Revenue-Sharing Allocation Strategies for Two-Sided Media Platforms: Pro-Rata versus User-Centric*

Saeed Alaei
saeed.a@gmail.com
Google Research, Mountain View

Ali Makhdoumi
ali.makhdoumi@duke.edu
Fuqua School of Business, Duke University

Azarakhsh Malekian
azarakhsh.malekian@rotman.utoronto.ca
Rotman School of Management, University of Toronto

Saša Pekeč
pekec@duke.edu
Fuqua School of Business, Duke University

We consider a two-sided streaming service platform which generates revenues by charging users a subscription fee for unlimited access to the content, and compensates content providers (artists) through a revenue-sharing allocation rule. Platform users are heterogeneous in both their overall consumption and the distribution of their consumption over different artists. We study two primary revenue allocation rules used by market-leading music streaming platforms— *pro-rata* and *user-centric*. With pro-rata, artists are paid proportionally to their share in the overall streaming volume, while with user-centric each user's subscription fee is divided proportionally among artists based on the consumption of that user. We characterize when these two allocation rules can sustain a set of artists on the platform and compare them from both the platform and the artists' perspectives. In particular, we show that, despite the cross-subsidization between low and high streaming volume users, the pro-rata rule can be preferred by both the platform and the artists. Further, the platform's problem of selecting an optimal portfolio of artists is NP-complete. However, by establishing connections to the Knapsack problem, we develop a Polynomial Time Approximation Scheme (PTAS) for the optimal platform's profit. In addition to determining the platform's optimal revenue allocation rule in the class of pro-rata and user-centric rules, we consider the optimal revenue allocation rule in the class of arbitrary rules. Building on duality theory, we develop a polynomial time algorithm which outputs a set of artists so that the platform's profit is within a single artist's revenue from the optimal profit.

Key words: Two-sided media platforms, subscription pricing, pro-rata and user-centric, digital goods, revenue-sharing

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* We would like to thank Leslie Marx for helpful comments, suggestions, and discussion.

1. Introduction

Streaming platforms (Spotify, Apple Music, Amazon Music, YouTube, Pandora, MelOn, Tencent, Deezer, TIDAL, etc.) established themselves as a primary way of listening to music, engaging 89% of music listeners globally (see e.g. IFPI, Music Listening in 2019). Consequently, these platforms now generate a majority of global music industry revenues and this trend is likely to intensify. For example, in the US, the music industry's largest market, streaming services generated $4.3 billion in the first half of 2019 (with 77% of that coming from paid subscriptions), which accounted for 80% of the entire US music industry revenues (see Friedlander (2019)). The rapid ascent to dominance of the subscription-based streaming platform model necessitates solutions for appropriately compensating content providers. This is a rather new issue for the music industry (and more generally for media streaming platforms and subscription-fee service platforms) as, historically, attributing revenue from sales on physical medium or from digital downloads to content providers have been straightforward and well-understood. In contrast, subscriptions to a streaming platform produces revenue that is not explicitly attributable to content providers. This decoupling of subscription fees that users pay for a streaming service and platforms' payments to content providers leaves the platform and the content providers with a problem of settling on the appropriate compensation model.

In this paper, we introduce a framework to design and evaluate performance of revenue-allocation schemes to content providers on a streaming platform where revenues are not directly attributable to any specific content consumption. Our framework captures the individual rationality of users and content providers to join the platform, as well as the value that platform provides in offering heterogeneous supply of available content to meet demand of its heterogeneous users. Our approach allows us to identify optimal compensation schemes for content providers that maintain platform sustainability, as well as to identify platform-optimal set of content providers to be joining the platform. The starting point of our analysis, motivated by a lively debate in the industry, is the evaluation of the performance of content providers' compensation schemes commonly used by the streaming music platforms.

In practice, streaming platforms and the music industry converged on a revenue-sharing payment model in which the platform compensates the content providers a prespecified proportion of its overall revenues. This is also known as *payout rate*, and is typically around 70% of overall revenues, attributed to various types of royalties (such as sound recording, performance, mechanicals; see e.g., Manatt, Phelps & Phillips (2016) and Soundcharts (2019)). The platform needs to further distribute this payout among content providers, which we refer to as *artists*.[1] The primary revenue-allocation model used by market-leading music streaming platforms is *pro-rata*: artists are paid proportionally to their share in the overall streaming volume (i.e., the payment which platform allocates to each artist is proportional to the number of streams of that artist on the platform). However, when consumption differs among users, pro-rata payments do not proportionally allocate revenues generated by users who subscribe to predominantly listen to a particular artist. Most analyses emphasize that this property of pro-rata results in a cross-subsidization between low streaming volume users and high streaming volume users, and introduces a fundamental inequity between the artist compensation and the revenue each artist brings to the platform by attracting more subscribers. Instead, it has been suggested and argued (e.g., Page and Safir (2018a), Jari (2018), and Page and Safir (2018b)) that streaming services should allocate revenues according to the *user-centric* model which applies the proportionality principle at the user level: each individual user's subscription fee should be divided proportionally among artists based on the consumption of that user. The pro-rata and user-centric revenue allocation rules yield identical payments to artists when the user consumption is homogeneous. However, payments under these two models differ when the user consumption is not homogeneous. The difference between the two rules stems from the subscription fee pricing that allows users unlimited consumption, thereby implicitly charging users a different price per unit of consumption.

In this paper we develop a model to study the performance of these two revenue allocation rules, as well as an arbitrary allocation rule, and their impact on artists' revenues and their decisions on whether to join the platform, and the sustainability and profitability of the platform. In particular, in our model users are heterogeneous in both their overall consumption (total listening) and their preferences for different artists. Based on the set of artists on the platform and the subscription fee, users decide whether they want to subscribe to the platform or not. If they subscribe, they will have (unlimited) access to the content on the platform for a period of time. The artists, on the other hand, based on the revenue allocation rule decide whether they want to join the platform or not. If an artist joins the platform, she will receive a payment based on the platform's revenue allocation rule. If she does not join the platform, then she receives an outside option value which again depends on how popular this artist is among users.

We first compare two prominent revenue allocation strategies: pro-rata and user-centric. We consider platform's sustainability under these two rules and compare them from the perspective of the artists and the platform. We show that relative popularity of an artist across different user

[1] In practice, artists only end up keeping a fraction of the payout attributed to the consumption of their content, because other parties (such as record labels, professional agencies ensuring proper attribution of the intellectual property, etc.) also get compensated from the same payout.

types is the critical parameter for determining the effects of these two allocation rules. Specifically, we establish that artists who are predominantly listened by users whose overall streaming volume (consumption) is high, receive higher payments with the pro-rata allocation than with the user-centric allocation. Similarly, artists who are predominantly listened by users whose overall streaming volume is low receive higher payments with the user-centric allocation than with the pro-rata allocation. When the user streaming volume is not too extreme, i.e., when the differences in artists' popularity across users are moderate, the analysis of which of the rules benefits an artist is more nuanced.

These findings have immediate consequences on the platform's choice of the allocation rule: if there is an artist that is extremely popular with users who have high consumption ("superstar" artist), the pro-rata rule is not just preferred but the platform might not even be sustainable with the user-centric rule. On the other hand, if there are artists that are extremely popular with users who have low consumption ("niche" artist), the user-centric rule could be preferred to the pro-rata rule, yet pro-rata could nevertheless ensure sustainability. Thus, the analysis in our model suggests that pro-rata should be the artist's compensation rule of choice if the users' behavior points to correlation between artists' popularity and consumption among users and/or if the platform might not be able to estimate heterogeneity of artists' popularity and consumption among its users (since pro-rata guarantees platform sustainability for a larger range of values of relative popularity of artists).

The reason why "niche" artists, whose revenues are lower with pro-rata than with user-centric, still benefit from joining a platform with the pro-rata rule is due to a positive externalities of "superstar" artists' users that have a high consumption. While such users join the platform to predominantly listen to the "superstar" artist, a small portion of their consumption on the platform will be directed at "niche" artists, hence they end up consuming "niche" artists' content only if available on the platform (i.e., they would not pay additional fees for the "niche" artists through their direct channel). Thus, the volume of consumption of "superstar" artists' fans creates sufficient revenue-share even under pro-rata rule for the "niche" artists to be better off on the platform, and thus being able to expand its reach to the superstar artists' fans. This positive externality is the value that platform adds to the streaming music market (or consumption of digital goods, more generally) and is sustainable with the pro-rata rule as the "superstar" artists need to be compensated in part for the value of this positive externality generated by the presence of their fans.[2]

[2] In Appendix B.2.1 we provide an example which illustrates these differences between pro-rata and user-centric allocation rules and their impact on the artists' and platform's decisions.

We also address the profit-maximization problem of a platform committed to using either a pro-rata or a user-centric allocation rule. We establish that the platform's profit maximization problem in the class of pro-rata and user-centric allocation rules is NP-complete (via a reduction from *subset sum*). However, with either of these rules, we establish a connection between the profit-maximization problem and a suitably defined two-parameter family of Knapsack problems, which enables us to develop a Polynomial Time Approximation Scheme (PTAS) for the platform's profit-maximization problem. These algorithms are based on a popularity-based score of artists, and are scalable. Further, we use the aforementioned connection to the Knapsack problem to establish that the optimal profit under pro-rata is at least as large as the optimal profit under user-centric rule, whenever consumption of any artist's content is correlated with overall consumption volume across user types.

We then consider an arbitrary revenue-sharing allocation strategy (that is not limited to be pro-rata or user-centric) and characterize the conditions under which the platform can sustain a set of artists on it, i.e., make a non-negative profit while compensating these artists so that it is profitable for them to join. We also study the profit-maximizing platform's problem in the class of arbitrary revenue allocation rules. A critical step in our analysis is a reformulation of the platform's problem in terms of choosing the set of artists the platform pays to join, which in turn uniquely specifies the subscription fee and artist payments that maximize the platform's profit. Using this reformulation, the platform's profit-maximization problem allows for an integer programming representation which in turn enables us to establish NP-completeness (again, via a reduction from subset sum). Further, we use a relaxation of this integer programming problem and employ a primal-dual analysis to develop a polynomial-time approximation algorithm for the platform's profit-maximization problem. In particular, the platform's profit guaranteed by our algorithm differs from the optimal profit by at most the outside option value of a single artist. Our algorithm assigns a popularity-based score to each artist and selects all artists with a sufficiently high score, which makes it readily scalable.

In summary, our contributions are threefold. First, we introduce a framework to study the impact of revenue-sharing allocation strategies on the performance of a two-sided streaming platform and its participants. Our model captures the heterogeneity of users in terms of their consumption volume, the heterogeneity of content providers in terms of the value they generate for different users, and the subscription-based business model on the user side (which is particularly relevant for streaming platforms). Second, we compare two widely used payment rules in music streaming industry: pro-rata and user-centric. The subscription-based business model on the user side (and users' heterogeneity in terms of consumption volume) creates a cross-subsidization between low and high volume users. We compare these two payment rules, in view of this cross-subsidization, and develop algorithms to find the optimal rule within these two classes of payment rules. Third, we formulate the optimal payment rule in the class of arbitrary revenue allocation rules (not restricted to pro-rata or user-centric rules) and develop an algorithm to find it. Finally, we note that while our work is motivated by music streaming platforms, our results are relevant for any service platform which generates revenues via a subscription-based service. As long as there are two dimensions of users' heterogeneity (i.e., in the consumption volume and the preferences over content), the choice of the service provider compensation scheme has a profound impact on platform's sustainability and profit maximization.

The rest of the paper proceeds as follows. We next discuss the related literature. In Section 2, we describe the model, formulate the platform's problem with an arbitrary revenue allocation rule, introduce pro-rata and user-centric allocation rules, and formulate the 's problem with these two rules. In Section 3 we consider sustainability of the platform with both pro-rata and user-centric rules and compare them from the perspective of the platform and the artists. We then develop a PTAS for profit maximization of a platform committed to using pro-rata or user-centric allocation rules. In Section 4 we consider arbitrary revenue-sharing allocation strategies and characterize when the platform can sustain all artists. We also address platform's profit maximization problem and develop a polynomial time approximation algorithm for it. In Section 5, we discuss our modeling choices and several extensions. Section 6 concludes, while Appendix A contains all the omitted proofs and Appendix B contains additional results, examples, and the detail of the extensions discussed in the paper.

1.1. Related Literature

Our work provides a novel analytical-modeling-based contribution to the discussion on the choice of the revenue-sharing allocation rule for payments to content providers in digital media platforms. In fact, the need for a methodological approach has been articulated by the most prominent legal experts and judges dealing with royalty rates (Strickler (2015)), and has been recognized by academics (e.g., Gans (2019)). This discussion has been particularly relevant in the context of music streaming platforms' practices and has revolved around comparing the impact of pro-rata and user-centric allocation rules; see e.g., Page and Safir (2018a), Jari (2018), Page and Safir (2018b), and Dimont (2017).

Our work relates to the literature on information goods (e.g., Lambrecht et al. (2014), Waldfogel (2017), Goldfarb and Tucker (2019)) and the design of media platforms (see, e.g., Carroni and Paolini (2019) for a survey). Li et al. (2020) study the interaction between media platforms and users and the implications of different pricing schemes. Belleflamme (2016) and Wu et al. (2019) consider a reduced-form model of the interaction between media platforms and users and study the role of piracy and the impact of streaming platforms on the distribution and consumption of digital goods. Also Lei and Swinney (2018) study the decision of digital good creators regarding the distribution of their products. In particular, they study how the creators should choose between *à la carte* and subscription services and establish that in the optimal solution the platform typically needs to shut down à la carte sales altogether in order to have high quality content, which is suggestive of the prevalent practice among streaming media platforms. Aguiar and Martens (2016) empirically study the effects of subscription-based music streaming platforms on the purchasing behavior of users. They show that the emergence of these platforms has increased the overall usage and also positively affected the sales of music through other channels. In another paper, Hendricks and Sorensen (2009) empirically study the role of product discovery in the demand for music and show that releasing a new album causes an increase in the sales of the artist's old albums. Other aspects of media platform's design such as the design of optimal pricing for (partially) ad-supported platforms and the effects of advertisements on quality of products and services, has been studied in the literature. For example, the impact of media platforms on the quality of digital goods is studied in Armstrong and Weeds (2007) for TV programs and Hiller and Walter (2017) for music. More recently, Holtz et al. (2020) study whether personalized recommendations increase or decrease the diversity of content that users consume. The ad-supported media platform's design choices have been studied in, e.g., DeValve and Pekeč (2016).[3]

More broadly, our paper is related to the literature on bundling and non-linear pricing. In settings in which a single buyer decides about a one-time purchase of either one or multiple products, Adams and Yellen (1976), McAfee et al. (1989), Armstrong (2013), Huang et al. (2013), Pavan et al. (2014), Menicucci et al. (2015), Daskalakis et al. (2017), Babaioff et al. (2014), McAfee et al. (1989), Manelli and Vincent (2006), Chen et al. (2019), and Haghpanah and Hartline (2021) provide conditions under which different types of bundling (such as pure bundling and mixed bundling) can be more or less profitable. Industries that use bundling, among others, are TV channels (e.g., Bergantiños and Moreno-Ternero (2020)), cultural venues such as museums (e.g., Ginsburgh and Zang (2003), Bergantiños and Moreno-Ternero (2015)), and matching markets (Birge et al. (2020)). New bundling strategies, such as subscription to unlimited streaming video or music from digital video merchants or music sellers (e.g., Netflix and Spotify) are emerging (see e.g., Shiller and Waldfogel (2013) and Danaher et al. (2014) for empirical studies of bundling strategies in music industry). Bundling of digital goods has been studied in Bakos and Brynjolfsson (1999), Geng

[3] In our model ad revenues are not considered separately as the focus is on the allocation of the total revenue shared by the platform, regardless of its source. Ad revenues are a small fraction of the overall revenue in music streaming platforms. For instance, it is estimated that the subscription fee accounts for about 90% of Spotify's revenue (Resnikoff (2016)).

et al. (2005), Abdallah et al. (2021), and Abdallah (2019), in which a buyer wants to make a one-time purchase decision of (possibly) large number of digital goods. These papers show the benefits of pure bundling as well as pricing based on the cardinality of bundles. More recently, Alaei et al. (2019) consider a setting in which buyers have multiple interactions with the platform and decide both about the subscription and renting individual goods. They find conditions under which grand subscription (i.e., offering a single subscription level that grants access to all items at a fee) is optimal and show the performance guarantee of size-based bundle pricing. Our model and analysis is different from these papers and also from the literature on bundling. First, all the above papers consider the user-side of the digital platforms, i.e., consider the interaction between the platform and the users as a one-sided market design problem. In this paper, however, we consider multiple interactions among users, the platform, and the content providers and study the corresponding two-sided market. Second, in addition to characterizing the optimal design of the platform with arbitrary revenue-allocation strategies, we consider pro-rata and user-centric rules, which are motivated by the business models of market-leading platforms. We compare these two rules from the perspective of the platform and the content providers and also find the optimal design within these two classes.[4]

2. Model

We model a platform that allows users to consume digital goods of the content providers, such as a music streaming platform. We refer to any content provider that the platform needs to compensate as an *artist*.[5] On the user side, the platform charges a subscription fee for unlimited access to content for a given subscription period T (which without loss of generality we normalize to $T=1$). On the artist side, the platform has to compensate the artists for making their content available on the platform. Therefore, the platform faces pricing decisions on both sides of the market: the platform needs to choose a subscription fee as well as a *revenue allocation rule* that specifies the payment to each of the artists who decide to join the platform.

The total population of artists are indexed by the set $\mathcal{M}=\{1,\dots,m\}$ where artists are heterogeneous in terms of the content they provide. Users are heterogeneous both in terms of their consumption volume and their preferences for different artists. In particular, we consider two user types indexed by the set $\mathcal{N}=\{1,2\}$. We let λ_i denote the *usage rate* of type i users for $i=1,2$. Each user has a heterogeneous preference over the artists and we let π_{ij} denote the probability

[4] For a study of other aspects of two-sided platforms see e.g. Bimpikis et al. (2020) for information provision, Ashlagi et al. (2019) for dynamic matching, Bimpikis et al. (2019) and Feng et al. (2020) for pricing and matching in the context of ride-sharing (see also Section 5.4), Hu et al. (2013) for pooling, and Chun et al. (2020) for loyalty programs.

[5] This includes individual content providers and copyright owners, but also companies representing them and/or owning media catalogues (e.g., Disney, Sony Music).

(per usage) that user type i wants to consume the content of artist $j \in \mathcal{M}$ (therefore, we have $\sum_{j=1}^{m} \pi_{ij} = 1$ for $i = 1,2$). Without loss of generality, throughout we assume $\lambda_2 > \lambda_1 > 0$ and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ denote the pair of usage rates. We also let $\Pi = [\pi_{ij}]$ denote the $2 \times m$ matrix whose (i,j)-th entry encodes the probability with which user type i wants to consume the content of artist j. Finally, we let $\mathbf{q} = (q_1, q_2) \in (0,1)^2$ denote the pair of user types' mass and normalize the total mass of users to one, i.e., $q_1 + q_2 = 1$. Throughout, we let the tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$ describe the setting.[6]

2.1. Equilibrium and timing

The platform first sets both the subscription fee for the users and the revenue allocation rule for the artists as a function of the set of artists and the user types that join the platform. The artists then decide whether they want to join the platform. Finally, the users decide whether they want to subscribe to the platform. Formally, the game proceeds in the following three stages:

1. In the first stage, the platform chooses the *revenue-sharing allocation strategy* (or *revenue allocation rule*), $p : 2^{\mathcal{M}} \times 2^{\mathcal{N}} \to \mathbb{R}^m$ such that $p_j(J,S)$ specifies the payment to artist j as a function of the set of artists who join the platform, denoted by J, and the set of user types who subscribe, denoted by S. The payment to artists who are not on the platform, naturally, is zero, i.e., $p_j(J,S) = 0$ for all $j \notin J$ and for all S. When an artist joins the platform she gives the platform the right to offer her content. Given the set of artists who join, the platform chooses a subscription fee which allows users to have access to the content on the platform. Specifically, the subscription fee set by the platform is a function $\textsc{Fee} : 2^{\mathcal{M}} \times 2^{\mathcal{N}} \to \mathbb{R}$ such that $\textsc{Fee}(J,S)$ specifies the fee that users need to pay to get access to the content of artists on the platform (i.e., artists in the set J) when set S of user types subscribe.

2. In the second stage, given the tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$, the revenue allocation rule $p(\cdot,\cdot)$, and subscription fee denoted by $\textsc{Fee}(\cdot,\cdot)$, the artists decide whether they want to join the platform or not. In their joining decision, the artists take into account the set of users who will subscribe to the platform.

3. In the third stage, given the tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$, the subscription fee denoted by $\textsc{Fee}(\cdot,\cdot)$, and the set of artists who have joined the platform, users decide whether they want to subscribe to the platform or not.

The equilibrium of the game is obtained by backward induction. First, we determine the user's subscription decision given the set of artists on the platform and the subscription fee.We then find the optimal decision of artists. Finally, we find the optimal revenue allocation rule as well as the subscription fee.

[6] Modeling heterogeneity in consumption volume with two values for the usage rates does not only allow for a tractable analysis, but is also sufficient to establish how the interplay between usage rate and users' preferences over artists determines the optimal allocation rule. As we show in Subsection 5.1, our analysis extends to more than two user types.

Notice that in choosing both the optimal revenue allocation rule and the subscription fee, in addition to the set of artists who will join, the platform takes into account the set of users who will subscribe. Thus, in equilibrium the set of users who subscribe is determined by the set of artists J and the subscription fee. To simplify the notation, in what follows we drop S in both the revenue allocation rule $p(J)$ and the subscription fee $\text{FEE}(J)$.

2.2. Users' utility and problem

For a given revenue allocation rule $p(\cdot)$ and subscription fee denoted by $\text{FEE}(\cdot)$, we let $J \subseteq \mathcal{M}$ be the set of artists who join the platform. The (expected) value of a type $i \in S$ user for subscribing to the platform is $\lambda_i \sum_{j\in J} \pi_{ij}$. This is because during the subscription period, a type i user seeks to consume the content of artists with rate λ_i and each time she finds a match on the platform with probability $\sum_{j\in J} \pi_{ij}$. User type i subscribes to the platform if and only if her utility from subscribing is non-negative, i.e.,

$$\lambda_i \sum_{j\in J} \pi_{ij} - \text{FEE}(J) \geq 0.$$

2.3. Artists' utility and problem

We first study the utility of each artist's outside option. The revenue that artist j can generate by *going solo*, i.e, by not joining the platform and offering her content through a direct channel, is $r\ d(j,r)$. The subscription price is denoted by r, while $d(j,r)$ denotes the demand, i.e., mass of users who would subscribe to artist j's direct channel:

$$d(j,r) = \begin{cases} 0, & \lambda_1\pi_{1j} < r \text{ and } \lambda_2\pi_{2j} < r, \\ q_1, & \lambda_1\pi_{1j} \geq r \text{ and } \lambda_2\pi_{2j} < r, \\ q_2, & \lambda_1\pi_{1j} < r \text{ and } \lambda_2\pi_{2j} \geq r, \\ 1, & \lambda_1\pi_{1j} \geq r \text{ and } \lambda_2\pi_{2j} \geq r. \end{cases}$$

This is because type i users subscribe to the direct channel of artist j if and only if their value from this solo artist, that is $\lambda_i\pi_{ij}$, is higher than the offered price r. Given the revenue allocation rule $p(\cdot)$, artist j joins the platform if the platform's payment is (weakly) larger than the revenue she could generate if she were to bypass the platform and offer her content through a direct channel. Thus, for the set J of artists who join the platform in equilibrium, we have

$$j \in J \text{ if and only if } p_j(J) \geq \max_r\ r\ d(j,r).$$

For each artist $j \in \mathcal{M}$ we let R_j denote the revenue that artist j obtains if she offers her content through a direct channel, i.e.,

$$R_j = \max_r\ r\ d(j,r). \tag{1}$$

There are two points worth noting. First, we explicitly model the outside option of artists (as opposed to a generic outside option) to be heterogeneous across artists. This outside option captures the fact that artists that are more popular (i.e., have higher likelihood of being the users' choice) will have a higher value from the outside option. Second, we assume artists are single-homing, i.e., once artists offer their content on the platform they cannot offer it through direct channels (users in our model can multi-home). This assumption is in line with the common strategy of platforms to sign exclusivity agreements with artists so as to attract and conserve users interested in their exclusive content (see also Lei and Swinney (2018) for an analysis of the optimality of exclusive contracts).[7] In Appendix B.1.3, we consider an extension of our model to a setting in which artists can multi-home—they can both join the platform and offer their content directly. We show that our results readily extend to this setting and find the platform's optimal revenue allocation rule.

2.4. Platform's problem

The platform's problem is to choose a revenue allocation rule together with a subscription fee that maximizes its profit defined as the total subscription fees collected from subscribed users (which we refer to as revenue) minus the payment made to the artists who join the platform. In general, the revenue allocation rule of the platform can be any function that maps the set of artists on the platform to a payment to each of them. We refer to such allocation rule as *arbitrary revenue allocation rule*. We next define the platform's problem with an arbitrary revenue allocation rule. We then formally introduce *user-centric* and *pro-rata*, the primary revenue allocation rules used by market-leading platforms, and define the platform's problem within these classes of allocation rules.

2.4.1. Platform's problem with arbitrary revenue allocation rule In the first-stage of the game, the platform can choose the set of artists who join the platform by paying them (at least) the revenue they obtain from directly offering their content and pay the rest of the artists 0. Therefore, with an arbitrary revenue-allocation rule, the platform's problem becomes

$$\max_{p(\cdot),\ \textsc{Fee}(\cdot), J} \sum_{i \in S} q_i \ \textsc{Fee}(J) - \sum_{j \in J} p_j(J) \tag{2}$$

$$\text{s.t. } p_j(J) \geq R_j \quad \text{ for all } j \in J \tag{3}$$

$$i \in S \text{ if and only if } \lambda_i \sum_{j \in J} \pi_{ij} - \textsc{Fee}(J) \geq 0, \tag{4}$$

where the objective (2) is the collected fees minus the payments to the artists on the platform. The inequalities in constraint (3) represent the fact that set J of artists are joining in the second-stage

[7] For instance, in 2016, Apple Music and Tidal secured an exclusive release of, respectively, Frank Ocean's and Beyoncé's latest album (see Behr (2016)). For the analyses of the impact of this strategy in competition, see, e.g., Armstrong and Wright (2007), Lee (2013), and Stennek (2014).

of the game. Finally, the inequalities in constraint (4) represent the fact that set $S \subseteq \{1,2\}$ of user types subscribe to the platform in the third-stage of the game.

2.4.2. Platform's problem with pro-rata and user-centric revenue allocation rules We let $\textsc{Rev} = \textsc{Fee}(J)\sum_{i\in S} q_i$ denote the platform's revenue when users of type $i \in S$ who join the platform are charged subscription fee denoted by $\textsc{Fee}(J)$. The platform payout rate is a proportion $\beta \in [0,1]$ of $\textsc{Rev}$ that is distributed among the artists on the platform. We next formally introduce pro-rata and user-centric allocation rules.

Pro-rata allocation rule gives each artist a share of the platform's revenue that is proportional to the rate with which her content is consumed by all users on the platform, formally defined next.

Definition 1 (Pro-rata revenue allocation rule). Let β be the platform payout rate. Let J be the set of artists on the platform and let S be set of users who subscribed to the platform. Pro-rata payment to artist $j \in J$ is

$$p_j(J) = \beta \ \textsc{Rev} \ \frac{\sum_{i\in S} q_i \lambda_i \pi_{ij}}{\sum_{i\in S} q_i \lambda_i \sum_{j\in J} \pi_{ij}}.$$

With pro-rata revenue allocation rule, the platform's problem comprises choosing a subset of artists to join with a payout rate β and a subscription fee and becomes

$$\max_{\beta,\ \textsc{Fee}(\cdot), J} (1-\beta)\ \textsc{Rev} \tag{5}$$

$$\text{s.t. } \beta\ \textsc{Rev}\ \frac{\sum_{i\in S} q_i \lambda_i \pi_{ij}}{\sum_{i\in S} q_i \lambda_i \sum_{j\in J} \pi_{ij}} \ge R_j \quad \text{for all } j \in J \tag{6}$$

$$i \in S \text{ if and only if } \lambda_i \sum_{j\in J} \pi_{ij} - \textsc{Fee}(J) \ge 0 \tag{7}$$

$$\textsc{Rev} = \textsc{Fee}(J) \sum_{i\in S} q_i.$$

where the objective (5) follows from the fact that the platform pays the artists β of its revenue, the inequalities in constraint (6) follow from the fact the minimum payment to retain artist j on the platform is $R_j = \max_r r\ d(j,r)$, and the inequalities in constraint (7) represent the fact that set $S \subseteq \{1,2\}$ of user types subscribe to the platform.

User-centric allocation rule distributes revenue collected by each user separately, giving each artist a share of the collected subscription fees proportional to the (expected) fraction of each user's consumption of the artist's content, defined next.

Definition 2 (User-centric revenue allocation rule). Let β be the platform payout rate. Let J be the set of artists on the platform and let S be set of users who subscribed to the platform. User-centric payment to artist $j \in J$ is

$$p_j(J) = \beta \sum_{i\in S} q_i \ \textsc{Fee}(J)\ \frac{\lambda_i \pi_{ij}}{\sum_{k\in J} \lambda_i \pi_{ik}}$$

$$=\beta \ \textsc{Rev} \ \frac{\sum_{i\in S} q_i \frac{\pi_{ij}}{\sum_{k\in J}\pi_{ik}}}{\sum_{i\in S} q_i}.$$

With user-centric revenue allocation rule, the platform's problem becomes

$$\begin{aligned} &\max_{\beta,\ \textsc{Fee}(\cdot), J} (1-\beta)\ \textsc{Rev} \\ &\text{s.t. } \beta \ \textsc{Rev} \ \frac{\sum_{i\in S} q_i \frac{\pi_{ij}}{\sum_{k\in J}\pi_{ik}}}{\sum_{i\in S} q_i} \geq R_j \quad \text{for all } j\in J \\ &\quad i\in S \text{ if and only if } \lambda_i \sum_{j\in J}\pi_{ij} - \textsc{Fee}(J) \geq 0, \\ &\quad \textsc{Rev} = \textsc{Fee}(J) \sum_{i\in S} q_i. \end{aligned} \tag{8}$$

Notice that in both pro-rata and user-centric rules, the payout rate β is constant, i.e., the same payout rate is applied to revenue-allocation rule for every artist. With the pro-rata rule, each artist is paid β fraction of platform's revenue attributable to the overall consumption of the artist's content on the platform. With the user-centric rule, each artist is paid β fraction of revenue collected from a platform's user attributable to the consumption of the artist's content by that user.

3. Pro-rata and user-centric revenue allocation rules

In this section, we compare the impact of pro-rata and user-centric allocation rules on both platform's performance and artists' payments. We first find the artists' solo market revenue and then address platform's sustainability by characterizing when all artists can be sustained on a platform that uses an arbitrary revenue allocation rule. Constraining platform's choice of the revenue allocation rule to pro-rata and user-centric classes affects sustainability. We discuss the impact of these rules both from the perspective of the artists and from the perspective of the platform. Finally, we address platform's optimal profit when it is committed to using either pro-rata or user-centric rule. We establish that profit optimization problem is NP-complete and develop a PTAS for it, under both the pro-rata and user-centric allocation rules.

3.1. Platform's sustainability

We start our analysis by establishing the revenue that each artist can obtain if she offers content directly to users (i.e., "going solo"). This revenue R_j, defined in (1), is the minimum payment the platform needs to pay artist j in order to keep her on the platform, and therefore plays a central role in characterizing conditions under which the platform can sustain all artists on the platform. It is also important in establishing payout rates and overall sustainability of platforms using pro-rata or user-centric allocation rules, which we discuss in Subsection 3.2.

Our analysis points to the important role of the ratio

$$\frac{\pi_{2j}}{\pi_{1j}},$$

which is the probability with which type 2 users (who have higher usage rate) listen to artist j relative to the probability with which type 1 users (who have lower usage rate) listen to the same artist. In particular, for an artist j, the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ characterizes the revenue she obtains when going solo (i.e., if she offers her content through a direct channel), which in turn impacts the platform's sustainability and profit. The following lemma establishes there are four distinct cases, building on the structure of the demand $d(j,r)$ for artist j's content offered at price r that was described in Section 2.3. It describes not just the revenue for an artist when she is going solo but also the optimal subscription fee she needs to charge and the user types that would subscribe to her.

Lemma 1. *The optimal subscription fee and revenue R_j for artist j offering content directly to users are as follows:*

1. *If $\frac{\pi_{2j}}{\pi_{1j}} \in \left[0, \frac{q_1\lambda_1}{\lambda_2}\right)$, then the optimal subscription fee is $\lambda_1\pi_{1j}$, only type 1 users subscribe to her, and $R_j = q_1\lambda_1\pi_{1j}$.*
2. *If $\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}\right)$, then the optimal subscription fee is $\lambda_2\pi_{2j}$, both user types subscribe to her and $R_j = \lambda_2\pi_{2j}$.*
3. *If $\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2 q_2}\right)$, then the optimal subscription fee is $\lambda_1\pi_{1j}$, both user types subscribe to her, and $R_j = \lambda_1\pi_{1j}$.*
4. *If $\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2 q_2}, \infty\right)$, then the optimal subscription fee is $\lambda_2\pi_{2j}$, only type 2 users subscribe to her, and $R_j = q_2\lambda_2\pi_{2j}$.*

Lemma 1 characterizes R_j and shows there are four distinct cases depending on the ratio $\frac{\pi_{2j}}{\pi_{1j}}$. In the first case, the small ratio suggests that the artist is predominantly listened by users who have a low usage rate (i.e., type 1 users). If such artist is going solo, the optimal subscription fee is such that only type 1 users subscribe. In the second case, the artist is more listened by users who have a low usage rate, but both user types are interested in her content. If such artist is going solo, the optimal subscription fee is such that both types subscribe. Now because type 2 users are less interested in her content, they are binding, i.e., the subscription fee is determined by the value of type 2 users get from subscribing which is $\lambda_2\pi_{2j}$. The intuition of the third and fourth cases are similar.

In view of Lemma 1, we partition the set of artists into the following four categories according to the ratio $\frac{\pi_{2j}}{\pi_{1j}}$:

$$W = \left\{ j \in \mathcal{M} : \frac{\pi_{2j}}{\pi_{1j}} \in \left[0, \frac{q_1\lambda_1}{\lambda_2}\right) \right\}, \quad Z = \left\{ j \in \mathcal{M} : \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}\right) \right\},$$
$$Y = \left\{ j \in \mathcal{M} : \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2 q_2}\right) \right\}, \quad X = \left\{ j \in \mathcal{M} : \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2 q_2}, \infty\right) \right\}.$$

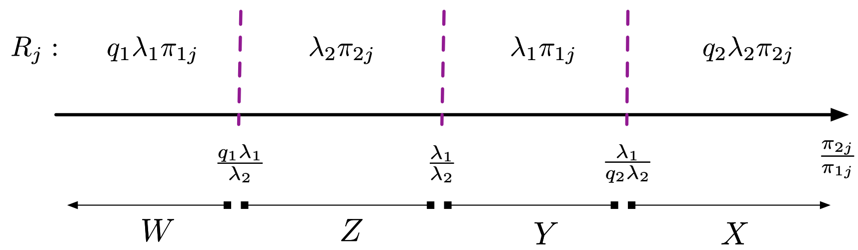


Figure 1 **The ratio $\frac{\pi_{2j}}{\pi_{1j}}$ determines the set of user types that join an artists' direct channel and the artists' revenue R_j. In particular, in the solo market of each artist one of the following happens: (i) in set W only type 1 users subscribe, (ii) in set Z both type of users subscribe and the subscription fee is determined by type 2 users (which have a higher usage rate), (iii) in set Y, again, both type of users subscribe and the subscription fee is determined by type 1 users (which have a lower usage rate), (iv) in set X only type 2 users subscribe.**

These sets are depicted in Figure 1.

The platform can sustain all artists if it can provide payments to artists that are at least as large as revenue that each artist can generate by a direct channel to users, and it needs to generate non-negative profits. We next find the sufficient and necessary condition under which the platform can sustain all artists on it.[8]

PROPOSITION 1. *For any tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$, the platform has a non-negative profit from sustaining all artists on it if and only if*

$$\max\{\lambda_1, \lambda_2 q_2\} - q_2\lambda_2 \sum_{j\in X} \pi_{2j} - \lambda_1 \sum_{j\in Y} \pi_{1j} - \lambda_2 \sum_{j\in Z} \pi_{2j} - \lambda_1 q_1 \sum_{j\in W} \pi_{1j} \geq 0. \tag{9}$$

In particular, if $\lambda_1 \geq \lambda_2 q_2$ and $\left(\max_{j\in\mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) \leq \frac{\lambda_1}{\lambda_2 q_2}$, the platform can sustain all artists on it.

The first term of (9), i.e., $\max\{\lambda_1, \lambda_2 q_2\}$, is the collected subscription fees by the platform and the rest of the terms are the payments of the platform to artists. Thus, in order to sustain all artists on the platform, the platform should compensate each artist for what she can obtain if she deviates and offers her content by herself.[9]

3.2. Platform's sustainability with pro-rata and user-centric rules

The platform can sustain all artists with pro-rata and user-centric revenue allocation rules if it can generate enough revenue to pay all artists according to these rules and still have a non-negative

[8] The problem of sustaining any subset $J \subseteq \mathcal{M}$ of artists follows from the same analysis. This is because for any set of artists J, we can normalize the probabilities $\{\pi_{ij}\}_{j\in J}$ for $i = 1, 2$ and modify the usage rates. The platform's problem of choosing the optimal set of artists is addressed in Subsection 3.3.

[9] In Appendix B.1.4 we extend this analysis by considering not just sustainability with respect to individual deviations, but also possible coalitional deviations. In other words, we discuss group strategy-proofness and provide conditions for having a non-empty core. Recall that by defining "artists" as any entity possibly representing a natural coalition of content providers (e.g., record label or media company), our platform sustainability result already accounts for possible deviations of such coalitions (e.g., in context of video streaming, Disney deciding to pull out its portfolio from Netflix and offer it through Disney+).

profit. As shown in Proposition 1, if $\lambda_1 < \lambda_2 q_2$, then it is optimal for the platform to choose the subscription fee equal to λ_2 and only user type 2 subscribes. In this case, user-centric and pro-rata allocation rules coincide because all subscribed users will have the same usage rate. Therefore, in what follows we adopt the following assumption which precludes this uninteresting setting.

ASSUMPTION 1. *The usage rate of user types and their mass is such that $\lambda_1 \geq \lambda_2 q_2$.*

Assumption 1 simply guarantees that, when all artists are on the platform, both types of users subscribe. This in turn implies that the platform's revenue is $\text{REV} = \lambda_1$.

In a pro-rata revenue allocation rule with payout rate β, the payment to artist j is

$$\beta\lambda_1 \frac{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}{q_1\lambda_1 + q_2\lambda_2}.$$

The platform can sustain artist j on it if this payment is (weakly) larger than the revenue she obtains if she offers her content by herself through a direct channel. Using Lemma 1 we derive minimum payout rate to keep artist j on the platform, and then take the maximum over these to determine the platform payout rate β^{pr} such that all artists stay on the platform that uses the pro-rata revenue allocation rule:

$$\begin{aligned}\beta^{\text{pr}} &= \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1} \max_{j\in\mathcal{M}} \left\{ \frac{R_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}} \right\} \\ &= \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1} \max\left\{ \max_{j\in X} \frac{q_2\lambda_2\pi_{2j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}, \max_{j\in Y} \frac{\lambda_1\pi_{1j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}, \right. \\ &\qquad\qquad\qquad \left. \max_{j\in Z} \frac{\lambda_2\pi_{2j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}, \max_{j\in W} \frac{\lambda_1 q_1\pi_{1j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}} \right\}. \end{aligned} \tag{10}$$

We call the artist j whose payout rate is the maximum among all the other artists the *pivot artist* for pro-rata.

In a user-centric revenue allocation rule with payout rate β, the payment to artist j is

$$\beta\lambda_1(q_1\pi_{1j} + q_2\pi_{2j}).$$

Analogous to the analysis of sustainability of the pro-rata allocation rule, using Lemma 1 we derive minimum payout rate to keep artist j on the platform, and then take the maximum over these to determine the platform payout rate β^{uc} such that all artists stay on the platform that uses the user-centric revenue allocation rule:

$$\begin{aligned}\beta^{\text{uc}} &= \frac{1}{\lambda_1} \max_{j\in\mathcal{M}} \left\{ \frac{R_j}{q_1\pi_{1j} + q_2\pi_{2j}} \right\} \\ &= \frac{1}{\lambda_1} \max\left\{ \max_{j\in X} \frac{q_2\lambda_2\pi_{2j}}{q_1\pi_{1j} + q_2\pi_{2j}}, \max_{j\in Y} \frac{\lambda_1\pi_{1j}}{q_1\pi_{1j} + q_2\pi_{2j}}, \right. \\ &\qquad\qquad \left. \max_{j\in Z} \frac{\lambda_2\pi_{2j}}{q_1\pi_{1j} + q_2\pi_{2j}}, \max_{j\in W} \frac{\lambda_1 q_1\pi_{1j}}{q_1\pi_{1j} + q_2\pi_{2j}} \right\}. \end{aligned} \tag{11}$$

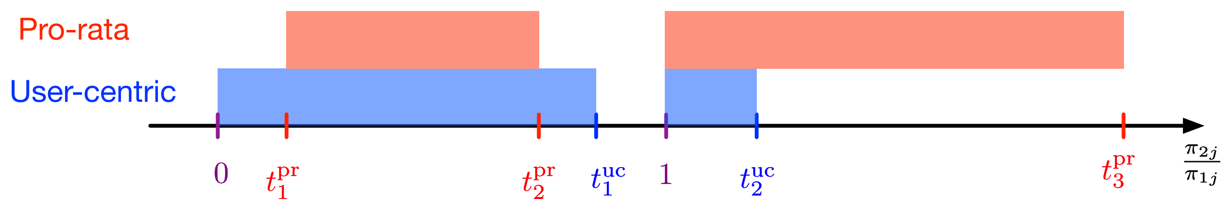


Figure 2 **The ratio $\frac{\pi_{2j}}{\pi_{1j}}$ for which the platform can sustain all artists with user-centric and pro-rata revenue allocation rules. The red intervals are the ratios sustainable by pro-rata and the blue intervals are the ratios sustainable by user-centric. In this figure, the thresholds for pro-rata rule are $t_1^{\rm pr} = \frac{q_1(\lambda_2-\lambda_1)}{\lambda_2}$, $t_2^{\rm pr} = \frac{q_1\lambda_1^2}{\lambda_2^2 q_2+\lambda_1\lambda_2(q_1-q_2)}$, and $t_3^{\rm pr} = \frac{q_1\lambda_1^2}{\lambda_2 q_2^2(\lambda_2-\lambda_1)}$ and for user-centric rule are $t_1^{\rm uc} = \frac{\lambda_1 q_1}{\lambda_2-\lambda_1 q_2}$ and $t_2^{\rm uc} = \frac{q_1\lambda_1}{q_2(\lambda_2-\lambda_1)}$.**

Similar to the pro-rata setting, we call the artist j whose payout rate is the maximum among all the other artists the pivot artist for user-centric.

The platform can sustain all artists on it with pro-rata (and similarly user-centric) revenue allocation rule, if its profit which is $(1-\beta^{\rm pr})\lambda_1$ (and similarly $(1-\beta^{\rm uc})\lambda_1$) is non-negative, i.e., $\beta^{\rm pr} \le 1$ (and similarly $\beta^{\rm uc} \le 1$). In the next theorem we characterize the platform's sustainability (i.e., whether the inequalities $\beta^{\rm pr} \le 1$ and $\beta^{\rm uc} \le 1$ hold) in terms of the ratios $\frac{\pi_{2j}}{\pi_{1j}}$ for both pro-rata and user-centric revenue allocation rules, and highlight the difference between the two allocation rules in terms of the platform's sustainability.

THEOREM 1. *Suppose Assumption 1 holds.*

(a) The platform is sustainable with pro-rata allocation rule if and only if

$$\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{q_1(\lambda_2-\lambda_1)}{\lambda_2}, \frac{q_1\lambda_1^2}{\lambda_2^2 q_2+\lambda_1\lambda_2(q_1-q_2)}\right] \bigcup \left[1, \frac{q_1\lambda_1^2}{\lambda_2 q_2^2(\lambda_2-\lambda_1)}\right], \quad \textit{for all } j \in \mathcal{M}.$$

(b) The platform is sustainable with user-centric allocation rule if and only if

$$\frac{\pi_{2j}}{\pi_{1j}} \in \left[0, \frac{q_1\lambda_1}{\lambda_2-q_2\lambda_1}\right] \bigcup \left[1, \frac{q_1\lambda_1}{q_2(\lambda_2-\lambda_1)}\right], \quad \textit{for all } j \in \mathcal{M}.$$

Figure 2 depicts the result of Theorem 1 and shows the ranges of the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ for which the platform is sustainable with pro-rata and user-centric revenue allocation rules.

Theorem 1 implies that there exists upper bounds on the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ for a platform to be able to sustain artist j with either pro-rata or user-centric revenue allocation rule. Therefore, the platform cannot sustain an artist with a sufficiently large $\frac{\pi_{2j}}{\pi_{1j}}$, i.e., an artist predominantly listened to by users who have a high usage rate. Such artists could generate higher revenues by going solo, as established in part 1 of Lemma 1 and noting that such artist belongs to the set X. This is because if they are going solo, type 2 users would follow them. On the platform, however, the artist would not receive the entire subscription fee of type 2 users because type 2 users listen to other artists as well; at the same time the positive externality of allocation of revenue from type 1 users is not large enough to compensate for this loss (because the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ is large).

Also note that there is no positive lower bound in terms of the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ of the artist j that the platform can sustain with a user centric rule, while a non-zero lower bound exists with pro-rata rule. This is because artists with sufficiently small $\frac{\pi_{2j}}{\pi_{1j}}$ belong to the set W and therefore part 4 of Lemma 1 establishes that if such artists are going solo, type 1 users will subscribe to them. On the platform with the pro-rata rule their share of the platform's subscription fee paid by type 1 users is diluted because type 1 users have a low usage rate.

Finally, note that if there is no discrepancy in popularity of artists among two user types (i.e., $\pi_{1j} = \pi_{2j}$ for all artists $j \in \mathcal{M}$), then pro-rata and user-centric rules are identical. However, if this discrepancy is significant and there is an artist j who is more popular with type 2 users, it follows from Theorem 1 (and can be seen from Figure 2) that the pro-rata rule can sustain such artist as long as $\frac{\pi_{2j}}{\pi_{1j}} \in [t_2^{\mathrm{uc}}, t_3^{\mathrm{pr}}]$, while the user-centric rule cannot. This is because the β of user-centric rule necessary to sustain the artist must be larger than 1, so the platform would not be profitable. The fact that the pro-rata rule can sustain artist j in the presence of the described discrepancy in popularity suggests that the pro-rata rule adjusts the payment for artist j, thereby compensating them for the positive externality type 2 users (who primarily follow artist j) bring by consuming the content of other artists on the platform. Thus, the pro-rata rule results in both paying artist j more than their user-centric rule, and in reducing user-centric payments to other artists who benefit from this positive externality (essentially compensating artist j for attracting fans to the platform, who then create value by consuming the content of other artists on the platform).

In the next two subsections we compare pro-rata and user-centric rules from the perspective of the artists and the platform.

3.2.1. Artist payments with pro-rata and user-centric rules The artists' preferences for pro-rata versus user-centric depend on their popularity among users with a higher usage rate. Intuitively, artists who are more popular among users with a higher usage rate prefer a pro-rata revenue allocation rule to a user-centric revenue allocation rule. The next proposition formalizes this intuition.

PROPOSITION 2. *Suppose Assumption 1 holds and pro-rata and user-centric payments can sustain all artists on the platform with β^{pr} and β^{uc}, respectively. The ratio of artists' payment with pro-rata to their payment with user-centric is increasing in $\frac{\pi_{2j}}{\pi_{1j}}$. In particular, if the platform's revenue is the same with user-centric and pro-rata rules (i.e., $\beta^{\mathrm{pr}} = \beta^{\mathrm{uc}}$), then the artists with $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$ prefer the pro-rata rule.*

Proposition 2 suggests the importance of the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ in comparing pro-rata and user-centric allocation rules. The statement further supports the discussion at the end of the previous subsection: artist j who is listened to by users with high usage rate (i.e., type 2 users) with higher probability than by type 1 users (e.g., the artist is not only popular but also her fans listen to the artist very frequently), prefers the pro-rata allocation rule to the user-centric allocation rule. On the other hand, Proposition 2 also indicates that artists whose fans (users who have a higher probability of listening to them) do not have a high usage rate (i.e., type 1 users) prefer the user-centric allocation rule since pro-rata revenue allocation dilutes their share of revenues generated by users with low usage rate.

3.2.2. Platform's profit with pro-rata and user-centric rules Recall that with β payout rate, the platform's profit is $(1-\beta)\textsc{Rev}$. Therefore, the platform prefers an allocation rule that ensures sustainability with a smaller payout rate β to artists. Whether the platform prefers pro-rata or user-centric rule depends on the popularity of artists among users with high and low usage rates, as shown in the next proposition.

PROPOSITION 3. *Suppose Assumption 1 holds and pro-rata and user-centric payments can sustain all artists on the platform with* β^{pr} *and* β^{uc}*, respectively.*

(a) If the pivot artist j for pro-rata is such that $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$, then the profit of the platform with pro-rata revenue allocation rule is weakly higher than its profit with user-centric.

(b) If the pivot artist j for user-centric is such that $\frac{\pi_{2j}}{\pi_{1j}} \leq 1$, then the profit of the platform with user-centric revenue allocation rule is weakly higher than its profit with pro-rata.

The first part of Proposition 3 shows that if an artist who is more likely to be listened to by users who have a higher usage rate is the pivot artist for pro-rata, then the platform prefers the pro-rata revenue allocation rule. The reason is that for artists j with $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$, the payout rate to sustain them on the platform with the pro-rata rule is smaller than with the user-centric rule. Therefore, if the pivot artist for pro-rata is among those artists, then β^{pr} is weakly smaller than β^{uc}. Similarly, the second part of Proposition 3 shows that if an artist who is less likely to be listened to by users who have a higher usage rate is the pivot artist for user-centric, then the platform prefers the user-centric rule. This is because for artists j with $\frac{\pi_{2j}}{\pi_{1j}} \leq 1$, the payout rate to sustain them on the platform with the user-centric rule is smaller than with the pro-rata rule. Therefore, if the pivot artist for user-centric is among those artists, then β^{uc} is weakly smaller than β^{pr}. In fact, as already discussed, the choice of allocation rule is not just affecting profitability but also platform sustainability.

The following corollary readily follows from Theorem 1 and Proposition 3.

Corollary 1. *Suppose Assumption 1 holds and let* $\underline{M} = \frac{q_1\lambda_1}{q_2(\lambda_2-\lambda_1)}$ *and* $\bar{M} = \frac{q_1\lambda_1^2}{\lambda_2 q_2^2(\lambda_2-\lambda_1)}$.

(a) If $\left(\max_{j\in\mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) \le \underline{M}$ *and the platform is sustainable with both pro-rata and user-centric revenue allocation rules, then pro-rata revenue allocation rule has a higher profit than user-centric revenue allocation rules.*

(b) If $\underline{M} < \left(\max_{j\in\mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) \le \bar{M}$, *then the platform is not sustainable with user-centric allocation rule, but could be sustainable with the pro-rata allocation rule.*

(c) If $\left(\max_{j\in\mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) > \bar{M}$, *a platform is not sustainable with neither user-centric nor pro-rata revenue allocation rule, but could be sustainable with an arbitrary revenue allocation rule rule.*

Some implications of the results in this section are illustrated in the following example.

Example 1. Suppose Assumption 1 holds and each artist $j \in \mathcal{M}$ either belongs to the set X or the set W (i.e., that $\frac{\pi_{2j}}{\pi_{1j}} \in [0, \frac{q_1\lambda_1}{\lambda_2}) \cup [\frac{\lambda_1}{q_2\lambda_2}, \infty)$ for all $j \in \mathcal{M}$). In other words, assume that each artist's content is predominantly consumed by a single user type: type 1 (low consumption) users predominantly consume content of each artist $j \in W$, while type 2 (high consumption) users predominantly consume content of each artist $j \in X$. Thus, as established by Proposition 2, if the platform is sustainable with the payout rate β using either pro-rata and user-centric rule (in which case both the platform's revenue, subscription fee, and consequently users' utility is the same under the two rules), the artists $j \in X$ prefer the pro-rata rule and artists $j \in W$ prefer user-centric rule.

Proposition 3 establishes that, when the platform optimizes payout rates, i.e., utilizes β^{pr} payout rate with the pro-rata allocation and β^{uc} payout rate with the user-centric allocation, its preference over the two rules is determined by the pivot artist j (i.e., the artist whose revenue from going solo determines the payout rate): if $j \in X$ then the platform prefers pro-rata ($\beta^{\mathrm{pr}} < \beta^{\mathrm{uc}}$) and if $j \in W$ then the platform prefers user-centric ($\beta^{\mathrm{pr}} > \beta^{\mathrm{uc}}$). Interestingly, even when $\beta^{pr} \neq \beta^{uc}$, artist preferences over the two allocation rules remain unchanged: artists $j \in X$ prefer the pro-rata payment and artists $j \in W$ prefer user-centric payment. (see Appendix B.2.2 for the detail).

If the overall consumption of type 2 users is large enough, i.e., if $\frac{q_2\lambda_2}{q_1\lambda_1} \ge \frac{1}{\sqrt{xw}}$ where $x = \max_{j\in X} \frac{\pi_{2j}}{\pi_{1j}}$ and $w = \min_{j\in W} \frac{\pi_{2j}}{\pi_{1j}}$, then the pivot artist is in the set X and the platform's profit with pro-rata allocation rule is higher than with user-centric rule. Note that this could be the case even with a small number of artists (those in the set X) preferring the pro-rata rule, while a large number of artists (those in the set W) would prefer the user-centric rule. A sustainable platform's choice is not based merely on comparing the number of artists who prefer one rule over another, but rather on comparing the overall consumption volume attributed to (possibly a small number of) artists in X who prefer pro-rata versus the overall consumption volume of artists in W who prefer user-centric. In this case, the artists in the set X demand a large payment by going solo. Pro-rata rule implicitly compensates these artists with a high payment (without the need to use a high payout rate β). Therefore, the platform generates higher revenue by using the pro-rata rule, and it can sustain all artists by ensuring that those with the highest outside option are compensated.

On the other hand, if the overall consumption of type 1 users is large enough, i.e., $\frac{q_1\lambda_1}{q_2\lambda_2} \geq \frac{q_1 x + q_2 x w}{q_1 + q_2 x}$, then the pivot artist is in the set W and the platform's profit with the user-centric allocation rule is higher than with the pro-rata rule. Here, unlike the previous case, artists in the set X who prefer pro-rata do not have a strong outside option, and demand a smaller payment. This points to a negative impact of the pro-rata allocation in such setting, as pro-rata favors artists whose content is predominantly consumed by type 2 users, at the expense of artists whose content is predominantly consumed by type 1 users. Therefore, without significant consumption volume attributed to type 2 users and to the artists they prefer, the platform would generate higher revenue by using the user-centric rule.

3.3. Optimal set of artists with pro-rata and user-centric rules

The platform's problem with the pro-rata or user-centric allocation rule, formulated in (5) and (8), respectively, comprises of choosing a set of artists, a subscription fee for the users, and a payout rate. Our next lemma provides an equivalent reformulation of the platform problem by characterizing the optimal solution in terms of the set of artists and the set of user types who join the platform.

Lemma 2. *For a tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$, with both pro-rata and user-centric revenue allocation rules the platform chooses a subscription fee so that both type of users subscribe, i.e., $S^* = \{1,2\}$. We also have:*

(a) With pro-rata rule, the set of artists J^{pr} that join the platform is the solution of

$$\max_{J \subseteq \mathcal{M}} \left\{ \left(\min_{i \in \{1,2\}} \lambda_i \sum_{j \in J} \pi_{ij} \right) - \left(q_1 \lambda_1 \sum_{j \in J} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J} \pi_{2j} \right) \max_{j \in J} \frac{R_j}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \right\}. \tag{12}$$

(b) With user-centric rule, the set of artists J^{uc} that join the platform is the solution of

$$\max_{J \subseteq \mathcal{M}} \left\{ \left(\min_{i \in \{1,2\}} \lambda_i \sum_{j \in J} \pi_{ij} \right) \left(1 - \max_{j \in J} \frac{R_j}{q_1 \frac{\pi_{1j}}{\sum_{j \in J} \pi_{1j}} + q_2 \frac{\pi_{2j}}{\sum_{j \in J} \pi_{2j}}} \right) \right\}. \tag{13}$$

In the next proposition, we establish the hardness of finding the optimal profit for the platform restricted to choosing an allocation rule in the class of pro-rata or user-centric rules by reducing *Subset Sum* problem (which is NP-complete, see e.g., Kleinberg and Tardos (2006, Chapter 6)) to these problems.

Proposition 4. *Problem (12) and problem (13) are both NP-complete.*

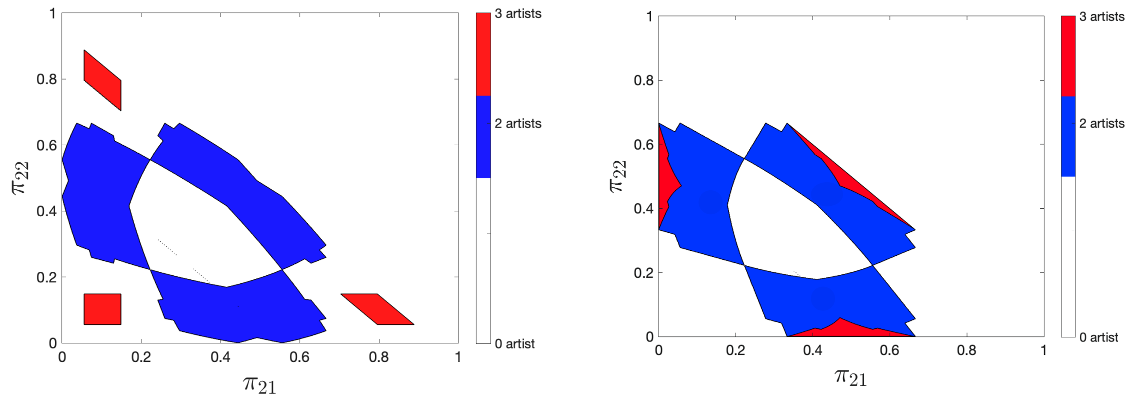
Before addressing approximation of the optimal profit for a platform using either pro-rata or user-centric rule, we illustrate and compare the optimal solutions of (12) and (13) in terms of sets J^{pr} and J^{uc}, respectively.

Example 2. We consider a setting with three artists with mass of user types $(q_1, q_2) = (1/2, 1/2)$ and usage rates $(\lambda_1, \lambda_2) = (1, 1.5)$. For type 1 users we let $(\pi_{11}, \pi_{12}, \pi_{13}) = (1/3, 1/3, 1/3)$ (i.e., consumption of type 1 users is uniformly distributed over all three artists), while for type 2 users we vary the probabilities π_{21} and π_{22} (therefore, π_{23} becomes $1 - \pi_{21} - \pi_{22}$). We track the number of artists on the platform in this two-dimensional parameter space. Figure 3 depicts the number of artists on the profit-maximizing platform using (a) pro-rata (i.e., $|J^{\text{pr}}|$) and (b) user-centric (i.e., $|J^{\text{uc}}|$) allocation rule (the upper right triangle is generically infeasible because we must have $\pi_{21} + \pi_{22} \leq 1$). Note that a profit-maximizing platform has either two or three artists, as a platform with a single artist generates zero profit (revenue is equivalent to artist's outside option which is a lower bound on the platform's payment to that artist).

Figures 3a and 3b depict $|J^{\text{pr}}|$ and $|J^{\text{uc}}|$, the number of artists on the profit-maximizing platform with the optimal pro-rata and user-centric allocation rules, respectively. These figures show the ranges for which the platform can obtain a non-zero profit. As we observe in Figure 3a, with the pro-rata rule for three regions the platform will have all the three artists on it (and has a higher profit than user-centric rule). These three regions correspond to one of the artists being much more popular for type 2 users (either π_{21} is large or π_{22} is large or both are small, meaning $\pi_{23} = 1 - \pi_{21} - \pi_{22}$ is large). In contrast, as we observe in Figure 3b, for these regions user-centric allocation rule cannot sustain any set of artists on it and has zero profit. This illustrates a shortcoming of the user-centric rule with respect to "super-star" artists: the platform can sustain an artist that is more popular among type 2 (high-consumption) users with the pro-rata rule, but not with the user-centric rule.

3.4. Profit maximization with pro-rata and user-centric rules

When a profit-maximizing platform committed to using pro-rata (or user-centric) allocation chooses the payout rate β, it needs to balance the impact of the β on its profit $(1-\beta)\textsc{Rev}$. While smaller β has a positive impact on the profit as proportion of the overall revenue, larger β allows for a larger set of artists J to join the platform, providing a higher value to its users and consequently generates a higher revenue. As established in Proposition 4, this is an NP-complete problem. Nevertheless, by using connections to the knapsack problem, we next develop a Polynomial Time Approximation Scheme (PTAS) for finding the optimal profit for a platform using pro-rata (or user-centric) revenue allocation rules. We present and discuss in detail our analysis and algorithm development for a platform using pro-rata allocation; the user-centric rule is handled analogously and along the way we report all parameter adjustments needed to obtain PTAS for that case.



(a) pro-rata allocation rule (b) user-centric allocation rule

Figure 3 The number of artists on the platform in the setting of Example 2. The x-axis is the probability of type 2 listening to artist 1 (i.e. π_{21}) and the y-axis is the probability of type 2 listening to artist 2 (i.e. π_{22}).

Using Lemma 2, the problem of finding the optimal set J^{pr} and payout rate β^{pr} with the pro-rata rule can be reformulated as

$$\max_{\substack{J\subseteq\mathcal{M}\\ \beta\in[0,1]}} \left(\min_{i\in\{1,2\}} \lambda_i \sum_{j\in J} \pi_{ij}\right)(1-\beta) \tag{14}$$

$$\text{s.t. } \beta\left(\min_{i\in\{1,2\}} \lambda_i \sum_{j\in J} \pi_{ij}\right) \frac{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}{q_1\lambda_1\sum_{j\in J}\pi_{1j}+q_2\lambda_2\sum_{j\in J}\pi_{2j}} \geq R_j \quad \text{for all } j\in J, \tag{15}$$

where the objective (14) is the revenue generated by the platform (i.e., $\min_{i\in\{1,2\}}\lambda_i\sum_{j\in J}\pi_{ij}$) multiplied by $1-\beta$ which is the fraction of revenue the platform keeps for itself, and the constraint (15) ensures that the payments to the artists on the platform are larger than their outside option R_j.

The relaxed version of problem (14) by setting indicator variables $x_j\in[0,1]$ for $j\in\mathcal{M}$ does not yield a linear program. This is in contrast with the setting in which the platform maximizes its profit over arbitrary revenue allocation rules, as we discuss in Section 4. Therefore, solving this problem requires a different approach. In particular, we first reformulate problem (14) into a suitably defined two-parameter family of knapsack problems. The knapsack constraint in each instance corresponds to a parametrized artist's *score*. Algorithm 1 then searches over this two-dimensional space by solving an instance of a knapsack problem in each step (Knapsack problem allows FPTAS; see e.g., Vazirani (2013)).

Let $(J^{\mathrm{pr}},\beta^{\mathrm{pr}})$ be the optimal solution of problem (14). Without loss of generality, suppose[10]

$$\lambda_1\sum_{j\in J^{\mathrm{pr}}}\pi_{1j}\leq\lambda_2\sum_{j\in J^{\mathrm{pr}}}\pi_{2j}.$$

[10] Algorithm can simply be re-run by swapping the indices denoting user types which covers the complementary case.

We define

$$\alpha^{\text{pr}} = \frac{\lambda_2 \sum_{j \in J^{\text{pr}}} \pi_{2j}}{\lambda_1 \sum_{j \in J^{\text{pr}}} \pi_{1j}}$$

as the ratio of type 2 users' consumption to type 1 users' consumption in the optimal solution. With this notation, we can rewrite constraint (15) succinctly in terms of β^{pr} and α^{pr} as

$$\frac{R_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}} \le \frac{\beta^{\text{pr}}}{q_1 + q_2\alpha^{\text{pr}}}, \quad \text{for all } j \in J^{\text{pr}}.$$

Motivated by this, for any β and α we let $\mathcal{M}^{\text{pr}}(\beta,\alpha)$ be the set of artists defined as

$$\mathcal{M}^{\text{pr}}(\beta,\alpha) = \left\{ j \in \mathcal{M} \ : \ \frac{R_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}} \le \frac{\beta}{q_1 + q_2\alpha} \right\}$$

and let $J^{\text{pr}}(\beta,\alpha)$ be the optimal solution of

$$\begin{aligned} \max_{J \subseteq \mathcal{M}^{\text{pr}}(\beta,\alpha)} & (1-\beta)\lambda_1 \sum_{j \in J} \pi_{1j} \\ \text{s.t. } & \sum_{j \in J} \left(\lambda_1\pi_{1j} - \alpha\lambda_2\pi_{2j}\right) \le 0. \end{aligned} \tag{16}$$

The optimal set of artists J^{pr} is the solution of problem (16) for $\beta = \beta^{\text{pr}}$ and $\alpha = \alpha^{\text{pr}}$, i.e., $J^{\text{pr}} = J^{\text{pr}}(\beta^{\text{pr}}, \alpha^{\text{pr}})$.

We next reformulate problem (16) into an instance of a Knapsack problem. The issue is that the summands $\lambda_1\pi_{1j} - \alpha^{\text{pr}}\lambda_2\pi_{2j}$ in the constraint of (16) (which correspond to weights in the Knapsack problem) could be negative, so reformulation into a Knapsack problem is not immediate. However, with notation

$$\mathcal{M}_+^{\text{pr}}(\beta,\alpha) = \left\{ j \in \mathcal{M}(\beta,\alpha) \ : \ \lambda_1\pi_{1j} - \alpha\lambda_2\pi_{2j} \ge 0 \right\},$$
$$\mathcal{M}_-^{\text{pr}}(\beta,\alpha) = \left\{ j \in \mathcal{M}(\beta,\alpha) \ : \ \lambda_1\pi_{1j} - \alpha\lambda_2\pi_{2j} < 0 \right\},$$

it is easy to see that any optimal solution of (16) should include all artists from $\mathcal{M}_-^{\text{pr}}(\beta,\alpha)$. Thus, the optimal solution of (16) can be partitioned as $\mathcal{M}_-^{\text{pr}}(\beta,\alpha) \cup J_+^{\text{pr}}$ where $J_+^{\text{pr}} \subseteq \mathcal{M}_+^{\text{pr}}(\beta,\alpha)$ is the solution of the following Knapsack problem

$$\begin{aligned} \max_{J \subseteq \mathcal{M}_+^{\text{pr}}(\beta,\alpha)} & (1-\beta)\lambda_1 \sum_{j \in J} \pi_{1j} \\ \text{s.t. } & \sum_{j \in J} \left(\lambda_1\pi_{1j} - \alpha\lambda_2\pi_{2j}\right) \le \sum_{j \in \mathcal{M}_-^{\text{pr}}(\beta^{\text{pr}},\alpha)} \left(\alpha\lambda_2\pi_{2j} - \lambda_1\pi_{1j}\right). \end{aligned} \tag{17}$$

Therefore, using FPTAS for the above Knapsack problem, in polynomial time we can find a solution of problem (17), and consequently problem (16), whose profit is at least $(1-\epsilon)$ of the optimal solution.

Input: $\mathbf{q}=(q_1,q_2)$, $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)$, Π, δ, ϵ

Initialization: profit $=0$, $\tilde{J}=\emptyset$, and $\tilde{\beta}=0$.

for $\ell=0,\dots,\lceil\frac{1}{\delta}\rceil$ **do**

  for $t=0,\dots,\left\lceil\frac{\sum_{j\in\mathcal{M}}\frac{\lambda_2\pi_{2j}}{\lambda_1\pi_{1j}}-1}{\delta}\right\rceil$ **do**

    $\alpha\leftarrow 1+t\delta,\quad \beta\leftarrow\ell\delta$

    Use FPTAS for the following Knapsack problem with approximation $(1-\epsilon/2)$

$$\max_{J\subseteq\mathcal{M}^{\mathrm{pr}}(\beta,\alpha)}(1-\beta)\lambda_1\sum_{j\in J}\pi_{1j}$$
$$\text{s.t.}\ \sum_{j\in J}(\lambda_1\pi_{1j}-\alpha\lambda_2\pi_{2j})\le 0$$

    Denote the solution by $J(\beta,\alpha)$ and its objective by profit(β,α)

    If profit$(\beta,\alpha)\ge$ profit **then** profit $\leftarrow$ profit(β,α), $\tilde{J}\leftarrow J(\beta,\alpha)$, $\tilde{\beta}\leftarrow\beta$

  end

end

Output: run the above algorithm one more time by swapping the user types and output $\tilde{J}$ and $\tilde{\beta}$ that generates the highest profit

Algorithm 1: PTAS for the optimal platform's profit with pro-rata (user-centric) rule

Finally, to provide a solution to problem (14), we need to determine the pair $(\beta^{\mathrm{pr}},\alpha^{\mathrm{pr}})$. We do that by searching in a two-dimensional grid over $[0,1]\times[1,\sum_{j\in\mathcal{M}}\frac{\lambda_2\pi_{2j}}{\lambda_1\pi_{1j}}]$ for the pair (β,α), which then yields an approximate solution of problem (14). Algorithm 1 summarizes this procedure.

The procedure for finding the optimal user-centric revenue allocation rule is identical to Algorithm 1 and is obtained by replacing the sets $\mathcal{M}^{\mathrm{pr}}$, $\mathcal{M}^{\mathrm{pr}}_+$, and $\mathcal{M}^{\mathrm{pr}}_-$ with their analogue in the user-centric rule defined as:

$$\mathcal{M}^{\mathrm{uc}}(\beta,\alpha)=\left\{j\in\mathcal{M}\ :\ \frac{R_j-\beta q_1\lambda_1\pi_{1j}}{\beta q_2\lambda_2\pi_{2j}}\le\frac{1}{\alpha}\right\},$$
$$\mathcal{M}^{\mathrm{uc}}_+(\beta,\alpha)=\left\{j\in\mathcal{M}(\beta,\alpha)\ :\ \lambda_1\pi_{1j}-\alpha\lambda_2\pi_{2j}\ge 0\right\},\ \text{and}$$
$$\mathcal{M}^{\mathrm{uc}}_-(\beta,\alpha)=\left\{j\in\mathcal{M}(\beta,\alpha)\ :\ \lambda_1\pi_{1j}-\alpha\lambda_2\pi_{2j}<0\right\}.$$

The following lemma provides the performance guarantee of Algorithm 1. In the lemma, we denote $\bar{\beta}=\max\{\beta_J\ :\ J\subseteq\mathcal{M},\ \beta_J<1\}$, where β_J is the minimum payout rate to sustain artists in set J on the platform. Also, poly$(\cdot)$ denotes a function that is polynomial in its inputs.

LEMMA 3. *For any $\epsilon>0$, Algorithm 1 with input $\delta=\frac{\epsilon(1-\bar{\beta})}{4}$ finds a set of artists and a payout rate for pro-rata (or user-centric) rule whose profit is at least $(1-\epsilon)$ of the optimal profit in time* poly$\left(m,\frac{1}{\epsilon},\frac{1}{\delta}\right)$.

The following theorem follows directly from Lemma 3.

Theorem 2. *For any tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$, Algorithm 1 is a polynomial time approximation scheme for finding the optimal profit of a platform using pro-rata (or user-centric) revenue-sharing allocation rule.*

In Appendix B.1.1 we show that our approach and Algorithm 1 performance guarantee readily extends to any constant $k > 2$ user types. Also, note that Algorithm 1 searches for the optimal payout rate β in a grid over $[0,1]$. When the optimal payout rate is close to 1, the profit becomes small and finding a $(1-\epsilon)$-approximation of it requires a very fine grid for β. However, as already noted, for payout rates close to 1 the platform's profit is small. Therefore, by introducing an additive error we can obtain a Fully Polynomial Time Approximation Scheme (FPTAS). This is formally stated in the next corollary.

Corollary 2. *For any tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$ and any $\epsilon > 0$, Algorithm 1 with input $\delta = \frac{\epsilon}{4}$ finds a set of artists and a payout rate for pro-rata (or user-centric) rule whose profit is at most $O(\epsilon)$ away from the optimal profit in time* $\text{poly}\left(m, \frac{1}{\epsilon}\right)$.

The reformulation of the optimal pro-rata and user-centric rules in terms of the Knapsack problems enables us to compare the optimal profits under the two rules. In particular, in the next proposition we provide a sufficient condition which guarantees that the platform generates higher profits when using pro-rata allocation than when using user-centric allocation rule.

Proposition 5. *For any tuple $(\boldsymbol{\lambda}, \mathbf{q}, \boldsymbol{\Pi})$, if $\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{\lambda_1}{\lambda_2}$ for all $j \in \mathcal{M}$, then the optimal profit with pro-rata rule is (weakly) higher than the optimal profit with user-centric rule.*

Proposition 5 is an analogue of Proposition 3 when the platform chooses the optimal set of artists and the optimal payout rate. In particular, it shows that if there is a correlation between artist's popularity and overall consumption among users types, then pro-rata revenue allocation rule generates a (weakly) higher profit for the platform.

4. Arbitrary revenue allocation rules

When a platform is committed to using pro-rata (or user-centric) allocation, it has a limited ability to choose payments for its artists, i.e., it can only choose a single parameter β^{pr} (or β^{uc}). In fact, the sustainability of the platform (characterized in Proposition 1) might not be achievable with either of these two rules (as established in Theorem 1), while it is achievable with an arbitrary allocation rule (i.e., choosing the artists' payments without constraints on its functional form). Besides sustainability, choosing an arbitrary allocation rule can improve the platform's profit. To that end, a profit-maximizing platform will pay each of the artists on the platform no more than the value of their outside option, i.e., the revenue the artist would generate by going solo. Therefore, with an arbitrary allocation rule, the platform's profit maximization problem can be reformulated as choosing an optimal set of artists. We provide this reformulation in Section 4.1, and go on to utilize it to establish the complexity of the profit-maximization problem and to develop a polynomial-time approximation algorithm for the platform's profit maximization problem in Section 4.2.

4.1. Optimal set of artists

Similar to Lemma 2, our next lemma provides a reformulation of platform's profit-maximization problem with arbitrary allocation rules (i.e., problem (2)) in terms of the set of artists J who join the platform, thereby reducing the profit-maximizing platform's problem to determining the optimal set of artists J^*. Recall that R_j, defined in (1), denotes the revenue artist j can generate if going solo, which is characterized in Lemma 1.

LEMMA 4. *For any tuple $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$, let J^* and S^* be the solution of the problem:*

$$\max_{\substack{J \subseteq \mathcal{M} \\ S \subseteq \{1,2\}}} \left\{ \left(\min_{i \in S} \lambda_i \sum_{j \in J} \pi_{ij} \right) \left(\sum_{i \in S} q_i \right) - \sum_{j \in J} R_j \right\}. \tag{18}$$

Then by choosing the revenue allocation rule and the subscription fee

$$p_j(J^*) = \begin{cases} R_j & j \in J^* \\ 0 & j \notin J^* \end{cases} \quad \textit{and} \quad \textsc{Fee}(J^*) = \min_{i \in S^*} \left\{ \lambda_i \sum_{j \in J^*} \pi_{ij} \right\},$$

the set J^* of artists join and set S^* of user types subscribe to the platform, which is the optimal solution of the platform's problem (2).

Lemma 4 follows from the fact that the minimum payment to have artists in a set $J \subseteq \mathcal{M}$ on the platform is $\sum_{j \in J} R_j$ and the maximum subscription fee to have user types in a set S on the platform is $\min_{i \in S}\{\lambda_i \sum_{j \in J} \pi_{ij}\}$.

As manifested by problem (18), expanding the set of artists has two effects on the platform's profit: the platform can charge a higher subscription fee but needs to pay to more artists. The optimal solution needs to find the optimal tradeoff between these two effects. To understand the main challenge in solving problem (18) and develop an algorithm for solving it, it is useful to distinguish four cases based on the set S of user types that join the platform listed below:

- $S = \emptyset$: In this case, for any set of artists J, the profit is zero.
- $S = \{1\}$: In this case, problem (18) becomes

$$\max_{J \subseteq \mathcal{M}} \left\{ q_1 \lambda_1 \sum_{j \in J} \pi_{1j} - \sum_{j \in J} R_j \right\}. \tag{19}$$

Therefore, in this case, the platform's profit decouples to the contribution of individual artists.[11] The solution of problem (19) is to include all artists whose contribution to the platform's profit is non-negative, i.e., all $j \in \mathcal{M}$ such that $q_1\lambda_1\pi_{1j} - R_j \geq 0$. We denote this set of artists by J_1, i.e.,

$$J_1 = \left\{ j \in \mathcal{M} \ : \ q_1\lambda_1\pi_{1j} - R_j \geq 0 \right\}. \tag{20}$$

- $S = \{2\}$: In this case, problem (18) becomes

$$\max_{J \subseteq \mathcal{M}} \left\{ q_2\lambda_2 \sum_{j \in J} \pi_{2j} - \sum_{j \in J} R_j \right\}. \tag{21}$$

Therefore, analogous to the previous case, the platform's profit decouples to the contribution of individual artists. [12] The solution of problem (21) is to include all artists whose contribution to the platform's profit is non-negative, i.e., all $j \in \mathcal{M}$ such that $q_2\lambda_2\pi_{2j} - R_j \geq 0$. We denote this set of artists by J_2, i.e.,

$$J_2 = \left\{ j \in \mathcal{M} \ : \ q_2\lambda_2\pi_{2j} - R_j \geq 0 \right\}. \tag{22}$$

- $S = \{1,2\}$: In this case, problem (18) becomes

$$\max_{J \subseteq \mathcal{M}} \left\{ \min \left\{ \sum_{j \in J} \lambda_1\pi_{1j}, \sum_{j \in J} \lambda_2\pi_{2j} \right\} - \sum_{j \in J} R_j \right\}. \tag{23}$$

In this case, unlike the previous cases, the platform's profit does not decouple to the contributions of the individual artists. Problem (23) highlights the difficulty of the platform's problem in this two-sided market. On the artist side, the platform needs to pay R_j to artist j in order to have her content on the platform. On the user side, the subscription fee is determined by the user type with the minimum utility. Because users have heterogeneous preferences over artists, this combines contributions of different artists to the platform's profit.

The case $S = \{1,2\}$ points to the following complexity result.

PROPOSITION 6. *Problem (18) is NP-complete.*

[11] Note that we do not need to impose the constraint $\lambda_1 \sum_{j \in J} \pi_{1j} > \lambda_2 \sum_{j \in J} \pi_{2j}$ in Problem (19) because if the maximum of (18) is achieved for $S^* = \{1\}$ and J^*, it follows from (18) that $q_1\lambda_1 \sum_{j \in J^*} \pi_{1j} - \sum_{j \in J^*} R_j \geq \min\left\{\lambda_1 \sum_{j \in J^*} \pi_{1j}, \lambda_2 \sum_{j \in J^*} \pi_{2j}\right\} - \sum_{j \in J^*} R_j$. Hence, cancelling out $\sum_{j \in J^*} R_j$ on both sides, if this inequality holds then we must have $\lambda_1 \sum_{j \in J^*} \pi_{1j} > \lambda_2 \sum_{j \in J^*} \pi_{2j}$.

[12] Analogous to the case $S^* = \{1\}$, we do not need to impose the constraint $\lambda_2 \sum_{j \in J} \pi_{2j} > \lambda_1 \sum_{j \in J} \pi_{1j}$ in Problem (21).

We establish this by focusing on the case $S = \{1,2\}$ and, similar to Proposition 4, by reducing Subset Sum problem to problem (18). As established in the following Proposition, however, if there is limited heterogeneity of artists' popularity among user types, Problem (18) has a simple solution: all artists join the platform and both types of users subscribe to the platform.

PROPOSITION 7. *For any tuple* $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$*, if*

$$\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right] \qquad \text{for all } j \in \mathcal{M},$$

then the optimal solution of problem (18) *is* $J^* = \mathcal{M}$ *and* $S^* = \{1,2\}$*.*

Note that the condition of Proposition 7 is automatically satisfied when $\frac{\lambda_2\pi_{2j}}{\lambda_1\pi_{1j}} \approx 1$ for all artists $j \in \mathcal{M}$, i.e., if the overall consumption and popularity of artists is similar across user types. In such a case, the user types are essentially identical from both platform and artist perspective. Consequently, an artist considering going solo would not be able to target just one user type, while only the platform could unlock positive externalities from users' interests in multiple artists. On the other hand, if $\frac{\lambda_2\pi_{2j}}{\lambda_1\pi_{1j}}$ is very small, it would be profitable for artist j to attract user type 1 by going solo, and if $\frac{\lambda_2\pi_{2j}}{\lambda_1\pi_{1j}}$ is very large, it would be profitable for artist j to attract user type 2 by going solo.

4.2. Profit maximization

Here, we develop a polynomial-time algorithm that finds a revenue allocation rule and a subscription fee whose profit is at most $\max_{j\in\mathcal{M}} R_j$ away from the optimal profit, i.e., our approximation bound is in terms of maximum revenue a single artist could generate by going solo.

First, note that by using Lemma 4, the platform's problem reduces to finding J^*, the set of artists on the platform, and S^*, the set of user types who subscribe. Recall that if only one user type joins the platform, $S^* = \{1\}$ or $S^* = \{2\}$, the optimal set of artists J^* is readily established by (20) and (22), respectively. However, when $S^* = \{1,2\}$, the problem becomes the max-min optimization problem (23). To guide this, we exploit a relaxed formulation of problem (23) with fractional variables in which integer constraints are dropped:

$$\begin{aligned} \max_{z, x_1, \ldots, x_m} \; & z - \sum_{j=1}^m x_j R_j \\ \text{s.t. } & z \le \sum_{j=1}^m x_j \lambda_i \pi_{ij}, \quad i = 1,2 \\ & 0 \le x_j \le 1, \quad j = 1, \ldots, m. \end{aligned} \tag{24}$$

The solutions of linear programming problem (24) are not necessarily integral and potentially could have multiple fractional variables. Nonetheless, we show that it has a solution with at most one fractional variable and we develop an algorithm to find such solution in a computationally efficient way. In order to find such solution, somewhat reminiscent of Algorithm 1, we first define a score for each artist as

$$S_j(w_1,w_2) = w_1\lambda_1\pi_{1j} + w_2\lambda_2\pi_{2j} - R_j, \tag{25}$$

for a pair of weights $(w_1,w_2)\in[0,1]^2$ where $w_1+w_2=1$. We also let $J(w_1,w_2)$ be the set of artists with non-negative scores, i.e.,

$$J(w_1,w_2) = \{j\in\mathcal{M}\ :\ S_j(w_1,w_2)\geq 0\}.$$

We then show that there exist $\hat{w}_1$ and $\hat{w}_2$, determined by the algorithm, such that the profit obtained by the platform with the set of artists $J(\hat{w}_1,\hat{w}_2)$ on it, is at most R_j away from the optimal profit for some $j\in\mathcal{M}$.

More precisely, in order to find the weights $\hat{w}_1$ and $\hat{w}_2$, we derive an iterative algorithm based on a primal-dual analysis, summarized in Algorithm 2. We start by an initial pair $(w_1,w_2)=(1/2,1/2)$ and form the set $J(1/2,1/2)$ (the choice of the initialization is arbitrary). One of the user types has a (weakly) lower value for the artists in $J(1/2,1/2)$. Without loss of generality, we assume type 1 users have a lower value for this initial set of artists.[13] In order to balance the contribution of type 1 and type 2 users (to increase the fee the users pay and therefore increase the profit), in the next round we increase w_1 (and decrease w_2), which brings an artist j with higher $\lambda_1\pi_{1j}$ compared to $\lambda_2\pi_{2j}$ to the platform. We continue this process as long as user type 1 still prefers to pay a lower fee for the set of artists on the platform (or when we can no longer increase w_1).

The output of Algorithm 2 is a pair $(\hat{w}_1,\hat{w}_2)$ which readily defines the set of artists $J(\hat{w}_1,\hat{w}_2)$ and consequently, using Lemma 4, the revenue allocation rule and the subscription fee for the platform. Importantly, we prove that there exists $j^*\in J(\hat{w}_1,\hat{w}_2)$ such that for some $x_{j^*}\in[0,1]$, the variable $\mathbf{x}=(x_1,\ldots,x_m)$ defined as

$$x_j = \begin{cases} 1 & j\in J(\hat{w}_1,\hat{w}_2)\setminus\{j^*\} \\ 0 & j\notin J(\hat{w}_1,\hat{w}_2), \end{cases}$$

together with $(\hat{w}_1,\hat{w}_2)$, satisfies the complementary slackness condition for the problem (24) and its dual. This shows that variable $\mathbf{x}$ together with $(\hat{w}_1,\hat{w}_2)$ are primal-dual optimal solutions. This in turn also yields that the platform's profit is at most R_{j^*} away from the optimal profit.

More formally, we have the following lemma.

[13] Algorithm can simply be re-run by swapping the indices denoting user types which covers the complementary case.

Input: $\mathbf{q}=(q_1,q_2)$, $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)$, Π

Initialization: $w_1^{(0)}=w_2^{(0)}=\frac{1}{2}$

for $t=0,\ldots$ **do**

Find the minimum $\delta>0$ such that the sets $J(w_1^{(t)}+\delta, w_2^{(t)}-\delta)$ and $J(w_1^{(t)}, w_2^{(t)})$ differ in one artist. If no such δ exists, let $\delta=1-w_1^{(t)}$

$(w_1^{(t+1)}, w_2^{(t+1)}) \leftarrow (w_1^{(t)}+\delta, w_2^{(t)}-\delta)$

If $w_1^{(t+1)}=1$ **or** $\sum_{j\in J(w_1^{(t+1)},w_2^{(t+1)})} \lambda_1\pi_{1j} > \sum_{j\in J(w_1^{(t+1)},w_2^{(t+1)})} \lambda_2\pi_{2j}$

Output $(\hat{w}_1,\hat{w}_2)=(w_1^{(t+1)}, w_2^{(t+1)})$

end

Algorithm 2: Primal-dual approximation of the optimal platform's profit

LEMMA 5. *For any tuple* $(\boldsymbol{\lambda},\mathbf{q},\Pi)$, *let* $(\hat{w}_1,\hat{w}_2)$ *be the output of Algorithm 2. If an optimal solution of problem* (18) *has* $S^*=\{1,2\}$, *then* $(J(\hat{w}_1,\hat{w}_2),\{1,2\})$ *is a feasible solution of problem* (18) *and the corresponding platform's profit is at most* $\max_{j\in\mathcal{M}} R_j$ *away from the optimal profit. Moreover, Algorithm 2 terminates in* $O(m^2\log m)$ *time.*

Implementing Algorithm 2 requires simply sorting the artists based on their scores (25) at each iteration and we have at most m iterations.[14]

Recall that when the optimal solution of problem (18) has $S^*=\{1\}$ or $S^*=\{2\}$, establishing the corresponding optimal set of artists follows from (20) and (22), respectively. Thus, in those two cases, we establish the optimal solution of problem (18) by sorting the artists, analogous to the above. Therefore, combined with Lemma 5, we have the following performance guarantee for the platform's profit-maximization problem.

THEOREM 3. *For any tuple* $(\boldsymbol{\lambda},\mathbf{q},\Pi)$, *a feasible solution to problem* (18) *which guarantees platform's profit at most* $\max_{j\in\mathcal{M}} R_j$ *away from the optimal, can be found in* $O(m^2\log m)$ *time.*

A couple of features of our algorithm are worth noting. First, our algorithm (and Theorem 3) holds for any outside option revenue that artists may have, i.e., even if values R_j are exogenously given. Second, our score-based algorithm readily extends and provides the same approximation of the optimal profit for a platform which is "locked in" (e.g., contractually) with a subset of artists already "established" on the platform and seeks to choose an optimal subset of "new" artists to add to its artists' portfolio (see Appendix B.1.7).

[14] In fact, as we show in Appendix B.1.2, the generalization of Algorithm 2 for any constant $k>2$ user types generates a profit which is at most $(k-1)\max_{j\in\mathcal{M}} R_j$ away from the optimal profit.

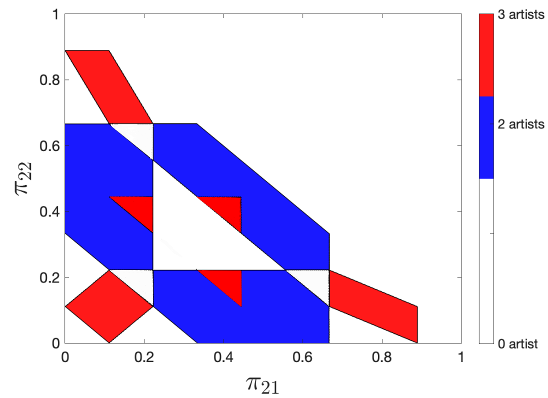


Figure 4 The number of artists on the platform with the optimal rule in the class of arbitrary allocation rules in the setting of Example 3. The x-axis is the probability of type 2 listening to artist 1 (i.e. π_{21}) and the y-axis is the probability of type 2 listening to artist 2 (i.e. π_{22}).

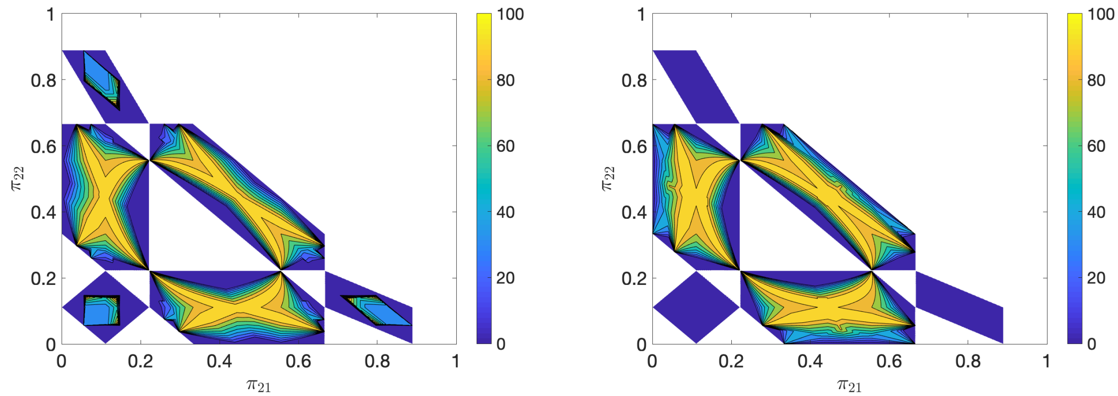
We conclude this section by comparing profit-maximizing platform's performance under pro-rata, user-centric, and arbitrary revenue allocation rules. The platform's ability to choose the optimal revenue allocation rule without any constraints, implies that the payment to each artist with either pro-rata or user-centric rules is (weakly) higher than her payment with the optimal rule in the class of arbitrary allocation rules. Consequently, the platform's profit with the optimal rule in the class of arbitrary allocation rules is (weakly) larger than the optimal profit with either pro-rata or user-centric rules. The next example illustrates the performance of pro-rata and user-centric rules compared to the optimal rule in the class of arbitrary allocation rules.

EXAMPLE 3. We consider the same setting as Example 2. Figure 4 depicts the number of artists on the profit-maximizing platform using an arbitrary allocation rule (i.e., $|J^*|$). Comparing this figure to Figure 3 of Example 2, we observe that with an arbitrary allocation rule the platform obtains non-zero profit in a wider region compared to either pro-rata and user-centric rules.

Figure 5 illustrates the performance of pro-rata and user-centric rules, by showing the optimal profit in these cases in terms of the percentage of the optimal profit in the class of arbitrary rules. We observe that the pro-rata allocation rule outperforms the user-centric for three regions corresponding to one of the artists being much more popular for type 2 users (either π_{21} is large or π_{22} is large or π_{23} is large). This again manifests the intuition that the platform prefers a pro-rata rule to a user-centric rule if the user behavior points to a correlation between artists' popularity and consumption among users.

5. Discussion of the model and extensions

In this section we present extensions of our analysis and results under generalizations of several modeling choices.



(a) (b)

Figure 5 **The percentage of the optimal profit (i.e., the profit obtained by the optimal rule in the class of arbitrary allocation rules) obtained by (a) the optimal pro-rata allocation rule and (b) the optimal user-centric allocation rule (in the white regions the profit of the platform with any arbitrary allocation rule is zero).**

5.1. Extension to more than two user types

In our baseline model, we consider two user types with different usage volumes. Here, we show that our results in Propositions 2 and 3 extend to more than two user types.

We order n user types according to their usage rates $\lambda_1 \leq \cdots \leq \lambda_n$ with the corresponding masses denoted by $q_1, \ldots, q_n$ (and $\sum_{i=1}^n q_i = 1$), and let

$$\bar{\lambda} = \sum_{i=1}^{n} q_i \lambda_i$$

be the average usage rate of all user types. We also let π_{ij} be the probability of type i users consuming content of artist j. In our baseline analysis with two types, the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ played an important rule. In the setting with n types, ratios denoting the popularity of artist j with type i users relative to the artist's popularity with type 1 users (the type with the lowest usage rate),

$$\frac{\pi_{2j}}{\pi_{1j}}, \ldots, \frac{\pi_{nj}}{\pi_{1j}}$$

are important to establish generalizations of our results.

We first provide the generalization of Assumption 1 that guarantees when all artists are on the platform, all users subscribe:

Assumption 2. The usage rate of user types and their masses is such that $\lambda_1 = \max_{k \in \mathcal{N}} \left\{ \lambda_k \sum_{i=k}^n q_i \right\}$.

The following generalizes Proposition 2 to a setting with $n \geq 2$ user types.

Proposition 8. *Suppose Assumption 2 holds and pro-rata and user-centric payments can sustain all artists on the platform with β^{pr} and β^{uc}, respectively. If the platform's revenue is the same with user-centric and pro-rata rules (i.e., $\beta^{\mathrm{pr}} = \beta^{\mathrm{uc}}$), then each artist j such that*

$$\sum_{i=1}^{n} q_i \pi_{ij} \left(\lambda_i - \bar{\lambda}\right) \geq 0 \tag{26}$$

prefers the pro-rata to user-centric rule. Moreover, for such artist j there exists a user type $i^* \in \mathcal{N}$ such that (26) remains to hold when π_{ij} is increased for $i \geq i^*$ and decreased for $i < i^*$.

Proposition 8 extends the result of Proposition 2 to $n > 2$ user types by establishing that artists who are more popular with high-volume users than with low-volume users prefer pro-rata to user-centric rule: $\left(\lambda_i - \bar{\lambda}\right)$ in (26) is positive if and only if the consumption of type i users is larger than the average consumption. For $n = 2$, (26) becomes $q_1\pi_{1j}\left(\lambda_1 - \bar{\lambda}\right) + q_2\pi_{2j}\left(\lambda_2 - \bar{\lambda}\right) \geq 0$, which holds if and only if $\pi_{2j} \geq \pi_{1j}$, the condition of Proposition 2 with $n = 2$ user types.

Th next example illustrates this result in a setting with $n = 3$ user types.

Example 4. Consider a setting with $n = 3$ user types and let Assumption 2 hold (i.e., $\lambda_1 \geq \max\{(q_2 + q_3)\lambda_2, q_3\lambda_3\}$). Unlike the setting with $n = 2$ user types, where artist j preference of pro-rata over user-centric rule is determined by the one-dimensional parameter $\frac{\pi_{2j}}{\pi_{1j}}$, with $n = 3$ we need to consider the two-dimensional parameter $\left(\frac{\pi_{2j}}{\pi_{1j}}, \frac{\pi_{3j}}{\pi_{1j}}\right)$. The (un)colored regions of Figure 6 depict the values of $\left(\frac{\pi_{2j}}{\pi_{1j}}, \frac{\pi_{3j}}{\pi_{1j}}\right)$ under which artist j prefers (user-centric) pro-rata rule. Observe that artist j prefers the pro-rata rule for a sufficiently large $\frac{\pi_{3j}}{\pi_{1j}}$, with the exact threshold depending on both the popularities and the usage rates of three user types. Finally, note that the threshold is decreasing (increasing) in $\frac{\pi_{2j}}{\pi_{1j}}$ if and only if λ_2 is larger (smaller) than the average listening rate $\bar{\lambda} = q_1\lambda_1 + q_2\lambda_2 + q_3\lambda_3$ (as noted in the discussion after Proposition 8, the critical consideration is whether the usage rate is larger than the average usage rate).

Finally, the following is a generalization of Proposition 3 to a setting with $n \geq 2$ user types.

Proposition 9. *Suppose Assumption 2 holds and pro-rata and user-centric payments can sustain all artists on the platform with β^{pr} and β^{uc}, respectively.*

(a) If the pivot artist j for pro-rata rule is such that $\sum_{i=1}^{n} q_i\pi_{ij}\left(\lambda_i - \bar{\lambda}\right) \geq 0$, then the platform's profit with pro-rata revenue allocation rule is weakly higher than its profit with user-centric.

(b) If the pivot artist j for user-centric rule is such that $\sum_{i=1}^{n} q_i\pi_{ij}\left(\lambda_i - \bar{\lambda}\right) \leq 0$, then the platform's profit with user-centric revenue allocation rule is weakly higher than its profit with pro-rata.

Note that the condition in part (a) of Proposition 9 is the same as (26). Therefore, the platform prefers pro-rata (user-centric) if and only if the pivot artist prefers pro-rata (user-centric), which is the same as what we established in Proposition 3 for the setting with $n = 2$ artists.

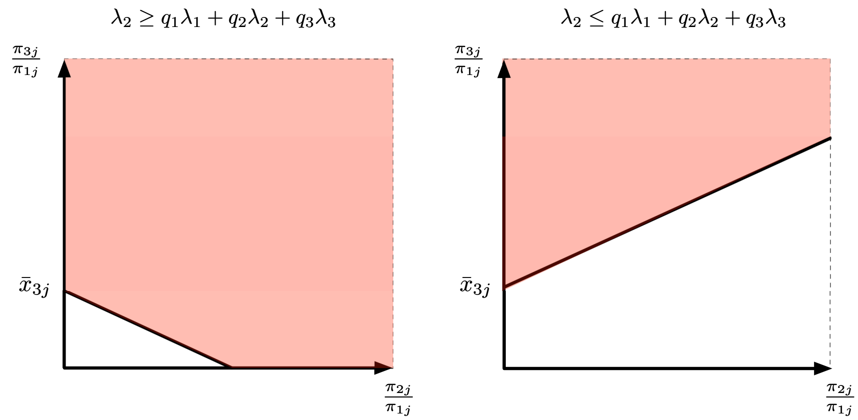


Figure 6 The colored region is the pairs $\left(\frac{\pi_{2j}}{\pi_{1j}}, \frac{\pi_{3j}}{\pi_{1j}}\right)$ for which artist j prefers pro-rata to user-centric. In this figure we have $\bar{x}_{3j} = \frac{q_1 q_2(\lambda_2-\lambda_1)+q_1 q_3(\lambda_3-\lambda_1)}{q_1 q_3(\lambda_3-\lambda_1)+q_2 q_3(\lambda_3-\lambda_2)}$ and the slope of the lines are $\mp\frac{q_1 q_2(\lambda_2-\lambda_1)-q_2 q_3(\lambda_3-\lambda_2)}{q_1 q_3(\lambda_3-\lambda_1)+q_2 q_3(\lambda_3-\lambda_2)}$.

5.2. Social welfare implications

We now address the impact of the platform on the social welfare. The (utilitarian) welfare is the sum of the utilities of users, the platform, and the artists. We refer to the maximum welfare as the *first-best*. Note that the maximum value for a type i user is λ_i and is achieved if the user is able to consume content from all artists $j \in \mathcal{M}$. The platform and the artists only generate value from transfers, so the user subscription fees and the artists' payments cancel out from the welfare. Consequently, the first-best is only achieved when all users consume the content from all artists, so it is equal to

$$q_1\lambda_1 + q_2\lambda_2.$$

In the next proposition we derive conditions under which we can implement the first-best with and without a platform.

PROPOSITION 10. *Suppose Assumption 1 holds.*

(a) Without a platform, the first-best is achievable if and only if

$$\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2 q_2}\right] \qquad \text{for all } j \in \mathcal{M}. \tag{27}$$

(b) With a platform, the first-best is achievable if and only if

$$\lambda_1 \geq \sum_{j \in \mathcal{M}} R_j \tag{28}$$

Consequently, the settings for which the first-best is achievable with a platform are a superset of the settings for which the first-best is achievable without a platform, i.e., if (27) holds, then (28) must also hold.

The proof of the proposition (in Appendix A) uses Lemma 1 which provides expressions for R_j. The intuition behind the proposition is that the platform creates value by coordinating two market sides, and could achieve the first-best (i.e., (28) holds) even if condition (27) is violated.

For a platform that uses pro-rata or user-centric, Theorem 1 provides the conditions under which the platform can achieve the first-best. Further, when the first-best is achievable with pro-rata (or user-centric), the payout rate $\beta \in [\beta^{\text{pr}}, 1]$ ($\beta \in [\beta^{\text{uc}}, 1]$). Among these first-best implementations, the platform's utility is maximized with payout rate β^{pr} (β^{uc}), while the artists' utilities are maximized with $\beta = 1$. As discussed after Theorem 1, pro-rata implements the first-best for a wider range of heterogeneity in artist popularity among user types (measured by $\frac{\pi_{2j}}{\pi_{ij}}$) than user-centric does. Also, as discussed in Proposition 2 and Corollary 1, pro-rata implementation of the first-best is preferred to user-centric by artist j and by the platform with the pivot artist j when $\frac{\pi_{2j}}{\pi_{ij}} \geq 1$. This indicates that pro-rata implementation of the first-best is preferred in presence of "super-star" artists, i.e., artists disproportionally popular with high consumption volume users.

5.3. The space of payment rules

5.3.1. Payments on the user side In our baseline model we consider a subscription-based unlimited-consumption service on the user side, which leads to decoupling of user prices and artists payments. Such user pricing model is the prevalent practice for streaming media platforms. This is because it allows locking-in users on the platform, generates a predictable revenue for the platform and a predictable expenditure on artist side, thereby attracting more users and increasing revenue (see e.g., Shiller and Waldfogel (2013), Danaher et al. (2014), Aguiar and Martens (2016), Talluri and Van Ryzin (2006, Chapter 10), and Belleflamme and Peitz (2015, Chapters 11 and 23) for a discussion of benefits of subscription-based services). Further, with our focus on the payment rules on the artist side, in our model we consider a single subscription fee on the user side (as opposed to a multi-tier subscription fee; the reasons for using a single-subscription fee and the problem of selecting the optimal multi-tier subscription service has been studied, among others, in Geng et al. (2005), Abdallah (2019), Lei and Swinney (2018), Alaei et al. (2019), and Abdallah et al. (2021)).[15]

An alternative to the subscription-based user pricing is a usage-based pricing (a.k.a, pay per view) in which users are charged per unit of consumption. The platform offering content from the set of artists J sets the rate it charges per unit of consumption: $\textsc{Rate}: 2^{\mathcal{M}} \to \mathbb{R}$. With the unit rate $\textsc{Rate}(\cdot)$, the utility of a type i user is

$$\lambda_i \sum_{j \in J} \pi_{ij} - \lambda_i \textsc{Rate}(J) \sum_{j \in J} \pi_{ij},$$

[15] Theoretically, a subscription fee tailored for every user type would allow for full revenue extraction from users. However, with large number of user types, this is infeasible and impractical, so a single subscription fee is set to cater to multiple user types.

and the artists' payments under pro-rata and user-centric rule coincide, as we formally establish in Appendix B.1.5. This is because the revenue generated is proportional to the consumption at the user level, which eliminates the cross-subsidization between low and high volume users.

However, even with usage-based subscription fee and the platform's function being reduced to facilitating transactions between users and artists on it, the platform still faces a revenue-maximization problem of determining the set of artists who will join the platform. In Appendix B.1.5, we show that a platform charging usage-based rates can find the optimal set of artists in at most $O(m \log m)$ time with an arbitrary allocation rule and with the allocation rule in the class of pro-rata/user-centric rules (as opposed to NP-completeness when charging subscription fee, as established in Propositions 4 and 6).

5.3.2. Payments on the artist side In our model, the value of artists' outside option, i.e., the revenue they can generate by going solo, is determining the minimum payment they need to receive if they are to join the platform. As we discussed in Section 3, some artists might not join a platform that uses pro-rata or user-centric allocation rules which are simple and widely used in practice. The simplicity of these allocation rules comes at the cost of not properly compensating some artists and consequently reducing the overall revenue potential for the platform. In Section 4, we consider the class of arbitrary allocation rules (not necessarily pro-rata or user-centric) and illustrate the potential loss due to commitment to either pro-rata or user-centric rules. However, even with arbitrary allocation rules, there are still settings in which the platform cannot be sustainable in the sense that the platform's revenue is insufficient to pay the outside option value R_j to each artist $j \in \mathcal{M}$.

Absent the outside option, revenue allocation and artists' payments could be based on their marginal contribution to the platform's revenue. In Appendix B.1.6, we show that our analysis extends to such setting with the outside option for an artist j suitably replaced by the marginal revenue that the platform can generate with artist j being on the platform. We also establish that artists' payments under such revenue allocation rule could be lower or higher than the artists' outside option in our baseline model, i.e., some artists are not incentivized to join the platform, while others may receive higher payments than their outside option.

Finally, if the outside option of artists is their private information, the platform would need to incentivize the artists to report it truthfully. In Appendix B.1.6 we present the VCG mechanism and the VCG payments to artists, and establish that with such privately held information, the VCG mechanism is not budget balanced, i.e., the platform is not sustainable with VCG payments to artists.

5.4. Costly service on the artist side

Media streaming platforms transact digital goods, implying zero marginal cost of user consumption imposed on content providers (artists). This is in contrast to other two-sided platforms where marginal cost to supply side might not be zero. For example, in ride-hailing platforms the suppliers, i.e., drivers, incur a cost every time they provide the service.[16] We next consider an extension of our model to a setting in which the content providers incur a cost for providing their services, and establish that our results extend to this setting. In particular, suppose that the content providers incur a cost which is a function of the set of users on the platform. Formally, we let

$$c_j : 2^{\mathcal{N}} \to \mathbb{R} \quad \text{ for all } j \in \mathcal{M}$$

represent a function that maps the set of users on the platform to artist j's cost for providing the service. Notice that this cost function also captures the volume and usage rate of different user types. For instance, if content provider j incurs a constant cost c_j per service, then we have $c_j(S) = c_j \sum_{i\in S} q_i \lambda_i \pi_{ij}$ for all $S \subseteq \mathcal{N}$.

We let $\tilde{R}_j$ be the revenue of going solo taking into account the cost of providing the service. The following is the analogue of Assumption 1 and guarantees that both user types subscribe.

Assumption 3. We assume $\lambda_1 - \sum_{j\in\mathcal{M}} c_j(\{1,2\}) \geq \lambda_2 q_2 - \sum_{j\in\mathcal{M}} c_j(\{2\})$.

The following proposition characterizes the conditions under which the platform can sustain all artists with costly services.

Proposition 11. *Suppose Assumption 3 holds.*

(a) The platform can sustain all content providers with an arbitrary rule if and only if

$$\max\left\{\lambda_1 - \sum_{j\in\mathcal{M}} c_j(\{1,2\}), \lambda_2 q_2 - \sum_{j\in\mathcal{M}} c_j(\{2\})\right\} - \sum_{j\in\mathcal{M}} \tilde{R}_j \geq 0,$$

(b) The platform can sustain all content providers with a pro-rata rule if and only if

$$\beta^{\text{pr}} = \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1} \max_{j\in\mathcal{M}} \left\{\frac{\tilde{R}_j + c_j(\{1,2\})}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}\right\} \leq 1,$$

(c) The platform can sustain all content providers with a user-centric rule if and only if

$$\beta^{\text{uc}} = \frac{1}{\lambda_1} \max_{j\in\mathcal{M}} \left\{\frac{\tilde{R}_j + c_j(\{1,2\})}{q_1\pi_{1j} + q_2\pi_{2j}}\right\} \leq 1.$$

The proof of the proposition is relegated to Appendix A. Further, as we prove in Appendix B.1.8, under Assumption 3, the results of Propositions 2 and 3 generalize to this setting with costly service. Also, in Appendix B.1.8 we show how our algorithms for finding the optimal pro-rata/user-centric rules extend to this setting by appropriately adjusting the outside option values.

[16] For ride-sharing platforms, different riders have mostly the same preference over drivers, i.e., we have $\pi_{1j} = \pi_{2j}$ for all $j \in \mathcal{M}$. This implies that user-centric and pro-rata rules coincide. In particular, for any set of artists $J \subset \mathcal{M}$ on the platform the payment of both pro-rata and user-centric rules for a payout rate β becomes $\beta \frac{\pi_{1j}}{\sum_{j\in J}\pi_{1j}}\text{Rev}$.

6. Concluding remarks

We develop a model to analyze revenue-sharing strategies for a two-sided media platform which generates revenues via the subscription fee it charges its users for unlimited consumption, and compensates content providers for making their content available. In our model we focus on music streaming platforms and compare the impact of pro-rata and user-centric revenue allocation rules (primary revenue-allocation rules used by market-leading music streaming platforms) on both platform's sustainability and artists' revenue. We find that the pro-rata allocation rule can be preferred to the user-centric rule, even though it does not allocate payments proportionally to the revenue generated by streaming of each artist. The reason for this is that pro-rata payments are better aligned with the added value that a platform featuring a large number of artists generates via positive externalities: by subscribing to a platform service, a user not only gets access to her favorite artist(s) but also to the content of all other artists on the platform.

We also consider the operational question of designing the optimal portfolio and optimal revenue-sharing allocation rules. When platform is free to select any revenue allocation rule, we use a relaxation of an integer programming together with a primal-dual approach to design and develop an easy-to-implement scheme that tightly approximates profit maximizing portfolio and payment rules. Similarly, for a platform that is committed to a pro-rata or user-centric payment allocation rule, we establish connections to Knapsack problem to design a Polynomial Time Approximation Scheme (PTAS) for platform's profit.

Our results suggest the adoption of pro-rata allocation rule if the user behavior points to correlation between artists' popularity and consumption among users, which is in contrast to the prevalent intuition due to the cross-subsidizing issue of pro-rata allocation rule. Finally, note that our work is motivated by music streaming, but the results (and in particular our algorithms) readily apply for any digital service platform setting (e.g.,video streaming) in which content/service consumption cannot readily be attributed to individual revenue sources and needs to be shared by service or content providers.

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Appendix A: Proofs

This Appendix contains all the omitted proofs.

Proof of Lemma 1 For any artist $j \in \mathcal{V}$, we have one of the following two cases:

1. $\pi_{2j}/\pi_{1j} \geq \lambda_1/\lambda_2$: In this case, the overall value of type 2 users for artist j is higher. Artist j will either offer a subscription fee equal to the value of type 2 users and incentivize them to subscribe or offers a subscription fee equal to the value of type 1 users to incentivize both types to subscribe. Therefore, the revenue of the artist is $\max\{\lambda_1\pi_{1j}, \lambda_2\pi_{2j}q_2\}$.
2. $\pi_{2j}/\pi_{1j} \leq \lambda_1/\lambda_2$: In this case, the overall value of type 1 users for artist j is higher. Artist j will either offer a subscription fee equal to the value of type 1 users and incentivize them to subscribe or offers a subscription fee equal to the value of type 2 users to incentivize both types to subscribe. Therefore, the revenue of the artist is $\max\{\lambda_2\pi_{2j}, \lambda_1\pi_{1j}q_1\}$.

Given this observation, we partition the artists into the following four subsets based on the characteristics of their monopoly market (i.e., when they offer their content through a direct channel):

- $\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{\lambda_1}{\lambda_2 q_2}$: For these artists, user type 2 has higher value and because $\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{\lambda_1}{\lambda_2 q_2}$, the subscription fee for the individual market is $\lambda_2\pi_{2j}$. The optimal revenue from the individual market is $\lambda_2 q_2 \pi_{2j}$ and in the individual market only users of type 2 subscribe.
- $\frac{\lambda_1}{\lambda_2} \leq \frac{\pi_{2j}}{\pi_{1j}} < \frac{\lambda_1}{\lambda_2 q_2}$: For these artists, users of type 2 still have higher value, but since $\frac{\pi_{2j}}{\pi_{1j}} < \frac{\lambda_1}{\lambda_2 q_2}$, the optimal revenue in individual market is $\lambda_1\pi_{1j}$ and in the individual market both user types subscribe.
- $\frac{q_1\lambda_1}{\lambda_2} \leq \frac{\pi_{2j}}{\pi_{1j}} < \frac{\lambda_1}{\lambda_2}$: For these artists users of type 1 have higher value, but since $\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{q_1\lambda_1}{\lambda_2}$, the optimal revenue in the individual market is $\lambda_2\pi_{2j}$ and in the individual market both user types subscribe.
- $\frac{\pi_{2j}}{\pi_{1j}} < \frac{q_1\lambda_1}{\lambda_2}$: For these artists, users of type 1 have higher values. Also, because $\frac{\pi_{2j}}{\pi_{1j}} < \frac{q_1\lambda_1}{\lambda_2}$, the optimal individual price is $\lambda_1\pi_{1j}$. The optimal revenue from individual market is $\lambda_1\pi_{1j}q_1$ and in the individual market only users of type 1 subscribe. ■

Proof of Proposition 1 For a subscription fee Fee, type i users subscribe to the platform if and only if $\text{Fee} \leq \lambda_i \sum_{j\in\mathcal{M}} \pi_{ij} = \lambda_i$. Any subscription fee will determine the subset of $\{1,2\}$ that subscribe to the platform. Therefore, instead of finding the optimal subscription fee, the platform can decide which subset of user types $\{1,2\}$ he wants to subscribe. In particular, if the platform decides to have set $S \subseteq \{1,2\}$ of user types to subscribe, then the optimal subscription fee would be $\min_{i\in S}\lambda_i$ and the mass of users who subscribe would be $\sum_{i\in S} q_i$. Given this and using $\lambda_2 \geq \lambda_1$, the optimal collected subscription fee of the platform becomes $\max\{\lambda_1, \lambda_2 q_2\}$. Using Lemma 1, with the minimum payment to artists the overall profit of the platform becomes

$$\max\{\lambda_1, \lambda_2 q_2\} - q_2\lambda_2 \sum_{j\in X}\pi_{2j} - \lambda_1 \sum_{j\in Y}\pi_{1j} - \lambda_2 \sum_{j\in Z}\pi_{2j} - \lambda_1 q_1 \sum_{j\in W}\pi_{1j}.$$

The platform can sustain all artists on it if and only if the above profit is non-negative. If $\lambda_1 \geq q_2\lambda_2$ and $\max_j \frac{\pi_{2j}}{\pi_{1j}} \leq \frac{\lambda_1}{\lambda_2 q_2}$, then there is no j in X and inequality (9) becomes

$$\lambda_1 - \lambda_1 \sum_{j\in Y}\pi_{1j} - \lambda_2 \sum_{j\in Z}\pi_{2j} - \lambda_1 q_1 \sum_{j\in W}\pi_{2j} \overset{(a)}{\geq} \lambda_1 - \lambda_1 \sum_{j\in Y}\pi_{1j} - \lambda_1 \sum_{j\in Z}\pi_{1j} - \lambda_1 q_1 \sum_{j\in W}\pi_{2j} \overset{(b)}{\geq} 0,$$

where (a) follows from the fact that for all $j \in Z$ we have $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{\lambda_1}{\lambda_2}$ and (b) follows from $\sum_{j\in\mathcal{M}}\pi_{2j} = \sum_{j\in Y}\pi_{1j} + \sum_{j\in Z}\pi_{1j} + \sum_{j\in W}\pi_{1j} = 1$. ■

Proof of Theorem 1 The theorem follows by combining the following two lemmas that characterize conditions to sustain all artists with either pro-rata and user-centric revenue allocation rules.

LEMMA 6. *For a given $\boldsymbol{\lambda}$ and $\mathbf{q}$, a platform with the pro-rata allocation rule is sustainable (i.e., $\beta^{\mathrm{pr}} \leq 1$) if and only if all of the following hold:*

- *For all artists $j \in X$, $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1 \lambda_1}{\lambda_2 q_2^2} \left(\frac{\lambda_1}{\lambda_2 - \lambda_1}\right)$.*
- *For all artists $j \in Y$, $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$.*
- *For all artists $j \in Z$, $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1 \lambda_1^2}{\lambda_2^2 q_2 + \lambda_1 \lambda_2 (q_1 - q_2)}$.*
- *For all artists $j \in W$, $\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{q_1(\lambda_2 - \lambda_1)}{\lambda_2}$.*

Proof of Lemma 6: The overall collected subscription fee by the platform is $\max\{\lambda_1, \lambda_2 q_2\} = \lambda_1$. This is because the platform either offers subscription fee λ_1 so that both user types subscribe ($\lambda_2 \geq \lambda_1$) or offers subscription $\lambda_2 q_2$ so that user type 2 subscribe. In order to be able to sustain all artists on the platform each artist j, $\lambda_1 \beta \frac{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}}{q_1 \lambda_1 + q_2 \lambda_2}$ should be larger than the revenue of the artist if she offers her content directly to users. Below, using Lemma 1 we list the four possible cases for the revenue of the artist if she offers her content by herself through direct channels and then find the minimum β:

- $j \in X$: To have $\beta \leq 1$, we must have

$$\frac{q_1 \lambda_1 + q_2 \lambda_2}{\lambda_1} \frac{\lambda_2 q_2 \pi_{2j}}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \leq 1.$$

This leads to $\frac{\pi_{21}}{\pi_{11}} \leq \frac{q_1 \lambda_1^2}{\lambda_2 q_2^2 (\lambda_2 - \lambda_1)}$.

- $j \in Y$: To have $\beta \leq 1$, we must have

$$\frac{q_1 \lambda_1 + q_2 \lambda_2}{\lambda_1} \frac{\lambda_1 \pi_{1j}}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \leq 1.$$

This leads to $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$.

- $j \in Z$: To have $\beta \leq 1$, we must have

$$\frac{q_1 \lambda_1 + q_2 \lambda_2}{\lambda_1} \frac{\lambda_2 \pi_{2j}}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \leq 1.$$

This leads to $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1}{\left(\frac{\lambda_2}{\lambda_1}\right)^2 q_2 + \frac{\lambda_2}{\lambda_1}(q_1 - q_2)}$.

- $j \in W$: To have $\beta \leq 1$, we must have

$$\frac{q_1 \lambda_1 + q_2 \lambda_2}{\lambda_1} \frac{\lambda_1 q_1 \pi_{1j}}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \leq 1.$$

This leads to $\frac{\pi_{2j}}{\pi_{1j}} \geq \frac{q_1(\lambda_2 - \lambda_1)}{\lambda_2}$, completing the proof. ■

LEMMA 7. *For a given $\boldsymbol{\lambda}$ and $\mathbf{q}$, a platform with the user-centric allocation rule is sustainable (i.e., $\beta^{\mathrm{uc}} \leq 1$) if and only if all of the following hold:*

- *For all artists $j \in X$, $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1}{q_2} \frac{\lambda_1}{\lambda_2 - \lambda_1}$.*
- *For all artists $j \in Y$, $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$.*
- *For all artists $j \in Z$, $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1}{\frac{\lambda_2}{\lambda_1} - q_2}$.*

Proof of Lemma 7: The overall collected subscription fee by the platform is $\max\{\lambda_1, \lambda_2 q_2\} = \lambda_1$. This is because the platform either offers subscription fee λ_1 so that both user types subscribe ($\lambda_2 \geq \lambda_1$) or offers subscription $\lambda_2 q_2$ so that user type 2 subscribe. In order to be able to sustain all artists on the platform each artist j, $\lambda_1 \beta \left(q_1 \pi_{1j} + q_2 \pi_{2j}\right)$ should be larger than the revenue of the artist if she offers her content by herself. Below, using Lemma 1, we list the four possible cases for the revenue of the artist if she offers her content by herself through direct channels and then find the minimum β:

- $j \in X$: To have $\beta \leq 1$, we must have

$$\frac{1}{\lambda_1} \frac{\lambda_2 q_2 \pi_{2j}}{q_1 \pi_{1j} + q_2 \pi_{2j}} \leq 1.$$

This leads to $\frac{\pi_{21}}{\pi_{11}} \leq \frac{q_1 \lambda_1}{q_2(\lambda_2 - \lambda_1)}$.

- $j \in Y$: To have $\beta \leq 1$, we must have

$$\frac{1}{\lambda_1} \frac{\lambda_1 \pi_{1j}}{q_1 \pi_{1j} + q_2 \pi_{2j}} \leq 1.$$

This leads to $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$.

- $j \in Z$: To have $\beta \leq 1$, we must have

$$\frac{1}{\lambda_1} \frac{\lambda_2 \pi_{2j}}{q_1 \pi_{1j} + q_2 \pi_{2j}} \leq 1.$$

This leads to $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{q_1}{\frac{\lambda_2}{\lambda_1} - q_2}$.

- $j \in W$: To have $\beta \leq 1$, we must have

$$\frac{1}{\lambda_1} \frac{\lambda_1 q_1 \pi_{1j}}{q_1 \pi_{1j} + q_2 \pi_{2j}} \leq 1.$$

This inequality always holds, completing the proof. ■

Proof of Proposition 2 Let β^{pr} and β^{uc} be the β's with which pro-rata and user-centric allocation rules can sustain all artists on the platform with pro-rata and user-centric allocation rules. Using equations (10) and (11), the ratio of the revenue that artist j obtains with pro-rata versus user-cenrtric is

$$\frac{\beta^{\text{pr}}}{\beta^{\text{uc}}(q_1\lambda_1 + q_2\lambda_2)} \frac{q_1 \lambda_1 \frac{\pi_{1j}}{\pi_{2j}} + q_2 \lambda_2}{q_1 \frac{\pi_{1j}}{\pi_{2j}} + q_2}.$$

The derivative of this ratio with respect to $\frac{\pi_{1j}}{\pi_{2j}}$ is

$$\frac{\beta^{\text{pr}}}{\beta^{\text{uc}}(q_1\lambda_1 + q_2\lambda_2)} \frac{q_1 q_2 (\lambda_1 - \lambda_2)}{(q_1 \frac{\pi_{1j}}{\pi_{2j}} + q_2)^2},$$

which is negative given $\lambda_2 \geq \lambda_1$. This shows that the ratio of artists' payment with pro-rata to their payment with user-centric is decreasing in $\frac{\pi_{1j}}{\pi_{2j}}$, showing that it is increasing in $\frac{\pi_{2j}}{\pi_{1j}}$. ■

Proof of Proposition 3 We first show that if β^{pr}, as given in (10), takes its maximum for some j with $\frac{\pi_{1j}}{\pi_{2j}} \leq 1$, then we have $\beta^{\text{pr}} \leq \beta^{\text{uc}}$. This follows because we have

$$\begin{aligned}
\beta^{\text{pr}} &= \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1} \max\left\{ \max_{j\in X} \frac{q_2 \lambda_2 \pi_{2j}}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}}, \max_{j \in Y \cup \{j:\, \frac{\pi_{1j}}{\pi_{2j}} \leq 1\}} \frac{\lambda_1 \pi_{1j}}{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}} \right\} \\
&\overset{(a)}{\leq} \frac{1}{\lambda_1} \max\left\{ \max_{j\in X} \frac{q_2 \lambda_2 \pi_{2j}}{q_1 \pi_{1j} + q_2 \pi_{2j}}, \max_{j \in Y \cup \{j:\, \frac{\pi_{1j}}{\pi_{2j}} \leq 1\}} \frac{\lambda_1 \pi_{1j}}{q_1 \pi_{1j} + q_2 \pi_{2j}} \right\} \\
&\leq \frac{1}{\lambda_1} \max\left\{ \max_{j\in X} \frac{q_2 \lambda_2 \pi_{2j}}{q_1 \pi_{1j} + q_2 \pi_{2j}}, \max_{j\in Y} \frac{\lambda_1 \pi_{1j}}{q_1 \pi_{1j} + q_2 \pi_{2j}}, \max_{j\in Z} \frac{\lambda_2 \pi_{2j}}{q_1 \pi_{1j} + q_2 \pi_{2j}}, \max_{j\in W} \frac{\lambda_1 q_1 \pi_{1j}}{q_1 \pi_{1j} + q_2 \pi_{2j}} \right\} = \beta^{\text{uc}},
\end{aligned}$$

where the inequality (a) holds because of the following: When we have $\frac{\pi_{1j}}{\pi_{2j}} \leq 1$, there are two possibilities for each artist j as listed below:

1. $j \in X$: In this case the β required to sustain this artist on the platform with pro-rata allocation rule (using equation (10)) is $\frac{q_1\lambda_1+q_2\lambda_2}{\lambda_1}\frac{q_2\lambda_2\pi_{2j}}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}$ and the β required to sustain this artist on the platform with user-centric allocation rule (using equation (11)) is $\frac{1}{\lambda_1}\frac{q_2\lambda_2\pi_{2j}}{q_1\pi_{1j}+q_2\pi_{2j}}$. The former is smaller than the latter because letting $x=\frac{\pi_{1j}}{\pi_{2j}}$ we have

$$\frac{q_1\lambda_1+q_2\lambda_2}{\lambda_1}\frac{q_2\lambda_2}{q_1\lambda_1 x+q_2\lambda_2}\leq\frac{1}{\lambda_1}\frac{q_2\lambda_2}{q_1 x+q_2},$$

where the inequality follows because after canceling out the common terms it becomes equivalent to $\lambda_1(1-x)\leq\lambda_2(1-x)$, which holds given $\lambda_1\leq\lambda_2$ and $x\leq 1$.

2. $j\in Y$: In this case the β required to sustain this artist on the platform with pro-rata allocation rule is $\frac{q_1\lambda_1+q_2\lambda_2}{\lambda_1}\frac{\lambda_1 y}{q_1\lambda_1 y+q_2\lambda_2}$ and the β required to sustain this artist on the platform with user-centric allocation rule is $\frac{1}{\lambda_1}\frac{\lambda_1 y}{q_1 y+q_2}$, where $y=\frac{\pi_{1j}}{\pi_{2j}}$. Again, it can be seen that the former is smaller than the latter.

Similarly, if β^{uc}, as given in (11), takes its maximum for some j with $\frac{\pi_{1j}}{\pi_{2j}}\geq 1$, then we have $\beta^{\mathrm{pr}}\geq\beta^{\mathrm{uc}}$. This again follows by comparing β^{uc} and β^{pr} for j such that $\frac{\pi_{1j}}{\pi_{2j}}\geq 1$. In particular, when we have $\frac{\pi_{1j}}{\pi_{2j}}\geq 1$, there are again two possibilities for each artist j as listed below:

1. $j\in Z$: In this case the β required to sustain this artist on the platform with pro-rata allocation rule is $\frac{q_1\lambda_1+q_2\lambda_2}{\lambda_1}\frac{\lambda_2}{q_1\lambda_1 z+q_2\lambda_2}$ and the β required to sustain this artist on the platform with user-centric allocation rule is $\frac{1}{\lambda_1}\frac{\lambda_2}{q_1 z+q_2}$ for $z=\frac{\pi_{1j}}{\pi_{2j}}$. The latter is smaller than the former because after canceling our the common terms it becomes equivalent to $\lambda_2(z-1)\geq\lambda_1(z-1)$, which holds given $\lambda_2\geq\lambda_1$ and $z\geq 1$.

2. $j\in W$: In this case the β required to sustain this artist on the platform with pro-rata allocation rule is $\frac{q_1\lambda_1+q_2\lambda_2}{\lambda_1}\frac{\lambda_1 q_1 w}{q_1\lambda_1 w+q_2\lambda_2}$ and the β required to sustain this artist on the platform with user-centric allocation rule is $\frac{1}{\lambda_1}\frac{\lambda_1 q_1 w}{q_1 w+q_2}$, where $w=\frac{\pi_{1j}}{\pi_{2j}}$. Again, it can be seen that the latter is smaller than the former. ■

Proof of Lemma 2

Proof of part (a): First note that if only one type of users subscribe, then the platform needs to pay all the revenue to the artists that join the platform (i.e., $\beta^{\mathrm{pr}}=1$). Therefore, both user types must join the platform.

With pro-rata allocation rule, the optimal β and J is the solution of the following optimization:

$$\max_{\substack{J\subseteq\mathcal{M}\\ \beta\in[0,1]}}\left(\min_{i\in\{1,2\}}\lambda_i\sum_{j\in J}\pi_{ij}\right)(1-\beta)$$
$$\text{s.t. }\beta\left(\min_{i\in\{1,2\}}\lambda_i\sum_{j\in J}\pi_{ij}\right)\frac{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}{q_1\lambda_1\sum_{j\in J}\pi_{1j}+q_2\lambda_2\sum_{j\in J}\pi_{2j}}\geq R_j,\qquad\text{for all } j\in J.$$

In order to have set J of artists joining the platform, for all $j\in J$ we must have

$$\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}\beta^{\mathrm{pr}}\frac{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}{\sum_{k\in J}q_1\lambda_1\pi_{1k}+\sum_{k\in J}q_2\lambda_2\pi_{2k}}\geq R_j.$$

This shows that the optimal β^{pr} is

$$\beta^{\mathrm{pr}}=\frac{1}{\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}}\max_{j\in J}\frac{R_j}{\frac{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}{\sum_{k\in J}q_1\lambda_1\pi_{1k}+\sum_{k\in J}q_2\lambda_2\pi_{2k}}}.$$

Therefore, the profit of the platform becomes

$$\begin{aligned}&\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}(1-\beta^{\text{pr}})\\&=\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}-\left(\sum_{k\in J}q_1\lambda_1\pi_{1k}+\sum_{k\in J}q_2\lambda_2\pi_{2k}\right)\max_{j\in J}\frac{R_j}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}.\end{aligned}$$

Proof of part (b): First, note that if only one type of users subscribe, then the platform needs to pay all the revenue to the artists that join the platform (i.e., $\beta^{\text{uc}}=1$). Therefore, both user types must join the platform.

With user-centric allocation rule, the optimal β and J is the solution of the following optimization:

$$\begin{aligned}&\max_{\substack{J\subseteq\mathcal{M}\\ \beta\in[0,1]}}\left(\min_{i\in\{1,2\}}\lambda_i\sum_{j\in J}\pi_{ij}\right)(1-\beta)\\&\text{s.t. }\beta\left(\min_{i\in\{1,2\}}\lambda_i\sum_{j\in J}\pi_{ij}\right)\left(q_1\frac{\pi_{1j}}{\sum_{j\in J}\pi_{1j}}+q_2\frac{\pi_{2j}}{\sum_{j\in J}\pi_{2j}}\right)\geq R_j,\quad\text{for all } j\in J.\end{aligned}$$

In order to have set J of artists joining the platform, for all $j\in J$ we must have

$$\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}\beta^{\text{uc}}\left(q_1\frac{\pi_{1j}}{q_1\sum_{k\in J}\pi_{1k}}+q_2\frac{\pi_{2j}}{q_2\sum_{k\in J}\pi_{2k}}\right)\geq R_j.$$

This shows that the optimal β^{uc} is

$$\beta^{\text{uc}}=\frac{1}{\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}}\max_{j\in J}\left\{\frac{R_j}{q_1\frac{\pi_{1j}}{q_1\sum_{k\in J}\pi_{1k}}+q_2\frac{\pi_{2j}}{q_2\sum_{k\in J}\pi_{2k}}}\right\}.$$

Therefore, the profit of the platform becomes

$$\begin{aligned}&\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}(1-\beta^{\text{uc}})\\&=\min\left\{\lambda_1\sum_{j\in J}\pi_{1j},\lambda_2\sum_{j\in J}\pi_{2j}\right\}-\max_{j\in J}\left\{\frac{R_j}{q_1\frac{\pi_{1j}}{q_1\sum_{k\in J}\pi_{1k}}+q_2\frac{\pi_{2j}}{q_2\sum_{k\in J}\pi_{2k}}}\right\}.\end{aligned}$$

This completes the proof. ■

Proof of Proposition 4 The proof of this proposition is identical to the proof of Proposition 6 by noting that the optimal set of artists that solves the subset problem generates the same profit with either pro-rata or user-centric allocation rules as shown next.

Pro-rata: Suppose set J of $\{1,\ldots,m\}$ is such that $\sum_{j\in J}w_j=W$. We show that a pro-rata allocation strategy with $\beta^{\text{pr}}=\frac{1}{1+\delta}$ generates profit δW.

With pro-rata allocation rule, the payment of an artist $j\in J$ becomes

$$\frac{\textsc{Rev}}{1+\delta}\frac{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}{\sum_{j\in J\cup\{m+1\}}q_1\lambda_1\pi_{1j}+\sum_{j\in J\cup\{m+1\}}q_2\lambda_2\pi_{2j}}\overset{(a)}{=}W\frac{\frac{1}{2}w_j+\frac{1}{2}\delta w_j}{\frac{1}{2}W(1+\delta)}=\frac{1}{2}w_j=R_j,$$

where (a) follows from the choices of probabilities and usage and $\textsc{Rev} = (1+\delta)W$. With pro-rata allocation rule, the payment of artist $m+1$ becomes

$$\frac{\textsc{Rev}}{1+\delta}\frac{q_1\lambda_1\pi_{1(m+1)} + q_2\lambda_2\pi_{2(m+1)}}{\sum_{j\in J\cup\{m+1\}} q_1\lambda_1\pi_{1j} + \sum_{j\in J\cup\{m+1\}} q_2\lambda_2\pi_{2j}} \stackrel{(a)}{=} W\frac{\frac{1}{2}\delta W + \frac{1}{2}W}{\frac{1}{2}W(1+\delta)} = \frac{1}{2}W = R_j.$$

User-centric: Suppose set J of $\{1,\dots,m\}$ is such that $\sum_{j\in J} w_j = W$. We show that a user-centric allocation strategy with $\beta^{\mathrm{uc}} = \frac{1}{1+\delta}$ generates profit δW.

Similarly, with user-centric allocation rule, the payment of an artist $j \in J$ becomes

$$\begin{aligned}\frac{\textsc{Rev}}{1+\delta}\left(q_1\frac{\pi_{1j}}{\sum_{j\in J\cup\{m+1\}}\pi_{1j}} + q_2\frac{\pi_{2j}}{\sum_{j\in J\cup\{m+1\}}\pi_{2j}}\right) &= W\left(\frac{1}{2}\frac{w_j}{(1+\delta)W} + q_2\frac{\delta w_j}{(1+\delta)W}\right)\\ &= \frac{1}{2}w_j = R_j.\end{aligned}$$

With user-centric allocation rule, the payment of artist $m+1$ becomes

$$\begin{aligned}\frac{\textsc{Rev}}{1+\delta}\left(q_1\frac{\pi_{1(m+1)}}{\sum_{j\in J\cup\{m+1\}}\pi_{1j}} + q_2\frac{\pi_{2(m+1)}}{\sum_{j\in J\cup\{m+1\}}\pi_{2j}}\right) &= W\left(\frac{1}{2}\frac{\delta W}{(1+\delta)W} + \frac{1}{2}\frac{W}{(1+\delta)W}\right)\\ &= \frac{1}{2}W = R_{m+1}.\end{aligned}$$

This completes the proof. ■

Proof of Lemma 3 We show the proof for pro-rata allocation rule. An identical argument shows the result for user-centric. For a given $\epsilon > 0$, we show the theorem for $\delta = \frac{\epsilon(1-\bar{\beta})}{4}$ where $\bar{\beta} \in [0,1)$ defined in the next claim.

Claim 1: When profit is non-zero, there exists a $\bar{\beta} < 1$ such that $\beta^{\mathrm{pr}} \le \bar{\beta}$.

Proof of Claim 1: For any $J \subseteq \mathcal{M}$, we let β_J be the minimum payout rate with which the platform can sustain artists in the set J on the platform. This claim follows by letting

$$\bar{\beta} = \max\left\{\beta_J \;:\; J\subseteq\mathcal{M},\ \beta_J < 1\right\},$$

completing the proof of Claim 1. ■

Claim 2: We have $\alpha^{\mathrm{pr}} \le \frac{\lambda_2}{\lambda_1}\sum_{j\in\mathcal{M}}\frac{\pi_{2j}}{\pi_{1j}}$.

Proof of Claim 2: Using Cauchy-Shwarz inequality we have

$$\left(\sum_{j\in J^*}\frac{\pi_{2j}}{\pi_{1j}}\right)\left(\sum_{j\in J^*}\pi_{1j}\right) \ge \left(\sum_{j\in J^*}\sqrt{\pi_{2j}}\right)^2 \ge \sum_{j\in J^*}\pi_{2j},$$

which leads to

$$\alpha^{\mathrm{pr}} = \frac{\lambda_2}{\lambda_1}\frac{\sum_{j\in J^*}\pi_{2j}}{\sum_{j\in J^*}\pi_{1j}} \le \frac{\lambda_2}{\lambda_1}\left(\sum_{j\in J^*}\frac{\pi_{2j}}{\pi_{1j}}\right) \le \frac{\lambda_2}{\lambda_1}\left(\sum_{j\in\mathcal{M}}\frac{\pi_{2j}}{\pi_{1j}}\right),$$

completing the proof of Claim 1. ■

This claim explains the choice of the grid of α over the interval

$$\left[1, \frac{\lambda_2}{\lambda_1}\left(\sum_{j\in\mathcal{M}}\frac{\pi_{2j}}{\pi_{1j}}\right)\right].$$

We denote $\frac{\lambda_2}{\lambda_1}\left(\sum_{j\in\mathcal{M}}\frac{\pi_{2j}}{\pi_{1j}}\right)$ by $\bar{\alpha}$.

We now proceed with the proof of theorem. By searching over the δ-grids we cover parameters β and α such that

$$\alpha^{\mathrm{pr}} \le \alpha \le \alpha^{\mathrm{pr}} + \delta, \tag{29}$$

and

$$\beta^{\mathrm{pr}} + \delta \le \beta \le \beta^{\mathrm{pr}} + 2\delta. \tag{30}$$

We first show that with these β and α, we have $\mathcal{M}(\beta,\alpha) \subseteq \mathcal{M}(\beta^{\mathrm{pr}},\alpha)$. This is because we have

$$\frac{\beta^{\mathrm{pr}}}{q_1+q_2\alpha^{\mathrm{pr}}} \overset{(a)}{\le} \frac{\beta^{\mathrm{pr}}+\delta}{q_1+q_2(\alpha^{\mathrm{pr}}+\delta)} \overset{(b)}{\le} \frac{\beta}{q_1+q_2\alpha},$$

where (a) follows from simple algebra and (b) follows from the choices in (29) and (30). Therefore, in the following Knapsack problem

$$\begin{aligned} &\max_{J\subseteq\mathcal{M}^{\mathrm{pr}}(\beta,\alpha)} (1-\beta)\lambda_1\sum_{j\in J}\pi_{1j} \\ &\text{s.t. } \lambda_1\sum_{j\in J}\pi_{1j} \le \alpha\lambda_2\sum_{j\in J}\pi_{2j}, \end{aligned}$$

the optimal set J^* is feasible. Letting $\tilde{J}$ denote the ϵ-approximation solution of this Knapsack problem and $\hat{J}$ the optimal solution of this Knapsack problem, we can write

$$\begin{aligned} (1-\beta)\lambda_1\sum_{j\in\tilde{J}}\pi_{1j} &\overset{(a)}{\ge} (1-\beta)\left((1-\frac{\epsilon}{2})\lambda_1\sum_{j\in\hat{J}}\pi_{1j}\right) \\ &\overset{(b)}{\ge} (1-\beta)\left((1-\frac{\epsilon}{2})\lambda_1\sum_{j\in J^*}\pi_{1j}\right) \\ &\overset{(c)}{\ge} (1-\beta^{\mathrm{pr}})(1-\frac{\epsilon}{2})\left((1-\frac{\epsilon}{2})\lambda_1\sum_{j\in J^*}\pi_{1j}\right) \\ &\ge (1-\beta^{\mathrm{pr}})\left((1-\epsilon)\lambda_1\sum_{j\in J^*}\pi_{1j}\right) \\ &= (1-\epsilon)\times(\text{optimal profit with pro-rata rule}), \end{aligned}$$

where (a) follows from the ϵ-approximation of solving the Knapsack problem, (b) follows from the fact that J^* is feasible for the Knapsack problem, and (c) follows from the choice of $\delta = \frac{\epsilon(1-\bar{\beta})}{4}$.

Notice that each instance of the Knapsack problem can be solved in time $O(\frac{\mathrm{poly}(m)}{\epsilon})$ and there are $\frac{1}{\delta}\frac{\bar{\alpha}}{\delta} = \frac{16\bar{\alpha}}{(1-\bar{\beta})^2\epsilon^2}$ many iterations, resulting in running time $O(\frac{\mathrm{poly}(m)}{\epsilon^3})$. Finally, we made the assumption

$$\lambda_1\sum_{j\in J^*}\pi_{1j} \le \lambda_2\sum_{j\in J^*}\pi_{2j}$$

and developed the algorithm. By swapping the user types and re-running the algorithm we cover the other case, i.e.,

$$\lambda_2\sum_{j\in J^*}\pi_{2j} \le \lambda_1\sum_{j\in J^*}\pi_{1j}.$$

This complete the proof. ■

Proof of Theorem 2 This theorem directly follows from Lemma 3. ∎

Proof of Corollary 2 Using the same argument as the proof of Theorem 2, we can write

$$\begin{aligned}
(1-\beta)\lambda_1 \sum_{j\in \tilde{J}} \pi_{1j} &\overset{(a)}{\geq} (1-\beta)\left((1-\frac{\epsilon}{2})\lambda_1 \sum_{j\in \hat{J}} \pi_{1j}\right) \\
&\overset{(b)}{\geq} (1-\beta)\left((1-\frac{\epsilon}{2})\lambda_1 \sum_{j\in J^*} \pi_{1j}\right) \\
&\overset{(c)}{\geq} (1-\beta^{\mathrm{pr}}-\frac{\epsilon}{2})\left((1-\frac{\epsilon}{2})\lambda_1 \sum_{j\in J^*} \pi_{1j}\right) \\
&\geq (1-\beta^{\mathrm{pr}})\left(\lambda_1 \sum_{j\in J^*} \pi_{1j}\right) - \epsilon\lambda_1 \sum_{j\in J^*} \pi_{1j} \\
&\overset{(d)}{\geq} (\text{optimal profit with pro-rata rule}) - \epsilon\lambda_2,
\end{aligned}$$

where (a) follows from the ϵ-approximation of solving the Knapsack problem, (b) follows from the fact that J^* is feasible for the Knapsack problem, (c) follows from $\beta \leq \beta^{\mathrm{pr}} + \epsilon/2$, and (d) follows from $\lambda_2 \geq \lambda_1$ and $\sum_{j\in J^*} \pi_{1j} \leq 1$. ∎

Proof of Proposition 5 The problem of finding the optimal pro-rata rule can be fomulated as the following Knapsack problem:

$$\begin{aligned}
\max_{J\subseteq \mathcal{M}_+^{\mathrm{pr}}(\beta^{\mathrm{pr}},\alpha^{\mathrm{pr}})} &(1-\beta^{\mathrm{pr}})\lambda_1 \sum_{j\in J} \pi_{1j} + (1-\beta^{\mathrm{pr}})\lambda_1 \sum_{j\in \mathcal{M}_-^{\mathrm{pr}}(\beta^{\mathrm{pr}},\alpha^{\mathrm{pr}})} \pi_{1j} \\
\text{s.t. } &\sum_{j\in J} \left(\lambda_1\pi_{1j} - \alpha^{\mathrm{pr}}\lambda_2\pi_{2j}\right) \leq \sum_{j\in \mathcal{M}_-^{\mathrm{pr}}(\beta^{\mathrm{pr}},\alpha^{\mathrm{pr}})} \left(\alpha^{\mathrm{pr}}\lambda_2\pi_{2j} - \lambda_1\pi_{1j}\right),
\end{aligned} \tag{31}$$

The problem of finding the optimal user-centric rule, on the other hand, can be formulated as the following Knapsack problem:

$$\begin{aligned}
\max_{J\subseteq \mathcal{M}_+^{\mathrm{uc}}(\beta^{\mathrm{uc}},\alpha^{\mathrm{uc}})} &(1-\beta^{\mathrm{uc}})\lambda_1 \sum_{j\in J} \pi_{1j} + (1-\beta^{\mathrm{uc}})\lambda_1 \sum_{j\in \mathcal{M}^{-}(\beta^{\mathrm{uc}},\alpha^{\mathrm{uc}})} \pi_{1j} \\
\text{s.t. } &\sum_{j\in J} \left(\lambda_1\pi_{1j} - \alpha^{\mathrm{uc}}\lambda_2\pi_{2j}\right) \leq \sum_{j\in \mathcal{M}_-^{\mathrm{uc}}(\beta^{\mathrm{uc}},\alpha^{\mathrm{uc}})} \left(\alpha^{\mathrm{uc}}\lambda_2\pi_{2j} - \lambda_1\pi_{1j}\right).
\end{aligned} \tag{32}$$

Comparing the formulations (31) and (32), we see that for a given β and α the objective as well as the constraints on α in these two formulations are the same. Therefore, to show the proposition we need to prove $\mathcal{M}^{\mathrm{uc}}(\beta,\alpha) \subseteq \mathcal{M}^{\mathrm{pr}}(\beta,\alpha)$. For a given $j \in \lambda_1\pi_{1j}$, we have

$$\frac{R_j}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}} \leq \frac{\beta}{q_1 + q_2\alpha},$$

which, by using $\alpha \geq 1$, is equivalent to

$$R_j \geq \frac{\alpha R_j - \beta(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j})}{(\alpha-1)q_1} \tag{33}$$

For a given $j \in \mathcal{M}^{\mathrm{uc}}(\beta,\alpha)$, we have

$$\frac{R_j - \beta q_1\lambda_1\pi_{1j}}{\beta q_2\lambda_2\pi_{2j}} \leq \frac{1}{\alpha},$$

which, by using $\alpha \geq 1$, is equivalent to

$$\beta\lambda_1\pi_{1j} \geq \frac{\alpha R_j - \beta(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j})}{(\alpha-1)q_1}. \tag{34}$$

Comparing (33) and (34) for $j \in Y \cup X$ we have $R_j \geq \lambda_1\pi_{1j} \geq \beta\lambda_1\pi_{1j}$. Therefore, if $j \in \mathcal{M}^{\mathrm{uc}}(\beta,\alpha)$, then $\mathcal{M}^{\mathrm{pr}}(\beta,\alpha)$, completing the proof. ■

Proof of Lemma 4 Consider the optimal allocation rule and subscription rule of problem (2). These will imply that a set $\hat{J} \subseteq \mathcal{M}$ of artists join the platform and set $\hat{S} \subseteq \{1,2\}$ of user types subscribe. The payment of the platform to each artist $j \in \hat{J}$ is at least the revenue they obtained by going solo that is $\max_r \ r\ d(j,r)$. Also, the maximum subscription fee to incentivize user types $\hat{S}$ to subscribe is $\min_{i\in\hat{S}} \lambda_i \sum_{j\in\hat{J}} \pi_{ij}$. Therefore, the profit of the platform, i.e., the objective in problem (2) is smaller than or equal to

$$\left(\min_{i\in\hat{S}} \lambda_i \sum_{j\in\hat{J}} \pi_{ij}\right)\left(\sum_{i\in\hat{S}} q_i\right) - \sum_{j\in\hat{J}} R_j,$$

where $R_j = \max_r \ r\ d(j,r)$. This establishes that the optimal objective in problem (2) is smaller than or equal to the optimal objective of problem (18).

We next show the other direction. Consider the optimal solution of problem (18) denoted by sets J^* and S^*. With the payment rule

$$p_j(J^*) = \begin{cases} R_j & j \in J^* \\ 0 & j \notin J^*, \end{cases}$$

and the subscription fee

$$\textsc{Fee}(J^*) = \min_{i\in S^*} \lambda_i \sum_{j\in J^*} \pi_{ij},$$

set J^* of artists join the platform and set S^* of user types subscribe. The profit of the platform is

$$\left(\min_{i\in S^*} \lambda_i \sum_{j\in J^*} \pi_{ij}\right)\left(\sum_{i\in S^*} q_i\right) - \sum_{j\in J^*} R_j.$$

This completes the proof. ■

Proof of Proposition 6 For completeness, we first include the definition of subset sum and then prove our hardness result.

Definition 3 (Subset sum). For a set of non-negative integers $\{w_1,\ldots,w_m\}$ and a given $W \in \mathbb{N}^+$, the subset sum problem is to find whether there exists a subset of this set whose summation is equal to W.

For a given set $\{w_1,\ldots,w_m\}$ and $W \in \mathbb{N}^+$, we consider the a set of artists $\{1,\ldots,m+1\}$ with the following probabilities and usage rates:

We suppose $q_1 = q_2 = 1/2$ and let

$$\pi_{11} = \frac{w_1}{\lambda_1},\ldots,\pi_{1m} = \frac{w_m}{\lambda_1},\pi_{1(m+1)} = \frac{\delta W}{\lambda_1},$$
$$\pi_{21} = \frac{\delta w_1}{\lambda_2},\ldots,\pi_{2m} = \frac{\delta w_m}{\lambda_2},\pi_{1(m+1)} = \frac{W}{\lambda_2},$$

for $\delta < \frac{1}{2}$ and

$$\lambda_1 = \delta W + \sum_{j=1}^{m} w_j, \quad \lambda_2 = W + \delta \sum_{j=1}^{m} w_j.$$

Claim 1: In the optimal solution of problem (18), both type of users subscribe.

Proof of Claim 1: first note that the revenue from offering their content solo through direct channels are

$$R_1 = q_1\lambda_1\pi_{11} = \frac{w_1}{2}, \ldots, R_m = q_1\lambda_1\pi_{1m} = \frac{w_m}{2}, R_{m+1} = q_2\lambda_2\pi_{2j} = \frac{W}{2}.$$

If only type 1 users subscribe and if set J of artists are on the platform then the platform's profit becomes

$$q_1\lambda_1 \sum_{j\in J} \pi_{1j} - \sum_{j\in J} R_j \overset{(a)}{\leq} \max\left\{0, q_1\lambda_1\pi_{1(m+1)} - q_2\lambda_2\pi_{2(m+1)}\right\} \overset{(b)}{=} \max\left\{0, (\delta-1)W\right\} \leq 0,$$

where (a) follows from the fact that for all artists in $\{1,\ldots,m\}$ we have $R_j = q_1\lambda_1\pi_{1j}$ and depending of whether $m+1 \in J$ or not the profit will be bounded by one of the terms in the maximum and (b) follows from the choice of λ_1, λ_2 and $\pi_{1(m+1)}, \pi_{2(m+1)}$.

Similarly, it is never optimal to have only user type 2 subscribing. ■.

Claim 2: If there exists a subset of $\{1,\ldots,m\}$ such that

$$\sum_{j\in J} w_j = W, \tag{35}$$

then the optimal platform's profit is equal to δW.

Proof of Claim 2: First note that if such a subset exists, then by choosing the set of artists $J \cup \{m+1\}$ the platform's profit becomes

$$\begin{aligned}
&\min\left\{\lambda_1 \sum_{j\in J\cup\{m+1\}} \pi_{1j}, \lambda_2 \sum_{j\in J\cup\{m+1\}} \pi_{2j}\right\} - \sum_{j\in J\cup\{m+1\}} R_j \\
&\overset{(a)}{=} \min\left\{\sum_{j\in J} w_j + \delta W, \delta\sum_{j\in J} w_j + W\right\} - \sum_{j\in J} \frac{1}{2}w_j - \frac{1}{2}W \\
&\overset{(b)}{=} (1+\delta)W - W = \delta W
\end{aligned}$$

where (a) follows form the choices of probabilities and usage rates and (b) follows from (35). We next show that the platform cannot have a profit more than δW. Suppose in the optimal solution, the set of artists who are on the platform is $J' \cup \{m+1\}$ where $J' \subseteq \{1,\ldots,m\}$. Using Claim 1, both type of users must subscribe. Suppose, without loss of generality, $W' = \sum_{j\in J'} w_j > W$. The profit of the platform can be written as

$$\begin{aligned}
&\min\left\{\lambda_1 \sum_{j\in J'\cup\{m+1\}} \pi_{1j}, \lambda_2 \sum_{j\in J'\cup\{m+1\}} \pi_{2j}\right\} - \sum_{j\in J'\cup\{m+1\}} R_j \\
&= \min\left\{\sum_{j\in J'} w_j + \delta W, \delta\sum_{j\in J'} w_j + W\right\} - \sum_{j\in J'} \frac{1}{2}w_j - \frac{1}{2}W \\
&= \min\left\{W' + \delta W, \delta W' + W\right\} - \frac{1}{2}W' - \frac{1}{2}W \\
&\overset{(a)}{=} \delta W' + W - \frac{1}{2}W' - \frac{1}{2}W \\
&\overset{(b)}{<} (\delta - \frac{1}{2})W + \frac{1}{2}W = \delta W,
\end{aligned}$$

where (a) follows from $W' > W$ (an identical proof works for the other direction) and (b) follows form $\delta < \frac{1}{2}$ and $W' > W$. ■

Using Claims 1 and 2, by solving problem (18) and evaluating whether the optimal profit is $(1+\delta)W$, we can solve the subset problem we started from. ■

Proof of Proposition 7 The optimal set S in the solution of problem 23 has four possibilities as follows: (i) $S = \emptyset$, (ii) $S = \{1,2\}$, (iii) $S = \{1\}$, and (iv) $S = \{2\}$. The first case gives a zero profit profit for the platform. We next show that in the second, the optimal set J is the entire set of artists $\mathcal{M}$. Finally, we show that the optimal set J in cases (iii) and (iv) gives a smaller profit than the one in case (ii).

With set $S = \{1,2\}$, the objective of problem (23) becomes

$$\min\left\{\sum_{j\in J}\lambda_1\pi_{1j}, \sum_{j\in J}\lambda_2\pi_{2j}\right\} - \sum_{j\in J} R_j.$$

Note that

$$\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right], \quad \text{for all } j \in \mathcal{M},$$

implies that $j \in Y \cup Z$ for all $j \in \mathcal{M}$. For $j \in Y \cup Z$, we have $\min\{\lambda_1\pi_{1j}, \lambda_2\pi_{2j}\} \geq R_j$. Therefore, it is optimal to include all artists (which belong in $Y \cup Z$) on the platform. Using Lemma 1 to replace the value of R_j, the optimal profit in this case becomes

$$\min\left\{\sum_{j\in\in Y\cup Z}\lambda_1\pi_{1j}, \sum_{j\in Y\cup Z}\lambda_2\pi_{2j}\right\} - \sum_{j\in Y}\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j}. \tag{36}$$

We next show that the optimal profit in case (iii) is smaller than (36) (similarly, the optimal profit in case (iv) is also smaller than (36)). In case (iii), the objective of problem (23) becomes

$$\sum_{j\in\cup Z} q_1\lambda_1\pi_{1j} - \sum_{j\in Y}\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j}.$$

Since for all artists $j \in Y$, we have $q_1\lambda_1\pi_{1j} < \lambda_1\pi_{1j}$, in the optimal set the optimal does not include artists from set Y. On the other hand, for all artists $j \in Z$, we have $q_1\lambda_1\pi_{1j} \geq \lambda_2\pi_{2j}$ and therefore, the platform will include those artists. Therefore, the optimal profit in this case becomes

$$\sum_{j\in Z} q_1\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j}, \tag{37}$$

which is smaller than (36). This is because if we consider either of the terms in the minimum in (36) it is larger than (37) as shown next:

$$\begin{aligned}
\sum_{j\in Y\cup Z}\lambda_1\pi_{1j} - \sum_{j\in Y}\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j} &= \sum_{j\in Z}\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j} \\
&\geq \sum_{j\in Z} q_1\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j\in Y\cup Z}\lambda_2\pi_{2j} - \sum_{j\in Y}\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j} &\overset{(a)}{\geq} \sum_{j\in Z}\lambda_2\pi_{2j} - \sum_{j\in Z}\lambda_2\pi_{2j} \\
&\overset{(b)}{\geq} \sum_{j\in Z} q_1\lambda_1\pi_{1j} - \sum_{j\in Z}\lambda_2\pi_{2j},
\end{aligned}$$

where (a) follows from the fact that for all $j \in Y$ we have $\lambda_2\pi_{2j} \geq \lambda_1\pi_{1j}$ and (b) follows from the fact for all $j \in Z$ we have $\lambda_2\pi_{2j} \geq q_1\lambda_1\pi_{1j}$. This completes the proof. ■

Proof of Lemma 5 In this proof, we let $\mathbf{0}$ and $\mathbf{1}$ denote vectors of zero and one, respectively where the size of the vectors are clear from the context. We show that when both types subscribe, Algorithm 2 finds a set of artists, i.e., a solution of the following problem

$$\max_{J\subseteq\mathcal{M}}\left\{\left(\min_{i\in\{1,2\}}\sum_{j\in J}\lambda_i\pi_{ij}\right)-\sum_{j\in J}R_j\right\} \tag{38}$$

whose profit is at most $\max_{j\in\mathcal{M}}R_j$ away from the optimal solution. In this regard, we first show that Algorithm 2 finds an optimal solution of the relaxed problem without the integer constraints and then show that this optimal solution will give an integer constraint solution that is $\max_{j\in\mathcal{M}}R_j$ away from the optimal profit.

We consider the relaxed version of Problem (38) in which the integer constraints are dropped:

$$\max_{z,\mathbf{x}=(x_1,\ldots,x_m)} z-\sum_{j=1}^m x_jR_j \tag{39}$$

$$\text{s.t. } z\le\sum_{j=1}^m x_j\lambda_1\pi_{1j} \tag{40}$$

$$z\le\sum_{j=1}^m x_j\lambda_2\pi_{2j} \tag{41}$$

$$x_j\le 1,\quad j=1,\ldots,m \tag{42}$$

$$x_j\ge 0,\quad j=1,\ldots,m. \tag{43}$$

We next write the dual of this linear programming. We let w_1 denote the dual variables corresponding to (40), w_2 denote the dual variables corresponding to (41), $\mathbf{y}=(y_1,\ldots,y_m)$ denote the dual variables corresponding to constraints (42), and $\mathbf{v}=(v_1,\ldots,v_m)$ denote the dual variables corresponding to constraints (42). The dual of Problem (39) becomes

$$\begin{aligned}
\min_{w_1,w_2,\mathbf{y},\mathbf{v}} &\sum_{j=1}^m y_j\\
\text{s.t. } & w_1,w_2\ge 0\\
& w_1+w_2=1\\
& \mathbf{y}\ge\mathbf{0}\\
& \mathbf{w}\ge\mathbf{0}\\
& w_1\lambda_1\pi_{1j}+w_2\lambda_2\pi_{2j}-y_j+v_j=R_j,\quad j=1,\ldots,m.
\end{aligned}$$

This dual problem simplifies to the following

$$\min_{w_1,w_2,\mathbf{y}}\sum_{j=1}^m y_j \tag{44}$$

$$\text{s.t. } w_1,w_2\ge 0 \tag{45}$$

$$w_1+w_2=1 \tag{46}$$

$$\mathbf{y}\ge\mathbf{0} \tag{47}$$

$$y_j\ge w_1\lambda_1\pi_{1j}+w_2\lambda_2\pi_{2j}-R_j,\quad j=1,\ldots,m. \tag{48}$$

The complementary slackness conditions for the primal-dual problems (39) and (44) becomes

$$\max\{0, S_j(w_1, w_2)\}(x_j - 1) = 0, \quad j = 1, \ldots, m \tag{49}$$

$$x_j \left(\max\{0, S_j(w_1, W_2)\} - S_j(w_1, w_2)\right) = 0, \quad j = 1, \ldots, m \tag{50}$$

$$w_1 \left(\min \left\{ \lambda_1 \sum_{j=1}^{m} \pi_{1j} x_j, \lambda_2 \sum_{j=1}^{m} \pi_{2j} x_j \right\} - \lambda_1 \sum_{j=1}^{m} \pi_{1j} x_j \right) = 0 \tag{51}$$

$$w_2 \left(\min \left\{ \lambda_1 \sum_{j=1}^{m} \pi_{1j} x_j, \lambda_2 \sum_{j=1}^{m} \pi_{2j} x_j \right\} - \lambda_2 \sum_{j=1}^{m} \pi_{2j} x_j \right) = 0 \tag{52}$$

If primal-dual variables $\mathbf{x}$ and w_1, w_2 satisfy these complementary slackness conditions together with feasibility constraints (i.e., $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \leq \mathbf{1}$, $w_1, w_2 \geq 0$, $w_1 + w_2 = 1$) then they are optimal solutions to primal and dual solutions, respectively.

Claim 1: Algorithm 2 finds an optimal solution of problem (39) that has at most one fractional x_j.

Proof of Claim 1: In Algorithm 2, the primal and dual feasibility always hold. We next show that the complementary slackness also holds. In particular, conditions (49) and (50) are equivalent to having:(i) $S_j(w_1, w_2) > 0$ implying $x_j = 1$, (ii) $S_j(w_1, w_2) < 0$ implying $x_j = 0$, and (iii) $x_j \in (0, 1)$ implying $S_j(w_1, w_2) = 0$. At the termination point of Algorithm 2 these conditions hold. We next show that conditions (51) and (52) hold as well. There are two possibilities for the termination point of Algorithm 2 as listed below:

- $\hat{w}_1 = 1$, $\hat{w}_2 = 0$: in this case, if $J^{(t)}$ is the terminal set, then we have $S_j(\hat{w}_1, \hat{w}_2) > 0$ for all $j \in J^{(t)}$, showing that conditions (49) and (50) holds. Condition (52) holds because $\hat{w}_2 = 0$ and condition (51) holds because

$$\min \left\{ \lambda_1 \sum_{j \in J^{(t)}} \pi_{1j}, \lambda_2 \sum_{j \in J^{(t)}} \pi_{2j} \right\} = \lambda_1 \sum_{j \in J^{(t)}} \pi_{1j}.$$

Therefore, the solution $x_j = 1$ for all $j \in J^{(t)}$ is an optimal solution.

- $\hat{w}_1, \hat{w}_2 \in (0, 1)$: in this case, suppose the algorithm has stopped at time t. For all $j \in J^{(t-1)}$ we have $S_j(w_1, w_2) > 0$ implying $x_j = 1$ and $S_j(w_1, w_2) < 0$ implying $x_j = 0$. For $j \in J^{(t)} \setminus J^{(t-1)}$, we have $S_j(\hat{w}_1, \hat{w}_2) = 0$ and $x_j \in (0, 1)$. This shows that conditions (49) and (50) hold. We next show that for some $x_j \in (0, 1)$ conditions (52) and (51) also hold. This is because by following the steps of the algorithm, we have

$$\sum_{j' \in J^{(t-1)}} \lambda_1 \pi_{1j'} < \sum_{j' \in J^{(t-1)}} \lambda_2 \pi_{2j'},$$

and

$$\lambda_1 \pi_{1j} + \sum_{j' \in J^{(t-1)}} \lambda_1 \pi_{1j'} > \lambda_2 \pi_{2j} + \sum_{j' \in J^{(t-1)}} \lambda_2 \pi_{2j'},$$

showing that there exists $x_j \in (0, 1)$ such that

$$x_j \lambda_1 \pi_{1j} + \sum_{j' \in J^{(t-1)}} \lambda_1 \pi_{1j'} = x_j \lambda_2 \pi_{2j} + \sum_{j' \in J^{(t-1)}} \lambda_2 \pi_{2j'}.$$

Therefore, the solution $x_{j'} = 1$ for all $j' \in J^{(t-1)}$ and $x_j \in (0, 1)$ is an optimal solution. ■

Claim 2: We next show that this solution that has at most one non-integral solution is at most $\max_{j\in\mathcal{M}} R_j$ away from the optimal solution.

Proof of Claim 2: This claim holds because we have

$$\begin{aligned}
&\min\left\{\lambda_1 \sum_{j'\in J^{(t)}} \pi_{1j'}, \lambda_2 \sum_{j'\in J^{(t)}} \pi_{2j}\right\} - \sum_{j'\in J^{(t)}} R_{j'} \\
&\geq \min\left\{x_j\lambda_1\pi_{1j} + \lambda_1 \sum_{j'\in J^{(t-1)}} \pi_{1j'}, x_j\lambda_2\pi_{2j} + \lambda_2 \sum_{j'\in J^{(t-1)}} \pi_{2j}\right\} - \sum_{j'\in J^{(t-1)}} R_{j'} - x_j R_j - R_j \\
&= \text{(optimal solution of relaxed problem (39))} - R_j \\
&\geq \text{(optimal solution of integer problem (38))} - R_j \\
&\geq \text{(optimal solution of integer problem (38))} - \max_{j\in\mathcal{M}} R_j,
\end{aligned}$$

proving the performance guarantee of Algorithm 2. ■

Finally, to see the running of Algorithm 2 note that each step of Algorithm 2 can be performed by letting

$$w_1^{(t+1)} \leftarrow \min_{j\in\mathcal{M}\setminus J^{(t)}} \frac{R_j - w_2^{(t)}\lambda_2\pi_{2j}}{\lambda_1\pi_{1j}} \text{ and } w_2^{(t+1)} \leftarrow w_2^{(t)} + \left(w_1^{(t)} - w_1^{(t+1)}\right)$$

or

$$w_2^{(t+1)} \leftarrow \min_{j\in J^{(t)}} \frac{R_j - w_1^{(t)}\lambda_1\pi_{1j}}{\lambda_2\pi_{2j}} \text{ and } w_1^{(t+1)} \leftarrow w_1^{(t)} + \left(w_2^{(t)} - w_2^{(t+1)}\right),$$

depending on which one has the minimum change. This takes $O(m\log m)$ and therefore the algorithm will terminate in time $O(m^2\log m)$. This completes the proof. ■

Proof of Theorem 3 When only one user type subscribes, i.e., $S^*=\{1\}$ and $S^*=\{2\}$, we can find the optimal set of artists in time $O(m\log m)$ time. This is because to find $J_1 = J(1,0)$ and $J_2 = J(0,1)$, we can sort the elements of $\{j\in\mathcal{M} : q_1\lambda_1\pi_{1j} - R_j\}$ and $\{j\in\mathcal{M} : q_2\lambda_2\pi_{2j} - R_j\}$, respectively and then keep the non-negative ones. This takes $O(m\log m)$ time. When both user types subscribe, then Lemma 5 finds a set of artists whose profit is at most $\max_{j\in\mathcal{M}} R_j$ away from the optimal profit in time $O(m^2\log m)$. ■

Proof of Proposition 8 For any artist $j\in\mathcal{M}$ we let

$$\mathbf{x}_j = (x_{1j} = \frac{\pi_{1j}}{\pi_{1j}} = 1, x_{2j} = \frac{\pi_{2j}}{\pi_{1j}}, \ldots, x_{nj} = \frac{\pi_{nj}}{\pi_{1j}}) \tag{53}$$

denote the relative popularity of different artists for a type i user. For a pro-rata allocation rule, the payment to artist j is

$$\beta^{\text{pr}}\lambda_1 \frac{\sum_{i=1}^n q_i\lambda_i\pi_{ij}}{\sum_{i=1}^n q_i\lambda_i}. \tag{54}$$

For a user-centric allocation rule, the payment to artist j is

$$\beta^{\text{uc}}\lambda_1 \sum_{i=1}^n q_i\pi_{ij}. \tag{55}$$

Comparing (54) and (55) and using the notation introduced in (53), for $\beta^{\text{pr}} = \beta^{\text{uc}}$ pro-rata is preffered to user centric if and only if

$$\sum_{i=1}^n x_{ij}q_i\lambda_i \geq \left(\sum_{\ell=1}^n q_\ell\lambda_\ell\right)\left(\sum_{i=1}^n x_{ij}q_i\right),$$

which is the same as the condition in the proposition. Finally, note that the coefficient of x_{nj} is $q_n\lambda_n - q_n\sum_{\ell=1}^n q_\ell\lambda_\ell$ which is non-negative, the coefficient of x_{1j} is $q_1\lambda_1 - q_1\sum_{\ell=1}^n q_\ell\lambda_\ell$ which is non-positive, the coefficients are increasing in i, and they sum up to 1. This implies that there exists i^* such that

$$q_i\lambda_i - q_i\sum_{\ell=1}^n q_\ell\lambda_\ell \geq 0 \text{ for } i \geq i^* \quad \text{ and } q_i\lambda_i - q_i\sum_{\ell=1}^n q_\ell\lambda_\ell < 0 \text{ for } i < i^*.$$

This completes the proof. ■

Proof of Proposition 9 In this proof we let

$$\mathbf{x}_j = (x_{1j} = \frac{\pi_{1j}}{\pi_{1j}} = 1, x_{2j} = \frac{\pi_{2j}}{\pi_{1j}}, \ldots, x_{nj} = \frac{\pi_{nj}}{\pi_{1j}})$$

denote the relative popularity of different artists for a type i user. Let j be the pivot artist for pro-rata allocation rule. We have

$$\beta^{\text{pr}} = \frac{R_j}{\lambda_1}\frac{\sum_{i=1}^n q_i\lambda_i}{\sum_{i=1}^n q_i\lambda_i\pi_{ij}}. \tag{56}$$

For a user-centric allocation rule we must have

$$\beta^{\text{uc}} \geq \frac{R_j}{\lambda_1}\frac{1}{\sum_{i=1}^n q_i\lambda_i\pi_{ij}}. \tag{57}$$

Comparing (56) and (57) and using the notation introduced in (53), we obtain $\beta^{\text{uc}} \geq \beta^{\text{pr}}$ if

$$\sum_{i=1}^n x_{ij}q_i\lambda_i \geq \left(\sum_{\ell=1}^n q_\ell\lambda_\ell\right)\left(\sum_{i=1}^n x_{ij}q_i\right),$$

which is the same as the condition in the proposition. The proof of part (b) follows from a similar argument, completing the proof. ■

Proof of Proposition 10 Without the platform, every user type i subscribes to the direct channel of artist j if she receives a non-negative utility from it. Using Lemma 1, the welfare becomes

$$\begin{aligned}&\sum_{j\in X} q_2\lambda_2\pi_{2j} + \sum_{j\in Y} q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} + \sum_{j\in Z} q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} + \sum_{j\in W} q_1\lambda_1\pi_{1j}\\ &= q_1\lambda_1 + q_2\lambda_2 - \sum_{j\in X} q_1\lambda_1\pi_{1j} - \sum_{j\in W} q_2\lambda_2\pi_{2j}.\end{aligned}$$

Therefore, without the platform, the first-best is achievable if and only if all artists belong to either the set Y or the set Z, i.e.,

$$\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2 q_2}\right] \qquad \text{for all } j \in \mathcal{M}. \tag{58}$$

Part (b) follows from the fact that the platform needs to choose $\text{FEE}(\mathcal{M}) = \lambda_1$ and $p_j(\mathcal{M}) = R_j$ to have all artists and users join it. The revenue of the platform becomes λ_1 and the platform is sustainable if and only if this revenue is larger than the sum of the payments to artists, i.e., $\sum_{j\in\mathcal{M}} R_j$.

If condition (58) holds then condition (28) of Part (a) of Proposition 10 also holds because

$$\sum_{j\in\mathcal{M}} R_j = \sum_{j\in Y\cup Z} R_j = \sum_{j\in Y}\lambda_1\pi_{1j} + \sum_{j\in Z}\lambda_2\pi_{2j} \overset{(i)}{\leq} \sum_{j\in Y}\lambda_1\pi_{1j} + \sum_{j\in Z}\lambda_1\pi_{1j} = \lambda_1,$$

where (i) follows from Lemma 1. This implies that when there exists a platform, first-best is implementable in a wider range of parameters. ■

Proof of Proposition 11 The proof of part (a) follows from a similar argument to Proposition 1. In particular, if type 1 users subscribe then type 2 users will also subscribe and the platform's optimal profit becomes

$$\lambda_1 - \sum_{j\in\mathcal{M}} \left(\tilde{R}_j + c_j(\{1,2\})\right).$$

If only type 2 users subscribe then the platform's optimal profit becomes

$$\lambda_2 q_2 - \sum_{j\in\mathcal{M}} \left(\tilde{R}_j + c_j(\{2\})\right).$$

Therefore, the platform's profit is the maximum of these two terms and the platform is sustainable if this maximum is non-negative.

The proof of part (b) (and similarly part (c)) follows from the fact that under Assumption 3 both user types subscribe and therefore the pro-rata (and similarly user-centric) payment to artist j needs to be larger than $\tilde{R}_j + c_j(\{1,2\})$. ■

Appendix B: Extensions, additional results, and examples

This section includes the extensions discussed in the main text, additional examples, and the detail of examples

B.1. Extensions and additional results

B.1.1. Extension of Algorithm 1 to constant number of user types Here, we show how Algorithm 1 extends to a setting with any constant $k > 2$ user types. We develop the algorithm for pro-rata rule and a similar algorithm works for user-centric. The problem of finding the optimal set with pro-rata revenue allocation rule becomes

$$\max_{\substack{J\subseteq\mathcal{M}\\ S\subseteq\{1,\dots,k\}\\ \beta\in[0,1]}} \left(\min_{i\in S}\lambda_i \sum_{j\in J}\pi_{ij}\right)\left(\sum_{i\in S} q_i\right)(1-\beta) \tag{59}$$

$$\text{s.t. } \beta\left(\min_{i\in S}\lambda_i \sum_{j\in J}\pi_{ij}\right)\left(\sum_{i\in S} q_i\right)\frac{\sum_{i\in S} q_i\lambda_i\pi_{ij}}{\sum_{i\in S} q_i\lambda_i\sum_{j\in J}\pi_{ij}} \geq R_j, \quad j\in J. \tag{60}$$

We solve the problem of choosing the optimal set of artists J and the optimal payout rate for a given set S which without loss of generality we assume is the entire set $\{1,\dots,k\}$. Problem (59) can then be solved by searching over 2^k subsets of $\{1,\dots,k\}$.

For $S = \{1,\dots,k\}$, suppose, without loss of generality, in the optimal solution J^* of problem (59), we have

$$1 = \arg\min_{i\in\{1,\dots,k\}} \lambda_i \sum_{j\in J^*}\pi_{ij}.$$

We also let β^{pr} be the optimal β and

$$\alpha_i^{\text{pr}} = \frac{\lambda_i\sum_{j\in J^*}\pi_{ij}}{\lambda_1\sum_{j\in J^*}\pi_{1j}}, \quad \text{for } i = 2,\dots,k,$$

be the ratio of type i users' utility to type 1 users' utility in the optimal solution. With these notations, we can write constraint (60) succinctly in terms of β^{pr} and α_i^{pr} as

$$\frac{R_j}{\sum_{i=1}^k q_i\lambda_i\pi_{ij}} \leq \frac{\beta^{\text{pr}}}{q_1 + \sum_{i=2,\dots,k} q_i\alpha_i^{\text{pr}}}, \quad \text{for all } j\in J^*.$$

Motivated by this observation, for any β and $\boldsymbol{\alpha} = (\alpha_2, \ldots, \alpha_k)$, we define the set $\mathcal{M}^{\text{pr}}(\beta, \boldsymbol{\alpha})$ as

$$\mathcal{M}^{\text{pr}}(\beta, \boldsymbol{\alpha}) = \left\{ j \in \mathcal{M} \;:\; \frac{R_j}{\sum_{i=1}^{k} q_i \lambda_i \pi_{ij}} \leq \frac{\beta}{q_1 + \sum_{i=2,\ldots,k} q_i \alpha_i} \right\}.$$

Therefore, knowing β^{pr} and $\boldsymbol{\alpha}^{\text{pr}} = (\alpha_2^{\text{pr}}, \ldots, \alpha_k^{\text{pr}})$, the optimal set of artists can be found by solving

$$\begin{aligned} \max_{J \subseteq \mathcal{M}^{\text{pr}}(\beta^{\text{pr}}, \boldsymbol{\alpha}^{\text{pr}})} & (1-\beta^{\text{pr}}) \lambda_1 \sum_{j \in J} \pi_{1j} \\ \text{s.t. } & \lambda_1 \sum_{j \in J} \pi_{1j} \leq \alpha_i^{\text{pr}} \lambda_i \sum_{j \in J} \pi_{ij}, \quad \text{for } i = 2, \ldots, k. \end{aligned} \tag{61}$$

A similar argument to the one we used in developing and analyzing algorithm 1 shows this problem can be formulated as an instance of a Knapsack problem (with multiple constraints). Therefore, for any constant k we can again find a PTAS of the optimal design within pro-rata class of revenue allocation strategies (and similarly within user-centric class of revenue allocation strategies).

B.1.2. Extension of Algorithm 2 to constant number of user types Here, we show that for any constant $k > 2$ user types, a similar algorithm finds a revenue allocation rule whose profit is at most $(k-1)\max_{j \in \mathcal{M}} R_j$ away from the optimal profit. First, note that, similar to Lemma 4, with k user types the problem can be formulated in terms of the sets of artists who join the platform and the set of user types who subscribe:

$$\max_{\substack{J \subseteq \mathcal{M} \\ S \subseteq \{1,\ldots,k\}}} \left\{ \left(\min_{i \in S} \lambda_i \sum_{j \in J} \pi_{ij} \right) \left(\sum_{i \in S} q_i \right) - \sum_{j \in J} R_j \right\}. \tag{62}$$

We next show that for any $S \subseteq \{1, \ldots, k\}$, a similar approach to Algorithm 2 finds a revenue allocation rule whose profit is at most $(k-1)\max_{j \in \mathcal{M}} R_j$ of the optimal profit. Without loss of generality, let us consider the subset $\{1, \ldots, k\}$. Problem (62) becomes

$$\max_{J \subseteq \mathcal{M}} \left\{ \left(\min_{i \in S} \lambda_i \sum_{j \in J} \pi_{ij} \right) \left(\sum_{i \in S} q_i \right) - \sum_{j \in J} R_j \right\}.$$

whose linear programming relaxation becomes

$$\begin{aligned} \max_{z, x_1, \ldots, x_m} \; & z - \sum_{j=1}^{m} x_j R_j \\ \text{s.t. } & z \leq \sum_{j=1}^{m} x_j \lambda_1 \pi_{ij}, \quad i = 1, \ldots, k \\ & 0 \leq x_j \leq 1, \quad j = 1, \ldots, m. \end{aligned} \tag{63}$$

The dual of problem (63), similar to the proof of Theorem 3, becomes

$$\begin{aligned} \min_{w_1, \ldots, w_k, \mathbf{y}} \; & \sum_{j=1}^{m} y_j \\ \text{s.t. } & w_i \geq 0, \quad i = 1, \ldots, k \\ & \sum_{i=1}^{k} w_i = 1 \\ & \mathbf{y} \geq \mathbf{0} \\ & y_j \geq \sum_{i=1}^{k} w_i \lambda_i \pi_{ij} - R_j, \quad j = 1, \ldots, m. \end{aligned}$$

A similar approach to Algorithm 2 gives the desired approximation by defining the score of artist j for the tuple of dual variables $\mathbf{w}^{(t)} = (w_1^{(t)}, \ldots, w_k^{(t)})$ by $S_j(\mathbf{w}^{(t)}) = \sum_{i=1}^{k} w_i^{(t)} \lambda_i \pi_{ij} - R_j$.

B.1.3. The optimal revenue allocation strategy when artists multi-home Here, we consider an extension of our model in which artists can join the platform and offer their content through direct channels at the same time. Similar to our baseline model, we have a three stage game as follows:

1. In the first stage, the platform chooses the payment to artists as well as the subscription fee denoted by $p: 2^{\mathcal{M}} \to \mathbb{R}^m$ and $\textsc{Fee}: 2^{\mathcal{M}} \to \mathbb{R}$, respectively. The artists also choose the subscription fee of their direct channels denoted by $\mathbf{r} = (r_1, \ldots, r_m)$.
2. In the second stage, given the tuple $(\lambda, \mathbf{q}, \Pi)$ and the payment rules the artists decide whether they want to join the platform.
3. In the third stage, the users decide which subset of subscriptions they want to subscribe to. In particular, the users decide about subscribing to the bundle J on the platform as well as subscribing to direct channels of different artists.

The equilibrium of this game, similar to our baseline model, is obtained by backward induction. We next find the optimal revenue allocation rule. We have the following cases depending on the user types that subscribe to the platform:

1. Both users types subscribe: In this case, the revenue of the artists who have joined the platform from offering their content outside of the platform is zero. Therefore, the platform need to pay each artist $j \in J$, $\max_r \; r \; d(j, r)$ which is the same as our baseline model. Therefore, the platform's problem in this case becomes the same as (23) and Algorithm 2 finds a set of artists whose profit is within one artist from the optimal set.
2. Type 1 users subscribe to the platform: In this case the platform's optimal allocation rule will make all artists join the platform and the platform compensates each artist for the difference between the revenue they obtain by offering their content solely through a direct channel and offering their content both on the platform and through a direct channel.
3. Type 2 users subscribe to the platform: this case is similar to the previous case.
4. None of the user types subscribe to the platform: in this case, the platform's profit is zero.

The optimal revenue allocation strategy is the one with the maximum of the above four cases.

B.1.4. Platform's monopoly and the existence of core Here, we derive conditions guaranteeing the emergence of a monopolist platform, i.e., conditions that no set $J \subseteq \mathcal{M}$ of artists would be better off creating a competing platform offering the content from artists in J. As suggested by the last statement of Corollary 1, neither pro-rata nor user-centric allocation can guarantee this. Thus, in this section we allow for an arbitrary payment rule. In particular, we let $f: 2^{\mathcal{M}} \to \mathbb{R}$ be a mapping from the set of artists who form a coalition and move to another platform to their maximum total collected subscription fee. That is

$$f(J) = \max_{S \subseteq \{1,2\}} \left\{ \left(\min_{i \in S} \lambda_i \sum_{j \in J} \pi_{ij} \right) \left(\sum_{i \in S} q_i \right) \right\}.$$

Note that the platform can sustain its monopoly if and only if there exists no coalition J of artists that generate a higher value for themselves on a competing platform offering artists $j \in J$, than what they collectively generate on the platform offering all artists. In other words, the platform monopoly emerges if and only if the *core* of the function $f(\cdot)$, defined next, is non-empty.

DEFINITION 4 (CORE). The vector $\mathbf{x} = (x_1, \ldots, x_m)$ forms the core for the function $f(\cdot)$ if we have

$$f(J) \leq \sum_{j \in J} x_j, \text{ for all } J \subseteq \mathcal{M},$$
$$f(\mathcal{M}) = \sum_{j \in \mathcal{M}} x_j.$$

The main result of our section provides a necessary condition in terms of the ratios $\frac{\pi_{1j}}{\pi_{2j}}$ for the existence of an allocation rule that belongs to the core.

THEOREM 4. *For a given $\boldsymbol{\lambda}$ and $\mathbf{q}$, if $\left(\max_{j \in \mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) \leq \frac{\lambda_1}{q_2 \lambda_2}$ then core is non-empty.*

Theorem 4 again indicates importance of the ratio $\frac{\pi_{2j}}{\pi_{1j}}$ for our analysis. It states that the core is non-empty if there are no artists who are considerably more popular with users who have high usage rate (i.e., type 2 users). This is because such artists have a potential to attract those users who have higher value for listening (joining some platform) than type 1 users. In the proof of Theorem 4, presented next, we also show that if $\left(\max_{j \in \mathcal{M}} \frac{\pi_{2j}}{\pi_{1j}}\right) > \frac{\lambda_1}{q_2 \lambda_2}$, then one can choose other parameters of $(\boldsymbol{\lambda}, \mathbf{q}, \Pi)$ so that the core does not exist. This is because if this condition does not hold, there exists a subset of artists to which only type 1 users would subscribe and there exists another subset of artists to which only type 2 users would subscribe. If these two subsets of artists form a coalition J, i.e., offer their content on the competing platform and move together, then can get both type of users to subscribe to the new platform offering artists from J, resulting in a higher collective revenue for those artists than on the platform offering all artists.

Proof of Theorem 4: We first show that if $\frac{\pi_{21}}{\pi_{11}} \leq \frac{\lambda_1}{q_2 \lambda_2}$, then $x_j = \lambda_1 \pi_{1j}$ for all $j \in \mathcal{M}$ defines a core. Consider set $J \subseteq \mathcal{M}$ and suppose these artists deviate altogether. We next list the possible revenues they can obtain and show that $\sum_{j \in J} x_j$ is (weakly) larger than that. We let $\pi_{iJ} = \sum_{j \in J} \pi_{ij}$ for $i = 1, 2$.

- $\frac{\pi_{2J}}{\pi_{1J}} > \frac{\lambda_1}{\lambda_2 q_2}$: This cannot happen as we have $\frac{\pi_{2j}}{\pi_{1j}} \leq \frac{\lambda_1}{\lambda_2 q_2}$ for all j.
- $\frac{\lambda_1}{\lambda_2} < \frac{\pi_{2J}}{\pi_{1J}} \leq \frac{\lambda_1}{\lambda_2 q_2}$: In this case the revenue of the artist in J is they deviate becomes $\lambda_1 \pi_{1J}$ which is equal to $\sum_{j \in J} x_j$.
- $\frac{\lambda_1 q_1}{\lambda_2} < \frac{\pi_{2J}}{\pi_{1J}} \leq \frac{\lambda_1}{\lambda_2}$: In this case the revenue of the artist in J is they deviate becomes $\lambda_2 \pi_{2J}$ which is (weakly) smaller than $\lambda_1 \pi_{1J} = \sum_{j \in J} x_j$.
- $\frac{\pi_{2J}}{\pi_{1J}} \leq \frac{\lambda_1 q_1}{\lambda_2}$: In this case the revenue of the artist in J is they deviate becomes $\lambda_1 q_1 \pi_{1J}$ which is (weakly) smaller than $\lambda_1 \pi_{1J} = \sum_{j \in J} x_j$.

We next show that if $\frac{\pi_{21}}{\pi_{11}} > \frac{\lambda_1}{q_2 \lambda_2}$ then there exists $\frac{\pi_{2j}}{\pi_{1j}}$ for which the platform cannot sustain the artists. In particular, we consider only two artists one in the set X and one in the set W. In particular, we let $\pi_{21} = 1 - \pi_{11} = a$ and $\pi_{12} = 1 - \pi_{22} = b$ and shows that there exists a and b such that the revenue of the platform is smaller than the sum of the individual revenues of these two artists. To have $1 \in X$ and $2 \in W$ the constraints on a and b become

$$1 - a \leq \frac{\lambda_2}{\lambda_1} q_2 b, \quad q_1 a \geq (1 - b) \frac{\lambda_2}{\lambda_1}. \tag{64}$$

The platform cannot sustain both artists on it if we have

$$\lambda_1 < \lambda_2 q_2 b + q_1 \lambda_1 a. \tag{65}$$

To satisfy (64), we let

$$a = \max\left\{(1-b)\frac{\lambda_2}{\lambda_1 q_1}, 1 - b\frac{\lambda_2 q_2}{\lambda_1}\right\},$$

which is less than 1 given

$$(1-b)\frac{\lambda_2}{\lambda_1 q_1} \leq 1. \tag{66}$$

Plugging a into (65), we see that this inequality holds if we have

$$1-(1-b)\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2 q_2}{\lambda_1} b, \quad 1 - q_1(1-b\frac{\lambda_2 q_2}{\lambda_1}) \leq \frac{\lambda_2 q_2}{\lambda_1} b. \tag{67}$$

The inequalities given in (66) and (67) hold if we have

$$\frac{\lambda_2}{\lambda_1} \geq \max\left\{1 + \frac{q_1}{q_2}, \frac{1-q_1 q_2}{q_2}\right\},$$

completing the proof. ■

B.1.5. Usage-based services Here, we extend our baseline model and analysis in a setting in which the platform offers a usage-based service. The payment to the artists is the same as our baseline model. The charged fee on the user side is a function $\text{RATE}: 2^{\mathcal{M}} \to \mathbb{R}$ where $\text{RATE}(J)$ determines the per consumption fee that users need to pay to get access to the content of artists in the set J. With this definition, the utility of a type i user from subscription becomes

$$\lambda_i \sum_{j \in J} \pi_{ij} - \lambda_i \sum_{j \in J} \pi_{ij} \text{RATE}(J).$$

LEMMA 8. *For any artist $j \in \mathcal{M}$, the optimal usage fee in going solo (i.e., offering the content directly to users) is 1 and revenue R'_j for artist j becomes $q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}$*

In the next proposition we compare different allocation rules when we have a usage-based service.

PROPOSITION 12. *For any set $J \subseteq \mathcal{M}$, the optimal payment to artists for pro-rata rule, user-centric rule, and arbitrary rule coincide. Moreover, the platform's profit (with all three rules) is always zero.*

Proof of Proposition 12: Consider a platform with an arbitrary allocation rule. The optimal usage fee is 1, both user types subscribe, and the revenue of the platform becomes

$$\text{REV} = q_1 \lambda_1 \sum_{j \in J} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J} \pi_{2j}.$$

The optimal payment to artist $j \in J$ with an arbitrary allocation rule is equal to $R'_j = q_1 \lambda_1 \pi_{1j}$. Therefore, the platform's profit is zero. We next describe pro-rata and user-centric rules.

Pro-rata: Let β be the platform payout rate. Pro-rata payment to artist $j \in J$ is

$$\beta \ \text{REV} \ \frac{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}}{q_1 \lambda_1 \sum_{j \in J} \pi_{1j} + q_2 \lambda_2 \sum_{j \in J} \pi_{2j}} = \beta \ \left(q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}\right).$$

This implies that the optimal platform's payout rate becomes

$$\beta^{\mathrm{pr}} = \max_{j\in J}\left\{\frac{R'_j}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}\right\} \overset{(i)}{=} 1,$$

where (i) follow from Lemma 8. This further establishes that the platform's profit with pro-rata rule is zero.

User-centric: Let β be the platform payout rate. User-centric payment to artist $j\in J$ is

$$\beta\left(q_1\lambda_1\sum_{j\in J}\pi_{1j}\frac{\lambda_1\pi_{1j}}{\sum_{j\in J}\lambda_1\pi_{1j}}+q_2\lambda_2\sum_{j\in J}\pi_{2j}\frac{\lambda_2\pi_{2j}}{\sum_{j\in J}\lambda_2\pi_{2j}}\right)=\beta\,\left(q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}\right).$$

This implies that the optimal platform's payout rate becomes

$$\beta^{\mathrm{uc}} = \max_{j\in J}\left\{\frac{R'_j}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}\right\} \overset{(i)}{=} 1,$$

where (i) follow from Lemma 8. This further establishes that the platform's profit with user-centric rule is zero. ■

Notice that if the outside option is given by Lemma 8, then for any set $J\subseteq\mathcal{M}$, the platform's profit with all the three allocation rules is zero and the platform can bring all artists on it. Our algorithms, however, work for any outside option that the artists may have. We next establish how we can modify our algorithms to find the optimal set of artists when for any given outsider options of artists denoted by R'_j for $j\in\mathcal{M}$.

Optimal set of artists with usage-based service and an arbitrary allocation rule: With an arbitrary revenue allocation rule, the platform's problem becomes

$$\max_{J\subseteq\mathcal{M}} q_1\lambda_1\sum_{j\in J}\pi_{1j}+q_2\lambda_2\sum_{j\in J}\pi_{1j}-\sum_{j\in J}R'_j$$

and therefore, the optimal set of artists is given by

$$J^* = \{j\in\mathcal{M}\ :\ q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{1j}\ge R'_j\}.$$

Optimal set of artists with usage-based service and a pro-rata/user-centric allocation rule: With both pro-rata and user-centric rules, the platform's problem becomes

$$\max_{J\subseteq\mathcal{M},\beta}(1-\beta)\left(q_1\lambda_1\sum_{j\in J}\pi_{1j}+q_2\lambda_2\sum_{j\in J}\pi_{2j}\right)$$
$$\text{s.t. } \beta\left(q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}\right)\ge R'_j \quad \text{ for all } j\in J.$$

In order to solve this problem we first sort the values $\frac{R'_j}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}$. Without loss of generality, let us assume

$$\frac{R'_1}{q_1\lambda_1\pi_{11}+q_2\lambda_2\pi_{21}}\le\cdots\le\frac{R'_m}{q_1\lambda_1\pi_{1m}+q_2\lambda_2\pi_{2m}}.$$

The optimal set of artists does not include any artist j for which

$$\frac{R'_j}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}\ge 1.$$

For some $\bar{j}\in\mathcal{M}$, we let $\mathcal{M}'=\{1,\ldots,\bar{j}\}$ be the set of artists j for which $\frac{R'_j}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}<1$ (if $\mathcal{M}'=\emptyset$, then the optimal set of artists is an empty set). Now notice that if the optimal set of artists includes $j'\le\bar{j}$ then it must include all artists $j\le j'$ as well. Therefore, the optimal set of artists is of the form $\{1,\ldots,j\}$ for some $j\in\mathcal{M}'$. In particular, the optimal set of artists is $\{1,\ldots,j^*\}$ where

$$j^*\in\arg\max_{j\in\mathcal{M}'}\left(1-\frac{R'_j}{q_1\lambda_1\pi_{1j}+q_2\lambda_2\pi_{2j}}\right)\left(q_1\lambda_1\sum_{\ell=1}^{j}\pi_{1\ell}+q_2\lambda_2\sum_{\ell=1}^{j}\pi_{2\ell}\right).$$

Thus, we can find the optimal set of artists in time $O(m\log m)$.

B.1.6. Paying artists their marginal contribution to the platform's revenue and VCG Here we consider a payment rule that allocates to an artist j her contribution to the platform's revenue. We first characterize this payment and then show that it can be below the outside option of the artists, implying that the platform is not sustainable with this payment rule.

Suppose Assumption 1 holds. The revenue of the platform when all artists are on the platform is λ_1. We take artist j moves out of the platform, and again find the platform's revenue. The payment to artist j is the former revenue minus the latter revenue. We next characterize this payment.

LEMMA 9. *Suppose Assumption 1 holds. We have the following cases:*

1. $\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[0, \frac{q_1\lambda_1}{\lambda_2}\right)$*: if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_1(1-\pi_{1j})$, only type 1 users subscribe to the platform, and platform's revenue becomes $q_1\lambda_1(1-\pi_{1j})$. Therefore, the payment to artist j is $\lambda_1 - q_1\lambda_1(1-\pi_{1j})$.*
2. $\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{q_1\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}\right)$*: if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_2(1-\pi_{2j})$, both type of users subscribe to the platform, and platform's revenue becomes $\lambda_2(1-\pi_{2j})$. Therefore, the payment to artist j is $\lambda_1 - \lambda_2(1-\pi_{2j})$.*
3. $\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{\lambda_2 q_2}\right)$*: if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_1(1-\pi_{1j})$, both type of users subscribe to the platform, and platform's revenue becomes $\lambda_1(1-\pi_{1j})$. Therefore, the payment to artist j is $\lambda_1 - \lambda_1(1-\pi_{1j})$.*
4. $\frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2 q_2}, \infty\right)$*: if artist j moves out of the platform, the optimal subscription fee becomes $\lambda_2(1-\pi_{2j})$, only type 2 users subscribe to the platform, and platform's revenue becomes $q_2\lambda_2(1-\pi_{2j})$. Therefore, the payment to artist j is $\lambda_1 - q_2\lambda_2(1-\pi_{2j})$.*

The proof of this lemma is similar to the proof of Lemma 1 and hence is omitted.

Comparing the payments characterized in Lemma 9 to the revenue of an artist in a solo market (characterized in Lemma 1), for π_{2j} and π_{1j} in the following regions the payment to the artists according to its marginal contribution to the revenue is smaller than the revenue of the solo market (see Figure 7):

$$\begin{gathered}
\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{q_2\lambda_2}, \infty\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ or} \\
\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{q_2\lambda_2}, \infty\right) \text{ or} \\
\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{\lambda_2}, \frac{\lambda_1}{q_2\lambda_2}\right) \text{ or} \\
\frac{1-\pi_{2j}}{1-\pi_{1j}} \in \left[0, \frac{q_1\lambda_1}{\lambda_2}\right) \text{ and } \frac{\pi_{2j}}{\pi_{1j}} \in \left[\frac{\lambda_1}{q_2\lambda_2}, \infty\right).
\end{gathered}$$

The following is the VCG (Vickrey (1961), Clarke (1971), Groves (1973)) mechanism in our setting:

PROPOSITION 13. *Let $\mathbf{R} = (R_1, \ldots, R_m)$ be the vector of reported outside options. Then the pricing scheme*

$$p_j(\mathbf{R}) = \begin{cases} q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} & q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} \geq R_j \\ 0 & q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} < R_j. \end{cases}$$

incentivizes artists to report their true outside option truthfully and maximizes the welfare.

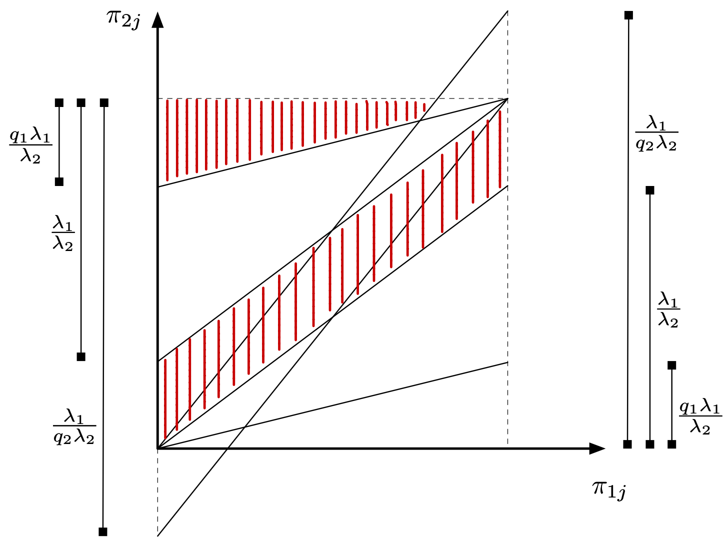


Figure 7 The pairs (π_{1j}, π_{2j}) for which the marginal contribution of artist j to platform's revenue is smaller than its revenue from a solo market.

Proof of Proposition 13: we first establish the functional form of the VCG payments and then simplify it.

For a vector of reported outside options $\mathbf{R}$, letting $a_j \in \{0,1\}$ for all $j \in \mathcal{M}$ denote whether artist j is on the platform or not, the VCG payment to each artist $\ell \in \mathcal{M}$ is given by

$$p_\ell(\mathbf{R}) = \left(\sum_{j\neq\ell} a_j(\mathbf{R})(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j)\right) + (q_1\lambda_1\pi_{1\ell} + q_1\lambda_1\pi_{1j\ell})a_\ell(\mathbf{R}) + h_\ell(\mathbf{R}_{-\ell}), \tag{68}$$

where $\mathbf{a}(\mathbf{R}) = (a_1(\mathbf{R}), \ldots, a_m(\mathbf{R}))$ is

$$\mathbf{a}(\mathbf{R}) \in \arg\max_{\mathbf{a}\in\{0,1\}^m} \sum_{j=1}^{m} a_j(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j), \tag{69}$$

and

$$h_\ell(\mathbf{R}_{-\ell}) = -\max_{\mathbf{a}\in\{0,1\}^{m-1}} \sum_{j\neq\ell} a_j(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j). \tag{70}$$

This incentivizes the artists to report truthfully. We show that each artist $\ell \in \mathcal{M}$ prefers to report truthfully by comparing her utility when she reports truthfully and when she reports $R'_\ell \neq R_\ell$:

$$\begin{aligned} p_\ell(R'_\ell, \mathbf{R}_{-\ell}) + (1 - a_\ell(R'_\ell, \mathbf{R}_{-\ell}))R_\ell &= \sum_{j=1}^{m} a_j(R'_\ell, \mathbf{R}_{-\ell})\left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j\right) + R_\ell + h_\ell(\mathbf{R}_{-\ell}) \\ &\overset{(a)}{\leq} \sum_{j=1}^{m} a_j(R_\ell, \mathbf{R}_{-\ell})\left(q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} - R_j\right) + R_\ell + h_\ell(\mathbf{R}_{-\ell}) \\ &= p_\ell(R_\ell, \mathbf{R}_{-\ell}) + (1 - a_\ell(R_\ell, \mathbf{R}_{-\ell}))R_\ell \end{aligned}$$

where (a) follows from (69). We next simplify this payment rule. Notice that in (69) and (70), we have $a_j = 1$ if and only if $q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j} \geq R_j$. Therefore, (68) simplifies to

$$p_\ell(\mathbf{R}) = \begin{cases} q_1\lambda_1\pi_{1\ell} + q_2\lambda_2\pi_{2\ell} & q_1\lambda_1\pi_{1\ell} + q_2\lambda_2\pi_{2\ell} \geq R_\ell \\ 0 & q_1\lambda_1\pi_{1\ell} + q_2\lambda_2\pi_{2\ell} < R_\ell. \end{cases}$$

This completes the proof. ■

B.1.7. Optimal additions to the set of artists on the platform Here, we show that Algorithm 2 readily extends and provides the same approximation of the optimal profit for a platform which is "locked in" (e.g., contractually) with a subset of artists already "established" on the platform and seeks to choose an optimal subset of "new" artists to add to its artists' portfolio. In particular, suppose set $\tilde{J}$ of artists already exists on the platform and the platform is choosing the optimal subset of $\mathcal{M} \setminus \tilde{J}$ to include. To solve this problem, we can update optimization (24) as follows

$$
\begin{aligned}
\max_{z,\ x_j\ :\ j \in \mathcal{M}\setminus \tilde{J}} \; & z - \sum_{j \in \mathcal{M}\setminus \tilde{J}} x_j R_j \\
\text{s.t. } & z \leq \sum_{j \in \mathcal{M}\setminus \tilde{J}} x_j \lambda_1 \pi_{1j} + \sum_{j \in \tilde{J}} \lambda_1 \pi_{1j} \\
& z \leq \sum_{j \in \mathcal{M}\setminus \tilde{J}} x_j \lambda_2 \pi_{2j} + \sum_{j \in \tilde{J}} \lambda_2 \pi_{2j} \\
& 0 \leq x_j \leq 1, \quad j \in \mathcal{M}\setminus \tilde{J},
\end{aligned}
$$

and then solve it with an identical algorithm to 2 to obtain a solution that is integral except for one variable. Similar to Theorem 3, rounding up the fraction variable gives us a solution whose profit is at most $\max_{j\in\mathcal{M}} R_j$ away from the optimal profit.

B.1.8. Extension to costly service The results of Propositions 2 and 3 continue to hold in a setting with costly services. We next present the proof of Proposition 2 in this setting with costly services (and note that the proof of Proposition 3 extends similarly).

Let β^{pr} and β^{uc} be the β's with which pro-rata and user-centric rules can sustain all artists on the platform with pro-rata and user-centric allocation rules. For a pro-rata rule, the payment to artist j is

$$
\beta^{\text{pr}} \textsc{Rev} \frac{q_1 \lambda_1 \pi_{1j} + q_2 \lambda_2 \pi_{2j}}{q_1 \lambda_1 + q_2 \lambda_2}. \tag{71}
$$

For a user-centric rule, the payment to artist j is

$$
\beta^{\text{uc}} \textsc{Rev} (q_1 \pi_{1j} + q_2 \pi_{2j}) \tag{72}
$$

Using equations (71) and (72), the ratio of the revenue that artist j obtains with pro-rata versus user-cenrtric is

$$
\frac{\beta^{\text{pr}}}{\beta^{\text{uc}}(q_1\lambda_1 + q_2\lambda_2)} \frac{q_1 \lambda_1 \frac{\pi_{1j}}{\pi_{2j}} + q_2 \lambda_2}{q_1 \frac{\pi_{1j}}{\pi_{2j}} + q_2}.
$$

Similar to the proof of Proposition 2, the ratio of artists' payment with pro-rata to their payment with user-centric is increasing in $\frac{\pi_{2j}}{\pi_{1j}}$.

Moreover, Algorithm 1 for finding the optimal pro-rata/user-cetric extend to this setting by noting that we developed them for arbitrary outside option of artists. More precisely, let us consider the algorithm for finding the optimal pro-rata rule. This algorithm continues to hold if we replace R_j by $\tilde{R}_j + c_j(\{1,2\})$ when the set $\{1,2\}$ of user types subscribe to the platform.

B.2. Examples

B.2.1. Introductory example with two artists We consider a setting with only two artists, $m=2$, and we also let $\pi_{11}=\pi_{22}=\pi$, i.e., each user type i listens to artist i's music with probability π and the other artist with probability $1-\pi$. In order to highlight how the usage rate of "fans" of an artist impact the payment of that artist, we further assume that π is sufficiently large. More precisely, we suppose $\frac{\pi}{1-\pi} \geq \max\{\frac{\lambda_2}{\lambda_1 q_1}, \frac{\lambda_1}{\lambda_2 q_2}\}$, which guarantees that user type i only subscribes to the platform if artist i is on the platform (as shown next). Additionally, we also suppose $\frac{\lambda_2}{\lambda_1} \leq \min\{\frac{1}{q_2}, \frac{1-\pi q_1}{\pi q_2}\}$, which guarantees platform sustainability, i.e., that both artists join the platform.

With these assumptions on user consumption we ensure that both artists join the platform and, consequently, that both user types subscribe. We next discuss artists' preferences over different payment allocation rules. We separate discussion in two cases:

1. $q_1 \leq q_2$: In this case, artist 2 whose fans have higher usage rate (recall that $\lambda_2 \geq \lambda_1$) has the majority of the users as her fans. We call artist 2 the "superstar" artist and artist 1 the "niche" artist. In this case, with both pro-rata and user-centric allocation rules, the platform pays the niche artist 1 more than what she could obtain through a direct channel. This is because of the positive externality of the presence of fans of the superstar artist on the platform as they listen to artist 1 with probability $1-\pi$. Furthermore, note that the niche artist 1 prefers the user-centric allocation rule to the pro-rata allocation rule (i.e., the payment to niche artist 1 with user-centric allocation rule is larger than her payment with pro-rata allocation rule) because fans of artist 1 have lower usage rate λ_1 and consequently, with unlimited consumption subscription fee model, end up paying more per unit of consumption than fans of artist 2. In Subsection 3.2.1, we show this result continues to hold with multiple artists.

2. $q_1 > q_2$: In this case, artist 2 preference depends on the ratio $\frac{\lambda_2}{\lambda_1}$. For $\frac{\lambda_2}{\lambda_1} \geq \frac{q_1}{q_2}$ (i.e., $\lambda_2 q_2 \geq \lambda_1 q_1$) artist 2 prefers the pro-rata allocation rule to the user-centric allocation rule. This is because the aggregate usage rate of fans of artist 2 is large enough that makes the pro-rata rule which is based on aggregate consumption more attractive. For $\frac{\lambda_2}{\lambda_1} < \frac{q_1}{q_2}$, however, artist 2 prefers user-centric allocation rule to pro-rata.

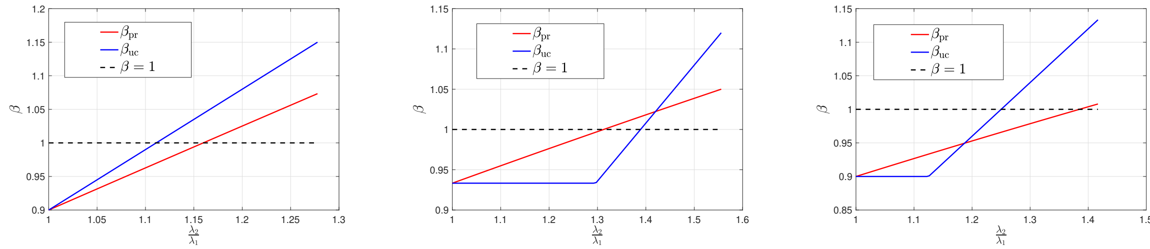
We next consider the platform's problem and characterize the preferred allocation rule of the platform. We denote by β^{pr} and β^{uc}, the smallest payout rate that guarantees the sustainability of the platform for pro-rata and user-centric allocation rules, respectively. Note that the platform's profit is $(1-\beta)R$ as any revenue-sharing rule mandates βR total payments to artists, hence the platform prefers the allocation rule that makes it sustainable with β as small as possible. Also note that platform is not sustainable if $\beta > 1$.

PROPOSITION 14. *There exists $\bar{q} \geq \frac{1}{2}$ such that:*

(a) If $q_1 \leq q_2$, then $\beta^{\mathrm{pr}} < \beta^{\mathrm{uc}}$ and the platform's profit is higher with the pro-rata allocation than with the user-centric allocation.

(b) If $q_1 \geq q_2$ and $q_1 \geq \bar{q}$, then $\beta^{\mathrm{pr}} > \beta^{\mathrm{uc}}$ and the platform's profit is higher with the user-centric allocation than with the pro-rata allocation.

(c) If $q_1 \geq q_2$ and $q_1 \in [\frac{1}{2}, \bar{q}]$, then there exists $\bar{r} \geq 1$ such that (i) if $\frac{\lambda_2}{\lambda_1} \geq \bar{r}$, then $\beta^{\mathrm{pr}} < \beta^{\mathrm{uc}}$ and the platform's profit is higher with the pro-rata allocation than with the user-centric allocation, and (ii) if $\frac{\lambda_2}{\lambda_1} < \bar{r}$, then $\beta^{\mathrm{pr}} > \beta^{\mathrm{uc}}$ and the platform's profit is higher with the user-centric allocation than with the pro-rata allocation.



(a) β^{uc} and β^{pr} for $q_1 = 0.4$ (b) β^{uc} and β^{pr} for $q_1 = 0.7$ (c) β^{uc} and β^{pr} for $q_1 = 0.6$

Figure 8 Illustration of Proposition 14 for $\pi = 0.85$.

Part (a) of Proposition 14 shows that in the presence of superstar artist 2 (i.e., an artist whose fans are not only majority of platform users, $q_2 > q_1$, but also have a high consumption rate, $\lambda_2 > \lambda_1$), the pro-rata allocation rule is always better than user-centric allocation rule. This is because in such situations the pro-rata allocation, unlike the user-centric allocation, implicitly compensates artist 2 for the positive externality her fans bring to the platform and to the consumption of the niche artist 1 content. Part (b) of Proposition 14 establishes that if artist 1 whose fans have a lower usage rate have a large enough majority of the users, then the platform's profits are higher with the user-centric allocation rule. This is because fans of artist 1 have a low usage rate and are a sufficiently large majority of platform users, so the positive externality of their presence on the platform is limited. Consequently, there is a limited effect of the pro-rata allocation that favors artists listened by high usage rate users. Furthermore, given low usage rate of its fans, artist 1 prefers the user-centric allocation, and since she is the artist with the better outside option (by potentially offering content through a direct channel and take away its fans from the platform) it is also in the interest of keeping all artists, that platform prefers the user-centric allocation rule. Finally, in part (c) of Proposition 14 the analysis is more nuanced and the preferred allocation rule depends on the ratio of the higher usage rate to the lower usage rate. In Subsection 3.2.2, we show that the results of Proposition 14 generalize to m artists with generic probabilities π_{ij}, for $i = 1, 2$ and $j = 1, \ldots, m$.

Figure 8 illustrates the results of Proposition 14. In this figure, β^{uc} and β^{pr} are the minimum β with which the platform can sustain both artists on it with user-centric and pro-rata allocation rules, respectively. Note that in part (a), corresponding to $q_1 < q_2$ and $\lambda_1 < \lambda_2$, the pro-rata allocation is not only preferred, but also there is a range of parameters for which the platform is sustainable only with the pro-rata and is not sustainable with the user-centric allocation (i.e., $\beta^{\text{pr}} \leq 1 < \beta^{\text{uc}}$).

Proof of Proposition 14: We use the following two lemmas in this proof.

LEMMA 10. *Suppose* $\frac{\pi}{1-\pi} \geq \max\{\frac{\lambda_2}{q_1\lambda_1}, \frac{\lambda_1}{\lambda_2 q_2}\}$ *and* $\frac{\lambda_2}{\lambda_1} \leq \min\{\frac{1}{q_2}, \frac{1-\pi q_1}{\pi q_2}\}$*. In equilibrium both artists join the platform and both user types subscribe and the profit of the platform becomes* $\lambda_1 - \pi q_1 \lambda_1 - \lambda_2 \pi q_2$*. Otherwise, in any equilibrium the profit of the platform is* 0*.*

Proof: We first show that with the assumptions of the lemma if an artist such as i offers her products through direct channels by herself, the fans of the other artist will not buy her content and her revenue will be $q_i \lambda_i \pi$. To see this, consider artist 1 and suppose the contrary, i.e., she sets the prices such that the fans of artist 2 subscribe. This means that the subscription fee is at most $\lambda_2(1-\pi)$. Now if this artist chooses the prices such that only her own fans subscribe, then her revenue will be at least $\lambda_1 \pi q_1$. Therefore, as long as we have $\lambda_1 \pi q_1 \geq \lambda_2(1-\pi)$ and $\lambda_2 \pi q_2 \geq \lambda_1(1-\pi)$, it is optimal for each artist to choose prices such that only her own fans join. Now consider having both artists on the platform. Because $\frac{\lambda_1}{\lambda_1} \leq \frac{1}{q_2}$, the optimal subscription fee is λ_1 and all fans subscribe. This generates revenue λ_1 for the platform. The payment that the platform need to pay artist i is the minimum payment that makes artist i indifferent between staying on the platform and offering her content by herself. As we have shows, the revenue she obtains from going solo is $\lambda_i \pi q_i$. Therefore, the overall profit of the platform is

$$\lambda_1 - \lambda_1 \pi q_1 - \lambda_2 \pi q_2.$$

If the platform wants only artist i to join it, then again as we showed only fans of artist i subscribe and the profit of platform becomes 0. Therefore, both artists joining and all users subscribing is an equilibrium if $\lambda_1 - \lambda_1 \pi q_1 - \lambda_2 \pi q_2$, which holds because of the bounds on $\frac{\lambda_2}{\lambda_1}$. ■

LEMMA 11. *Suppose* $\frac{\pi}{1-\pi} \geq \max\{\frac{\lambda_2}{q_1 \lambda_1}, \frac{\lambda_1}{\lambda_2 q_2}\}$ *and* $\frac{\lambda_2}{\lambda_1} \leq \min\{\frac{1}{q_2}, \frac{1-\pi q_1}{\pi q_2}\}$. *We have the following:*

(a) ***Pro-rata:*** *For* $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2}$ *we have* $\beta^{\mathrm{pr}} = \pi q_1 \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_1 \pi + q_2 \frac{\lambda_2}{\lambda_1}(1-\pi)}$ *in which case the platform is paying artist* 2 *more than the revenue she would obtain if she deviates and offers her music by herself. For* $\frac{\lambda_2}{\lambda_1} \geq \frac{q_1}{q_2}$ *we have* $\beta^{\mathrm{pr}} = \pi q_2 \frac{\lambda_2}{\lambda_1} \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_2 \pi \frac{\lambda_2}{\lambda_1} + q_1(1-\pi)}$ *in which case the platform is paying artist* 1 *more than the revenue she would obtain if she deviates and offers her music by herself.*

(b) ***User-centric:*** *For* $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2} \frac{q_2 \pi + q_1(1-\pi)}{q_1 \pi + q_2(1-\pi)}$ *we have* $\beta^{\mathrm{uc}} = \frac{\pi q_1}{q_1 \pi + q_2(1-\pi)}$ *in which case the platform is paying artist* 2 *more than the revenue she would obtain if she deviates and offers her music by herself through a direct channel. For* $\frac{\lambda_2}{\lambda_1} \geq \frac{q_1}{q_2} \frac{q_2 \pi + q_1(1-\pi)}{q_1 \pi + q_2(1-\pi)}$ *we have* $\beta^{\mathrm{uc}} = \frac{\pi q_2 \frac{\lambda_2}{\lambda_1}}{q_2 \pi + q_1(1-\pi)}$ *in which case the platform is paying artist* 1 *more than the revenue she would obtain if she deviates and offers her music by herself.*

Proof: For pro-rata allocation rule we must have

$$\lambda_1 \beta^{\mathrm{pr}} \frac{\lambda_1 q_1 \pi + q_2 \lambda_2 (1-\pi)}{\lambda_1 q_1 + \lambda_2 q_2} \geq \lambda_1 \pi q_1,$$

where we used Lemma 10 to write the right-hand side, i.e., the revenue of an artist for offering her content directly. Similarly, we have

$$\lambda_1 \beta^{\mathrm{pr}} \frac{\lambda_2 q_2 \pi + q_1 \lambda_1 (1-\pi)}{\lambda_1 q_1 + \lambda_2 q_2} \geq \lambda_2 \pi q_2.$$

Therefore, the minimum β to have both inequalities is

$$\beta^{\mathrm{pr}} = \max\left\{\pi q_1 \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_1 \pi + q_2 \frac{\lambda_2}{\lambda_1}(1-\pi)}, p q_2 \frac{\lambda_2}{\lambda_1} \frac{q_1 + q_2 \frac{\lambda_2}{\lambda_1}}{q_2 \pi \frac{\lambda_2}{\lambda_1} + q_1(1-\pi)}\right\}.$$

By comparing the two terms of the maximum and canceling out the common terms, we see that the first term of the maximum is higher than the second term if and only if $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2}$

For user-centric allocation rule we must have

$$\lambda_1 \beta^{\mathrm{uc}} (q_1 \pi + q_2(1-\pi)) \geq \lambda_1 \pi q_1,$$

and

$$\lambda_1 \beta^{\mathrm{uc}} (q_2 \pi + q_1(1-\pi)) \geq \lambda_2 \pi q_2,$$

which leads to

$$\beta^{\mathrm{uc}} = \max\left\{\pi q_1 \frac{1}{q_1\pi + q_2(1-\pi)}, \pi q_2 \frac{\lambda_2}{\lambda_1}\frac{1}{q_2\pi + q_1(1-\pi)}\right\}.$$

By comparing the two terms of the maximum and canceling out the common terms, we see that the first term of the max is higher than the second term if and only if $\frac{\lambda_2}{\lambda_1} \leq \frac{q_1}{q_2}\frac{q_2\pi+q_1(1-\pi)}{q_1\pi+q_2(1-\pi)}$. ■

We now proceed with the proof of proposition. Given $q_1 \leq \frac{1}{2}$, using Lemma 11, we obtain

$$\beta^{\mathrm{pr}} = \pi q_2 \frac{\lambda_2}{\lambda_1}\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_2\pi\frac{\lambda_2}{\lambda_1} + q_1(1-\pi)}, \quad \beta^{\mathrm{uc}} = \pi q_2 \frac{\lambda_2}{\lambda_1}\frac{1}{q_2\pi + q_1(1-\pi)}.$$

Comparing these two expressions we find out $\beta^{\mathrm{pr}} \leq \beta^{\mathrm{uc}}$. This proves part (a). We next prove parts (b) and (c). Given, $q_1 \geq q_2$, we have the following cases:

- $\frac{\lambda_2}{\lambda_1} \in [1, \frac{q_1}{q_2}\frac{q_2\pi+q_1(1-\pi)}{q_1\pi+q_2(1-\pi)}]$: In this interval, we have

$$\beta^{\mathrm{uc}} = \frac{q_1\pi}{q_1\pi + q_2(1-\pi)} \leq q_1\pi\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_1\pi + q_2\frac{\lambda_2}{\lambda_1}(1-\pi)} = \beta^{\mathrm{pr}}.$$

- $\frac{\lambda_2}{\lambda_1} \in [\frac{q_1}{q_2}\frac{q_2\pi+q_1(1-\pi)}{q_1\pi+q_2(1-\pi)}, \frac{q_1}{q_2}]$: In this interval, we have

$$\beta^{\mathrm{uc}} = \frac{q_2\pi\frac{\lambda_2}{\lambda_1}}{q_2\pi + q_1(1-\pi)} \leq q_1\pi\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_1\pi + q_2\frac{\lambda_2}{\lambda_1}(1-\pi)} = \beta^{\mathrm{pr}},$$

if and only if $\frac{\lambda_2}{\lambda_1} \leq \bar{r}$, where $\bar{r}$ is the larger solution of $\frac{q_2\pi\frac{\lambda_2}{\lambda_1}}{q_2\pi+q_1(1-\pi)} = q_1\pi\frac{q_1+q_2\frac{\lambda_2}{\lambda_1}}{q_1\pi+q_2\frac{\lambda_2}{\lambda_1}(1-\pi)}$.

- $\frac{\lambda_2}{\lambda_1} \in [\frac{q_1}{q_2}, \infty)$: In this range, using $2p-1 \geq 0$, we have

$$\beta^{\mathrm{uc}} = \frac{q_2\pi\frac{\lambda_2}{\lambda_1}}{q_2\pi + q_1(1-\pi)} \geq q_2\pi\frac{\lambda_2}{\lambda_1}\frac{q_1 + q_2\frac{\lambda_2}{\lambda_1}}{q_2\frac{\lambda_2}{\lambda_1}\pi + q_1(1-\pi)} = \beta^{\mathrm{pr}}.$$

Therefore, for $\frac{\lambda_2}{\lambda_1} \geq \bar{r}$ we have $\beta^{\mathrm{uc}} \geq \beta^{\mathrm{pr}}$ and for $\frac{\lambda_2}{\lambda_1} \leq \bar{r}$ we have $\beta^{\mathrm{uc}} \leq \beta^{\mathrm{pr}}$. Now the question is whether pro-rata or user-centric can sustain $\frac{\lambda_2}{\lambda_1} \geq \bar{r}$ in equilibrium. This depends on whether β^{uc} is larger than 1 at $\bar{r}$ or not. There exists a $\bar{q}$ such that for $q_1 \geq \bar{q}$, β^{uc} (which is equal to β^{pr} at $\bar{r}$) is larger than 1 at $\frac{\lambda_2}{\lambda_1} = \bar{r}$. Finally, for $q_1 \leq \bar{q}$, β^{uc} and β^{pr} cross each other within the range that these two payments can sustain equilibrium. This completes the proof. ■

B.2.2. Detail of Example 1 Assuming all artists either belong to X or W, using (10), we obtain

$$\begin{aligned}\beta^{\mathrm{pr}} &= \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1}\max\left\{\max_{j\in X}\frac{q_2\lambda_2\pi_{2j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}, \max_{j\in W}\frac{\lambda_1 q_1\pi_{1j}}{q_1\lambda_1\pi_{1j} + q_2\lambda_2\pi_{2j}}\right\}\\ &= \frac{q_1\lambda_1 + q_2\lambda_2}{\lambda_1}\max\left\{\frac{q_2\lambda_2 x}{q_1\lambda_1 + q_2\lambda_2 x}, \frac{\lambda_1 q_1}{q_1\lambda_1 + q_2\lambda_2 w}\right\}.\end{aligned}$$

Therefore, the pivot artist is in the set X (for which $\frac{\pi_{2j}}{\pi_{1j}} \geq 1$), if and only if we have $\frac{q_2\lambda_2 x}{q_1\lambda_1+q_2\lambda_2 x} \geq \frac{\lambda_1 q_1}{q_1\lambda_1+q_2\lambda_2 w}$. After some simplification, this results in $\frac{\lambda_2 q_2}{\lambda_1 q_1} \geq \frac{1}{xw}$ which is the same as part (a) of Example 1. Therefore, Proposition 3 establishes that the platform prefers pro-rata rule to user-centric rule.

Using (11), we obtain

$$\begin{aligned}\beta^{\mathrm{uc}} &= \frac{1}{\lambda_1}\max\left\{\max_{j\in X}\frac{q_2\lambda_2\pi_{2j}}{q_1\pi_{1j} + q_2\pi_{2j}}, \max_{j\in W}\frac{\lambda_1 q_1\pi_{1j}}{q_1\pi_{1j} + q_2\pi_{2j}}\right\}\\ &= \frac{1}{\lambda_1}\max\left\{\frac{q_2\lambda_2 x}{q_1 + q_2 x}, \frac{\lambda_1 q_1}{q_1 + q_2 w}\right\}.\end{aligned}$$

Therefore, the pivot artist is in the set W (for which $\frac{\pi_{2j}}{\pi_{1j}} \leq 1$), if and only if we have $\frac{\lambda_1 q_1}{q_1+q_2 w} \geq \frac{q_2\lambda_2 x}{q_1+q_2 x}$. After some simplification, this results in the same condition as part (b) of Example 1. Therefore, Proposition 3 establishes that the platform prefers user-centric rule to pro-rata rule.