Research Article

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Strong limit of processes constructed from a renewal process

https://doi.org/10.1515/math-2023-0112
received January 25, 2023; accepted August 13, 2023

Abstract: We construct a family of processes, from a renewal process, that have realizations that converge almost surely to the Brownian motion, uniformly on the unit time interval. Finally, we compute the rate of convergence in a particular case.

Keywords: strong convergence, renewal process, Brownian motion

MSC 2020: 60F17, 60G15

1 Introduction

In this article, we study realizations of processes that converge almost surely, uniformly on the unit time interval, to the standard Brownian motion. In the mathematical literature, we can find papers studying the strong convergence of random walks or the process usually called as uniform transport processes. Our aim is to deal with extensions of the uniform transport process.

The uniform transport process, introduced by Kac in [1], can be written as

$$y_n(t) = \frac{1}{n}(-1)^A \int_0^{n^2 t} (-1)^{N(u)} \mathrm{d}u,$$

where $N = \{N(t), t \geq 0\}$ is a standard Poisson process and $A \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ independent of the Poisson process N.

Griego et al. [2] showed that these processes converge strongly and uniformly on bounded time intervals to Brownian motion. Gorostiza and Griego [3] and Csörgő and Horváth [4] obtained a rate of convergence. More precisely, in [3] it is proved that there exist versions of the transport processes $\tilde{y}_n$ on the same probability space as a given Brownian motion $(y(t))_{t\geq 0}$ such that, for each $q > 0$,

$$P\left(\sup_{a\leq t\leq b} |y_n(t) - y(t)| > Cn^{\frac{1}{2}}(\log n)^{\frac{5}{2}}\right) = o(n^{-q}),$$

as $n \to \infty$ and where C is a positive constant depending on a, b and q. These bounds are improved in [5] using an explicit computation in the Skorohod embedding problem. Furthermore, we can find several papers (see for instance [6–11]) where the authors defined a sequence of processes, obtained as modifications of the uniform transport process, that converges strongly to some Gaussian processes uniformly on bounded intervals.

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Nevertheless, all these papers are based on processes built from a Poisson process. Let us recall that Poisson process has jump times with exponential laws and so we are able to use all the particular properties of this distribution.

It is well known that the Gaussian distribution serves as a universal approximation for the sum of a large number of independent random variables, thanks to the central limit theorem. The aim of our result, using renewal process instead of the Poisson process, is to show that the Brownian motion plays a similar role as a general approximation beyond the Donsker invariance principle.

Our aim is to deal with jump times that do not have exponential law. We consider extensions of the uniform transport process using a reward renewal process instead of a Poisson process. As far as the authors know, these types of processes have not been studied.

We consider

$$x_n(t) = h(n) \int_0^{g(n)t} (-1)^{T(u)} \mathrm{d}u,$$

where $T = \{T(t), t \geq 0\}$ is a renewal reward process and h and g are nonnegative functions defined on $\mathbb{N}$.

We will show that for a wide class of renewal reward processes we have, when n goes to ∞, the strong convergence of these processes to a standard Brownian motion. We also deal with the rate of convergence. Unfortunately, we are not able to obtain a general result since the proofs heavily rely on the specific distribution of the jump times. We will compute the rate of convergence when the jump times have uniform distribution, showing that the method used in [3] can be adapted for nonexponential times. All these results give us new ways to simulate the behavior of the standard Brownian motion.

The article is organized in the following way. Section 2 is devoted to define the processes and to give the main results. In Section 3, we prove the strong convergence theorem. The study of the rate of convergence is given in Section 4.

2 Definitions and main result

Consider $(U_m)_{m\geq 1}$ be a sequence of independent random variables which take on only nonnegative values. We also assume that they are identically distributed with $P(U_1 = 0) < 1$ and $E((U_1)^4) < \infty$.

For each $k \geq 1$ consider the renewal sequence $S_k = U_1 + \cdots + U_k$ and the counting renewal function

$$L(t) = \sum_{k=1}^{\infty} \mathbb{1}_{[0,t]}(S_k),$$

that is the counting function of the number of renewals in $[0, t]$.

Set $\{\eta_m\}_{m\geq 0}$ a sequence of independent identically distributed random variables with law Bernoulli($\frac{1}{2}$), independent of $\{U_m\}_{m\geq 1}$. Then, we will deal with the renewal reward process defined as

$$T(t) = \eta_0 + \sum_{k=1}^{\infty} \eta_k \mathbb{1}_{[0,t]}(S_k) = \sum_{l=0}^{L(t)} \eta_l.$$

Then, given a strictly positive function β we define

$$T_{\beta(n)}(t) = T\left(\frac{t}{\beta(n)}\right) = \eta_0 + \sum_{k=1}^{\infty} \eta_k \mathbb{1}_{[0,\frac{t}{\beta(n)}]}(S_k) = \eta_0 + \sum_{k=1}^{\infty} \eta_k \mathbb{1}_{[0,t]}(\beta(n) S_k). \tag{1}$$

Note that putting $U_m^n = \beta(n) U_m$ for all $m \geq 1$, we have that

$$\beta(n) S_k = U_1^n + \cdots + U_k^n.$$

Our aim is to study the convergence of the processes

$$x_n(t) = \left(\beta(n)\frac{\mathbf{E}((U_1)^2)}{\mathbf{E}(U_1)}\right)^{-\frac{1}{2}} \int_0^t (-1)^{T_{\beta(n)}(u)} \mathrm{d}u = \frac{1}{G(n)} \int_0^t (-1)^{T_{\beta(n)}(u)} \mathrm{d}u, \tag{2}$$

where

$$G(n) = \left(\beta(n)\frac{\mathbf{E}((U_1)^2)}{\mathbf{E}(U_1)}\right)^{\frac{1}{2}},$$

with

$$\sum_{n\geq 1} \beta(n) < \infty.$$

Obviously, we can write

$$x_n(t) = \frac{1}{G(n)} \int_0^t (-1)^{T(\frac{u}{\beta(n)})} \mathrm{d}u = \frac{1}{G(n)} \beta(n) \int_0^{\frac{t}{\beta(n)}} (-1)^{T(v)} \mathrm{d}v = \left(\frac{\mathbf{E}(U_1)}{\mathbf{E}((U_1)^2)}\right)^{\frac{1}{2}} \beta(n)^{\frac{1}{2}} \int_0^{\frac{t}{\beta(n)}} (-1)^{T(v)} \mathrm{d}v. \tag{3}$$

Our next result gives the strong convergence result and states as follows:

Theorem 2.1. *There exist realizations* $\{x_n(t), t \in [0,1]\}$ *of the processes defined in* (3) *on the same probability space as a standard Brownian motion* $\{x(t), t \geq 0\}$ with $x(0) = 0$ *such that*

$$\lim_{n\to\infty} \max_{0\leq t\leq 1} |x_n(t) - x(t)| = 0 \quad a.s.$$

Proof. See Section 3. □

Observe that we are assuming that the jumps of $\{x_n(t), t \in [0,1]\}$ occur with times that follows a family of nonnegative independent identically distributed random variables $\{U_m^n\}_{m\geq 1}$.

3 Proof of strong convergence

In this section, we will prove the strong convergence when n tends to ∞ of the processes $\{x_n(t); t \in [0,1]\}$ defined in Section 2.

Proof of Theorem 2.1. We will follow the methodology used in [2].

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the probability space for a standard Brownian motion $\{x_t, t \geq 0\}$ with $x(0) = 0$ and let us define:

(1) for each $n > 0$, $\{\xi_m^n\}_{m\geq 1}$ a sequence of nonnegative independent identically distributed random variables, independent of the Brownian motion x, such that
$$G(n) \times \xi_m^n \sim U_m^n. \tag{4}$$
(2) $\{k_m\}_{m\geq 1}$ a sequence of independent identically distributed random variables such that $P(k_1 = 1) = P(k_1 = -1) = \frac{1}{2}$, independent of x and $\{\xi_m^n\}_{m\geq 1}$ for all n.

Note that

$$\xi_m^n \sim \frac{\beta(n)}{G(n)} \times U_m = \beta(n)^{\frac{1}{2}} \frac{\mathbf{E}(U_1)^{\frac{1}{2}}}{E((U_1)^2)^{\frac{1}{2}}} \times U_m.$$

So

$$\mathbf{E}(\xi_m^n) = \beta(n)^{\frac{1}{2}} \frac{\mathbf{E}(U_1)^{\frac{3}{2}}}{E((U_1)^2)^{\frac{1}{2}}}, \quad \mathbf{E}((\xi_m^n)^2) = \beta(n)\mathbf{E}(U_1),$$

and

$$\mathbf{E}((\xi_m^n)^4) = \beta(n)^2 \frac{\mathbf{E}((U_1)^4)\mathbf{E}(U_1)^2}{\mathbf{E}((U_1)^2)^2}.$$

By Skorokhod's theorem ([12], p. 163) for each $n \geq 1$ there exists a sequence $\sigma_1^n, \sigma_2^n, \dots$ of nonnegative independent random variables on $(\Omega, \mathcal{F}, \mathcal{P})$ so that the sequence $x(\sigma_1^n), x(\sigma_1^n + \sigma_2^n), \dots,$ has the same distribution as $k_1\xi_1^n, k_1\xi_1^n + k_2\xi_2^n, \dots,$ and, for each m,

(1)

$$\mathbf{E}(\sigma_m^n) = \operatorname{Var}(k_m\xi_m^n) = \mathbf{E}((\xi_m^n)^2) = \beta(n)\mathbf{E}(U_1).$$

(2) There exists a constant L_2 such that

$$\operatorname{Var}(\sigma_m^n) \leq \mathbf{E}((\sigma_m^n)^2) \leq L_2\mathbf{E}((\xi_m^n)^4) = L_2\beta(n)^2 \frac{\mathbf{E}((U_1)^4)\mathbf{E}(U_1)^2}{\mathbf{E}((U_1)^2)^2}.$$

For each n we define $\gamma_0^n \equiv 0$ and for each m

$$\gamma_m^n = G(n) \left| x\left(\sum_{j=0}^{m} \sigma_j^n\right) - x\left(\sum_{j=0}^{m-1} \sigma_j^n\right) \right|,$$

where $\sigma_0^n \equiv 0$.

Then, from (4) it follows that the random variables $\gamma_1^n, \gamma_2^n, \dots,$ are independent with the same distribution that $U_1^n, U_2^n, \dots,$ and

$$E(\gamma_m^n) = E(U_m^n) = \beta(n)E(U_1)$$

and

$$\operatorname{Var}(\gamma_m^n) = \beta(n)^2 \operatorname{Var}(U_1).$$

Now, we define $x_n(t)$, $t \geq 0$ to be piecewise linear satisfying

$$x_n\left(\sum_{j=1}^{m} \gamma_j^n\right) = x\left(\sum_{j=1}^{m} \sigma_j^n\right), \quad m \geq 1 \tag{5}$$

and $x_n(0) \equiv 0$. Observe that the process x_n has slope $\pm|G(n)|^{-1}$ in the interval $[\sum_{j=1}^{m-1}\gamma_j^n, \sum_{j=1}^{m}\gamma_j^n]$.

On the other hand, let $\Gamma_m^n = \sum_{j=1}^{m}\gamma_j^n$. We obtain that the increments $\Gamma_m^n - \Gamma_{m-1}^n$, for each m, with $\Gamma_0^n \equiv 0$, are independent and have law $G(n) \times \xi_1^m \sim U_1^n$. Moreover, the probability that $x(\sum_{j=0}^{m}\sigma_j^n) - x(\sum_{j=0}^{m-1}\sigma_j^n)$ is positive is $\frac{1}{2}$, independent of the past up to time $\sum_{j=0}^{m-1}\sigma_j^n$. Thus, x_n is a realization of the process (2).

Set $H(n) \coloneqq \beta(n)E(U_1)$. Recalling that $\gamma_0^n \equiv \sigma_0^n \equiv 0$, by (5) and the uniform continuity of Brownian motion on $[0, 1]$, we have almost surely

$$\begin{aligned}\lim_{n\to\infty}\max_{0\leq t\leq 1}|x_n(t) - x(t)| &= \lim_{n\to\infty}\max_{0\leq m\leq\frac{1}{H(n)}} \left| x_n\left(\sum_{j=0}^{m}\gamma_j^n\right) - x\left(\sum_{j=0}^{m}\gamma_j^n\right) \right| \\ &= \lim_{n\to\infty}\max_{0\leq m\leq\frac{1}{H(n)}} \left| x\left(\sum_{j=0}^{m}\sigma_j^n\right) - x\left(\sum_{j=0}^{m}\gamma_j^n\right) \right|,\end{aligned}$$

and it reduces the proof to check that

$$\lim_{n\to\infty}\max_{1\le m\le\frac{1}{H(n)}}|\gamma_1^n+\cdots+\gamma_m^n-mH(n)|=0\quad\text{a.s.},$$

and that

$$\lim_{n\to\infty}\max_{1\le m\le\frac{1}{H(n)}}|\sigma_1^n+\cdots+\sigma_m^n-mH(n)|=0\quad\text{a.s.}$$

The first limit can be obtained easily by Borel-Cantelli lemma since by Kolmogorov's inequality, for each $\alpha>0$, we have

$$P\left(\max_{1\le m\le\frac{1}{H(n)}}|\gamma_1^n+\cdots+\gamma_m^n-mH(n)|\ge\alpha\right)\le\frac{1}{\alpha^2}\sum_{m=1}^{[\frac{1}{H(n)}]}\mathrm{Var}(\gamma_k^n)\le\frac{1}{\alpha^2}\sum_{m=1}^{\infty}\beta(n)^2\mathrm{Var}(U_1)<\infty.$$

We can study the second limit repeating the same arguments as before. Using the bounds obtained from Skorohod's theorem, for each $\alpha>0$, we have

$$P\left(\max_{1\le m\le\frac{1}{H(n)}}|\sigma_1^n+\cdots+\sigma_m^n-mH(n)|\ge\alpha\right)\le\frac{1}{\alpha^2}\sum_{m=1}^{[\frac{1}{H(n)}]+1}\mathrm{Var}(\sigma_m^n)<\infty.$$ □

4 Rate of convergence

In this section, we will prove the rate of convergence of the processes $x_n(t)$ in a particular case. We consider $U_m\sim U(0,1)$ for all $m\ge1$ and $\beta(n)=n^{-k}$ with $k>1$. Then

$$G(n)=\frac{2^{\frac12}}{3^{\frac12}n^{\frac{k}{2}}},\qquad H(n)=\frac{1}{2n^k},$$

and

$$U_m^n\sim U(0,n^{-k}),\qquad\gamma_m^n\sim U(0,n^{-k}),\qquad\xi_m^n\sim U\left(0,\frac{3^{\frac12}}{2^{\frac12}}n^{-\frac{k}{2}}\right).$$

Theorem 4.1. *Assume $U_m\sim U(0,1)$ for all $m\ge1$ and $\beta(n)=n^{-k}$ with $k>1$. Then, for all $q>0$,*

$$P\left(\max_{0\le t\le1}|x_n(t)-x(t)|>an^{-\frac{k}{4}}(\log n)^{\frac32}\right)=o(n^{-q})\quad\text{as } n\to\infty,$$

where a is a positive constant depending on q.

Since the proof follows the structure of part b) of Theorem 1 in [3], we give a sketch of the proof.

Proof of Theorem 4.1. Recall that $\gamma_0^n\equiv\sigma_0^n\equiv0$ and define

$$\Gamma_m^n=\sum_{j=0}^m\gamma_j^n\quad\text{and}\quad\Lambda_m^n=\sum_{j=0}^m\sigma_j^n.$$

Note that $x_n(\Gamma_m^n)=x(\Lambda_m^n)$. Set

$$J^n\equiv\max_{0\le m\le\frac{1}{H(n)}}\max_{0\le r\le\gamma_{m+1}^n}|x_n(\Gamma_m^n+r)-x(\Gamma_m^n+r)|.$$

Since x_n is piecewise linear and using the definition of γ_m^n, note that

$$x_n(\Gamma_m^n + r) = x(\Lambda_m^n) + \frac{x(\Lambda_{m+1}^n) - x(\Lambda_m^n)}{\gamma_{m+1}^n}\, r$$
$$= x(\Lambda_m^n) + \frac{1}{G(n)} \times \mathrm{sgn}(x(\Lambda_{m+1}^n) - x(\Lambda_m^n)) r.$$

Thus,

$$\begin{aligned} J^n \le{} & \max_{0\le m\le \frac{1}{H(n)}} |x(\Lambda_m^n) - x(mH(n))| + \max_{0\le m\le \frac{1}{H(n)}} |x(\Gamma_m^n) - x(mH(n))| \\ & + \max_{0\le m\le \frac{1}{H(n)}} \max_{0\le r\le \gamma_{m+1}^n} |x(\Gamma_m^n) - x(\Gamma_m^n + r)| + \max_{1\le m\le \frac{1}{H(n)}+1} \frac{1}{G(n)}\gamma_m^n \\ \coloneqq{} & J_1^n + J_2^n + J_3^n + J_4^n, \end{aligned}$$

and for any $a_n > 0$,

$$P(J^n > a_n) \le \sum_{j=1}^4 P\left(J_j^n > \frac{a_n}{4}\right) \coloneqq I_1^n + I_2^n + I_3^n + I_4^n.$$

We will study the four terms separately.

1. Study of the term I_4^n. Since γ_m^ns are independent variables with law $\sim U(0, n^{-k})$,

$$I_4^n \le P\left(\max_{1\le m\le \frac{1}{H(n)}+1} \gamma_m^n > \frac{a_n}{2^{\frac{3}{2}} 3^{\frac{1}{2}} n^{\frac{k}{2}}}\right) = 1 - P\left(\gamma_m^n \le \frac{a_n}{2^{\frac{3}{2}} 3^{\frac{1}{2}} n^{\frac{k}{2}}}\right)^{[\frac{1}{H(n)}]+1} = 0,$$

when n is big enough for a_n of the type $a n^{-\frac{k}{4}} (\log n)^\beta$, with α and β positive arbitrary fixed constants.

2. Study of the term I_1^n. Let $\delta_n > 0$. We can write

$$\begin{aligned} I_1^n \le{} & P\left(\max_{0\le m\le \frac{1}{H(n)}} \max_{|s|\le \delta_n} |x(mH(n) + s) - x(mH(n))| > \frac{a_n}{4}\right) + P\left(\max_{1\le m\le \frac{1}{H(n)}} |\Lambda_m^n - mH(n)| > \delta_n\right) \\ ={} & I_{11}^n + I_{12}^n. \end{aligned}$$

2.1. Study of the term I_{12}^n. Note that

$$I_{12}^n = P\left(\max_{1\le m\le \frac{1}{H(n)}} \left| \sum_{j=1}^m \left(\frac{1}{H(n)}\sigma_j^n - 1\right)\right| > \frac{\delta_n}{H(n)}\right) \le \left(\frac{H(n)}{\delta_n}\right)^{2p} \mathbf{E}\left[\left(\sum_{m=1}^{[\frac{1}{H(n)}]} \left(\frac{1}{H(n)}\sigma_m^n - 1\right)\right)^{2p}\right], \tag{6}$$

for any $p \ge 1$, by Doob's martingale inequality.

Set $Y_m \coloneqq \frac{1}{H(n)}\sigma_m^n - 1$. For any integer $p \ge 1$, using Hölder's inequality, we obtain

$$\begin{aligned} \mathbf{E}\left[\left(\sum_{m=1}^{[\frac{1}{H(n)}]} Y_m\right)^{2p}\right] &= \sum_{\substack{|u|=2p \\ u_m\ne 1\ \forall m}} \binom{2p}{u} \mathbf{E}\left(Y_1^{u_1}\cdots Y_{[\frac{1}{H(n)}]}^{u_{[\frac{1}{H(n)}]}}\right) \\ &\le \sum_{\substack{|u|=2p \\ u_m\ne 1\ \forall m}} \binom{2p}{u} [\mathbf{E}(Y_1^{2p})]^{u_1/2p} \cdots [\mathbf{E}\left(Y_{[\frac{1}{H(n)}]}^{2p}\right)]^{u_{[\frac{1}{H(n)}]}/2p}, \end{aligned} \tag{7}$$

where $u = (u_1, \ldots, u_{[\frac{1}{H(n)}]})$ with $|u| = u_1 + \cdots + u_{[\frac{1}{H(n)}]}$ and

$$\binom{2p}{u} = \frac{(2p)!}{u_1! \cdots u_{[\frac{1}{H(n)}]}!}.$$

Note that in the first equality we have used that if $u_m = 1$ for any m, then $\mathbf{E}\left(Y_1^{u_1}\cdots Y_{[\frac{1}{H(n)}]}^{u_{[\frac{1}{H(n)}]}}\right) = 0$. On the other hand, by the estimates given by Skorohod's theorem (see [3]), we have

$$\mathbf{E}[(\sigma_m^n)^{2p}] \leq 2(2p)!\mathbf{E}[(k_i\xi_m^n)^{4p}] \leq 2(2p)!\frac{1}{(4p+1)}3^{2p}\left(\frac{1}{2n^k}\right)^{2p}.$$

So, using the inequality $|a+b|^{2p} \leq 2^{2p}(|a|^{2p}+|b|^{2p})$, we obtain

$$\mathbf{E}(Y_m^{2p}) \leq (2p)!6^{2p}. \tag{8}$$

Finally, from a lemma in page 298 in [3] (see also Lemma 5-1 in [7]) we obtain that for

$$p \leq 1 + \frac{\log 2}{\log[1+(2n^{-k}-n^{-2k})^{\frac{1}{2}}]}, \tag{9}$$

we obtain that

$$\sum_{\substack{|u|=2p\\ u_i\neq 1\ \forall i}} \binom{2p}{u} \leq 2^{2p}(2p)!(2n^k)^p. \tag{10}$$

Therefore, for p as above, putting together (6), (7), (8), and (10) and applying Stirling formula, $k! = \sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}e^{\frac{a}{12k}}$, with $0<a<1$, we obtain

$$I_{12}^n \leq (\delta_n)^{-2p}n^{-kp}6^{2p}2^p[\sqrt{2\pi}(2p)^{2p+\frac{1}{2}}e^{-2p}e^{\frac{a}{24p}}]^2 \leq K_1^p(\delta_n)^{-2p}n^{-kp}p^{4p+1},$$

where K_1 is a constant.

Let us impose now $K_1^p(\delta_n)^{-2p}n^{-kp}p^{4p+1} = n^{-2q}$ and $p = [\log n]$. Observe that this p fulfills condition on p of inequality (9). We obtain

$$\delta_n = K_2 n^{q/[\log n]-\frac{k}{2}}[\log n]^{2+1/(2[\log n])}, \tag{11}$$

where $K_2 = \sqrt{K_1}$ is a constant. Clearly, with this δ_n, it follows that $I_{12}^n = o(n^{-q})$.

2.2. Study of the term I_{11}^n. As in Theorem 1 in [3], for big n and using a Doob's martingale inequality for Brownian motion we obtain

$$I_{11}^n \leq \frac{1}{H(n)}P\left(\max_{|s|\leq\delta_n}|x(s)| > \frac{a_n}{4}\right) \leq 8n^k P\left(\max_{0\leq s\leq\delta_n} x(s) > \frac{a_n}{4}\right) \leq 8n^k \exp\left(-\left(\frac{a_n}{4}\right)^2\frac{1}{2\delta_n}\right).$$

Taken δ_n as in equation (11), condition $8n^k\exp(-(a_n)^2/(32\delta_n)) = 8n^{-2q}$ yields that

$$a_n = K_3 n^{-k/4}n^{q/2[\log n]}(\log n)^{1+1/4[\log n]}(\log n)^{1/2},$$

where K_3 is a constant depending on q. Note that

$$a_n = an^{-k/4}(\log n)^{3/2},$$

for big n, where a is a constant that depends on q, satisfies such a condition. Thus, with δ_n as in (11), it follows that $I_{11}^n = o(n^{-q})$.

3. Study of the term I_2^n. For our $\delta_n > 0$, we have

$$I_2^n \leq P\left(\max_{0\leq m\leq\frac{1}{H(n)}}\max_{|s|\leq\delta_n}|x(mH(n)+s)-x(mH(n))| > \frac{a_n}{4}\right)$$
$$+ P\left(\max_{0\leq m\leq\frac{1}{H(n)}}|\Gamma_m^n - mH(n)| > \delta_n\right) = I_{21}^n + I_{22}^n.$$

On one hand, observe that $I_{21}^n = I_{11}^n$, thus $I_{21}^n = o(n^{-q})$.

On the other hand, applying Doob's martingale inequality

$$I_{22}^n = P\left(\max_{0\le m\le \frac{1}{H(n)}} \left|\sum_{j=0}^{m}\left(\frac{1}{H(n)}\gamma_j^n - 1\right)\right| > \frac{\delta_n}{H(n)}\right)$$
$$\le \left(\frac{H(n)}{\delta_n}\right)^{2p} \mathbf{E}\left[\left(\sum_{m=1}^{[\frac{1}{H(n)}]}\left(\frac{1}{H(n)}\gamma_m^n - 1\right)\right)^{2p}\right].$$

Set $V_m \coloneqq \frac{1}{H(n)}\gamma_m^n - 1$. Note that V_m's are independent and centered random variables with

$$\mathbf{E}(V_m^{2p}) \le 2^{2p}((2n^k)^{2p}\mathbf{E}[(\gamma_m^n)^{2p}] + 1) \le 2\cdot 4^{2p}\,\frac{1}{(2p+1)}.$$

Then using an inequality of the type of (7) and following the same arguments that in the study of I_{12}^n, we obtain that $I_{22}^n = o(n^{-q})$.

4. Study of the term I_3^n. For $\delta_n > 0$ defined in (11) and a_n of the type $an^{-\frac{k}{4}}(\log n)^{\frac{3}{2}}$

$$I_3^n \le P\left(\max_{0\le m\le\frac{1}{H(n)}}\max_{|r|\le\delta_n}|x(\Gamma_m^n) - x(\Gamma_m^n + r)| > \frac{a_n}{4}\right) + P\left(\max_{1\le m\le\frac{1}{H(n)}+1}\gamma_m^n > \delta_n\right) \coloneqq I_{31}^n + I_{32}^n.$$

On one hand, $I_{31}^n = o(n^{-q})$ is proved in the same way as I_{11}^n. On the other hand, for n big enough, $I_{32}^n = 0$, similarly as we have proved for I_4^n.

We have checked now that all the terms in our decomposition are of order n^{-q}. The proof of Theorem 4.1 can be completed following the same computations that in [3] (see also Theorem 3.2 in [7]). □

Remmark 4.2. In the computation of the rate of convergence of Theorem 4.1, the role of the law of the uniform random variables is important in order to prove the result. Nevertheless, the proof could be adapted to obtain a similar result for other sequences of independent identically distributed positive interarrival times with moments of any order.

Funding information: X. Bardina and C. Rovira are supported by the grant PID2021-123733NB-I00 from SEIDI, Ministerio de Economia y Competividad.

Conflict of interest: The authors declare no conflicts of interest.

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