3 THE ITÔ INTEGRAL

- Kiyoshi Itô → rigurous meaning to some differential equations driven by a B.m related with continuous time Markov process.
- Roughly speaking, analogue of classical calculus for stochastic process.
- In classical mathematical analysis, there are several extensions of the Riemann integral. However, before Itô's development, no theory of integration of random mappings with respect to nowhere differentiable random integrators existed.
- One of the motivations: the modeling of evolutions of phenomena with an external random forcing.. There also exist other external perturbations as processes with jumps.

In this course, we will consider "stochastic differential equations driven by a Brownian motion

$$\begin{cases} dX_t = \sigma(t, X_t)\, dB_t + b(t, X_t)\, dt, & t > 0, \\ X_0 = x_0 \in \mathbb{R}^d, \end{cases}$$

where $\{B_t, t \geq 0\}$ is a Brownian motion and σ and b are some (deterministic) functions.

The above equation is interpreted thought its integral form:

$$X_t = x_0 + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds \qquad (1)$$

In order to understand the meaning of (1), we need to make precise the type of integrals of the equation. First, we can consider the integral $\int_0^t b(s, X_s) ds$ defined ω by ω (pathwise). That means, since $\{X_t, t \geq 0\}$ is random, we can fix $\omega \in \Omega$ and set

$$\left(\int_0^t b(s, X_s)\, ds\right)(\omega) := \int_0^t b(s, X_s(\omega))\, ds,$$

assuming that the right-~~hand~~ side of this equality exists as a Lebesgue integral (for example).

The pathwise approach cannot be used to give a meaning to the other case.

3.1 Itô's integral

We now consider $\{B_t, t \geq 0\}$ a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We will also consider a (complete, right-continuous) filtration $(\mathcal{F}_t, t \geq 0)$ satisfying:

1) B is adapted to $(\mathcal{F}_t, t \geq 0)$, that is, for any $t \geq 0$, the random variable B_t is $\mathcal{F}_t$-measurable.

2) For any $0 \leq s \leq t$, the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$.

These two properties are satisfied if $(\mathcal{F}_t, t \geq 0)$ is the natural filtration associated to B.

Definition: We fix a finite time horizon T and define $L^2_{a,T}$ as the set of stochastic processes $u = \{u_t, t \in [0,T]\}$ satisfying the following conditions:

i) u is adapted and jointly measurable in (t, ω) with respect to the product σ-field $\mathcal{B}([0,T]) \otimes \mathcal{F}$, that is

1) for any $t \in [0,T]$, the random variable u_t is $\mathcal{F}_t$-measurab

2) the mapping $[0,T] \times \Omega \ni (t,\omega) \longmapsto u(t,\omega)$ is measurab with respect to the σ-field $\mathcal{B}([0,T]) \otimes \mathcal{F}$.

ii) $\int_0^T E(u_t^2)\,dt < +\infty$, that is, $u \in L^2([0,T] \times \Omega, \lambda \times P)$, where λ denotes Lebesgue measure.

Observation: $L^2_{a,T}$ is a Hilbert space with the norm

$$\|u\|_{L^2_{a,T}} = \left[\int_0^T E(u_t^2)\,dt \right]^{1/2}.$$

Itô's integral of step processes

Let $\mathcal{E}$ denote the subset of $L^2_{a,T}$ consisting of stochastic processes that can be written as

$$u_t = \sum_{j=1}^{n} u_j \mathbb{1}_{[t_{j-1}, t_j)}(t), \qquad (0')$$

with $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$ and where u_j, $j = 1, \dots, n$, are $\mathcal{F}_{t_{j-1}}$-measu rable square integrable random variables.

For step processes, the Itô stochastic integral is defined by the very natural formula

$$\int_0^T u_t \, dB_t = \sum_{j=1}^{m} u_j (B_{t_j} - B_{t_{j-1}}), \qquad (1).$$

that we can compare with Lebesgue integral of simple functions. Note that

$$\int_0^T u_s \, dB_t$$

is random.

We need some tools in order to extend this type of integral. We first prove some properties of (1)

<u>Proposition</u> (Isometry property). For any $u \in \mathcal{E}$,

$$E\left[\left(\int_0^T u_t \, dB_t\right)^2\right] = E\left(\int_0^T u_t^2 \, dt\right).$$

Consequently, the mapping

$$\mathcal{E} \ni u \longmapsto \int_0^T u_t \, dB_t$$

is continuous from $\mathcal{E} \subset L^2([0,T] \times \Omega, \lambda \times P)$ into $L^2(\Omega)$.

<u>Proof</u>. Developping the right-hand side of (*) yields

$$E\left[\left(\int_0^T u_t \, dB_t\right)^2\right] = \sum_{j=1}^{m} E(u_j^2 (\Delta_j B)^2) + 2 \sum_{j<k} E(u_j u_k (\Delta_j B)(\Delta_k B))$$

the measurability property of the random u_j, $j = 1, \dots, m$, implies that the random variables u_j^2 are independent of $(\Delta_j B)^2$. So

$$\sum_{j=1}^{m} E[u_j^2 (\Delta_j B)^2] = \sum_{j=1}^{m} E(u_j^2)(t_j - t_{j-1}) = \int_0^T E(u_t^2) \, dt.$$

Since $u_j u_k \Delta_j B$ and $\Delta_k B$, $j<k$, are independent

$$E[u_j u_k (\Delta_j B)(\Delta_k B)] = E[u_j u_k (\Delta_j B)] E(\Delta_k B) = 0.$$

$\blacksquare$

Other properties of the stochastic integral of step processes

1) the stochastic integral is a centered r.v..

First $L^2(\Omega) \subset L^1(\Omega)$

$$E\left(\int_0^T u_t\, dB_t\right) = E\left(\sum_{j=1}^{m} u_j (B_{t_j} - B_{t_{j-1}})\right) = \sum_{j=1}^{m} E(u_j) \underset{\text{indep.}}{E}(B_{t_j} - B_{t_{j-1}}) = 0.$$

2) Linearity. If u_t^1 and u_t^2 belong to $\mathcal{E}$ and $a, b \in \mathbb{R}$, then $a u_t^1 + b u_t^2 \in \mathcal{E}$ and

$$\int_0^T (a u^1 + b u^2)(t)\, dB_t = a \int_0^T u_t^1\, dB_t + b \int_0^T u_t^2\, dB_t.$$

First, we assume

$$u_t^1 = \sum_{j=1}^{m} u_j^1 \mathbb{1}_{[t_{j-1}, t_j)}(t), \quad u_t^2 = \sum_{j=1}^{m} u_j^2 \mathbb{1}_{[t_{j-1}, t_j)}(t),$$

where u_j^1 and u_j^2 as before. Then,

$$v_t = a u_t^1 + b u_t^2 = \sum_{j=1}^{m} (a u_j^1 + b u_j^2) \mathbb{1}_{[t_{j-1}, t_j)}(t).$$

Indefinite Itô integral of step processes

The indefinite Itô stochastic integral of a process $u \in \mathcal{E}$ is defined as follows:

$$\int_0^t u_s\, dB_s := \int_0^T u_s \mathbb{1}_{[0,t]}(s)\, dB_s, \quad t \in [0, T]$$

From (1) and taking into account that u has the representation (0'), we have that

$$\int_0^t u_s\, dB_s = \int_0^T u_s \mathbb{1}_{[0,t]}(s)\, dB_s = \sum_{j;\, t_j \le t} u_j (B_{t_j \wedge t} - B_{t_{j-1} \wedge t})$$

<u>Proposition</u>: For any $u \in \mathcal{E}$, the process $\{I_t = \int_0^t u_s\, dB_s,\ t \in [0,T]\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \ge 0\}$ associated to the Brownian motion.

<u>Proof</u>: Let u_t, $t \in [0,T]$, be defined by (0'). Fix $0 \le s \le t \le T$ and assume first that $t_{k-1} < s \le t_k \le t_\ell < t \le t_{\ell+1}$.

then we have

$$I_s = \sum_{j=0}^{k-1} u_j(B_{t_j} - B_{t_{j-1}}) + u_k(B_s - B_{t_{k-1}})$$

$$I_t = I_s + \sum_{j=k+1}^{\ell} u_j(B_{t_j} - B_{t_{j-1}}) + u_{\ell+1}(B_t - B_\ell),$$

and therefore

$$I_t - I_s = \sum_{j=k+1}^{\ell} u_j(B_{t_j} - B_{t_{j-1}}) + u_{\ell+1}(B_t - B_{t_\ell}) + u_k(B_{t_k} - B_s) \quad (*)$$

Our first aim is to prove that $E(I_t - I_s | \mathcal{F}_s) = 0$. Since the conditional expectation is a linear operator, it will be proved by conditioning each term on the right-hand of $(*)$ and checking that they all are zero.

Indeed, the random variable u_k is $\mathcal{F}_{t_{k-1}}$-measurable and then $\mathcal{F}_s$-measurable ($\mathcal{F}_{t_{k-1}} \subset \mathcal{F}_s$). Because the increment $B_{t_k} - B_s$ is independent of $\mathcal{F}_s$, by applying the factorization property of the conditional expectation, we obtain

$$E(u_k(B_{t_k} - B_s)|\mathcal{F}_s) = u_k E(B_{t_k} - B_s | \mathcal{F}_s) = u_k E(B_{t_k} - B_s) = 0.$$

Now, for $j \in \{k+1, \dots, \ell\}$,

$$E(u_j(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_s) = E(E(u_j(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{j-1}})|\mathcal{F}_s)$$

$$= E[(u_j E(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{j-1}})|\mathcal{F}_s] = E[(u_j E(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_s) = 0,$$

where we have used that u_j is $\mathcal{F}_{t_{j-1}}$-measurable and $B_{t_j} - B_{t_{j-1}}$ is independent of $\mathcal{F}_{t_{j-1}}$. Using the same argument,

$$E(u_{\ell+1}(B_t - B_{t_\ell})|\mathcal{F}_s) = 0.$$

Putting toghether of these equalities we obtain the desired result. To finish we should study the cases $t_{k-1} < s \le t < t_k$ and $t_{k-1} < s \le t_k < t \le t_{k+1}$.

$\blacksquare$

Extension of the Itô's integral of step processes

~~With the~~ definition (1) in mind, previous results and the linearity property, we establish a linear continuous mapping and isometry,

$$I_T : \mathcal{E} \subset L^2_{a,T} \longrightarrow L^2(\Omega)$$
$$u \longrightarrow I_T(u) = \int_0^T u_t \, dB_t = \sum_{j=1}^{m} u_j (B_{t_j} - B_{t_{j-1}}).$$

Recall that the space $L^2_{a,T}$ is a Hilbert space subset of $L^2([0,T]\times\Omega, N\times P)$ and that the scalar product in $L^2_{a,T}$ is

$$\langle u, v \rangle_{L^2_{a,T}} = \int_0^T E(u_t v_t)\, dt, \quad u, v \in L^2_{a,T}. \qquad (*)$$

Denote by $\bar{\mathcal{E}}$ the closure of $\mathcal{E}$ in $L^2_{a,T}$ with respect to the norm derived of $(*)$. Since $L^2_{a,T}$ is complete with respect to this norm (Hilbert space), we have that $\bar{\mathcal{E}} \subseteq L^2_{a,T}$. The converse inclusion is also true. This is proved in the following proposition, which is fundamental in Itô's theory.

Proposition: For any $u \in L^2_{a,T}$, there exists a sequence $\{u^n, n \geq 1\} \subset \mathcal{E}$ such that

$$\lim_{n\to+\infty} \int_0^T E(u_t^n - u_t)^2 dt = 0. \qquad (**)$$

Proof: We divide the proof into three steps.

Step 1: Assume that $u \in L^2_{a,T}$ is bounded and has continuous sample paths, a.s. We define the approximation sequence

$$u^n(t) = \sum_{k=0}^{[nT]} u\left(\frac{k}{n}\right) \mathbb{1}_{\left[\frac{k}{n}, \frac{k+1}{n}\right)}(t), \quad n \geq 1,$$

with the convention $\frac{[nT]+1}{n} = T$. Since the process u is bounded and adapted, $u^n \in L^2_{a,T}$. Furthermore, by the continuity of the sample paths of u,

$$\int_0^T |u^n(t) - u(t)|^2 dt = \sum_{k=0}^{[nT]} \int_{k/n}^{\frac{k+1}{n}\wedge T} \left|u\left(\frac{k}{n}\right) - u(t)\right|^2 dt$$

$$\leq T \sup_k \sup_{t\in\Delta_k} |u^n(t) - u(t)|^2 \to 0, \text{ as } n\to\infty, \text{ a.s.}$$

The approximation of this proposition $(**)$ follows by the bounded convergence theorem on the measure space $L^1(\Omega, \mathcal{F}, P)$ applied to the sequence of random variables $Y_n \in L^1(\Omega, \mathcal{F}, P)$,

$$Y_n = \int_0^T |u_n(t) - u(t)|^2 dt.$$

Indeed, we have checked that $\lim_{n\to+\infty} Y_n = 0$, and by assumptions on u $\sup_{n\geq 1} |Y_n| \leq Y$ with $Y \in L^1(\Omega, \mathcal{F}, P)$. Thus, $\{Y_n, n\geq 1\}$ converges in $L^1(\Omega)$ to zero, that is

$$E(|Y_n - 0|) = E\left(\int_0^T |u_n(t) - u(t)|^2\right) \longrightarrow 0, \text{ as } n \to +\infty.$$

Step 2. Assume that $u \in L^2_{a,T}$ is bounded (but not necessarily continuous). For any $n\geq 1$, let $\psi_n(t) = n \mathbb{1}_{[0,\frac{1}{n}]}(t)$. The sequence $\{\psi_n, n\geq 1\}$ is an approximation of the identity. Define

It's wrong

$$u^n(t) := (\psi_n * u)(t) = \int_{-\infty}^{+\infty} \psi_n(t-s) u(s)\, ds = \int_{-\infty}^{\infty} n \mathbb{1}_{[0,\frac{1}{n}]}(t-s) u(s)\, ds$$

$$= \int_{t-\frac{1}{n}}^{t} u(s)\, ds,$$

where in the last integral we put $u(r) = 0$ if $r<0$. So, we have defined a stochastic process u^n with continuous and bounded sample paths, a.s. and by the properties of the convolution, we have

$$\int_0^T |u^n(s) - u(s)|^2 ds \longrightarrow 0, \text{ a.s.}$$

Now, as in the preceding step, by bounded convergence theorem we conclude.

Step 3. Let $u \in L^2_{a,T}$ and define the sequence of truncated processes

$$u^n(t) = \begin{cases} 0, & u(t) < -n \\ u(t), & -n \leq u(t) \leq n, \\ 0, & u(t) > n. \end{cases}$$

Clearly, $\sup_{\omega, t} |u^n(t)| \leq n$, and $u^n \in L^2_{a,T}$. Moreover,

$$E\int_0^T |u^n(s) - u(s)|^2 ds = E\int_0^T |u(s)|^2 \mathbb{1}_{\{|u(s)|>n\}}\, ds,$$

and this tends to zero as $n\to+\infty$. Indeed, it is known that for a function $f \in L^1([0,T]\times\Omega, dN\times P)$,

$$\lim \int^T E(|f(t)| \mathbb{1}_{\dots}) dt = \dots$$

Definition: the Itô stochastic integral of a process $u \in L^2_{a,T}$ is

$$\int_0^T u_t \, dB_t := L^2(\Omega)-\lim_{n\to\infty} \int_0^T u^n_t \, dB_t \quad , \; u^n \in \mathcal{E}.$$

In order to make sense, we need to ensure that if u is approximated by two different sequences $u^{n,1}$ and $u^{n,2}$, this definition of the stochastic integral coincides. The isometry property implies that

$$E\left[\left(\int_0^T u_t^{n,1} dB_s - \int_0^T u_t^{n,2} dB_s\right)^2\right] = E\left[\left(\int_0^T (u_t^{n,1} - u_t^{n,2}) dB_s\right)^2\right]$$

$$= \int_0^T E(u_t^{n,1} - u_t^{n,2})^2 dt \le 2\int_0^T E(u_t^{n,1} - u_t)^2 dt + 2\int_0^T E(u_t^{n,2} - u_t)^2 dt \longrightarrow 0, \text{ as } n \to \infty.$$

It is not difficult to show that

$$\| I^1(u) - I^2(u) \|_{L^2(\Omega)} \Rightarrow 0,$$

using the triangle inequality.

Properties of the stochastic integral of processes of $L^2_{a,T}$.

For any $u \in L^2_{a,T}$ we use the notation $I(u) = \int_0^T u_t \, dB_t$, with the last definition.

a) $I(u)$ satisfies the isometry property

$$E\left[I(u)^2\right] = E\left(\int_0^t u_t^2 dt\right) = \|u\|^2_{L^2(\Omega\times[0,T])}$$

b) Stochastic integrals are centered random variables, $E\, I(u) = 0$

c) $L^2_{a,T} \ni u \longmapsto I(u)$ is a linear operator. That is, for any $u, v \in L^2_{a,T}$, $a, b \in \mathbb{R}$

$$\int_0^T (a u_t + b v_t) dB_t = a\int_0^T u_t \, dB_t + b\int_0^T v_t \, dB_t.$$

Proof of the properties:

a) Let $\{u^n, n \ge 1\} \subset \mathcal{E}$ be an approximation sequence of u in the sense of the previous proposition. Then

$$\|I(u)\|_{L^2(\Omega)} = \lim_n \|I(u^n)\|_{L^2(\Omega)} = \lim_n \|u^n\|_{L^2(\Omega\times[0,T])} = \lim_n \|u\|_{L^2(\Omega\times[0,T])}$$

b) Follow with the same notation. Since the convergence implies the convergence in $L^1(\Omega)$, we have

$$\| I(u) \|_{L^1(\Omega)} = \lim_n \| I(u^n) \|_{L^1(\Omega)} = 0.$$

c) Linearity of the stochastic integral for processes in $L^2_{a,T}$ follows from the same property for processes in $\mathcal{E}$.

Except for very particular cases, stochastic integrals cannot be computed explicitly. We give an exemple where we can do it.

EXEMPLE :

For the Brownian motion B,

$$\int_0^T B_t \, dB_t = \frac{1}{2}(B_T^2 - T)$$

Let us remark that we would rather expect "$\int_0^T B_t dB_t = \frac{1}{2} B_T^2$" by analogy with rules of deterministic calculus. We consider a particular approximation process based on the partition $\{ \frac{jT}{n}, j = 0, \ldots, n \}$.

$$u_t^n = \sum_{j=1}^n B_{t_{j-1}} \mathbb{1}_{(t_{j-1}, t_j]}(t), \quad \text{with } t_j = \frac{jT}{n}.$$

Clearly $u^n \in L^2_{a,T}$. So,

$$\int_0^T E(u_t^n - B_t)^2 dt = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} E(B_{t_{j-1}} - B_t)^2 dt \leq \frac{T}{n} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} dt = \frac{T^2}{n}.$$

Therefore, $\{u^n, n \geq 1\}$ is an approximating sequence of B in the norm of $L^2(\Omega \times [0,T])$. According to the definition

$$\int_0^T B_t \, dB_t = \lim_n \sum_{j=1}^n B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}) \text{ in } L^2(\Omega).$$

Fix $\omega \in \Omega$ and using that $(a-b)^2 = a^2 + b^2 - 2ab$ with $a = B_{t_j}(\omega)$ and $b = B_{t_{j-1}}(\omega)$, then

$$\sum_{j=1}^n B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}) = \frac{1}{2} \sum_{j=1}^n (B_{t_j}^2 - B_{t_{j-1}}^2) - \frac{1}{2} \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}})^2$$

$$= \frac{1}{2} B_T^2 - \frac{1}{2} \sum_{j=1}^n (B_{t_j} - B_{t_{j-1}})^2, \tag{3.2}$$

and we finish using the proposition on the quadratic variation of Bm.

3.2 THE ITÔ INTEGRAL AS A STOCHASTIC PROCESS

The indefinite Itô stochastic integral of $u \in L^2_{a,T}$ is defined by

$$I_t(u) = \int_0^t u_s \, dB_s := \int_0^T u_s \mathbb{1}_{[0,t]}(s) \, dB_s, \quad t \in [0,T]. \qquad (3.2.1)$$

It makes sense, since for any $t \in [0,T]$, $\{u_s \mathbb{1}_{[0,t]}(s), s \in [0,T]\}$ belongs to $L^2_{a,T}$.
The properties of isometry, zero mean and linearity also hold for the infinite integral.
In this section we will give important properties but not rigorous and complete proofs. We would like to give a general vision of the Itô integral.

Proposition (Martingale property).
The process $\{I_t = \int_0^t u_s \, dB_s, t \in [0,T]\}$ is a martingale with respect to the natural filtration associated with B.m.
Proof:
From a previous proposition, it is true for $u \in \mathcal{E}$. For the general case, if $\{M_t^n, t \in [0,T]\}, n \geq 1$, is a sequence of martingales with respect to a given filtration $\{\mathcal{G}_t, t \in [0,T]\}$ such that for each $t \in [0,T]$, there exists a random variable M_t satisfying

Not very rigorous

$$L^1(\Omega) - \lim_n M_t^n = M_t,$$

so, the process $\{M_t, t \in [0,T]\}$ is also a martingale with respect to $\{\mathcal{G}_t, t \in [0,T]\}$.
$\blacksquare$

Proposition (Quadratic variation of the stochastic integral)
For any process $u \in L^2_{a,T}$, the stochastic process

$$\left(\int_0^t u_s \, dB_s\right)^2 - \int_0^t u_s^2 \, ds, \quad t \in [0,T],$$

is a martingale with respect to the natural filtration associated to the B.m.
By analogy with the B.m where $\{B_t^2 - t, t \in [0,T]\}$ is a martingale, the process $\{\int_0^t u_s^2 ds, t \in [0,T]\}$ is called the quadratic variation of the stochastic integral. With the same conditions on u,

we can check that

$$L^1(\Omega)-\lim_n \sum_{j=1}^n \left(\int_{t_{j-1}}^{t_j} u_s \, dB_s\right)^2 = \int_0^t u_s^2 \, ds ,$$

where the sum refers to a sequence of partitions of $[0,t]$ whose mesh tends to zero.

Burkhölder's inequality

The isometry property of the stochastic integral can be extended in the following sense. Let $p \in [2,+\infty)$, there exists a positive constant $C(p)$ depending on p s.t.

$$E\left(\int_0^t u_s \, dB_s\right)^p \leq C(p) E\left(\int_0^t u_s^2 \, ds\right)^{p/2}$$

Hölder continuity of the indefinite stochastic integral

A combination of Burkholder's inequality and Kolmogorov continuity criterion allows to deduce it. ~~Indeed~~, assume that $\int_0^T E(u_r)^p dr < +\infty$ for any $p \in [2,\infty)$. Using de last inequality implies

$$E\left(\int_s^t u_r dB_r\right)^p \leq C(p) E\left(\int_s^t u_r^2 \, dr\right)^{p/2} \leq C(p) |t-s|^{\frac{p}{2}-1} \int_s^t E(u_r)^p dr$$

$$\leq C(p) |t-s|^{\frac{p}{2}-1}.$$

$1+\frac{p}{2}-2$

$\frac{1}{2}-\frac{2}{p}$

Since $p \geq 2$ is arbitrary, we have that the sample paths of $\{\int_0^t u_s dB_s, t \in [0,T]\}$ are γ-Hölder continuous with $\gamma \in (0,\frac{1}{2})$.

3.3. AN EXTENSION OF ITÔ INTEGRAL

Previously, we have introduced the set $L^2_{a,T}$ in order to define the stochastic integral for processes of this class with respect to the B.m. In this section, we shall consider a larger class of integrands.

Let $\Lambda^2_{a,T}$ be the set of real valued processes u adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$ jointly measurable in (t,ω) with respect to the product σ-field $\mathcal{B}([0,T]) \times \mathcal{F}$ and satisfying

$$P\left(\int_0^T u_t^2 \, dt < \infty\right) = 1$$

Clearly, $L^2_{a,T} \subset \Lambda^2_{a,T}$ because of

$$E\left(\int_0^T u_t^2 dt\right) < +\infty \Rightarrow \int_0^T u_t^2 dt < +\infty, \text{ a.s.}$$

Our aim is to define the stochastic integral for processes in $\Lambda^2_{a,T}$. It will be similar to section 3.1. We start with step processes $\{u^m, n \geq 1\}$ as in (0')

$$u_t^m = \sum_{j=1}^{m} u_j \mathbb{1}_{[t_{j-1}, t_j)}(t). \qquad (3.2.2)$$

where $0 \leq t_0 < t_1 < \dots < t_n = T$ and u_j being $\mathcal{F}_{t_{j-1}}$-measurable and finite almost surely. Hence, $u^m \in \Lambda^2_{a,T}$. For this class of integrals, the stochastic integral is defined as in section 3.1

$$\int_0^T u_t \, dB_t = \sum_{j=1}^{M} u_j (B_{t_j} - B_{t_{j-1}}).$$

The extension to processes in $\Lambda^2_{a,T}$ needs two ingredients.

Proposition:
Let $u \in \Lambda^2_{a,T}$. There exists a sequence of step processes $\{u^m, n \geq 1\}$ of the form (3.2.2) belonging to $\Lambda^2_{a,T}$ s.t.

$$\lim_n \int_0^T |u_t^n - u_t|^2 dt = 0 \quad \text{in probability} \qquad (3.2.3)$$

Proof [Baldi].

Proposition:
Let u be a step processes in $\Lambda^2_{a,T}$. Then, for any $\varepsilon > 0$, $N > 0$,

$$P\left\{\left|\int_0^T u_t \, dB_t\right| > \varepsilon\right\} \leq P\left\{\int_0^T u_t^2 dt > N\right\} + \frac{N}{\varepsilon^2}.$$

Proof: [Sanz-Sole].

Extension of the stochastic integral
Fix $u \in \Lambda^2_{a,T}$ and consider $\{u^m, n \geq 1\}$ a sequence of step processes as in (3.2.2) belonging to $\Lambda^2_{a,T}$ s.t. (3.2.3) is true. By the previous proposition applied to the step process $u_t^n - u_t^m$, for any $\varepsilon > 0$, $N > 0$ we have

$$P\left\{\left|\int_0^T (u_t^n - u_t^m)\,dB_t\right| > \varepsilon\right\} \le P\left\{\int_0^T (u_t^n - u_t^m)^2\,dt > N\right\} + \frac{N}{\varepsilon^2}.$$

Using (3.2.3), we can choose ε such that for any $N>0$ and n, m big enough

$$P\left\{\left|\int_0^T (u_t^n - u_t^m)^2\,dt > N\right\} \le \frac{\varepsilon}{2}.$$

In particular, we may take N small enough so that $\frac{N}{\varepsilon^2} \le \frac{\varepsilon}{2}$ and obtain, for any n, m big enough

$$P\left\{\left|\int_0^T (u_t^n - u_t^m)\,dB_t\right| > \varepsilon\right\} \le \varepsilon.$$

this proves that the sequence of stochastic integrals of step processes

$$\left\{\int_0^T u_t^n\,dB_t,\ n \ge 1\right\} \qquad (A)$$

is Cauchy for the convergence in probability.

the space $L^0(\Omega)$ of classes of finite random variables (a.s.) endowed with the convergence in probability is a complete metric space. For example, the distance

$$d(X,Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right)$$

metrizes the convergence in probability and is such that $L^0(\Omega)$ endowed with d is a complete metric space. Hence, (A) does have a limit in probability, then we can define

$$\int_0^T u_t\,dB_t := P\text{-}\lim_n \int_0^T u_t^n\,dB_t.$$

It is not difficult to check that it does not depend on the approximation used.

3.4 STRATONOVICH INTEGRAL: A BRIEF INTRODUCTION

One of the problems of the Itô's stochastic integral is its lack of robustness with respect to approximations. Assume that we "naively" want to give a meaning to the stochastic integral $\int_0^T B_t \, dB_t$, where $\{B_t, t \geq 0\}$ is a B.m. Since almost all sample paths of B are continuous, one can think of using equally one of the approximating process

$$B_t^{n,1} = \sum_{j=1}^{n} B_{t_{j-1}} \mathbb{1}_{[t_{j-1}, t_j)}(t),$$

$$B_t^{n,2} = \sum_{j=1}^{n} B_{t_j} \mathbb{1}_{[t_{j-1}, t_j)}(t),$$

and define

$$\int_0^T B_t \, dB_t = \lim_{n} \sum_{j=1}^{n} B_{t_j^*} (B_{t_j} - B_{t_{j-1}}),$$

where t_j^* is either the point t_{j-1} or t_j and the limit is taken in one of the usual convergences in probability theory for example, in L^2 or in probability.

However, the limit will depend on the particular choice of t_j^*. Indeed, Case I, $t_j^* = t_{j-1}$,

$$E(I^{n,1}) = E\left[\sum_{j=1}^{n} B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}})\right] \underset{\text{independence}}{=} \sum_{j=1}^{n} E[B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}})] = 0.$$

Case II, $t_j^* = t_j$,

$$E(I^{n,2}) = E\left[\sum_{j=1}^{n} B_{t_j} (B_{t_j} - B_{t_{j-1}})\right] = \sum_{j=1}^{n} E[B_{t_j} (B_{t_j} - B_{t_{j-1}})]$$

$$= \sum_{j=1}^{n} (t_j - t_{j-1}) = T.$$

The two convergences above tell us that, if for example we expect to construct the stochastic integral as a limit in $L^2(\Omega)$, this limit will depend on whether we take as approximation of the integrand. This is a bad sign.

Proposition. Consider a sequence of partitions

$$\Pi_n = \{0 = t_0 < t_1 < \cdots < t_{k_n} = T\},\ n \geq 1.$$

For any $\lambda \in [0,1]$, set $t^{\lambda}_{n,j} = \lambda t_j + (1-\lambda) t_{j-1}$. Then,

$$I(\lambda) := L^2\text{-}\lim_{|\Pi_n| \to 0} \sum_{j=1}^{k_n} B_{t^{\lambda}_{n,j}} (B_{t_j} - B_{t_{j-1}}) = \int_0^T B_t \, dB_t + \lambda T,$$

where $\int_0^T B_t \, dB_t$ is the Itô's stochastic integral.

So, from the computations in previous example, we deduce

$$I(\lambda) = \frac{1}{2}(B_T^2 + (2\lambda - 1)T).$$

Remark: taking $\lambda = \frac{1}{2}$, we obtain

$$I\left(\tfrac{1}{2}\right) = \frac{1}{2} B_T^2 := \int_0^T B_t \circ dB_t,$$

where $\int_0^T B_t \circ dB_t$ refers to the Stratonovich integral