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Prova 1, Curso: Física. Gabarito

Disciplina: Métodos Matemáticos. Prof. Paulo Freitas Gomes.

1a) 1 ponto Temos que $\vec{V}_1 = \hat{x}V_{1x} + \hat{y}V_{1y} + \hat{z}V_{1z}$, logo:

$$V_{1x} = x^2 \qquad \therefore \qquad V_{1y} = 3xyz \qquad \therefore \qquad V_{1z} = 2x^2\sqrt{z}$$

O divergente fica então:

$$\vec{\nabla} \cdot \vec{V}_1 = \frac{dV_{1x}}{dx} + \frac{dV_{1y}}{dy} + \frac{dV_{1z}}{dz} = \frac{d(x^2)}{dx} + 3\frac{d(xzy)}{dy} + 2\frac{d(x^2\sqrt{z})}{dz} = \boxed{2x + 3xz + \frac{x^2}{\sqrt{z}}}$$

1b) 1 ponto Temos que $\vec{V}_2 = \hat{x}V_{2x} + \hat{y}V_{2y} + \hat{z}V_{2z}$, logo:

$$V_{2x} = \ln x \qquad \therefore \qquad V_{2y} = z\cos y \qquad \therefore \qquad V_{2z} = 2xe^z$$

O divergente fica então:

$$\vec{\nabla} \cdot \vec{V}_2 = \frac{dV_{2x}}{dx} + \frac{dV_{2y}}{dy} + \frac{dV_{2z}}{dz} = \frac{d(\ln x)}{dx} + z\frac{d(\cos y)}{dy} + 2x\frac{d(e^z)}{dz} = \boxed{\frac{1}{x} - z\sin y + 2xe^z}$$

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2) O laplaciano fica 0.5 ponto:

$$\nabla^2 f_1 = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\cos x + 3\frac{xz^2}{y}\right)$$

Calculando cada derivada separadamente, temos 0.5 ponto:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(\cos x + 3\frac{xz^2}{y}\right) &= \frac{\partial}{\partial x}\left(3\frac{z^2}{y} - \sin x\right) = -\cos x \\
\frac{\partial^2}{\partial y^2}\left(\cos x + 3\frac{xz^2}{y}\right) &= -3\frac{\partial}{\partial y}\left(\frac{xz^2}{y^2}\right) = 6\frac{xz^2}{y^3} \\
\frac{\partial^2}{\partial z^2}\left(\cos x + 3\frac{xz^2}{y}\right) &= 6\frac{\partial}{\partial z}\left(\frac{xz}{y}\right) = 6\frac{x}{y}
\end{aligned}$$

Somando tudo 0.5 ponto:

$$\nabla^2 f_1 = 6\frac{xz^2}{y^3} - \cos x + 6\frac{x}{y} = \boxed{6\frac{x}{y}\left(\frac{z^2}{y^2} + 1\right) - \cos x}$$

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3) Primeiro vou calcular o gradiente 0.5 ponto:

$$\vec{v} = \vec{\nabla}p = \hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z} = \hat{i}(3yx^2 + \sin y) + \hat{j}(x^3 + x\cos y + e^z) + \hat{k}ye^z = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z$$

Temos então que 0.5 ponto:

$$v_x = 3yx^2 + \sin y \qquad \therefore \qquad v_y = x^3 + x\cos y + e^z \qquad \therefore \qquad v_z = ye^z$$

Agora, vou calcular o divergente desse gradiente 0.5 ponto:

$$\begin{aligned} h = \vec{\nabla}\cdot\vec{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial x}(3yx^2 + \sin y) + \frac{\partial}{\partial y}(x^3 + x\cos y + e^z) + \frac{\partial}{\partial z} ye^z \\ &= 6yx - x\sin y + ye^z \end{aligned}$$

Para o laplaciano, calculo primeiro cada derivada separadamente:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} h &= \frac{\partial^2}{\partial x^2}(6yx - x\sin y) = \frac{\partial}{\partial x}(6y - \sin y) = 0 \\ \frac{\partial^2}{\partial y^2} h &= \frac{\partial^2}{\partial y^2}(6yx - x\sin y + ye^z) = \frac{\partial}{\partial y}(6x - x\cos y + e^z) = x\sin y \\ \frac{\partial^2}{\partial z^2} h &= \frac{\partial^2}{\partial z^2}(ye^z) = \frac{\partial}{\partial z}(ye^z) = ye^z \end{aligned}$$

Por fim, somando os três termos 0.5 ponto:

$$\nabla^2 h = \nabla^2 \left[\nabla\cdot\left(\vec{\nabla} p\right)\right] = \boxed{x\sin y + ye^z}$$

4a) 0.5 ponto Da figura 1a), temos que $x = r\cos\theta$ e $y = r\sin\theta$. b) 0.5 ponto Ainda da figura temos que $\tan\theta = \dfrac{y}{x}$, logo $\theta = \arctan\dfrac{y}{x}$. Também do triângulo $r = \sqrt{x^2 + y^2}$.

4c) 0.5 ponto Definição de complexo conjugado: $\bar{z} = x - iy$. Logo:

$$|z|^2 = x^2 + y^2 = (x - iy)(x + iy) = z\bar{z}$$

4d) 0.5 ponto Temos que $\mathrm{Re}(z) = x$ e $\mathrm{Im}(z) = y$, logo:

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(x + iy + x - iy) = x = \mathrm{Re}(z)$$

Da mesma forma:

$$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}(x + iy - x + iy) = y = \mathrm{Im}(z)$$

5a) 0.5 ponto Seja $z_0 = 1 - i = r_0 e^{i\theta_0}$, logo $z_1 = z_0^8 = r_0^8 e^{8i\theta_0}$. Temos que $r_0 = \sqrt{1+1} = \sqrt{2}$ e $\tan\theta_0 = -\dfrac{1}{1} = -1$, logo $\theta_0 = -\dfrac{\pi}{4}$ radianos (veja figura 1b). Então:

$$z_1 = (\sqrt{2})^8 \exp\left(-8i\frac{\pi}{4}\right) = 2^4[\cos(-2\pi) + i\sin(-2\pi)] = \boxed{16}$$

onde usamos a fórmula de Euler $e^{i\theta} = \cos\theta + i\sin\theta$.

5b) 0.5 ponto Da definição de cosseno de um número complexo, temos que:

$$z_2 = \cos(i\pi) = \frac{e^{ii\pi} + e^{-ii\pi}}{2} = \frac{1}{2}\left(e^{-\pi} + e^{\pi}\right)$$

5c) 0.5 ponto Temos que $-e = ee^{i\pi} = e^{1+i\pi}$ (veja figura 1c). Assim:

$$\ln(-e) = \ln e^{1+i\pi} = 1 + i\pi$$

5d) 0.5 ponto Seja $a = i - 1 = r_a e^{i\theta_a}$ e $b = i + 1 = r_b e^{i\theta_b}$, logo

$$r_a = r_b = r = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \therefore \quad \theta_a = 3\pi/4 \quad \therefore \quad \theta_b = \pi/4$$

Veja figura 1d para definição dos ângulos. Temos que $z_4 = a^b = e^{b \ln a}$. Calculando o logaritmo:

$$\ln a = \ln\left(r_a e^{i\theta_a}\right) = i\theta_a + \ln r_a = \frac{3}{4}\pi i + \ln(\sqrt{2})$$

Usando a fórmula de Euler temos $b = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)]$. Logo, calculando o expoente de z_4 temos:

$$\begin{aligned}
b\ln a &= \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)]\left(\frac{3}{4}\pi i + \ln(\sqrt{2})\right) \\
&= \sqrt{2}\left[\cos(\pi/4)\frac{3}{4}\pi i + i\sin(\pi/4)\frac{3}{4}\pi i + \cos(\pi/4)\ln(\sqrt{2}) + i\sin(\pi/4)\ln(\sqrt{2})\right] \\
&= \sqrt{2}\cos(\pi/4)\ln(\sqrt{2}) - \sqrt{2}\sin(\pi/4)\frac{3}{4}\pi + \left[\sqrt{2}\cos(\pi/4)\frac{3}{4}\pi + \sqrt{2}\sin(\pi/4)\ln(\sqrt{2})\right] i \\
&= \sqrt{2}\frac{\sqrt{2}}{2}\ln(\sqrt{2}) - \sqrt{2}\frac{\sqrt{2}}{2}\frac{3}{4}\pi + \left[\sqrt{2}\frac{\sqrt{2}}{2}\frac{3}{4}\pi + \sqrt{2}\frac{\sqrt{2}}{2}\ln(\sqrt{2})\right] i \\
&= \frac{1}{2}\ln 2 - \frac{3}{4}\pi + \left[\frac{3}{4}\pi + \frac{1}{2}\ln 2\right] i \\
&= \alpha + i\beta
\end{aligned}$$

pois $\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$, onde:

$$\alpha = \frac{1}{2}\ln 2 - \frac{3}{4}\pi \qquad \therefore \qquad \beta = \frac{3}{4}\pi + \frac{1}{2}\ln 2$$

Voltando ao enunciado do exercício. Ainda não acabou, continuando:

$$z_4 = e^{b\ln a} = e^{(\alpha + i\beta)} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$$

Assim, as partes real e imaginária são:

$$\begin{aligned}
\mathrm{Re}(z_4) &= e^{\alpha}\cos\beta = \exp\left(\frac{1}{2}\ln 2 - \frac{3}{4}\pi\right)\cos\left(\frac{3}{4}\pi + \frac{1}{2}\ln 2\right) \\
\mathrm{Im}(z_4) &= e^{\alpha}\sin\beta = \exp\left(\frac{1}{2}\ln 2 - \frac{3}{4}\pi\right)\sin\left(\frac{3}{4}\pi + \frac{1}{2}\ln 2\right)
\end{aligned}$$

5e) 0.5 ponto Fazendo $a = 2$ e $b = i$, temos

$$z_5 = a^b = e^{b\ln a} = e^{i\ln 2} = \cos(\ln 2) + i\sin(\ln 2)$$

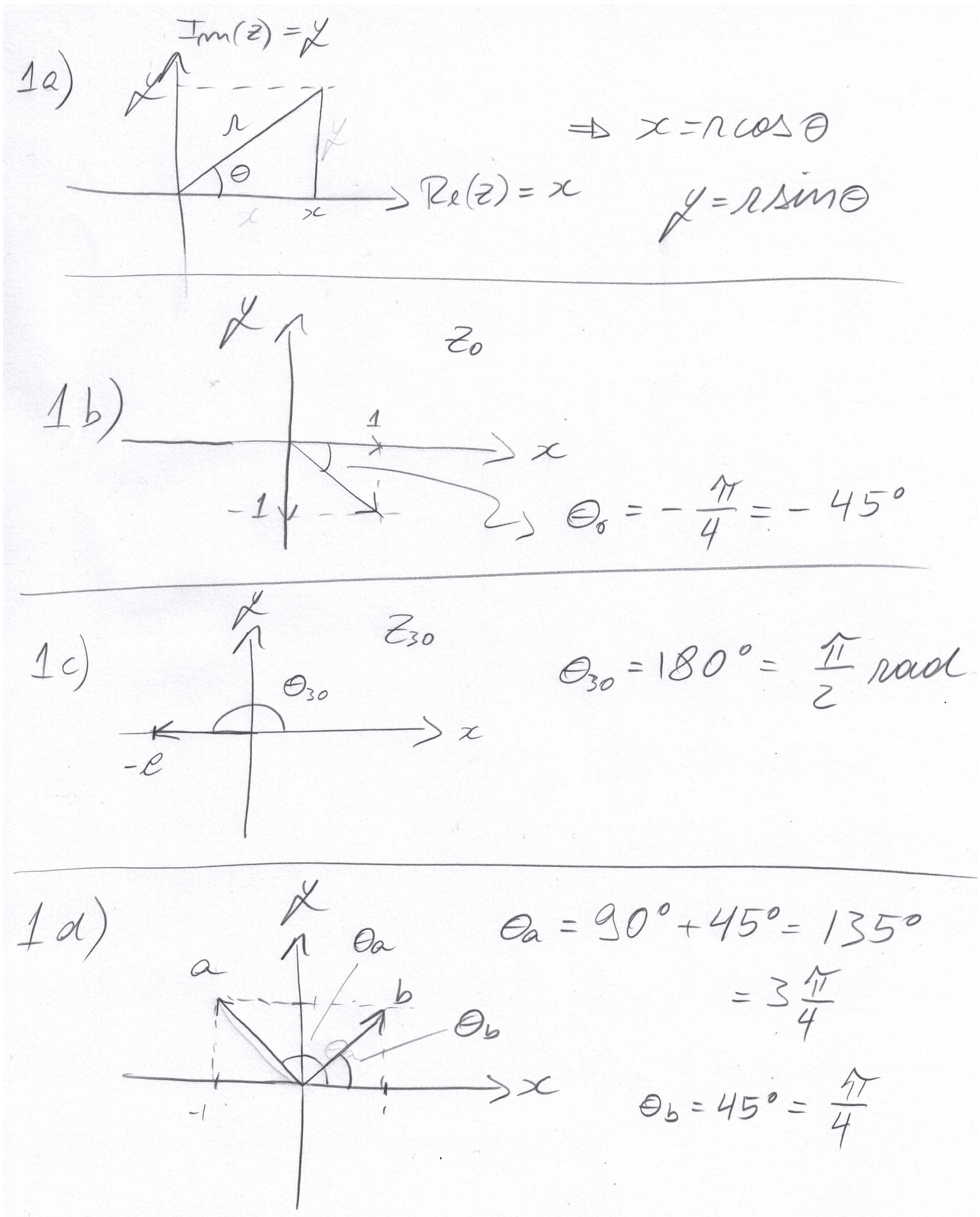

Figura 1: *Desenhos utilizados no gabarito. a) Referente ao problema 4a. b) Referente ao problema 5a. c) Referente ao problema 5c. d) Referente ao problema 5d.*