, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant \overline{C}\} \geqslant \mathrm{Cr}\{C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant \overline{C}\} \tag{8}$$

for all spanning tree $\boldsymbol{x} \in \Gamma$, where $\overline{C}$ is a predetermined cost supremum and $\mathrm{Cr}\{C(\boldsymbol{x}^*, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant \overline{C}\}$ is called the most credibility.

4. Fuzzy quadratic minimum spanning tree model

In this section, we use the concepts of EMST, α-PMST and MMST as decision criteria, and formulate the FQMST problem as expected value model, chance-constrained programming model and dependent-chance programming model, respectively.

Fuzzy expected value model, which was presented by Liu and Liu [15], is to optimize the expected objective subject to some constraints. In order to find the EMST with fuzzy costs, we have the following expected value model:

$$\begin{cases} \min & E\left[\sum_{i=1}^{m} \sum_{k=1}^{m} \eta_{ik} x_i x_k + \sum_{i=1}^{m} \xi_i x_i\right] \\ \text{s.t.:} & x_i = 0 \text{ or } 1, \\ & \boldsymbol{x} \in \Gamma. \end{cases} \tag{9}$$

Chance-constrained programming offers us a powerful means for modelling stochastic decision systems [4] and fuzzy decision systems [13]. The essential idea of chance-constrained programming is to optimize the critical value of the fuzzy objective with certain confidence level subject to some chance constraints. In order to find the α-PMST, where α is a confidence level provided by the decision maker, we propose the following chance-constrained programming model:

$$\begin{cases} \min & \overline{C} \\ \text{s.t.:} & \mathrm{Cr}\left\{\sum_{i=1}^{m}\sum_{k=1}^{m}\eta_{ik}x_ix_k + \sum_{i=1}^{m}\xi_ix_i \leqslant \overline{C}\right\} \geqslant \alpha, \\ & x_i = 0 \text{ or } 1, \\ & \boldsymbol{x} \in \Gamma. \end{cases} \tag{10}$$

Sometimes the decision-maker wishes to maximize the chance functions of some events (i.e. the credibility of satisfying these fuzzy events). In order to model this type of fuzzy decision system, Liu [14] provided one type of fuzzy programming model: dependent-chance programming, in which the underlying philosophy is based on selecting the decision with maximal chance to meet the fuzzy event. Now let us model the FQMST problem by dependent-chance programming. Suppose that the decision maker sets a cost supremum $\overline{C}$, and hopes to find the MMST, we have the dependent-chance programming model as follows:

$$\begin{cases} \min & \mathrm{Cr}\left\{\sum_{i=1}^{m}\sum_{k=1}^{m}\eta_{ik}x_ix_k + \sum_{i=1}^{m}\xi_ix_i \leqslant \overline{C}\right\} \\ \text{s.t.:} & x_i = 0 \text{ or } 1, \\ & \boldsymbol{x} \in \Gamma. \end{cases} \tag{11}$$

5. Crisp equivalents

In this section, based on the credibility theory, we propose the crisp equivalents of the proposed FQMST models under some assumptions.

Theorem 1. *Let ξ_i, η_{ij}, $i, j = 1, 2, \ldots, m$ be independent fuzzy variables. Then the crisp equivalent of the FQMST model (9) is*

$$\begin{cases} \min & \sum_{i=1}^{m}\sum_{k=1}^{m}E[\eta_{ik}]x_ix_k + \sum_{i=1}^{m}E[\xi_i]x_i \\ \text{s.t.:} & x_i = 0 \textit{ or } 1, \\ & \boldsymbol{x} \in \Gamma. \end{cases} \tag{12}$$

Now, we suppose that the direct costs and the interactive costs are independent trapezoidal fuzzy numbers, and give the crisp equivalents of FQMST models (10) and (11), respectively.

By *trapezoidal fuzzy variables* we mean the fuzzy variables fully determined by quadruples (r_1, r_2, r_3, r_4) of crisp numbers with $r_1 < r_2 \leqslant r_3 < r_4$, whose membership functions can be denoted by

$$\mu(x) = \begin{cases} \frac{x-r_1}{r_2-r_1} & \text{if } r_1 \leqslant x \leqslant r_2, \\ 1 & \text{if } r_2 \leqslant x \leqslant r_3, \\ \frac{x-r_4}{r_3-r_4} & \text{if } r_3 \leqslant x \leqslant r_4, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

We note that the trapezoidal fuzzy variable is a *triangular fuzzy variable* if $r_2 = r_3$, denoted by a triple (r_1, r_2, r_4).

Lemma 2. *Let $\xi = (r_1, r_2, r_3, r_4)$ and $\eta = (q_1, q_2, q_3, q_4)$ be two trapezoidal fuzzy variables, and a, b two nonnegative numbers. Then we have that*

$$a\xi + b\eta = (ar_1 + bq_1, ar_2 + bq_2, ar_3 + bq_3, ar_4 + bq_4).$$

Proof. The lemma can be proved directly by Zadeh's *Extension Principle*. The proof is omitted here for simplicity. □

Lemma 3. *Let $\xi = (r_1, r_2, r_3, r_4)$ be a trapezoidal fuzzy variable. Then the credibility distribution of ξ is continuous and defined by*

$$\mathrm{Cr}\{\xi \leqslant \overline{C}\} = \begin{cases} 1 & \text{if } r_4 \leqslant \overline{C}, \\ \frac{\overline{C}+r_4-2r_3}{2(r_4-r_3)} & \text{if } r_3 \leqslant \overline{C} \leqslant r_4, \\ \frac{1}{2} & \text{if } r_2 \leqslant \overline{C} \leqslant r_3, \\ \frac{\overline{C}-r_1}{2(r_2-r_1)} & \text{if } r_1 \leqslant \overline{C} \leqslant r_2, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Proof. From the definition of credibility, we can obtain the credibility distribution of ξ defined by (14). It is obvious that the credibility distribution function is continuous. □

Lemma 4. *Let $\xi = (r_1, r_2, r_3, r_4)$ be a trapezoidal fuzzy variable, and α a given confidence level. Then we have*

(a) *when* $\alpha \leqslant 1/2$, $\mathrm{Cr}\{\xi \leqslant \overline{C}\} \geqslant \alpha$ *if and only if* $(1-2\alpha)r_1 + 2\alpha r_2 \leqslant \overline{C}$;
(b) *when* $\alpha > 1/2$, $\mathrm{Cr}\{\xi \leqslant \overline{C}\} \geqslant \alpha$ *if and only if* $(2-2\alpha)r_3 + (2\alpha-1)r_4 \leqslant \overline{C}$.

Proof. Part (a) If $\alpha \leqslant 1/2$ and $\mathrm{Cr}\{\xi \leqslant \overline{C}\} \geqslant \alpha$, then we have $r_2 \leqslant \overline{C} \leqslant r_3$ or $(r_2-\overline{C})/2(r_2-r_1) \geqslant \alpha$. When $(r_2-\overline{C})/2(r_2-r_1) \geqslant \alpha$, we have $(1-2\alpha)r_1 + 2\alpha r_2 \leqslant \overline{C}$. When $r_2 \leqslant \overline{C} \leqslant r_3$, since $\alpha \leqslant 1/2$, $(1-2\alpha)r_1 + 2\alpha r_2 \leqslant \overline{C}$ holds too. Conversely, if $r_2 \leqslant \overline{C}$, then $\mathrm{Cr}\{\xi \leqslant \overline{C}\} \geqslant 1/2 \geqslant \alpha$. If $(1-2\alpha)r_1 + 2\alpha r_2 \leqslant \overline{C}$, then $\overline{C} - r_1/2(r_2-r_1) \geqslant \alpha$. Thus $\mathrm{Cr}\{\xi \leqslant \overline{C}\} \geqslant \alpha$.

Part (b) If $\alpha > 1/2$ and $\mathrm{Cr}\{\xi \leqslant \overline{C}\} \geqslant \alpha$, then we have $r_4 \leqslant \overline{C}$ or $(\overline{C} + r_4 - 2r_3)/2(r_4-r_3) \geqslant \alpha$. It is easy to verify that $(2-2\alpha)r_3 + (2\alpha-1)r_4 \leqslant \overline{C}$. Conversely, if $r_4 \leqslant \overline{C}$, then $\mathrm{Cr}\{\xi \leqslant \overline{C}\} = 1 \geqslant \alpha$. If $(2-2\alpha)r_3 + (2\alpha-1)r_4 \leqslant \overline{C}$, then we also have $(2r_3 - r_4)/2(r_3 - r_4) \geqslant \alpha$. Thus $\mathrm{Cr}\{\xi \leqslant \overline{C}\} \geqslant \alpha$. □

Theorem 2. *Let* ξ_i, η_{ij} $(i, j = 1, 2, \ldots, m)$ *be independent trapezoidal fuzzy numbers.* $\xi_i = (s_{i1}, s_{i2}, s_{i3}, s_{i4})$, $\eta_{ij} = (t_{ij1}, t_{ij2}, t_{ij3}, t_{ij4})$, $i, j = 1, 2, \ldots, m$. *If* $\alpha > 0.5$, *then the crisp equivalent of the FQMST model* (10) *is given by*

$$\begin{cases} \min & (2-2\alpha)\left(\sum_{i=1}^{m}\sum_{j=1}^{m} t_{ij3}x_i x_j + \sum_{i=1}^{m} s_{i3}x_i\right) \\ & +\,(2\alpha-1)\left(\sum_{i=1}^{m}\sum_{j=1}^{m} t_{ij4}x_i x_j + \sum_{i=1}^{m} s_{i4}x_i\right) \\ \text{s.t.:} & x_i = 0 \text{ or } 1, \\ & \boldsymbol{x} \in \Gamma. \end{cases} \tag{15}$$

Proof. Since $x_i \geqslant 0$ for $i = 1, 2, \ldots, m$, it follows from Lemma 2 that the cost function

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \eta_{ij}x_i x_j + \sum_{i=1}^{m} w_i x_i$$

is also a trapezoidal fuzzy number, and determined by the quadruple

$$\begin{pmatrix} g_1(\boldsymbol{x}) \\ g_2(\boldsymbol{x}) \\ g_3(\boldsymbol{x}) \\ g_4(\boldsymbol{x}) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} \sum_{i=1}^{m}\sum_{j=1}^{m} t_{ij1}x_i x_j + \sum_{i=1}^{m} s_{i1}x_i \\ \sum_{i=1}^{m}\sum_{j=1}^{m} t_{ij2}x_i x_j + \sum_{i=1}^{m} s_{i2}x_i \\ \sum_{i=1}^{m}\sum_{j=1}^{m} t_{ij3}x_i x_j + \sum_{i=1}^{m} s_{i3}x_i \\ \sum_{i=1}^{m}\sum_{j=1}^{m} t_{ij4}x_i x_j + \sum_{i=1}^{m} s_{i4}x_i \end{pmatrix}^{\mathrm{T}}. \tag{16}$$

Then it follows from Lemma 4 that the chance constraint

$$\mathrm{Cr}\left\{\sum_{i=1}^{m}\sum_{k=1}^{m}\eta_{ik}x_ix_k+\sum_{i=1}^{m}\xi_ix_i\leqslant\overline{C}\right\}\geqslant\alpha$$

is equivalent to

$$\begin{aligned}\overline{C}\geqslant&(2-2\alpha)\left(\sum_{i=1}^{m}\sum_{j=1}^{m}t_{ij3}x_ix_j+\sum_{i=1}^{m}s_{i3}x_i\right)\\&+(2\alpha-1)\left(\sum_{i=1}^{m}\sum_{j=1}^{m}t_{ij4}x_ix_j+\sum_{i=1}^{m}s_{i4}x_i\right).\end{aligned}\tag{17}$$

That is, the FQMS model (10) is equivalent to model (15). □

Theorem 3. *Let ξ_i, η_{ij} $(i,j=1,2,\ldots,m)$ be independent trapezoidal fuzzy numbers. $\xi_i=(s_{i1},s_{i2},s_{i3},s_{i4})$, $\eta_{ij}=(t_{ij1},t_{ij2},t_{ij3},t_{ij4})$, $i,j=1,2,\ldots,m$. Then the crisp equivalent of the fuzzy dependent-chance programming (11) is given by*

$$\begin{cases}\min & f(\boldsymbol{x})\\ \text{s.t.:} & x_i=0\ \mathit{or}\ 1,\\ & \boldsymbol{x}\in\Gamma,\end{cases}\tag{18}$$

where $f(\mathbf{x})$ is a real function defined by

$$f(\boldsymbol{x})=\begin{cases}1 & \mathit{if}\ g_4(\boldsymbol{x})\leqslant\overline{C},\\ \frac{\overline{C}+g_4(\boldsymbol{x})-2g_3(\boldsymbol{x})}{2(g_4(\boldsymbol{x})-g_3(\boldsymbol{x}))} & \mathit{if}\ g_3(\boldsymbol{x})\leqslant\overline{C}\leqslant g_4(\boldsymbol{x}),\\ \frac{1}{2} & \mathit{if}\ g_2(\boldsymbol{x})\leqslant\overline{C}\leqslant g_3(\boldsymbol{x}),\\ \frac{\overline{C}-g_1(\boldsymbol{x})}{2(g_2(\boldsymbol{x})-g_1(\boldsymbol{x}))} & \mathit{if}\ g_1(\boldsymbol{x})\leqslant\overline{C}\leqslant g_2(\boldsymbol{x}),\\ 0 & \mathit{otherwise},\end{cases}\tag{19}$$

and $g_i(\boldsymbol{x})$, $i=1,2,3,4$, are defined by (16).

Proof. Since $x_i\geqslant 0$ for $i=1,2,\ldots,m$, it follows from Lemma 2 that the cost function

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\eta_{ij}x_ix_j+\sum_{i=1}^{m}w_ix_i$$

is also a trapezoidal fuzzy number, and determined by the quadruple $(g_1(\boldsymbol{x}),g_2(\boldsymbol{x}),g_3(\boldsymbol{x}),g_4(\boldsymbol{x}))$ defined in Eq. (16). Then it follows from Lemma 3 that the chance function

$$\mathrm{Cr}\left\{\sum_{i=1}^{m}\sum_{k=1}^{m}\eta_{ik}x_ix_k+\sum_{i=1}^{m}\xi_ix_i\leqslant\overline{C}\right\}$$

is equivalent to the real function $f(\boldsymbol{x})$ defined by Eq. (19). That is, the FQMST model (11) is equivalent to model (18). □

6. Genetic algorithm approach

Xu [20] proved that the quadratic MST problem is NP-hard and gave two heuristic algorithms which can get some local optima. Zhou and Gen [24] showed that genetic algorithm can find much better solutions than Xu's heuristic algorithms. As a fuzzy extension of quadratic MST problem, the FQMST problem is more difficult to solve. In the following, we will design a fuzzy simulation-based genetic algorithm for solving the proposed FQMST models as well as their crisp equivalents.

6.1. Chromosome representation

When talking of genetic algorithm, we must start with the representation of chromosome. In the FQMST problem, a chromosome should represent a spanning tree.

Firstly, we refer to Cayley's theorem in graphical enumeration, which shows that there are n^{n-2} distinct labelled trees on a complete graph with n vertices. Prüfer [19] proved this theorem by constructing one-to-one correspondence between such trees and the set of permutation of $n-2$ digit. And this permutation is known as the Prüfer number. Then based on the one-to-one correspondence, we use the Prüfer number as chromosome, which uniquely represents a spanning tree. For detailed discussion of encoding method of tree, the reader may consult the book [9] by Gen and Cheng.

For any tree, there are at least two leaf vertices. By leaf vertex we mean that there is only one edge connected to the vertex. The following encoding and decoding procedure were proposed by Zhou and Gen [24].

Encoding procedure

Step 1. Find the smallest labelled leaf vertex, say i, in a labelled tree.
Step 2. Let j be the first digit in the encoding if j is incident to vertex i. Here we build the encoding by appending digits to the right, and thus the encoding is built from left to right.
Step 3. Remove vertex i and the edge from i to j, we get a tree with $n-1$ vertices.

Step 4. Repeat the above steps until one edge is left and produce the Prüfer number with $n-2$ digits between 1 and n inclusive.

Decoding procedure

Step 1. Let P be the original Prüfer number, $\overline{P}$ be the set of all vertices not included in P, and be designated as eligible for consideration.

Step 2. Find the smallest labelled leaf vertex, say i, in $\overline{P}$, and the leftmost digit of P, say j. Add the edge from i to j in to the tree. Remove i from $\overline{P}$ and j from P. If j does not occur anywhere in the remain part of P, designate j as eligible and put it in to $\overline{P}$.

Step 3. Repeat the above two steps until there are only two vertices, r and s, eligible for consideration. Add the edge from r to s into the tree and form a tree with $n-1$ edges.

Now we convert a spanning tree (Fig. 1) to a Prüfer number for illustrating the procedure of encoding process. In Fig. 1, the leaf vertex with smallest label is 4, and vertex 3 is incident to it. So we remove the vertex 4 and the edge from vertex 4 to 3 from the tree, and let 3 be the leftmost digit in the Prüfer number. Nextly, find the smallest leaf vertex 3, and the vertex incident to it in the subtree, vertex 2. Let 2 be the second digit in the Prüfer number, and remove the vertex 3 and edge from vertex 3 to 2 from the tree. Repeat the above process on the subtree, we finally get a Prüfer number with four digits $(3,2,2,1)$ and leave only one edge from vertex 1 to 6.

The decoding procedure of the Prüfer number $(3,2,2,1)$ will be demonstrated in Section 7.

6.2. Fuzzy simulation

If the fuzzy chance-constrained programming can be converted to its deterministic equivalent, then it is easy for us to compute the objective function. Otherwise, we may resort to the fuzzy simulation technique [16,17]. It has been discussed at length by Liu [16,17] that how to compute the credibility, pessimistic value and expected value defined by

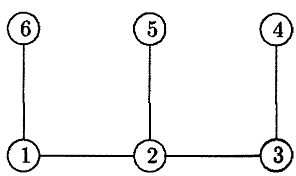

Fig. 1. The spanning tree associated with Prüfer number (3,2,2,1).

$$\mathrm{Cr}\left\{\sum_{i=1}^{m}\sum_{j=1}^{m}\eta_{ij}x_ix_j+\sum_{i=1}^{m}\xi_ix_i\leqslant\overline{C}\right\},\tag{20}$$

$$\inf\left\{f\,|\,\mathrm{Cr}\left\{\sum_{i=1}^{m}\sum_{j=1}^{m}\eta_{ij}x_ix_j+\sum_{i=1}^{m}\xi_ix_i\leqslant\overline{C}\right\}\geqslant\alpha\right\}\tag{21}$$

and

$$E\left[\sum_{i=1}^{m}\sum_{j=1}^{m}\eta_{ij}x_ix_j+\sum_{i=1}^{m}\xi_ix_i\right],\tag{22}$$

respectively. Here we only give the procedures for computing them.

Fuzzy simulation for credibility

Step 1. Randomly generate u_{ijk} from the ε-level set of η_{ij}, and v_{ik} from the ε-level set of ξ_i, $i, j = 1, 2, \ldots, m$, respectively, where $k = 1, 2, \ldots, N$ and ε is a sufficiently small positive number.

Step 2. Set $v_k=\min_{i,j}\{\mu_{\eta_{ij}}(u_{ijk})\}\wedge\min_i\{\mu_{w_i}(v_{ik})\}$ for $k = 1, 2, \ldots, N$.

Step 3. Return $L(\overline{C})$ via the following estimation formula (23):

$$L(b)=\frac{1}{2}\left(\max_{1\leqslant k\leqslant N}\{v_k|C_k(\boldsymbol{x})\leqslant b\}+\min_{1\leqslant k\leqslant N}\{1-v_k|C_k(\boldsymbol{x})>b\}\right),\tag{23}$$

where

$$C_k(\boldsymbol{x})=\sum_{i=1}^{m}\sum_{j=1}^{m}u_{ijk}x_ix_j+\sum_{i=1}^{m}v_{ik}x_i.\tag{24}$$

Fuzzy simulation for α-pessimistic value

Step 1. Randomly generate u_{ijk} from the ε-level set of η_{ij}, and v_{ik} from the ε-level set of w_i, $i,j = 1, 2, \ldots, m$, respectively, where $k = 1, 2, \ldots, N$ and ε is a sufficiently small positive number.

Step 2. Set $v_k=\min_{i,j}\{\mu_{\eta_{ij}}(u_{ijk})\}\wedge\min_i\{\mu_{w_i}(v_{ik})\}$ for $k = 1, 2, \ldots, N$.

Step 3. Find the maximal value b such that $L(b)\geqslant\alpha$ holds, where $L(b)$ is defined by (23).

Step 4. Return b.

Fuzzy simulation for expected value

Step 1. Set $e = 0$.
Step 2. Randomly generate u_{ijk} from the ε-level set of η_{ij}, and v_{ik} from the ε-level set of w_i, $i, j = 1, 2, \ldots, m$, respectively, where $k = 1, 2, \ldots, N$ and ε is a sufficiently small positive number.
Step 3. Set $v_k = \min_{i,j}\{\mu_{\eta_{ij}}(u_{ijk})\} \wedge \min_i\{\mu_{w_i}(v_{ik})\}$.
Step 4. Randomly generate b from $[b_1, b_2]$.
Step 5. If $b \geqslant 0$, then $e \leftarrow e + \mathrm{Cr}\{C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) \leqslant b\}$.
Step 6. If $b < 0$, then $e \leftarrow e - \mathrm{Cr}\{C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) \geqslant b\}$.
Step 7. Repeat the fourth to sixth steps for N times.
Step 8. Return $\mathrm{E}[C(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta})] = b_1 \vee 0 + b_2 \wedge 0 + e \cdot (b_2 - b_1)/N$.

6.3. Crossover and mutation operation

Crossover and mutation operation are two crucial factors in the biological evolutionary process. In the genetic algorithm approach, they may guarantee the diversity of the population. Hence the population have a great chance to be evolved to the optimal solution. Since a Prüfer number can always represent a labelled tree, we select a simple way for crossover and mutation operation. For two chromosomes to crossover, we just exchange their digits at randomly selected positions. And for a chromosome to mutate, randomly select a position and randomly generate an integer between 1 and n including 1 and n to replace the original one.

6.4. Evaluation and selection process

Evaluation and selection process play an important role in genetic algorithm. They may be regarded as the exploration for genetic algorithm to converge to the optimal or near-optimal solution. In our genetic algorithm approach for FQMST problem, the evaluation perform the following functions: (i) decoding all the chromosomes and calculating their expected cost, α-pessimistic cost and chance function; (ii) assigning each chromosome a fitness by a rank-based method according to its objective value. Then in the selection process, by spinning the roulette wheel *pop_size* times, we get a new population to go further.

6.5. Genetic algorithm procedure

Following selection, crossover, and mutation, the new population is ready for its next evaluation. The genetic algorithm will terminate after a given number of cyclic repetitions of the above steps or a suitable solution has been found. A general genetic algorithm procedure is given as follows.

Genetic algorithm procedure

Step 1. Initialize *pop_size* chromosomes randomly.
Step 2. Update the chromosomes by crossover and mutation operations.
Step 3. Calculate the objective values for all chromosomes.
Step 4. Compute the fitness of each chromosome according to the objective values.
Step 5. Select the chromosomes by spinning the roulette wheel.
Step 6. Repeat the second to fifth steps for a given number of cycles.
Step 7. Report the best chromosome as the optimal solution.

7. Numerical example

In this section, we give a numerical example that is performed on a personal computer to illustrate the effectiveness of the simulation-based genetic algorithm.

Example. Consider an FQMST problem with 6 vertices. For a complete graph, we label its 6 vertices with integers 1, 2, ..., 6, respectively. Then it has $C_6^2 = 15$ edges. The direct costs and interactive costs are independent trapezoidal fuzzy numbers. $\xi_i = (s_{i1}, s_{i2}, s_{i3}, s_{i4})$, $\eta_{ij} = (t_{ij1}, t_{ij2}, t_{ij3}, t_{ij4})$, where s_{ik} are randomly generated from [2, 10], t_{ijk} are randomly generated from [0, 4], $i, j = 1, 2, \ldots, 15$, $k = 1, 2, 3, 4$.

In order to find the EMST and 0.90-PMST and MMST (cost supremum is set as 40 by the decision maker), we formulate the FQMST problem as expected value model (9), chance-constrained programming (10) and dependent-chance programming (11), respectively. Then, based on Theorems 1–3, we convert them to their crisp equivalents (12), (15) and (18), respectively. After a run of the genetic algorithm with *pop_size* = 30, we get that the EMST, 0.90-MST and MMST are the same spanning with Prüfer number $(3, 2, 2, 1)$. The corresponding expected minimum cost, 0.90-pessimistic minimum cost and most credibility are 28.98, 43.28 and 0.78, respectively. Now, we convert the Prüfer number $(3, 2, 2, 1)$ to a spanning tree by the decoding procedure.

1. Let $P = (3, 2, 2, 1)$ and $\overline{P} = \{4, 5, 6\}$.
2. Remove the smallest integer 4 from $\overline{P}$, and the leftmost integer 3 from P. Add the edge from 3 to 4 to the tree. Since 3 does not occur at other places of P, we have $P = (2, 2, 1)$, $\overline{P} = \{3, 5, 6\}$.

3. Remove the smallest integer 3 from $\overline{P}$, and the leftmost integer 2 from P. Add the edge from 2 to 3 to the tree. Since 2 occurs at other places of P, we have $P = (2, 1)$ and $\overline{P} = \{5, 6\}$.
4. Remove the smallest integer 5 from $\overline{P}$, and the leftmost integer 2 from P. Add the edge from 2 to 5 to the tree. Since 2 does not occur at other places of P, we have $P = (1)$, $\overline{P} = \{2, 6\}$.
5. Remove the smallest integer 2 from $\overline{P}$, and the leftmost integer 1 from P. Add the edge from 1 to 2 to the tree. Now we have $\overline{P} = \{1, 6\}$, $P = \emptyset$.
6. Add the edge from vertex 1 to 6 to the tree, and we get a tree with five edges, i.e., a spanning tree in Fig. 1.

8. Conclusions

In this paper, the FQMST problem was formulated as expected value model, chance-constrained programming and dependence-chance programming. Their crisp equivalent models were also proposed based on the credibility theory. Moreover, a genetic algorithm approach was proposed for solving the proposed FQMST models as well as their crisp equivalents. A numerical example was also provided for illustrating the effectiveness of the genetic algorithm.

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