The Complex Brownian Motion as a Weak Limit of Processes Constructed from a Poisson Process

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ABSTRACT In this paper we show an approximation in law of the complex Brownian motion by processes constructed from a unique standard Poisson process.

1 Introduction and main result

Consider the processes

$$\{x_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} e^{i\theta N_s} ds, \quad t \in [0,T]\} \tag{1.1}$$

where $\{N_s, s \geq 0\}$ is a standard Poisson process, and $i^2 = -1$. The aim of this paper is to study the weak limits of these processes depending on the value of θ when ε tends to zero.

In the trivial case, when $\theta = 0$, the processes $x_\varepsilon^\theta(t)$ are deterministic and obviously they go to infinity when ε tends to zero. On the other hand, when $\theta = \pi$, the processes x_ε^θ are real and

$$x_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N_s} ds.$$

This case was studied by Stroock in [2], who proved that the laws of these processes in the space of continuous functions on $[0,T]$ converge weakly toward the law of $\sqrt{2}W_t$, where $\{W_t;\, t \in [0,T]\}$ is a standard Brownian motion.

Then we are interested in studying weak convergence when $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Our result also gives an alternative proof of the case $\theta = \pi$ proved by Stroock in [2], reads as follows.

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Theorem 1.1. *Define for any* $\varepsilon > 0$

$$\{x_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} e^{i\theta N_s} ds, \quad t \in [0,T]\},$$

where $\{N_s,\, s \geq 0\}$ *is a standard Poisson process.*

Consider P_ε^θ *the image law of* x_ε^θ *in the Banach space* $\mathcal{C}([0,T],\mathbb{C})$ *of continuous functions on* $[0,T]$. *Then, if* $\theta \in (0,\pi)\cup(\pi,2\pi)$, P_ε^θ *converges weakly as* ε *tends to zero, toward the law on* $\mathcal{C}([0,T],\mathbb{C})$ *of a complex Brownian motion. On the other hand, when* $\theta = \pi$, P_ε^θ *converges weakly toward the law of* $\sqrt{2}W_t$, *where* $\{W_t;\, t \in [0,T]\}$ *is a standard Brownian motion.*

It is said that $\{B_t,\, t \in [0,T]\}$ is a complex Brownian motion if it takes values on $\mathbb{C}$ and its real part and its imaginary part are two independent standard Brownian motions. Note that the processes real part and imaginary part of the integrand process defining x_ε^θ are both functionally dependent. Nevertheless, integrating and taking limits, we obtain independent processes.

These processes have very different properties from the classical examples that converge in law to the Wiener process. For instance, the processes $\{\cos(\theta N_s)\}$ and $\{\sin(\theta N_s)\}$ are neither stationary nor do they involve sums of independent random variables. Moreover, their jumps occur at random times and, depending on the value of θ, the size of the jumps is not constant.

In order to prove the theorem, we have to check that the family P_ε^θ is tight (see Section 2) and that the law of all possible weak limits of P_ε^θ is the law of a complex Brownian motion if $\theta \in (0,\pi)\cup(\pi,2\pi)$, and the law of $\sqrt{2}W_t$ if $\theta = \pi$ (see Section 3).

Throughout the paper, K denotes any positive constant, not depending on ε, which may change from one expression to another.

2 Proof of tightness

We can write

$$x_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \cos(\theta N_s)ds + i\varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \sin(\theta N_s)ds.$$

We need to prove that the laws corresponding to the real part and the imaginary part of the processes x_ε^θ are tight. Using the Billingsley criterion (see Theorem 12.3 of [1]) and that our processes are null on the origin, it suffices to prove the following lemma.

Lemma 2.1. *There exists a constant K such that for any $s < t$*

$$\sup_{\varepsilon} \big(E(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \cos(\theta N_x)dx)^4 + E(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \sin(\theta N_x)dx)^4\big) \leq K(t-s)^2.$$

Proof. We denote by $\mathcal{I}$ the 4-dimensional cube $[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}]^4$.

$$\begin{aligned}
&E(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \cos(\theta N_x)dx)^4 + E(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \sin(\theta N_x)dx)^4 \\
&= 24\varepsilon^4 E \int_{\mathcal{I}} I_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} \big(\cos(\theta N_{x_4}) \cos(\theta N_{x_3}) \cos(\theta N_{x_2}) \cos(\theta N_{x_1}) \\
&\qquad\qquad + \sin(\theta N_{x_4}) \sin(\theta N_{x_3}) \sin(\theta N_{x_2}) \sin(\theta N_{x_1})\big) \otimes_{i=1}^4 dx_i \\
&= 12\varepsilon^4 E \int_{\mathcal{I}} I_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} \cos(\theta(N_{x_4} - N_{x_3})) \cos(\theta(N_{x_2} - N_{x_1})) \otimes_{i=1}^4 dx_i \\
&+ 12\varepsilon^4 E \int_{\mathcal{I}} I_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} \cos(\theta(N_{x_4} + N_{x_3})) \cos(\theta(N_{x_2} + N_{x_1})) \otimes_{i=1}^4 dx_i \\
&= I_1 + I_2.
\end{aligned}$$

When $\theta = \pi$, I_1 and I_2 are equal because, in this case, for any $m \in \mathbb{N}$, $\cos(m\theta) = (-1)^m$.

Using that the Poisson process has independent increments we obtain

$$\begin{aligned}
I_1 &= 12\varepsilon^4 \int_{[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}]^4} I_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} E(\cos(\theta(N_{x_4} - N_{x_3}))) \\
&\quad \times E(\cos(\theta(N_{x_2} - N_{x_1})))dx_1 \dots dx_4 \\
&\leq 12\big(\varepsilon^2 \int_{[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}]^2} I_{\{x_1 \leq x_2\}} |E(\cos(\theta(N_{x_2} - N_{x_1})))| dx_1 dx_2\big)^2 \\
&\leq 12\big(\varepsilon^2 \int_{[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}]^2} I_{\{x_1 \leq x_2\}} \| E(e^{i\theta(N_{x_2} - N_{x_1})}) \| dx_1 dx_2\big)^2
\end{aligned}$$

where $\|z\|$ is the modulus of the complex number z. The last expression is equal to

$$\begin{aligned}
&12\big(\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-(x_2 - x_1)(1-\cos(\theta))} dx_1 dx_2\big)^2 \\
&\qquad\qquad \leq \frac{48}{(1-\cos(\theta))^2}(t-s)^2.
\end{aligned}$$

We have proved the lemma for $\theta = \pi$. So, from now on, we assume that $\theta \in (0, \pi) \cup (\pi, 2\pi)$. For this case we need to prove the bound for I_2. Using that $\{x_1 \le x_2 \le x_3 \le x_4\}$ and that the Poisson process has independent increments we have

$$
\begin{aligned}
&E\big[\cos(\theta(N_{x_4}+N_{x_3}))\cos(\theta(N_{x_2}+N_{x_1}))\big] \\
=&E\big[\cos\big(\theta((N_{x_4}-N_{x_3})+2(N_{x_3}-N_{x_2})+2N_{x_2})\big)\cos(\theta(N_{x_2}+N_{x_1}))\big] \\
=&E\big[\cos\big(\theta((N_{x_4}-N_{x_3})+2(N_{x_3}-N_{x_2}))\big)\big]E\big[\cos(2\theta N_{x_2})\cos(\theta(N_{x_2}+N_{x_1}))\big] \\
-&E\big[\sin\big(\theta((N_{x_4}-N_{x_3})+2(N_{x_3}-N_{x_2}))\big)\big]E\big[\sin(2\theta N_{x_2})\cos(\theta(N_{x_2}+N_{x_1}))\big]
\end{aligned}
$$

and we can bound the last expression by

$$
\begin{aligned}
&|E\big[\cos\big(\theta((N_{x_4}-N_{x_3})+2(N_{x_3}-N_{x_2}))\big)\big]| \\
&\qquad\qquad + |E\big[\sin\big(\theta((N_{x_4}-N_{x_3})+2(N_{x_3}-N_{x_2}))\big)\big]| \\
&\le \big(|E(\cos(\theta(N_{x_4}-N_{x_3})))|+|E(\sin(\theta(N_{x_4}-N_{x_3})))|\big) \\
&\qquad\qquad \times \big(|E(\cos(2\theta(N_{x_3}-N_{x_2})))|+|E(\sin(2\theta(N_{x_3}-N_{x_2})))|\big).
\end{aligned}
$$

Thus, $I_2/48$ is less than or equal to

$$
\begin{aligned}
&\varepsilon^4 \int_{[\frac{2s}{\varepsilon^2},\frac{2t}{\varepsilon^2}]^4} I_{\{x_1\le\cdots\le x_4\}} \|E(e^{i\theta(N_{x_4}-N_{x_3})})\|\cdot\|E(e^{2i\theta(N_{x_3}-N_{x_2})})\| dx_1\ldots dx_4 \\
\le\,& \varepsilon^4 \int_{[\frac{2s}{\varepsilon^2},\frac{2t}{\varepsilon^2}]^4} I_{\{x_1\le\cdots\le x_4\}} e^{-(x_4-x_3)(1-\cos(\theta))} e^{-(x_3-x_2)(1-\cos(2\theta))} dx_1\ldots dx_4.
\end{aligned}
$$

Integrating with respect to x_2 and x_3, we obtain that the previous expression can be bounded by

$$
\frac{48\varepsilon^4}{(1-\cos(\theta))(1-\cos(2\theta))}\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_4} dx_1 dx_4 \le \frac{192(t-s)^2}{(1-\cos(\theta))(1-\cos(2\theta))}.
$$

This completes the proof of the lemma. □

3 Identification of the limit law

Let $\{P^\theta_{\varepsilon_n}\}_n$ be a subsequence of $\{P^\theta_\varepsilon\}_\varepsilon$ (which we will also denote by $\{P^\theta_\varepsilon\}$) weakly convergent to some probability P^θ. We want to see that if $\theta \in (0,\pi)\cup(\pi,2\pi)$, the canonical process $X = \{X_t(x) =: x(t)\}$ is a complex Brownian motion under the probability P^θ, that is, the real part and the imaginary part of this process are two independent Brownian motions.

Using Paul Lévy's theorem it suffices to prove that under P^θ, the real and imaginary parts (denoted by $Re[\cdot]$ and $Im[\cdot]$) of the canonical process are both martingales with respect to the natural filtration, $\{\mathcal{F}_t\}$, with quadratic variations $< Re[X], Re[X] >_t = t$, $< Im[X], Im[X] >_t = t$ and covariation $< Re[X], Im[X] >_t = 0$.

We also have to see that if $\theta = \pi$ the canonical process has the law of $\sqrt{2}W_t$ where $\{W_t; t \in [0,T]\}$ is a standard Brownian motion. So, in this case, we need to check that X is martingale with quadratic variation $< X, X >_t = 2t$.

3.1 Martingale property

When $\theta \in (0,\pi) \cup (\pi, 2\pi)$, to see that under P^θ the real part and the imaginary part of the canonical process X are martingales with respect to its natural filtration $\{\mathcal{F}_t\}$, we have to prove that for any $s_1 \le s_2 \le \cdots \le s_n \le s$ and for any bounded continuous function $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$

$$E_{P^\theta}\left[\varphi(X_{s_1}, \dots, X_{s_n})(Re[X_t] - Re[X_s])\right] = 0,$$

$$E_{P^\theta}\left[\varphi(X_{s_1}, \dots, X_{s_n})(Im[X_t] - Im[X_s])\right] = 0.$$

When $\theta = \pi$, the process X is real and we only check the first condition. Since $P_\varepsilon^\theta \overset{w}{\Rightarrow} P^\theta$, and taking into account Lemma 2.1, we have that,

$$\begin{aligned}&\lim_{\varepsilon \to 0} E_{P_\varepsilon^\theta}\left[\varphi(x(s_1), \dots, x(s_n))(Re[x(t)] - Re[x(s)])\right] \\ &\qquad = E_{P^\theta}\left[\varphi(x(s_1), \dots, x(s_n))(Re[x(t)] - Re[x(s)])\right],\end{aligned}$$

and we get the same with the imaginary part. So, it suffices to see that

$$E\big(\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \cos(\theta N_x) dx\big)$$

and

$$E\big(\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \sin(\theta N_x) dx\big)$$

converge to zero when ε tends to zero. Thus, it is enough to prove that

$$\|E\big(\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} e^{i\theta N_x} dx\big)\|$$

converges to zero when ε tends to zero. But this expression is equal to

$$
\begin{aligned}
\|E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))e^{i\theta N_{\frac{2s}{\varepsilon^2}}}\big)\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} E(e^{i\theta(N_x-N_{\frac{2s}{\varepsilon^2}})})dx\|
&\leq K\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \|e^{-(x-\frac{2s}{\varepsilon^2})(1-e^{i\theta})}\|dx \\
&= K\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} e^{-(x-\frac{2s}{\varepsilon^2})(1-\cos(\theta))}dx \\
&\leq K\varepsilon\int_0^{\frac{2(t-s)}{\varepsilon^2}} e^{-x(1-\cos(\theta))}dx \\
&= \frac{K\varepsilon}{1-\cos(\theta)}\big(1-e^{-\frac{2(t-s)}{\varepsilon^2}(1-\cos(\theta))}\big),
\end{aligned}
$$

which converges to zero when ε tends to zero.

3.2 Quadratic variations and covariation

We have to prove the following result

Proposition 3.1. *Consider $\{P_\varepsilon^\theta\}$ the laws on $\mathcal{C}([0,T],\mathbb{C})$ of the processes x_ε^θ defined by (1.1), and assume that $P_{\varepsilon_n}^\theta$ is a subsequence weakly convergent to P^θ. Let X be the canonical process and let $\{\mathcal{F}_t\}$ be its natural filtration. Then, under P^θ, if $\theta\in(0,\pi)\cup(\pi,2\pi)$ we have that the quadratic variations $<Re[X],Re[X]>_t=t$, $<Im[X],Im[X]>_t=t$ and the covariation $<Re[X],Im[X]>_t=0$. On the other hand, if $\theta=\pi$, the quadratic variation $<X,X>_t$ is equal to $2t$.*

To prove Proposition 3.1 it will be useful the following lemma.

Lemma 3.2. *Consider $\{\mathcal{F}_t^{\varepsilon,\theta}\}$ the natural filtration of the processes x_ε^θ. Then, for any $s<t$ and for any real $\mathcal{F}_s^{\varepsilon,\theta}-$measurable and bounded random variable Y, we have that if $\theta\in(0,2\pi)$,*

a) $$\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E[e^{i\theta(N_{x_2}-N_{x_1})}]dx_1dx_2 = (t-s)(1+i\frac{\sin(\theta)}{1-\cos(\theta)})+o(\varepsilon),$$
and for any $\theta\in(0,\pi)\cup(\pi,2\pi)$,

b) $$\lim_{\varepsilon\to0}\|\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E[e^{i\theta(N_{x_2}+N_{x_1})}Y]dx_1dx_2\|=0.$$

Proof. Let us prove a):

$$\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E[e^{i\theta(N_{x_2}-N_{x_1})}]dx_1 dx_2$$

$$
\begin{aligned}
&= \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-(x_2-x_1)(1-e^{i\theta})} dx_1 dx_2 \\
&= \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \frac{1}{1-e^{i\theta}}(1-e^{-(x_2-\frac{2s}{\varepsilon^2})(1-e^{i\theta})})dx_2 \\
&= \frac{2(t-s)}{1-e^{i\theta}} + \frac{\varepsilon^2}{(1-e^{i\theta})^2}(e^{-\frac{2(t-s)}{\varepsilon^2}(1-e^{i\theta})}-1) \\
&= (t-s)(1+i\frac{\sin(\theta)}{1-\cos(\theta)}) + o(\varepsilon).
\end{aligned}
$$

This finishes the proof. In order to see b), suppose that $\theta \in (0,\pi)\cup(\pi,2\pi)$. Using that the Poisson process has independent increments we obtain

$$
\begin{aligned}
& \|\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E[e^{i\theta(N_{x_2}+N_{x_1})}Y]dx_1 dx_2\| \\
= \; & \|\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E[e^{i\theta(N_{x_2}-N_{x_1})}]E[e^{i2\theta(N_{x_1}-N_{\frac{2s}{\varepsilon^2}})}]E[Ye^{i2\theta N_{\frac{2s}{\varepsilon^2}}}]dx_1 dx_2\| \\
\leq \; & K\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \|e^{-(x_2-x_1)(1-e^{i\theta})}\| \cdot \|e^{-(x_1-\frac{2s}{\varepsilon^2})(1-e^{2i\theta})}\| dx_1 dx_2 \\
= \; & K\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-(x_2-x_1)(1-\cos(\theta))} e^{-(x_1-\frac{2s}{\varepsilon^2})(1-\cos(2\theta))} dx_1 dx_2.
\end{aligned}
$$

Notice that if $\cos(\theta) = \cos(2\theta)$ (that is, when $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$) it is easy to check that this integral converges to zero when ε tends to zero. Otherwise, we obtain that the last integral is equal to

$$
\begin{aligned}
& K\varepsilon^2 \frac{1}{\cos(\theta)-\cos(2\theta)} e^{\frac{2s}{\varepsilon^2}(1-\cos(\theta))} \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} e^{-x_2(1-\cos(\theta))} dx_2 \\
& \qquad - K\varepsilon^2 \frac{1}{\cos(\theta)-\cos(2\theta)} e^{\frac{2s}{\varepsilon^2}(1-\cos(2\theta))} \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} e^{-x_2(1-\cos(2\theta))} dx_2.
\end{aligned}
$$

The first integral is equal to

$$K\varepsilon^2 \frac{1}{(\cos(\theta)-\cos(2\theta))(1-\cos(\theta))}(1-e^{-\frac{2(t-s)}{\varepsilon^2}(1-\cos(\theta))}),$$

and converges to zero when ε tends to zero. And the second one is equal to

$$K\varepsilon^2 \frac{1}{(\cos(\theta) - \cos(2\theta))(1 - \cos(2\theta))}(1 - e^{-\frac{2(t-s)}{\varepsilon^2}(1-\cos(2\theta))}),$$

and also converges to zero. □

Proof of Proposition 3.1. When $\theta \in (0,\pi) \cup (\pi, 2\pi)$ we have to prove that $< Re[X], Re[X] >_t = t$ and $< Im[X], Im[X] >_t = t$. It is enough to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_n \leq s$ and for any bounded continuous function $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$,

$$E\big[\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))\big((Re[x_\varepsilon^\theta(t)] - Re[x_\varepsilon^\theta(s)])^2 - (t-s)\big)\big]$$

and

$$E\big[\varphi(x_\varepsilon^\theta(s_1) \dots, x_\varepsilon^\theta(s_n))\big((Im[x_\varepsilon^\theta(t)] - Im[x_\varepsilon^\theta(s)])^2 - (t-s)\big)\big]$$

converge to zero when ε tends to zero. When $\theta = \pi$ we want to see that

$$E\big[\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))\big((x_\varepsilon^\theta(t) - x_\varepsilon^\theta(s))^2 - 2(t-s)\big)\big]$$

converges to zero when ε tends to zero. But,

$$\begin{aligned}
&E(\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \cos(\theta N_x)dx)^2) \\
=&2\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n)) \cos(\theta N_{x_1}) \cos(\theta N_{x_2})\big) dx_1 dx_2 \\
=&\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E(\cos(\theta(N_{x_2} - N_{x_1})))dx_1 dx_2 E[\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))] \\
+&\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n)) \cos(\theta(N_{x_2} + N_{x_1}))\big) dx_1 dx_2 \\
=&E[\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))] Re\big[\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E(e^{i\theta(N_{x_2} - N_{x_1})}) dx_1 dx_2\big] \\
+&Re\big[\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n)) e^{i\theta(N_{x_2} + N_{x_1})}\big) dx_1 dx_2\big].
\end{aligned} \tag{3.2}$$

If $\theta = \pi$ this expression is equal to

$$2E[\varphi(x_\varepsilon^\theta(s_1), \dots, x_\varepsilon^\theta(s_n))] Re\big[\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} E(e^{i\theta(N_{x_2} - N_{x_1})}) dx_1 dx_2\big]$$

and using that P_ε^θ converges weakly to P^θ and Lemma 3.2 we know that this expression converges to $2(t-s)E[\varphi(X_{s_1}, \dots, X_{s_n})]$. This completes the proof when $\theta = \pi$. Thus, we can suppose now that $\theta \in (0,\pi) \cup (\pi, 2\pi)$. In this case, using again Lemma 3.2 and that P_ε^θ converge weakly to P^θ we obtain that (3.2) converges to $(t-s)E[\varphi(X_{s_1},\dots,X_{s_n})]$. Similarly,

$$
\begin{aligned}
&E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\sin(\theta N_x)dx)^2\big)\\
&=2\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))\sin(\theta N_{x_1})\sin(\theta N_{x_2})\big)dx_1dx_2\\
&=\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E(\cos(\theta(N_{x_2}-N_{x_1})))dx_1dx_2E[\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))]\\
&-\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))\cos(\theta(N_{x_2}+N_{x_1}))\big)dx_1dx_2;
\end{aligned}
$$

and this expression converges to $(t-s)E[\varphi(X_{s_1},\cdots,X_{s_n})]$.

Finally we have to prove that $< Re[X], Im[X] >_t = 0$. It suffices to prove that for any $s_1 \le s_2 \le \cdots \le s_n \le s$ and for any bounded continuous function $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$,

$$
E\big[\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))(Re[x_\varepsilon^\theta(t)]-Re[x_\varepsilon^\theta(s)])(Im[x_\varepsilon^\theta(t)]-Im[x_\varepsilon^\theta(s)])\big],
$$

converges to zero when ε tends to zero.

But this expression is equal to

$$
\begin{aligned}
&E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\cos(\theta N_x)dx)(\varepsilon\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\sin(\theta N_x)dx)\big)\\
=\;&\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))\cos(\theta N_{x_1})\sin(\theta N_{x_2})\big)dx_1dx_2\\
+\;&\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))\sin(\theta N_{x_1})\cos(\theta N_{x_2})\big)dx_1dx_2\\
=\;&\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))\sin(\theta(N_{x_2}+N_{x_1}))\big)dx_1dx_2\\
=\;&Im\big[\varepsilon^2\int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}}\int_{\frac{2s}{\varepsilon^2}}^{x_2} E\big(\varphi(x_\varepsilon^\theta(s_1),\dots,x_\varepsilon^\theta(s_n))e^{i\theta(N_{x_2}+N_{x_1})}\big)dx_1dx_2\big]
\end{aligned}
$$

and we have proved that this expression converges to zero when ε tends to zero in Lemma 3.2. □

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