A linearization method for quadratic minimum spanning tree problem

Jing-Rung Yu, Ren-Xiang Shi, and Her-Jiun Sheu

Abstract

The crisp and fuzzy quadratic minimum spanning tree (Q-MST) problem can be formulated as a linear model, and thus, the global optimum can be obtained by the proposed method. Conventionally, the Q-MST problem, which contains a quadratic term in the objective function, is solved by genetic algorithm and other heuristic methods. However, these methods cannot guarantee to obtain a global optimal solution. To address this issue, the proposed method transforms the quadratic term into linear formulations for crisp and fuzzy Q-MST problems, and yields the global optimum solutions by linear integer programming. Two examples are given to demonstrate the proposed method in greater detail.

Keywords: Linear integer programming, Network optimization, Quadratic minimum spanning tree.

1. Introduction

The quadratic minimum spanning tree (Q-MST) problem, which is an extension of minimum spanning tree, was first discussed by Xu [9]. The Q-MST problem focuses on the interactive cost of a pair of edges in an undirected graph. Xu also proved that the Q-MST problem is NP-hard, and proposed two heuristic algorithms solutions [10]. Zhou and Gen found that Xu's two heuristics algorithms could only obtain local optimal solutions, which are far from the optimal solution [11]. The genetic algorithm approach, as proposed by Zhou and Gen [11] can address Q-MST problem more effectively, and thus, obtain a better local optimum solution than the two algorithms of Xu. However, the approach of Zhou and Gen cannot guarantee to obtain the global optimum [3].

In order to deal with vague parameters found in real applications, Gao and Lu proposed three models for solving the fuzzy Q-MST problem [3]. Gao and Lu considered that the decision-makers may face a fuzzy decision environment, thus the fuzzy Q-MST problem was formulated. The three models, with simulation-based genetic algorithms, are more efficient than the approach of Zhou and Gen, which cannot guarantee achieving the global optimum.

Whether a crisp or fuzzy Q-MST problem, all must be solved by genetic or other heuristic algorithms because of the nonlinear issues. Thus, the proposed method adopts Chang and Chang's concept [1], which is to linearize the nonlinear terms of Gao and Lu's models [3]. Both crisp and fuzzy Q-MST problems can be reformulated into a linear integer-programming problem, and thus, the global optimal solution can be easily obtained.

The remainder of this paper is organized as follows. Section 2 introduces the crisp and fuzzy Q-MST problem. Section 3 presents Chang's linearization method [1]. Section 4 discusses the proposed method. Section 5 provides two examples to illustrate the proposed method. Section 6 is the conclusion.

2. Crisp and fuzzy quadratic minimum spanning tree problem

This section introduces a crisp and fuzzy quadratic minimum spanning tree method, as proposed by Xu [9], Gao and Lu [3]. First, an undirected graph $G = (V, E)$ consists of a vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and edge set $E = \{e_1, e_2, \ldots, e_m\}$, and $T = (V, S)$ is signified as a spanning tree, which is a sub-graph of G such that $S \subseteq E$, $|S| = n - 1$. The direct cost ζ_j occurs when edge e_j is selected, $j = 1, 2, \ldots, m$, and interactive cost η_{jk} occurs when edges e_j and e_k are selected simultaneously, where $j, k = 1, 2, \ldots, m$. Let x_j be a binary variable, $x_j = 1$ when e_j is selected; otherwise, $x_j = 0$. Γ is the set of all spanning trees $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ corresponding to graph G.

The problem of the crisp Q-MST can be formulated as follows:

$$\min \quad \sum_{j=1}^{m} \zeta_j x_j + \sum_{j=1}^{m}\sum_{k=1}^{m} \eta_{jk} x_j x_k$$

$$\text{s.t.} \quad x_j = 0 \text{ or } 1, j = 1, 2, \ldots, m, \mathbf{x} \in \Gamma. \tag{1}$$

The uncertainties of direct and interactive costs might occur in real cases. Therefore, according to the credibility theory [6] [7], Gao and Lu proposed the

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concepts of an expected-minimum spanning tree, α-pessimistic-minimum spanning tree, and most-minimum spanning tree for the fuzzy Q-MST problem [3]. The expected-minimum spanning tree, α-pessimistic-minimum spanning tree, and most-minimum spanning tree can all be solved by the expected-value model, chance-constrained programming, and dependent-chance programming, respectively.

The three models are described in the following.

Expected value model:

To obtain the expected-minimum spanning tree with fuzzy costs, Gao and Lu [3] proposed the following expected-value model:

$$\min \quad E\Big(\sum_{j=1}^{m}\zeta_j x_j + \sum_{j=1}^{m}\sum_{k=1}^{m}\eta_{jk} x_j x_k\Big)$$

$$\text{s.t.} \quad x_j = 0 \text{ or } 1, j = 1, 2, \ldots, m, \mathbf{x} \in \Gamma . \qquad (2)$$

Chance-constrained programming model:

The chance-constrained programming model aims to optimize the critical value of the objective function with confidence level α.

$$\min \quad \bar{C}$$

$$\text{s.t.} \quad Cr\Big(\sum_{j=1}^{m}\zeta_j x_j + \sum_{j=1}^{m}\sum_{k=1}^{m}\eta_{jk} x_j x_k \le \bar{C}\Big) \ge \alpha$$

$$x_j = 0 \text{ or } 1, j = 1, 2, \ldots, m, \mathbf{x} \in \Gamma , \qquad (3)$$

where α is a confidence level, which is provided by the decision-maker.

Dependent-chance programming model:

The dependent-chance programming model focuses on selecting the decision with maximal chance to meet the fuzzy event.

$$\min \quad Cr\Big(\sum_{j=1}^{m}\zeta_j x_j + \sum_{j=1}^{m}\sum_{k=1}^{m}\eta_{jk} x_j x_k \le \bar{C}\Big)$$

$$\text{s.t.} \quad x_j = 0 \text{ or } 1, j = 1, 2, \ldots, m, \mathbf{x} \in \Gamma , \qquad (4)$$

where $\bar{C}$ is assigned by the decision-maker, and is an upper cost boundary.

Since the above three models are employed to solve the fuzzy Q-MST problem, suppose the trapezoidal fuzzy costs η and ζ are defined as quadruples (r_1, r_2, r_3, r_4) of crisp numbers, with $r_1 < r_2 \le r_3 < r_4$, and the membership function $u(r)$, are as follows.

$$u(r) = \begin{cases} \dfrac{r - r_1}{r_2 - r_1} & \text{if } r_1 \le r \le r_2 \\ 1 & \text{if } r_2 \le r \le r_3 \\ \dfrac{r - r_4}{r_3 - r_4} & \text{if } r_3 \le r \le r_4 \\ 0 & \text{otherwise} \end{cases} \qquad (5)$$

where r represents the direct, or interactive, costs. For simplicity, the cut is set to be 0 here.

According to some assumptions, Gao and Lu turned their three models into crisp equivalents of the fuzzy Q-MST, and then solved the problem using a simulation-based genetic algorithm [3]. For example, the crisp equivalent of chance-constrained programming is as follows:

$$\min \quad (2 - 2\alpha)\Big(\sum_{j=1}^{m}\sum_{k=j}^{m} t_{jk3}x_j x_k + \sum_{j=1}^{m} s_{j3}x_j\Big) + (2\alpha - 1)\Big(\sum_{j=1}^{m}\sum_{k=j}^{m} t_{jk4}x_j x_k + \sum_{j=1}^{m} s_{j4}x_j\Big)$$

$$\text{s.t.} \quad x_j = 0 \text{ or } 1, \qquad (6)$$

$$j = 1, 2, \ldots, m, \mathbf{x} \in \Gamma ,$$

where η_{jk} and ζ_j are independent trapezoidal fuzzy numbers. The fuzzy direct and interactive costs are equal to $\zeta_j = (s_{j1}, s_{j2}, s_{j3}, s_{j4})$ and $\eta_{jk} = (t_{jk1}, t_{jk2}, t_{jk3}, t_{jk4})$, respectively, where $j, k = 1, 2, \ldots, m$.

For crisp and fuzzy Q-MST problems, the genetic algorithm might obtain the local optimal solution [3][5][11] because of nonlinear properties. Therefore, we employ the proposed method to linearize the quadratic terms of the objective functions of the Q-MST problem, and then solve it by linear programming. Thus, the global optimal solution can be guaranteed. In Section 3, Chang and Chang's linearization method is introduced.

3. Chang and Chang's method for linearization[1]

Chang and Chang proposed a linear optimization method for solving mixed 0-1 nonlinear programming. The general model, which they solve, is as follows:

$$\min \quad \eta\, x_1 x_2 \ldots x_n y$$

$$\text{s.t.} \quad L(x, y), \qquad (7)$$

where x_j (j = 1, 2, …, n) is a 0-1 variable, y is a continuous variable ($0 \le y \le \bar{u}$), $\bar{u}$ is the upper bound, and $L(x, y)$ is a set of linear constraints. In addition to having the same linear type constraints, Q-MST fits the general model, where n is equal to 2, without continuous variable y and η as positive coefficients. Thus, only the nonlinear term of the objective function requires linearization, because the nonlinear term of the Q-MST problem exists only in the objective function. Model 8 is a linear problem, which is described as follows:

$$\min \quad z$$

$$\text{s.t.} \quad z \ge \eta y - \eta\bar{u}\ \Big(n - \sum_{j=1}^{n} x_j\Big) \qquad (8)$$

$$z \ge 0, \quad L(x, y),$$

where $z = \eta\, x_1 x_2 \ldots\, x_n y$. For more details on the transfer procedure, refer to Chang's linearization method [1]. The linearization procedure is suitable for the quadratic term in the Q-MST problem, where only two 0-1 variables are adopted, and y is equal to one. Therefore, this technique is useful for linearization issues inherit in the Q-MST problem.

4. The proposed method

Each model of Section 2 include the quadratic term $\sum_{j=1}^{m}\sum_{k=1}^{m}\eta_{jk}\, x_j\, x_k$ in its objective function, which is equal to the total interactive costs of all pair edges in the observed graph. According to Model 8, a quadratic term $\eta_{jk}x_jx_k$ can be linearized as the following model:

$$
\begin{aligned}
\min\quad & z_{jk} \\
\text{s.t.}\quad & z_{jk} \ge \eta_{jk} - \eta_{jk}(2 - x_j - x_k), \qquad (9) \\
& z_{jk} \ge 0,
\end{aligned}
$$

where z_{jk} represents $\eta_{jk}x_jx_k$; $\bar{u}$ is equal to one; η_{jk} is the interactive cost; x_j and x_k are 0-1 variables, for $j = 1, 2, \ldots, m - 1$, $k = j + 1, j + 2, \ldots, m$.

Proof. (i) if $x_j = 1$ and $x_k = 1$ then $z_{jk} = \eta_{jk}$; (ii) if $x_j = 1$ and $x_k = 0$ then $z_{jk} = 0$; (ii) if $x_j = 0$ and $x_k = 1$ then $z_{jk} = 0$; (ii) if $x_j = 0$ and $x_k = 0$ then $z_{jk} = 0$.

Therefore, this technique is capable of linearizing the objective quadratic term of both crisp and fuzzy Q-MST problems. Take a crisp Q-MST model as an example, the linearization procedure of the cost function $\sum_{j=1}^{m}\zeta_j x_j + \sum_{j=1}^{m}\sum_{k=1}^{m}\eta_{jk}\, x_j\, x_k$, in Model 1, is presented as follows:

$$
\begin{aligned}
\min\quad & \sum_{j=1}^{m}\zeta_j x_j + \sum_{j=1}^{m-1}\sum_{k=j+1}^{m} z_{jk} \\
\text{s.t.}\quad & \sum_{j=1}^{m} x_j \ge 1 \quad \text{for each vertex,} \\
& \sum_{j=1}^{m} x_j = n - 1 \quad \text{for connecting all } n \text{ vertices} \\
& \qquad \text{by } n\text{-1 edges} \qquad (10) \\
& z_{jk} \ge \eta_{jk} - \eta_{jk}(2 - x_j - x_k), \\
& z_{jk} \ge 0, \\
& x_j, x_k = 0 \text{ or } 1, \\
& j = 1, 2, \ldots, m - 1,\ k = j + 1, j + 2, \ldots, m,
\end{aligned}
$$

where there are n vertices and m edges. The first two constraints are the constraints of the minimum spanning trees; the other constraints are the constraints of linearization.

For fuzzy Q-MST, the linearization methods of the expected-value model, the chance-constrained, and dependent-chance programming models are the same. Therefore, we take a chance-constrained programming model (Model 6) as a linearized example of a fuzzy Q-MST problem, and thus, obtain the α-pessimistic-minimum spanning tree. Its linearized expression is represented as follows:

$$
\begin{aligned}
\min\quad & (2 - 2\alpha)\Big(\sum_{j=1}^{m} s_{j3}x_j + \sum_{j=1}^{m-1}\sum_{k=j+1}^{m} z_{jk3}\Big) \\
& \quad + (2\alpha - 1)\Big(\sum_{j=1}^{m} s_{j4}x_j + \sum_{j=1}^{m-1}\sum_{k=j+1}^{m} z_{jk4}\Big) \\
\text{s.t.}\quad & \sum_{j=1}^{m} x_j \ge 1 \quad \text{for each vertex,} \\
& \sum_{j=1}^{m} x_j = n - 1 \quad \text{for connecting all } n \text{ vertices by} \\
& \qquad n\text{-1 edges} \\
& z_{jk3} \ge t_{jk3} - t_{jk3}(2 - x_j - x_k), \\
& z_{jk3} \ge 0, \\
& z_{jk4} \ge t_{jk4} - t_{jk4}(2 - x_j - x_k), \\
& z_{jk4} \ge 0, \\
& x_j, x_k = 0 \text{ or} 1, \\
& j = 1, 2, \ldots, m - 1,\ k = j + 1, j + 2, \ldots, m. \qquad (11)
\end{aligned}
$$

The chance-constrained programming chooses the third and fourth quadruple values of the fuzzy directed costs $\zeta_j = (s_{j1}, s_{j2}, s_{j3}, s_{j4})$ and interactive costs $\eta_{jk} = (t_{jk1}, t_{jk2}, t_{jk3}, t_{jk4})$ for calculating α-pessimistic-minimum spanning tree. For more details on criterion selection, refer to Lu's proposed models [3]. As shown in [2][3][8][9][11], all heuristic methods, including the genetic algorithm approach, provide comparisons to show better solution quality by their approaches because of the nonlinear properties in the crisp and fuzzy Q-MST problem. However, we have proved that the crisp and fuzzy Q-MST problem can be reformulated as a linear formulation, and the global optimum can be obtained by our proposed method. In the following section, two examples are given to illustrate the proposed method; the issue of the optimal solution problem is solved by LINGO [4].

5. Illustrated Examples

In this section, we solve two examples, using the proposed method for illustration. The first example is a crisp Q-MST problem, and the second is a fuzzy Q-MST problem.

Example 1:

In Figure 1, there are six vertices (n = 6) and nine edges (m = 9), with 36 pairs of edges.

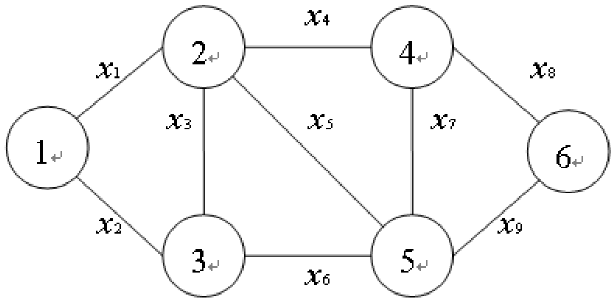

Figure 1. A given Q-MST problem.

The claimed costs are presented as follows:

Direct cost:

$\zeta_1 = 4$, $\zeta_2 = 6$, $\zeta_3 = 1$, $\zeta_4 = 5$, $\zeta_5 = 3$, $\zeta_6 = 5$, $\zeta_7 = 7$, $\zeta_8 = 8$, $\zeta_9 = 2$

Interactive cost:

$\eta_{12} = 1$, $\eta_{13} = 2$, $\eta_{14} = 3$, $\eta_{15} = 5$, $\eta_{16} = 4$, $\eta_{17} = 3$, $\eta_{18} = 1$, $\eta_{19} = 2$, $\eta_{23} = 9$, $\eta_{24} = 3$, $\eta_{25} = 1$, $\eta_{26} = 1$, $\eta_{27} = 2$, $\eta_{28} = 3$, $\eta_{29} = 2$, $\eta_{34} = 1$, $\eta_{35} = 2$, $\eta_{36} = 4$, $\eta_{37} = 3$, $\eta_{38} = 5$, $\eta_{39} = 1$, $\eta_{45} = 3$, $\eta_{46} = 4$, $\eta_{47} = 4$, $\eta_{48} = 2$, $\eta_{49} = 1$, $\eta_{56} = 3$, $\eta_{57} = 5$, $\eta_{58} = 1$, $\eta_{59} = 4$, $\eta_{67} = 2$, $\eta_{68} = 2$, $\eta_{69} = 4$, $\eta_{78} = 4$, $\eta_{79} = 1$, $\eta_{89} = 1$

With Model 10, this example can be formulated as a linear problem. The cost of the crisp Q-MST problem is 39, where x_1, x_3, x_4, x_5 and x_9 are the selected paths. The result is presented in Figure 2.

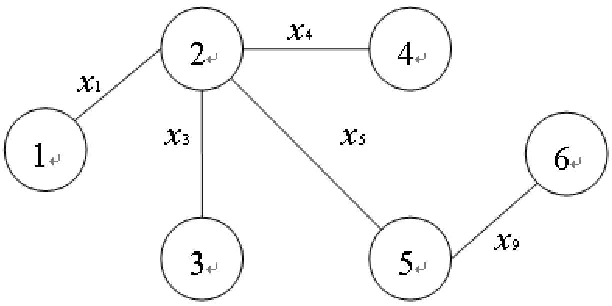

Figure 2. The crisp Q-MST.

Example 2:

The undirected graph of this fuzzy Q-MST problem is equivalent to Figure 1, with the directed and interactive costs substituted into the quadruples of the crisp numbers. The claimed costs are presented as follows:

Direct cost:

$\zeta_1 = (1, 2, 3, 4)$, $\zeta_2 = (1, 3, 4, 5)$, $\zeta_3 = (2, 3, 6, 7)$, $\zeta_4 = (2, 5, 7, 9)$, $\zeta_5 = (2, 4, 8, 9)$, $\zeta_6 = (1, 4, 6, 8)$, $\zeta_7 = (4, 5, 6, 7)$, $\zeta_8 = (2, 4, 6, 8)$, $\zeta_9 = (2, 3, 7, 9)$.

Interactive cost:

$\eta_{12} = (1, 2, 3, 4)$, $\eta_{13} = (2, 5, 7, 9)$, $\eta_{14} = (1, 3, 5, 8)$, $\eta_{15} = (3, 4, 7, 9)$, $\eta_{16} = (1, 2, 3, 8)$, $\eta_{17} = (3, 5, 7, 9)$, $\eta_{18} = (1, 2, 3, 9)$, $\eta_{19} = (1, 2, 5, 6)$, $\eta_{23} = (3, 4, 5, 8)$, $\eta_{24} = (3, 4, 6, 7)$, $\eta_{25} = (1, 4, 5, 8)$, $\eta_{26} = (3, 4, 5, 6)$, $\eta_{27} = (1, 2, 4, 5)$, $\eta_{28} = (2, 4, 6, 8)$, $\eta_{29} = (3, 5, 8, 9)$, $\eta_{34} = (1, 3, 4, 7)$, $\eta_{35} = (1, 2, 4, 6)$, $\eta_{36} = (1, 5, 6, 7)$, $\eta_{37} = (1, 3, 4, 6)$, $\eta_{38} = (3, 4, 5, 8)$, $\eta_{39} = (2, 3, 5, 7)$, $\eta_{45} = (1, 3, 6, 8)$, $\eta_{46} = (1, 2, 5, 7)$, $\eta_{47} = (2, 4, 6, 9)$, $\eta_{48} = (2, 3, 4, 8)$, $\eta_{49} = (1, 2, 3, 8)$, $\eta_{56} = (1, 6, 7, 9)$, $\eta_{57} = (2, 3, 5, 8)$, $\eta_{58} = (1, 2, 5, 8)$, $\eta_{59} = (2, 3, 4, 8)$, $\eta_{67} = (2, 4, 6, 8)$, $\eta_{68} = (1, 2, 6, 8)$, $\eta_{69} = (2, 4, 6, 8)$, $\eta_{78} = (1, 2, 3, 5)$, $\eta_{79} = (2, 3, 4, 7)$, $\eta_{89} = (2, 5, 7, 9)$.

With Model 11, this example can be reformulated as a linear problem. The cost of the fuzzy Q-MST problem is 98.9, dependant on a 95% confidence level (α = 0.95), where x_1, x_2, x_6, x_7 and x_8 are the selected paths. The selected edges are presented in Figure 3.

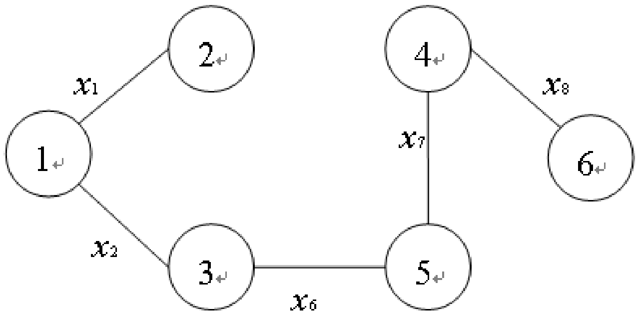

Figure 3. The fuzzy Q-MST.

By employing the proposed method, which differs from other methods, addressing the issues of nonlinear terms, in both Q-MST and fuzzy Q-MST problems, is simplified [3][11]; and a global optimal solution, rather than an approximated solution, can be easily achieved.

6. Conclusions

The crisp and fuzzy Q-MST problems can be easily linearized by the proposed method, as a result, a global optimal solution, rather than one approximated, is obtained for both crisp and fuzzy Q-MST problems. Thus, it does not require heuristic methods to achieve an approximated solution because of nonlinear properties. However, during our experiments, a cycle occasionally occurred when interactive costs were the same. Future research will focus on the cycle problems of Q-MST.

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