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Prova 1 Eletromagnetismo

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Gabarito

1) $V_d(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

$$V_f(r,\theta) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

Solução geral $V(r,\theta) = \begin{cases} V_d(r,\theta) & \text{se } r \leq R \\ V_f(r,\theta) & \text{se } r \geq R \end{cases}$

O potencial na superfície é

$$V(r=R,\theta) = V_0(\theta) = \frac{K}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right]$$

a) Vou calcular os coeficientes A_n para um potencial $V_0(\theta)$ genérico. O potencial deve ser contínuo em todo o espaço. Por exemplo

$$V_d(r=R,\theta) = V_0(\theta)$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n R^n P_n = V_0(\theta) \quad (1)$$

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Irei explorar o fato de P_n formar um conjunto completo.

$$\int_0^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & \text{se } n \neq m \\ \dfrac{2}{2n+1} & \text{se } n = m \end{cases} \quad (2)$$

Para isso multiplico a eq. (1) por $\int P_m(\cos\theta)\sin\theta\, d\theta$, Temos

$$\sum_{n=0}^{\infty} A_n R^n \int P_n(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta$$

$$= \int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta\, d\theta$$

Da Eq. (2), apenas o termo $n=m$ sobrevive

$$\Rightarrow A_m R^m \frac{2}{2m+1} = \int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow A_n = \frac{2n+1}{2R^n}\int_0^{\pi} V_0(\theta) P_n(\cos\theta)\sin\theta\, d\theta$$

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b) A eq. desejada envolve A_n e B_n, logo vou explorar a continuidade do potencial envolvendo V_d e V_f. Em $r=R$ ambas as funções tem que ser iguais

$$V_d(R,\theta) = V_f(R,\theta)$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = \sum_{n=0}^{\infty} \frac{B_n}{R^{n+1}} P_n(\cos\theta)$$

Duas séries em P_n são iguais se os respectivos coeficientes são iguais

$$A_n R^n = \frac{B_n}{R^{n+1}} \Rightarrow \underline{B_n = A_n R^{2n+1}} \quad (3)$$

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c) Vou calcular explicitamente todos os coeficientes A_n e B_n. Do item a) temos

$$A_n = \frac{2n+1}{2R^n}\frac{K}{5}\left(8\int P_3 P_n - 3\int P_1 P_n\right) \text{ sinodo}$$

Cada integral $\int P_q P_n$ sinodo é resolvida de acordo com a Eq. (2)

$$A_0 = \frac{1}{2}\frac{K}{5}\left(8\int P_3 P_0 - 3\int P_1 P_0\right)\sin\theta\, d\theta$$

$$\boxed{A_0 = 0}$$

$$A_1 = \frac{3}{2R}\frac{K}{5}\left(8\int P_3 P_1 - 3\int P_1 P_1\right)\sin\theta\, d\theta$$

$$A_1 = \frac{-3K}{10R}\, 3\int \left[P_1(\cos\theta)\right]^2 \sin\theta\, d\theta$$

$$A_1 = -\frac{9K}{10R}\cdot\frac{2}{3} = \boxed{-\frac{3K}{5R}}$$

$$A_2 = \frac{5}{2R^2}\frac{K}{5}\left(8\int P_3 P_2 - 3\int P_1 P_2\right)\sin\theta\, d\theta$$

$$\Rightarrow \boxed{A_2 = 0}$$

$$A_3 = \frac{7}{2R^3}\frac{K}{5}\left(8\int \left[P_3(\cos\theta)\right]^2 - 3\int P_1 P_3\right)\sin\theta\, d\theta$$

$$A_3 = \frac{7K}{10R^3}\, 8\, \frac{2}{7} = \boxed{\frac{8K}{5R^3}}$$

Da mesma forma que A_0 e A_2, A_n para $n = 4, 5, 6, 7, \ldots$, irão envolver integrais cruzados, que serão zero de acordo com a eq. 2. Assim

$$A_n = 0 \quad \text{para } n = 0, 2, 4, 5, 6, 7, \ldots$$

Os únicos termos diferentes de zero são A_1 e A_3. Da mesma forma, apenas B_1 e B_3 são diferente de zero. Do item b

$$B_1 = A_1 R^3 = -\frac{3KR^2}{5}$$

$$B_3 = A_3 R^7 = \frac{8K}{5R^3} R^7 = \frac{8}{5} K R^4$$

A solução geral é para $r \leq R$

$$V_d(r,\theta) = A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta)$$

$$= -\frac{3K}{5R} r P_1(\cos\theta) + \frac{8K}{5R^3} r^3 P_3(\cos\theta)$$

Só para $r \geq R$

$$V_f = \frac{B_1}{r^2} P_1(\cos\theta) + \frac{B_3}{r^4} P_1(\cos\theta)$$

$$V_f(r,\theta) = -\frac{3KR^2}{5r^2} P_1(\cos\theta) + \frac{8KR^4}{5r^4} P_3(\cos\theta)$$

d) A densidade de carga é dado por

$$\left(\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n}\right)\Bigg|_S = \frac{1}{\varepsilon_0} \sigma_0(\theta)$$

Em nosso caso temos $V_{out} = V_f$

$V_{in} = V_d$, $n = r$ e S implica $r = R$. Logo

$$\left(\frac{\partial V_f}{\partial r} - \frac{\partial V_d}{\partial r}\right)\Bigg|_{r=R} = -\frac{1}{\varepsilon_0} \sigma_0(\theta)$$

No caso geral

$$-\sum_{n=0}^{\infty} (l+1) \frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = \frac{-\sigma_0(\theta)}{\varepsilon_0}$$

Usando a Eq 3

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$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\varepsilon_0}\sigma_0(\theta)$$

Considerando apenas $l=1$ e $l=3$

$$(2+1)A_1 P_1 + (6+1)A_3 R^2 P_3 = \frac{1}{\varepsilon_0}\sigma_0(\theta)$$

$$\boxed{\sigma_0(\theta) = \frac{\varepsilon_0 K}{5R}\left[56P_3(\cos\theta) - 9P_1(\cos\theta)\right]}$$

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2) $\rho(r,\theta) = K\frac{R}{r^2}(R-2r)\sin\theta$

a) Expansão de Multipolos

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta)\rho(\vec{r}')d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[V_0(\vec{r}) + V_1(\vec{r}) + V_2(\vec{r}) + \dots\right]$$

$V_0(\vec{r})$, é o termo de monopolo

$$V_0(\vec{r}) = \frac{1}{r}\int P_0(\cos\theta)\rho(\vec{r}')d\tau' = \frac{1}{r}\int \rho(\vec{r}')d\tau'$$

pois $P_0(\cos\theta) = 1$

$$V_0 = \frac{1}{r} K R \int \frac{R - 2r'}{r'^2} \sin\theta' \, r'^2 \sin\theta' \, dr' d\theta' d\varphi'$$ (8)

$$V_0 = \frac{KR}{r} \int_0^R (R - 2r') dr \int_0^{\pi} \sin\theta'^2 d\theta' \int_0^{2\pi} d\varphi'$$

$$\int_0^R (R - 2r') dr' = \left(Rr - 2\frac{r'^2}{2}\right)\Big|_0^R = 0$$

$$\Rightarrow \boxed{V_0(\vec{r}) = 0}$$

b) Il termo de dipolo é $V_1(\vec{r})$

$$V_1(\vec{r}) = \frac{1}{r^2} \int (r')^1 P_1(\cos\theta) \rho(\vec{r}') d\tau'$$

$$V_1(\vec{r}) = \frac{1}{r^2} \int r' \cos\theta' \, K \frac{R}{r'^2} (R - 2r') \sin\theta' \, r'^2 \sin\theta' \, dr' d\theta' d\varphi'$$

$$V_1(\vec{r}) = \frac{KR}{r^2} \int_0^R r'(R - 2r') dr' \int_0^{2\pi} d\varphi' \int \sin^2\theta' \cos\theta' d\theta'$$

$$\int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta' = \frac{1}{3} \sin^3\theta \Big|_0^{2\pi} = 0$$

$$\Rightarrow \boxed{V_1(\vec{r}) = 0}$$

c) $V_2(\vec{r})$ = termo de quadrupolo

$$V_2(\vec{r}) = \frac{1}{r^3}\int (r')^2 P_3(\cos\theta')\rho(\vec{r}')\,d\tau'$$

$$V_2(\vec{r}) = \frac{KR}{2r^3}\int r'^2\left(3\cos^2\theta' - 1\right)\frac{R-2r}{r^2}\,r'^2\sin\theta'\,dr'd\theta'd\varphi'$$

$$V_2(\vec{r}) = \frac{KR}{2r^3}\, I_r\, I_\theta\, I_\varphi$$

$$I_\varphi = \int_0^{2\pi} d\varphi' = 2\pi$$

$$I_r = \int_0^R (R - 2r')\,r'^2\,dr' = \int_0^R \left(Rr'^2 - 2r'^2\right)dr'$$

$$I_r = \left(\frac{1}{3}Rr'^3 - \frac{2}{4}r'^4\right)\Big|_0^R = -\frac{R^4}{6}$$

$$I_\theta = \int_0^\pi \left(3\cos^2\theta' - 1\right)\sin\theta'\,d\theta' = -\frac{\pi}{8}$$

$$\Rightarrow V_2(\vec{r}) = \frac{KR}{\cancel{2}r^3}\left(-\frac{1}{6}R^4\right)\left(-\frac{\pi}{8}\right)\cdot\cancel{2}\pi$$

$$V_2(\vec{r}) = \frac{5K\pi^2R^5}{48r^3}$$