2 THE BROWNIAN MOTION

this chapter is devoted to the study of Brownian motion. this motion is the irregular movements of pollen particles in suspension in a liquid.

the first physical description (and mathematical) was given by Einstein in 1905 (also Smoluchowski).

2.1 EQUIVALENT DEFINITIONS OF BROWNIAN MOTION

Definition: the stochastic process $\{B_t, t \geq 0\}$ is a one-dimensional brownian motion if its is a gaussian, zero mean and with covariance function given by $E(B_t B_s) = t \wedge s$.

Since $EB_0^2 = 0 \Rightarrow$ the random variable $B_0 = 0$ a.s.
According to the definition, each random variable B_t, $t>0$, has the law $N(0,t)$ with density

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad x \in \mathbb{R}.$$

Remark: Differentiating $f_t(x)$ once with respect to t and twice with respect to x yield

$$\begin{cases} \partial_t f_t(x) = \frac{1}{2} \partial_x^2 f_t(x), & t>0, x \in \mathbb{R} \\ f(0) = \delta_{\{0\}} \end{cases}$$

this is the heat equation on $\mathbb{R}$ with i.c. $f_0(x) = \delta_{\{0\}}(x)$.
that means, the density of BM behaves like a diffusive physical phenomenom.

Proposition: A stochastic process $X = \{X_t, t \geq 0\}$ is a Brownian motion if and only if

i) $X_0 = 0$, a.s,

ii) for any $s \in [0,t)$, the random variable $X_t - X_s$ is independent of X_r, $r \in [0,s]$, and $X_t - X_s$ is a $N(0, t-s)$ random variable.

Proof: Assume first that $\{X_t, t \geq 0\}$ satisfies the definition given before. So, $E X_0^2 = 0$ and $X_0 = 0$, a.s.

Let any $0 \leq r \leq s < t$, then using the covariance function of Brownian motion, we deduce

$$E(X_r(X_t - X_s)) = r \wedge t - r \wedge s = r - r = 0.$$

Since X is zero mean, this implies

$$E(X_r(X_t - X_s)) = E(X_r)\, E(X_t - X_s).$$

Because the joint distribution of (X_r, X_s, X_t) is gaussian, this implies that $X_t - X_s$ is independent of X_r and, so, the statement on independence in (ii) holds.

Since the linear combinations of gaussian random variabl are also gaussian, $X_t - X_s$ is Normal. It is easy to check that

$$E(X_t - X_s) = 0,$$

$$E[(X_t - X_s)^2] = E X_t^2 - 2 E X_t X_s + E X_s^2 = t - 2s + s = t - s.$$

Assume now that (i) and (ii) are true. Then the finite dimensional distributions of $\{X_t, t \geq 0\}$ are multidimensional normal, sin they are obtained by a linear transformations of r.v with gaussian independent components. Indeed, for any $m \geq 1$ an $0 \leq t_1 \leq \ldots \leq t_m$, set

$$Y = (X_{t_1}, \ldots, X_{t_m}), \quad Z = (X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}).$$

Clearly, there exists a matrix A such that $Y = AZ$. The law of Z is gaussian. Thus, Y is gaussian, too.

For $0 \leq s \leq t$, we have

$$E(X_t X_s) = E[(X_t - X_s + X_s) X_s] = E[(X_t - X_s) X_s] + E X_s^2$$
$$= E(X_t - X_s) E X_s + E X_s^2 = 0 + s = s \wedge t.$$

■

Remark: Later we will prove that Brownian motion has continuous sample paths. For the moment, it is a model for a random evolution now which starts from $x = 0$ at $t = 0$, such that the qualitative change on time increments only depends on their length and the future evolution is independ of its past (Markov property)

It is not difficult to show that:

i) If $\{B_t, t\geq 0\}$ is a Brownian motion, $\{-B_t, t\geq 0\}$ is also a B.m.

ii) the process $\{\frac{1}{N} B_{N^2 t}, t\geq 0\}$ is also a B.m. "scaling property"

iii) $\{B_{t+a} - B_a, t\geq 0\}, a>0$, is a B.m.

2.2 A construction of Brownian motion.

Kolmogorov's theorem ensures the existence of B.m. this existence is in law. Now we give an explicit description of the sample paths.

Possible aproachs:

- B.m as limit of a random walk.
- Paul Lévy's construction of B.m

Brownian motion as limit of random walk

Let $\{\xi_j, j\in\mathbb{N}\}$ be a sequence of independent, identically distributed random variables, with zero mean and variance $\sigma^2>0$. Consider $S_0=0$ and $S_n=\sum_{i=1}^{n}\xi_i$. the sequence $\{S_n, n\geq 0\}$ is a Markov chaing, and also a martingale.

Let us consider the continuous time stochastic process defined defined by linear interpolation of $\{S_n, n\geq 0\}$, as follows. For $t\geq 0$, let $[t]$ denote its integer value. then, set

$$Y_t = S_{[t]} + (t-[t])\xi_{[t]+1}, \qquad t\geq 0.$$

the next step consists in scaling the sample paths of $\{Y_t, t\geq 0\}$. By analogy with the scaling in the central limit theorem, we set

$$B_t^{(n)} = \frac{1}{\sigma\sqrt{n}} Y_{nt}, \quad t\geq 0.$$

the Donsker theorem tell us that the sequence of process $\{B_t^{(n)}, t\geq 0\}, n\geq 1$, converges in law to the Brownian motion. the reference sample space is the set of continuous functions vanishing at zero. We obtain continuity of the sample paths of the limit.

Donsker theorem is the infinite dimensional version of the central limit theorem

PAUL LÉVY'S CONSTRUCTION OF BROWNIAN MOTION

The space $L^2([0,1])$ consists of Borel measurable functions $f:[0,1]\to\mathbb{R}$ such that

$$\int_0^1 f^2(s)\,ds < \infty.$$

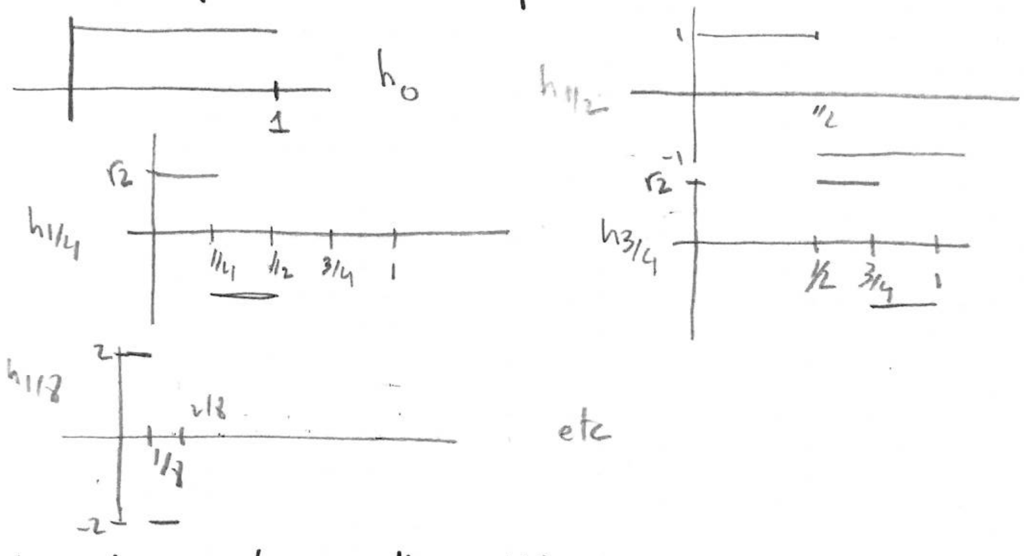
We identify those functions that coincide except on a set of points of $[0,1]$ of null length. So, we can define a scalar product

$$\langle f, g\rangle_{L^2([0,1])} = \int_0^1 f(s)g(s)\,ds,$$

giving the norm

$$\|f\|^2_{L^2([0,1])} = \int_0^1 f^2(s)\,ds.$$

Lévy gave the construction in 1939 using the Haar functions. It was inspired in an interpolation idea.



Consider $h_0(t) = \mathbb{1}_{[0,1]}(t)$,

$$h_{k\cdot 2^{-n}}(t) = \begin{cases} 2^{\frac{n-1}{2}}, & \text{if } (k-1)2^{-n} \le t < k\cdot 2^{-n}, \\ -2^{\frac{n-1}{2}}, & \text{if } k\cdot 2^{-n} \le t < (k+1)\cdot 2^{-n}, \end{cases}$$

defined by $n \geq 1$, $1 \le k \le 2^n$, k odd.

The family $\{h_0, h_{k\cdot 2^{-n}}\}$ is a complete orthonormal system (CONS) or a basis of the Hilbert space $L^2([0,1])$. Indeed,

Proposition: the Haar functions $\{h_{k \cdot 2^{-m}}, \overbrace{k \in \{0, \dots, 2^m\}}^{"h_{k,m}" \quad I(m)}_{\text{odd}}, m \geq 0\}$ is a complete orthonormal basis of $L^2([0,1])$.

Proof: First we prove that $h_{k \cdot 2^{-m}}$ are orthonormal. Let $m \geq 0$, $k \in I(m)$. We have

$$\langle h_{k,m}, h_{k,m} \rangle = \int_0^1 h_{k,m}(t) h_{k,m}(t)\, dt = \int_{\frac{k-1}{2^m}}^{\frac{k+1}{2^m}} 2^{m-1}\, dt = 1.$$

Now, we suppose $m_1, m_2 \geq 0$, $k_1, k_2 \in I(m)$ with $m_1 \neq m_2$ or $k_1 \neq k_2$. Assume whithout loss of generality $m_1 \leq m_2$. then, we have or

$\left(\frac{k_1-1}{2^{m_1}}, \frac{k_1+1}{2^{m_1}}\right) \cap \left(\frac{k_2-1}{2^{m_2}}, \frac{k_2+1}{2^{m_2}}\right) = \emptyset$ or $\left(\frac{k_1-1}{2^{m_1}}, \frac{k_1+1}{2^{m_1}}\right) \cap \left(\frac{k_2-1}{2^{m_2}}, \frac{k_2+1}{2^{m_2}}\right) = \left(\frac{k_2-1}{2^{m_2}}, \frac{k_2-1}{2^{m_2}}\right)$

In the first cas, $\langle h_{k_1,m_1}, h_{k_2,m_2} \rangle = 0$, in the second one, h_{k,m_1} takes a constant value and

$$\int_{\frac{k_2-1}{2^{m_2}}}^{\frac{k_2+1}{2^{m_2}}} h_{k_2,m_2}(t)\, dt = 0,$$

and so $\langle h_{k_1,m_1}, h_{k_2,m_2} \rangle = 0$.

Now, we will prove that, for any $f \in L^2([0,1])$, if $\langle f, h_{k,m} \rangle = 0$ for all $m \geq 0$, $k \in I(m)$, then $f = 0$. We need to check that, for any couple of diadic numbers $a, b \in \mathcal{D} = \{\frac{i}{2^n}, i \in \{0, 1, \dots, 2^n\}, n \in \mathbb{N}\}$, the integral

$$\int_a^b f(t)\, dt = 0.$$

as $\mathcal{D}$ is dens in $[0,1]$, f is also zero.

Let $f \in L^2([0,1])$. Define for $m \geq 1$, and $i \in \{0, 1, \dots, 2^m - 1\}$,

$$F_i^m = \int_{i/2^m}^{i+1/2^m} f(t)\, dt.$$

For each $m \geq 0$ and $k \in I(m)$, we have

$$\langle f, h_{k,m} \rangle = \int_0^1 f(t) h_{k,m}(t)\, dt = \int_{k-1/2^m}^{k/2^m} f(t) 2^{\frac{m-1}{2}}\, dt + \int_{k/2^m}^{k+1/2^m} -f(t)\, 2^{\frac{m-1}{2}}\, dt = 2^{\frac{m-1}{2}} (F_{k-1}^m - F_k^m)$$

$$\langle f, h_{k,m-1} \rangle = \int_{k-1/2^{m-1}}^{k/2^{m-1}} f(t) 2^{\frac{m-2}{2}}\, dt + \int_{k/2^{m-1}}^{k+1/2^m} -f(t) 2^{\frac{m-2}{2}}\, dt$$

$$= 2^{\frac{m-2}{2}} (F_{2k-2}^m + F_{2k-1}^m - F_{2k}^m - F_{2k+1}^m), \quad k \in I(m-1). \quad (*)$$

Since $\langle f, h_{k,m} \rangle = 0$, we get $F_{k-1}^m = F_k^m$, $\forall k \in I(m)$. Applying that in $(*)$,

$$F_{2k-2}^m + F_{2k-1}^m - F_{2k}^m - F_{2k+1}^m = 2(F_{2k-1}^m - F_{2k+1}^m) = 0.$$

So, $\forall i \in \{0, 1, \dots, 2^m - 1\}$, $F_i^m = F_0^m$. Moreover $\langle f, h_{0,m} \rangle = 0$. ▨

Consequently, for any $f, g \in L^2([0,1])$, we have the Parseval identity,

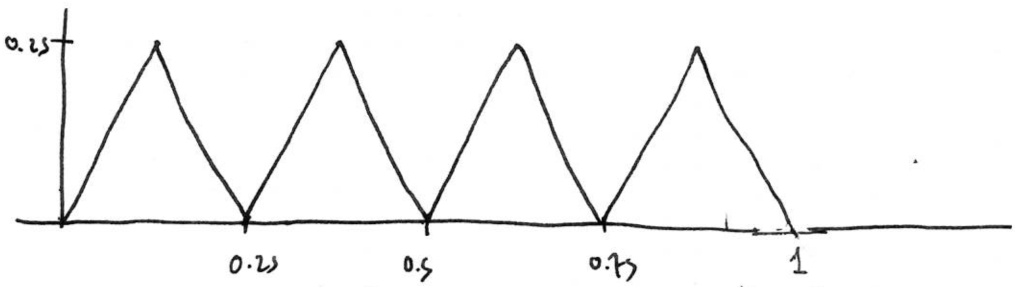
$$\langle f, g \rangle = \sum_{n=0}^{\infty} \sum_{k \in I(n)} \langle f, h_{k,n} \rangle \langle g, h_{k,n} \rangle . \qquad (*)$$

Now, we define the Schauder functions:

$$S_{k,n}(t) = \int_0^t h_{k,n}(u)\,du \;,\; 0 \le t \le 1,\; n \geq 0,\; k \in I(n).$$

Observe that $S_{0,0}(t) = t$ and, for $n \geq 1$, the graphics of $S_{k,n}$ are small tents, centered on $k/2^n$ and

$$\max_{0 \le t \le 1} S_{k,n}(t) = \frac{1}{2^{n+1/2}} .$$



We can check that

$$S_{k,n}(t) = \int_0^1 h_{k,n}(u)\, \mathbb{1}_{[0,t]}(u)\,du = \langle \mathbb{1}_{[0,t]}, h_{k,n} \rangle$$

and

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle = \int_0^1 \mathbb{1}_{[0,s]} \mathbb{1}_{[0,t]}\,du = \int_0^{t \wedge s} du = t \wedge s .$$

Applying $(*)$ for $f = \mathbb{1}_{[0,t]}$ and $g = \mathbb{1}_{[0,s]}$ we obtain

$$\sum_{n=0}^{\infty} \sum_{k \in I(n)} S_{k,n}(t)\, S_{k,n}(s) = t \wedge s . \qquad (3)$$

Let $\{ \xi_{k,n},\, k \in I(n),\, n = 0, 1, \dots, \}$ be a numerable colection of standand gaussian independent random variables. Consider the functions $\{ B_n(t),\, 0 \le t \le 1 \}$ defined by

$$B_n(t) = \sum_{m=0}^{n} \sum_{k \in I(m)} \xi_{k,m}(w)\, S_{k,m}(t), \quad 0 \le t \le 1,\; n \geq 0 . \qquad (1)$$

they will converge a Brownian motion. the idea is for each n, determine the points of the process at time $t = k/2^n \in [0,1]$ for each $k \in I(n)$

these processes are gaussian, centered with continuous paths.
First we calculate

$$\| B_n(t) - B_{n-1}(t) \|_\infty = \sup_{0\le t\le 1} | B_n(t,w) - B_{n-1}(t,w)|$$

$$= \sup_{0\le t\le 1} \left| \sum_{\substack{1\le k\le 2^n \\ k \text{ odd}}} \xi_{k,n} S_{k,n}(t) \right|$$

$$\le 2^{-\frac{n+1}{2}} \max_{\substack{1\le k\le 2^n \\ k \text{ odd}}} |\xi_{k,n}(w)|$$

We need the following bound

$$\int_a^\infty e^{-u^2/2} du < \int_a^\infty \frac{u}{a} e^{-u^2/2} du = \frac{1}{a} e^{-a^2/2}.$$

So, we have

$$P\left(\| B_n(t) - B_{n-1}(t)\|_\infty > \sqrt{2 \cdot 2^{-n} \lg 2^n} \right) \le P\left(\max_{\substack{1\le k\le 2^n \\ k \text{ odd}}} |\xi_{k,n}| > 2\sqrt{n \lg 2} \right)$$

$$= P\left(\bigcup_{\substack{k=1 \\ k \text{ odd}}}^{2^n} \{ |\xi_{k,n}| > 2\sqrt{n\lg 2} \} \right) \le 2^{n-1} P\left(|\xi_{k,n}| > 2\sqrt{n \lg 2}\right)$$

$$= 2^{n-1} \, 2 \int_{2\sqrt{n\lg 2}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \le \frac{2^{n-1} \exp\{-\frac{1}{2} 4 n \lg 2\}}{\sqrt{2\pi} \sqrt{n \lg 2}} = \frac{2^{-n-1}}{\sqrt{2\pi n \lg 2}}.$$

We take $A_n = \{ w ; \| B_n(w) - B_{n-1}(w) \|_\infty > \sqrt{2^{-n+1} \lg 2^n} \}$. Since $\sum_{n=1}^\infty P(A_n) < +\infty$, Borel-Cantelli's Lemma we have that $P(\liminf_n A_n^c) = 1$

$$\Big(P(\underbrace{\limsup_n A_n}_{N}) = 0 \Rightarrow P(\liminf A_n^c) = 1 \Big)$$

$\forall w \notin N$, $\exists n_0(w)$ s.t. $\| B_n(w) - B_{n-1}(w) \|_\infty \le \sqrt{2^{-n+1} \lg 2^n}$ if $n \ge n_0(w)$.

So, the sequence $\sum_n \| B_n(t) - B_{n-1}(t) \| < +\infty$, that's mean, the succession $\{ B_n(t,w), n \ge 0 \}$ is convergent in $C([0,1])$ if $w \notin N$.

Define

$$B(t,w) = \begin{cases} \lim_n B_n(t,w), & \text{if } w \notin N, \\ 0, & \text{if } w \in N. \end{cases}$$

then, the process $\{B(t,\omega),\ t\in[0,1]\}$ has continuous path (it is known that if f_n are continuous functions converging uniformly to a function f, this function f is continuous). This process is also gaussian because $B(t)$ belongs to the closed linear envelope of the family $\{\xi_{k,m}\}$.

On the other hand:

a) $\lim_n E[B_n(t)] = 0,\ \forall t\in[0,1]$,

b) $\lim_n E[B_n(s)B_n(t)] = s\wedge t$ (thanks to (3)) $\forall s,t\in[0,1]$.

Using characteristic functions the properties a) and b) imply that for any instants times $t_1,\dots,t_k\in[0,1]$, the random vectors $(B_n(t_1),\dots,B_n(t_k))$ converge in law to a k-dimensional normal random vector, centered and covariance matrix $\{(t_i\wedge t_j)\}$. We know that this succession converges a.s to $\{B(t_1),\dots,B(t_k)\}$ and then we can deduce that $\{B(t),\ 0\le t\le 1\}$ is centered, has covariance $s\wedge t$ and it is Gaussian.

Observation: Another possibility is the following: for any $0<t_0<t_1<\dots t_k<1$ the increments $\{B_{t_j}-B_{t_{j-1}}\}_{j=1}^{k}$ are independents, with normal distribution, centered and variance t_j-t_{j-1}. This means that given $\lambda_j\in\mathbb{R}$, $j=1,\dots,m$,

$$E\left(e^{i\sum_{j=1}^{k}\lambda_j(B_{t_j}-B_{t_{j-1}})}\right)=\prod_{j=1}^{k}e^{-\frac{1}{2}\lambda_j^2(t_j-t_{j-1})}.$$

~~Finally, we can extend this construction to $\mathbb{R}_+$. Using the succession $\{B_n(t),\ 0\le t\le 1\}$ we can constuct another sequence $\{\tilde{B}_n(t),\ 0\le t\le n+1\}$ as follows:~~

~~$0\le t\le 1$,~~

~~for $n\ge 1$~~

~~B_t~~

Finally, using the before construction we take a succession $\{X_t^n, t\in[0,1], n\geq 0\}$ of Brownian motions, independents with continuous paths. We define:

For $n=0$, $\tilde{X}_t^{(0)} = X_t^0$, if $0\leq t\leq 1$,

$$\tilde{X}_t^{(n)} = \begin{cases} \tilde{X}_t^{(n-1)}, & \text{if } 0\leq t\leq n \\ \tilde{X}_n^{(n-1)} + X_{t-n}^{(n)}, & \text{if } n\leq t\leq n+1. \end{cases}$$

If we define the process $\{X_t, 0\leq t<+\infty\}$ as $\lim_n \tilde{X}_t^{(n)} = X_t$. It is defined for any $t\in[0,\infty)$, it is continuous because X_t^n are continuous and $\tilde{X}_t^{(n-1)} = \tilde{X}_t^{(n)} = \tilde{X}_t^{(n-1)} + X_0^n = \tilde{X}_t^{(n-1)} + 0$. The other properties of Brownian motion are not difficult to prove.

2.3 Properties of Brownian motion (Path).

Hölder continuity of the sample paths

A function $g:\mathbb{R}\longrightarrow\mathbb{R}$ is locally Hölder continuous with exponent $\gamma\in(0,1]$, if for any bounded set $\Theta\subset\mathbb{R}$, there exists a finite constant $\|g\|_{C^\gamma(\Theta)}$ satisfying

$$\sup_{x,y\in\Theta, x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\gamma} = \|g\|_{C^\gamma(\Theta)}.$$

If $\gamma=1$ the function g is said to be locally Lipschitz continuous.

The sample paths of Brownian motion are γ-Hölder continuous for any $\gamma\in(0,\frac{1}{2})$. It is a consequence of Kolmogorov's continuity criterion

Let $\{X_t, t\geq 0\}$ be a stoch. process satisfying: For any bounded set of indices $\Theta\subset\mathbb{R}_+$ there exist positive real numbers $\alpha\geq 1$, β and C, and for any $s,t\in\Theta$,

$$E[|X_t-X_s|^\alpha) \leq C|t-s|^{1+\beta}.$$

Then, almost surely, the sample paths of the process are γ-Hölder continuous with $\gamma\in(0,\frac{\beta}{\alpha})$.

Consider a Brownian motion $\{B_t, t\geq 0\}$. For any $0<s<t$, $B_t - B_s \sim N(0, t-s)$. So, for any $k\in\mathbb{N}$, we obtain

$$E[|B_t - B_s|^{2k}] = \frac{(2k)!}{2^k k!}(t-s)^k.$$

then,

$\alpha = 2k$, $\beta = k-1$, we deduce that a.s the sample path of Brownian motion are Hölder continuous with exponent

$$\gamma\in\left(0,\frac{\beta}{\alpha}\right) = \left(0,\frac{k-1}{2k}\right),$$

letting $k\to+\infty$, $\gamma\in(0,\frac{1}{2})$.

<u>Nowhere differentiability</u>

Now we give a result of Dvoretzky, Erdös and Kakutani.

<u>theorem</u>: For any $\gamma\in(\frac{1}{2},1]$, then a.s the sample paths of $\{B_t, t\geq 0\}$ are Hölder continuous with exponent γ.

Proof: Let $\gamma\in(\frac{1}{2},1]$ and assume that a sample path $t\longrightarrow B_t(\omega)$ is γ-Hölder continuous at $s\in[0,1)$. then for any $t\in[0,1]$ there exists a constant $c>0$ such that

$$|B_t(\omega) - B_s(\omega)| \leq c|t-s|^\gamma.$$

By triangular inequality

$$|B_{\frac{j}{n}}(\omega) - B_{\frac{j+1}{n}}| \leq |B_s(\omega) - B_{\frac{j}{n}}(\omega)| + |B_s(\omega) - B_{\frac{j+1}{n}}(\omega)|$$

$$\leq c\left[\left(s-\frac{j}{n}\right)^\gamma + \left|s-\frac{j+1}{n}\right|^\gamma\right].$$

Let n big enough and let $i=[ns]+1$, by restricting $j=i, i+1, \dots, i+N-1$, we obtain

$$\left|B_{\frac{j}{n}}(\omega) - B_{\frac{j+1}{n}}(\omega)\right| \leq c\left(\frac{N}{n}\right)^\gamma = \frac{M}{n^\gamma}.$$

Define

$$A^i_{M,n} = \left\{\left|B_{\frac{j}{n}}(\omega) - B_{\frac{j+1}{n}}(\omega)\right| \leq \frac{M}{n^\gamma}, j=i, i+1, \dots, i+N-1\right\}.$$

the set of trajectories where $t\longrightarrow B_t(\omega)$ is γ-Hölder continuous at s is included in

$$\bigcup_{M=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{+\infty}\bigcup_{i=1}^{n} A^i_{M,n}.$$

Now we prove that this set has null probability.

Indeed,

$$P\left(\bigcap_{m=k}^{\infty}\bigcup_{i=1}^{m} A^{i}_{M,m}\right) \le \liminf_{n} P\left(\bigcup_{i=1}^{m} A^{i}_{M,m}\right) \le \liminf_{n} \sum_{i=1}^{m} P(A^{i}_{M,m})$$

$$\le \liminf_{n} m \left[P\left(|B_{\frac{1}{m}}| \le \frac{M}{m^{\gamma}}\right)\right]^{N},$$

where we have used that $B_{\frac{i}{m}} - B_{\frac{i+1}{m}} \sim N(0, \frac{1}{m})$ and they are independent.

But

$$P\left(|B_{\frac{1}{n}}| \le \frac{M}{m^{\gamma}}\right) = \sqrt{\frac{m}{2\pi}} \int_{-Mm^{-\gamma}}^{Mm^{-\gamma}} e^{-\frac{mx^2}{2}}dx = \frac{1}{\sqrt{2\pi}} \int_{-Mm^{\frac{1}{2}-\gamma}}^{Mm^{\frac{1}{2}-\gamma}} e^{-x^2/2}dx$$

$$\le C m^{\frac{1}{2}-\gamma}$$

So, taking N such that $N(\gamma-\frac{1}{2}) > 1$,

$$P\left(\bigcap_{m=k}^{\infty}\bigcup_{i=1}^{m} A^{i}_{M,m}\right) \le \liminf_{n} m\, C^{N}\left(m^{\frac{1}{2}-\gamma}\right)^{N} = 0.$$

Since this holds for any k, M, we get the desired result. ∎

Notice that, if a.s the sample path of Brownian motion were differentiable at some point, they would also be locally γ-Hölder continuous of degree $\gamma = 1$. This contradicts the precedent theorem.

Sharpness of the Hölder exponent

What happens for $\gamma = \frac{1}{2}$?

For a function $g: [0,+\infty) \to \mathbb{R}$, the modulus of continuity is a way to describe its local smoothness. More precisely, let $\sigma: [0,\infty) \to [0,\infty)$ be such $\sigma(0) = 0$ and strictly increasing. The function σ is a modulus of continuity for the function f if for any $0 \le s \le t$,

$$|f(t) - f(s)| \le \sigma(|t-s|).$$

Observe that for γ-Hölder continuous functions, except the constant $\sigma(t) = t^{\gamma}$.

Theorem: Let $\{B_t, t \ge 0\}$ be a Brownian motion. Then

$$P\left(\limsup_{h\to 0} \frac{\sup_{0\le t\le 1-h} |B(t+h) - B(t)|}{\sqrt{2h|\lg h|}} = 1\right) = 1.$$

Paul Lévy

We have

$$\frac{\sup_{0\le t\le 1-h} |B(t+h)-B(t)|}{\sqrt{h}} = \frac{\sup_{0\le t\le 1-h} |B(t+h)-B(t)|}{\sqrt{2h|\ln h|}} \sqrt{2|\ln h|} \xrightarrow[h\to 0]{} +\infty$$

If the sample paths were Hölder continuous with exponent $\gamma = \frac{1}{2}$, this limit would be a constant.

Quadratic variation

The notion of quadratic variation provides a mesure of the roughness of a function. Existence of variation of different orders are also important in approximation by taylor's expansion and also in the developement of infinitesimal calculus: Itô's sthocastic calculus

Fix a finite interval $[0,T]$ and consider the sequence of partitions defined by $\Pi_n : \{t_0^n = 0 \le t_1^n \le \dots \le t_{k_n}^n = T\}$, $n\ge 1$. We assume

$$\lim_n |\Pi_n| = 0,$$

where $|\Pi_n|$ denotes the norm of the partition Π_n, i.e.

$$|\Pi_n| = \sup_{j=0,\dots,k_n-1} |t_{j+1}-t_j|.$$

Set $\Delta_k B = B_{t_k^n} - B_{t_{k-1}^n}$.

Proposition: the sequence $\{\sum_{k=1}^{k_n} (\Delta_k B)^2, n\ge 1\}$ converge in $L^2(\Omega)$ to T.

Proof: We will prove

$$\lim_n E\left[\left(\sum_{k=1}^{k_n} (\Delta_k B)^2 - T\right)^2\right] = 0.$$

For the sake of simplicity we omit the dependence on n. Notice that the random variable $(\Delta_k B)^2 - \Delta_k t$, $k=1,\dots,n$, are independent and centred. Indeed, the independence follows from the property of independent increment. Moreover $\Delta_k B \approx N(0,\Delta_k t)$ and so $E(\Delta_k B)^2 = \Delta_k t$. Then,

$$E\left[\left(\sum_{k=1}^{k_n} (\Delta_k B)^2 - T\right)^2\right] = E\left[\left(\sum_{k=1}^{k_n}\left((\Delta_k B)^2 - \Delta_k t\right)\right)^2\right]$$

$$= \sum_{k=1}^{k_n} E\left[\left((\Delta_k B)^2 - \Delta_k t\right)^2\right] = \sum_{k=1}^{k_n}\left[\underbrace{E\left[(\Delta_k B)^4\right]}_{3(\Delta_k B)^2} - 2(\Delta_k t)\, E\left[(\Delta_k B)^2\right] + (\Delta_k t)^2\right]$$

$$= \sum_{k=1}^{k_n}\left(3(\Delta_k t)^2 - 2(\Delta_k t)^2 + (\Delta_k t)^2\right) = 2\sum_{k=1}^{k_n} (\Delta_k t)^2 \le 2T|\Pi_n|$$

However, sample paths of Brownian motion are infinite variation.

$$V = \sup_n \sum_{k=1}^{r_n} |\Delta_k B| = \infty, \text{ a.s.}$$

Indeed, assume $V < \infty$. Then,

$$\sum_{k=1}^{r_n} (\Delta_k B)^2 \leq \sup_k (\Delta_k B) \left(\sum_{k=1}^{r_n} |\Delta_k B| \right) \leq V . \sup_k |\Delta_k B|$$

We obtain $\lim_n \sum_{k=1}^{r_n} (\Delta_k B)^2 = 0$, a.s. which contradicts the precedent result.

Observations

1) In the particular case $\Pi_n \subset \Pi_{n+1}$, $n \geq 1$, we can obtain

$$\lim_n \sum_{k=1}^{r_n} (\Delta_k B)^2 = T, \text{ a.s.} \qquad (1)$$

For instance, $\Pi_n = \{ (kT) 2^{-n}, k = 0, \dots, 2^n \}$.

2) Assuming that Π_n satisfies the hypothesis of precedent Proposition and that there exists $\gamma \in (0,1)$ such that $\sum_{n \geq 1} |\Pi_n|^\gamma < \infty$. Then (1) holds.
The proof is based on Chebychev's inequality,

$$P\left\{ \left| \sum_{k=1}^{r_n} (\Delta_k B)^2 - T \right| > \lambda \right\} \leq \lambda^{-2} E\left(\left| \sum_{k=1}^{r_n} (\Delta_k B)^2 - T \right|^2 \right) \leq c \lambda^{-2} |\Pi_n|$$

Choose $\lambda = |\Pi_n|^{\frac{1-\gamma}{2}}$, then

$$\sum_{n \geq 1} P\left\{ \left| \quad \right| > \lambda \right\} \leq c \sum_{n \geq 1} |\Pi_n|^\gamma < \infty.$$

So, Borel-Cantelli's Lemma implies (1).

2.4 The martingale property of Brownian motion

We introduce the notion of filtration. A family $\{\mathcal{F}_t, t \geq 0\}$ of sub-σ-fields of $\mathcal{F}$ is said a filtration if

a) $\mathcal{F}_0$ contains all the sets of $\mathcal{F}$ of null probability

b) For any $0 \leq s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$.

If in addition

$$\bigcap_{s > t} \mathcal{F}_s = \mathcal{F}_t, \qquad \text{for any } t \geq 0,$$

the filtration is said to be right-continuous. (in the sequel filtration means a right-continuous filtration)

Given a stochastic process $\{X_t, t \geq 0\}$, there is a simple way to define a filtration by considering

$$\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t), \ t \geq 0,$$

that is, the smallest σ-field generated by the random variables X_s, $0 \le s \le t$ (the smallest σ-field $\mathcal{G}$ such that all random variables X_s, $0 \le s \le t$, are $\mathcal{G}$-measurable). In this case, $\{\mathcal{F}_t, t \geq 0\}$ is called the natural σ-field associated with $\{X_t, t \geq 0\}$.

In general there is no reason why the natural filtration associated with a process should satisfy the conditions 1 and 2, above. Property a) can be achieved by completing the σ-field $\mathcal{F}_0$ with all set of $\mathcal{F}$ of null probability. The natural filtration of the Brownian motion posseses property b).

<u>Definition</u>: A stochastic process $\{X_t, t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if each random variable X_t belongs to $L^1(\Omega)$ and moreover,

a) X_t is $\mathcal{F}_t$-measurable for any $t \geq 0$.

b) for any $0 \le s \le t$, $E(X_t \mid \mathcal{F}_s) = X_s$.

If b) $E(X_t \mid \mathcal{F}_s) \le X_s$ is a supermartingale.

b) $E(X_t \mid \mathcal{F}_s) \geq X_s$ is a submartingale.

<u>Observation</u>: We can also change the property b) by

b) for any $0 \le s \le t$, $E(X_t - X_s \mid \mathcal{F}_s) = 0$.

<u>Interpretation of the martingale condition</u> (pg 35)

<u>Proposition</u>: A stochastic process $\{X_t, t \geq 0\} \subset L^1(\Omega)$ with X_0 constant, constant mean and independent increments possesses the martingale property with respect to the natural filtration.

<u>Proof</u>: the property a) is obvious. On the other hand, the independence between $X_t - X_s$, $0 \le s \le t$, and $\mathcal{F}_s$ implies that

$$E(X_t - X_s \mid \mathcal{F}_s) = E(X_t - X_s) = 0.$$

$\boxtimes$

<u>Corollary</u>: A Brownian motion has the martingale property with respect to the natural filtration.

Examples

① $X_t = \{B_t^2 - t,\ t \geq 0\}$ is a martingale.

For $0 \leq s \leq t$,

$$E\left(B_t^2 - t - (B_s^2 - s) \mid \mathcal{F}_s\right) = E(B_t^2 - B_s^2 \mid \mathcal{F}_s) - (t-s)$$

$$= E\left((B_t - B_s)^2 \mid \mathcal{F}_s\right) + 2E\left(B_s(B_t - B_s) \mid \mathcal{F}_s\right) - (t-s)$$

$$= E\left((B_t - B_s)^2\right) + 2B_s E(B_t - B_s \mid \mathcal{F}_s) - (t-s)$$

$$= t - s + 0 - (t-s) = 0.$$

② $Y = \{\exp\{aB_t - \frac{a^2t}{2}\},\ t \geq 0\}$ is a martingale.

$$E\left(e^{aB_t - \frac{a^2t}{2}} \mid \mathcal{F}_s\right) = e^{aB_s} E\left(e^{a(B_t - B_s) - \frac{a^2t}{2}} \mid \mathcal{F}_s\right)$$

$$= e^{aB_s} E\left(e^{a(B_t - B_s) - \frac{a^2t}{2}}\right).$$

$$= e^{aB_s} \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} e^{ax - \frac{a^2t}{2} - \frac{x^2}{2(t-s)}} \, dx$$

$$= e^{aB_s} e^{\frac{a^2(t-s)}{2} - \frac{a^2t}{2}} = e^{aB_s} e^{-\frac{a^2s}{2}},$$

where we have used

$$\frac{x^2}{2(t-s)} - ax + \frac{a^2t}{2} = \frac{(x - a(t-s))^2}{2(t-s)} + \frac{a^2t}{2} - \frac{a^2(t-s)}{2}.$$

2.5 MARKOV PROPERTY OF BROWNIAN MOTION

The conditional probability with respect to a σ-field $\mathcal{G} \subset \mathcal{F}$ is given by

$$P(F \mid \mathcal{G}) = E(\mathbb{1}_F \mid \mathcal{G}), \quad F \in \mathcal{F}.$$

For any $0 \leq s \leq t$, $x \in \mathbb{R}$, we set

$$p(s,t,x,A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{|x-y|^2}{2(t-s)}} \, dy \qquad (-1)$$

Actually $p(s,t,x,A)$ is the probability that a random variable $Z \sim N(x, t-s)$ takes values on a fixed set A, $P(Z \in A)$.

We observe that

$$\begin{aligned} p(s,t,x,\cdot) : \mathcal{F} &\longrightarrow [0,1] \\ A &\longrightarrow p(s,t,x,A) \end{aligned} \quad \text{is a probability}$$

Proposition: Let $\{B_t, t \geq 0\}$ be a B.m. and $\{\mathcal{F}_t, t \geq 0\}$ its natural filtration.

1) For $0 \leq s \leq t$, $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$,

$$P(B_t \in A \mid \mathcal{F}_s) := E(\mathbb{1}_{\{B_t \in A\}} \mid \mathcal{F}_s) = p(s, t, B_s, A), \text{ a.s.} \quad (1)$$

where the expression is defined by ω as follows

$$p(s,t, B_s, A)(\omega) = p(s,t, B_s(\omega), A) = p(s, t, x, A)\big|_{x = B_s(\omega)}.$$

2) Let $\sigma(B_s)$ denote the σ-field generated by the random variable B_s, $s \geq 0$. Then

$$P(B_t \in A \mid \sigma(B_s)) =: E(\mathbb{1}_{\{B_t \in A\}} \mid \sigma(B_s)) = p(s,t, B_s, A), \text{ a.s.}$$

Consequently,

$$P(B_t \in A \mid \mathcal{F}_s) = P(B_t \in A \mid \sigma(B_s)) = p(s,t, B_s, A), \text{ a.s.}$$

Proof: We will prove a slightly stronger result: For any bounded measurable function $f: \mathbb{R} \to \mathbb{R}$

$$E(f(B_t) \mid \mathcal{F}_s) = \int_{\mathbb{R}} f(y)\, p(s,t, B_s, dy), \text{ a.s.}$$

So, taking $f = \mathbb{1}_A$ we obtain (1).
Using that B_s is $\mathcal{F}_s$-measurable and $B_t - B_s$ independent of $\mathcal{F}_s$ we have that

$$E(f(B_t) \mid \mathcal{F}_s) = E(f(B_s + B_t - B_s) \mid \mathcal{F}_s) = E(f(x + B_t - B_s))\big|_{x = B_s}.$$

The random variable $x + B_t - B_s$ has the law $N(x, t-s)$,

$$E(f(x + B_t - B_s)) = \int_{\mathbb{R}} f(y)\, p(s, t, x, dy)$$

and then

$$E(f(B_t) \mid \mathcal{F}_s) = \int_{\mathbb{R}} f(y)\, p(s,t, B_s, dy).$$

Similar arguments give the other consequence. ■

Proposition. [Chapman - Kolmogorov equation]. Fix $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$. For any $0 \leq s \leq u \leq t$, the mapping $(s,t) \longrightarrow p(s, t, x, A)$ satisfies the relation

$$p(s,t, x, A) = \int_{\mathbb{R}} p(u, t, y, A)\, p(s, u, x, dy), \quad (2)$$

where, for any $0 \leq s \leq u$ and $x, y \in \mathbb{R}$,

$$p(s, u, x, dy) = \frac{1}{(2\pi(u-s))^{1/2}} \exp\left(-\frac{|x-y|^2}{2(u-s)}\right) dy = f_{N(x, u-s)}(y)\, dy$$

Proof: Using (-1),

$$\int_{\mathbb{R}} p(u,t,y,A)\, p(s,u,x,dy) = \int_{\mathbb{R}} p(s,u,x,dy) \int_A p(u,t,y,dz)$$

$$= \int_A dz \left(\int_{\mathbb{R}} dy\, f_{N(x,u-s)}(y)\, f_{N(0,t-u)}(y-z) \right)$$

$$= \int_A dz \left(f_{N(x,u-s)} * f_{N(0,t-u)} \right)(z)$$

$$= \int_A dz\, f_{N(x,t-s)}(z) = p(s,t,x,A),$$

where we have used that

a) X and Y (independent) random variables with densities f_1 and f_2, respectively. then, the random variable $X+Y$ has a density given by

$$f_{X_1+X_2}(z) = \int_{\mathbb{R}} f_1(x) f_2(x-z)\, dx := (f_1 * f_2)(z).$$

b) From b) we deduce that the sum of two independent normal random variables is a normal random variable whose mean and variance is the sum of the respective means and varianc

The formule of the last proposition can be understood as a general statement of the "total probabilities" principle. ▨

We are now ready to give the definition of a Markov process.

Definition: Consider a mapping

$$p: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathcal{B}(\mathbb{R}) \longrightarrow \mathbb{R}_+$$

satisfying the properties:

i) For any fixed $s,t \in \mathbb{R}_+$, $A \in \mathcal{B}(\mathbb{R})$, $x \longmapsto p(s,t,x,A)$ is $\mathcal{B}(\mathbb{R})$-measurable

ii) For any fixed $s,t \in \mathbb{R}_+$, $x \in \mathbb{R}$, $A \longmapsto p(s,t,x,A)$ is a probability.

iii) Equation (2) holds.

Such a function p is termed a Markovian transition probability function. Le us also fix a probability μ on $\mathcal{B}(\mathbb{R})$.

A real valued process $\{X_t, t \geq 0\}$ is a Markov process with initial law μ and transition probability function p if

a) the law of X_0 is μ

b) For any $0 \leq s \leq t$,

$$P(X_t \in A \mid \mathcal{F}_s) = p(s,t,X_s,A)$$

Proposition: A Brownian motion $\{B_t, t \geq 0\}$ is a Markov process with initial law $\mu = \delta_{\{0\}}$ (a Dirac delta function at 0) and transition probability function p defined in (-1).

$$p(s,t,x,A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A \exp\left(-\frac{(x-y)^2}{2(t-s)}\right) dy \qquad (3)$$

Proof: Since $B_0 = 0$ a.s., the law of B_0 is a Dirac delta function at 0. Moreover, from the above considerations and, in particular, the previous propositions we deduce that (3) satisfies the conditions of the definition.

2.6 STRONG MARKOV PROPERTY OF THE BROWNIAN MOTION

Throughout this section, $(\mathcal{F}_t, t \geq 0)$ will denote the natural filtration associated with a Brownian motion $\{B_t, t \geq 0\}$ and stopping times will always refer to this filtration.

Theorem: Let T be a stopping time. Then, conditionally to $\{T < \infty\}$, the process defined by

$$B_t^T = B_{T+t} - B_T, \; t \geq 0,$$

is a Brownian motion independent of $\mathcal{F}_T$.

Proposition: For any $t > 0$, set $S_t = \sup_{s \leq t} B_s$. Then, for any $a \geq 0$ and $b \leq a$,

$$P(S_t \geq a, B_t \leq b) = P(B_t \geq 2a - b).$$

As a consequence, the probability law of S_t and $|B_t|$ are the same.