



# The complex Brownian motion as a strong limit of processes constructed from a Poisson process



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## ABSTRACT

We construct a family of processes, from a single Poisson process, that converges in law to a complex Brownian motion. Moreover, we find realizations of these processes that converge almost surely to the complex Brownian motion, uniformly on the unit time interval. Finally the rate of convergence is derived.

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## 1. Introduction

Kac [17] in 1956 to obtain a solution from a Poisson of the telegraph equation

$$\frac{1}{v} \frac{\partial^2 F}{\partial t^2} = v \frac{\partial^2 F}{\partial x^2} - \frac{2a}{v} \frac{\partial F}{\partial t}, \quad (1)$$

with  $a, v > 0$ , introduced the processes

$$x(t) = v \int_0^t (-1)^{N_a(r)} dr,$$

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where  $N_a = \{N_a(t), t \geq 0\}$  is a Poisson process of intensity  $a$ . He noticed that if in equation (1) the parameters  $a$  and  $v$  tend to infinity with  $\frac{2a}{v^2}$  constant and equal to  $\frac{1}{D}$ , then the equation converges to the heat equation:

$$\frac{1}{D} \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}. \quad (2)$$

Let  $x_\varepsilon(t)$  be the processes considered by Kac with  $a = \frac{1}{\varepsilon^2}$ ,  $v = \frac{1}{\varepsilon}$ . These values satisfy that  $\frac{2a}{v^2}$  is constant and  $D = \frac{1}{2}$  and we get in (2) an equation whose solution is a standard Brownian motion.

Stroock [20] proved in 1982 that the processes  $x_\varepsilon$  converge in law to a standard Brownian motion. That is, if we consider  $(P^\varepsilon)$  the image law of the process  $x_\varepsilon$  in the Banach space  $\mathcal{C}([0, T])$  of continuous functions on  $[0, T]$ , then  $(P^\varepsilon)$  converges weakly, when  $\varepsilon$  tends to zero, towards the Wiener measure. Doing a change of variables, these processes can be written as

$$x_\varepsilon = \left\{ x_\varepsilon(t) := \varepsilon \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(u)} du, t \in [0, T] \right\},$$

where  $\{N(t), t \geq 0\}$  is a standard Poisson process.

In the mathematical literature we find generalizations with regard to the Stroock result which can be channeled in three directions:

- (i) modifying the processes  $x_\varepsilon$  in order to obtain approximations of other Gaussian processes,
- (ii) proving convergence in a stronger sense than the convergence in law in the space of continuous functions,
- (iii) weakening the conditions of the approximating processes.

In direction (i), a first generalization is also made by Stroock [20] who modified the processes  $x_\varepsilon$  to obtain approximations of stochastic differential equations. There are also generalizations, among others, to the fractional Brownian motion (fBm) [18], to a general class of Gaussian processes (that includes fBm) [7], to a fractional stochastic differential equation [3], to the stochastic heat equation driven by Gaussian white noise [2] or to the Stratonovich heat equation [8].

On the other hand, there is some literature where the authors present realizations of the processes that converge almost surely, uniformly on the unit time interval. These processes are usually called uniform transport processes. Since the approximations always start increasing, a modification of the processes as

$$\tilde{x}_\varepsilon(t) = \varepsilon(-1)^A \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(u)} du,$$

has to be considered where  $A \sim \text{Bernoulli}(\frac{1}{2})$  independent of the Poisson process  $N$ .

Griego, Heath and Ruiz-Moncayo [16] showed that these processes converge strongly and uniformly on bounded time intervals to Brownian motion. In [14] Gorostiza and Griego extended the result to diffusions. Again Gorostiza and Griego [15] and Csörgő and Horváth [6] obtained a rate of convergence. More precisely, in [15] it is proved that there exist versions of the transport processes  $\tilde{x}_\varepsilon$  on the same probability space as a given Brownian motion  $(W(t))_{t \geq 0}$  such that, for each  $q > 0$ ,

$$P \left( \sup_{a \leq t \leq b} |W(t) - \tilde{x}_\varepsilon(t)| > C\varepsilon \left( \log \frac{1}{\varepsilon^2} \right)^{\frac{5}{2}} \right) = o(\varepsilon^{2q}),$$

as  $\varepsilon \rightarrow 0$  and where  $C$  is a positive constant depending on  $a$ ,  $b$  and  $q$ . Garzón, Gorostiza and León [10] defined a sequence of processes that converges strongly to fractional Brownian motion uniformly on bounded intervals, for any Hurst parameter  $H \in (0, 1)$  and computed the rate of convergence. In [12] and [11] the same authors deal with subfractional Brownian motion and fractional stochastic differential equations. Garzón, Torres and Tudor [13] also studied the Rosenblatt process.

Since

$$(-1)^{N(u)} = e^{i\pi N(u)} = \cos(\pi N(u)),$$

the question that if the convergence is also true with other angles appears. Bardina [1] showed that if we consider

$$\bar{x}_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} e^{i\theta N_s} ds \quad (3)$$

where  $\theta \neq 0, \pi$ , the laws of the processes converge weakly towards the law of a complex Brownian motion, i.e., the laws of the real and imaginary parts

$$\bar{z}_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \cos(\theta N_s) ds$$

and

$$\bar{y}_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \sin(\theta N_s) ds,$$

converge weakly towards the law of two independent Brownian motions. The approximating processes are functionally dependent because we use a single Poisson process but, in the limit, we obtain two independent processes. Later, in [4] it is shown that for different angles  $\theta_i$  the corresponding processes converge in law towards independent Brownian motions despite using only one Poisson process. Finally, in [5], we prove that we can use a Lévy process instead of a Poisson process in the definition of the sequence of approximations.

In this paper we present an extension of the Kac–Stroock result in the directions (ii) and (iii). Our aim is to define a modification of the processes  $\bar{x}_\varepsilon$  used by Stroock, similar to (3) proposed in [1]. These complex processes, that we will denote by  $x_\varepsilon^\theta = z_\varepsilon^\theta + iy_\varepsilon^\theta$ , will depend on a parameter  $\theta \in (0, \pi) \cup (\pi, 2\pi)$  and will be defined from an unique standard Poisson process and a sequence of independent random variables with common distribution  $\text{Bernoulli}(\frac{1}{2})$ . We will check that if we consider  $\theta_1, \theta_2, \dots, \theta_m$  such that for all  $i \neq j$ ,  $1 \leq i, j \leq m$ ,  $\theta_i, \theta_j \in (0, \pi) \cup (\pi, 2\pi)$ ,  $\theta_i + \theta_j \neq 2\pi$  and  $\theta_i \neq \theta_j$ , the law of the processes

$$(z_\varepsilon^{\theta_1}, \dots, z_\varepsilon^{\theta_m}, y_\varepsilon^{\theta_1}, \dots, y_\varepsilon^{\theta_m})$$

converges weakly in the space of the continuous functions towards the joint law of  $2m$  independent Brownian motions. Moreover, we also prove that there exist realizations of  $x_\varepsilon^\theta$  that converge almost surely to a complex Brownian motion and we are able to obtain the rate of convergence that does not depend on  $\theta$ . As a consequence, simulating a sequence of independent random variables with common distribution  $\text{exponential}(1)$  and a sequence of independent random variables with common distribution  $\text{Bernoulli}(\frac{1}{2})$ , we can get sequences of almost sure approximations of  $d$  independent Brownian motions for any  $d$ .

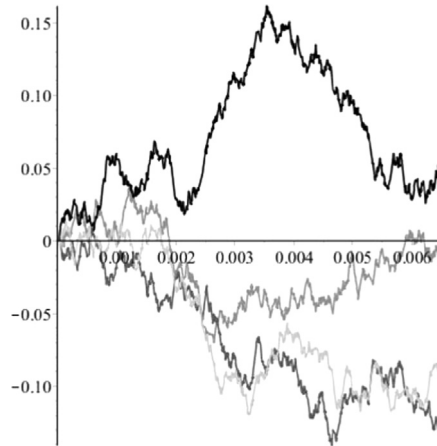


Fig. 1. Simulation of the trajectories of the processes  $x_\varepsilon^\theta$  and  $y_\varepsilon^\theta$  for the values of the parameters  $\varepsilon = \frac{1}{200}$ ,  $\theta = 2$  and  $\theta = 7$ .

For simplicity's sake, we only consider  $\theta \in (0, \pi) \cup (\pi, 2\pi)$  for which does not exist any  $m \in \mathbb{N}$  such that  $\cos(m\theta) = 0$  or  $\sin(m\theta) = 0$ .

As usual, the weak convergence is proved using tightness and the identification of the law of all possible weak limits (see, for instance [1,4]). The almost sure convergence is inspired by [16] while the computation of the rate of convergence follows the method given in [15] and [14].

The paper is organized in the following way. Section 2 is devoted to define the processes and to give the main results. In Section 3 we prove the weak convergence theorems. In Section 4 we prove the strong convergence theorem. The proof of the rate of convergence is given in Section 5.

Throughout the paper  $K$ ,  $C$  will denote any positive constant, not depending on  $\varepsilon$ , which may change from one expression to another.

## 2. Main results

Let  $\{M_t, t \geq 0\}$  be a Poisson process of parameter 2. We define  $\{N_t, t \geq 0\}$  and  $\{N'_t, t \geq 0\}$  two other counting processes that, at each jump of  $M$ , each of them jumps or does not jump with probability  $\frac{1}{2}$ , independently of the jumps of the other process and of its past. It is well-known that  $N$  and  $N'$  are Poisson processes of parameter 1 with independent increments on disjoint intervals.

Consider now, for  $\theta \in (0, \pi) \cap (\pi, 2\pi)$ , the following processes:

$$\left\{ z_\varepsilon^\theta(t) = (-1)^G \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_r} e^{i\theta N_r} dr, \quad t \in [0, T] \right\}, \quad (4)$$

where  $N$  and  $N'$  are the processes defined above and  $G$  is a random variable, independent of  $N$  and  $N'$ , with Bernoulli distribution of parameter  $\frac{1}{2}$ . We can write the process  $z_\varepsilon^\theta(t)$  as  $z_\varepsilon^\theta(t) = x_\varepsilon^\theta(t) + iy_\varepsilon^\theta(t)$ , where

$$x_\varepsilon^\theta(t) := \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_s+G} \cos(\theta N_s) ds, \quad y_\varepsilon^\theta(t) := \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_s+G} \sin(\theta N_s) ds$$

are the real part and the imaginary part, respectively. In Fig. 1 we can see a simulation of the trajectories of these processes for different values of  $\theta$ .

Our first result gives us the weak convergence to a complex Brownian motion.

**Theorem 2.1.** Let  $P_\varepsilon^\theta$  be the image law of  $z_\varepsilon^\theta$  in the Banach space  $\mathcal{C}([0, T], \mathbb{C})$  of continuous functions on  $[0, T]$ . Then  $P_\varepsilon^\theta$  converges weakly when  $\varepsilon$  tends to zero to the law  $P^\theta$  on  $\mathcal{C}([0, T], \mathbb{C})$  of a complex Brownian motion.

**Proof.** See Section 3.  $\square$

We can also get the following extension of Theorem 2.1, that is the equivalent of the result obtained in [4] for our processes.

**Theorem 2.2.** Consider  $\theta_1, \theta_2, \dots, \theta_m$  such that for all  $i \neq j$ ,  $1 \leq i, j \leq m$ ,  $\theta_i, \theta_j \in (0, \pi) \cup (\pi, 2\pi)$ ,  $\theta_i + \theta_j \neq 2\pi$  and  $\theta_i \neq \theta_j$ . Then the laws of the processes

$$(x_\varepsilon^{\theta_1}, \dots, x_\varepsilon^{\theta_m}, y_\varepsilon^{\theta_1}, \dots, y_\varepsilon^{\theta_m})$$

converge weakly, in the space of the continuous functions, towards the joint law of  $2m$  independent Brownian motions.

**Proof.** See the end of Section 3.  $\square$

Our next result gives the strong convergence of realizations of our processes  $\{z_\varepsilon^\theta(t); t \in [0, 1]\}$  and states as follows:

**Theorem 2.3.** There exist realizations of the process  $z_\varepsilon^\theta$  on the same probability space as a complex Brownian motion  $\{z(t), t \geq 0\}$  such that

$$\lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq 1} |z_\varepsilon^\theta(t) - z(t)| = 0 \quad a.s.$$

**Proof.** See Section 4.  $\square$

Notice that combining the results of Theorem 2.2 and Theorem 2.3 we get that from our two Poisson processes  $N$  and  $N'$  and a random variable  $G$  with Bernoulli law, we are able to construct approximations to  $d$  standard independent Brownian motions for  $d$  as large as we want.

In our last result we give the rate of convergence of these processes.

**Theorem 2.4.** For all  $q > 0$ ,

$$P \left( \max_{0 \leq t \leq 1} |z_\varepsilon^\theta(t) - z(t)| > \alpha^* \varepsilon^{\frac{1}{2}} \left( \log \frac{1}{\varepsilon} \right)^{\frac{5}{2}} \right) = o(\varepsilon^q), \quad \text{as } \varepsilon \rightarrow 0$$

where  $\alpha^*$  is a positive constant depending on  $q$ .

**Proof.** See Section 5.  $\square$

### 3. Proof of weak convergence

In order to prove Theorem 2.1 we have to check that the family  $P_\varepsilon^\theta$  is tight and that the law of all possible limits of  $P_\varepsilon^\theta$  is the law of a complex Brownian motion. Following the same method as in [1], the proof is based on the following lemma:

**Lemma 3.1.** For any  $0 \leq x_1 \leq x_2$

$$\mathbf{E}[(-1)^{N'_{x_2}-N'_{x_1}} e^{i\theta(N_{x_2}-N_{x_1})}] = e^{-2(x_2-x_1)}.$$

**Proof.** From the definition of  $N$  and  $N'$  it follows that

$$\begin{aligned} & \mathbf{E}[(-1)^{N'_{x_2}-N'_{x_1}} e^{i\theta(N_{x_2}-N_{x_1})}] \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n e^{i\theta m} P(N'_{x_2} - N'_{x_1} = n, N_{x_2} - N_{x_1} = m) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n e^{i\theta m} \sum_{k=n \vee m}^{\infty} P(N'_{x_2} - N'_{x_1} = n, N_{x_2} - N_{x_1} = m | M_{x_2} - M_{x_1} = k) \\ & \quad \times P(M_{x_2} - M_{x_1} = k) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n e^{i\theta m} \sum_{k=n \vee m}^{\infty} \binom{k}{n} \binom{k}{m} \frac{1}{2^k} \frac{1}{2^k} \frac{[2(x_2 - x_1)]^k e^{-2(x_2-x_1)}}{k!} \\ &= e^{-2(x_2-x_1)} \sum_{k=0}^{\infty} \left(\frac{x_2-x_1}{2}\right)^k \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} (-1)^n \sum_{m=0}^k \binom{k}{m} e^{i\theta m}. \end{aligned}$$

Notice that  $\sum_{n=0}^k \binom{k}{n} (-1)^n = 0$  when  $k \neq 0$ , therefore the above expression is different from zero only when  $k = 0$  and, as a consequence, when  $n = 0$  and  $m = 0$ . Hence, as the series is absolutely convergent,

$$\mathbf{E}[(-1)^{N'_{x_2}-N'_{x_1}} e^{i\theta(N_{x_2}-N_{x_1})}] = e^{-2(x_2-x_1)},$$

as we wanted to prove.  $\square$

Using Lemma 3.1, we can also get a version of Lemma 3.2 in [1] well adapted to our processes.

**Lemma 3.2.** Consider  $\{\mathcal{F}_t^{\varepsilon, \theta}\}$  the natural filtration of the processes  $z_\varepsilon^\theta$ . Then, for any  $s < t$  and for any real  $\{\mathcal{F}_s^{\varepsilon, \theta}\}$ -measurable and bounded random variable  $Y$ , we have that, for any  $\theta \in (0, \pi) \cup (\pi, 2\pi)$ ,

$$\begin{aligned} \text{a) } & \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbf{E} \left[ (-1)^{N'_{x_2}-N'_{x_1}} e^{i\theta(N_{x_2}-N_{x_1})} \right] dx_1 dx_2 = (t-s) + \frac{\varepsilon^2}{4} (e^{-\frac{4}{\varepsilon^2}(t-s)} - 1) \\ \text{b) } & \lim_{\varepsilon \rightarrow 0} \left| \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbf{E} \left[ (-1)^{N'_{x_2}+N'_{x_1}} e^{i\theta(N_{x_2}+N_{x_1})} Y \right] dx_1 dx_2 \right| = 0. \end{aligned}$$

**Proof.** Follow the same ideas as in [1] using Lemma 3.1.  $\square$

**Proof of Theorem 2.1.** We will give only the skeleton of the proof.

We need to prove that the laws corresponding to  $\{x_\varepsilon^\theta(t), t \geq 0\}$  and  $\{y_\varepsilon^\theta(t), t \geq 0\}$  are tight, using Billingsley criterion and that our processes are null in the origin, it is sufficient to check that there exists a constant  $K$  such that for any  $s < t$

$$\sup_{\varepsilon} \left[ \mathbf{E} \left( \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x+G} \cos(\theta N_x) dx \right)^4 + \mathbf{E} \left( \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x+G} \sin(\theta N_x) dx \right)^4 \right] \leq K(t-s)^2.$$

Following the proof of lemma 2.1 in [1] and applying Lemma 3.1 we obtain that

$$\begin{aligned} & \mathbf{E} \left( \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x+G} \cos(\theta N_x) dx \right)^4 + \mathbf{E} \left( \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x+G} \sin(\theta N_x) dx \right)^4 \\ & \leq 12 \left( \varepsilon^2 \int_{\left[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}\right]^2} \mathbb{1}_{\{x_1 \leq x_2\}} \left| \mathbf{E} \left[ (-1)^{N'_{x_2}-N'_{x_1}} e^{i\theta(N_{x_2}-N_{x_1})} \right] \right| dx_1 dx_2 \right)^2 \\ & \quad + 48 \varepsilon^4 \int_{\mathcal{I}} \left| \mathbf{E} \left[ (-1)^{N'_{x_4}-N'_{x_3}} e^{i\theta(N_{x_4}-N_{x_3})} \right] \right| \left| \mathbf{E} \left[ e^{2i\theta(N_{x_3}-N_{x_2})} \right] \right| dx_1 \dots dx_4 \\ & \leq 12 \left( \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-2(x_2-x_1)} dx_1 dx_2 \right)^2 \\ & \quad + 48 \varepsilon^4 \int_{\mathcal{I}} e^{-2(x_4-x_3)} e^{-(x_3-x_2)(1-\cos(2\theta))} dx_1 \dots dx_4 \\ & \leq 12(t-s)^2 + \frac{48(t-s)^2}{1-\cos(2\theta)} \end{aligned}$$

where  $\mathcal{I} := \{(x_1, x_2, x_3, x_4) \in [\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}]^4 : x_1 \leq x_2 \leq x_3 \leq x_4\}$ . So the family of laws is tight.

Now we need to identify all the possible limit laws. Consider a subsequence, which we will also denote by  $\{P_\varepsilon^\theta\}$ , weakly convergent to some probability  $P^\theta$ . We want to prove that the canonical process  $Z^\theta = \{z^\theta(t), t \geq 0\}$  is a complex Brownian motion under  $P^\theta$ , that is the real part  $X$  and the imaginary part  $Y$  of this process are two independent Brownian motions. Using Paul Lévy's theorem it is sufficient to prove that, under  $P^\theta$ ,  $X$  and  $Y$  are both martingale with respect to the natural filtration  $\{\mathcal{F}_t\}$  with quadratic variation  $\langle \operatorname{Re}[Z^\theta], \operatorname{Re}[Z^\theta] \rangle_t = t$ ,  $\langle \operatorname{Im}[Z^\theta], \operatorname{Im}[Z^\theta] \rangle_t = t$  and null covariation.

To prove the martingale property with respect to the natural filtration  $\{\mathcal{F}_t\}$ , following the Section 3.1 in [1], it is enough to see that for any  $s_1 \leq s_2 \leq \dots \leq s_n \leq s < t$  and for any bounded continuous function  $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$

$$\left| \mathbf{E} \left( \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{G+N'_x} e^{i\theta N_x} dx \right) \right|$$

converges to zero as  $\varepsilon$  tends to zero. But,

$$\begin{aligned}
& \left| \mathbf{E} \left( \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{G+N'_x} e^{i\theta N_x} dx \right) \right| \\
&= \left| \mathbf{E} \left( \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) (-1)^{G+N'_{2s/\varepsilon^2}} e^{i\theta N_{2s/\varepsilon^2}} \right) \right. \\
&\quad \left. \times \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \mathbf{E} \left( (-1)^{N'_x - N'_{2s/\varepsilon^2}} e^{i\theta(N_x - N_{2s/\varepsilon^2})} \right) dx \right| \\
&\leq K\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} e^{-2(x - \frac{2s}{\varepsilon^2})} dx = \frac{K\varepsilon}{2} (1 - e^{-\frac{4}{\varepsilon^2}(t-s)}),
\end{aligned}$$

that converges to zero as  $\varepsilon$  tends to zero. Therefore, martingale property is proved.

To prove that  $\langle \operatorname{Re}[Z^\theta], \operatorname{Re}[Z^\theta] \rangle_t = t$  and  $\langle \operatorname{Im}[Z^\theta], \operatorname{Im}[Z^\theta] \rangle_t = t$  we will check that for any  $s_1 \leq \dots \leq s_n \leq s < t$  and for any bounded continuous function  $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$ ,

$$\mathbf{E} \left[ \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) ((x_\varepsilon^\theta(t) - x_\varepsilon^\theta(s))^2 - (t-s)) \right]$$

and

$$\mathbf{E} \left[ \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) ((y_\varepsilon^\theta(t) - y_\varepsilon^\theta(s))^2 - (t-s)) \right]$$

converge to zero as  $\varepsilon$  tends to zero. Notice that, in our case,

$$\begin{aligned}
& \mathbf{E} \left[ \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) (x_\varepsilon^\theta(t) - x_\varepsilon^\theta(s))^2 \right] \\
&= \mathbf{E} \left[ \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) \left( \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{G+N'_x} \cos(\theta N_x) dx \right)^2 \right] \\
&= 2\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbf{E} \left[ \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) (-1)^{N'_{x_2} + N'_{x_1}} \right. \\
&\quad \left. \times \cos(\theta N_{x_1}) \cos(\theta N_{x_2}) \right] dx_1 dx_2.
\end{aligned}$$

The proof of the real part follows again the structure of Section 3.2 in [1] using Lemma 3.2. The imaginary part can be done similarly.

Finally we have to prove that  $\langle \operatorname{Re}[Z^\theta], \operatorname{Im}[Z^\theta] \rangle_t = 0$ . It is sufficient to show that for any  $s_1 \leq \dots \leq s_n \leq s < t$  and for any bounded continuous function  $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$ ,

$$\mathbf{E} \left[ \varphi(z_\varepsilon^\theta(s_1), \dots, z_\varepsilon^\theta(s_n)) (x_\varepsilon^\theta(t) - x_\varepsilon^\theta(s)) (y_\varepsilon^\theta(t) - y_\varepsilon^\theta(s)) \right]$$

converges to zero as  $\varepsilon$  tends to zero. But we obtain this convergence using similar calculations and statement b) of Lemma 3.2.  $\square$



**Proof of Theorem 2.2.** Taking into account the proof of Theorem 2.1 it remains only to check that for  $i \neq j$ , and  $\theta_i, \theta_j$  in the conditions of Theorem 2.2,  $\langle \operatorname{Re}[Z^{\theta_i}], \operatorname{Re}[Z^{\theta_j}] \rangle_t = 0$ ,  $\langle \operatorname{Im}[Z^{\theta_i}], \operatorname{Im}[Z^{\theta_j}] \rangle_t = 0$  and  $\langle \operatorname{Im}[Z^{\theta_i}], \operatorname{Re}[Z^{\theta_j}] \rangle_t = 0$ . But it can be proved following the proof of Theorem 2 in [4] and taking into account our Lemma 3.1.  $\square$

#### 4. Proof of strong convergence

In this section, we will prove the strong convergence when  $\varepsilon$  tends to zero of the processes  $\{z_\varepsilon^\theta(t); t \in [0, 1]\}$  defined in Section 2.

**Proof of Theorem 2.3.** We will study the strong convergence of  $x_\varepsilon^\theta$ ,

$$x_\varepsilon^\theta(t) = \frac{2}{\varepsilon}(-1)^G \int_0^t (-1)^{\frac{N'_{2r}}{\varepsilon^2}} \cos(\theta N_{\frac{2r}{\varepsilon^2}}) dr, \quad (5)$$

the real part of the processes  $z_\varepsilon^\theta$ , to a standard Brownian motion  $\{x_t; t \in [0, 1]\}$  when  $\varepsilon$  tends to 0. More precisely, we will prove that there exist realizations  $\{x_\varepsilon^\theta(t), t \geq 0\}$  of the above process on the same probability space of a Brownian motion process  $\{x(t), t \geq 0\}$  such that

$$\lim_{\varepsilon \rightarrow \infty} \max_{0 \leq t \leq 1} |x_\varepsilon^\theta(t) - x(t)| = 0 \quad \text{a.s.} \quad (6)$$

The convergence of the imaginary part of  $z_\varepsilon$  to another standard Brownian motion  $\{y_t; t \in [0, 1]\}$ , independent of  $\{x_t, t \geq 0\}$ , follows the same proof. We will follow the method used in [16] to prove the strong convergence to a standard Brownian motion. We will divide the proof in five steps.

*Step 1: Definitions of the processes.* Let  $(\Omega, \mathcal{F}, \mathcal{P})$  be the probability space for a standard Brownian motion  $\{x_t, t \geq 0\}$  with  $x(0) = 0$  and let us define:

1. for each  $\varepsilon > 0$ ,  $\{\epsilon_m^\varepsilon\}_{m \geq 1}$  a sequence of independent identically distributed random variables with law exponential of parameter  $\frac{2}{\varepsilon}$ , independent of the Brownian motion  $x$ ,
2.  $\{\eta_m\}_{m \geq 1}$  a sequence of independent identically distributed random variables with law Bernoulli( $\frac{1}{2}$ ), independent of  $x$  and  $\{\epsilon_m^\varepsilon\}_{m \geq 1}$  for all  $\varepsilon$ ,
3.  $\{k_m\}_{m \geq 1}$  a sequence of independent identically distributed random variables such that  $P(k_1 = 1) = P(k_1 = -1) = \frac{1}{2}$ , independent of  $x$ ,  $\{\eta_m\}_{m \geq 1}$  and  $\{\epsilon_m^\varepsilon\}_{m \geq 1}$  for all  $\varepsilon$ .

Using these random variables we are able to introduce the following ones:

1.  $\{b_m\}_{m \geq 0}$  such that  $b_0 = 0$  and  $b_m = \sum_{j=1}^m \eta_j$  for  $m \geq 1$ . Clearly  $b_m$  has a Binomial distribution of parameters  $(m, \frac{1}{2})$  and, for all  $n \in \{0, 1, \dots, m\}$ ,  $P(b_{m+1} = n | b_m = n) = P(b_{m+1} = n + 1 | b_m = n) = \frac{1}{2}$ .
2.  $\{\xi_m^{\varepsilon, \theta}\}_{m \geq 1} = \{|\cos(b_{m-1}\theta)|\epsilon_m^\varepsilon\}_{m \geq 1}$ . This family of random variables is clearly independent of  $x$ .

Let  $\mathcal{B}$  be the  $\sigma$ -algebra generated by  $\{b_m\}_{m \geq 1}$ . The sequence of random variables  $\{k_m \xi_m^{\varepsilon, \theta}\}_{m \geq 0}$  satisfies

$$\mathbf{E}(k_m \xi_m^{\varepsilon, \theta} | \mathcal{B}) = 0,$$

$$\operatorname{Var}(k_m \xi_m^{\varepsilon, \theta} | \mathcal{B}) = \mathbf{E}[(\xi_m^{\varepsilon, \theta})^2 | \mathcal{B}] = \frac{\varepsilon^2}{2} [\cos(b_{m-1}\theta)]^2.$$

By Skorokhod's theorem ([19] p. 163 or Lemma 2 in [14]) for each  $\varepsilon > 0$  there exists a sequence  $\sigma_1^{\varepsilon, \theta}, \sigma_2^{\varepsilon, \theta}, \dots$  of nonnegative random variables on  $(\Omega, \mathcal{F}, \mathcal{P})$  so that the sequence  $x(\sigma_1^{\varepsilon, \theta}), x(\sigma_1^{\varepsilon, \theta} + \sigma_2^{\varepsilon, \theta}), \dots$ , has the same distribution as  $k_1 \xi_1^{\varepsilon, \theta}, k_1 \xi_1^{\varepsilon, \theta} + k_2 \xi_2^{\varepsilon, \theta}, \dots$ , and, for each  $m$ ,

$$\mathbf{E}(\sigma_m^{\varepsilon, \theta} | \mathcal{B}) = \text{Var}(k_m \xi_m^{\varepsilon, \theta} | \mathcal{B}) = \frac{\varepsilon^2}{2} [\cos(b_{m-1} \theta)]^2.$$

For each  $\varepsilon$  we define  $\gamma_0^{\varepsilon, \theta} \equiv 0$  and for each  $m$

$$\gamma_m^{\varepsilon, \theta} = |\beta_m^{\varepsilon, \theta}|^{-1} \left| x \left( \sum_{j=0}^m \sigma_j^{\varepsilon, \theta} \right) - x \left( \sum_{j=0}^{m-1} \sigma_j^{\varepsilon, \theta} \right) \right|,$$

where  $\sigma_0^{\varepsilon, \theta} \equiv 0$  and

$$\beta_m^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos(b_{m-1} \theta).$$

Then, the random variables  $\gamma_1^{\varepsilon, \theta}, \gamma_2^{\varepsilon, \theta}, \dots$ , are independent with common exponential distribution with parameter  $\frac{4}{\varepsilon^2}$ . Indeed for any  $0 \leq n < m$ ,  $x, y \in \mathbb{R}$

$$\begin{aligned} & P(\gamma_{n+1}^{\varepsilon, \theta} \leq x, \gamma_{m+1}^{\varepsilon, \theta} \leq y) \\ &= \sum_{j=0}^n \sum_{k=j}^{j+(m-n)} P(\gamma_{n+1}^{\varepsilon, \theta} \leq x, \gamma_{m+1}^{\varepsilon, \theta} \leq y \mid b_n = j, b_m = k) P(b_n = j, b_m = k) \\ &= (1 - e^{-\frac{4x}{\varepsilon^2}}) (1 - e^{-\frac{4y}{\varepsilon^2}}) \sum_{j=0}^n \sum_{k=j}^{j+(m-n)} P(b_n = j, b_m = k) \\ &= P(\gamma_{n+1}^{\varepsilon, \theta} \leq x) P(\gamma_{m+1}^{\varepsilon, \theta} \leq y), \end{aligned}$$

using that when  $b_n$  and  $b_m$  are known,  $\gamma_{n+1}^{\varepsilon, \theta}$  and  $\gamma_{m+1}^{\varepsilon, \theta}$  are independent.

Now, we define  $x_\varepsilon^\theta(t)$ ,  $t \geq 0$  to be piecewise linear satisfying

$$x_\varepsilon^\theta \left( \sum_{j=1}^m \gamma_j^{\varepsilon, \theta} \right) = x \left( \sum_{j=1}^m \sigma_j^{\varepsilon, \theta} \right), \quad m \geq 1 \quad (7)$$

and  $x_\varepsilon^\theta(0) \equiv 0$ . Observe that the process  $x_\varepsilon^\theta$  has slope  $\pm |\beta_m^{\varepsilon, \theta}|$  in the interval  $[\sum_{j=1}^{m-1} \gamma_j^{\varepsilon, \theta}, \sum_{j=1}^m \gamma_j^{\varepsilon, \theta}]$ .

Let  $\rho_m^{\varepsilon, \theta}$  be the time of the  $m$ th change of the absolute values of  $\beta_j^{\varepsilon, \theta}$ 's, i.e. the time when  $\beta_j^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos[(m-1)\theta]$  and  $\beta_{j+1}^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos(m\theta)$ , that is when the slope of  $x_\varepsilon^\theta(\cdot)$  changes from  $\pm \frac{2}{\varepsilon} |\cos[(m-1)\theta]|$  to  $\pm \frac{2}{\varepsilon} |\cos(m\theta)|$ . Then the increments  $\rho_m^{\varepsilon, \theta} - \rho_{m-1}^{\varepsilon, \theta}$ ,  $m \geq 1$ , with  $\rho_0^{\varepsilon, \theta} \equiv 0$  are independent and exponentially distributed with a parameter  $\frac{2}{\varepsilon^2}$ . Indeed, since

$$P \left( \beta_m^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos(n\theta) \mid \beta_{m-1}^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos[(n-1)\theta] \right) = \frac{1}{2}$$

for every  $n = 0, 1, \dots, m$ , we can write  $\rho_1^{\varepsilon, \theta} = \gamma_1^{\varepsilon, \theta} + \dots + \gamma_{\widehat{N}}^{\varepsilon, \theta}$ , where  $P(\widehat{N} = n) = 2^{-n}$  for  $n \in \mathbb{N}$  and  $\widehat{N}$  is independent of  $\{\gamma_j^{\varepsilon, \theta}\}_{j \geq 1}$ . Therefore  $\rho_1^{\varepsilon, \theta}$  has an exponential distribution with parameter  $\frac{2}{\varepsilon^2}$  (see (5.6) in Chap. 2 of [9]). Likewise, each increment  $\rho_m^{\varepsilon, \theta} - \rho_{m-1}^{\varepsilon, \theta}$  has an exponential distribution with parameter  $\frac{2}{\varepsilon^2}$  and the increments are independent since they are sum of disjoint blocks of the  $\gamma_m^{\varepsilon, \theta}$ 's.

On the other hand, let  $\tau_m^\varepsilon$  be the time of the  $m$ th change of the sign of the slopes. Following the same arguments for the times  $\rho_m^{\varepsilon,\theta}$  we get that the increments  $\tau_m^\varepsilon - \tau_{m-1}^\varepsilon$ , for each  $m$ , with  $\tau_0^\varepsilon \equiv 0$ , are independent and exponentially distributed with a parameter  $\frac{2}{\varepsilon^2}$ . Moreover the increments are sum of disjoint blocks of the  $\gamma_m^{\varepsilon,\theta}$ 's.

Thus  $x_\varepsilon^\theta$  is a realization of the process (5).

*Step 2: Decomposition of the convergence.* Let us come back to the proof of (6). Recalling that  $\gamma_0^{\varepsilon,\theta} \equiv \sigma_0^{\varepsilon,\theta} \equiv 0$ , by (7) and the uniform continuity of Brownian motion on  $[0, 1]$ , we have almost surely

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq 1} |x_\varepsilon^\theta(t) - x(t)| &= \lim_{\varepsilon \rightarrow 0} \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| x_\varepsilon^\theta \left( \sum_{j=1}^m \gamma_j^{\varepsilon,\theta} \right) - x \left( \sum_{j=1}^m \gamma_j^{\varepsilon,\theta} \right) \right| \\ &= \lim_{\varepsilon \rightarrow 0} \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| x \left( \sum_{j=1}^m \sigma_j^{\varepsilon,\theta} \right) - x \left( \sum_{j=1}^m \gamma_j^{\varepsilon,\theta} \right) \right|, \end{aligned}$$

and it reduces the proof to check that,

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \gamma_1^{\varepsilon,\theta} + \cdots + \gamma_m^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| = 0 \quad \text{a.s.},$$

and that

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sigma_1^{\varepsilon,\theta} + \cdots + \sigma_m^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| = 0 \quad \text{a.s.} \quad (8)$$

The first limit can be obtained easily by Borel–Cantelli lemma since by Kolmogorov's inequality, for each  $\alpha > 0$ , we have

$$\begin{aligned} P \left( \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \gamma_1^{\varepsilon,\theta} + \cdots + \gamma_m^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| \geq \alpha \right) &\leq \frac{1}{\alpha^2} \sum_{m=1}^{\frac{4}{\varepsilon^2}} \text{Var}(\gamma_m^{\varepsilon,\theta}) \\ &= \frac{\varepsilon^2}{4\alpha^2}. \end{aligned}$$

In order to deal with (8) we can use the decomposition

$$\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \sigma_j^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| \leq \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m (\sigma_j^{\varepsilon,\theta} - \alpha_j^{\varepsilon,\theta}) \right| + \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \alpha_j^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right|, \quad (9)$$

where  $\alpha_j^{\varepsilon,\theta} := \mathbf{E}(\sigma_j^{\varepsilon,\theta} | \mathcal{B}) = \frac{\varepsilon^2}{2} [\cos(b_{j-1}\theta)]^2$  and

$$\begin{aligned} \left| \sum_{j=1}^m \alpha_j^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| &= \left| \sum_{j=1}^m \frac{\varepsilon^2}{2} [\cos(b_{j-1}\theta)]^2 - m \frac{\varepsilon^2}{4} \right| \\ &= \frac{\varepsilon^2}{4} \left| \sum_{j=1}^m [1 + \cos(2b_{j-1}\theta)] - m \right| = \frac{\varepsilon^2}{4} \left| \sum_{j=1}^m \cos(2b_{j-1}\theta) \right|. \end{aligned} \quad (10)$$

For a fixed  $m$ , consider the random variable  $\hat{n} := b_{m-1} - 1$ . Then, fixed  $m$ , notice that

$$\sum_{j=1}^m \cos(2b_{j-1}\theta) = \sum_{\substack{0 \leq k \leq \hat{n} \\ B_m}} T_k \cos(2k\theta) + Z_{\hat{n}+1} \cos[2(\hat{n}+1)\theta], \quad (11)$$

where  $B_m := \{k \in \{0, \dots, \hat{n}\} \text{ s.t. } T_0 + \dots + T_{\hat{n}} < m, T_0 + \dots + T_{\hat{n}+1} \geq m\}$ ,  $T_k$  are independent identically distributed random variables with  $T_k \sim \text{Geom}(\frac{1}{2})$  for each  $k = 0, 1, 2, \dots$ , that is  $P(T_k = j) = 2^{-j}$  for  $j \geq 1$  and  $\mathbf{E}(T_k) = \text{Var}(T_k) = 2$ , and  $0 \leq Z_{\hat{n}+1} \leq T_{\hat{n}+1}$ . Hence, putting together (9), (10) and (11), it follows that

$$\begin{aligned} & \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \sigma_j^{\varepsilon, \theta} - m \frac{\varepsilon^2}{4} \right| \leq \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta}) \right| \\ & + \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \frac{\varepsilon^2}{4} \left| \sum_{\substack{0 \leq k \leq \hat{n} \\ B_m}} T_k \cos(2k\theta) \right| + \max_{\substack{1 \leq m \leq \frac{4}{\varepsilon^2} \\ B_m}} \frac{\varepsilon^2}{4} |Z_{\hat{n}+1} \cos[2(\hat{n}+1)\theta]| \\ & := L_1^\varepsilon + L_2^\varepsilon + L_3^\varepsilon, \end{aligned}$$

and reduces the proof of (8) to check that  $\lim_{\varepsilon \rightarrow 0} (L_1^\varepsilon + L_2^\varepsilon + L_3^\varepsilon) = 0$  a.s.

*Step 3: Study of  $L_1^\varepsilon$ .* Let  $M_n := \sum_{j=1}^n (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})$ . Let  $\mathcal{B}_n$  denote the  $\sigma$ -algebra generated by  $\{\mathcal{B}, \sigma_k^{\varepsilon, \theta}; k \leq n\}$ . Clearly,  $M_n$  is  $\mathcal{B}_n$ -measurable and, since  $\mathbf{E}(\sigma_n^{\varepsilon, \theta} - \alpha_n^{\varepsilon, \theta} | \mathcal{B}) = 0$ ,  $\mathbf{E}(M_n | \mathcal{B}_{n-1}) = M_{n-1}$ . So  $|M_n|$  is a submartingale. By Doob's martingale inequality, for each  $\alpha > 0$

$$P\left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} |M_m| \geq \alpha\right) \leq \frac{1}{\alpha^2} \mathbf{E}\left(|M_{[\frac{4}{\varepsilon^2}]}|^2\right), \quad (12)$$

where  $[v]$  is the integer part of  $v$ . Fixed  $b_j$ 's,  $\{\sigma_j^{\varepsilon, \theta}\}$ 's are independent, and so for  $j \neq k$

$$\mathbf{E}[(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})(\sigma_k^{\varepsilon, \theta} - \alpha_k^{\varepsilon, \theta})] = \mathbf{E}[\mathbf{E}(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta} | \mathcal{B}) \mathbf{E}(\sigma_k^{\varepsilon, \theta} - \alpha_k^{\varepsilon, \theta} | \mathcal{B})] = 0. \quad (13)$$

On the other hand

$$\mathbf{E}[\sigma_j^{\varepsilon, \theta} \alpha_j^{\varepsilon, \theta}] = \mathbf{E}[\mathbf{E}(\sigma_j^{\varepsilon, \theta} \alpha_j^{\varepsilon, \theta} | \mathcal{B})] = \mathbf{E}[\alpha_j^{\varepsilon, \theta} \mathbf{E}(\sigma_j^{\varepsilon, \theta} | \mathcal{B})] = \mathbf{E}[(\alpha_j^{\varepsilon, \theta})^2]. \quad (14)$$

Using (13) and (14), we get that

$$\begin{aligned} & \mathbf{E}\left(|M_{[\frac{4}{\varepsilon^2}]}|^2\right) \\ &= \sum_{j=1}^{[\frac{4}{\varepsilon^2}]} \mathbf{E}[(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})^2] + 2 \sum_{j=1}^{[\frac{4}{\varepsilon^2}]-1} \sum_{k=j+1}^{[\frac{4}{\varepsilon^2}]} \mathbf{E}[(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})(\sigma_k^{\varepsilon, \theta} - \alpha_k^{\varepsilon, \theta})] \\ &= \sum_{j=1}^{[\frac{4}{\varepsilon^2}]} \mathbf{E}[(\sigma_j^{\varepsilon, \theta})^2] - \sum_{j=1}^{[\frac{4}{\varepsilon^2}]} \mathbf{E}[(\alpha_j^{\varepsilon, \theta})^2]. \end{aligned} \quad (15)$$

Recalling that by Skorokhod's theorem ([19], p. 163) there exists a positive constant  $C_1$  such that

$$\mathbf{E}[(\sigma_j^{\varepsilon, \theta})^2 | \mathcal{B}] \leq C_1 \mathbf{E}[(\xi_j^{\varepsilon, \theta})^4 | \mathcal{B}] = C_1 4! \left(\frac{\varepsilon}{2}\right)^4 [\cos(b_{j-1}\theta)]^4,$$

(15) can be bounded by

$$\begin{aligned} \mathbf{E}\left(|M_{[\frac{4}{\varepsilon^2}]}|^2\right) &\leq C_1 4! \left(\frac{\varepsilon}{2}\right)^4 \sum_{j=1}^{[\frac{4}{\varepsilon^2}]} \mathbf{E}[\cos(b_{j-1}\theta)^4] - \left(\frac{\varepsilon^2}{2}\right)^2 \sum_{j=1}^{[\frac{4}{\varepsilon^2}]} \mathbf{E}[\cos(b_{j-1}\theta)^4] \\ &= C\varepsilon^4 \sum_{j=1}^{[\frac{4}{\varepsilon^2}]} \mathbf{E}[\cos(b_{j-1}\theta)^4] \leq 4C\varepsilon^2, \end{aligned}$$

where  $C$  is a positive constant. So, from (12) we obtain that for any  $\alpha$

$$P\left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} |M_m| \geq \alpha\right) \leq \frac{4C\varepsilon^2}{\alpha^2},$$

and by Borel–Cantelli lemma it follows that  $\lim_{\varepsilon \rightarrow 0} L_1^\varepsilon = 0$  a.s.

*Step 4: Study of  $L_2^\varepsilon$ .* Since  $\hat{n} \leq m - 1$ , we have

$$\begin{aligned} L_2^\varepsilon &\leq \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} \left| \sum_{k=0}^n T_k \cos(2k\theta) \right| \\ &\leq \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} \left| \sum_{k=0}^n (T_k - 2) \cos(2k\theta) \right| + \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{2} \left| \sum_{k=0}^n \cos(2k\theta) \right| \\ &:= L_{21}^\varepsilon + L_{22}^\varepsilon. \end{aligned}$$

Let us prove that  $L_{21}^\varepsilon$  vanishes when  $\varepsilon$  goes to 0. Let  $\mathcal{F}^n$  denote the  $\sigma$ -algebra generated by  $T_k$  for  $k \leq n$ . Define  $M'_n := \frac{\varepsilon^2}{4} \sum_{k=0}^n (T_k - 2) \cos(2k\theta)$ . It is easy to see that  $M'_n$  is a martingale. By Doob's martingale inequality, for each  $\alpha > 0$

$$\begin{aligned} &P\left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} |M'_n| > \alpha\right) \\ &\leq \frac{1}{\alpha^2} \mathbf{E}\left(|M'_{[\frac{4}{\varepsilon^2}] - 1}|^2\right) = \frac{1}{\alpha^2} \mathbf{E}\left(\frac{\varepsilon^4}{16} \left| \sum_{k=0}^{[\frac{4}{\varepsilon^2}] - 1} (T_k - 2) \cos(2k\theta) \right|^2\right) \\ &= \frac{1}{\alpha^2} \frac{\varepsilon^4}{16} \sum_{k=0}^{[\frac{4}{\varepsilon^2}] - 1} \mathbf{E}[(T_k - 2)^2] \cos^2(2k\theta) \leq \frac{1}{\alpha^2} \frac{\varepsilon^4}{16} \frac{4}{\varepsilon^2} = \frac{\varepsilon^2}{2\alpha^2}, \end{aligned}$$

where we have used that  $(T_k - 2)$ 's are independent and centered. Therefore, by Borel–Cantelli lemma  $\lim_{\varepsilon \rightarrow 0} L_{21}^\varepsilon = 0$  a.s.

On the other hand, since for any  $n$ , we get that

$$\begin{aligned} &\sum_{k=0}^n \cos(2k\theta) \\ &= \frac{1}{2} \left( \sum_{k=0}^n e^{i2k\theta} + \sum_{k=0}^n e^{-i2k\theta} \right) = \frac{1}{2} \left( \frac{1 - e^{i2(n+1)\theta}}{1 - e^{i2\theta}} + \frac{1 - e^{-i2(n+1)\theta}}{1 - e^{-i2\theta}} \right) \\ &= \frac{1}{2} \left( 1 + \frac{-\cos(2(n+1)\theta) + \cos(2n\theta)}{1 - \cos(2\theta)} \right). \end{aligned}$$

Thus

$$\left| \sum_{k=0}^n \cos(2k\theta) \right| \leq \frac{1}{2} \left( 1 + \frac{2}{1 - \cos(2\theta)} \right), \quad (16)$$

and it yields that  $\lim_{\varepsilon \rightarrow 0} L_{22}^\varepsilon = 0$ , a.s.

*Step 5: Study of  $L_3^\varepsilon$ .* Since  $\hat{n} \leq m - 1$  and  $Z_{\hat{n}+1} \leq T_{\hat{n}+1}$ ,

$$L_3^\varepsilon \leq \max_{\substack{1 \leq m \leq \frac{4}{\varepsilon^2} \\ B_m}} \frac{\varepsilon^2}{4} T_{\hat{n}+1} \leq \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} T_{n+1}. \quad (17)$$

Thus, it is sufficient to see that

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq n} \frac{T_k}{n} = 0 \quad \text{a.s.}$$

It is enough to observe that

$$\left( \max_{1 \leq k \leq n} \frac{T_k}{n} \right)^2 \leq \frac{1}{n} \frac{T_1^2 + \dots + T_n^2}{n}$$

and apply the strong law of large numbers.  $\square$

## 5. Proof of rate of convergence

In this section we will prove the rate of convergence of the processes  $z_\varepsilon^\theta(t)$ . Although the proof follows the structure of part b) of Theorem 1 in [15], some of the terms appearing have to be computed in a new way.

First of all, for completeness we recall a technical lemma of [15] (p. 298).

**Lemma 5.1.** *Let  $F(k, n)$  = number of ways of putting  $k$  balls into  $n$  boxes so that no box contains exactly one ball, i.e.,*

$$F(k, n) = \sum_{\substack{\alpha_1 + \dots + \alpha_n = k \\ \alpha_i \neq 1 \forall i}} \frac{k!}{\alpha_1! \dots \alpha_n!}.$$

*Then  $F(k, n) \leq 2^k k! n^{k/2}$  for  $k \leq 2 + \log 4 / \log[1 + 2(1/n - 1/n^2)^{1/2}]$ .*

**Proof of Theorem 2.4.** To prove the theorem it is sufficient to check that, for any  $q > 0$ ,

$$P \left( \max_{0 \leq t \leq 1} |x_\varepsilon^\theta(t) - x(t)| > \alpha \varepsilon^{\frac{1}{2}} \left( \log \frac{1}{\varepsilon} \right)^{\frac{5}{2}} \right) = o(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0$$

and

$$P \left( \max_{0 \leq t \leq 1} |y_\varepsilon^\theta(t) - y(t)| > \alpha' \varepsilon^{\frac{1}{2}} \left( \log \frac{1}{\varepsilon} \right)^{\frac{5}{2}} \right) = o(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0$$

where  $\alpha$  and  $\alpha'$  are two positive constants depending on  $q$ . We will analyze the rate of convergence for the real part. The results for the imaginary part can be obtained by similar computations.

Recall that  $\gamma_0^{\varepsilon,\theta} \equiv \sigma_0^{\varepsilon,\theta} \equiv 0$  and define

$$\Gamma_m^{\varepsilon,\theta} = \sum_{j=0}^m \gamma_j^{\varepsilon,\theta} \quad \text{and} \quad \Lambda_m^{\varepsilon,\theta} = \sum_{j=0}^m \sigma_j^{\varepsilon,\theta}.$$

Set

$$J^\varepsilon \equiv \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq r \leq \gamma_{m+1}^{\varepsilon,\theta}} |x_\varepsilon^\theta(\Gamma_m^{\varepsilon,\theta} + r) - x(\Gamma_m^{\varepsilon,\theta} + r)|.$$

Since  $x_\varepsilon^\theta$  is piecewise linear and using the definition of  $\gamma_m^{\varepsilon,\theta}$ , notice that

$$\begin{aligned} x_\varepsilon^\theta(\Gamma_m^{\varepsilon,\theta} + r) &= x(\Lambda_m^{\varepsilon,\theta}) + \frac{x(\Lambda_{m+1}^{\varepsilon,\theta}) - x(\Lambda_m^{\varepsilon,\theta})}{\gamma_{m+1}^{\varepsilon,\theta}} r \\ &= x(\Lambda_m^{\varepsilon,\theta}) + |\beta_{m+1}^{\varepsilon,\theta}| \cdot \operatorname{sgn} \left( x(\Lambda_{m+1}^{\varepsilon,\theta}) - x(\Lambda_m^{\varepsilon,\theta}) \right) r. \end{aligned}$$

Thus,

$$\begin{aligned} J^\varepsilon &\leq \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} |x(\Lambda_m^{\varepsilon,\theta}) - x(\Gamma_m^{\varepsilon,\theta})| + \max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} |\beta_m^{\varepsilon,\theta}| \gamma_m^{\varepsilon,\theta} \\ &\quad + \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq r \leq \gamma_{m+1}^{\varepsilon,\theta}} |x(\Gamma_m^{\varepsilon,\theta}) - x(\Gamma_m^{\varepsilon,\theta} + r)| \\ &\leq \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| x(\Lambda_m^{\varepsilon,\theta}) - x\left(\frac{m\varepsilon^2}{4}\right) \right| + \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| x(\Gamma_m^{\varepsilon,\theta}) - x\left(\frac{m\varepsilon^2}{4}\right) \right| \\ &\quad + \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq r \leq \gamma_{m+1}^{\varepsilon,\theta}} |x(\Gamma_m^{\varepsilon,\theta}) - x(\Gamma_m^{\varepsilon,\theta} + r)| + \max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} |\beta_m^{\varepsilon,\theta}| \gamma_m^{\varepsilon,\theta} \\ &:= J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon, \end{aligned}$$

and for any  $a_\varepsilon > 0$ ,

$$P(J^\varepsilon > a_\varepsilon) \leq \sum_{j=1}^4 P\left(J_j^\varepsilon > \frac{a_\varepsilon}{4}\right) := I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon.$$

We will study the four terms separately.

1. *Study of the term  $I_4^\varepsilon$ .* Since  $\beta_m^{\varepsilon,\theta} = \frac{2}{\varepsilon} \cos(b_{m-1}\theta)$  and  $\gamma_m^{\varepsilon,\theta}$ 's are independent exponentially distributed variables with parameter  $\frac{4}{\varepsilon^2}$ , this term can be handled as term D in Theorem 1 in [15]. Indeed

$$\begin{aligned} I_4^\varepsilon &= P\left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} |\beta_m^{\varepsilon,\theta}| \gamma_m^{\varepsilon,\theta} > \frac{a_\varepsilon}{4}\right) \leq P\left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} \gamma_m^{\varepsilon,\theta} > \frac{a_\varepsilon \varepsilon}{8}\right) \\ &= 1 - P\left(\gamma_m^{\varepsilon,\theta} \leq \frac{a_\varepsilon \varepsilon}{8}\right)^{\frac{4}{\varepsilon^2} + 1} = 1 - \left(1 - e^{-\frac{a_\varepsilon}{2\varepsilon}}\right)^{\frac{4}{\varepsilon^2} + 1}. \end{aligned}$$

For  $a_\varepsilon$  of the type  $\alpha \varepsilon^{\frac{1}{2}} (\log \frac{1}{\varepsilon})^\beta$ , with  $\alpha$  and  $\beta$  are positive arbitrary fixed constants, we can see that

$$\lim_{\varepsilon \rightarrow 0} \frac{1 - \left(1 - e^{-\frac{a_\varepsilon}{2\varepsilon}}\right)^{\frac{4}{\varepsilon^2} + 1}}{\varepsilon^q} = 0 \quad \text{a.s.} \quad (18)$$

and so  $I_4^\varepsilon = o(\varepsilon^q)$ . Thus, the result is true with  $\beta = \frac{5}{2}$ .

2. *Study of the term  $I_1^\varepsilon$ .* Let  $\delta_\varepsilon > 0$ . Using the same decomposition as in the previous Section (see (10) and (11)) we can write

$$\begin{aligned} I_1^\varepsilon &\leq P \left( \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|s| \leq \delta_\varepsilon} \left| x \left( \frac{m\varepsilon^2}{4} + s \right) - x \left( \frac{m\varepsilon^2}{4} \right) \right| > \frac{a_\varepsilon}{4} \right) \\ &\quad + P \left( \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \Lambda_m^{\varepsilon, \theta} - \frac{m\varepsilon^2}{4} \right| > \delta_\varepsilon \right) \\ &\leq P \left( \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|s| \leq \delta_\varepsilon} \left| x \left( \frac{m\varepsilon^2}{4} + s \right) - x \left( \frac{m\varepsilon^2}{4} \right) \right| > \frac{a_\varepsilon}{4} \right) \\ &\quad + P \left( \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta}) \right| > \frac{\delta_\varepsilon}{2} \right) \\ &\quad + P \left( \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \frac{\varepsilon^2}{4} \left| \sum_{\substack{0 \leq k \leq \hat{n} \\ B_m}} T_k \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{4} \right) \\ &\quad + P \left( \max_{\substack{1 \leq m \leq \frac{4}{\varepsilon^2} \\ B_m}} \frac{\varepsilon^2}{4} \left| Z_{\hat{n}+1} \cos[2(\hat{n}+1)\theta] \right| > \frac{\delta_\varepsilon}{4} \right) \\ &= I_{11}^\varepsilon + I_{12}^\varepsilon + I_{13}^\varepsilon + I_{14}^\varepsilon, \end{aligned}$$

recalling that  $\alpha_j^{\varepsilon, \theta} = \mathbf{E}(\sigma_j^{\varepsilon, \theta} | \mathcal{B}) = \frac{\varepsilon^2}{2} [\cos(b_{j-1}\theta)]^2$  and where  $\hat{n} := b_{m-1} - 1$ ,  $B_m := \{k \in \{0, \dots, \hat{n}\} \text{ s.t. } T_0 + \dots + T_{\hat{n}} < m, T_0 + \dots + T_{\hat{n}+1} \geq m\}$ ,  $T_k$  are independent identically distributed random variables with  $T_k \sim \text{Geom}(\frac{1}{2})$  for each  $k = 0, 1, 2, \dots$  and  $0 \leq Z_{\hat{n}+1} \leq T_{\hat{n}+1}$ . We will study again the four terms separately.

2.1. *Study of the term  $I_{12}^\varepsilon$ .* In Section 4 we have seen that  $|M_n| = |\sum_{j=1}^m (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})|$  is a submartingale, so

$$\begin{aligned} I_{12}^\varepsilon &= P \left( \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \left( \frac{4}{\varepsilon^2} \sigma_j^{\varepsilon, \theta} - 2[\cos(b_{j-1}\theta)]^2 \right) \right| > \frac{2\delta_\varepsilon}{\varepsilon^2} \right) \\ &\leq \left( \frac{\varepsilon^2}{2\delta_\varepsilon} \right)^{2p} \mathbf{E} \left[ \left( \sum_{m=1}^{\lfloor 4/\varepsilon^2 \rfloor} \left( \frac{4}{\varepsilon^2} \sigma_m^{\varepsilon, \theta} - 2[\cos(b_{m-1}\theta)]^2 \right) \right)^{2p} \right], \end{aligned} \quad (19)$$

for any  $p \geq 1$ , by Doob's martingale inequality.

Set  $Y_m := \frac{4}{\varepsilon^2} \sigma_m^{\varepsilon, \theta} - 2[\cos(b_{m-1}\theta)]^2$ . Using Hölder's inequality, we obtain

$$\begin{aligned} \mathbf{E} \left[ \left( \sum_{m=1}^{\lfloor 4/\varepsilon^2 \rfloor} Y_m \right)^{2p} \right] &= \sum_{\substack{|u|=2p \\ u_m \neq 1 \forall m}} \binom{2p}{u} \mathbf{E} \left( Y_1^{u_1} \dots Y_{\lfloor 4/\varepsilon^2 \rfloor}^{u_{\lfloor 4/\varepsilon^2 \rfloor}} \right) \\ &\leq \sum_{\substack{|u|=2p \\ u_m \neq 1 \forall m}} \binom{2p}{u} [\mathbf{E}(Y_1^{2p})]^{u_1/2p} \dots [\mathbf{E}(Y_{\lfloor 4/\varepsilon^2 \rfloor}^{2p})]^{u_{\lfloor 4/\varepsilon^2 \rfloor}/2p}, \end{aligned} \quad (20)$$



where  $u = (u_1, \dots, u_{[4/\varepsilon^2]})$  with  $|u| = u_1 + \dots + u_{[4/\varepsilon^2]}$  and

$$\binom{2p}{u} = \frac{(2p)!}{u_1! \dots u_{[4/\varepsilon^2]}!}.$$

Notice that in the first equality we have used that if  $u_m = 1$  for any  $m$ , then  $\mathbf{E}(Y_1^{u_1} \dots Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}}) = 0$ . Indeed, assume that  $u_m = 1$ , then

$$\begin{aligned} \mathbf{E}(Y_1^{u_1} \dots Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}}) &= \mathbf{E}[\mathbf{E}(Y_1^{u_1} \dots Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}} | \mathcal{B})] \\ &= \mathbf{E}[\mathbf{E}(Y_1^{u_1} | \mathcal{B}) \dots \mathbf{E}(Y_{m-1}^{u_{m-1}} | \mathcal{B}) \mathbf{E}(Y_m | \mathcal{B}) \mathbf{E}(Y_{m+1}^{u_{m+1}} | \mathcal{B}) \dots \mathbf{E}(Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}} | \mathcal{B})] \end{aligned}$$

that is clearly zero since we have used that fixed  $\{b_j\}$ 's,  $\{\sigma_j\}$ 's are independent and  $\mathbf{E}(Y_m | \mathcal{B}) = 0$ .

On the other hand, by Skorokhod's theorem, we have

$$\begin{aligned} \mathbf{E}[(\sigma_m^{\varepsilon, \theta})^{2p}] &= \mathbf{E}[\mathbf{E}[(\sigma_m^{\varepsilon, \theta})^{2p} | \mathcal{B}]] \leq \mathbf{E}[2(2p)! \mathbf{E}[(k_i \xi_m^{\varepsilon, \theta})^{4p} | \mathcal{B}]] \\ &\leq 2(2p)! \mathbf{E}\left[(4p)! \left(\frac{\varepsilon}{2}\right)^{4p} (\cos(b_{m-1}\theta))^{4p}\right] \leq 2(2p)! (4p)! \left(\frac{\varepsilon}{2}\right)^{4p}. \end{aligned}$$

So, using the inequality  $|a + b|^{2p} \leq 2^{2p}(|a|^{2p} + |b|^{2p})$ , we obtain

$$\begin{aligned} \mathbf{E}(Y_m^{2p}) &\leq 2^{2p} \left[ \left(\frac{4}{\varepsilon^2}\right)^{2p} \mathbf{E}[(\sigma_m^{\varepsilon, \theta})^{2p}] + 2^{2p} \mathbf{E}[(\cos(b_{m-1}\theta))^{4p}] \right] \\ &\leq 2^{2p} \left[ \left(\frac{4}{\varepsilon^2}\right)^{2p} 2(2p)! (4p)! \left(\frac{\varepsilon}{2}\right)^{4p} + 2^{2p} \right] \\ &\leq 2^{2p} [2(2p)! (4p)! + 2^{2p}] \leq 2^{2p+1} \cdot 2(2p)! (4p)! \\ &= 4 \cdot 2^{2p} (2p)! (4p)!. \end{aligned} \tag{21}$$

Finally Lemma 5.1 yields that, for  $p \leq 1 + \frac{\log 2}{\log [1 + \varepsilon(1 - \varepsilon^2/4)^{\frac{1}{2}}]}$ ,

$$\sum_{\substack{|u|=2p \\ u_i \neq 1 \forall i}} \binom{2p}{u} \leq 2^{2p} (2p)! \left(\frac{4}{\varepsilon^2}\right)^p. \tag{22}$$

Therefore, for  $p$  as above, putting together (19), (20), (21) and (22) and applying Stirling formula,  $k! = \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} e^{\frac{a}{12k}}$ , with  $0 < a < 1$ , we obtain

$$\begin{aligned} I_{12}^\varepsilon &\leq 4 (\delta_\varepsilon)^{-2p} \varepsilon^{2p} 2^{4p} ((2p)!)^2 (4p)! \\ &\leq 4 (\delta_\varepsilon)^{-2p} \varepsilon^{2p} 2^{4p} \left[ \sqrt{2\pi} (2p)^{2p+\frac{1}{2}} e^{-2p} e^{\frac{a}{24p}} \right]^2 \left[ \sqrt{2\pi} (4p)^{4p+\frac{1}{2}} e^{-4p} e^{\frac{a}{48p}} \right] \\ &= (\delta_\varepsilon)^{-2p} \varepsilon^{2p} 2^{16p+4} (2\pi)^{\frac{3}{2}} e^{-8p+\frac{a}{12p}+\frac{a}{48p}} p^{8p+\frac{3}{2}} \\ &\leq K_1^p (\delta_\varepsilon)^{-2p} \varepsilon^{2p} p^{8p+3/2} \end{aligned}$$

where  $K_1$  is a constant.

Let us impose now  $K_1^p (\delta_\varepsilon)^{-2p} \varepsilon^{2p} p^{8p+3/2} = \varepsilon^{2q}$  and  $p = \lceil \log \frac{1}{\varepsilon} \rceil$ . Observe that  $p = \lceil \log \frac{1}{\varepsilon} \rceil$  fulfills the condition on  $p$  of inequality (22). We get

$$\delta_\varepsilon = K_2 \varepsilon^{1-q/\lceil \log 1/\varepsilon \rceil} \left[ \log \frac{1}{\varepsilon} \right]^{4+3/(4\lceil \log 1/\varepsilon \rceil)}, \quad (23)$$

where  $K_2 = \sqrt{K_1}$  is a constant. Clearly, with this  $\delta_\varepsilon$ , it follows that  $I_{12}^\varepsilon = o(\varepsilon^q)$ .

**2.2. Study of the term  $I_{13}^\varepsilon$ .** As in Section 4, since  $\hat{n} \leq m-1$ , we can write

$$\begin{aligned} I_{13}^\varepsilon &\leq P \left( \max_{0 \leq n \leq \frac{4}{\varepsilon^2}-1} \frac{\varepsilon^2}{4} \left| \sum_{k=0}^n (T_k - 2) \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{8} \right) \\ &\quad + \mathbb{1}_{\left\{ \max_{0 \leq n \leq \frac{4}{\varepsilon^2}-1} \frac{\varepsilon^2}{2} \left| \sum_{k=0}^n \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{8} \right\}} := I_{131}^\varepsilon + I_{132}^\varepsilon. \end{aligned}$$

Let us begin studying  $I_{132}^\varepsilon$ . By (23) and (16),

$$\begin{aligned} I_{132}^\varepsilon &\leq \mathbb{1}_{\left\{ \frac{\varepsilon^2}{2} K > \frac{\delta_\varepsilon}{8} \right\}} = \mathbb{1}_{\left\{ \frac{\varepsilon^2}{2} K > \frac{K_2^2}{8} \varepsilon^{1-q/\lceil \log 1/\varepsilon \rceil} \left[ \log \frac{1}{\varepsilon} \right]^{4+3/(4\lceil \log 1/\varepsilon \rceil)} \right\}} \\ &= \mathbb{1}_{\left\{ \varepsilon^{-1-q/\lceil \log 1/\varepsilon \rceil} \left[ \log \frac{1}{\varepsilon} \right]^{4+3/(4\lceil \log 1/\varepsilon \rceil)} < \frac{4K}{K_2} \right\}}, \end{aligned}$$

and clearly  $I_{132}^\varepsilon = 0$  for small  $\varepsilon$ .

On the other hand, since  $M'_n := \frac{\varepsilon^2}{4} \sum_{k=0}^n (T_k - 2) \cos(2k\theta)$  is a martingale (see Section 4), by Doob's martingale inequality,

$$\begin{aligned} I_{131}^\varepsilon &= P \left( \max_{0 \leq n \leq \frac{4}{\varepsilon^2}-1} \left| \sum_{k=0}^n (T_k - 2) \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{2\varepsilon^2} \right) \\ &\leq \left( \frac{2\varepsilon^2}{\delta_\varepsilon} \right)^{2p} \mathbf{E} \left[ \left( \sum_{k=0}^{\lceil 4/\varepsilon^2 \rceil - 1} (T_k - 2) \cos(2k\theta) \right)^{2p} \right]. \end{aligned}$$

Set  $U_k := (T_k - 2) \cos(2k\theta)$ .  $U_k$ 's are independent and centered random variables and by Hölder's inequality, we have

$$\mathbf{E} \left[ \left( \sum_{k=0}^{\lceil 4/\varepsilon^2 \rceil - 1} U_k \right)^{2p} \right] \leq \sum_{\substack{|u|=2p \\ u_i \neq 1 \forall i}} \binom{2p}{u} [\mathbf{E}(U_0^{2p})]^{u_0/2p} \cdots [\mathbf{E}(U_{\lceil 4/\varepsilon^2 \rceil - 1}^{2p})]^{u_{\lceil 4/\varepsilon^2 \rceil - 1}/2p}.$$

Let us recall that  $T_k \sim \text{Geom}(\frac{1}{2})$ . It is well-known that  $T_k = \lceil \tilde{T}_k \rceil + 1$  where  $\tilde{T}_k \sim \text{Exp}(\log 2)$ . Hence

$$\mathbf{E}(T_k^{2p}) \leq \mathbf{E}[(\tilde{T}_k + 1)^{2p}] \leq 2^{2p} [\mathbf{E}(\tilde{T}_k^{2p}) + 1] \leq 2^{2p} \left[ \frac{(2p)!}{(\log 2)^{2p}} + 1 \right] \leq 2(2p)!(4p)!,$$

and it follows that

$$\begin{aligned} \mathbf{E}(U_k^{2p}) &\leq 2^{2p} \left[ \mathbf{E}[(T_k \cos(2k\theta))^{2p}] + 2^{2p} [\cos(2k\theta)]^{2p} \right] \\ &\leq 2^{2p} [\cos(2k\theta)]^{2p} (\mathbf{E}(T_k^{2p}) + 2^{2p}) \leq 4 \cdot 2^{2p} (2p)!(4p)!. \end{aligned}$$

Some inequalities are very crude, but they are helpful since we get the same bounds as in the study of  $I_{12}^\varepsilon$ . Thus, with  $\delta_\varepsilon$  as in (23) we get that  $I_{131}^\varepsilon = o(\varepsilon^q)$ .

2.3. *Study of the term  $I_{14}^\varepsilon$ .* Since  $T_k$ 's are independent identically distributed random variables with  $T_k \sim \text{Geom}(\frac{1}{2})$  for each  $k$ , by (17),

$$\begin{aligned} I_{14}^\varepsilon &\leq P\left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2}-1} \frac{\varepsilon^2}{4} T_{n+1} > \frac{\delta_\varepsilon}{4}\right) = 1 - P\left(\max_{1 \leq n \leq \frac{4}{\varepsilon^2}} T_n \leq \frac{\delta_\varepsilon}{\varepsilon^2}\right) \\ &= 1 - P\left[\left(T_n \leq \frac{\delta_\varepsilon}{\varepsilon^2}\right)\right]^{4/\varepsilon^2} = 1 - \left(\sum_{k=1}^{[\delta_\varepsilon/\varepsilon^2]} \frac{1}{2^k}\right)^{4/\varepsilon^2} = 1 - \left(1 - \left(\frac{1}{2}\right)^{[\delta_\varepsilon/\varepsilon^2]}\right)^{4/\varepsilon^2} \\ &= 1 - \left(1 - e^{-\log 2 [\delta_\varepsilon/\varepsilon^2]}\right)^{4/\varepsilon^2} \leq 1 - \left(1 - e^{-\frac{a_\varepsilon}{2\varepsilon}}\right)^{4/\varepsilon^2+1}. \end{aligned}$$

Notice that in the last inequality we have used that  $\frac{a_\varepsilon}{2\varepsilon} \leq \log 2[\frac{\delta_\varepsilon}{\varepsilon^2}]$ . The bound for  $I_{14}^\varepsilon$  is the same that we got in the study of  $I_4^\varepsilon$ . So, by the same arguments, we can conclude that  $I_{14}^\varepsilon = o(\varepsilon^q)$ .

2.4. *Study of the term  $I_{11}^\varepsilon$ .* As in Theorem 1 in [15], for small  $\varepsilon$  and using a Doob's martingale inequality for Brownian motion we get

$$\begin{aligned} I_{11}^\varepsilon &\leq \sum_{m=0}^{[4/\varepsilon^2]} P\left(\max_{|s| \leq \delta_\varepsilon} \left|x\left(\frac{m\varepsilon^2}{4} + s\right) - x\left(\frac{m\varepsilon^2}{4}\right)\right| > \frac{a_\varepsilon}{4}\right) \\ &= \left(\frac{4}{\varepsilon^2} + 1\right) P\left(\max_{|s| \leq \delta_\varepsilon} |x(s)| > \frac{a_\varepsilon}{4}\right) \leq \frac{32}{\varepsilon^2} P\left(\max_{0 \leq s \leq \delta_\varepsilon} x(s) > \frac{a_\varepsilon}{4}\right) \\ &\leq \frac{32}{\varepsilon^2} \exp\left(-\left(\frac{a_\varepsilon}{4}\right)^2 \frac{1}{2\delta_\varepsilon}\right) = \frac{32}{\varepsilon^2} \exp\left(-\frac{(a_\varepsilon)^2}{32\delta_\varepsilon}\right). \end{aligned}$$

Condition  $(32/\varepsilon^2) \exp\left(-\frac{(a_\varepsilon)^2}{32\delta_\varepsilon}\right) \leq 32\varepsilon^{2q}$  yields that

$$a_\varepsilon \geq K_3 \varepsilon^{1/2} \varepsilon^{-q/2[\log 1/\varepsilon]} \left[\log \frac{1}{\varepsilon}\right]^{2+3/8[\log 1/\varepsilon]} \left(\log \frac{1}{\varepsilon}\right)^{1/2},$$

where  $K_3$  is a constant depending on  $q$ . Notice that

$$a_\varepsilon = \alpha \varepsilon^{1/2} \left(\log \frac{1}{\varepsilon}\right)^{5/2},$$

for small  $\varepsilon$ , where  $\alpha$  is a constant that depends on  $q$ , satisfies such a condition. Thus, with  $\delta_\varepsilon$  as in (23), it follows that  $I_{11}^\varepsilon = o(\varepsilon^q)$ .

3. *Study of the term  $I_2^\varepsilon$ .* For our  $\delta_\varepsilon > 0$ , we have

$$\begin{aligned} I_2^\varepsilon &\leq P\left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|s| \leq \delta_\varepsilon} \left|x\left(\frac{m\varepsilon^2}{4} + s\right) - x\left(\frac{m\varepsilon^2}{4}\right)\right| > \frac{a_\varepsilon}{4}\right) \\ &\quad + P\left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left|\Gamma_m^{\varepsilon, \theta} - \frac{m\varepsilon^2}{4}\right| > \delta_\varepsilon\right) = I_{21}^\varepsilon + I_{22}^\varepsilon. \end{aligned}$$

On one hand, observe that  $I_{21}^\varepsilon = I_{11}^\varepsilon$ , thus  $I_{21}^\varepsilon = o(\varepsilon^q)$ .

On the other hand, it is easy to check that  $\sum_{j=0}^m (\gamma_j^{\varepsilon, \theta} - \frac{\varepsilon^2}{4})$  is a martingale. So applying Doob's martingale inequality

$$I_{22}^\varepsilon = P \left( \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=0}^m \left( \frac{4}{\varepsilon^2} \gamma_j^{\varepsilon, \theta} - 1 \right) \right| > \frac{4\delta_\varepsilon}{\varepsilon^2} \right) \\ \leq \left( \frac{\varepsilon^2}{4\delta_\varepsilon} \right)^{2p} \mathbf{E} \left[ \left( \sum_{m=1}^{4/\varepsilon^2} \left( \frac{4}{\varepsilon^2} \gamma_m^{\varepsilon, \theta} - 1 \right) \right)^{2p} \right].$$

Set  $V_m := \frac{4}{\varepsilon^2} \gamma_m^{\varepsilon, \theta} - 1$ . Notice that  $V_m$ 's are independent and centered random variables with

$$\mathbf{E}(V_m^{2p}) \leq 2^{2p} \left( \left( \frac{4}{\varepsilon^2} \right)^{2p} \mathbf{E}[(\gamma_m^{\varepsilon, \theta})^{2p}] + 1 \right) \\ \leq 2^{2p} ((2p)! + 1) \leq 2^{2p+1} (2p)! \leq 4 \cdot 2^{2p} (2p)! (4p)!.$$

Then using an inequality of the type of (20) and following the same arguments as in the study of  $I_{12}^\varepsilon$ , we get that  $I_{22}^\varepsilon = o(\varepsilon^q)$ .

4. Study of the term  $I_3^\varepsilon$ . For  $\delta_\varepsilon > 0$  defined in (23) and  $a_\varepsilon$  of the type  $\alpha \varepsilon^{\frac{1}{2}} (\log \frac{1}{\varepsilon})^{\frac{5}{2}}$

$$I_3^\varepsilon \leq P \left( \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|r| \leq \delta_\varepsilon} |x(\Gamma_m^{\varepsilon, \theta}) - x(\Gamma_m^{\varepsilon, \theta} + r)| > \frac{a_\varepsilon}{4} \right) \\ + P \left( \max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} \gamma_m^{\varepsilon, \theta} > \delta_\varepsilon \right) := I_{31}^\varepsilon + I_{32}^\varepsilon.$$

On one hand,  $I_{31}^\varepsilon = o(\varepsilon^q)$  is proved in the same way as  $I_{11}^\varepsilon$ .

On the other hand,

$$I_{32}^\varepsilon = 1 - [P(\gamma_m^{\varepsilon, \theta} \leq \delta_\varepsilon)]^{(4/\varepsilon^2)+1} = 1 - \left( 1 - e^{-4\delta_\varepsilon/\varepsilon^2} \right)^{(4/\varepsilon^2)+1}.$$

Thus  $I_{32}^\varepsilon = o(\varepsilon^q)$ , similarly as we have proved for  $I_4^\varepsilon$ .

We have checked now that all the terms in our decomposition are of order  $\varepsilon^q$ . More precisely we have proved that for  $a_\varepsilon = \alpha \varepsilon^{\frac{1}{2}} (\log \frac{1}{\varepsilon})^{\frac{5}{2}}$  and  $q > 0$ ,

$$E := P \left( \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq r \leq \gamma_{m+1}^{\varepsilon, \theta}} |x_\varepsilon^\theta(\Gamma_m^{\varepsilon, \theta} + r) - x(\Gamma_m^{\varepsilon, \theta} + r)| > a_\varepsilon \right) = o(\varepsilon^q).$$

Let us fix  $0 < v < 1$ . Then

$$P \left( \max_{0 \leq t \leq 1-v} |x_\varepsilon^\theta(t) - x(t)| > a_\varepsilon \right) \\ \leq P \left( \max_{0 \leq t \leq 1-v} |x_\varepsilon^\theta(t) - x(t)| > a_\varepsilon, \Gamma_{4/\varepsilon^2}^{\varepsilon, \theta} \geq 1-v \right) + P \left( \Gamma_{4/\varepsilon^2}^{\varepsilon, \theta} < 1-v \right) \\ := E_1 + E_2.$$

On one hand, since  $E_1 \leq E$ , we get that  $E_1 = o(\varepsilon)$ . On the other hand,

$$E_2 \leq P \left( \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| \Gamma_m^{\varepsilon, \theta} - \frac{m\varepsilon^2}{4} \right| > v \right) \leq I_{22}^\varepsilon,$$

for small  $\varepsilon$ . Thus,  $E_2 = o(\varepsilon^q)$ .

We have proved the rate of convergence results in the interval  $[0, 1 - v]$ , but we can extend the argument for any compact interval. So, the proof of [Theorem 2.4](#) is completed.  $\square$

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