The University of Sheffield
ACS6101 Foundations of Control Systems
Week 4 Assignment

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1 Question 1

The aim of this task is to design a Phase-lead compensator using the Bode analysis in the frequency domain, and evaluate it in Matlab. Although the plant $G(s)$ is a third-order system Eq.(1), the design requirements can be approximated in terms of the natural frequency (ω_n) and damping ratio (ζ) of a second-order system.

$$G(s) = \frac{K}{s(s+a)(s+b)} \tag{1}$$
$$G(s) = \frac{K}{s(s+2.5)(s+27)}$$

1.1 Determine the gain K for a step response overshoot no more than 10%

There are two steps to follow in order to obtain the required gain K, the first one uses the second-order performance approximation in the frequency domain, and the second step uses the gain and angle condition.

The relation between the Percentage Overshoot ($P.O.$) in response to a unity step input of a second-order system in the **time domain**, and the Phase Margin (PM) of the Bode analysis in the **frequency domain**, can be written as follows,

$$P.O. = 100\ e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{2}$$
$$PM \approx 100\ \zeta \tag{3}$$

applying the requirement of $P.O. \leq 10\%$,

$$\zeta = \frac{\ln\left(\frac{100}{P.O.}\right)}{\sqrt{\pi^2 + \left[\ln\left(\frac{100}{P.O.}\right)\right]}} \tag{4}$$
$$\zeta = 0.59 \approx 0.60$$

Using Eq.(3), the desired Phase Margin (PM_d) the system should have in order to satisfy the $P.O.$ is defined as,

$$PM_d = 100 \times 0.6 = 60°$$

Now, it is necessary to obtain the new crossover frequency (ω'_c) where the PM_d is satisfied. Using the PM_d equation and the phase angle condition ($\phi(\omega'_c)$) as follows,

$$\phi(\omega'_c) = -180° + PM_d \tag{5}$$

applying the rule of angles into the Eq.(1),

$$\phi(\omega_c') = -90° - \arctan\frac{\omega_c'}{a} - \arctan\frac{\omega_c'}{b} \tag{6}$$

and after some algebraic operations ω_c',

$$PM_d - 180° = -90° - \arctan\frac{\omega_c'}{a} - \arctan\frac{\omega_c'}{b}$$
$$\frac{\omega_c'(a+b)}{a\ b} = \left(1 - \frac{\omega_c'^2}{a\ b}\right)\tan(180° - 90° - PM_d)$$

solving the quadratic equation with the corresponding values a, b, and PM_d, and using the high value of the solution,

$$\omega_c'^2 + \frac{a+b}{\tan(180° - 90° - PM_d)}\omega_c' - a\ b = 0 \tag{7}$$
$$\omega_c' = 1.29\ rad/sec$$

Finally, the gain K can be obtained from the **gain condition**, where it is established that the new crossover frequency ω_c' should be at 0 dB of magnitude, in other words,

$$|G(j\omega_c')| = 1 \tag{8}$$

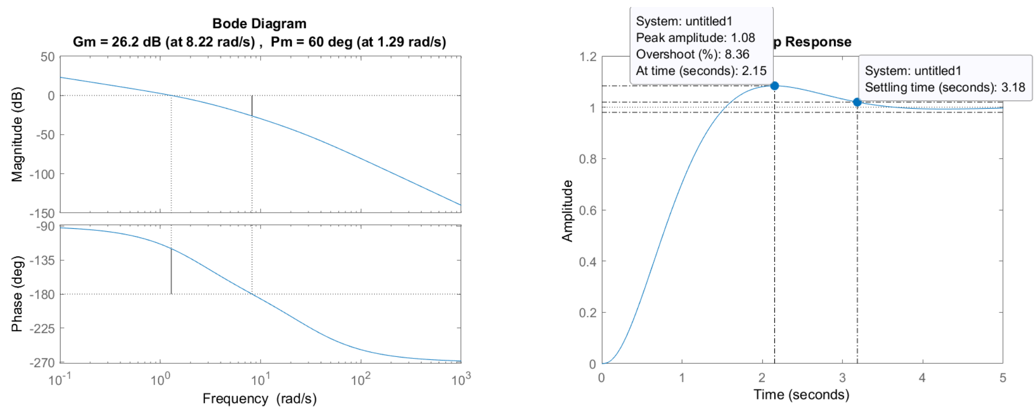
applying into the plant G, Eq.(1), and replacing the values

$$\frac{K}{\omega_c'\sqrt{\omega_c'^2 + a^2}\sqrt{\omega_c'^2 + b^2}} = 1$$
$$K = \omega_c'\sqrt{\omega_c'^2 + a^2}\sqrt{\omega_c'^2 + b^2}$$
$$K = 97.96$$

therefore, the plant becomes,

$$G(s) = \frac{97.96}{s(s+2.5)(s+27)}$$

In Fig. 1a it can be seen that the desired $PM_d = 60°$ is satisfied with the new gain, and step response has an $P.O. = 8.36$.



(a) Bode plot

(b) Step response

Figure 1: Evaluation of the system with the new gain

The following script in Matlab simulates the previous results.

```
%% ==========================================================================
% ---- ACS6101 Assignment week 4
% ---- Registration number: 180123717
% ---- Name: Paulo Roberto Loma Marconi
% ---- 23/10/2018
%% === Question 1 ==========================================================
clear; clc; close all;
%% plant G parameters
s = tf('s');
a = 2.5; b = 27;
%% a) Gain K for PO=10
PO = 10; % percentage overshoot
zeta = log(100/PO)/sqrt(pi^2+ (log(100/PO))^2 ); % damping ratio
PM_d = round(100*zeta)+1; % PM desired at the nearest round value
h = tand(180-90-PM_d);
omega_c = roots([1 (a+b)/h -a*b]); % new omega_c
K = omega_c(2)*sqrt(omega_c(2)^2+a^2)*sqrt(omega_c(2)^2+b^2); % new gain K
G = K/( s*(s+a)*(s+b) ); % Plant G
fig = figure(1);
margin(G);
saveas(fig,'Q1_a_margin.png');
fig = figure(2);
step(feedback(G,1));
saveas(fig,'Q1_a_stepp.png');
```

1.2 Design of the Phase-lead compensator

The Phase-Lead compensator has the following transfer function,

$$Gc = \frac{s+z}{s+p} \tag{9}$$

where the location of the zero is to the left of the pole, $|z| < |p|$.

The design requirements are the velocity error constant $K_v \leq 25$, and the step response $P.O. \leq 10\%$

The first step is to calculate the loop gain that satisfies the K_v is calculated as follows,

$$\begin{aligned} K_v &= \lim_{s\to 0} s\frac{K}{s(s+2.5)(s+27)} \\ K_g &= 1687 \end{aligned} \tag{10}$$

therefore, the new plant is,

$$\begin{aligned} G(s) &= \frac{K_g}{s(s+a)(s+b)} \\ G(s) &= \frac{1687}{s(s+2.5)(s+27)} \end{aligned} \tag{11}$$

Using the Eq.(2), Eq.(4), Eq.(3), it is obtained that for an overshoot of $P.O. = 10\%$, the damping ratio is $\zeta = 0.6$, and the desired phase margin is $PM_d = 60°$.

In order to obtain the ω_c that satisfies the gain condition with the new gain K, Eq.(8) is applied as follows,

$$\begin{aligned} |G(j\omega_c)| &= 1 \\ \left|\frac{K_g}{j\omega_c(j\omega_c+a)(j\omega_c+b)}\right| &= 1 \end{aligned}$$

after some algebraic operation,

$$\omega_c^2(\omega_c^2 + a^2)(\omega_c^2 + b^2) - K_g^2 = 0$$
$$\omega_c = 7.55\ rad/sec$$

Now, the actual phase margin PM_{act} is needed to calculate the additional phase (ϕ_m) required from the compensator,

$$\begin{aligned} PM_act &= 180° + \phi(\omega_c) \\ PM_act &= 180° - 90° - \arctan\frac{\omega_c}{a} - \arctan\frac{\omega_c}{b} \\ PM_act &= 2.65 \end{aligned} \tag{12}$$

so, the additional phase angle is,

$$\phi_m = PM_d + \theta - PM_act \tag{13}$$

where θ is a factor of correction. If $\theta = 0$,

$$\phi_m = 60° + 0° - 2.65$$
$$\phi_m = 57°$$

The additional phase margin is related with the zero and the pole of the Phase-Lead compensator as follows,

$$\begin{aligned} \alpha &= \frac{\sin\phi_m + 1}{1 - \sin\phi_m} \\ \alpha &= 11 \end{aligned} \tag{14}$$

where α is,

$$\alpha = \frac{p}{z} \tag{15}$$

The next step is to determine the new ω_c' using the following logarithm gain condition,

$$20\log|G(j\omega_c')| = -10\log\alpha \tag{16}$$
$$\frac{K_g}{\omega_c'\sqrt{\omega_c'^2 + a^2}\sqrt{\omega_c'^2 + b^2}} = \frac{1}{\sqrt{\alpha}} \tag{17}$$

after some operations,

$$\omega_c'^2(\omega_c'^2 + a^2)(\omega_c'^2 + b^2) - K_g^2\ \alpha = 0$$
$$\omega_c' = 13.50\ rad/sec$$

now, calculating the zero of the compensator,

$$\begin{aligned} \omega_c' &= \sqrt{p\ z} \\ z &= \frac{\omega_c'}{\alpha} \\ z &= 4.07 \end{aligned} \tag{18}$$

and the pole of the compensator using Eq.(15) is,

$$p = \alpha\ z$$
$$p = 44.78$$

The last step is to obtain the new gain of the **compensated system,**

$$\begin{aligned} K_{new} &= \sqrt{\alpha} K_g \\ K_{new} &= 5596.80 \end{aligned} \tag{19}$$

therefore, the compensated open-loop transfer function is as follows,

$$\begin{aligned} G_{ol} &= \frac{s+z}{s+p} \quad \frac{K_{new}}{s(s+a)(s+b)} \\ G_{ol} &= \frac{s+4.07}{s+44.78} \quad \frac{5596.80}{s(s+2.5)(s+27)} \end{aligned} \tag{20}$$

The following Matlab script calculates the Phase-Lead compensator with the design requirements.

```matlab
%% b) Phase-lead compensator
Kv = 25; % velocity error
Kg = Kv*a*b; % loop gain to satisfy Kv
G = Kg/( s*(s+a)*(s+b) ); % Plant G
% omega_c for the uncompensated system
omega_c = real( sqrt( roots([1 a^2+b^2 a^2*b^2 -Kg^2]) ) );
% obtaining actual PM
PM_act = 180-90-atand(omega_c(3)/a)-atand(omega_c(3)/b);
% additional phase angle from the compensator PM
theta = 0; % factor of correction
PM_c = round(PM_d+theta-PM_act);
% calculating alpha
alpha = round( (sind(PM_c)+1)/(1-sind(PM_c)) );
% Determine the new cross over frequency
omega_c_new = real( sqrt( roots([1 a^2+b^2 a^2*b^2 -Kg^2*alpha]) ) );
% Calculating the zero of the compensator
z = omega_c_new(3)/sqrt(alpha);
% Calculating the pole of the compensator
p = alpha*z;
% lead compensator
Gc = (s+z)/(s+p);
% new gain of the compensated system
K_new = sqrt(alpha)*Kg;
% compensated open-loop
Gol = K_new*Gc*G/Kg;
% closed-loop
Gcl = feedback(Gol,1);
```

1.3 Evaluation of the compensated system to a unit ramp input

The unit ramp in the LaPlace domain has the following equation,

$$R(s) = \frac{1}{s^2} \tag{21}$$

and the steady-state error to that unit ramp is,

$$ess_v = \frac{1}{K_v} \tag{22}$$

where,

$$K_v = \lim_{s \to 0} s\ G(s) \tag{23}$$

evaluating K_v for the compensated system G_{ol},

$$K_v = \lim_{s \to 0} s \left[\frac{s+z}{s+p} \quad \frac{K_{new}}{s(s+a)(s+b)} \right]$$
$$K_v = 7.53$$

therefore,

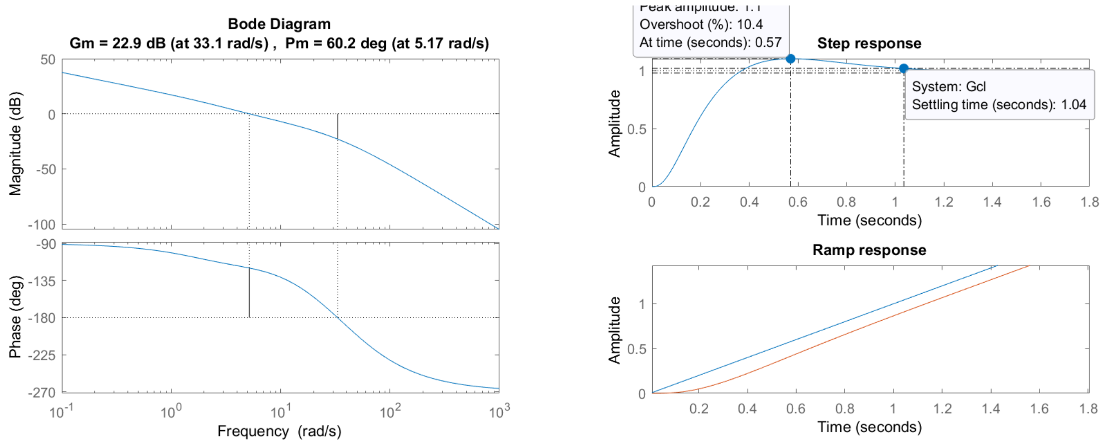
$$ess_v = 0.13$$

which is a relatively small steady-state error.

$$\tag{24}$$

1.4 Evaluation the performance of the compensated system

Fig. 2 shows the Bode plot, unit step response, and the unit ramp response. It can be seen that the desired $PM_d = 60°$ and the $P.O. = 10\%$ is satisfied, and the system has an improved the settling time, $t_s = 1.04$, see Table 1.



(a) Bode plot. (b) Step and ramp response.

Figure 2: Evaluation of the compensated system.

The following Matlab script was used to obtain Fig. 2.

```
%% c) evaluating the system to unit step and unit ramp
figure(2);
margin(Gol);
figure(3);
subplot(2,1,1);
step = 1/s; % step input
impulse(step,Gcl*step);
title('Step response');
subplot(2,1,2);
ramp = 1/s^2; % ramp input
impulse(ramp,Gcl*ramp);
title('Ramp response');
```

Quantity	**Value**
Steady state error to a unit ramp	0.13
Rise Time	0.24
Settling Time	1.04
Percentage Overshoot	10.38
Phase Margin	60.20
Gain Margin	22.90
Bandwidth	8.19
Peak Magnitude	0.57
Resonant frequency	3.42

Table 1: Performance evaluation.

```
% evaluating Kv for the new compensated system
Kv_new = (z/p)*K_new/(a*b);
% steady-state error to a unity ramp
ess_ramp = 1/Kv_new;
stepinfo(Gcl)
% bandwidth of Gcl
BW = bandwidth(Gcl);
% resonant frequency
omega_r = 4/(0.6*1.03) *sqrt(1-2*0.6^2);
```

1.5 Conclusion

Comparing the uncompensated system in Fig. 1, and the Phase-Lead compensated system in Fig. 2, the desired phase margin $PM_d = 60°$ and the overshoot $P.O. \leq 10\%$ were satisfied. Moreover, the settling time t_s was improved three times in the Phase-lead compensated, which demonstrates this compensator improves the transient response without altering the desired phase margin.

2 Question 2

The objective of this task is to design a Phase-Lead compensator, and a Phase-Lag compensator in series with a the first Phase-Lead compensator using two different approaches.

The plant to be studied is,

$$G(s) = \frac{K}{s^2(s+a)(s+b)} \tag{25}$$
$$G(s) = \frac{K}{s^2(s+9)(s+50)}$$

2.1 Determine the location of closed-loop dominant poles

The design requirements are the settling time $t_s \leq 2.9\ sec$, and the overshoot $P.O. \leq 20\%$. Although the plant is a fourth-order system, the compensator can be designed using the properties of a second-order system.

The settling time is,

$$t_s = \frac{4}{\zeta\omega_n} \tag{26}$$

where ζ is the damping ration and ω_n is the natural frequency. Using Eq.(4), and Eq.(26),

$$\zeta = 0.45, \quad \omega_n = 3.06\ rad/sec$$

therefore, the desired closed-loop dominant poles are,

$$\begin{aligned} r_{1,2} &= -\omega_n\zeta \pm j\ \omega_n\sqrt{1-\zeta^2} \\ r_{1,2} &= -1.38 \pm j\ 2.73 \end{aligned} \tag{27}$$

The following Matlab script calculates the previous results.

```
%% ======================================================================
% ---- ACS6101 Assignment week 4
% ---- Registration number: 180123717
% ---- Name: Paulo Roberto Loma Marconi
% ---- 27/10/2018
%% === Question 2 ========================================================
clear; clc; close all;
%% plant G parameters
s = tf('s');
a = 9; b = 50;
%% a) Location of the dominant poles for PO=20 and ts=2.9
PO = 20; % percentage overshoot
ts = 2.9; % settling time
zeta = log(100/PO)/sqrt(pi^2+ (log(100/PO))^2 ); % damping ratio
zeta = round(zeta,2)-0.01;
omega_n = 4/(zeta*ts);
% desired location of dominant poles
s1 = -omega_n*zeta+omega_n*sqrt(1-zeta^2)*1i;
```

2.2 Demonstrate the desired poles do not belong to the root locus

In order to demonstrate that the desired dominant poles $r_{1,2}$ do not belong to the root locus of the plant $G(s)$, the angle condition must not be satisfied,

Choosing on pole, $r_1 = -1.38 \pm j\ 2.73$,

$$\begin{aligned} &\measuredangle(G(r_1)) = -180° \\ &\measuredangle\left(\frac{K}{r_1^2(r_1+9)(r_1+50)}\right) = 180° \\ &-\measuredangle r_1 - \measuredangle r_1 - \measuredangle(r_1+9) - \measuredangle(r_1+50) = -80° \\ &-2\ \arctan\frac{2.73}{-1.38} - \arctan\frac{2.73}{9-1.38} - \arctan\frac{2.73}{50-1.38} = -180° \\ &103.44 \neq -180° \end{aligned} \tag{28}$$

if the second pole r_2 is evaluated,

$$-103.44 \neq -180°$$

2.3 Design of the Phase-Lead compensator

The Phase-Lead compensator has the following transfer function,

$$G_{lead} = \frac{s+z_{lead}}{s+p_{lead}}, \quad |z_{lead}| < |p_{lead}| \tag{29}$$

The design requirement is that the location of the compensator's zero is 1. Once again, the angle criteria is applied with the desired pole r_1 as follows,

$$\begin{aligned} &\measuredangle\left(\frac{s+z_{lead}}{s+p_{lead}} \quad \frac{K}{s^2(s+a)(s+b)}\right) = -180° \\ &\measuredangle\left(\frac{r_1+z_{lead}}{r_1+p_{lead}} \quad \frac{K}{r_1^2(r_1+a)(r_1+b)}\right) = -180° \\ &\measuredangle z_{lead} - \measuredangle p_{lead} - 2\ \measuredangle s - \measuredangle a - \measuredangle b = -180° \end{aligned} \tag{30}$$

if, $r_1 = -x + j\ y$

$$\left(180^\circ - \arctan \frac{y}{x - z_{lead}}\right) - \arctan \frac{y}{p_{lead} - x} - 2\left(180^\circ - \arctan \frac{y}{x}\right) ...$$
$$- \arctan \frac{y}{a - x} - \arctan \frac{y}{b - x} = 180^\circ$$

after some operations,

$$p_{lead} = 8.36$$

so, the Phase-Lead compensator is,

$$G_{lead} = \frac{s + 1}{s + 8.36}$$

Now, the gain K of the compensated system can be found using the gain condition,

$$\left| \frac{s + z_{lead}}{s + p_{lead}} \quad \frac{K}{s^2(s + 9)(s + 50)} \right| = 1$$
$$\left| \frac{r_1 + 1}{r_1 + 8.38} \quad \frac{K}{r_1^2(r_1 + 9)(r_1 + 50)} \right| = 1$$
$$K = 10047$$

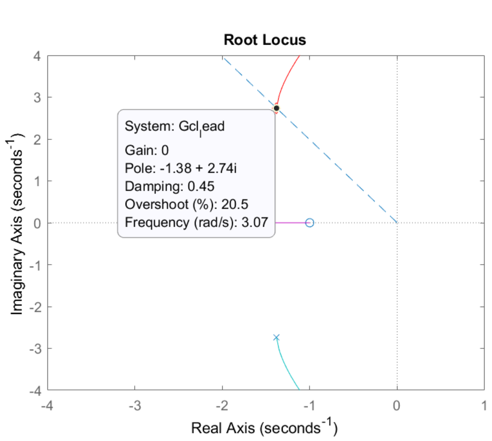
Therefore, evaluating the following open-loop compensated system,

$$G_{ol} = G_{lead}\ G(s) \tag{31}$$
$$G_{ol} = \frac{s + 1}{s + 8.36}\ \frac{10047}{s^2(s + 9)(s + 50)}$$

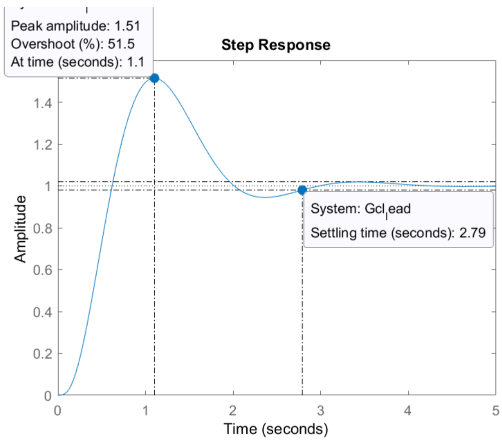
in Fig. 3 it can be seen that the close-loop desired poles belong to the root locus of the compensated system, but the overshoot is too high, so, it is recommended to use a Pre-filter to reduce the overshoot.

The script below simulates the Phase-Lead compensator.

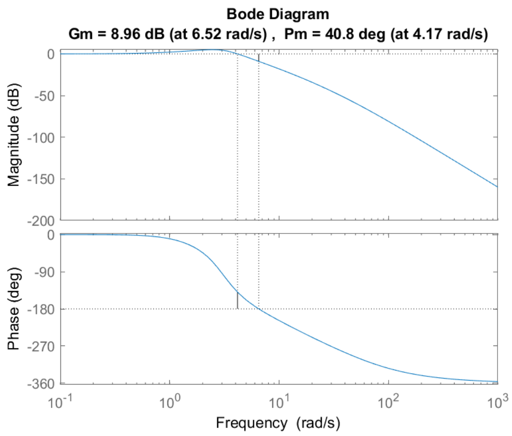
```
%% c) Phase lead compensator
z_lead = 1; % location of the desired zero
% using the angle condition to determine the location of the pole
x = -real(s1); y = imag(s1);
h = 180 + 180 - atand(y/(x-z_lead))-2*( 180-atand(y/x) )-atand(y/(a-x))...
-atand(y/(b-x));
p_lead = y/tand(h)+x;
% Gc compensator TF
Gc_lead = (s+z_lead)/(s+p_lead);
% obtaining the gain K with the gain condition equation
K = ((-x)^2+y^2)*sqrt((-x+a)^2+y^2)*sqrt((-x+b)^2+y^2)*...
sqrt((-x+p_lead)^2+y^2)/sqrt((-x+z_lead)^2+y^2);
% the open-loop system
G = K/(s^2*(s+a)*(s+b));% plant G
Gol_lead = Gc_lead*G; % open-loop with the Lead compensator
Gcl_lead = feedback(Gol_lead,1); % closed-loop with the Lead compensator

fig = figure(1);
rlocus(Gcl_lead);
hold;
% ploting the s1 and zeta in the rlocus
n = 0:1:160; m = n*sqrt(zeta^2/(1-zeta^2));
axis ([ -4 1 -4 4]);
plot (-m,n,'--'); % zeta
```



(a) Root locus.



(b) Step response.



(c) Bode plot.

Figure 3: Closed-loop system with the Lead compensator.

```
plot (-x,y,'rd');
saveas(fig,'Q2_Lead_rlocus.png');

fig = figure(2);
step(Gcl_lead);
saveas(fig,'Q2_Lead_step.png');

fig = figure(3);
margin(Gcl_lead);
saveas(fig,'Q2_Lead_margin.png');
BW_lead = bandwidth(Gcl_lead); % bandwidth
```

2.4 Adding a Pre-filter

The use of a Pre-filter in series with the closed-loop systems can reduce the overshoot canceling the effect of the Phase-Lead's zero.

The Pre-filter is written as follows,

$$G_{pf} = \frac{p}{s+p} \tag{32}$$

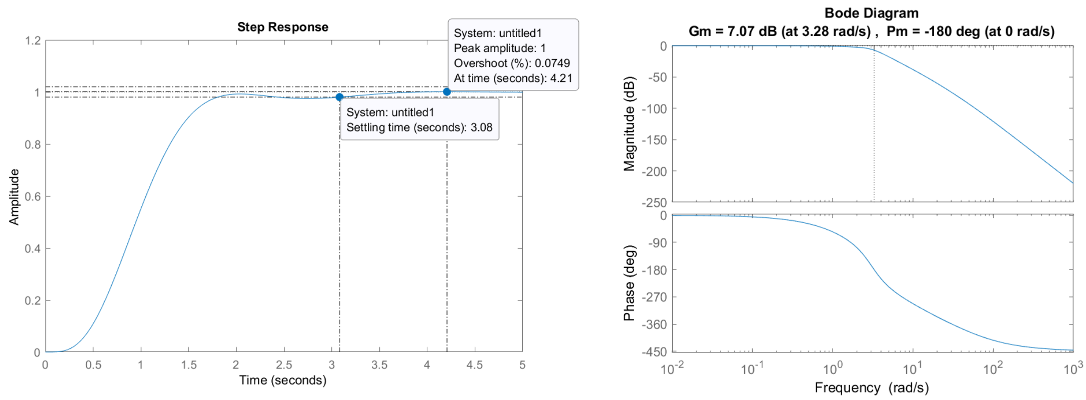
where p can be at the exact point of the $z_{lead} = 1$,

$$G_{pf} = \frac{1}{s+1}$$

applying to the closed-loop Phase-Lead compensated system and evaluating it,

$$G_{lead+prefilter} = G_{pf}\ \frac{G_{ol}}{1+G_{ol}}$$
$$G_{lead+prefilter} = \frac{1}{s+1}\ \frac{10047\ (s+1)}{(s+50)(s+13.07)(s+1.64)(s^2+2.76\ s+9.40)}$$

Fig. 4 shows the result of implementing a Pre-filter, where the overshoot decreases dramatically with a small change in the settling time.



(a) Step response.

(b) Bode plot

Figure 4: Closed-loop Lead compensated in series with a Pre-filter.

The Matlab script below calculates and implements the Pre-filter.

```
%% Using prefilter to reduce the overshoot
pf = z_lead; % selecting the zero of the lead compensator (z_lead)
Gpf = pf/(s+pf);

fig = figure(3);
step(Gpf*Gcl_lead);
saveas(fig,'Q2_Lead+prefilter_step.png');

fig = figure(4);
margin(Gpf*Gcl_lead);
saveas(fig,'Q2_Lead+prefilter_margin.png');
BW_lead_prefilter = bandwidth(Gpf*Gcl_lead); % bandwidth
```

2.5 Design of the Phase-Lag compensator

The Phase-Lag compensator has the following transfer function,

$$G_{lag} = \frac{s+z_{lag}}{s+p_{lag}}, \quad |p_{lag}| < |z_{lag}| \tag{33}$$

This compensator will be used in series with the Phase-Lead compensator designed in the previous task. And now, the design requirement is the steady-state error (ess_a) for a parabolic input $0.5At^2 \leq 2.5\%$.

First, obtaining the LaPlace transform of the parabolic input,

$$r(t) = 0.5At^2, \quad R(s) = \frac{A}{s^3}$$

according to the final value theorem,

$$ess = \lim_{s \to 0} \; s \; \frac{A}{s^3} \frac{1}{1+G} \tag{34}$$

if $K_a = \lim_{s \to 0} s^2 \; G$, the steady-state error is,

$$ess_a = \frac{A}{K_a} \tag{35}$$

where K_a is the acceleration error constant.

If $ess_a = 0.025$ and $A = 1$, the **desired** acceleration error constant (K_{a_d}) is obtained using Eq.(35),

$$K_{a_d} = \frac{1}{ess_a}$$
$$K_{a_d} = 40$$

and again, with $A = 1$ and $G = G_{ol}$ from Eq.(31), the **actual** acceleration error constant $K_{a_{act}}$ is,

$$K_{a_{act}} = \lim_{s \to 0} s^2 \; G_{ol} = K_{a_{act}} = 2.67$$

The relation between the zero z_{lag} and the pole p_{lag} of the Phase-Lag compensator can be written as follows,

$$\alpha = \frac{K_{a_d}}{K_{a_{act}}} = \frac{z_{lag}}{p_{lag}} \tag{36}$$
$$\alpha = 14.97$$

choosing a z_{lag} ten times smaller than the real part of the desired dominant pole r_1,

$$z_{lag} = \left| \frac{-1.38}{10} \right|$$
$$z_{lag} = 0.14$$

and the pole should be,

$$p_{lag} = \frac{z_{lag}}{\alpha}$$
$$p_{lag} = 0.0092$$

therefore, the Phase-Lag compensator becomes,

$$G_{lag} = \frac{s + 0.14}{s + 0.0092}$$

The Matlab script of the Phase-Lag compensator is presented below.

```
%% d) Phase-lag compensator
ess_a = 0.025; % steady-state error for a parabolic input 0.5At^2
Ka_d = 1/ess_a; % Ka desired
Ka_act = (z_lead/p_lead)*(K/(a*b));
% calculating the zero and pole of the compensator
alpha = Ka_d/Ka_act;
z_lag = abs(x)/10;
p_lag = z_lag/alpha;
% Gc phase-lag compensator
Gc_lag = (s+z_lag)/(s+p_lag);
```

2.6 Evaluation of the performance

Finally, the open-loop of the Phase-Lead compensator in series with the Phase-Lag compensator is written as follows,

$$\begin{aligned} G_{ol1} &= G_{lag}\ G_{lead}\ G(s) \\ G_{ol1} &= \frac{s+0.14}{s+0.0092}\ \frac{s+1}{s+8.36}\ \frac{10047}{s^2(s+9)(s+50)} \end{aligned} \tag{37}$$

and the Pre-filter with the closed-loop systems is,

$$\begin{aligned} G_{lag+lead+prefilter} &= G_{pf}\ \frac{G_{ol1}}{1+G_{ol1}} \\ G_{lag+lead+prefilter} &= G_{pf}\ \frac{10047(s+1)(s+0.14)}{(s+50)(s+13.05)(s+1.78)(s+0.14)(s^2+2.51\ s+8.75)} \end{aligned} \tag{38}$$

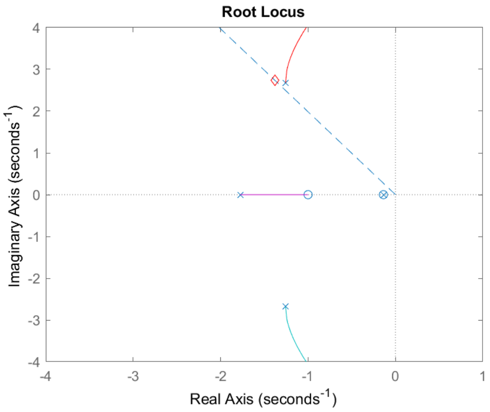
Fig. 5c shows the step response of the system with and without the Pre-filter, it is clear that the Pre-filter reduces the overshoot in both cases with good settling time, the performance results can be seen in Table 2. On the other hand, in Fig. 5a, the desired dominant poles are slightly out of the root locus, it is recommended to add a second Phase-lead compensator in series in order correct that gap.

Quantity	**Lead**	**Prefilter+Lead**	**Lead+Lag**	**Prefilter+Lead+Lag**
Rise Time	0.37	1.00	0.36	0.94
Settling Time	2.79	3.08	3.59	3.31
Percentage Overshoot	51.46	0.07	57.96	1.79
Phase Margin	40.80	-180	36.60	36.60
Gain Margin	8.96	7.07	8.51	8.51
Bandwidth	4.87	2.31	4.89	2.58
Peak Magnitude	1.51	1.00	1.58	1.02

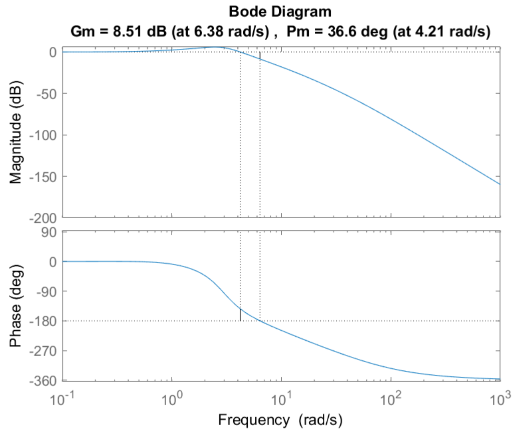
Table 2: Performance evaluation to unit step response

The following Matlab script was used to evaluate the performance of the previous systems.

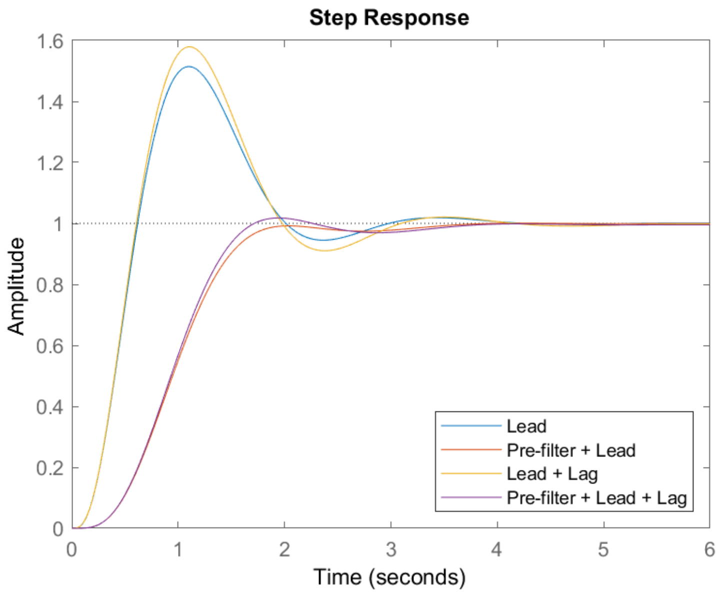
```
%% Evaluating the performance with the phase-lead, phase-lag and prefilter
Gol_lead_lag = Gc_lead*Gc_lag*G;
Gcl_lead_lag = feedback(Gol_lead_lag,1);

fig = figure(5);
rlocus(Gcl_lead_lag);
hold;
% ploting the s1 and zeta in the rlocus
n = 0:1:160; m = n*sqrt(zeta^2/(1-zeta^2));
axis ([ -4 1 -4 4]);
plot (-m,n,'--'); % zeta
```



(a) Root locus of the Lead and Lag closed-loop compensator system.



(b) Bode plot of the closed-loop system.



(c) Step response of the closed-loop system.

Figure 5: Phase-lead, Phase-lag in series, and the Pre-filter.

```
plot (-x,y,'rd');
saveas(fig,'Q2_Lead+Lag_rlocus.png');

fig = figure(6);
step(Gcl_lead,Gpf*Gcl_lead,Gcl_lead_lag,Gpf*Gcl_lead_lag)
legend('Lead','Pre-filter + Lead','Lead + Lag','Pre-filter + Lead + Lag'...
,'Location','southeast');
saveas(fig,'Q2_All_step_1.png');
stepinfo(Gpf*Gcl_lead_lag)

fig = figure(7);
margin(Gcl_lead_lag);
saveas(fig,'Q2_Lead+lag_margin.png');
BW_lead_lag = bandwidth(Gcl_lead_lag); % bandwidth

fig = figure(8);
margin(Gcl_lead_lag);
```

```
saveas(fig,'Q2_Lead+lag+prefilter_margin.png');
BW_lead_lag_prefilter = bandwidth(Gpf*Gcl_lead_lag); % bandwidth

%% Verifing Ka = 40
Ka = (z_lag/p_lag) * (z_lead/p_lead) * ( K/(a*b) );
ess_parabolic = a/Ka;
```

2.7 New Phase-Lead compensator

Fig. 5a demonstrates that the desired dominant closed-loop poles have moved slightly from the root locus, therefore, a new Phase-Lead compensator is calculated to correct that shift.

This time, the location of the zero is exactly below the real part of the desired pole, $z_{lead\ 2} = 1.38$. Applying the angle condition with the desired pole r_1 as follows,

$$\begin{aligned} &\measuredangle\left(\frac{s+z_{lead\ 2}}{s+p_{lead\ 2}}\ \frac{s+z_{lag}}{s+p_{lag}}\ \frac{s+z_{lead}}{s+p_{lead}}\ \frac{K}{s^2(s+a)(s+b)}\right) = -180^\circ \\ &\measuredangle\left(\frac{r_1+z_{lead\ 2}}{r_1+p_{lead\ 2}}\ \frac{r_1+z_{lag}}{r_1+p_{lag}}\ \frac{r_1+z_{lead}}{r_1+p_{lead}}\ \frac{K}{r_1^2(r_1+a)(r_1+b)}\right) = -180^\circ \\ &\measuredangle z_{lead\ 2} + \measuredangle z_{lag} + \measuredangle z_{lead} - \measuredangle p_{lead\ 2} - \measuredangle p_{lag} - \measuredangle p_{lead} - 2\ \measuredangle s - \measuredangle a - \measuredangle b = -180^\circ \end{aligned} \tag{39}$$

if, $r_1 = -x + j\ y$

$$\begin{aligned} &\arctan\frac{y}{x-z_{lead\ 2}} + \left(180^\circ - \arctan\frac{y}{x-z_{lag}}\right) + \left(180^\circ - \arctan\frac{y}{x-z_{lead}}\right) \ldots \\ &\quad - \arctan\frac{y}{p_{lead\ 2}} - \left(180^\circ - \arctan\frac{y}{x-p_{lag}}\right) - \arctan\frac{y}{p_{lead}-x} - 2\left(180^\circ - \arctan\frac{y}{x}\right) \ldots \\ &\quad\quad - \arctan\frac{y}{a-x} - \arctan\frac{y}{b-x} = 180^\circ \end{aligned}$$

after some operations,

$$p_{lead\ 2} = 1.48$$

so, the second Phase-Lead compensator is,

$$G_{lead\ 2} = \frac{s+1.38}{s+1.48}$$

Now, the new gain K_{new} of the compensated system can be found using the gain condition,

$$\begin{aligned} &\left|K_{new}\ \frac{s+z_{lead\ 2}}{s+p_{lead\ 2}}\ \frac{s+z_{lag}}{s+p_{lag}}\ \frac{s+z_{lead}}{s+p_{lead}}\ \frac{K}{s^2(s+a)(s+b)}\right| = 1 \\ &\left|K_{new}\ \frac{r_1+z_{lead\ 2}}{r_1+p_{lead\ 2}}\ \frac{r_1+z_{lag}}{r_1+p_{lag}}\ \frac{r_1+z_{lead}}{r_1+p_{lead}}\ \frac{K}{r_1^2(r_1+a)(r_1+b)}\right| = 1 \\ &K_{new} = 1 \end{aligned} \tag{40}$$

which means that the gain has not changed.

The following script in Matlab simulates the previous results.

```
%% New phase-lead compensator
z_lead2 = x; % given zero lead right below the desired pole
f = 180 + atand(y/(x-z_lead2))+(180-atand(y/(x-z_lag)))...
+(180-atand(y/(x-z_lead))) - (180-atand(y/(x-p_lag)))-atand(y/(p_lead-x))...
-2*(180-atand(y/x))-atand(y/(a-x))-atand(y/(b-x));
% p_lead2 = y/tand(180+f) - x;
```

```
p_lead2 = y/tand(f) + x;
Gc_lead2 = (s+z_lead2)/(s+p_lead2);
% Calculating the new gain K with the lead 2 compensator
K_new = sqrt((-x+p_lead2)^2+y^2)*sqrt((-x+p_lag)^2+y^2)*...
sqrt((-x+p_lead)^2+y^2)*((-x)^2+y^2)*sqrt((-x+a)^2+y^2)*sqrt((-x+b)^2+y^2)/...
( sqrt((-x+z_lead)^2+y^2)*sqrt((-x+z_lag)^2+y^2)*sqrt((-x+z_lead)^2+y^2)*K );
Gol_lead2_lead_lag = K_new*Gc_lead2*Gol_lead_lag;
Gcl_lead2_lead_lag = feedback(Gol_lead2_lead_lag,1);
%% Verifing Ka = 40
Ka_new = K_new*(z_lead2/p_lead2)*(z_lag/p_lag)*(z_lead/p_lead)*(K/(a*b));
ess_parabolic_new = a/Ka_new;
```

2.8 Evaluation of the performance with the second Phase-Lead compensator

The open-loop of the new Phase-Lead compensator in series with the previous compensators is written as follows,

$$\begin{aligned} G_{ol2} &= G_{lead\ 2}\ G_{lag}\ G_{lead}\ G(s) \\ G_{ol2} &= \frac{s+1.38}{s+1.48}\ \frac{s+0.14}{s+0.0092}\ \frac{s+1}{s+8.36}\ \frac{10047}{s^2(s+9)(s+50)} \end{aligned} \tag{41}$$

and the Pre-filter with the closed-loop systems is,

$$\begin{aligned} G_{lag+lead+prefilter} &= G_{pf}\ \frac{G_{ol2}}{1+G_{ol2}} \\ G_{lag+lead+prefilter} &= G_{pf}\ \frac{10047(s+1.38)(s+1)(s+0.14)}{(s+8.36)(s+1.48)(s+0.009)s^2(s+9)(s+50)} \end{aligned} \tag{42}$$

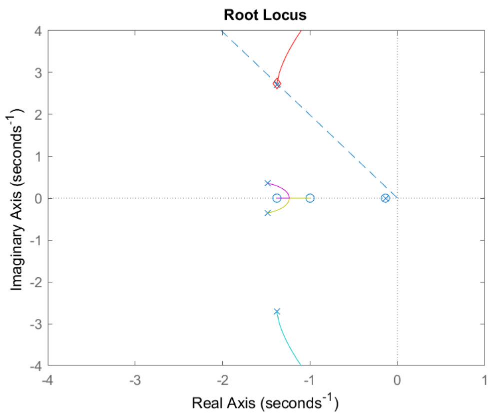
From Fig. 6a it can be seen that the desired closed-loop poles are exactly in the root locus. Also, the PM has changed a little with respect of the previous compensated system, from 36.60° to 39.50°, Fig. 6b. On the other hand, the step response has been improved with the Pre-filter and the new Phase-Lead compensator, see Table 3.

Quantity	**Lead**	**Prefilter +Lead**	**Lead +Lag**	**Prefilter +Lead +Lag**	**Lead_2 +Lead +Lag**	**Prefilter +Lead_2 +Lead +Lag**
Rise Time	0.37	1.00	0.36	0.94	0.37	0.97
Settling Time	2.79	3.08	3.59	3.31	2.87	1.68
Percentage Overshoot	51.46	0.07	57.96	1.79	55.00	1.25
Phase Margin	40.80	-180	36.60	36.60	39.50	-180
Gain Margin	8.96	7.07	8.51	8.51	8.79	6.79
Bandwidth	4.87	2.31	4.89	2.58	4.88	2.42
Peak Magnitude	1.51	1.00	1.58	1.02	1.55	1.01

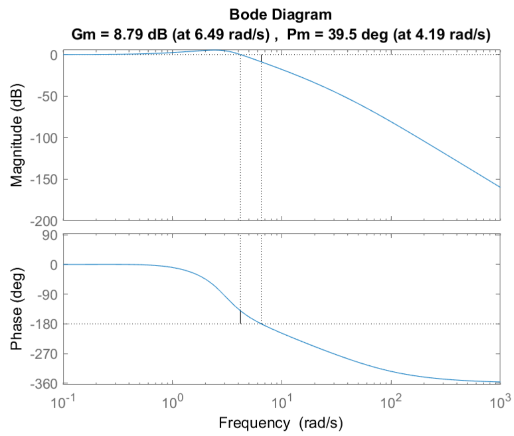
Table 3: Performance evaluation to unit step response

The following script in Matlab simulates the previous results.

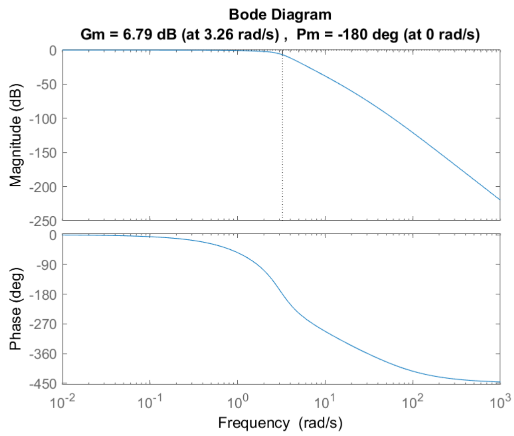
```
%% Comparing the new compensator of the previous designs.
fig = figure(9);
rlocus(Gcl_lead2_lead_lag)
hold on;
% ploting the s1 and zeta in the rlocus
n = 0:1:160; m = n*sqrt(zeta^2/(1-zeta^2));
```



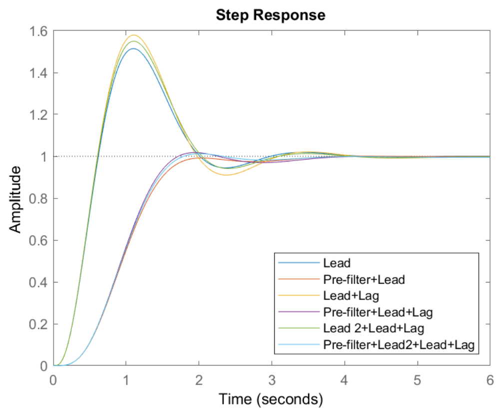
(a) Root locus



(b) Bode plot of the open-loop system without the Pre-filter.



(c) Bode plot of the closed-loop system with the Pre-filter



(d) Step response of the closed-loop system.

Figure 6: Phase-lead 2, Phase-lead, Phase-lag in series, and the Pre-filter.

```
axis ([ -4 1 -4 4]);
plot (-m,n,'--'); % zeta
plot (-x,y,'rd');
saveas(fig,'Q2_Lead2+Lead+Lag_rlocus.png');

fig = figure(10);
% step(Gcl_lead2_lead_lag,Gpf*Gcl_lead2_lead_lag)
step(Gcl_lead,Gpf*Gcl_lead,Gcl_lead_lag,Gpf*Gcl_lead_lag,...
Gcl_lead2_lead_lag,Gpf*Gcl_lead2_lead_lag)
legend('Lead','Pre-filter+Lead','Lead+Lag','Pre-filter+Lead+Lag'...
,'Lead 2+Lead+Lag','Pre-filter+Lead2+Lead+Lag','Location','southeast');
saveas(fig,'Q2_All_step_2.png');
stepinfo(Gpf*Gcl_lead2_lead_lag)

fig = figure(11);
margin(Gcl_lead2_lead_lag);
saveas(fig,'Q2_Lead2+lead+lag_margin.png');
BW_lead2_lead_lag = bandwidth(Gcl_lead2_lead_lag); % bandwidth

fig = figure(12);
margin(Gpf*Gcl_lead2_lead_lag);
saveas(fig,'Q2_Lead2+lead+lag+prefilter_margin.png');
BW_lead2_lead_lag_prefilter = bandwidth(Gpf*Gcl_lead2_lead_lag); % bandwidth
```

2.9 Conclusion

The settling time requirement $t_s \leq 2.9$ was satisfied by three of the six compensators, and only the last one with the Pre-filter achieved the desired overshoot $P.O. \leq 20\%$. It is clear that the use of the Pre-filter reduces the overshoot drastically.