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Complete description for the spanning tree problem with one linearised quadratic term

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ABSTRACT

Given an edge-weighted graph the minimum spanning tree problem (MSTP) asks for a spanning tree of minimal weight. The complete description of the associated polytope is well-known. Recently, Buchheim and Klein suggested studying the MSTP with one quadratic term in the objective function resp. the polytope arising after linearisation of that term, in order to better understand the MSTP with a general quadratic objective function. We prove a conjecture by Buchheim and Klein (2013) concerning the complete description of the associated polytope.

1. Introduction and models

Let $G = (V, E)$ be an undirected, simple, complete, edge-weighted graph with node set V, $|V| = n$, set of edges E and weight function $c\colon E \to \mathbb{R}$. Then the *minimum spanning tree problem* (MSTP) asks for a spanning tree in G with minimal total weight,

$$\begin{aligned} &\text{minimise} \quad c(T) \\ &\text{subject to} \quad T \subseteq G \text{ is a spanning tree,} \end{aligned}$$

where $c(X) := \sum_{e \in E(X)} c(e)$, $X \subseteq G$, with $E(X) := \{e = \{u, v\}\colon u, v \in X, u \neq v\}$. It is well-known that using a binary variable for each edge $e \in E$ indicating whether the edge is contained in the spanning tree or not a linear integer formulation reads

$$\begin{aligned} &\text{minimise} \quad \sum_{e \in E} c(e) \cdot x(e) \\ &\text{subject to} \quad -x(E) = 1 - |V|, && (1) \\ &-x(E(S)) \geq 1 - |S|, \quad \emptyset \neq S \subsetneq V, && (2) \\ &x(e) \in \{0, 1\}, \quad e \in E. && (3) \end{aligned}$$

Edmonds [5] proved that replacing $x(e) \in \{0, 1\}$, $e \in E$, by $x(e) \geq 0$ yields a complete description of the associated polytope. So we get a linear optimisation problem formulation (LP) for the MSTP. Its corresponding dual linear program (DP) reads

$$\begin{aligned} &\text{maximise} \quad \sum_{\emptyset \neq S \subseteq V} (1 - |S|) z_S \\ &\text{subject to} \quad -\sum_{S: e \subseteq S \subseteq V} z_S \leq c(e), \quad e \in E, && (4) \\ &z_V \text{ free}, \quad z_S \geq 0, \emptyset \neq S \subsetneq V. \end{aligned}$$

Although the linear spanning tree problem and its associated polytope are well understood, not much is known if the objective function depends on products of edge-variables, *i.e.*, if we want to optimise

$$\sum_{e \in E} c(e) \cdot x(e) + \sum_{e, f \in E, e \neq f} c_q(e, f) \cdot x(e) \cdot x(f)$$

with additional weight function $c_q\colon E \times E \to \mathbb{R}$. The so called *Quadratic Minimum Spanning Tree Problem* (QMSTP) is known to be $\mathcal{NP}$-hard [1]. This is analogous to the *Assignment Problem*, which can be solved efficiently and whose polyhedral structure is well-known, and the *Quadratic Assignment Problem* (see, *e.g.*, [8]), which is one of the computationally most challenging combinatorial optimisation problems. Some branch-and-bound algorithms and heuristics for the QMSTP were presented, *e.g.*, in [1,9,4]. However, not much is known about the structure of the polytope that arises after a linearisation of $x(e)x(f)$, $e, f \in E$, $e \neq f$, by introducing new variables $y(e, f)$, $e, f \in E$, $e \neq f$. In order to better

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understand the polyhedral structure of the QMSTP and of combinatorial optimisation problems with a quadratic objective function in general, Buchheim and Klein [3,2] suggested considering the special case of the QMSTP resp. of a combinatorial optimisation problem with exactly one quadratic term in the objective function. Because the MSTP is polynomially-solvable QMSTP-1 (QMSTP with one quadratic term) can be solved in polynomial time, too, and by the well-known "optimisation equals separation" result [6], we can hope to fully characterise the polytope of the linearised version of QMSTP-1. Furthermore, the separation algorithms for valid inequalities or facets of QMSTP-1 may also be useful for solving the general QMSTP because valid inequalities of the one-monomial-case remain valid for the general case. First computational experiments in [3,2] also indicate this behaviour.

QMSTP-1 can be formally described as follows. Let $u_1, v_1, u_2, v_2 \in V$ with $u_1v_1, u_2v_2 \in E$, $u_1v_1 \neq u_2v_2$, either $\{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset$ or $v_1 = v_2$, $u_1 \neq u_2$, and $\bar{c} \in \mathbb{R}$ be the *monomial weight*. Then QMSTP-1 reads

$$\text{minimise} \quad q(T) := c(T) + \begin{cases} \bar{c}, & u_1v_1, u_2v_2 \in T, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{subject to} \quad T \subseteq G \text{ is a spanning tree.}$$

In [3] the case $v_1 = v_2$ is called the *connected case* because the two edges u_1v_1 and u_2v_2 share a common node, otherwise it is called the *unconnected case*. In most parts we will not distinguish between the two cases.

In this article we will prove that the following equations and inequalities are a complete description of the integer polytope if we linearise the monomial $x(u_1v_1) \cdot x(u_2v_2)$ by introducing a new variable y. Let

$$\mathcal{S} := \{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1v_1, u_2v_2 \in E(S, S')\}$$

with $E(X, Y) := \{e = \{u, v\} \in E : u \in X, v \in Y\}$. Then (QP) reads

$$\text{minimise} \quad \sum_{e \in E} c(e) \cdot x(e) + \bar{c} \cdot y$$

subject to (1), (2), $x \geq 0$

$$-x(E(S) \cup E(S')) - y \geq 2 - |S| - |S'|, \quad (S, S') \in \mathcal{S}, \tag{5}$$

$$x_{u_iv_i} - y \geq 0, \quad i \in \{1, 2\}, \tag{6}$$

$$y - x_{u_1v_1} - x_{u_2v_2} \geq -1, \tag{7}$$

$$y \geq 0. \tag{8}$$

Let $\mathcal{F}$ be a family of sets, then we write $z(\mathcal{F}) = \sum_{F \in \mathcal{F}} z_F$ and $\bar{z}(\mathcal{F}) = \sum_{F \in \mathcal{F}} \bar{z}_F$, respectively. So the dual problem (DQP) is

$$\text{maximise} \quad \sum_{\emptyset \neq S \subseteq V} (1 - |S|) z_S + \sum_{(S,S') \in \mathcal{S}} (2 - |S| - |S'|) \bar{z}_{(S,S')} - \zeta_y \tag{9}$$

$$\text{subject to} \quad - \sum_{S : e \subseteq S \subseteq V} z_S - \sum_{\substack{(S,S') \in \mathcal{S}: \\ e \in E(S) \cup E(S')}} \bar{z}_{(S,S')} \leq c(e),$$

$$e \in E \setminus \{u_1v_1, u_2v_2\}, \tag{10}$$

$$- \sum_{S : u_iv_i \subseteq S \subseteq V} z_S + \zeta_{u_iv_i} - \zeta_y \leq c(u_iv_i), \quad i \in \{1, 2\}, \tag{11}$$

$$-\bar{z}(\mathcal{S}) - \zeta_{u_1v_1} - \zeta_{u_2v_2} + \zeta_y \leq \bar{c}, \tag{12}$$

$$z_S \geq 0, \quad \emptyset \neq S \subsetneq V, \quad \bar{z}_{(S,S')} \geq 0, \quad (S, S') \in \mathcal{S}, \quad z_V \text{ free}, \tag{13}$$

$$\zeta_{u_1v_1}, \zeta_{u_2v_2}, \zeta_y \geq 0. \tag{14}$$

Indeed, Buchheim and Klein conjectured that in the unconnected case the model (QP) above provides a complete description of QMSTP-1. In the connected case, their conjecture looks a bit different. It says that apart from the standard linearisation (6)–(8) and the formulation of the MSTP (1)–(2), $x \geq 0$ one only needs

$$-x(E(S)) - y \geq 1 - |S|, \quad S \subset V, u_1, u_2 \in S, v_1 = v_2 \notin S,$$

for a complete description. If we can show that (QP) is a complete description this conjecture follows because then $\{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1, v_2 \in S, u_2, v_1 \in S'\} = \emptyset$ and inequalities (5) with $(S, S') \in \{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1, u_2 \in S, v_1 = v_2 \in S', |S'| > 1\}$ are implied by (5) with $(S, S') \in \{(S, S') : S, S' \subset V, S \cap S' = \emptyset, u_1, u_2 \in S, S' = \{v_1\}\}$ and (2). Note that in the meantime Buchheim and Klein independently proved the above-mentioned conjectures. A complete proof for the connected case can be found in [2].

2. Notation and previous results

In the following we write $[k]$ instead of $\{1, \ldots, k\}$, $k \in \mathbb{N}$. We denote the objective value of a spanning tree T w.r.t. $\tilde{c} : E \to \mathbb{R}$ by

$$v_{\text{LP}}(\tilde{c}, T) = \sum_{e \in E(T)} \tilde{c}(e) \quad \text{and} \quad v_{\text{DP}}(z) := \sum_{S : \emptyset \neq S \subseteq V} (1 - |S|) z_S$$

denotes the value of a solution z of (DP) with $z = (z_S)_{S : \emptyset \neq S \subseteq V}$. The following result follows from [5] and can, *e.g.*, be found in [7] (proof of Theorem 6.13).

Lemma 1 ([5,7]). *Let T be a minimum spanning tree (MST) in G w.r.t. $\tilde{c} : E \to \mathbb{R}$ computed by the greedy algorithm. Let $f_1, \ldots, f_{|V|-1}$ be the edges selected by the (best-in) greedy algorithm in order and denote by $X_k \subseteq V$, $k \in [|V| - 1]$, the nodes of the connected component of $(V, \{f_1, \ldots, f_k\})$ that contains f_k. Furthermore, let $s(k) \in [|V| - 1]$, $k \in [|V| - 2]$, denote the smallest index greater than k so that $f_{s(k)} \cap X_k \neq \emptyset$. Then the dual solution*

$$z^*(\tilde{c}, T) = (z_S)_{S : \emptyset \neq S \subseteq V},$$

$$\textit{with} \quad z_S := \begin{cases} \tilde{c}(f_{s(k)}) - \tilde{c}(f_k), & S = X_k, k < |V| - 1, \\ -\tilde{c}(f_{|V|-1}), & S = X_{|V|-1} = V, \\ 0, & \textit{otherwise}, \end{cases}$$

is an optimal solution of (DP). In particular, for any edge $e \in E$ there holds $lhs(z, e) := -\sum_{S : e \subseteq S \subseteq V} z_S = \tilde{c}(f_i)$, where $i \in [|V| - 1]$ is the smallest index so that $e \subseteq X_i$.

Remark 2. Note that we may assume, w.l.o.g., that each variable $z_{\{u\}}$, $u \in V$, of the solution $z^*(\tilde{c}, T)$ has an arbitrarily large value, because these variables do not contribute to the objective value and do not appear in any constraint except for $z_{\{u\}} \geq 0$. We will make use of this property later in Corollary 5.

We denote the value of spanning tree T w.r.t. $\tilde{c} : E \to \mathbb{R}$ and weight $\bar{c}$ by

$$v_{\text{QP}}(\tilde{c}, T) = \sum_{e \in E(T)} \tilde{c}(e) + \begin{cases} \bar{c}, & u_1v_1, u_2v_2 \in T, \\ 0, & \text{otherwise,} \end{cases}$$

and the value of a solution $(z, \bar{z}, \zeta)$ of (DQP) with $z = (z_S)_{S : \emptyset \neq S \subseteq V}$, $\bar{z} = (\bar{z}_{(S,S')})_{(S,S') \in \mathcal{S}}$ and $\zeta = (\zeta_y, \zeta_{u_1v_1}, \zeta_{u_2v_2})$ by

$$v_{\text{DQP}}(z, \bar{z}, \zeta) = \sum_{S : \emptyset \neq S \subseteq V} (1 - |S|) z_S + \sum_{(S,S') \in \mathcal{S}} (2 - |S| - |S'|) \bar{z}_{(S,S')} - \zeta_y.$$

3. Complete description

In this section we will prove that (QP) is indeed a complete description of the integer polytope for QMSTP-1. We start with a

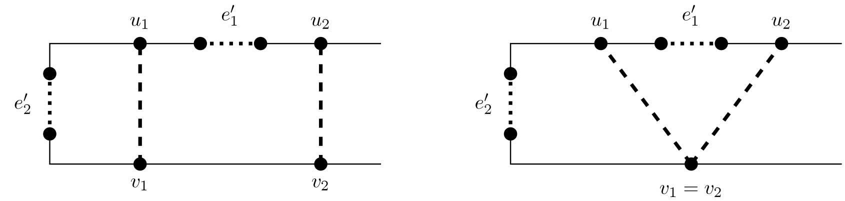

Fig. 1. Visualisation of assumption (A). The solid edges are contained in the forest $\tilde{T}$.

lemma about the structure of optimal spanning trees if we adapt the coefficients of the edges u_1v_1, u_2v_2.

Lemma 3. *Let $G = (V, E)$ be a complete undirected graph and $c\colon E \to \mathbb{R}$ a weight function. Then there exist a forest $\tilde{T} \subseteq G$ and four edges $e_i, e'_i \in E \setminus E(\tilde{T})$, $i = 1, 2$, with $\{e_1, e_2\} \subseteq \{e'_1, e'_2\}$, $e'_1 \neq e'_2$, $c(e'_1) \leq c(e'_2)$ and either $e_1 \neq e_2$ or $e_1 = e_2 = e'_1$, so that the four graphs $T_0 := \tilde{T} + e'_1 + e'_2$, $T_1 := \tilde{T} + u_1v_1 + e_2$, $T_2 := \tilde{T} + u_2v_2 + e_1$, and $T_{12} := \tilde{T} + u_1v_1 + u_2v_2$ are spanning trees, and for each weight function $\tilde{c}\colon E \to \mathbb{R}$ with $\tilde{c}(e) = c(e)$ for all $e \in E \setminus \{u_1v_1, u_2v_2\}$ one of these four trees is an MST in G w.r.t. $\tilde{c}$.*

Proof. Consider the tree T generated by the greedy algorithm, but starting with the forest containing the two edges u_1v_1, u_2v_2. Let $f_1, \ldots, f_{|E|-2}$ be the sequence of edges $E \setminus \{u_1v_1, u_2v_2\}$ in the order they have been considered by the greedy algorithm, in particular $c(f_1) \leq \cdots \leq c(f_{|E|-2})$. We set $\tilde{T} := T - u_1v_1 - u_2v_2$, and choose

$$e_i := f_{k_i},\ k_i = \min\left\{j \in [|E| - 2]\colon f_j \notin T, T - u_iv_i + f_j \text{ is a tree}\right\},\quad i = 1, 2.$$

First assume $e_1 \neq e_2$ and set $\{e'_1, e'_2\} := \{e_1, e_2\}$. Then by the choice of e_1, e_2 the graph $\tilde{T}+e'_1+e'_2 = \tilde{T}+e_1+e_2 = T-u_1v_1+e_1-u_2v_2+e_2$ forms a tree, too. Next consider the greedy algorithm for $\tilde{c}$. We can assume that all edges in $E \setminus \{u_1v_1, u_2v_2\}$ are considered in the same order because their costs remain unchanged. Then the greedy algorithm selects all edges $E(T) \setminus \{u_1v_1, u_2v_2\}$, and the first of u_1v_1 and e_1, and the first of u_2v_2 and e_2 by the choice of e_1, e_2. So the greedy algorithm for $\tilde{c}$ will generate one of the trees T_0, T_1, T_2, T_{12}.

Now assume $e_1 = e_2$. Then set $e'_1 := e_1 = e_2$, choose

$$m = \min\left\{i \in [|E| - 2]\colon f_i \notin T, T - u_1v_1 - u_2v_2 + e'_1 + f_i \text{ is a tree}\right\},$$

and set $e'_2 := f_m$. We show that $k_1 = k_2 < m$. Let X, Y, Z be the three components of $\tilde{T}$ and assume, w.l.o.g., that u_1v_1 connects X and Y and u_2v_2 connects X and Z. Because $\tilde{T} + u_1v_1 + e'_1$ and $\tilde{T}+u_2v_2+e'_1$ are both trees, e'_1 must connect Y and Z. Furthermore, because $\tilde{T} + e'_1 + e'_2$ is a tree, e'_2 must connect X with either Y or Z. Consequently, either $T - u_1v_1 + e'_2$ or $T - u_2v_2 + e'_2$ is a tree, proving $k_1 = k_2 < m$. Regarding the greedy algorithm with $\tilde{c}$, similar to above, depending on the values of $\tilde{c}(u_1v_1)$ and $\tilde{c}(u_2v_2)$, the algorithm generates one of the four trees T_0, T_1, T_2, T_{12}. □

Note that we will use the notation $T, \tilde{T}, e_1, e_2, e'_1, e'_2$ throughout the article. The previous lemma implies the following.

Corollary 4. *The objective value of an MST in G w.r.t. $\tilde{c}\colon E \to \mathbb{R}$ with $\tilde{c}(e) = c(e)$ for all $e \in E \setminus \{u_1v_1, u_2v_2\}$ will be*

$$\begin{aligned}\min\{\tilde{c}(T_0), \tilde{c}(T_1), \tilde{c}(T_2), \tilde{c}(T_{12})\} &= c(\tilde{T}) + \min\{c(e'_1) + c(e'_2), \tilde{c}(u_1v_1) + c(e_2),\\ &\quad c(e_1) + \tilde{c}(u_2v_2), \tilde{c}(u_1v_1) + \tilde{c}(u_2v_2)\}.\end{aligned}$$

Throughout the rest of the article we will assume, w.l.o.g. (otherwise rename the nodes):

$$\text{If } e_1 = e_2 = e'_1\text{, then } e'_1 \text{ lies on the path between } u_1 \text{ and } u_2 \text{ not using } v_1, v_2 \text{ on the cycle in } T_{12} + e'_1. \tag{A}$$

Note that (A) automatically holds in the connected case, see also Fig. 1.

Corollary 5. *Let $T \in \{T_0, T_1, T_2, T_{12}\}$ be an MST w.r.t. $\tilde{c}\colon E \to \mathbb{R}$ with $\tilde{c}(e) = c(e)$ for all $e \in E \setminus \{u_1v_1, u_2v_2\}$ and $z = z^*(\tilde{c}, T)$. We consider the following condition*

$$e_1 \neq e_2 \quad or \quad c(e'_1) \geq \min\{\tilde{c}(u_1v_1), \tilde{c}(u_2v_2)\}. \tag{C}$$

1. *If (C) holds, then*

$$lhs(z, u_1v_1) + lhs(z, u_2v_2) = \tilde{c}(T) - c(\tilde{T}). \tag{15}$$

2. *Otherwise, if (C) does not hold and assuming (A), then $T \neq T_{12}$ and*

$$\begin{aligned}&lhs(z, u_1v_1) + lhs(z, u_2v_2)\\ &\quad = 2 \cdot \min\{\tilde{c}(u_1v_1), \tilde{c}(u_2v_2), c(e'_2)\},\end{aligned} \tag{16}$$

and there exist two families of sets S_p, $p \in [k]$, S'_q, $q \in [k']$, satisfying the following conditions

$$\begin{aligned}&u_1, u_2 \in \bigcap_{p\in[k]} S_p, \qquad v_1, v_2 \in \bigcap_{q\in[k']} S'_q,\\ &\forall\, p \in [k], q \in [k'] : S_p \cap S'_q = \emptyset,\end{aligned} \tag{S}$$

so that $z(\{S_p\colon p \in [k]\}) = \min\{\tilde{c}(u_1v_1), \tilde{c}(u_2v_2), c(e'_2)\} - c(e'_1) \leq z(\{S'_q\colon q \in [k']\})$.

Proof. First observe that $u_iv_i \in T$ for some $i \in \{1, 2\}$ implies $\mathrm{lhs}(z, u_iv_i) = \tilde{c}(u_iv_i)$ by Lemma 1. Assume (C) holds. Let $e_1 \neq e_2$ and, w.l.o.g., $u_1v_1 \notin T$ then $e_1 \in T$ and $\mathrm{lhs}(z, u_1v_1) = c(e_1)$. This proves (15) for all cases if $e_1 \neq e_2$. Otherwise, if $e_1 = e_2$ and $c(e'_1) \geq \min\{\tilde{c}(u_1v_1), \tilde{c}(u_2v_2)\}$, then, w.l.o.g., $u_1v_1 \in T$. If $u_2v_2 \in T$ then $\mathrm{lhs}(z, u_iv_i) = \tilde{c}(u_iv_i)$, $i = 1, 2$, as above and (15) follows. Because $e'_1 = e_2 \in T$ we know that e'_1 is the most expensive edge on the cycle in $T + u_2v_2$ except for u_2v_2, hence e'_1 is the edge that connects u_2 and v_2 first in the greedy algorithm, so $\mathrm{lhs}(z, u_2v_2) = c(e'_1)$. This implies again (15).

Next, assume (C) does not hold, *i.e.*, $e_1 = e_2 = e'_1$ and $c(e'_1) < \tilde{c}(u_iv_i)$, $i = 1, 2$. In this case $e'_1 \in T$ and $f \in T$ where $f \in \operatorname{Arg\,min}\left\{\tilde{c}(e)\colon e \in \{u_1v_1, u_2v_2, e'_2\}\right\}$. So by (A), u_1, u_2 are connected for the first time when e'_1 is selected, hence u_i, v_i, $i = 1, 2$, are connected for the first time when f is selected. Consequently $\mathrm{lhs}(z, u_iv_i) = \tilde{c}(f)$, $i = 1, 2$, proving (16).

It remains to prove the existence of sets satisfying (S) and the last condition. For this assume first $v_1 \neq v_2$ and let X_{i_j}, $j \in [k]$, $k \in \mathbb{N}$, be the components in order that contain u_1, u_2 but not v_1, v_2, and $X_{i'_j}$, $j \in [k']$, $k' \in \mathbb{N}$, the components that contain v_1, v_2 but not u_1, u_2 according to Lemma 1. Note that $i_1 > i'_1$ and $f_{i_1} = e'_1$ (because e'_1 is considered before u_1v_1, u_2v_2 and e'_2 and because of assumption (A)), and with $i_{k+1} = i'_{k'+1}$ defined so that $f_{i_{k+1}}$

$= f_{i'_{k'+1}} = f$ it follows that $z(\{X_{i_p}: p \in [k]\}) = \sum_{p=1}^{k} \tilde{c}(f_{i_{p+1}}) - \tilde{c}(f_{i_p}) = \tilde{c}(f) - \tilde{c}(e'_1)$, as well as $z(\{X_{i'_q}: q \in [k']\}) = \tilde{c}(f) - \tilde{c}(f_{i'_1})$. Because $i_1 > i'_1$ we know that $\tilde{c}(e'_1) \geq \tilde{c}(f_{i'_1})$ and because of $X_{i_p} \cap X_{i'_q} = \emptyset$ for all $p \in [k], q \in [k']$, we can choose $S_p := X_{i_p}$ and $S'_q := X_{i'_q}$.

Finally, if $v_1 = v_2$ we may simply use $S_p := X_{i_p}$ as before and $S'_1 := \{v_1\}, k' = 1$, by Remark 2. □

Corollary 6. *Let $(z, \bar{z}, \zeta)$ be a point satisfying constraints (10), (11) and (13), (14) of (DQP). Assume that there exist two families of sets $S_p \subseteq V, p \in [k]$, and $S'_q \subseteq V, q \in [k']$, satisfying (S) with $z(\{S_p: p \in [k]\}) \leq z(\{S'_q: q \in [k']\})$. Then there exists a point $(z', \bar{z}', \zeta')$ with $\zeta' = \zeta$ also satisfying constraints (10), (11) and (13), (14) with $v_{DQP}(z, \bar{z}, \zeta) = v_{DQP}(z', \bar{z}', \zeta')$ and $\bar{z}'(\mathbb{S}) \geq z(\{S_p: p \in [k]\})$.*

Proof. If $z_{S_p} = 0$ for all $p \in [k]$ then $(z', \bar{z}', \zeta') = (z, \bar{z}, \zeta)$ satisfies the required conditions because $\bar{z} \geq 0$ by (13), so we may assume that there is at least one $i \in [k]$ with $z_{S_i} > 0$ and, consequently, a $j \in [k']$ with $z_{S'_j} > 0$. Let $d := \min\{z_{S_i}, z_{S'_j}\}$. We define $\zeta^{(1)} := \zeta$ and

$$z_S^{(1)} := \begin{cases} z_S - d, & S \in \{S_i, S'_j\}, \\ z_S, & \text{otherwise,} \end{cases} \quad \text{and}$$

$$\bar{z}_{(S,S')}^{(1)} := \begin{cases} \bar{z}_{(S,S')} + d, & (S, S') = (S_i, S'_j), \\ \bar{z}_{(S,S')}, & \text{otherwise.} \end{cases}$$

Note that at least one of $z_{S_i}^{(1)}, z_{S'_j}^{(1)}$ is zero, so the number of non-zero coefficients decreased. Because z_{S_i} and $z_{S'_j}$ do not appear in the left-hand side of constraints (11), these are satisfied by $(z^{(1)}, \bar{z}^{(1)}, \zeta^{(1)})$, too, and constraints (13) and (14) trivially hold. Furthermore it is

$$z(\{S_p: p \in [k]\}) + \bar{z}(\mathbb{S}) = z^{(1)}(\{S_p: p \in [k]\}) + \bar{z}^{(1)}(\mathbb{S}). \tag{17}$$

For (10) let $e \in E \setminus \{u_1v_1, u_2v_2\}$ be an arbitrary edge. If $e \not\subseteq S_i$ and $e \not\subseteq S'_j$ then the left-hand side of (10) cannot be increased, so assume, w.l.o.g., $e \subseteq S_i$. Then $e \not\subseteq S'_j$ by assumption and direct computation shows that the left-hand side of (10) for $(z^{(1)}, \bar{z}^{(1)}, \zeta^{(1)})$ remains the same. Similarly, direct computation shows $v_{DQP}(z, \bar{z}, \zeta) = v_{DQP}(z^{(1)}, \bar{z}^{(1)}, \zeta^{(1)})$. Iterating the argument we get a feasible solution $(z', \bar{z}', \zeta') := (z^{(r)}, \bar{z}^{(r)}, \zeta^{(r)}), r \in \mathbb{N}$, with $z_{S_p}^{(r)} = 0$ for all $p \in [k]$. Hence, by assumption and (17) there holds

$$\bar{z}'(\mathbb{S}) = z'(\{S_p: p \in [k]\}) + \bar{z}'(\mathbb{S}) = z(\{S_p: p \in [k]\}) + \bar{z}(\mathbb{S}) \geq z(\{S_p: p \in [k]\}).$$

This completes the proof. □

Now we prove our main result.

Theorem 7. *(QP) is a formulation of QMSTP-1.*

Proof. Let T be an MST for (LP) w. r. t. c; by Lemma 3 this is one of the trees T_0, T_1, T_2, T_{12}. In the whole proof $\tilde{T}$ denotes the subgraph as defined in Lemma 3.

$\bar{c} \geq 0$ and $T \in \{T_0, T_1, T_2\}$: Then T fulfils $v_{LP}(c, T) = v_{QP}(c, T)$ and an optimal solution $z = z^*(c, T)$ of (DP) can be extended to a feasible solution $(z, \bar{z}, \zeta)$ of (DQP) by setting $\bar{z} = 0$ and $\zeta_{u_1v_1} = \zeta_{u_2v_2} = \zeta_y = 0$ with $v_{DQP}(z, \bar{z}, \zeta) = v_{DP}(z) = v_{LP}(c, T) = v_{QP}(c, T)$, so T is an optimal solution of (QP).

$\bar{c} \geq 0$ and $T = T_{12}$: By Corollary 4 there holds $c(u_1v_1) \leq c(e_1)$ and $c(u_2v_2) \leq c(e_2)$. Assume, w.l.o.g., $c(e_1) - c(u_1v_1) \leq c(e_2) - c(u_2v_2)$. Set $\varepsilon := \min\{c(e_1) - c(u_1v_1), \bar{c}\}$, then $\varepsilon \leq c(e_2) - c(u_2v_2)$, and set

$$\hat{c}(e) := \begin{cases} c(e) + \varepsilon, & e \in \{u_1v_1, u_2v_2\}, \\ c(e), & \text{otherwise.} \end{cases}$$

Let T' be an MST w. r. t. $\hat{c}$ and let $z = z^*(\hat{c}, T')$ be an optimal solution of (DP). Direct computation using Corollary 4 shows that either T_{12} (if $\varepsilon = \bar{c}$) or T_2 (if $\varepsilon < \bar{c}$) is an MST w. r. t. $\hat{c}$, and in both cases $v_{QP}(c, T') = v_{LP}(\hat{c}, T') - \varepsilon$. Furthermore, because z is feasible for (DP) with $\hat{c}$ there holds $\text{lhs}(z, e) \leq \hat{c}(e) = c(e), e \in E \setminus \{u_1v_1, u_2v_2\}$, as well as $\text{lhs}(z, e) \leq \hat{c}(e) = c(e) + \varepsilon$ for $e \in \{u_1v_1, u_2v_2\}$. Hence, with $\bar{z} = 0$, $\zeta_{u_1v_1} = \zeta_{u_2v_2} = 0$ and $\zeta_y = \varepsilon$, solution $(z, \bar{z}, \zeta)$ satisfies (10) and (11), and because $\bar{c} \geq 0$ constraint (12) as well, so it is feasible for (DQP). Furthermore the objective value fulfils $v_{DQP}(z, \bar{z}, \zeta) = v_{DP}(z) - \zeta_y = v_{QP}(c, T')$ proving the optimality of T'.

$\bar{c} < 0$ and $c(u_1v_1) + c(u_2v_2) + \bar{c} \geq c(T) - c(\tilde{T})$: Note that in this case $T \neq T_{12}$ (otherwise $c(u_1v_1) + c(u_2v_2) + \bar{c} < c(u_1v_1) + c(u_2v_2) = c(T_{12}) - c(\tilde{T})$). Let $z = z^*(c, T)$ be an optimal solution of (DP) and set $\bar{z} = 0$, $\zeta_y = 0$ and $\zeta_{u_iv_i} = c(u_iv_i) - \text{lhs}(z, u_iv_i)$, $i = 1, 2$. Then $(z, \bar{z}, \zeta)$ satisfies (10) and (11). To check (12) we consider two cases.

1. Condition (C) holds for $\tilde{c} = c$. Corollary 4 and (15) imply $\text{lhs}(z, u_1v_1) + \text{lhs}(z, u_2v_2) = c(T) - c(\tilde{T}) \leq c(u_1v_1) + c(u_2v_2) + \bar{c}$. Thus, $-\zeta_{u_1v_1} - \zeta_{u_2v_2} \leq \bar{c}$ and (12) is satisfied. Because $v_{DQP}(z, \bar{z}, \zeta) = v_{DP}(z) = v_{LP}(c, T) = v_{QP}(c, T)$ tree T is optimal for (QP).
2. Condition (C) does not hold for $\tilde{c} = c$. Then by Corollaries 5 and 6 there exist a solution $(z', \bar{z}', \zeta')$, $\zeta' = \zeta$, satisfying (10) and (11) and $v_{DQP}(z', \bar{z}', \zeta') = v_{DQP}(z, \bar{z}, \zeta)$, and two families $S_p, p \in [k]$, and $S'_q, q \in [k']$, such that $\bar{z}'(\mathbb{S}) \geq z(\{S_p: p \in [k]\}) = \min\{c(u_1v_1), c(u_2v_2), c(e'_2)\} - c(e'_1)$. Consequently, direct computation with (16) shows that (12) is satisfied by $(z', \bar{z}', \zeta')$ for $T \in \{T_0, T_1, T_2\}$ and because $T \neq T_{12}$ it is $v_{DQP}(z', \bar{z}', \zeta') = v_{DQP}(z, \bar{z}, \zeta) = v_{DP}(z) = v_{LP}(c, T) = v_{QP}(c, T)$, hence T is optimal.

$\bar{c} < 0$ and $c(u_1v_1) + c(u_2v_2) + \bar{c} \leq c(T) - c(\tilde{T})$: Because T is optimal w. r. t. c, Corollary 4 implies

$$c(u_1v_1) + c(u_2v_2) + \bar{c} \leq \min\{c(e_1) + c(u_2v_2), c(e'_1) + c(e'_2), c(u_1v_1) + c(e_2)\}. \tag{18}$$

Assume, w.l.o.g., $c(u_1v_1) \leq c(u_2v_2)$. We consider two cases.

1. Condition (C) holds for $\tilde{c} = c$. We set

$$\hat{c}(e) := \begin{cases} c(u_1v_1) + c(u_2v_2) + \bar{c} - \min\{c(u_2v_2), c(e_2)\}, \\ \quad e = u_1v_1, \\ \min\{c(u_2v_2), c(e_2)\}, \quad e = u_2v_2, \\ c(e), \quad \text{otherwise.} \end{cases}$$

Then $\hat{c}(u_1v_1) \leq \min\{c(e_1) + c(u_2v_2), c(e'_1) + c(e'_2), c(u_1v_1) + c(e_2)\} - \min\{c(u_2v_2), c(e_2)\} \leq c(e_1)$, where the last inequality follows from the first term in the min in (18) if $c(u_2v_2) \leq c(e_2)$, from the second if $e_1 \neq e_2$ or the third if $c(u_1v_1) \leq c(e'_1) \leq c(e_1)$ (because (C) holds this is exhaustive). By definition $\hat{c}(u_2v_2) \leq c(e_2)$, so T_{12} is an MST w. r. t. $\hat{c}$ with $v_{LP}(\hat{c}, T_{12}) = c(\tilde{T}) + c(u_1v_1) + c(u_2v_2) + \bar{c}$. In particular, condition (C) holds for $\tilde{c} = \hat{c}$. Let $z = z^*(\hat{c}, T_{12})$ and set $\bar{z} = 0$, $\zeta_y = 0$, $\zeta_{u_iv_i} = c(u_iv_i) - \text{lhs}(z, u_iv_i)$, $i = 1, 2$. Because $\text{lhs}(z, u_1v_1) \leq \hat{c}(u_1v_1) \leq c(u_1v_1)$ by $\bar{c} < 0$ and (18) as well as $\text{lhs}(z, u_2v_2) \leq \hat{c}(u_2v_2) \leq c(u_2v_2)$ we have $\zeta_{u_iv_i} \geq 0$ for $i = 1, 2$. Hence, $(z, \bar{z}, \zeta)$ satisfies (10) and (11) and, because $-\zeta_{u_1v_1} - \zeta_{u_2v_2} \leq \bar{c}$ by (15), constraint (12) is satisfied as well. So, $(z, \bar{z}, \zeta)$ is feasible for (DQP) and $v_{QP}(c, T_{12}) = v_{LP}(\hat{c}, T_{12}) = v_{DP}(z) = v_{DQP}(z, \bar{z}, \zeta)$ proves optimality of T_{12} for (QP).
2. Condition (C) does not hold for $\tilde{c} = c$, in particular, $e'_1 = e_1 = e_2$ and $T \neq T_{12}$. We set

$$\hat{c}(e) := \begin{cases} c(u_1v_1) + c(u_2v_2) + \bar{c} - c(e_2), & e = u_1v_1, \\ c(e), & \text{otherwise.} \end{cases}$$

By direct computations and by assumption we get $\hat{c}(u_1v_1) \leq c(u_1v_1) \leq c(u_2v_2) = \hat{c}(u_2v_2)$, $\hat{c}(u_1v_1) \leq c(e_2')$, and, because $T \neq T_{12}$ it holds $c(e_1') = c(e_2) \leq c(u_2v_2) = \hat{c}(u_2v_2)$. So Corollary 4 implies that T_1 is an MST w. r. t. $\hat{c}$ with $v_{\mathrm{LP}}(\hat{c}, T_1) = v_{\mathrm{QP}}(c, T_{12})$. Let $z = z^*(\hat{c}, T_1)$ and set $\bar{z} = 0$, $\zeta_y = 0$, $\zeta_{u_iv_i} = c(u_iv_i) - \mathrm{lhs}(z, u_iv_i)$, $i = 1, 2$. Because z is feasible for (DP) w. r. t. $\hat{c}$ we have $\mathrm{lhs}(z, u_iv_i) \leq \hat{c}(u_iv_i) \leq c(u_iv_i)$, so $\zeta_{u_iv_i} \geq 0$ for $i = 1, 2$, and $(z, \bar{z}, \zeta)$ satisfies (10) and (11).

(a) If (C) holds for $\tilde{c} = \hat{c}$, then (15) implies $\mathrm{lhs}(z, u_1v_1) + \mathrm{lhs}(z, u_2v_2) = \hat{c}(T_1) - c(\tilde{T}) = \hat{c}(u_1v_1) + c(e_2) = \bar{c} + c(u_1v_1) + c(u_2v_2)$. Hence (12) is satisfied and $(z, \bar{z}, \zeta)$ is feasible for (DQP) with $v_{\mathrm{DQP}}(z, \bar{z}, \zeta) = v_{\mathrm{DP}}(z) = v_{\mathrm{LP}}(\hat{c}, T_1) = v_{\mathrm{QP}}(c, T_{12})$, so T_{12} is optimal for (QP).

(b) If (C) does not hold for $\tilde{c} = \hat{c}$, then by Corollaries 5 and 6 there exist a solution $(z', \bar{z}', \zeta')$, $\zeta' = \zeta$, satisfying (10) and (11) and $v_{\mathrm{DQP}}(z', \bar{z}', \zeta') = v_{\mathrm{DQP}}(z, \bar{z}, \zeta)$, and two families S_p, $p \in [k]$ and S_q', $q \in [k']$, such that $\bar{z}'(\mathcal{S}) \geq \hat{c}(u_1v_1) - c(e_1')$. Direct computation with (16) shows that $(z', \bar{z}', \zeta')$ satisfies (12), so it is feasible for (DQP), and $v_{\mathrm{DQP}}(z', \bar{z}', \zeta') = v_{\mathrm{DQP}}(z, \bar{z}, \zeta) = v_{\mathrm{DP}}(z) = v_{\mathrm{LP}}(\hat{c}, T_1) = v_{\mathrm{QP}}(c, T_{12})$ proves optimality of T_{12} for (QP). $\square$

Remark 8. Note that in the construction of the optimal dual solution of (DQP) in the proof of Theorem 7 we only used non-zero values for the variables $\bar{z}_{(S,S')}$, $(S, S') \in \mathcal{S}$, if $\bar{c} < 0$ and if condition (C) does not hold. But this implies that the inequalities (5) in the primal problem (QP) are only required in these cases and can be dropped otherwise.

So we have found a complete description of QMSTP-1. It remains for future work to consider further optimisation problems in the same manner or the QMSTP with two or more quadratic terms in the objective function. Furthermore, it might be interesting to look at higher degree polynomials.

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