Spring 2026

Problem 1. Let $f \in \mathcal{C}_{[0,1]}$ be continuous. Suppose

$$\int_0^1 f(x)\, x^n \, dx = 0 \quad \text{for every integer } n \geqslant 0$$

Prove that $f \equiv 0$ on $[0, 1]$.
You may use without proof that polynomials are uniformly dense in $\mathcal{C}_{[0,1]}$.

Problem 2. Let (f_n) be a sequence of differentiable functions on $[0, 1]$ with $f_n(0) = 0$ for all n. Suppose that (f_n') converges uniformly on $[0, 1]$ to a continuous function g.

1. Prove that (f_n) converges uniformly on $[0, 1]$ to a function f.
2. Show that f is C^1 on $[0, 1]$, that $f(0) = 0$, and that $f' = g$.

Problem 3. Let f be an entire function such that

$$|f(z)| \leqslant A + B|z|^m$$

for some constants $A, B \geqslant 0$ and a nonnegative integer m.

1. Show that f is a polynomial of degree at most m.
2. Deduce that if $|f(z)| \leqslant C(1 + |z|^{1/2})$ for some $C > 0$, then f is constant.

Problem 4. Let $0 < \alpha < 1$. Evaluate

$$I(\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x$$

using contour integration (do not use Beta/Gamma function identities).

Problem 5. Let A be a real $n \times n$ matrix such that $A^3 = A$. Describe all possible Jordan canonical forms of A.

Problem 6. Orthogonally diagonalize the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Specifically:

1. Find the eigenvalues and eigenspaces of A.

2. Within the eigenspace of the repeated eigenvalue, use the *Gram–Schmidt process* to produce an orthonormal basis.

3. Form an orthogonal matrix Q from your orthonormal eigenvectors and verify that $Q^\top AQ$ is diagonal. Give Q and $Q^\top AQ$ explicitly.

Problem 7. Let G be a finite p-group, i.e. $|G| = p^n$ for some prime p and integer $n \geqslant 1$.

1. Show that the center $Z(G)$ of G is nontrivial.

2. Deduce that G has a normal subgroup of order p.

Problem 8.

1. Let R be a finite integral domain, i.e, a finite commutative ring with 1 and no zero divisors. Prove that R is a field.

2. Show that if p is the characteristic of R then the Frobenius map $F : R \to R$, $F(x) = x^p$ is an automorphism (and in particular is bijective).

Problem 9. Let c_n be the number of ways of giving n cents change using coins of values $1, 5, 10, 25$ cents. Find $\lim_{n\to\infty} c_n/n^3$.

Problem 10. The Catalan numbers $1, 1, 2, 5, 14, \ldots$ satisfy $C_n = 0$ if $n < 0$, $C_0 = 1$, $C_n = \sum_i C_i C_{n-1-i}$. Find the function $f(x) = \sum_n C_n x^n$, and use this to calculate C_n as the quotient of a binomial coefficient by a linear function.

Problem 11. Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}$$

Hint: Use the substitution $z = e^{i\theta}$.

Problem 12. Let f be analytic on the disk $D = \{z \mid |z| < 1\}$ and suppose that $|f(z)| \leqslant 1$ for all $z \in D$ and that $f(0) = 0$.

1. Show that $|f'(0)| \leqslant 1$.

2. Find all functions f for which equality holds.

Problem 13. Let $M = (m_{i,j}), 0 \leqslant i, j < n$ be an $n \times n$ complex matrix with $m_{i,j} = m_{i+j}$ for constants m_i satisfying $m_i = m_{n+i}$. Find the eigenvalues of M.

Problem 14. For complex square matrices A_i, put

$$F_n(A_1, \ldots, A_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)}$$

The sum is over the symmetric group S_n, and $\operatorname{sgn}(\sigma)$ is 1 or -1 if σ is an even or odd permutation.

Show that F_n vanishes if all its arguments are $m \times m$ matrices and $n > m^2$. Hint: first show F_n vanishes if two arguments are equal.

Problem 15.

1. Let G be a finite group and $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ a group homomorphism. Show that $\ker \varphi = \{g \in G \mid \operatorname{tr} \varphi(g) = n\}$.

2. Show that this fails for infinite G.

Problem 16. Prove that if all elements of a group satisfy $g^2 = 1$ then the group is abelian. Given an example of a group such that $g^3 = 1$ for all elements but the group is not abelian.