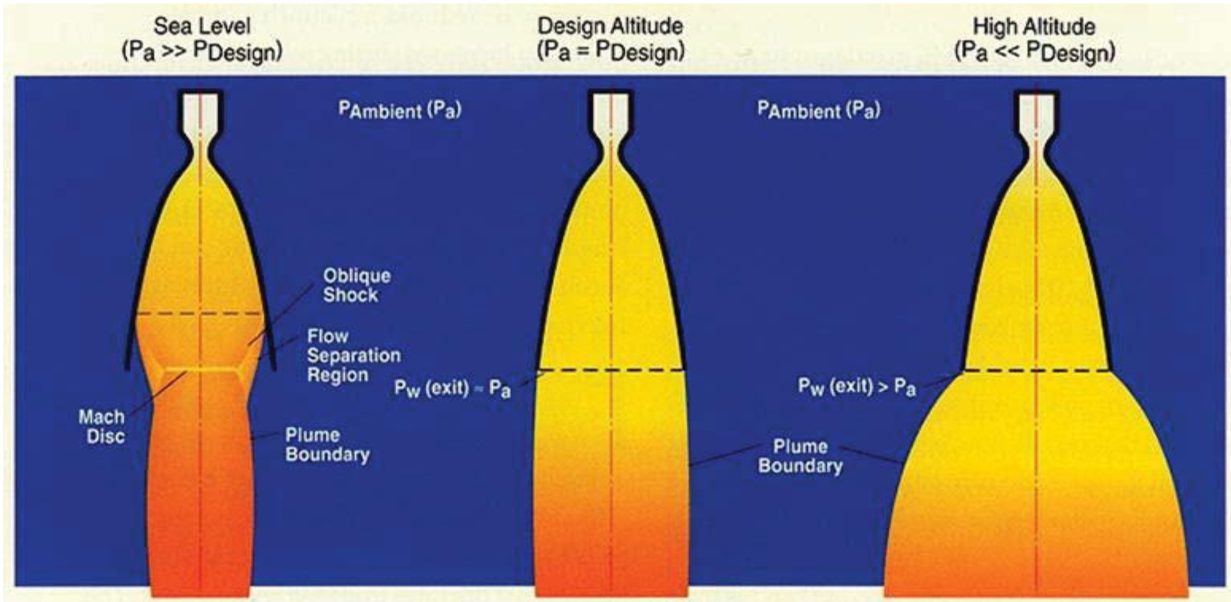
Escoamento compressível
Cálculo de Nozzles de foguetes



Francisco Brito

francisco@dem.uminho.pt

Laboratório de Mobilidade e Termodinâmica Aplicada

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Simplificações: Gás Ideal

- Partículas pontuais
- não há interação entre elas: Efeito global é a soma do efeito de cada uma em geral,
- aproximação aceitável para:
  - T ~ elevado
  - P ~ baixa

$$pV = nRT$$

$$pV = m\frac{R}{M}T$$

$$pV = m\overline{R}T$$

$$p\upsilon = \overline{R}T$$

$$p = \rho\overline{R}T$$

$$M = \frac{m}{n}$$

$$\overline{R} = \frac{R}{M}$$

$$\upsilon = \frac{V}{m}$$

$$\rho = \frac{m}{V} = \frac{1}{\upsilon}$$

$p \; [Pa] \equiv \left[N/m^2 \right]$

$V \; [m^3]$

$T \; [K]$

$m \; [kg]$

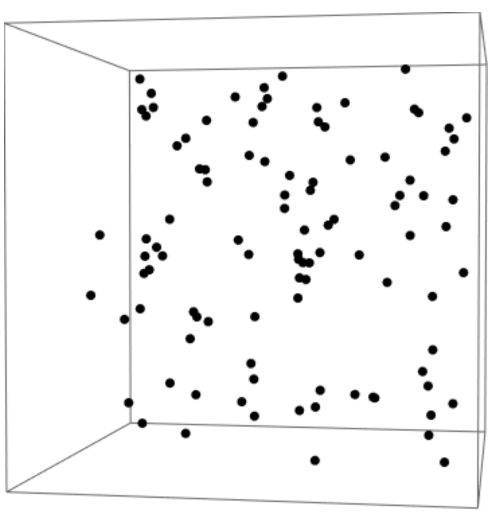
$\upsilon \; \left[m^3/kg \right]$

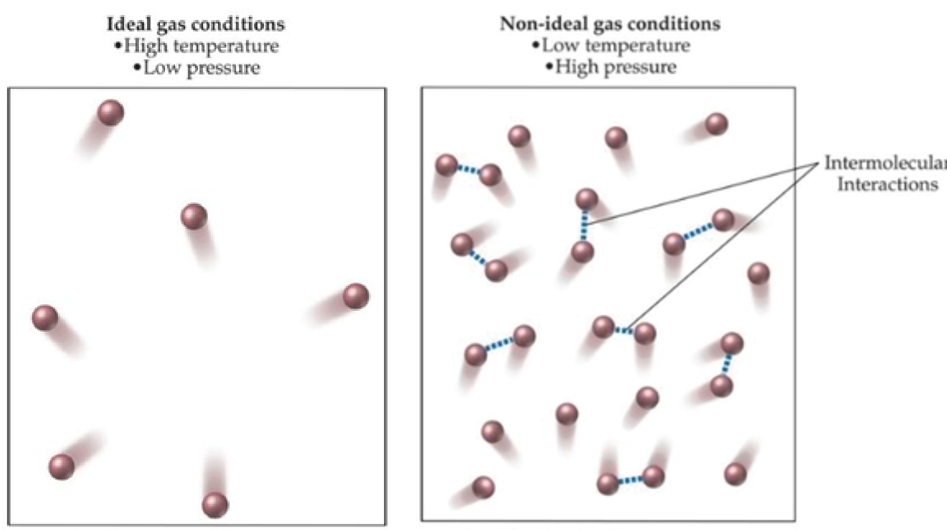
$n \; [kmol]$

$R \; \left[= 8314 \; J/kmol \cdot K \right]$

$\overline{R} \; \left[J/kg \cdot K \right]$

$M \; \left[kg/kmol \right]$





Simplificações: cap. caloríficas constantes, isentrópico

- Energia interna, *U* e Entalpia, *H*, capacidades caloríficas constantes:

$$dU = m c_v dT \qquad c_v = \left(\frac{\delta q}{dT}\right)_v \quad e \quad c_p = \left(\frac{\delta q}{dT}\right)_p \qquad u = {}^{U}/_{m} \qquad \gamma = \frac{c_p}{c_v}$$

$$dH = m c_p dT \qquad c_v = \frac{du}{dT} \qquad c_p = \frac{dh}{dT} \qquad h = {}^{H}/_{m} \qquad \bar{R} = \frac{R}{M} = c_p - c_v$$

- Processos isentrópicos: “perfeitos”, sem perdas térmicas

$$pV^{\gamma} = Cte \qquad TV^{\gamma-1} = Cte \qquad Tp^{\frac{1-\gamma}{\gamma}} = Cte$$

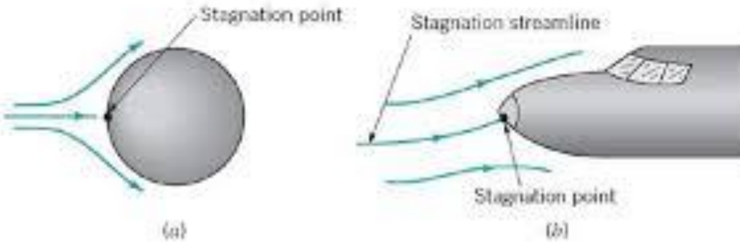
- Velocidade do som com gás ideal, e nº de Mach (gás perfeito)

$$a_0 = \sqrt{\gamma \bar{R} T_0} \quad ; \quad M_0 = \frac{V_0}{a_0}$$

Análise das relações de estagnação

- Equação de Bernoulli: na ausência de ganhos ou perdas, a pressão total do escoamento mantém-se. É composta da pressão estática, p_e, e da pressão dinâmica, q. Também é designada por pressão de estagnação, p_0

$$p_0 = p_e + q = p_e + \frac{1}{2}\rho V^2 = cte$$



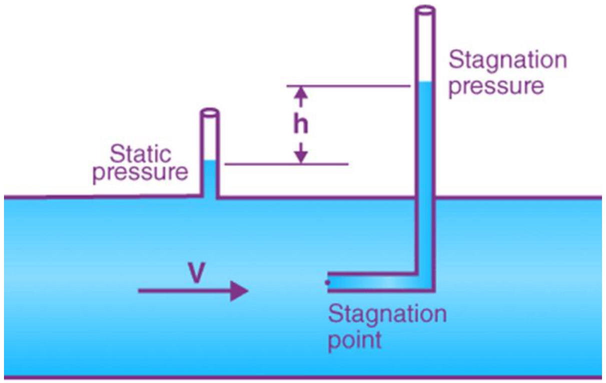
Stagnation points on bodies in flowing fluids.

- A pressão dinâmica é uma forma de ver a energia cinética do escoamento em unidades de pressão. É a pressão que o escoamento ganharia com a redução da sua velocidade até a zero (conversão integral da energia cinética em energia de pressão). Pode ser obtida de várias formas:

$$q = \frac{1}{2}\rho V^2$$

- Estagnação: situação em que o escoamento não possui energia cinética por ainda não ter sido acelerado ou por já ter sido desacelerado e perdido toda a sua energia cinética. Nestas condições, a pressão dinâmica é nula e a estática iguala a pressão total / de estagnação

$$p_0 = p_e$$



- As condições de estagnação são também caraterizadas pela a temperatura de estagnação T_0 ou a massa volúmica de estagnação ρ_0.

Objectives

- Develop the **general relations for compressible flows** encountered when gases flow at high speeds.
- Introduce the concepts of ***stagnation*** *state*, *speed of sound*, and *Mach number* for a compressible fluid.
- Develop the **relationships between the static and stagnation** fluid properties for **isentropic flows** of ideal gases.
- Derive the relationships between the static and stagnation fluid properties as functions of specific-heat ratios and Mach number.
- Derive the **effects of area changes** for one-dimensional isentropic subsonic and supersonic flows.
- Solve problems of isentropic flow through **converging and converging–diverging nozzles**.
- Discuss the **shock wave** and the variation of flow properties across the shock wave.
- Develop the concept of duct flow with heat transfer and negligible friction known as *Rayleigh flow*.
- Examine the operation of steam nozzles commonly used in steam turbines.

STAGNATION PROPERTIES

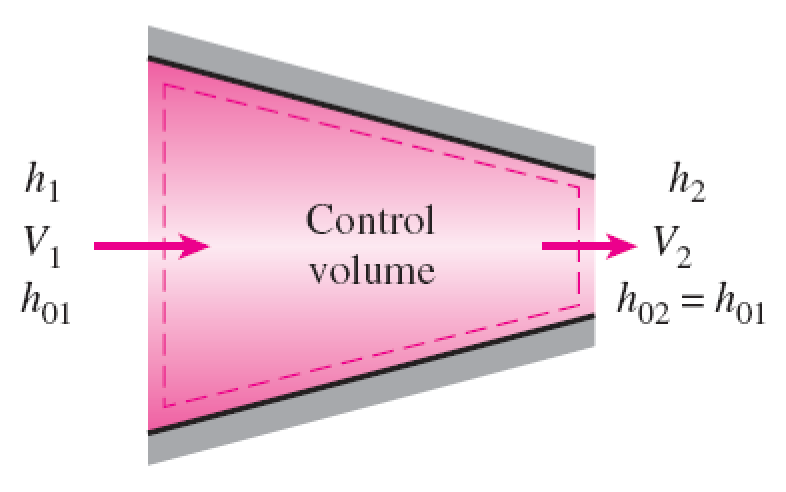
Stagnation (or **total**) **enthalpy**

$$h_0 = h + \frac{V^2}{2} \qquad \text{(kJ/kg)}$$

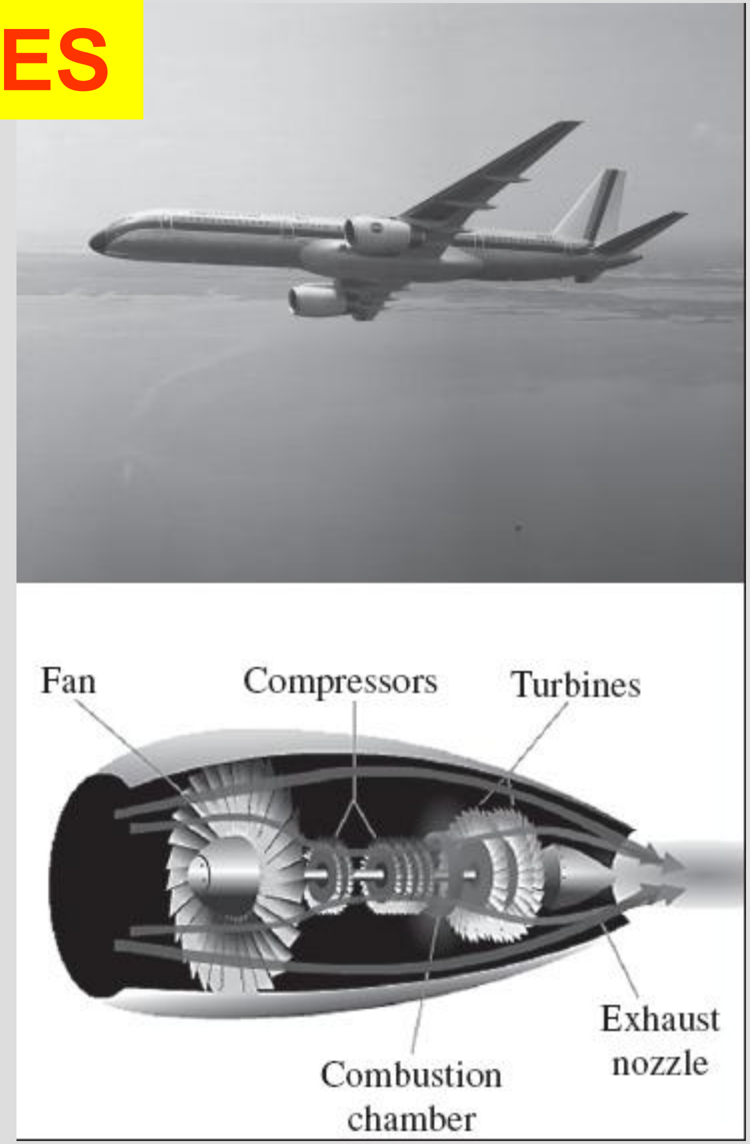
Static enthalpy: the ordinary enthalpy *h*

Energy balance (with no heat or work interaction, no change in potential energy):

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \rightarrow h_{01} = h_{02}$$



Steady flow of a fluid through an adiabatic duct.



Aircraft and jet engines involve high speeds, and thus the kinetic energy term should always be considered when analyzing them.

If the fluid were brought to a complete stop, the energy balance becomes

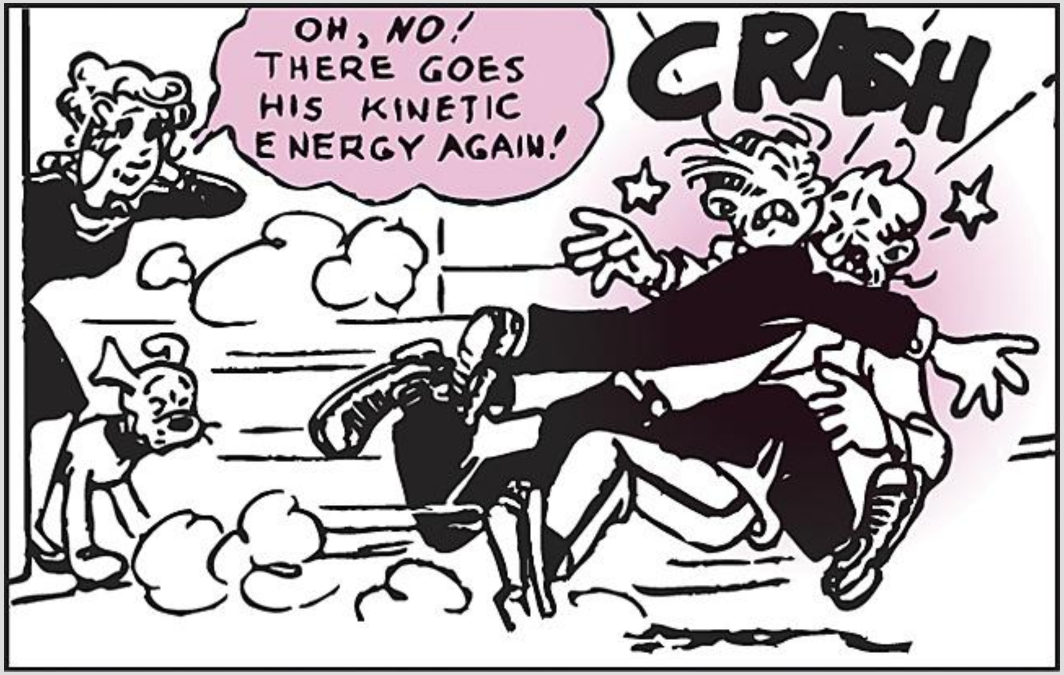
$$h_1 + \frac{V_1^2}{2} = h_2 = h_{02}$$

Stagnation enthalpy: The *enthalpy of a fluid when it is brought to rest adiabatically.*

During a stagnation process, **the kinetic energy of a fluid is converted to enthalpy**, which results in an increase in the fluid temperature and pressure.

The properties of a fluid at the stagnation state are called **stagnation properties** (stagnation temperature, stagnation pressure, stagnation density, etc.).

The stagnation state is indicated by the subscript 0.



Kinetic energy is converted to enthalpy during a stagnation process.

Isentropic stagnation state: When the stagnation process is **reversible** as well as adiabatic (i.e., isentropic).

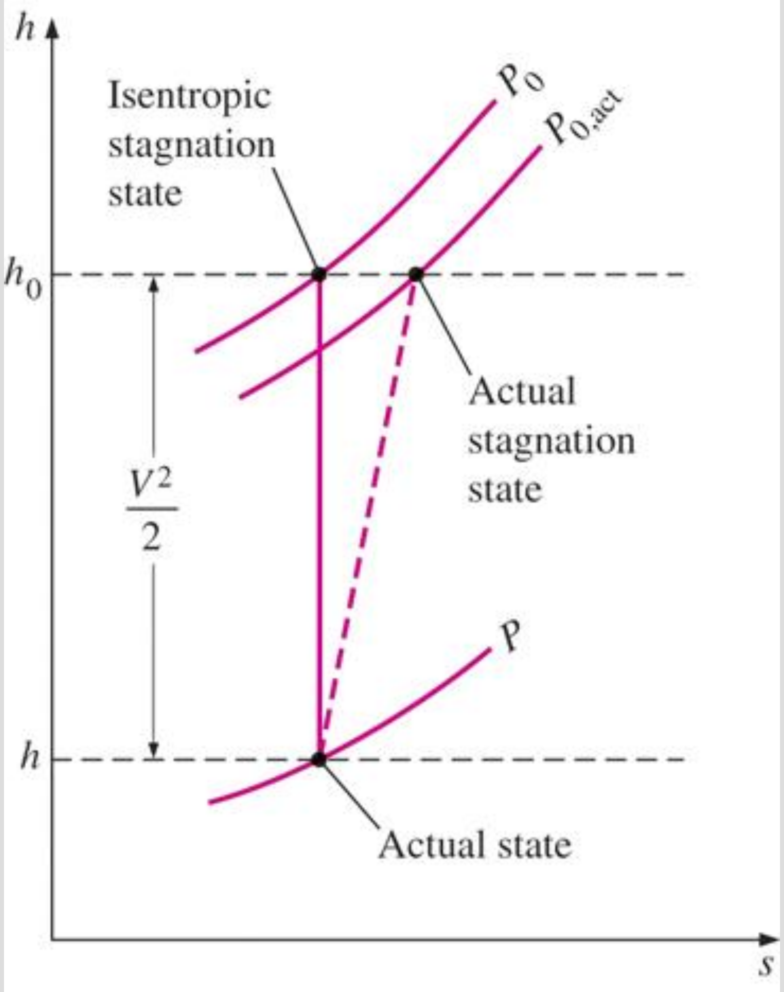
The stagnation processes are often approximated to be **isentropic**, and the isentropic stagnation properties are simply referred to as stagnation properties.

When the fluid is approximated as an ***ideal gas* with constant specific heats**

$$c_p T_0 = c_p T + \frac{V^2}{2} \quad \rightarrow \quad T_0 = T + \frac{V^2}{2c_p}$$

T_0 is called the **stagnation** (or **total**) **temperature**, and it represents *the temperature an ideal gas attains when it is brought to rest adiabatically.*

The term $V^2/2c_p$ corresponds to the temperature rise during such a process and is called the **dynamic temperature**.



The actual state, actual stagnation state, and isentropic stagnation state of a fluid on an *h*-*s* diagram.

The pressure a fluid attains when brought to rest isentropically is called the **stagnation pressure** P_0.

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{k/(k-1)}$$

Stagnation density ρ_0

$$\rho = 1/v$$

$$Pv^k = P_0 v_0^k$$

$$\rightarrow \quad \frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{1/(k-1)}$$

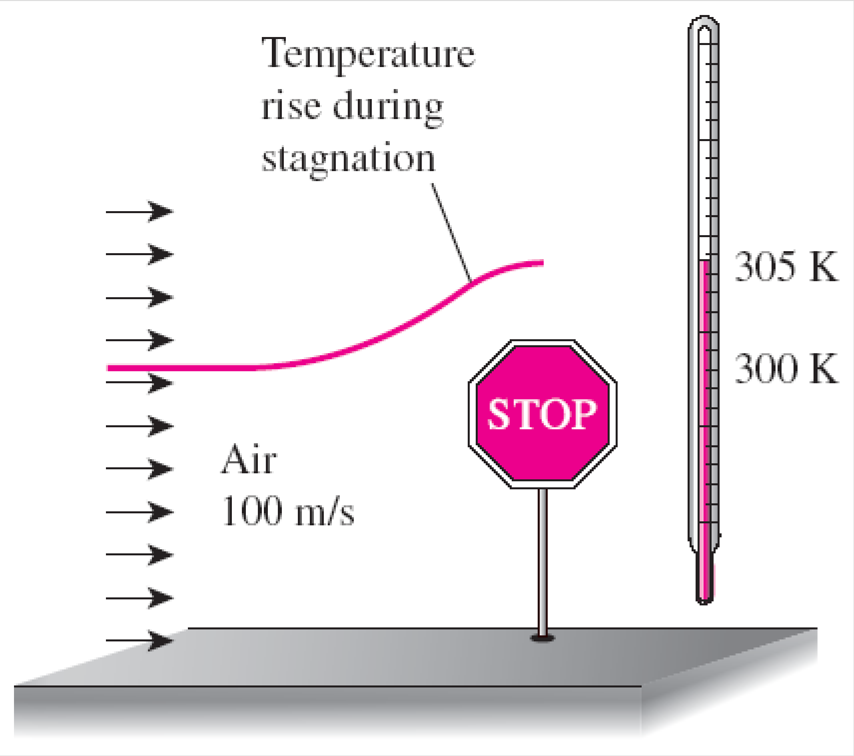
When stagnation enthalpies are used, the energy balance for a single-stream, steady-flow device

$$\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$$

$$q_{\text{in}} + w_{\text{in}} + (h_{01} + gz_1) = q_{\text{out}} + w_{\text{out}} + (h_{02} + gz_2)$$

When the fluid is approximated as an *ideal gas* with constant specific heats

$$(q_{\text{in}} - q_{\text{out}}) + (w_{\text{in}} - w_{\text{out}}) = c_p(T_{02} - T_{01}) + g(z_2 - z_1)$$



The temperature of an ideal gas flowing at a velocity V rises by $V^2/2c_p$ when it is brought to a complete stop.

EXAMPLE 12–1 Compression of High-Speed Air in an Aircraft

An aircraft is flying at a cruising speed of 250 m/s at an altitude of 5000 m where the atmospheric pressure is 54.05 kPa and the ambient air temperature is 255.7 K. The ambient air is first decelerated in a diffuser before it enters the compressor (Fig. 12–6). Assuming both the diffuser and the compressor to be isentropic, determine (*a*) the stagnation pressure at the compressor inlet and (*b*) the required compressor work per unit mass if the stagnation pressure ratio of the compressor is 8.

SOLUTION High-speed air enters the diffuser and the compressor of an aircraft. The stagnation pressure of the air and the compressor work input are to be determined.

Assumptions **1** Both the diffuser and the compressor are isentropic. **2** Air is an ideal gas with constant specific heats at room temperature.

Properties The constant-pressure specific heat c_p and the specific heat ratio k of air at room temperature are

$$c_p = 1.005 \text{ kJ/kg} \cdot \text{K} \qquad \text{and} \qquad k = 1.4$$

Analysis (*a*) Under isentropic conditions, the stagnation pressure at the compressor inlet (diffuser exit) can be determined from Eq. 12–5. However, first we need to find the stagnation temperature T_{01} at the compressor inlet. Under the stated assumptions, T_{01} can be determined from Eq. 12–4 to be

$$T_{01} = T_1 + \frac{V_1^2}{2c_p} = 255.7\text{ K} + \frac{(250\text{ m/s})^2}{(2)(1.005\text{ kJ/kg}\cdot\text{K})}\left(\frac{1\text{ kJ/kg}}{1000\text{ m}^2/\text{s}^2}\right)$$
$$= 286.8\text{ K}$$

Then from Eq. 12–5,

$$P_{01} = P_1\left(\frac{T_{01}}{T_1}\right)^{k/(k-1)} = (54.05\text{ kPa})\left(\frac{286.8\text{ K}}{255.7\text{ K}}\right)^{1.4/(1.4-1)}$$
$$= \mathbf{80.77\text{ kPa}}$$

That is, the temperature of air would increase by 31.1°C and the pressure by 26.72 kPa as air is decelerated from 250 m/s to zero velocity. These increases in the temperature and pressure of air are due to the conversion of the kinetic energy into enthalpy.

(*b*) To determine the compressor work, we need to know the stagnation temperature of air at the compressor exit T_{02}. The stagnation pressure ratio across the compressor P_{02}/P_{01} is specified to be 8. Since the compression process is assumed to be isentropic, T_{02} can be determined from the ideal-gas isentropic relation (Eq. 12–5):

$$T_{02} = T_{01}\left(\frac{P_{02}}{P_{01}}\right)^{(k-1)/k} = (286.8\text{ K})(8)^{(1.4-1)/1.4} = 519.5\text{ K}$$

Disregarding potential energy changes and heat transfer, the compressor work per unit mass of air is determined from Eq. 12–8:

$$\begin{aligned} w_{in} &= c_p(T_{02} - T_{01}) \\ &= (1.005\text{ kJ/kg}\cdot\text{K})(519.5\text{ K} - 286.8\text{ K}) \\ &= \mathbf{233.9\text{ kJ/kg}} \end{aligned}$$

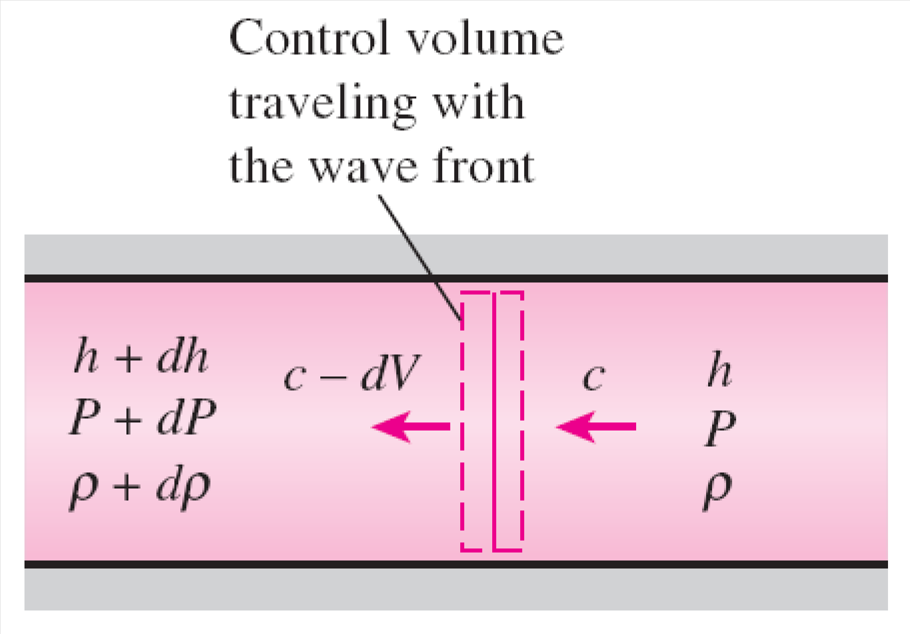
Thus the work supplied to the compressor is 233.9 kJ/kg.

Discussion Notice that using stagnation properties automatically accounts for any changes in the kinetic energy of a fluid stream.

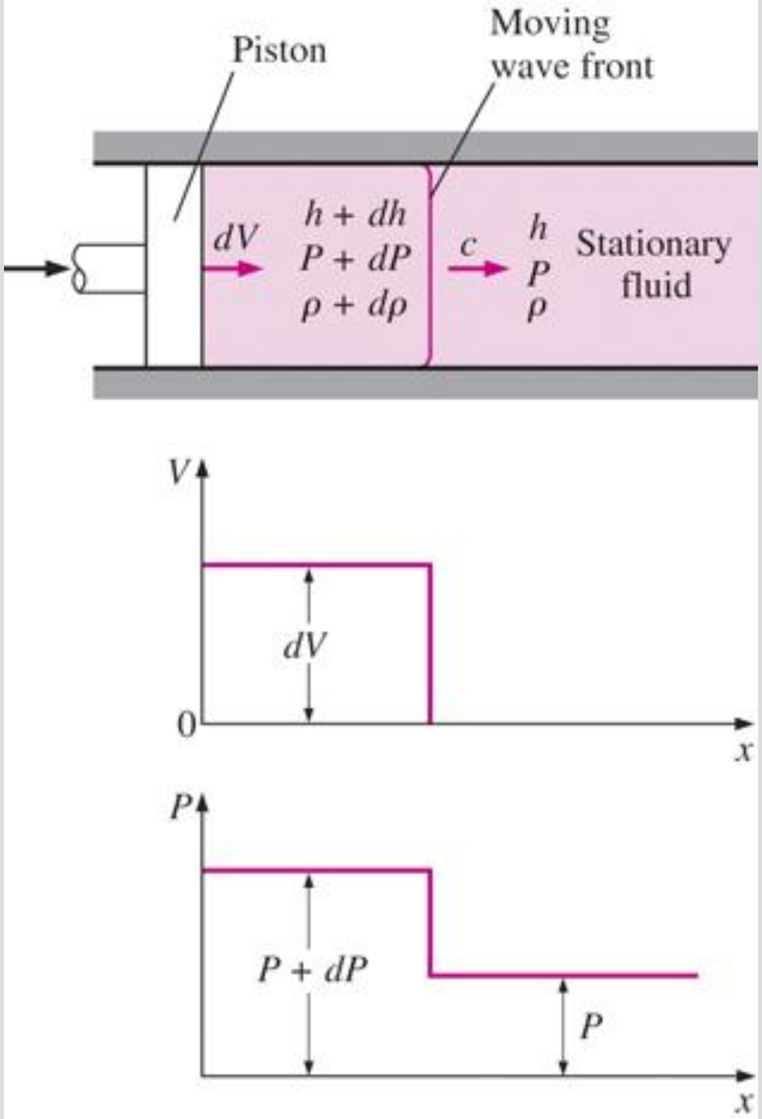
SPEED OF SOUND AND MACH NUMBER

Speed of sound (or the **sonic speed**): The speed at which an infinitesimally small pressure wave travels through a medium.

To obtain a relation for the speed of sound in a medium, the systems in the figures are considered.



Control volume moving with the small pressure wave along a duct.



Propagation of a small pressure wave along a duct.

$$c^2 = k\left(\frac{\partial P}{\partial \rho}\right)_T$$

Speed of sound

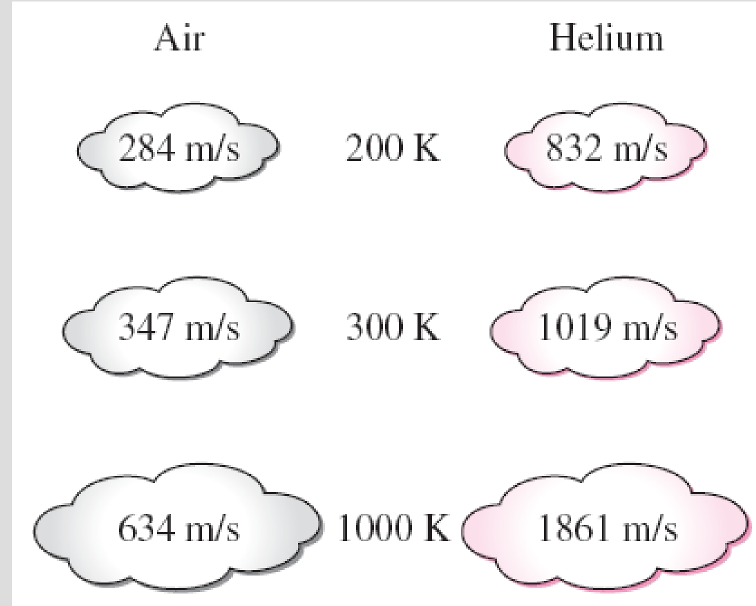
For an ideal gas $P = \rho RT$

$$c^2 = k\left(\frac{\partial P}{\partial \rho}\right)_T = k\left[\frac{\partial(\rho RT)}{\partial \rho}\right]_T = kRT$$

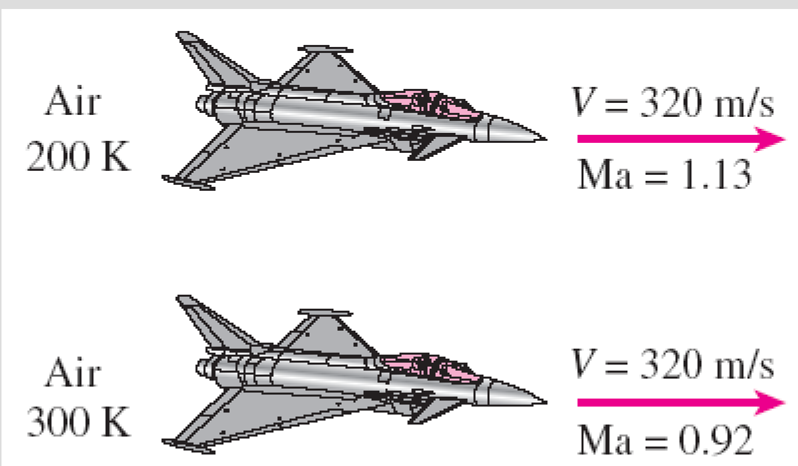
$$c = \sqrt{kRT}$$

$$\mathrm{Ma} = \frac{V}{c}$$

Mach number



The speed of sound changes with temperature and varies with the fluid.



The Mach number can be different at different temperatures even if the velocity is the same.

Ma = 1	Sonic flow
Ma < 1	Subsonic flow
Ma > 1	Supersonic flow
Ma >> 1	Hypersonic flow
Ma ≅ 1	Transonic flow

EXAMPLE 12–2 **Mach Number of Air Entering a Diffuser**

Air enters a diffuser shown in Fig. 12–11 with a velocity of 200 m/s. Determine (*a*) the speed of sound and (*b*) the Mach number at the diffuser inlet when the air temperature is 30°C.

SOLUTION Air enters a diffuser with a high velocity. The speed of sound and the Mach number are to be determined at the diffuser inlet.

Assumption Air at specified conditions behaves as an ideal gas.

Properties The gas constant of air is $R = 0.287$ kJ/kg · K, and its specific heat ratio at 30°C is 1.4.

Analysis We note that the speed of sound in a gas varies with temperature, which is given to be 30°C.

(*a*) The speed of sound in air at 30°C is determined from Eq. 12–11 to be

$$c = \sqrt{kRT} = \sqrt{(1.4)(0.287\ \mathrm{kJ/kg \cdot K})(303\ \mathrm{K})\left(\frac{1000\ \mathrm{m^2/s^2}}{1\ \mathrm{kJ/kg}}\right)} = \mathbf{349\ m/s}$$

(*b*) Then the Mach number becomes

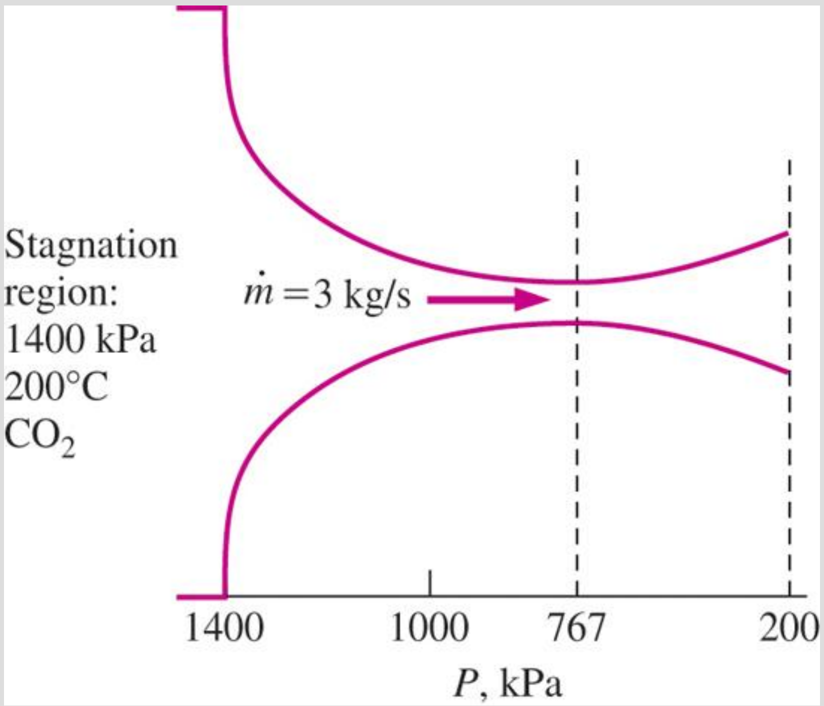
$$\mathrm{Ma} = \frac{V}{c} = \frac{200\ \mathrm{m/s}}{349\ \mathrm{m/s}} = \mathbf{0.573}$$

Discussion The flow at the diffuser inlet is subsonic since Ma < 1.

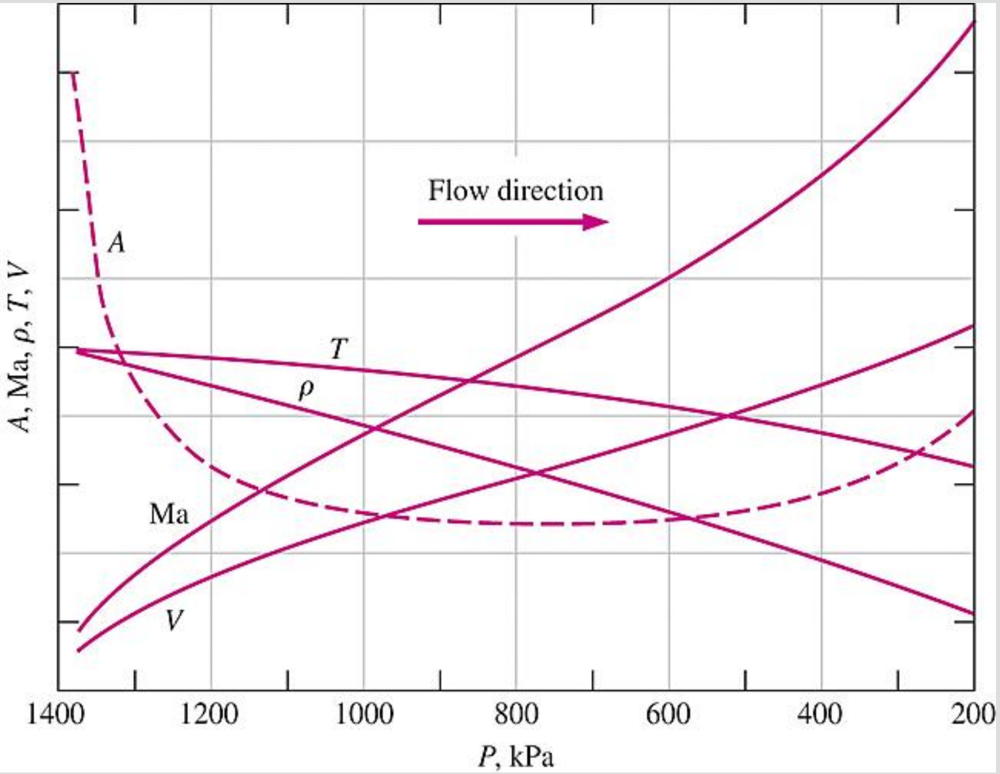
ONE-DIMENSIONAL ISENTROPIC FLOW

During fluid flow through many devices such as nozzles, diffusers, and turbine blade passages, flow quantities vary primarily in the flow direction only, and the flow can be approximated as one-dimensional isentropic flow with good accuracy.

EXAMPLE



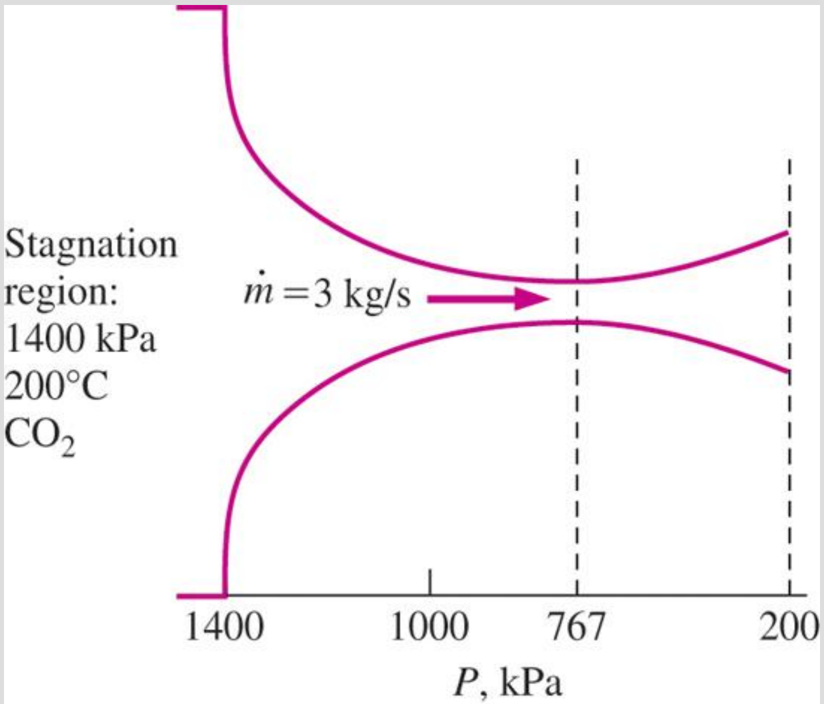
A converging–diverging nozzle.



Variation of normalized fluid properties and cross-sectional area along a duct as the pressure drops from 1400 to 200 kPa.

EXAMPLE 12–3 Gas Flow through a Converging–Diverging Duct

Carbon dioxide flows steadily through a varying cross-sectional area duct such as a nozzle shown in Fig. 12–12 at a mass flow rate of 3 kg/s. The carbon dioxide enters the duct at a pressure of 1400 kPa and 200°C with a low velocity, and it expands in the nozzle to a pressure of 200 kPa. The duct is designed so that the flow can be approximated as isentropic. Determine the density, velocity, flow area, and Mach number at each location along the duct that corresponds to a pressure drop of 200 kPa.



SOLUTION Carbon dioxide enters a varying cross-sectional area duct at specified conditions. The flow properties are to be determined along the duct.

Assumptions **1** Carbon dioxide is an ideal gas with constant specific heats at room temperature. **2** Flow through the duct is steady, one-dimensional, and isentropic.

Properties For simplicity we use $c_p = 0.846$ kJ/kg · K and $k = 1.289$ throughout the calculations, which are the constant-pressure specific heat and specific heat ratio values of carbon dioxide at room temperatures. The gas constant of carbon dioxide is $R = 0.1889$ kJ/kg · K.

Analysis We note that the inlet temperature is nearly equal to the stagnation temperature since the inlet velocity is small. The flow is isentropic, and thus the stagnation temperature and pressure throughout the duct remain constant. Therefore,

$$T_0 \cong T_1 = 200°\text{C} = 473 \text{ K}$$

and

$$P_0 \cong P_1 = 1400 \text{ kPa}$$

To illustrate the solution procedure, we calculate the desired properties at the location where the pressure is 1200 kPa, the first location that corresponds to a pressure drop of 200 kPa.

From Eq. 12–5,

$$T = T_0\left(\frac{P}{P_0}\right)^{(k-1)/k} = (473\text{ K})\left(\frac{1200\text{ kPa}}{1400\text{ kPa}}\right)^{(1.289-1)/1.289} = 457\text{ K}$$

From Eq. 12–4,

$$V = \sqrt{2c_p(T_0 - T)}$$

$$= \sqrt{2(0.846\text{ kJ/kg}\cdot\text{K})(473\text{ K} - 457\text{ K})\left(\frac{1000\text{ m}^2/\text{s}^3}{1\text{ kJ/kg}}\right)}$$

$$= \mathbf{164.5\text{ m/s}}$$

From the ideal-gas relation,

$$\rho = \frac{P}{RT} = \frac{1200\text{ kPa}}{(0.1889\text{ kPa}\cdot\text{m}^3/\text{kg}\cdot\text{K})(457\text{ K})} = \mathbf{13.9\text{ kg/m}^3}$$

From the mass flow rate relation,

$$A = \frac{\dot{m}}{\rho V} = \frac{3\ \text{kg/s}}{(13.9\ \text{kg/m}^3)(164.5\ \text{m/s})} = 13.1 \times 10^{-4}\ \text{m}^2 = \mathbf{13.1\ cm^2}$$

From Eqs. 12–11 and 12–12,

$$c = \sqrt{kRT} = \sqrt{(1.289)(0.1889\ \text{kJ/kg}\cdot\text{K})(457\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)} = 333.6\ \text{m/s}$$

$$\text{Ma} = \frac{V}{c} = \frac{164.5\ \text{m/s}}{333.6\ \text{m/s}} = \mathbf{0.493}$$

The results for the other pressure steps are summarized in Table 12–1 and are plotted in Fig. 12–13.

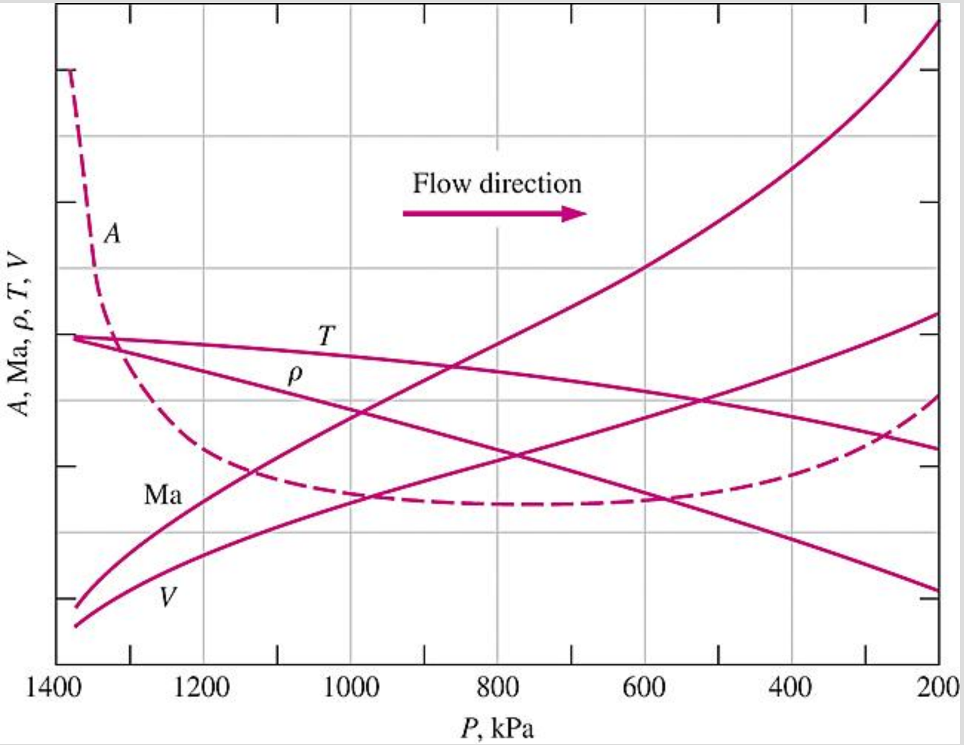
Discussion Note that as the pressure decreases, the temperature and speed of sound decrease while the fluid velocity and Mach number increase in the flow direction. The density decreases slowly at first and rapidly later as the fluid velocity increases.

TABLE 12–1

Variation of fluid properties in flow direction in the duct described in Example 12–3 for $\dot{m}$ = 3 kg/s = constant

P, kPa	T, K	V, m/s	ρ, kg/m^3	c, m/s	A, cm^2	Ma
1400	473	0	15.7	339.4	∞	0
1200	457	164.5	13.9	333.6	13.1	0.493
1000	439	240.7	12.1	326.9	10.3	0.736
800	417	306.6	10.1	318.8	9.64	0.962
767*	413	317.2	9.82	317.2	9.63	1.000
600	391	371.4	8.12	308.7	10.0	1.203
400	357	441.9	5.93	295.0	11.5	1.498
200	306	530.9	3.46	272.9	16.3	1.946

* 767 kPa is the critical pressure where the local Mach number is unity.



Variation of normalized fluid properties and cross-sectional area along a duct as the pressure drops from 1400 to 200 kPa.

We note from Example that the flow area decreases with decreasing pressure up to a critical-pressure value (Ma = 1), and then it begins to increase with further reductions in pressure.

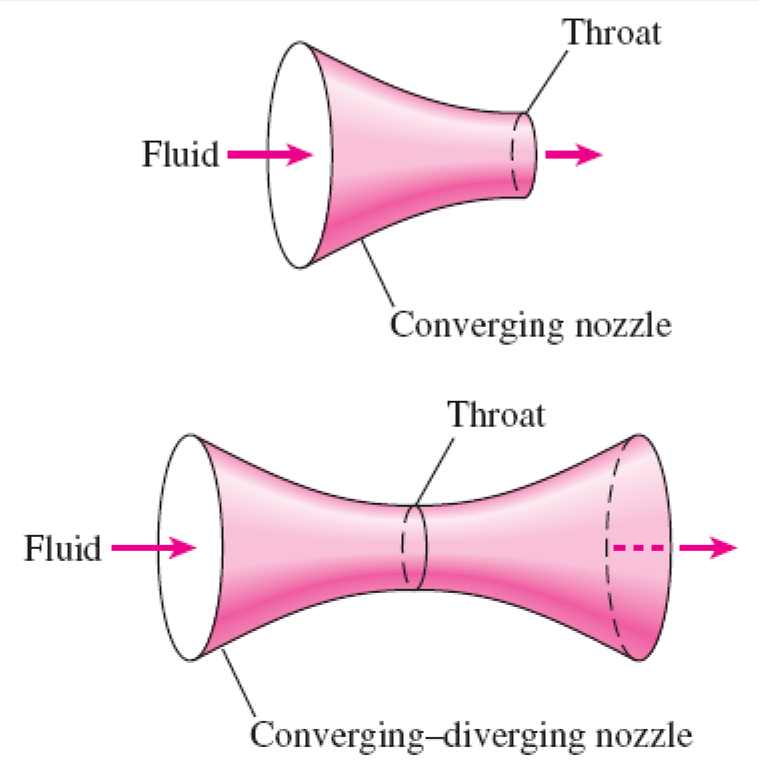
The Mach number is unity at the location of smallest flow area, called the **throat.**

The velocity of the fluid keeps increasing after passing the throat although the flow area increases rapidly in that region.

This increase in velocity past the throat is due to the rapid decrease in the fluid density.

The flow area of the duct considered in this example first decreases and then increases. Such ducts are called **converging–diverging nozzles**.

These nozzles are used to accelerate gases to supersonic speeds and should not be confused with *Venturi nozzles*, which are used strictly for incompressible flow.



The cross section of a nozzle at the smallest flow area is called the *throat*.

Variation of Fluid Velocity with Flow Area

In this section, the relations for the variation of static-to-stagnation property ratios with the Mach number for pressure, temperature, and density are provided.

$$\frac{dA}{A} = \frac{dP}{\rho V^2}\left(1 - \mathrm{Ma}^2\right)$$

This relation describes the variation of pressure with flow area.

At subsonic velocities, the pressure decreases in converging ducts (subsonic nozzles) and increases in diverging ducts (subsonic diffusers).

At supersonic velocities, the pressure decreases in diverging ducts (supersonic nozzles) and increases in converging ducts (supersonic diffusers).

Conservation of Energy
(steady flow, w = 0, $q = 0$, Δpe = 0)

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

or

$$h + \frac{V^2}{2} = \text{constant}$$

Differentiate,

$$dh + V\,dV = 0$$

Also,

$$T\,\overset{0\ (\text{isentropic})}{ds} = dh - v\,dP$$

$$dh = v\,dP = \frac{1}{\rho}\,dP$$

Substitute,

$$\frac{dP}{\rho} + V\,dV = 0$$

Derivation of the differential form of the energy equation for steady isentropic flow.

$$\frac{dA}{A} = -\frac{dV}{V}(1 - \mathrm{Ma}^2)$$

This equation governs the shape of a nozzle or a diffuser in subsonic or supersonic isentropic flow.

For subsonic flow ($\mathrm{Ma} < 1$), $\dfrac{dA}{dV} < 0$

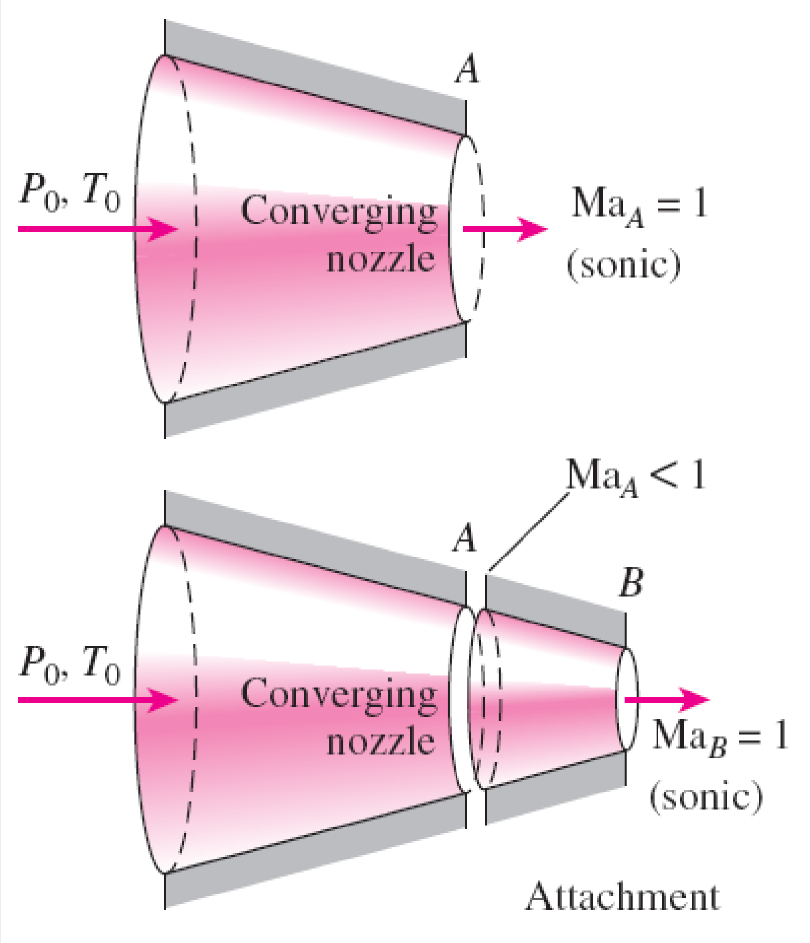
For supersonic flow ($\mathrm{Ma} > 1$), $\dfrac{dA}{dV} > 0$

For sonic flow ($\mathrm{Ma} = 1$), $\dfrac{dA}{dV} = 0$

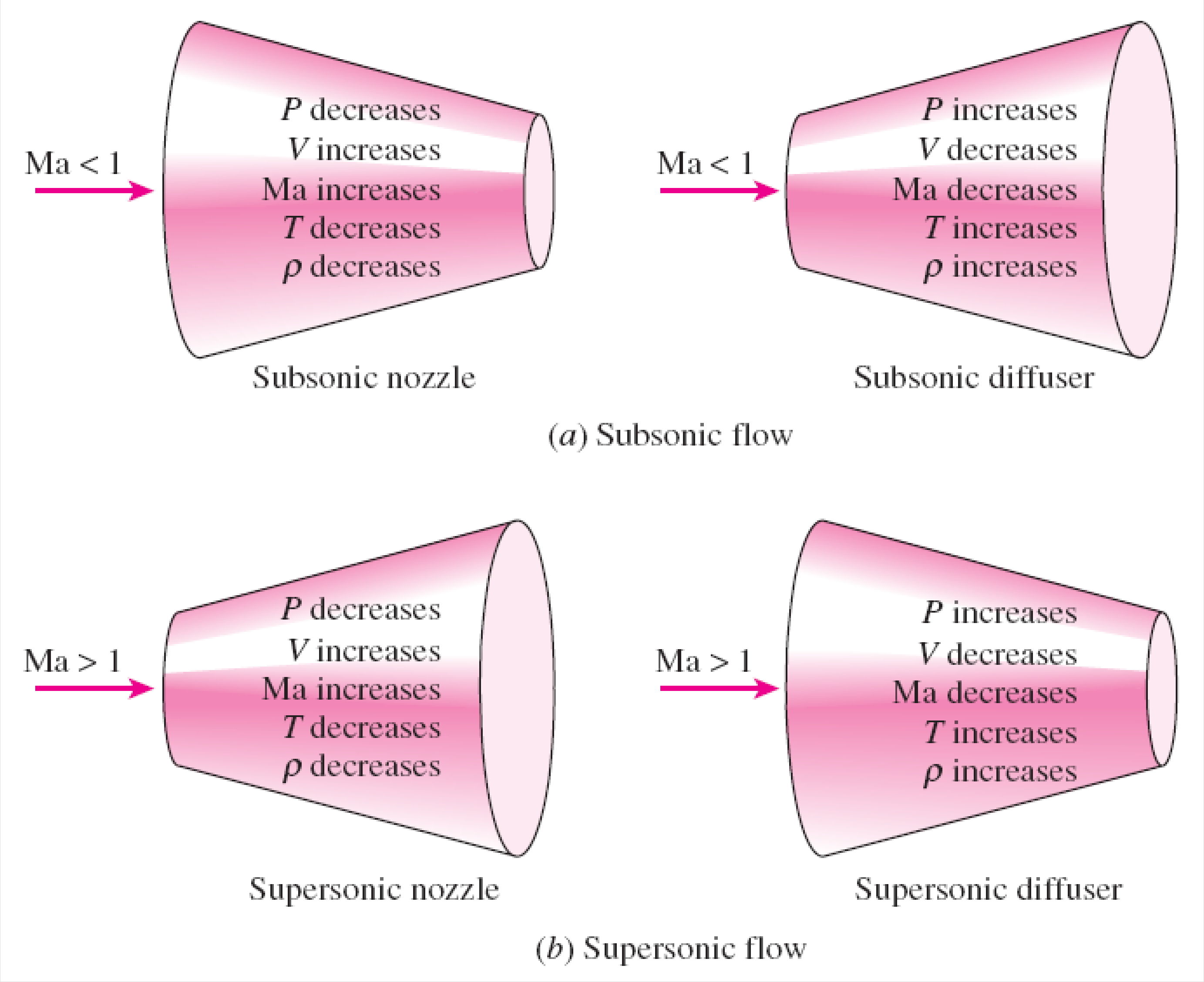
The proper shape of a nozzle depends on the highest velocity desired relative to the sonic velocity.

To accelerate a fluid, we must use a converging nozzle at subsonic velocities and a diverging nozzle at supersonic velocities.

To accelerate a fluid to supersonic velocities, we must use a converging–diverging nozzle.



We cannot obtain supersonic velocities by attaching a converging section to a converging nozzle. Doing so will only move the sonic cross section farther downstream and decrease the mass flow rate.



Variation of flow properties in subsonic and supersonic nozzles and diffusers.

Property Relations for Isentropic Flow of Ideal Gases

The relations between the static properties and stagnation properties of an ideal gas with constant specific heats

$$\frac{T_0}{T} = 1 + \left(\frac{k-1}{2}\right)\text{Ma}^2$$

$$\frac{P_0}{P} = \left[1 + \left(\frac{k-1}{2}\right)\text{Ma}^2\right]^{k/(k-1)}$$

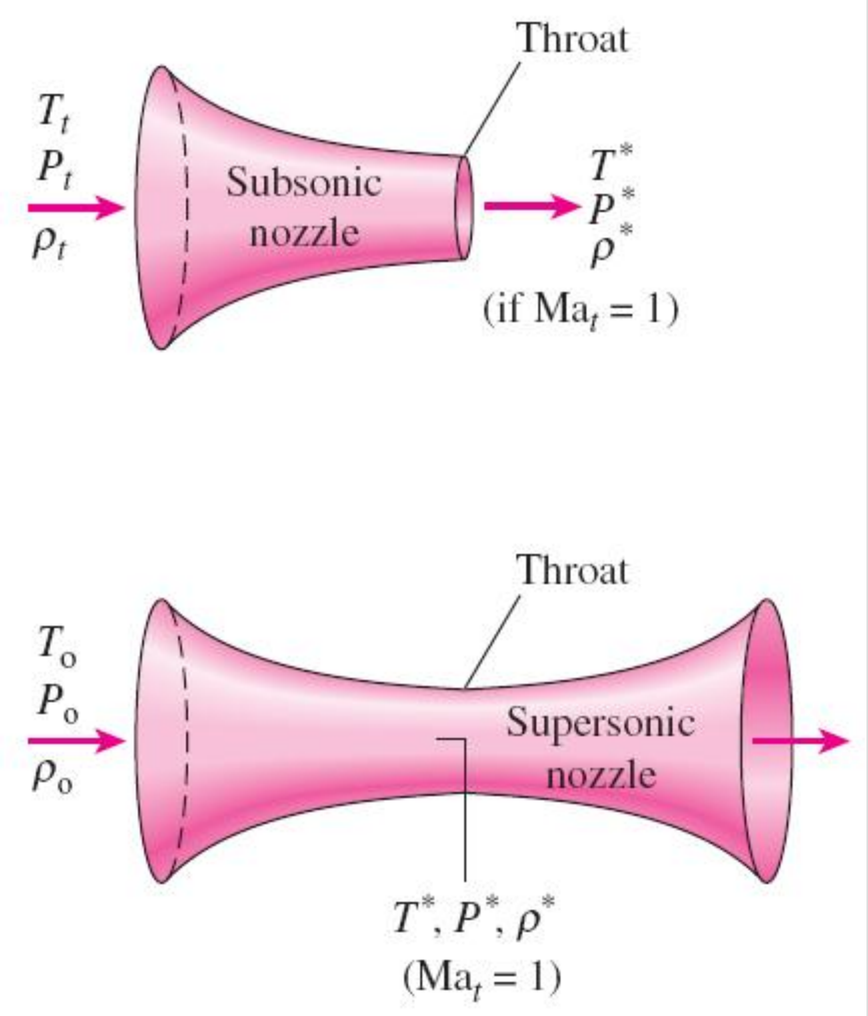
$$\frac{\rho_0}{\rho} = \left[1 + \left(\frac{k-1}{2}\right)\text{Ma}^2\right]^{1/(k-1)}$$

$$\frac{T^*}{T_0} = \frac{2}{k+1}$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{1/(k-1)}$$

Critical ratios (Ma=1)



When $\text{Ma}_t = 1$, the properties at the nozzle throat become the critical properties.

TABLE 17–2

The critical-pressure, critical-temperature, and critical-density ratios for isentropic flow of some ideal gases

	Superheated steam, $k = 1.3$	Hot products of combustion, $k = 1.33$	Air, $k = 1.4$	Monatomic gases, $k = 1.667$
$\frac{P^*}{P_0}$	0.5457	0.5404	0.5283	0.4871
$\frac{T^*}{T_0}$	0.8696	0.8584	0.8333	0.7499
$\frac{\rho^*}{\rho_0}$	0.6276	0.6295	0.6340	0.6495

EXAMPLE 12–4 Critical Temperature and Pressure in Gas Flow

Calculate the critical pressure and temperature of carbon dioxide for the flow conditions described in Example 12–3 (Fig. 12–19).

SOLUTION For the flow discussed in Example 12–3, the critical pressure and temperature are to be calculated.

Assumptions **1** The flow is steady, adiabatic, and one-dimensional. **2** Carbon dioxide is an ideal gas with constant specific heats.

Properties The specific heat ratio of carbon dioxide at room temperature is $k = 1.289$.

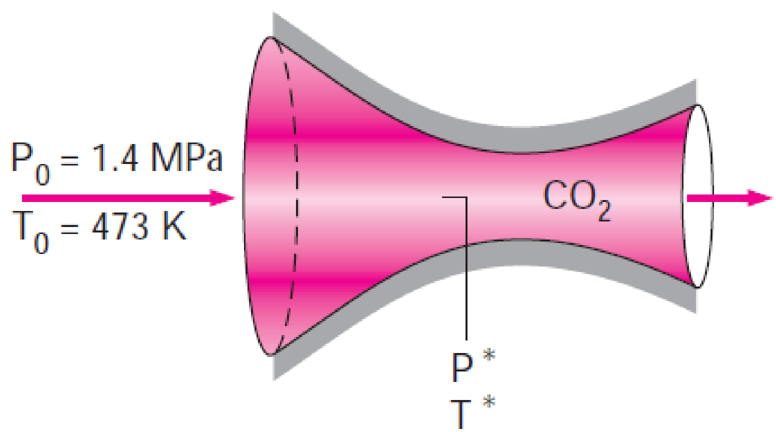


FIGURE 12–19
Schematic for Example 12–4.

Analysis The ratios of critical to stagnation temperature and pressure are determined to be

$$\frac{T^*}{T_0} = \frac{2}{k+1} = \frac{2}{1.289+1} = 0.87337$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} = \left(\frac{2}{1.289+1}\right)^{1.289/(1.289-1)} = 0.5477$$

$$\frac{T^*}{T_0} = \frac{2}{k+1}$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{1/(k-1)}$$

Noting that the stagnation temperature and pressure are, from Example 12–3, $T_0 = 473$ K and $P_0 = 1400$ kPa, we see that the critical temperature and pressure in this case are

$$T^* = 0.87337T_0 = (0.87337)(473\ \text{K}) = \mathbf{413\ K}$$

$$P^* = 0.5477P_0 = (0.5477)(1400\ \text{kPa}) = \mathbf{767\ kPa}$$

Discussion Note that these values agree with those listed in Table 12–1, as expected. Also, property values other than these at the throat would indicate that the flow is not critical, and the Mach number is not unity.

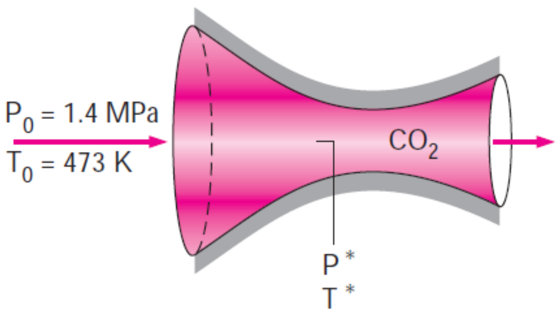


FIGURE 12–19
Schematic for Example 12–4.

TABLE 12–1

Variation of fluid properties in flow direction in the duct described in Example 12–3 for $\dot{m}$ = 3 kg/s = constant

P, kPa	*T*, K	*V*, m/s	*ρ*, kg/m³	*c*, m/s	*A*, cm²	Ma
1400	473	0	15.7	339.4	∞	0
1200	457	164.5	13.9	333.6	13.1	0.493
1000	439	240.7	12.1	326.9	10.3	0.736
800	417	306.6	10.1	318.8	9.64	0.962
767*	413	317.2	9.82	317.2	9.63	1.000
600	391	371.4	8.12	308.7	10.0	1.203

ISENTROPIC FLOW THROUGH NOZZLES

Converging or converging–diverging nozzles are found in steam and gas turbines and aircraft and spacecraft propulsion systems.

In this section we consider the effects of **back pressure** (i.e., the pressure applied at the nozzle discharge region) on the exit velocity, the mass flow rate, and the pressure distribution along the nozzle.

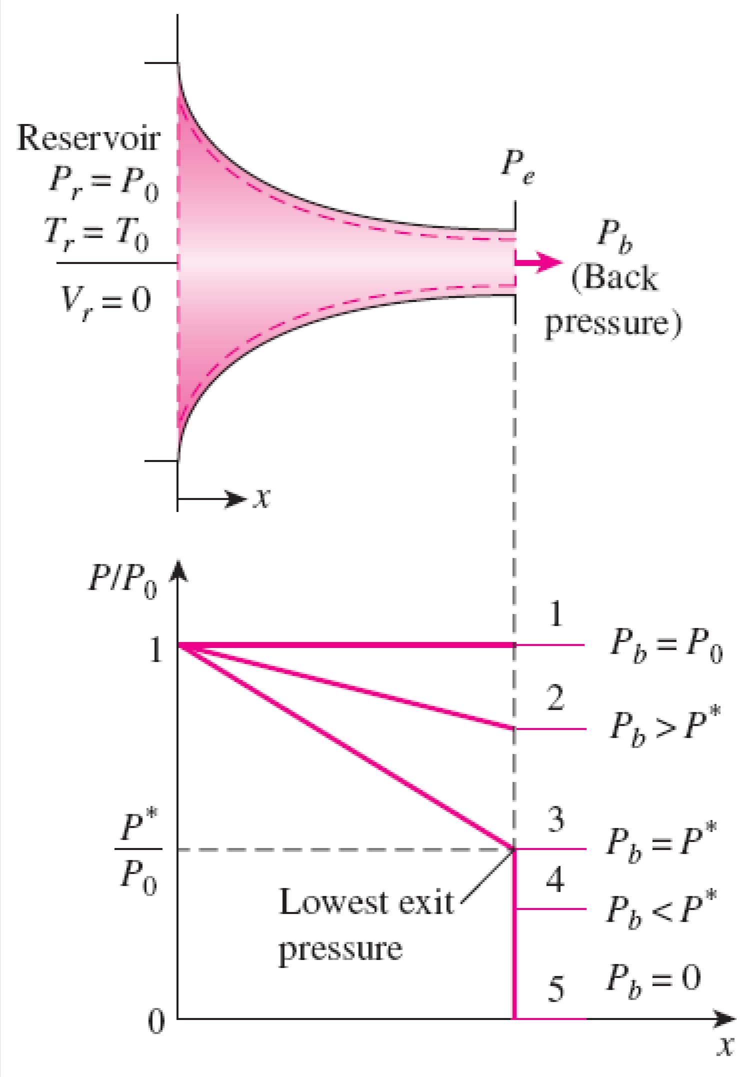
Converging Nozzles

Mass flow rate through a nozzle

$$\dot{m} = \frac{A\mathrm{Ma}P_0\sqrt{k/(RT_0)}}{[1 + (k-1)\mathrm{Ma}^2/2]^{(k+1)/[2(k-1)]}}$$

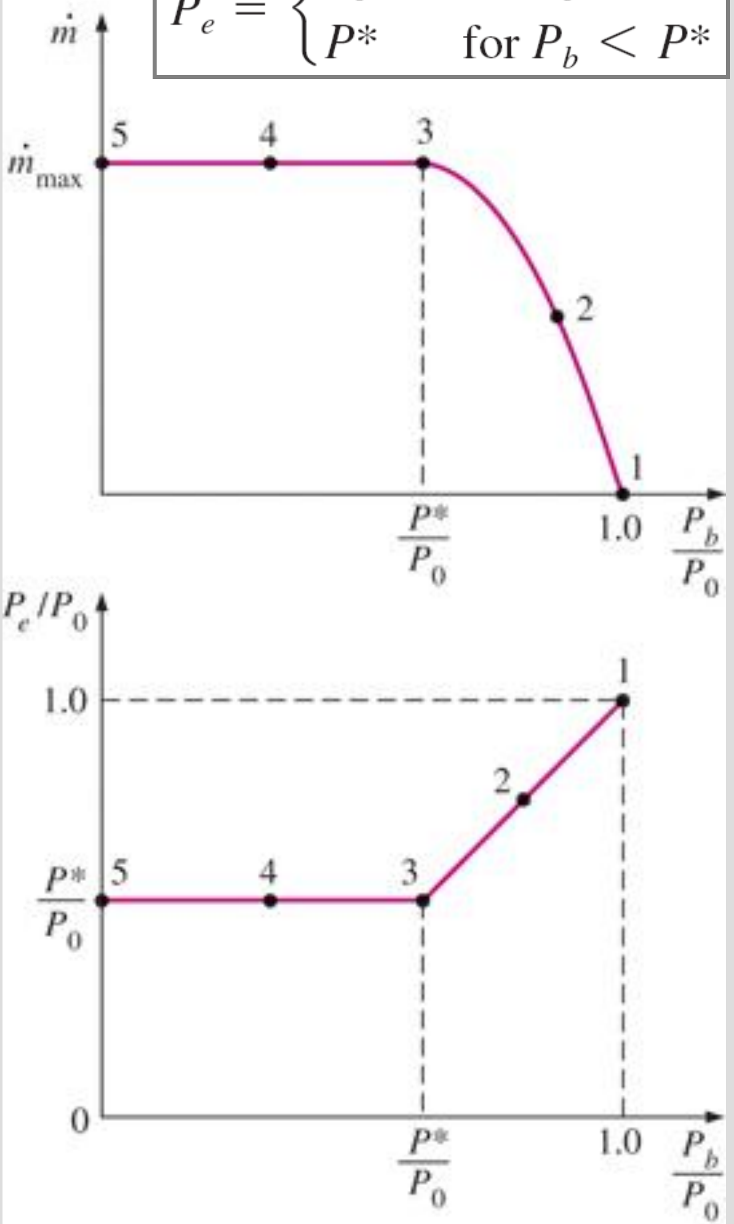
Maximum mass flow rate

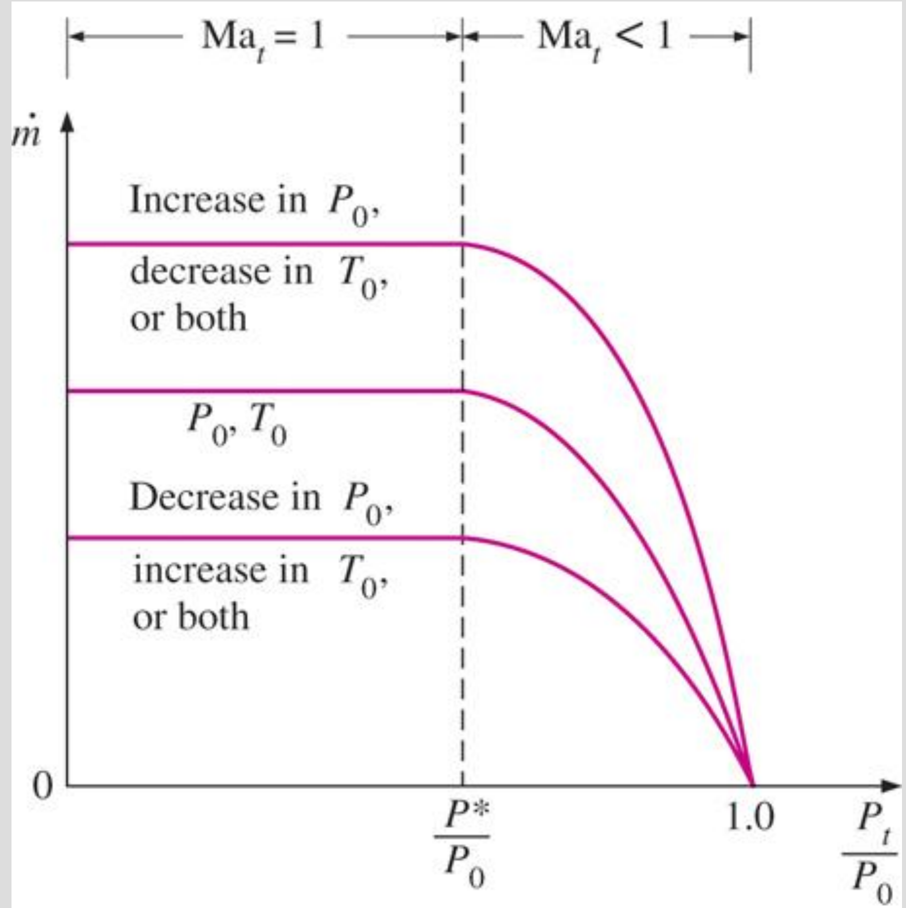
$$\dot{m}_{\max} = A^* P_0 \sqrt{\frac{k}{RT_0}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}}$$



The effect of back pressure on the pressure distribution along a converging nozzle.

$$P_e = \begin{cases} P_b & \text{for } P_b \geq P^* \\ P^* & \text{for } P_b < P^* \end{cases}$$





The variation of the mass flow rate through a nozzle with inlet stagnation properties.

The effect of back pressure P_b on the mass flow rate and the exit pressure P_e of a converging nozzle.

$$\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{(k+1)/[2(k-1)]}$$

$$\mathrm{Ma}^* = \frac{V}{c^*}$$

$$\mathrm{Ma}^* = \frac{V}{c}\frac{c}{c^*} = \frac{\mathrm{Ma}c}{c^*} = \frac{\mathrm{Ma}\sqrt{kRT}}{\sqrt{kRT^*}} = \mathrm{Ma}\sqrt{\frac{T}{T^*}}$$

$$\mathrm{Ma}^* = \mathrm{Ma}\sqrt{\frac{k+1}{2+(k-1)\mathrm{Ma}^2}}$$

Ma* is the local velocity nondimensionalized with respect to the sonic velocity at the *throat*.

Ma is the local velocity nondimensionalized with respect to the *local* sonic velocity.

Ma	Ma^*	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{\rho}{\rho_0}$	$\frac{T}{T_0}$
⋮	⋮	⋮	⋮	⋮	⋮
0.90	0.9146	1.0089	0.5913	⋮	⋮
1.00	1.0000	1.0000	0.5283	⋮	⋮
1.10	1.0812	1.0079	0.4684	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮

Various property ratios for isentropic flow through nozzles and diffusers are listed in Table A–32 for $k = 1.4$ for convenience.

$$Ma^* = Ma\sqrt{\frac{k+1}{2+(k-1)Ma^2}}$$

$$\frac{A}{A^*} = \frac{1}{Ma}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}Ma^2\right)\right]^{0.5(k+1)/(k-1)}$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}Ma^2\right)^{-k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}Ma^2\right)^{-1/(k-1)}$$

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}Ma^2\right)^{-1}$$

TABLE A–13

One-dimensional isentropic compressible flow functions for an ideal gas with $k = 1.4$

Ma	Ma^*	A/A^*	P/P_0	ρ/ρ_0	T/T_0
0	0	∞	1.0000	1.0000	1.0000
0.1	0.1094	5.8218	0.9930	0.9950	0.9980
0.2	0.2182	2.9635	0.9725	0.9803	0.9921
0.3	0.3257	2.0351	0.9395	0.9564	0.9823
0.4	0.4313	1.5901	0.8956	0.9243	0.9690
0.5	0.5345	1.3398	0.8430	0.8852	0.9524
0.6	0.6348	1.1882	0.7840	0.8405	0.9328
0.7	0.7318	1.0944	0.7209	0.7916	0.9107
0.8	0.8251	1.0382	0.6560	0.7400	0.8865
0.9	0.9146	1.0089	0.5913	0.6870	0.8606
1.0	1.0000	1.0000	0.5283	0.6339	0.8333
1.2	1.1583	1.0304	0.4124	0.5311	0.7764
1.4	1.2999	1.1149	0.3142	0.4374	0.7184
1.6	1.4254	1.2502	0.2353	0.3557	0.6614
1.8	1.5360	1.4390	0.1740	0.2868	0.6068
2.0	1.6330	1.6875	0.1278	0.2300	0.5556
2.2	1.7179	2.0050	0.0935	0.1841	0.5081
2.4	1.7922	2.4031	0.0684	0.1472	0.4647
2.6	1.8571	2.8960	0.0501	0.1179	0.4252
2.8	1.9140	3.5001	0.0368	0.0946	0.3894
3.0	1.9640	4.2346	0.0272	0.0760	0.3571
5.0	2.2361	25.000	0.0019	0.0113	0.1667
∞	2.2495	∞	0	0	0

EXAMPLE 12–5 **Effect of Back Pressure on Mass Flow Rate**

Air at 1 MPa and 600°C enters a converging nozzle, shown in Fig. 12–24, with a velocity of 150 m/s. Determine the mass flow rate through the nozzle for a nozzle throat area of 50 cm^2 when the back pressure is (*a*) 0.7 MPa and (*b*) 0.4 MPa.

SOLUTION Air enters a converging nozzle. The mass flow rate of air through the nozzle is to be determined for different back pressures.

Assumptions **1** Air is an ideal gas with constant specific heats at room temperature. **2** Flow through the nozzle is steady, one-dimensional, and isentropic.

Properties The constant pressure specific heat and the specific heat ratio of air are c_p = 1.005 kJ/kg · K and k = 1.4.

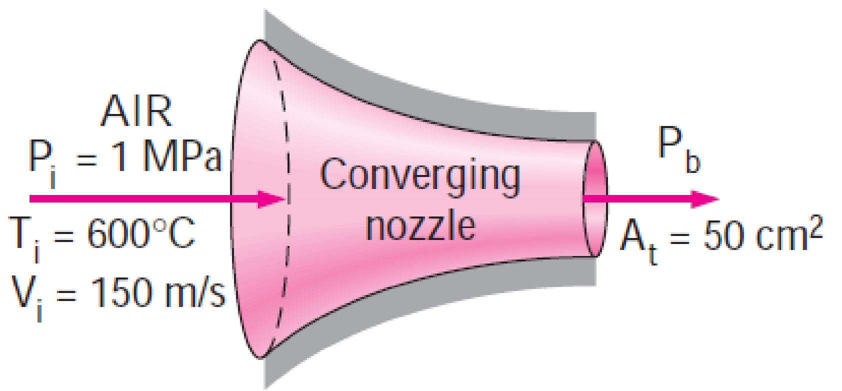


FIGURE 12–24
Schematic for Example 12–5.

Analysis We use the subscripts *i* and *t* to represent the properties at the nozzle inlet and the throat, respectively. The stagnation temperature and pressure at the nozzle inlet are determined from Eqs. 12–4 and 12–5:

$$T_{0i} = T_i + \frac{V_i^2}{2c_p} = 873\text{ K} + \frac{(150\text{ m/s}^2)^2}{2(1.005\text{ kJ/kg}\cdot\text{K})}\left(\frac{1\text{ kJ/kg}}{1000\text{ m}^2/\text{s}^2}\right) = 884\text{ K}$$

$$P_{0i} = P_i\left(\frac{T_{0i}}{T_i}\right)^{k/(k-1)} = (1\text{ MPa})\left(\frac{884\text{ K}}{873\text{ K}}\right)^{1.4/(1.4-1)} = 1.045\text{ MPa}$$

These stagnation temperature and pressure values remain constant throughout the nozzle since the flow is assumed to be isentropic. That is,

$$T_0 = T_{0i} = 884\text{ K} \qquad \text{and} \qquad P_0 = P_{0i} = 1.045\text{ MPa}$$

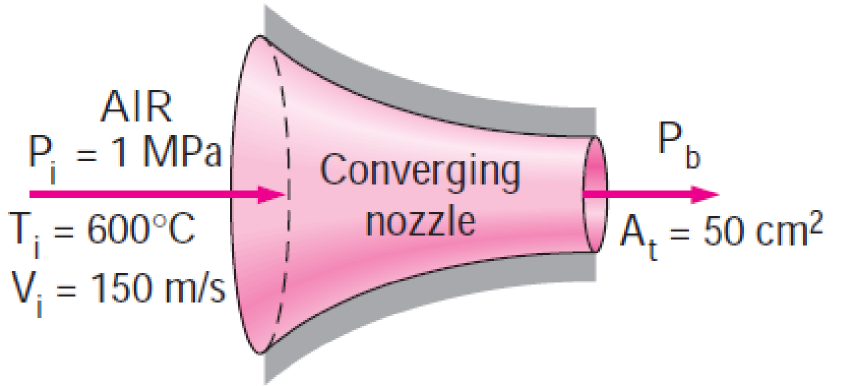


FIGURE 12–24
Schematic for Example 12–5.

The critical-pressure ratio is determined from Table 12–2 (or Eq. 12–22) to be $P^*/P_0 = 0.5283$.

TABLE A–13

One-dimensional isentropic compressible flow functions for an ideal gas with $k = 1.4$

Ma	Ma*	A/A^*	P/P_0	ρ/ρ_0	T/T_0
0	0	∞	1.0000	1.0000	1.0000
0.1	0.1094	5.8218	0.9930	0.9950	0.9980
0.2	0.2182	2.9635	0.9725	0.9803	0.9921
0.3	0.3257	2.0351	0.9395	0.9564	0.9823
0.4	0.4313	1.5901	0.8956	0.9243	0.9690
0.5	0.5345	1.3398	0.8430	0.8852	0.9524
0.6	0.6348	1.1882	0.7840	0.8405	0.9328
0.7	0.7318	1.0944	0.7209	0.7916	0.9107
0.8	0.8251	1.0382	0.6560	0.7400	0.8865
0.9	0.9146	1.0089	0.5913	0.6870	0.8606
1.0	1.0000	1.0000	0.5283	0.6339	0.8333

(*a*) The back pressure ratio for this case is

$$\frac{P_b}{P_0} = \frac{0.7 \text{ MPa}}{1.045 \text{ MPa}} = 0.670$$

which is greater than the critical-pressure ratio, 0.5283. Thus the exit plane pressure (or throat pressure P_t) is equal to the back pressure in this case. That is, $P_t = P_b = 0.7$ MPa, and $P_t/P_0 = 0.670$. Therefore, the flow is not choked. From Table A–13 at $P_t/P_0 = 0.670$, we read $\text{Ma}_t = 0.778$ and $T_t/T_0 = 0.892$.

The mass flow rate through the nozzle can be calculated from Eq. 12–24.

$$\dot{m} = \frac{A\mathrm{Ma}P_0\sqrt{k/(RT_0)}}{[1 + (k - 1)\mathrm{Ma}^2/2]^{(k+1)/[2(k-1)]}}$$

But it can also be determined in a step-by-step manner as follows:

$$T_t = 0.892T_0 = 0.892(884\ \mathrm{K}) = 788.5\ \mathrm{K}$$

$$\rho_t = \frac{P_t}{RT_t} = \frac{700\ \mathrm{kPa}}{(0.287\ \mathrm{kPa \cdot m^3/kg \cdot K})(788.5\ \mathrm{K})} = 3.093\ \mathrm{kg/m^3}$$

$$V_t = \mathrm{Ma}_t c_t = \mathrm{Ma}_t\sqrt{kRT_t}$$

$$= (0.778)\sqrt{(1.4)(0.287\ \mathrm{kJ/kg \cdot K})(788.5\ \mathrm{K})\left(\frac{1000\ \mathrm{m^2/s^2}}{1\ \mathrm{kJ/kg}}\right)}$$

$$= 437.9\ \mathrm{m/s}$$

Thus,

$$\dot{m} = \rho_t A_t V_t = (3.093\ \mathrm{kg/m^3})(50 \times 10^{-4}\ \mathrm{m^2})(437.9\ \mathrm{m/s}) = \mathbf{6.77\ kg/s}$$

(*b*) The back pressure ratio for this case is

$$\frac{P_b}{P_0} = \frac{0.4\ \text{MPa}}{1.045\ \text{MPa}} = 0.383$$

which is less than the critical-pressure ratio, 0.5283. Therefore, sonic conditions exist at the exit plane (throat) of the nozzle, and Ma = 1. The flow is choked in this case, and the mass flow rate through the nozzle can be calculated from Eq. 12–25:

$$\dot{m}_{\max} = A^* P_0 \sqrt{\frac{k}{RT_0}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}}$$

$$\begin{aligned}\dot{m} &= A^* P_0 \sqrt{\frac{k}{RT_0}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}} \\ &= (50 \times 10^{-4}\ \text{m}^2)(1045\ \text{kPa})\sqrt{\frac{1.4}{(0.287\ \text{kJ/kg}\cdot\text{K})(884\ \text{K})}\left(\frac{2}{1.4+1}\right)^{2.4/0.8}} \\ &= \mathbf{7.10\ kg/s}\end{aligned}$$

since $\text{kPa}\cdot\text{m}^2/\sqrt{\text{kJ/kg}} = \sqrt{1000}\ \text{kg/s}$.

Discussion This is the maximum mass flow rate through the nozzle for the specified inlet conditions and nozzle throat area.

EXAMPLE 12–6 **Gas Flow through a Converging Nozzle**

Nitrogen enters a duct with varying flow area at $T_1 = 400$ K, $P_1 = 100$ kPa, and $Ma_1 = 0.3$. Assuming steady isentropic flow, determine T_2, P_2, and Ma_2 at a location where the flow area has been reduced by 20 percent.

SOLUTION Nitrogen gas enters a converging nozzle. The properties at the nozzle exit are to be determined.

Assumptions **1** Nitrogen is an ideal gas with $k = 1.4$. **2** Flow through the nozzle is steady, one-dimensional, and isentropic.

Analysis The schematic of the duct is shown in Fig. 12–25. For isentropic flow through a duct, the area ratio A/A^* (the flow area over the area of the throat where Ma = 1) is also listed in Table A–13. At the initial Mach number of Ma = 0.3, we read

$$\frac{A_1}{A^*} = 2.0351 \qquad \frac{T_1}{T_0} = 0.9823 \qquad \frac{P_1}{P_0} = 0.9395$$

With a 20 percent reduction in flow area, $A_2 = 0.8A_1$, and

$$\frac{A_2}{A^*} = \frac{A_2}{A_1}\frac{A_1}{A^*} = (0.8)(2.0351) = 1.6281$$

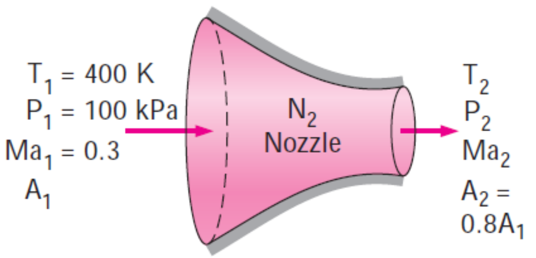


FIGURE 12–25
Schematic for Example 12–6 (not to scale).

For this value of A_2/A^* from Table A–13, we read

$$\frac{T_2}{T_0} = 0.9703 \qquad \frac{P_2}{P_0} = 0.9000 \qquad \mathrm{Ma}_2 = 0.391$$

Here we chose the subsonic Mach number for the calculated A_2/A^* instead of the supersonic one because the duct is converging in the flow direction and the initial flow is subsonic. Since the stagnation properties are constant for isentropic flow, we can write

$$\frac{T_2}{T_1} = \frac{T_2/T_0}{T_1/T_0} \quad \rightarrow \quad T_2 = T_1\left(\frac{T_2/T_0}{T_1/T_0}\right) = (400\ \mathrm{K})\left(\frac{0.9703}{0.9823}\right) = \mathbf{395\ K}$$

$$\frac{P_2}{P_1} = \frac{P_2/P_0}{P_1/P_0} \quad \rightarrow \quad P_2 = P_1\left(\frac{P_2/P_0}{P_1/P_0}\right) = (100\ \mathrm{kPa})\left(\frac{0.9000}{0.9395}\right) = \mathbf{95.8\ kPa}$$

which are the temperature and pressure at the desired location.

Discussion Note that the temperature and pressure drop as the fluid accelerates in a converging nozzle.

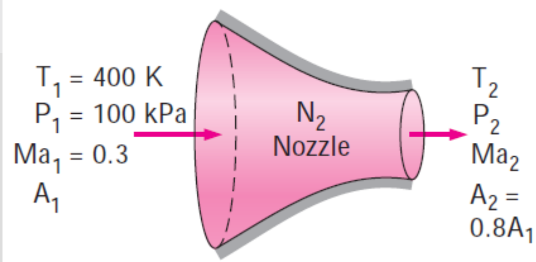
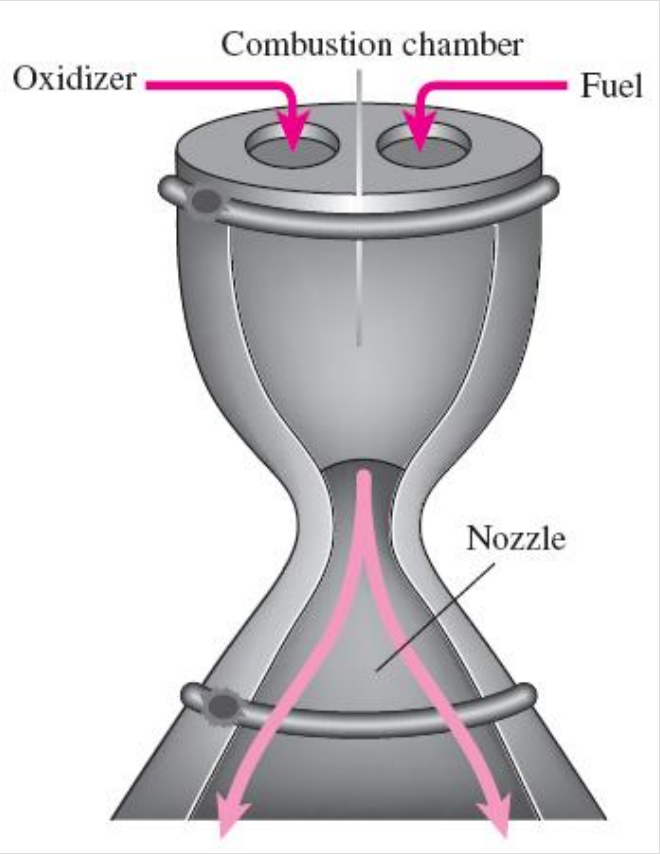


FIGURE 12–25
Schematic for Example 12–6 (not to scale).

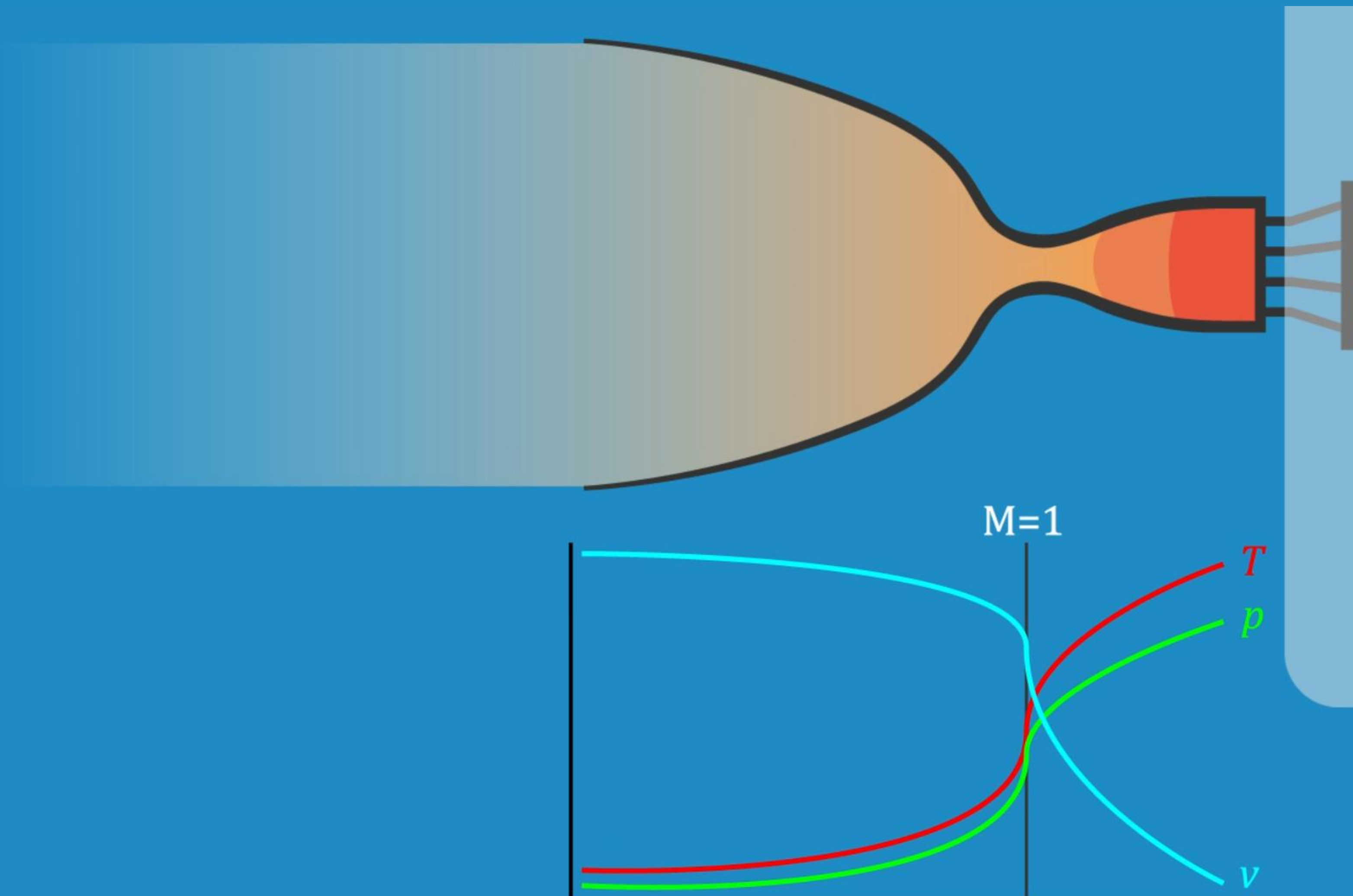
Converging–Diverging Nozzles

The highest velocity in a converging nozzle is limited to the sonic velocity (Ma = 1), which occurs at the exit plane (throat) of the nozzle.

Accelerating a fluid to supersonic velocities (Ma > 1) can be accomplished only by attaching a diverging flow section to the subsonic nozzle at the throat (a converging–diverging nozzle), which is standard equipment in supersonic aircraft and rocket propulsion.

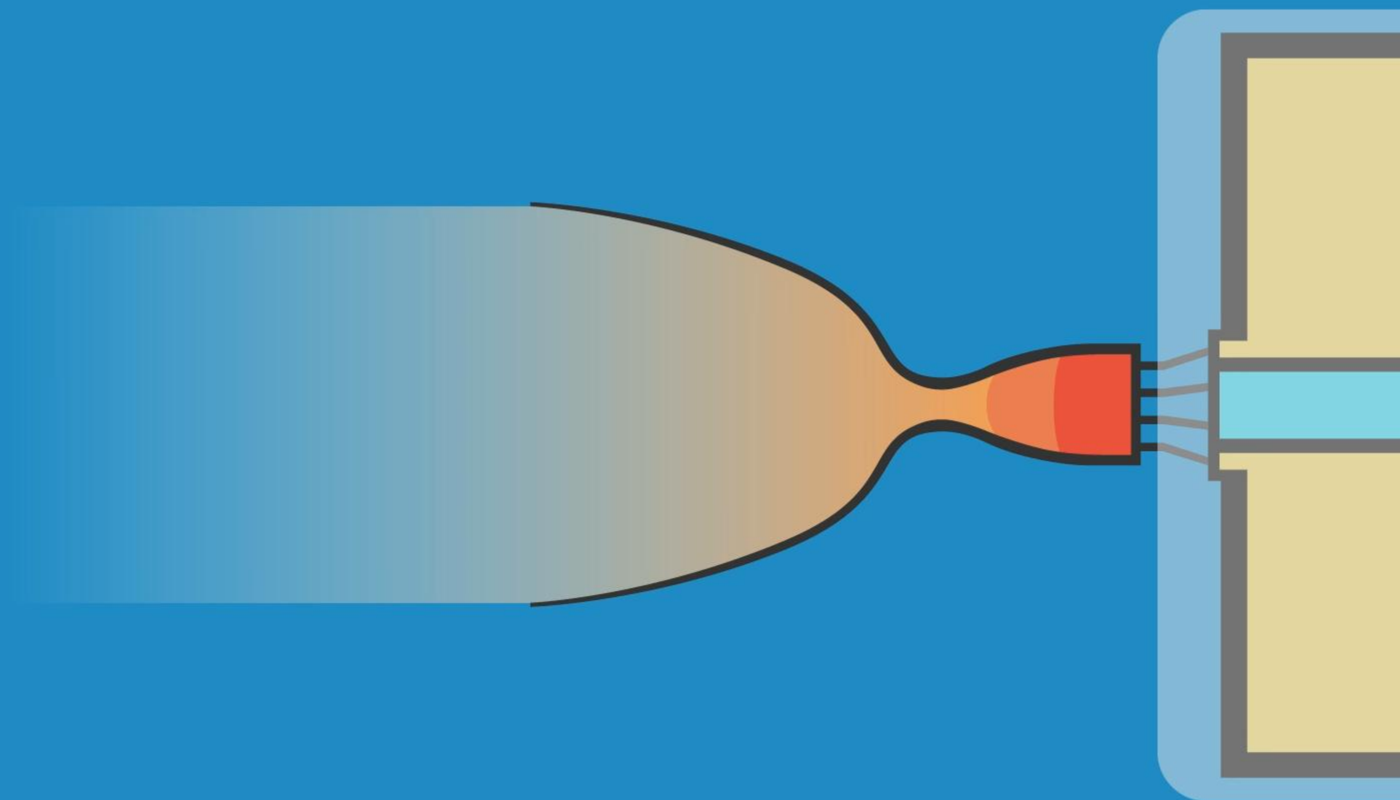


Converging–diverging nozzles are commonly used in rocket engines to provide high thrust.



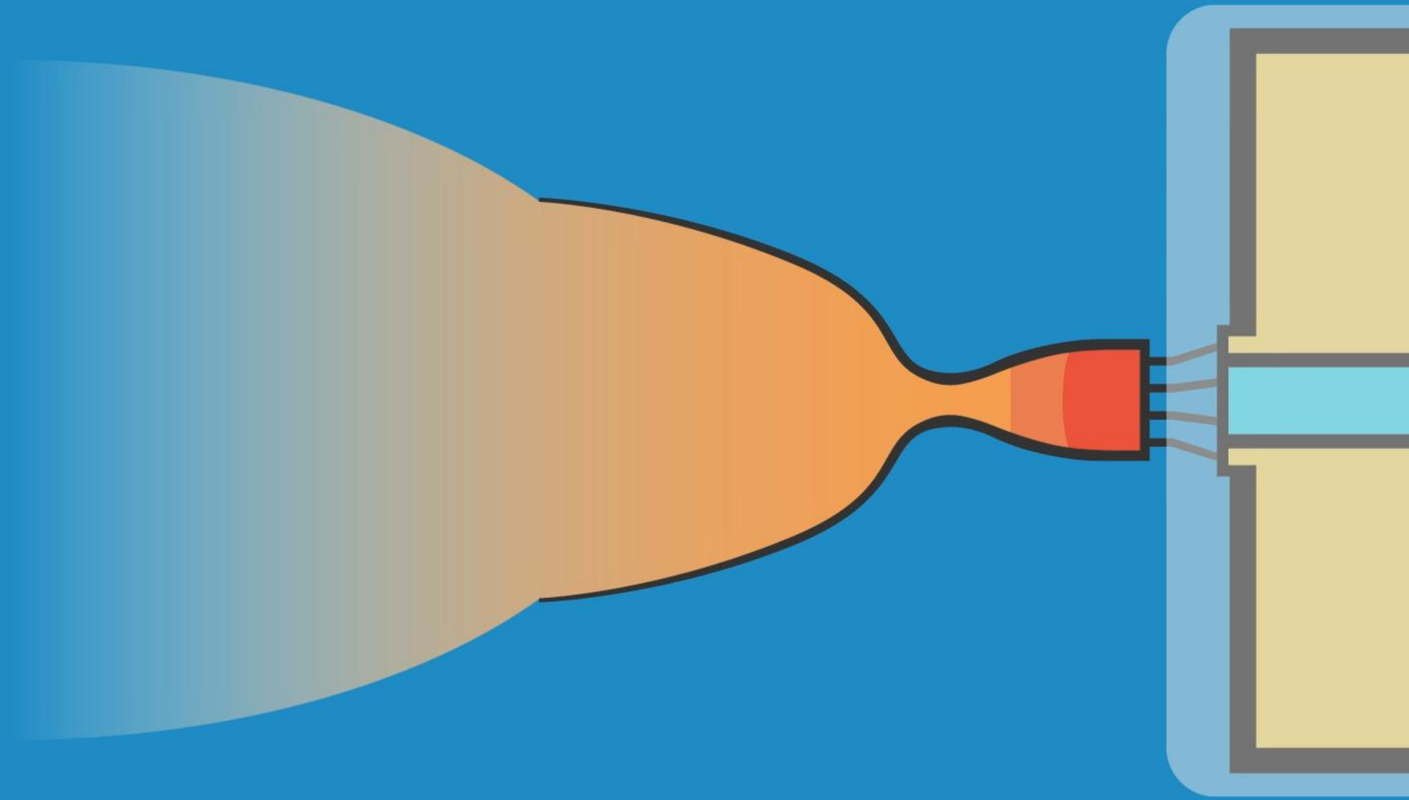
M=1
T
p
v

$$p_e = p_a$$



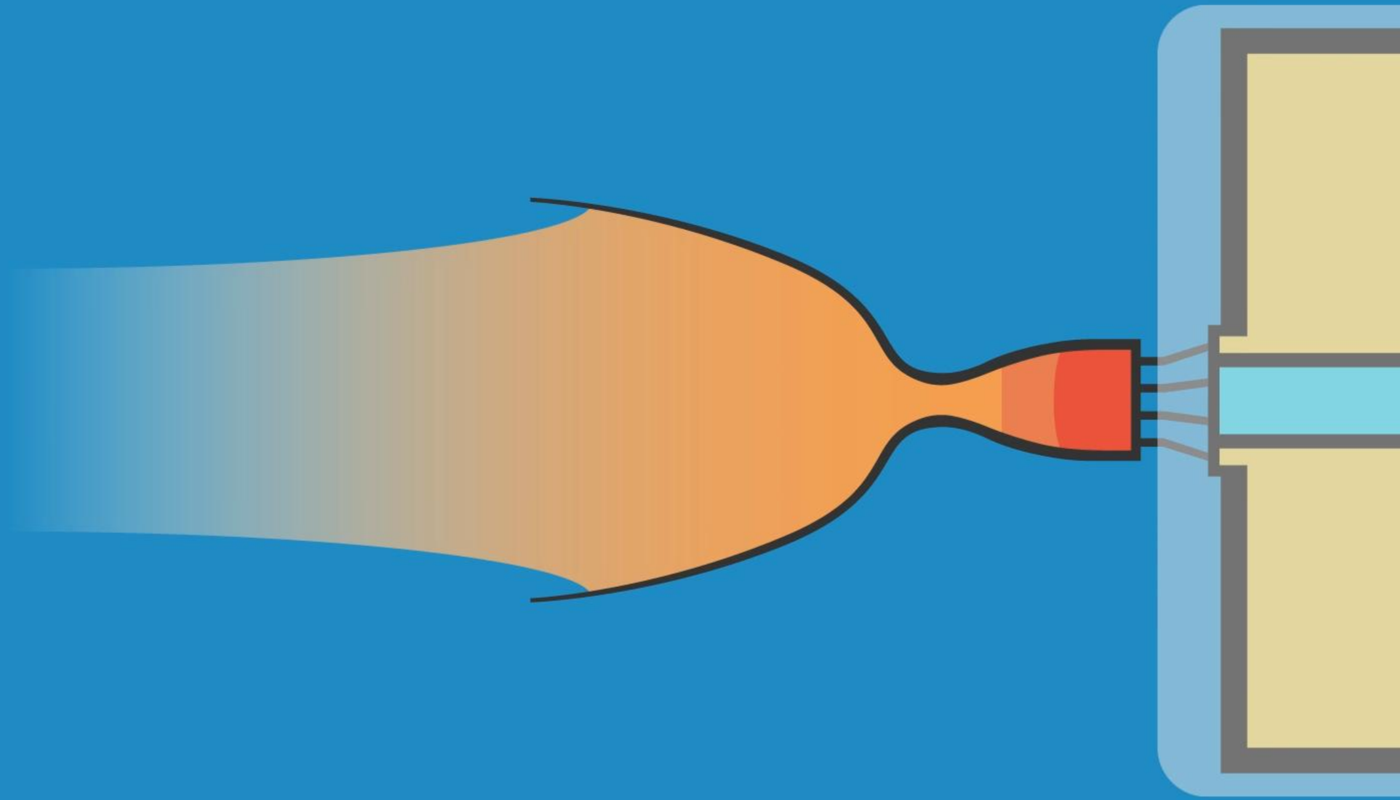
Under-expanded

$$p_e > p_a$$

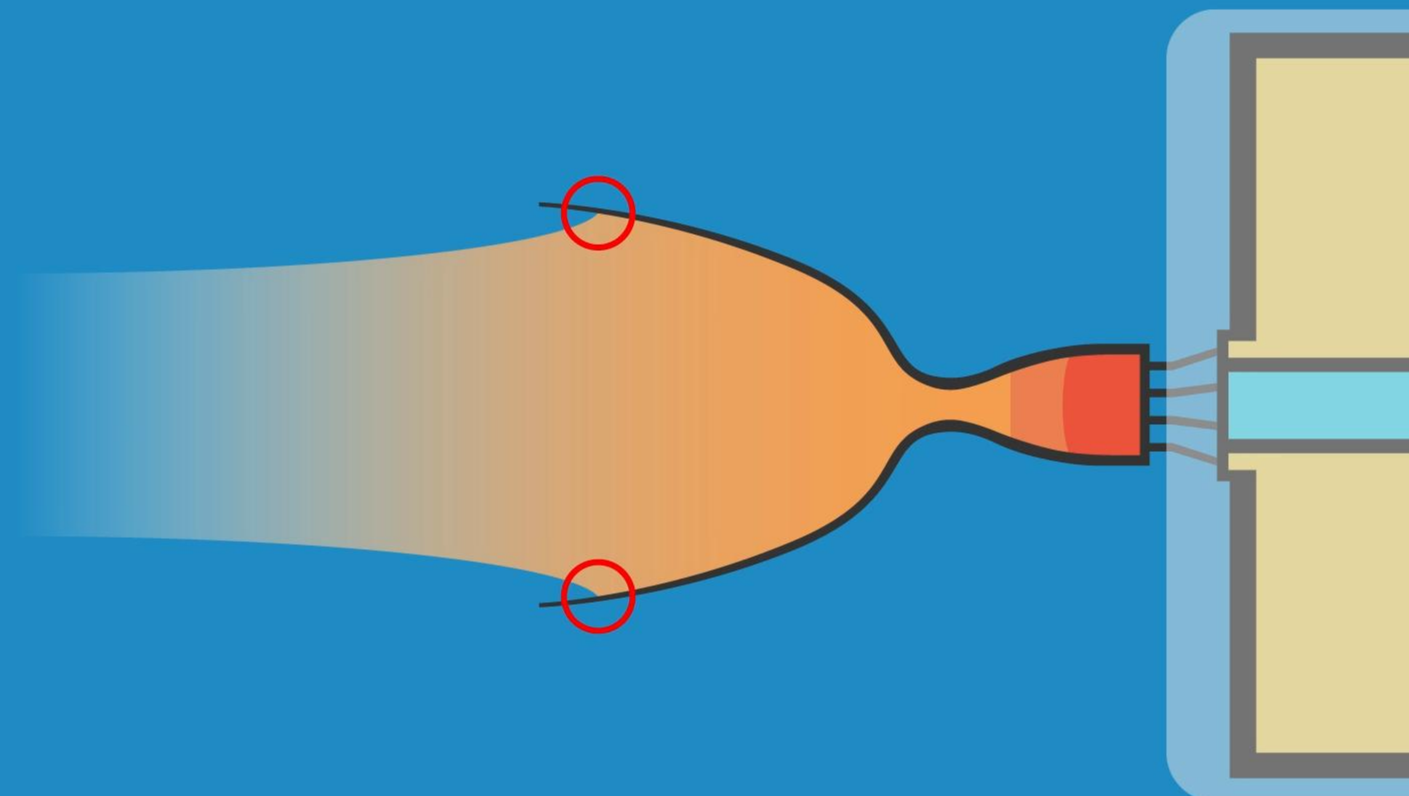


Over-expanded

$p_e < p_a$



$p_e < p_a$

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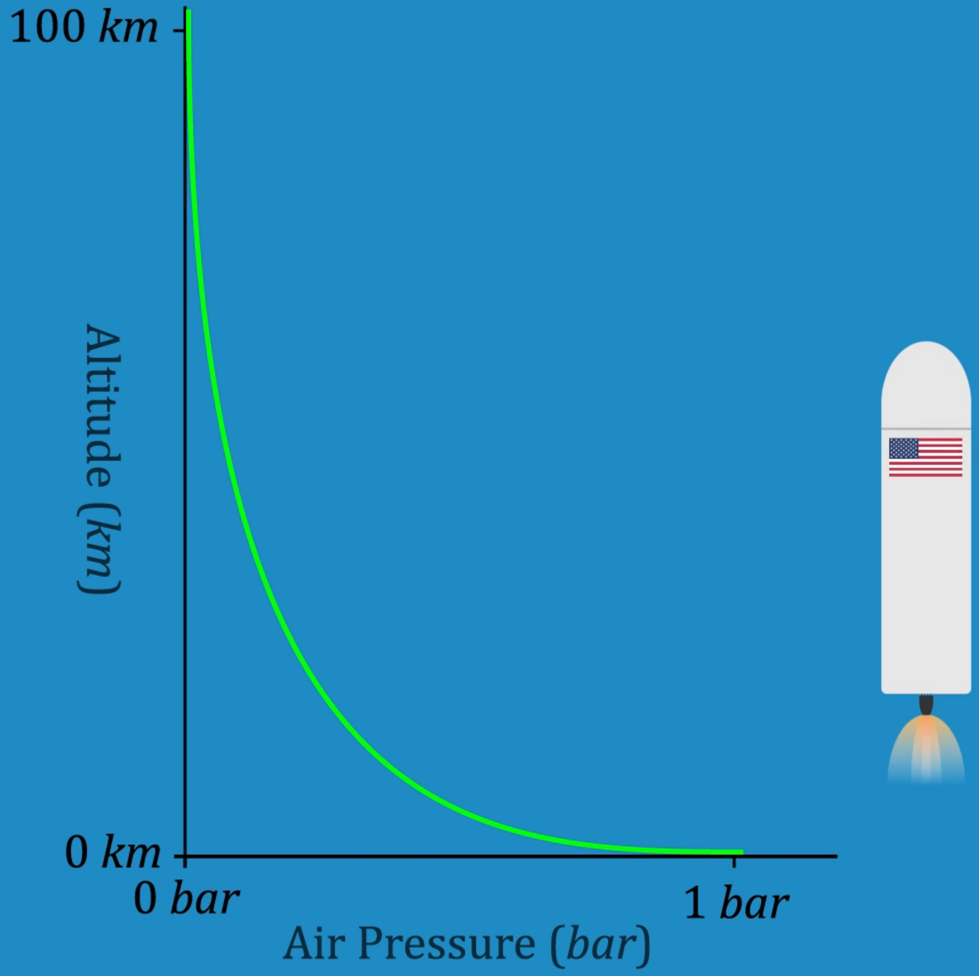
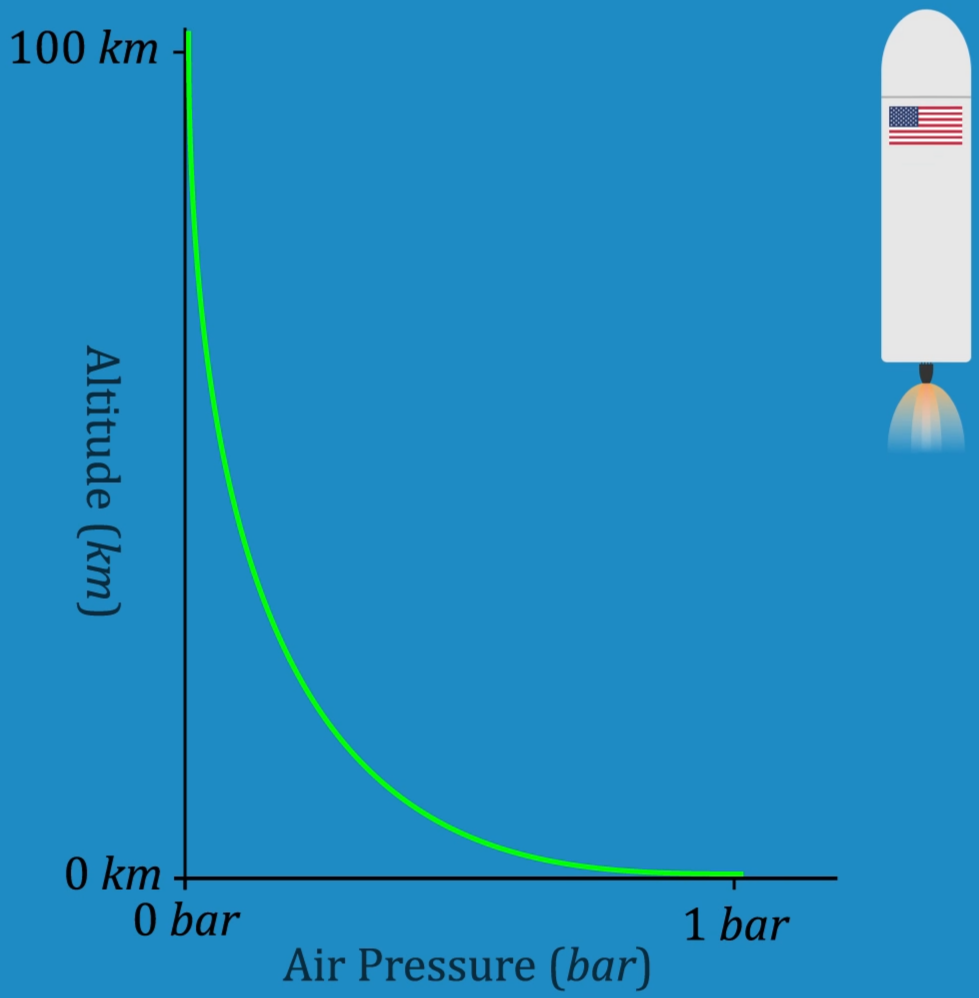
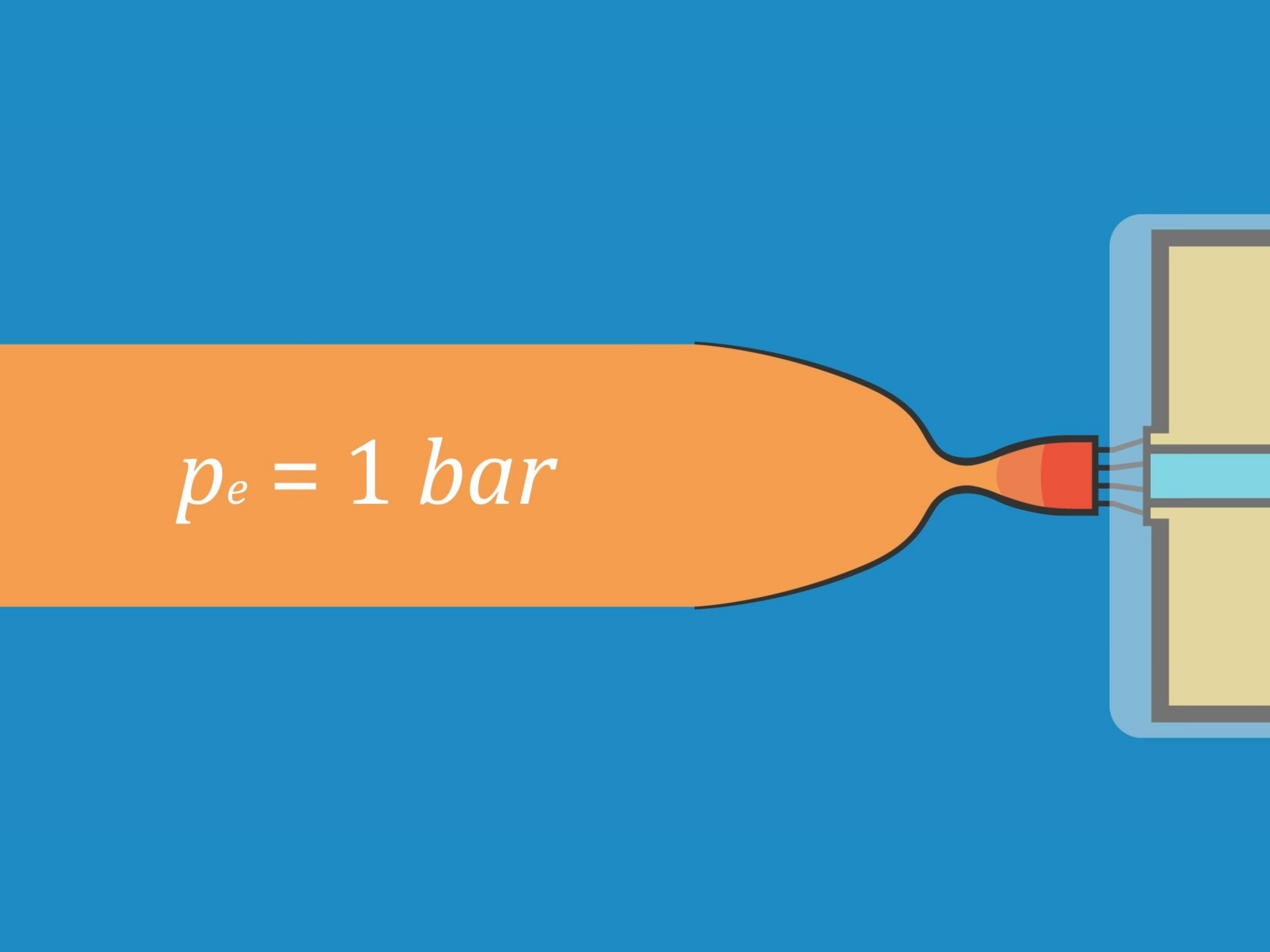
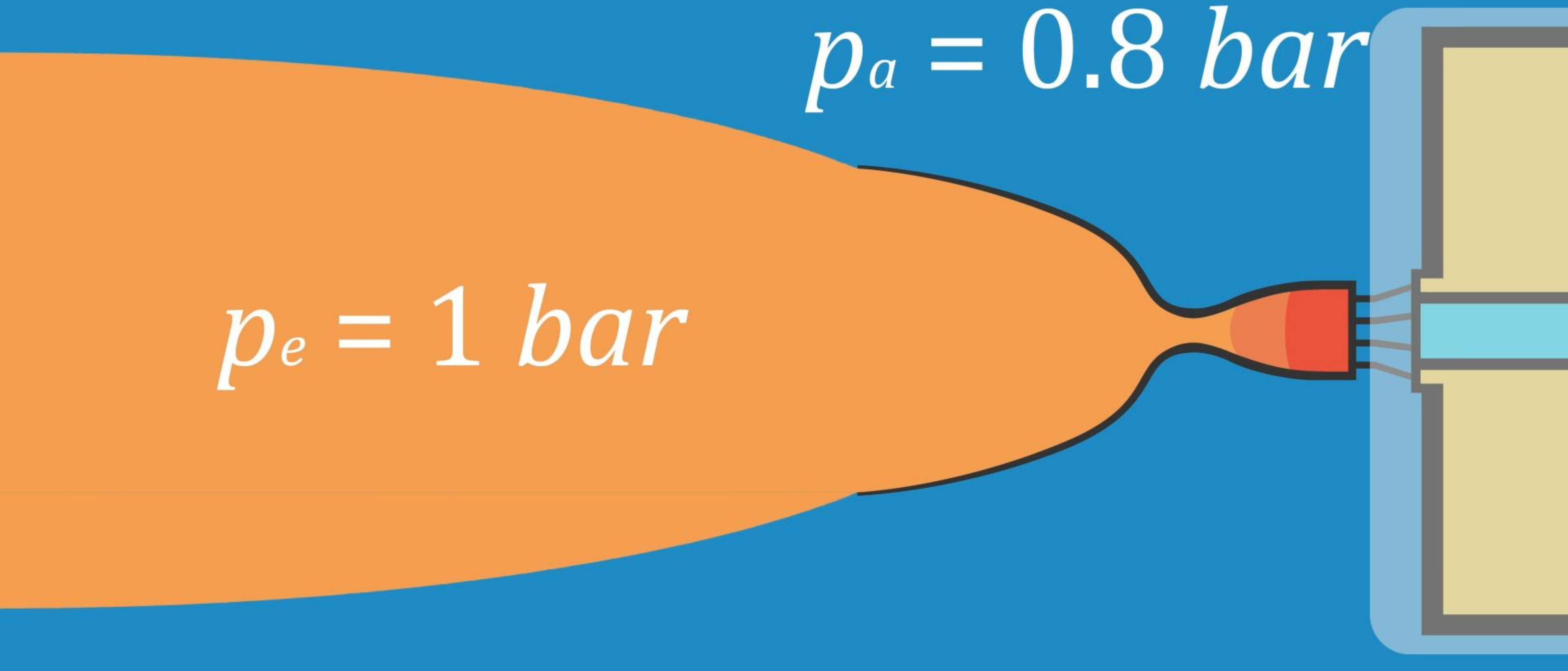


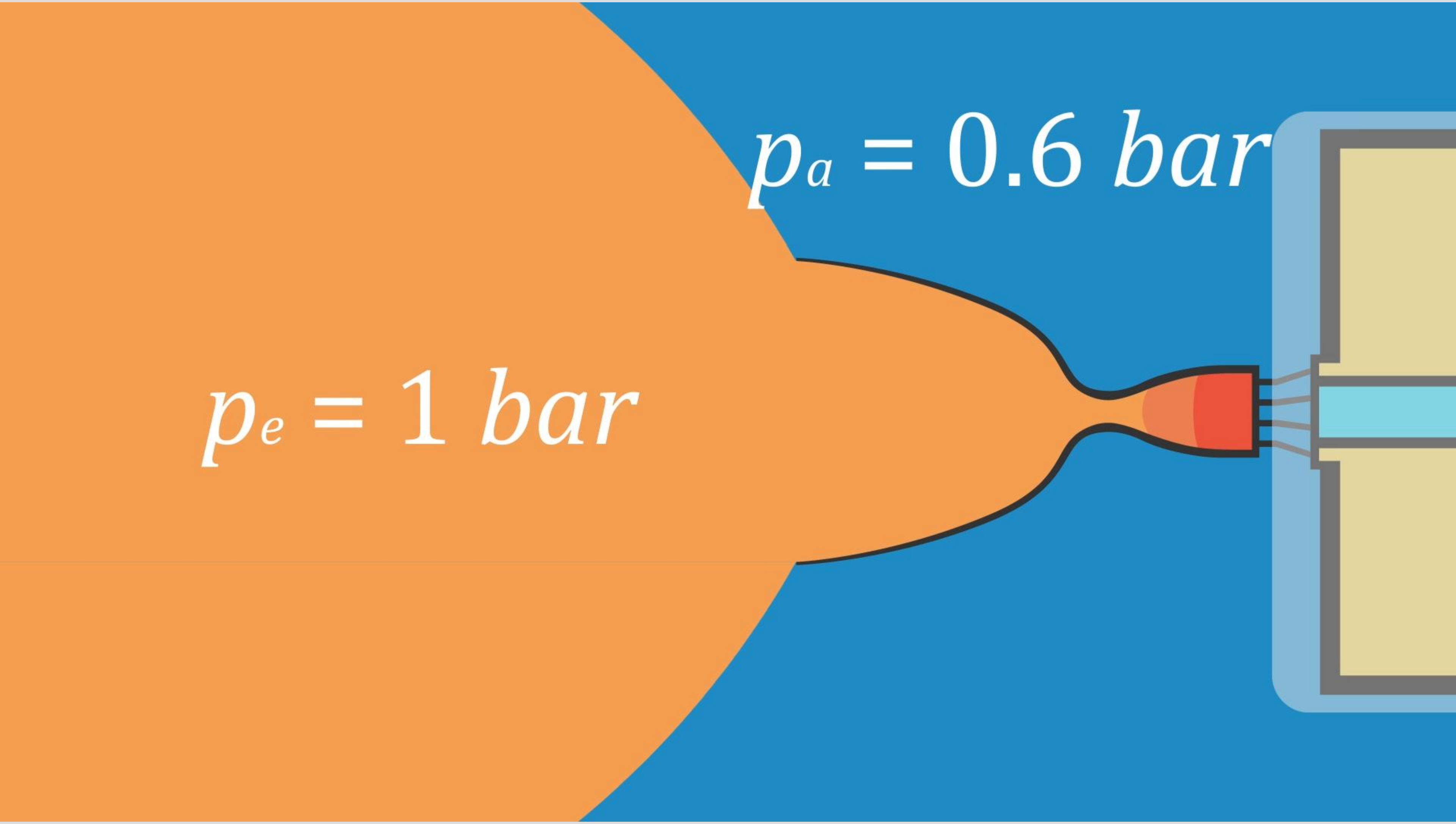
100 km
0 km
Altitude (km)
0 bar
1 bar
Air Pressure (bar)

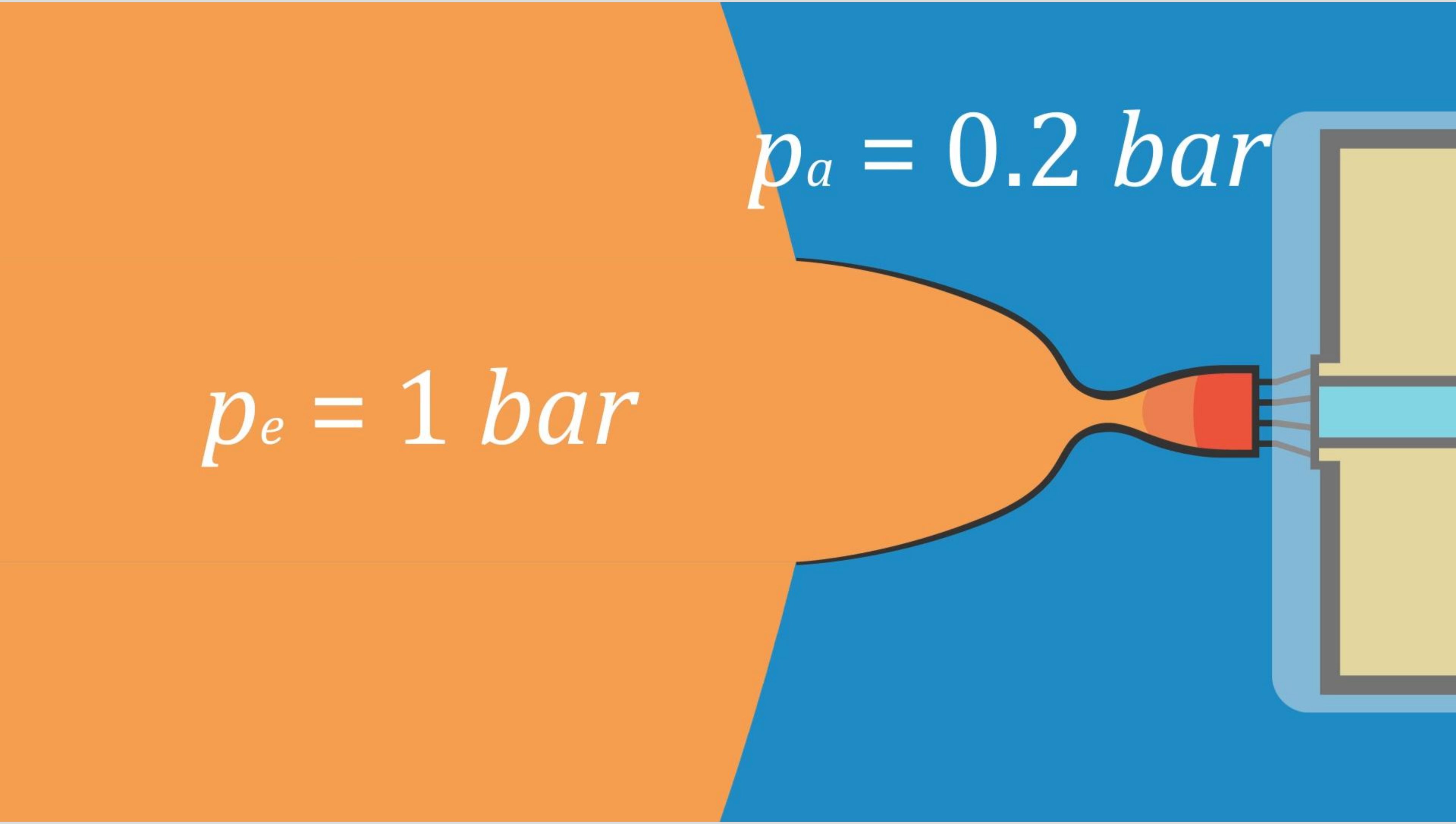

100 km
Altitude (km)
0 km
0 bar
1 bar
Air Pressure (bar)

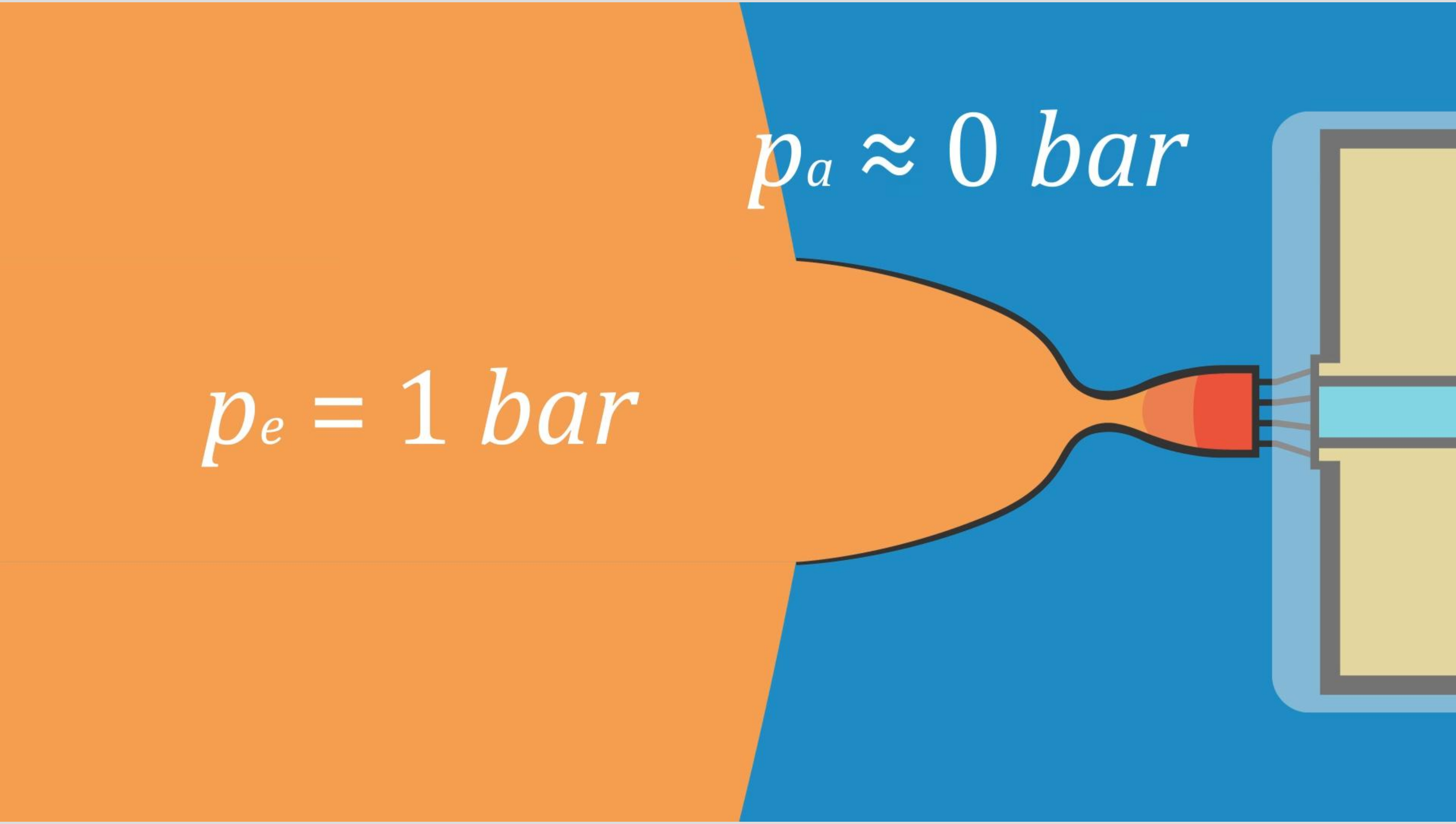

$p_e = 1\ bar$



$p_a = 0.8\ bar$
$p_e = 1\ bar$

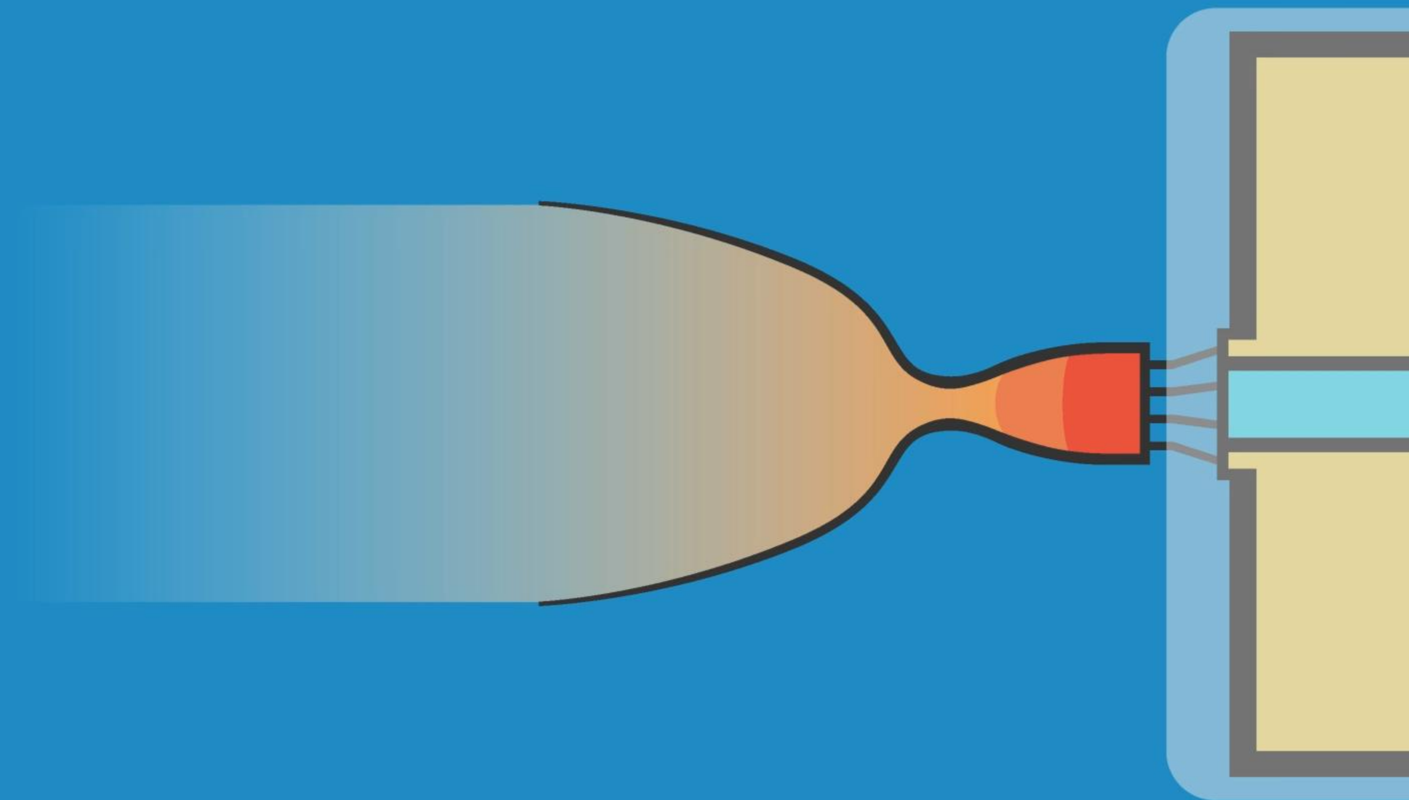

$p_a = 0.6\ bar$
$p_e = 1\ bar$


$p_a = 0.2\ bar$
$p_e = 1\ bar$



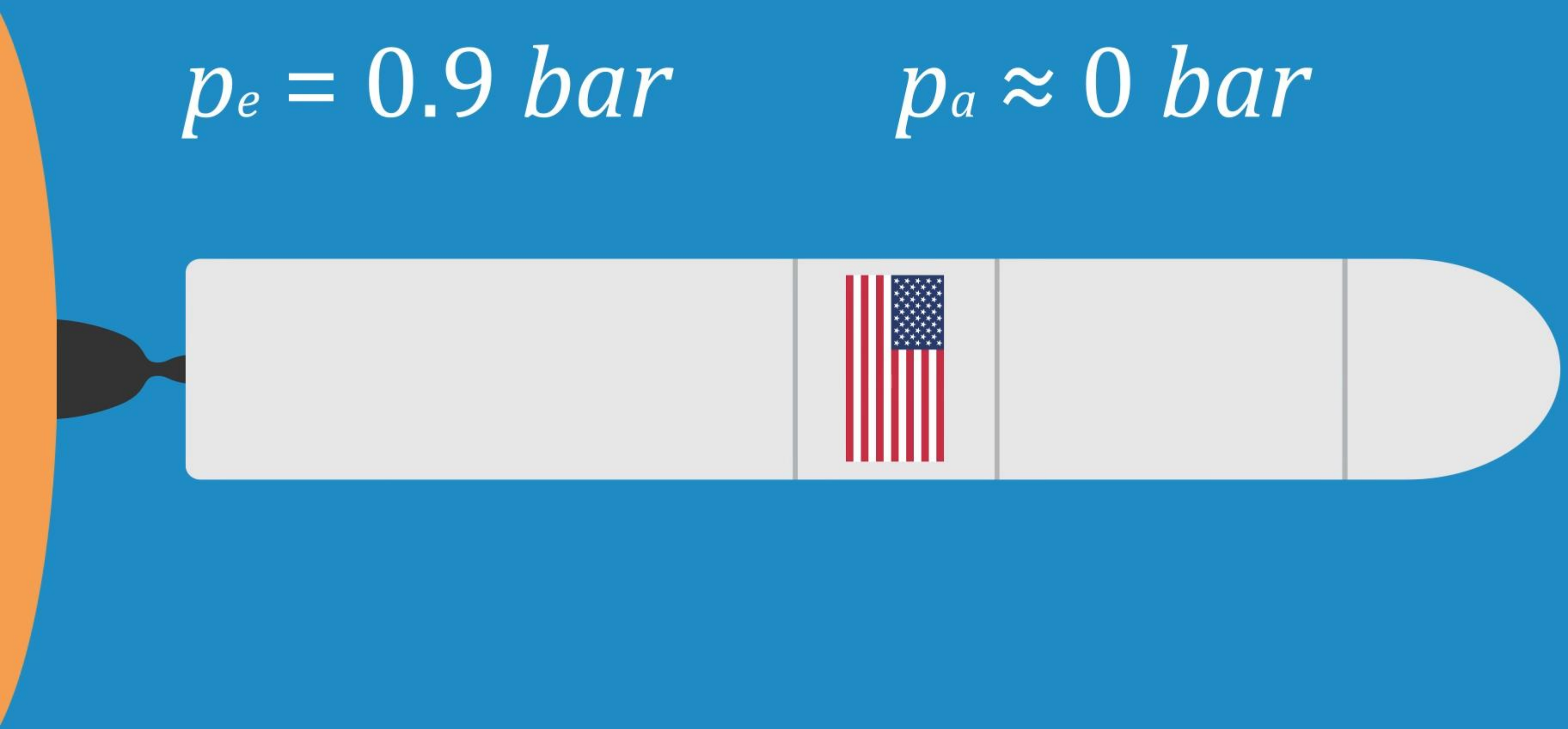
$p_a \approx 0\ bar$
$p_e = 1\ bar$

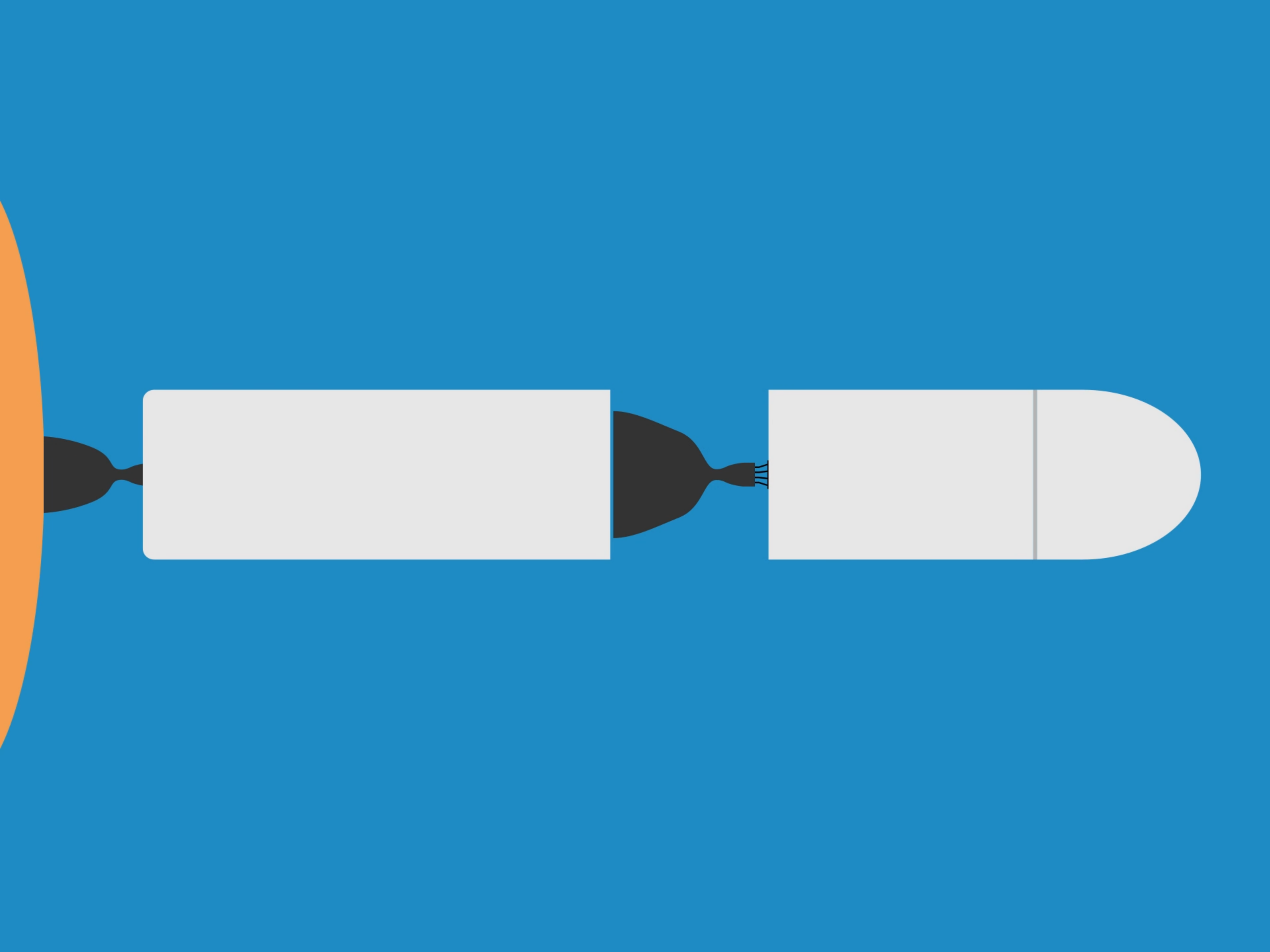
$$p_e = ?$$



$$p_e = p_a = 0.9\ bar$$

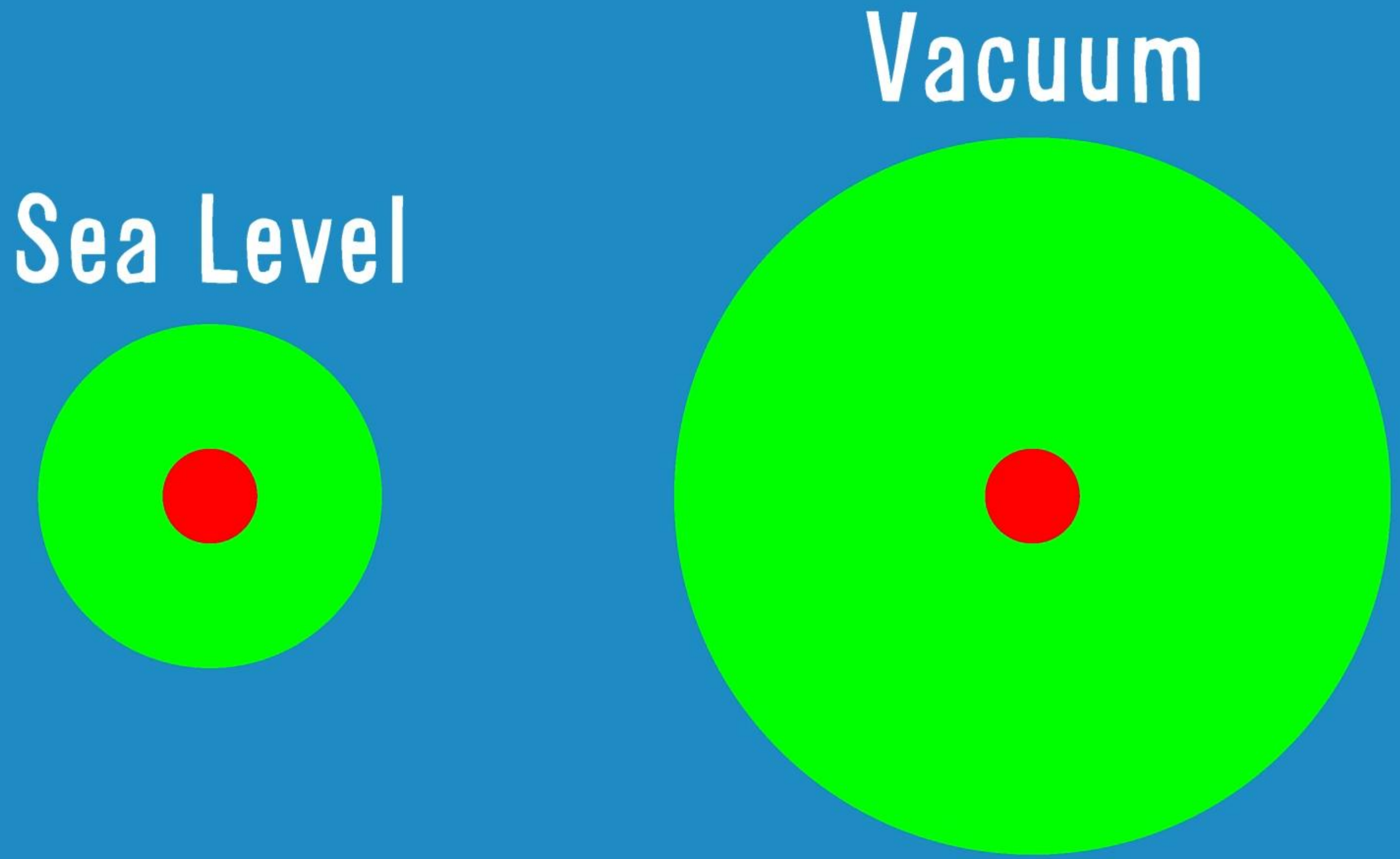
$p_e = 0.9\ bar$ $\qquad$ $p_a \approx 0\ bar$






Merlin 1D

Merlin 1D Vac

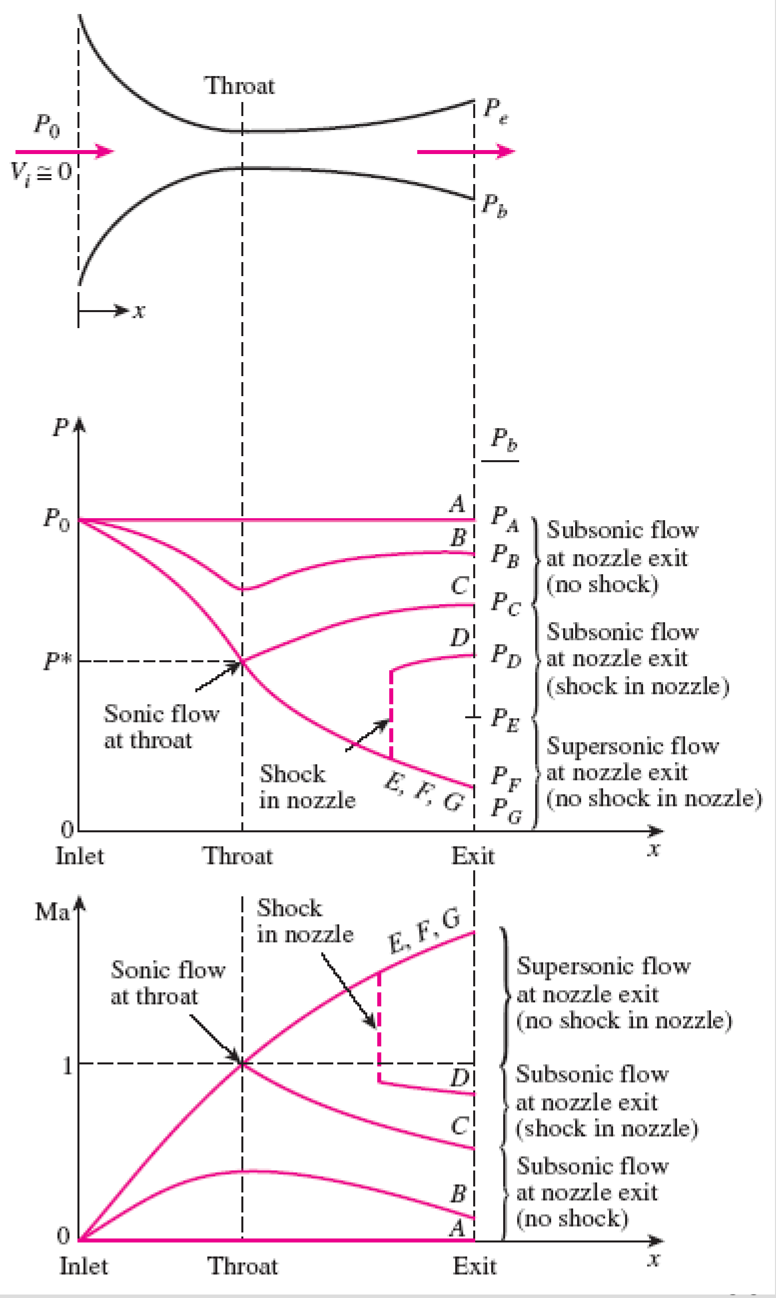


Merlin 1D:
Expansion Ratio 16:1

Merlin 1D-V:
Expansion Ratio 165:1

When $P_b = P_0$ (case *A*), there will be no flow through the nozzle.

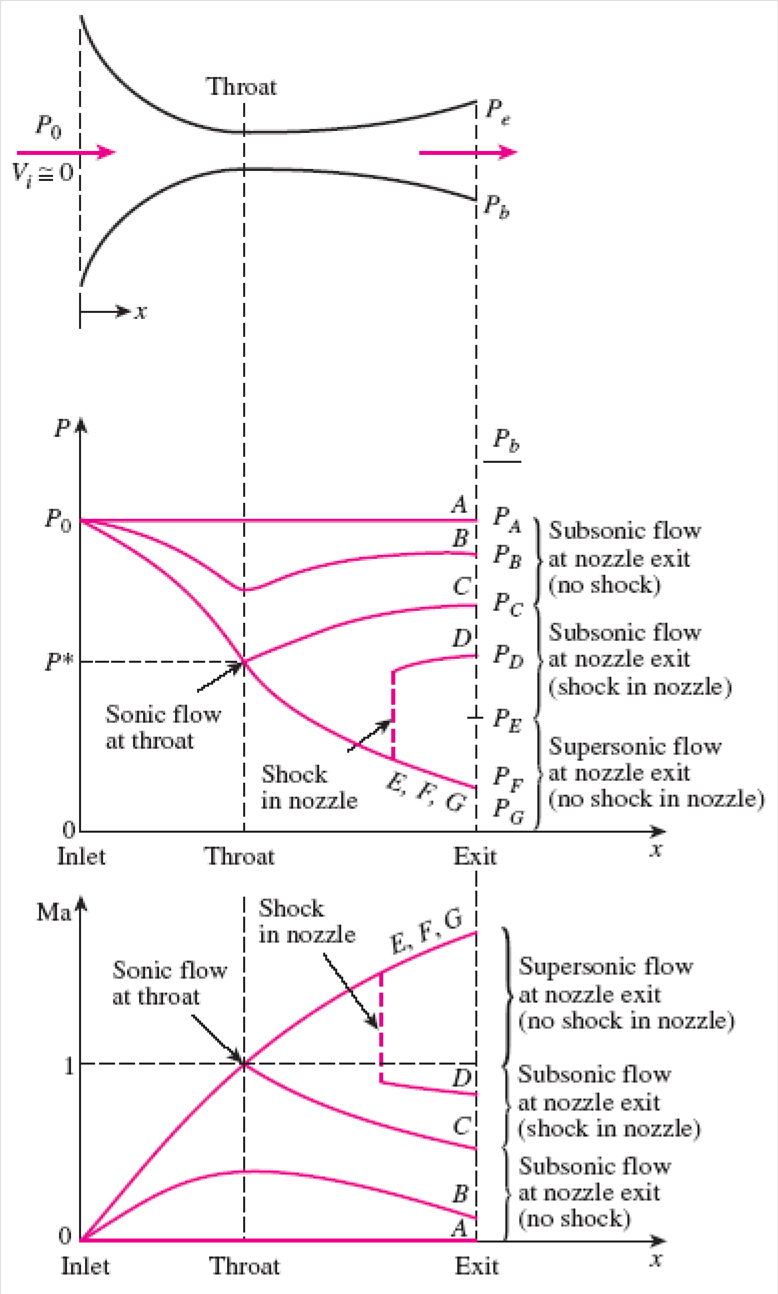
1. When $P_0 > P_b > P_C$, the flow remains subsonic throughout the nozzle, and the mass flow is less than that for choked flow. The fluid velocity increases in the first (converging) section and reaches a maximum at the throat (but Ma < 1). However, most of the gain in velocity is lost in the second (diverging) section of the nozzle, which acts as a diffuser. The pressure decreases in the converging section, reaches a minimum at the throat, and increases at the expense of velocity in the diverging section.



The effects of back pressure on the flow through a converging–diverging nozzle.

2. When $P_b = P_C$, the throat pressure becomes P^* and the fluid achieves sonic velocity at the throat. But the diverging section of the nozzle still acts as a diffuser, slowing the fluid to subsonic velocities. The mass flow rate that was increasing with decreasing P_b also reaches its maximum value.

3. When $P_C > P_b > P_E$, the fluid that achieved a sonic velocity at the throat continues accelerating to supersonic velocities in the diverging section as the pressure decreases. This acceleration comes to a sudden stop, however, as a **normal shock** develops at a section between the throat and the exit plane, which causes a sudden drop in velocity to subsonic levels and a sudden increase in pressure. The fluid then continues to decelerate further in the remaining part of the converging–diverging nozzle.

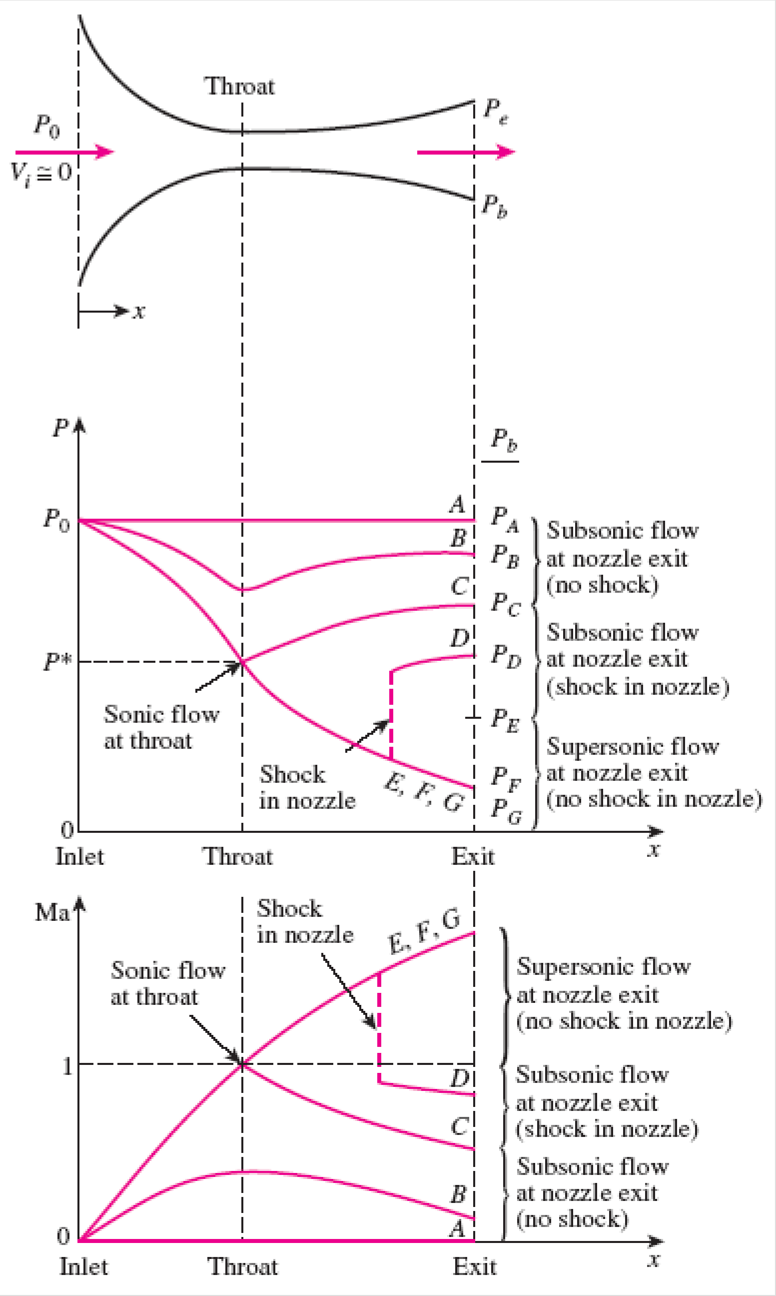


4. When P_E > Pb > 0, the flow in the diverging section is supersonic, and the fluid expands to P_F at the nozzle exit with no normal shock forming within the nozzle. Thus, the flow through the nozzle can be approximated as isentropic.

When $P_b = P_F$, no shocks occur within or outside the nozzle.

When $P_b < P_F$, irreversible mixing and expansion waves occur downstream of the exit plane of the nozzle.

When $P_b > P_F$, however, the pressure of the fluid increases from P_F to P_b irreversibly in the wake of the nozzle exit, creating what are called *oblique shocks.*



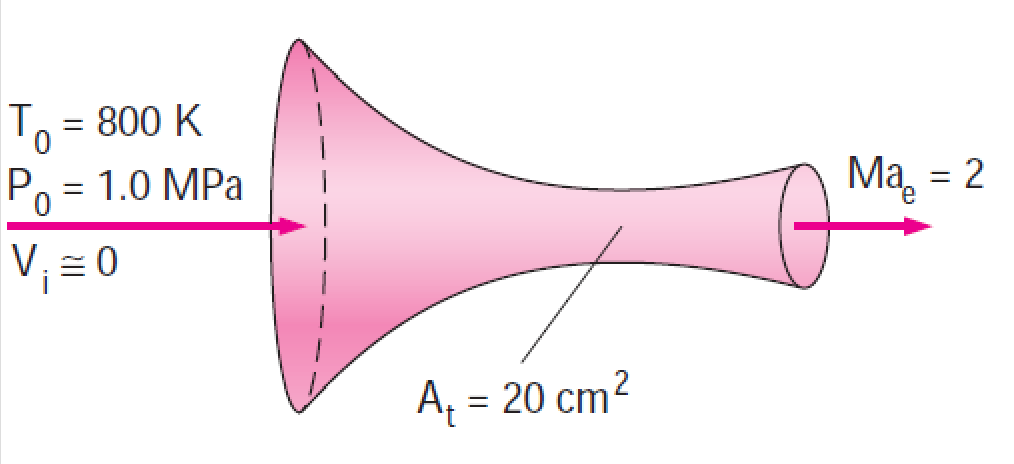
EXAMPLE 12–7 **Airflow through a Converging–Diverging Nozzle**

Air enters a converging–diverging nozzle, shown in Fig. 12–28, at 1.0 MPa and 800 K with a negligible velocity. The flow is steady, one-dimensional, and isentropic with $k = 1.4$. For an exit Mach number of Ma = 2 and a throat area of 20 cm^2, determine (*a*) the throat conditions, (*b*) the exit plane conditions, including the exit area, and (*c*) the mass flow rate through the nozzle.

SOLUTION Air flows through a converging–diverging nozzle. The throat and the exit conditions and the mass flow rate are to be determined.

Assumptions **1** Air is an ideal gas with constant specific heats at room temperature. **2** Flow through the nozzle is steady, one-dimensional, and isentropic.

Properties The specific heat ratio of air is given to be $k = 1.4$. The gas constant of air is 0.287 kJ/kg · K.



Analysis The exit Mach number is given to be 2. Therefore, the flow must be sonic at the throat and supersonic in the diverging section of the nozzle. Since the inlet velocity is negligible, the stagnation pressure and stagnation temperature are the same as the inlet temperature and pressure, $P_0 = 1.0$ MPa and $T_0 = 800$ K. Assuming ideal-gas behavior, the stagnation density is

$$\rho_0 = \frac{P_0}{RT_0} = \frac{1000 \text{ kPa}}{(0.287 \text{ kPa} \cdot \text{m}^3/\text{kg} \cdot \text{K})(800 \text{ K})} = 4.355 \text{ kg/m}^3$$

(*a*) At the throat of the nozzle Ma = 1, and from Table A–13 we read

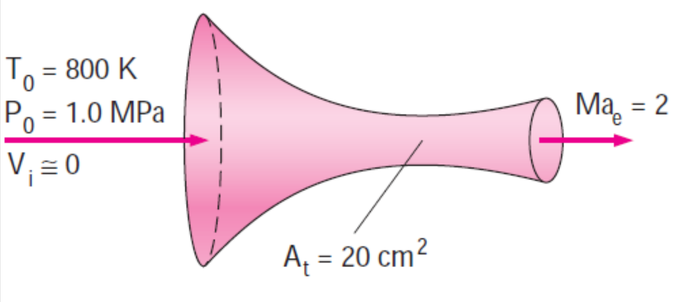
$$\frac{P^*}{P_0} = 0.5283 \qquad \frac{T^*}{T_0} = 0.8333 \qquad \frac{\rho^*}{\rho_0} = 0.6339$$

Thus,

$$P^* = 0.5283P_0 = (0.5283)(1.0 \text{ MPa}) = \mathbf{0.5283 \text{ MPa}}$$

$$T^* = 0.8333T_0 = (0.8333)(800 \text{ K}) = \mathbf{666.6 \text{ K}}$$

$$\rho^* = 0.6339\rho_0 = (0.6339)(4.355 \text{ kg/m}^3) = \mathbf{2.761 \text{ kg/m}^3}$$



Also,

$$V^* = c^* = \sqrt{kRT^*} = \sqrt{(1.4)(0.287\ \text{kJ/kg}\cdot\text{K})(666.6\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)}$$

$$= \mathbf{517.5\ m/s}$$

(*b*) Since the flow is isentropic, the properties at the exit plane can also be calculated by using data from Table A–13. For Ma = 2 we read

$$\frac{P_e}{P_0} = 0.1278 \quad \frac{T_e}{T_0} = 0.5556 \quad \frac{\rho_e}{\rho_0} = 0.2300 \quad \text{Ma}_e^* = 1.6330 \quad \frac{A_e}{A^*} = 1.6875$$

Thus,

$$P_e = 0.1278P_0 = (0.1278)(1.0\ \text{MPa}) = \mathbf{0.1278\ MPa}$$

$$T_e = 0.5556T_0 = (0.5556)(800\ \text{K}) = \mathbf{444.5\ K}$$

$$\rho_e = 0.2300\rho_0 = (0.2300)(4.355\ \text{kg/m}^3) = \mathbf{1.002\ kg/m^3}$$

$$A_e = 1.6875A^* = (1.6875)(20\ \text{cm}^2) = \mathbf{33.75\ cm^2}$$

and

$$V_e = \text{Ma}_e^* c^* = (1.6330)(517.5\ \text{m/s}) = \mathbf{845.1\ m/s}$$

The nozzle exit velocity could also be determined from $V_e = \mathrm{Ma}_e c_e$, where c_e is the speed of sound at the exit conditions:

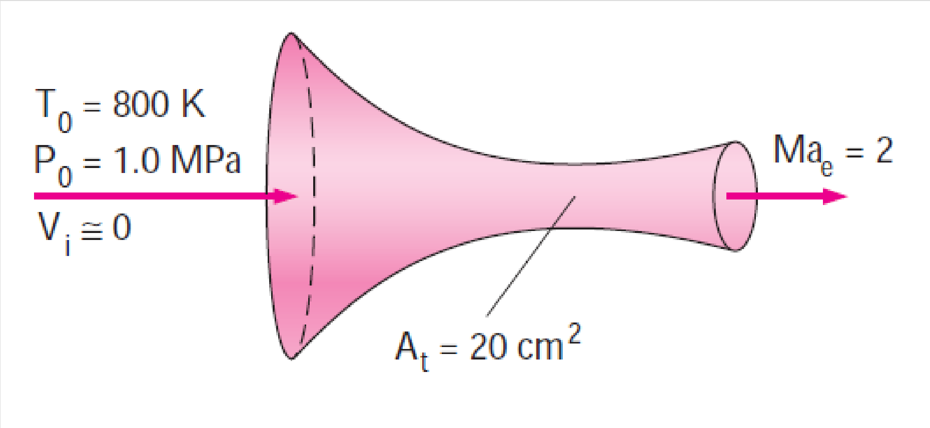
$$V_e = \mathrm{Ma}_e c_e = \mathrm{Ma}_e \sqrt{kRT_e} = 2\sqrt{(1.4)(0.287 \text{ kJ/kg}\cdot\text{K})(444.5 \text{ K})\left(\frac{1000 \text{ m}^2/\text{s}^2}{1 \text{ kJ/kg}}\right)}$$

$$= 845.2 \text{ m/s}$$

(c) Since the flow is steady, the mass flow rate of the fluid is the same at all sections of the nozzle. Thus it may be calculated by using properties at any cross section of the nozzle. Using the properties at the throat, we find that the mass flow rate is

$$\dot{m} = \rho^* A^* V^* = (2.761 \text{ kg/m}^3)(20 \times 10^{-4} \text{ m}^2)(517.5 \text{ m/s}) = \mathbf{2.86 \text{ kg/s}}$$

Discussion Note that this is the highest possible mass flow rate that can flow through this nozzle for the specified inlet conditions.



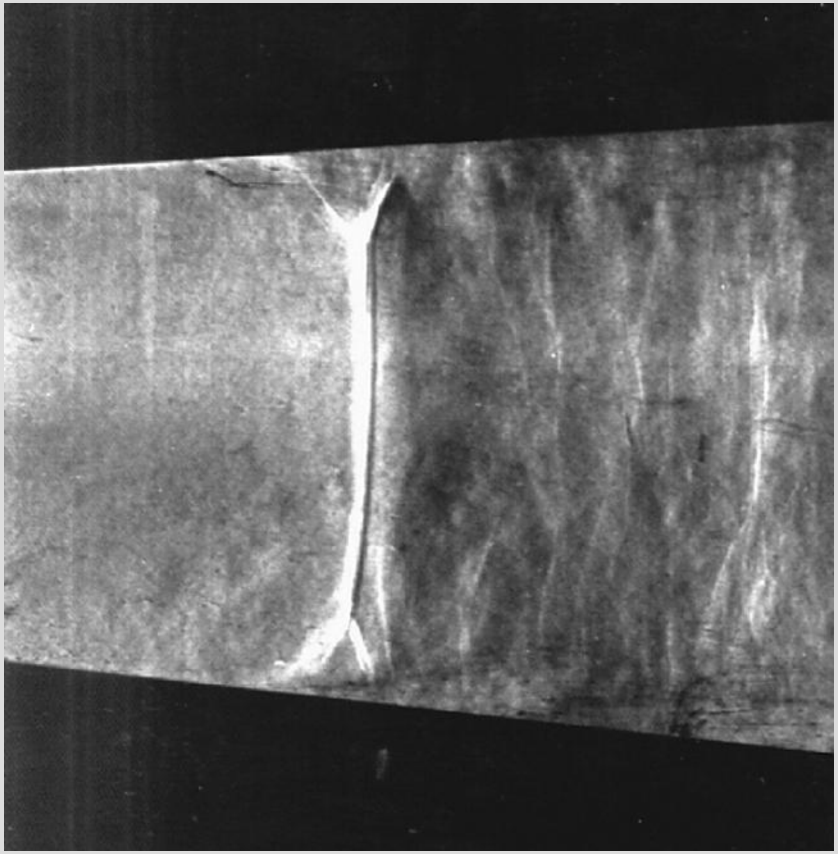
SHOCK WAVES AND EXPANSION WAVES

For some back pressure values, abrupt changes in fluid properties occur in a very thin section of a converging–diverging nozzle under supersonic flow conditions, creating a **shock wave**.

We study the conditions under which shock waves develop and how they affect the flow.

Normal Shocks

Normal shock waves: The shock waves that occur in a plane normal to the direction of flow. The flow process through the shock wave is highly irreversible.



Schlieren image of a normal shock in a Laval nozzle. The Mach number in the nozzle just upstream (to the left) of the shock wave is about 1.3. Boundary layers distort the shape of the normal shock near the walls and lead to flow separation beneath the shock.

Conservation of mass

$$\rho_1 A V_1 = \rho_2 A V_2 \rightarrow \rho_1 V_1 = \rho_2 V_2$$

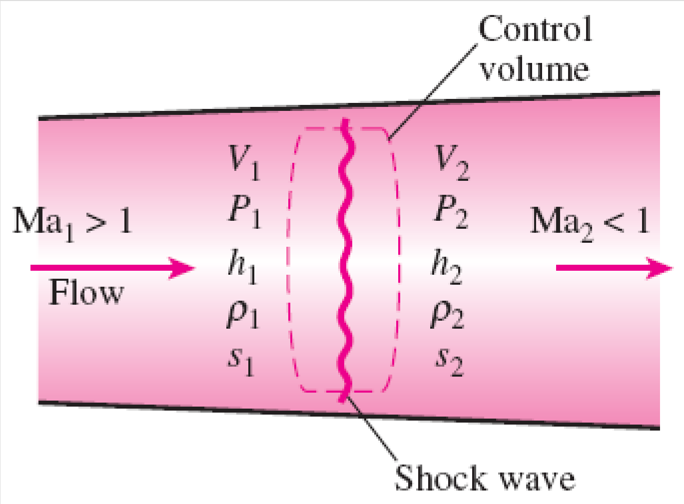
Conservation of energy

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \rightarrow h_{01} = h_{02}$$

Conservation of momentum

$$A(P_1 - P_2) = \dot{m}(V_2 - V_1)$$

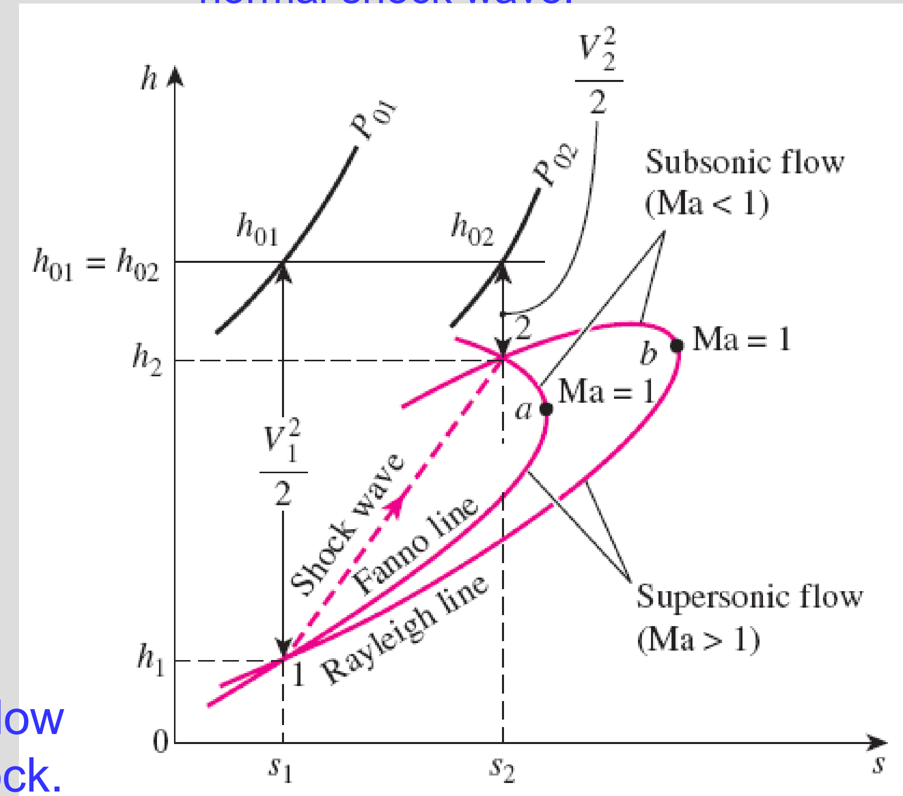
$s_2 - s_1 \geq 0$ Increase of entropy



Control volume for flow across a normal shock wave.

Fanno line: Combining the conservation of mass and energy relations into a single equation and plotting it on an *h*-*s* diagram yield a curve. It is the locus of states that have the same value of stagnation enthalpy and mass flux.

Rayleigh line: Combining the conservation of mass and momentum equations into a single equation and plotting it on the *h*-*s* diagram yield a curve.



The *h*-*s* diagram for flow across a normal shock.

The relations between various properties before and after the shock for an ideal gas with constant specific heats.

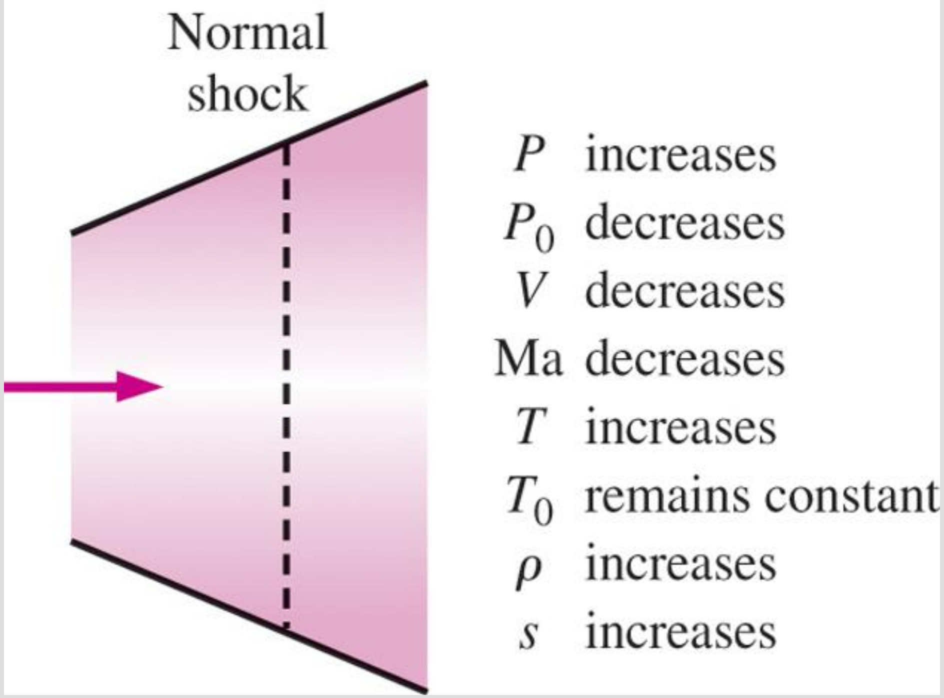
$$\frac{T_2}{T_1} = \frac{P_2 V_2}{P_1 V_1} = \frac{P_2 \mathrm{Ma}_2 c_2}{P_1 \mathrm{Ma}_1 c_1} = \frac{P_2 \mathrm{Ma}_2 \sqrt{T_2}}{P_1 \mathrm{Ma}_1 \sqrt{T_1}} = \left(\frac{P_2}{P_1}\right)^2 \left(\frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\right)^2$$

$$\frac{P_2}{P_1} = \frac{\mathrm{Ma}_1 \sqrt{1 + \mathrm{Ma}_1^2 (k-1)/2}}{\mathrm{Ma}_2 \sqrt{1 + \mathrm{Ma}_2^2 (k-1)/2}}$$

Various flow property ratios across the shock are listed in Table A–33.

$$\mathrm{Ma}_2^2 = \frac{\mathrm{Ma}_1^2 + 2/(k-1)}{2\mathrm{Ma}_1^2 k/(k-1) - 1}$$

This represents the intersections of the Fanno and Rayleigh lines.



Variation of flow properties across a normal shock.

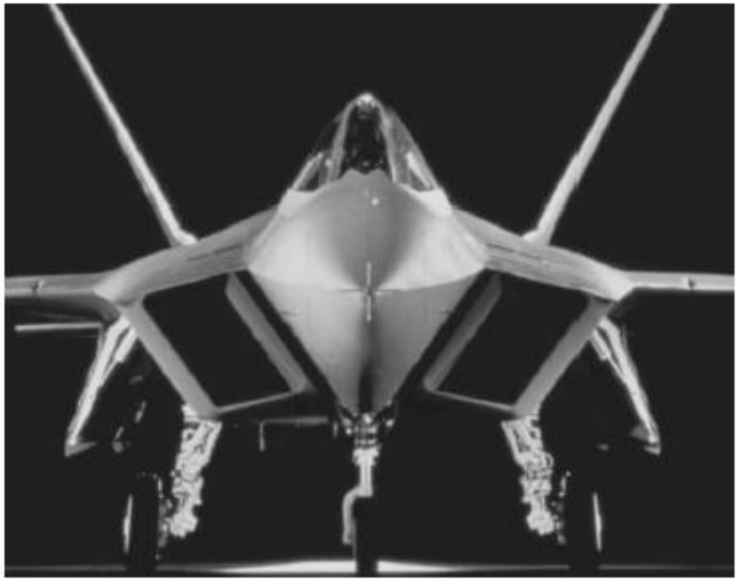


FIGURE 17–33
The air inlet of a supersonic fighter jet is designed such that a shock wave at the inlet decelerates the air to subsonic velocities, increasing the pressure and temperature of the air before it enters the engine.

Schlieren image of the blast wave (expanding spherical normal shock) produced by the explosion of a firecracker detonated inside a metal can that sat on a stool. The shock expanded radially outward in all directions at a supersonic speed that decreased with radius from the center of the explosion. The microphone at the lower right sensed the sudden change in pressure of the passing shock wave and triggered the microsecond flashlamp that exposed the photograph.

$$T_{01} = T_{02}$$

$$Ma_2 = \sqrt{\frac{(k-1)Ma_1^2 + 2}{2kMa_1^2 - k + 1}}$$

$$\frac{P_2}{P_1} = \frac{1 + kMa_1^2}{1 + kMa_2^2} = \frac{2kMa_1^2 - k + 1}{k + 1}$$

$$\frac{\rho_2}{\rho_1} = \frac{P_2/P_1}{T_2/T_1} = \frac{(k+1)Ma_1^2}{2 + (k-1)Ma_1^2} = \frac{V_1}{V_2}$$

$$\frac{T_2}{T_1} = \frac{2 + Ma_1^2(k-1)}{2 + Ma_2^2(k-1)}$$

$$\frac{P_{02}}{P_{01}} = \frac{Ma_1}{Ma_2}\left[\frac{1 + Ma_2^2(k-1)/2}{1 + Ma_1^2(k-1)/2}\right]^{(k+1)/[2(k-1)]}$$

$$\frac{P_{02}}{P_1} = \frac{(1 + kMa_1^2)[1 + Ma_2^2(k-1)/2]^{k/(k-1)}}{1 + kMa_2^2}$$

TABLE A–14

One-dimensional normal shock functions for an ideal gas with $k = 1.4$

Ma_1	Ma_2	P_2/P_1	ρ_2/ρ_1	T_2/T_1	P_{02}/P_{01}	P_{02}/P_1
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.8929
1.1	0.9118	1.2450	1.1691	1.0649	0.9989	2.1328
1.2	0.8422	1.5133	1.3416	1.1280	0.9928	2.4075
1.3	0.7860	1.8050	1.5157	1.1909	0.9794	2.7136
1.4	0.7397	2.1200	1.6897	1.2547	0.9582	3.0492
1.5	0.7011	2.4583	1.8621	1.3202	0.9298	3.4133
1.6	0.6684	2.8200	2.0317	1.3880	0.8952	3.8050
1.7	0.6405	3.2050	2.1977	1.4583	0.8557	4.2238
1.8	0.6165	3.6133	2.3592	1.5316	0.8127	4.6695
1.9	0.5956	4.0450	2.5157	1.6079	0.7674	5.1418
2.0	0.5774	4.5000	2.6667	1.6875	0.7209	5.6404
2.1	0.5613	4.9783	2.8119	1.7705	0.6742	6.1654
2.2	0.5471	5.4800	2.9512	1.8569	0.6281	6.7165
2.3	0.5344	6.0050	3.0845	1.9468	0.5833	7.2937
2.4	0.5231	6.5533	3.2119	2.0403	0.5401	7.8969
2.5	0.5130	7.1250	3.3333	2.1375	0.4990	8.5261
2.6	0.5039	7.7200	3.4490	2.2383	0.4601	9.1813
2.7	0.4956	8.3383	3.5590	2.3429	0.4236	9.8624
2.8	0.4882	8.9800	3.6636	2.4512	0.3895	10.5694
2.9	0.4814	9.6450	3.7629	2.5632	0.3577	11.3022
3.0	0.4752	10.3333	3.8571	2.6790	0.3283	12.0610
4.0	0.4350	18.5000	4.5714	4.0469	0.1388	21.0681
5.0	0.4152	29.000	5.0000	5.8000	0.0617	32.6335
∞	0.3780	∞	6.0000	∞	0	∞

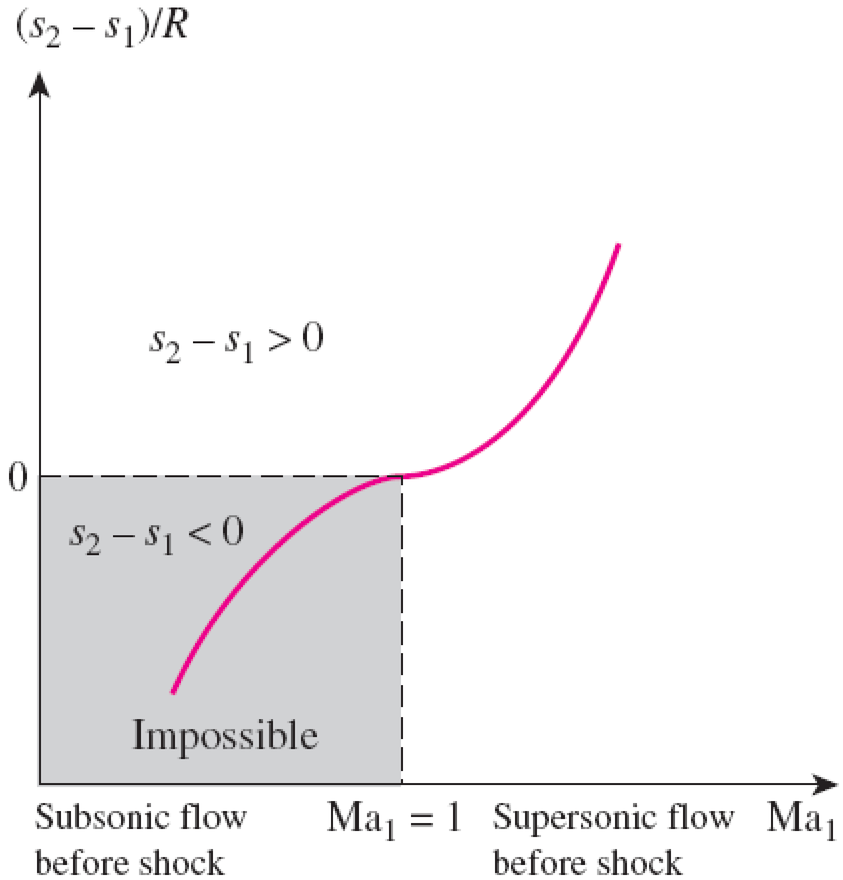


FIGURE 17–35

Entropy change across the normal shock.

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

Since the flow across the shock is adiabatic and irreversible, the second law requires that the entropy increase across the shock wave.

Thus, a shock wave cannot exist for values of Ma_1 less than unity where the entropy change would be negative.

For adiabatic flows, shock waves can exist only for supersonic flows, $Ma_1 > 1$.

FIGURE 17–37

When a lion tamer cracks his whip, a weak spherical shock wave forms near the tip and spreads out radially; the pressure inside the expanding shock wave is higher than ambient air pressure, and this is what causes the crack when the shock wave reaches the lion's ear.

EXAMPLE 12–9 **Shock Wave in a Converging–Diverging Nozzle**

If the air flowing through the converging–diverging nozzle of Example 12–7 experiences a normal shock wave at the nozzle exit plane (Fig. 12–35), determine the following after the shock: (*a*) the stagnation pressure, static pressure, static temperature, and static density; (*b*) the entropy change across the shock; (*c*) the exit velocity; and (*d*) the mass flow rate through the nozzle. Assume steady, one-dimensional, and isentropic flow with $k = 1.4$ from the nozzle inlet to the shock location.

SOLUTION Air flowing through a converging–diverging nozzle experiences a normal shock at the exit. The effect of the shock wave on various properties is to be determined.

Assumptions **1** Air is an ideal gas with constant specific heats at room temperature. **2** Flow through the nozzle is steady, one-dimensional, and isentropic before the shock occurs. **3** The shock wave occurs at the exit plane.

Properties The constant-pressure specific heat and the specific heat ratio of air are $c_p = 1.005$ kJ/kg · K and $k = 1.4$. The gas constant of air is 0.287 kJ/kg · K.

Analysis (*a*) The fluid properties at the exit of the nozzle just before the shock (denoted by subscript 1) are those evaluated in Example 12–7 at the nozzle exit to be

$$P_{01} = 1.0 \text{ MPa} \qquad P_1 = 0.1278 \text{ MPa} \qquad T_1 = 444.5 \text{ K} \qquad \rho_1 = 1.002 \text{ kg/m}^3$$

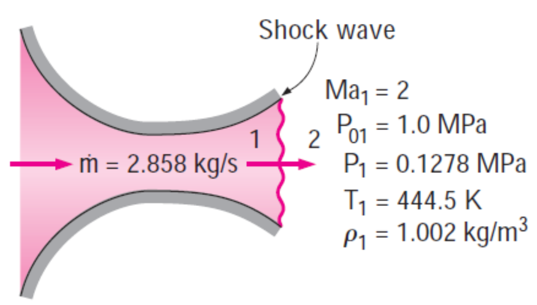


FIGURE 12–35
Schematic for Example 12–9.

The fluid properties after the shock (denoted by subscript 2) are related to those before the shock through the functions listed in Table A–14. For $\mathrm{Ma}_1 = 2.0$, we read

$$\mathrm{Ma}_2 = 0.5774 \qquad \frac{P_{02}}{P_{01}} = 0.7209 \qquad \frac{P_2}{P_1} = 4.5000 \qquad \frac{T_2}{T_1} = 1.6875 \qquad \frac{\rho_2}{\rho_1} = 2.6667$$

Then the stagnation pressure P_{02}, static pressure P_2, static temperature T_2, and static density ρ_2 after the shock are

$$P_{02} = 0.7209P_{01} = (0.7209)(1.0 \text{ MPa}) = \mathbf{0.721\ MPa}$$

$$P_2 = 4.5000P_1 = (4.5000)(0.1278 \text{ MPa}) = \mathbf{0.575\ MPa}$$

$$T_2 = 1.6875T_1 = (1.6875)(444.5 \text{ K}) = \mathbf{750\ K}$$

$$\rho_2 = 2.6667\rho_1 = (2.6667)(1.002 \text{ kg/m}^3) = \mathbf{2.67\ kg/m^3}$$

(*b*) The entropy change across the shock is

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1}$$

$$= (1.005 \text{ kJ/kg}\cdot\text{K}) \ln(1.6875) - (0.287 \text{ kJ/kg}\cdot\text{K}) \ln(4.5000)$$

$$= \mathbf{0.0942\ kJ/kg\cdot K}$$

Thus, the entropy of the air increases as it experiences a normal shock, which is highly irreversible.

(*c*) The air velocity after the shock can be determined from $V_2 = Ma_2 c_2$, where c_2 is the speed of sound at the exit conditions after the shock:

$$V_2 = Ma_2 c_2 = Ma_2 \sqrt{kRT_2}$$

$$= (0.5774)\sqrt{(1.4)(0.287\ \text{kJ/kg}\cdot\text{K})(750.1\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)}$$

$$= \mathbf{317\ m/s}$$

(*d*) The mass flow rate through a converging–diverging nozzle with sonic conditions at the throat is not affected by the presence of shock waves in the nozzle. Therefore, the mass flow rate in this case is the same as that determined in Example 12–7:

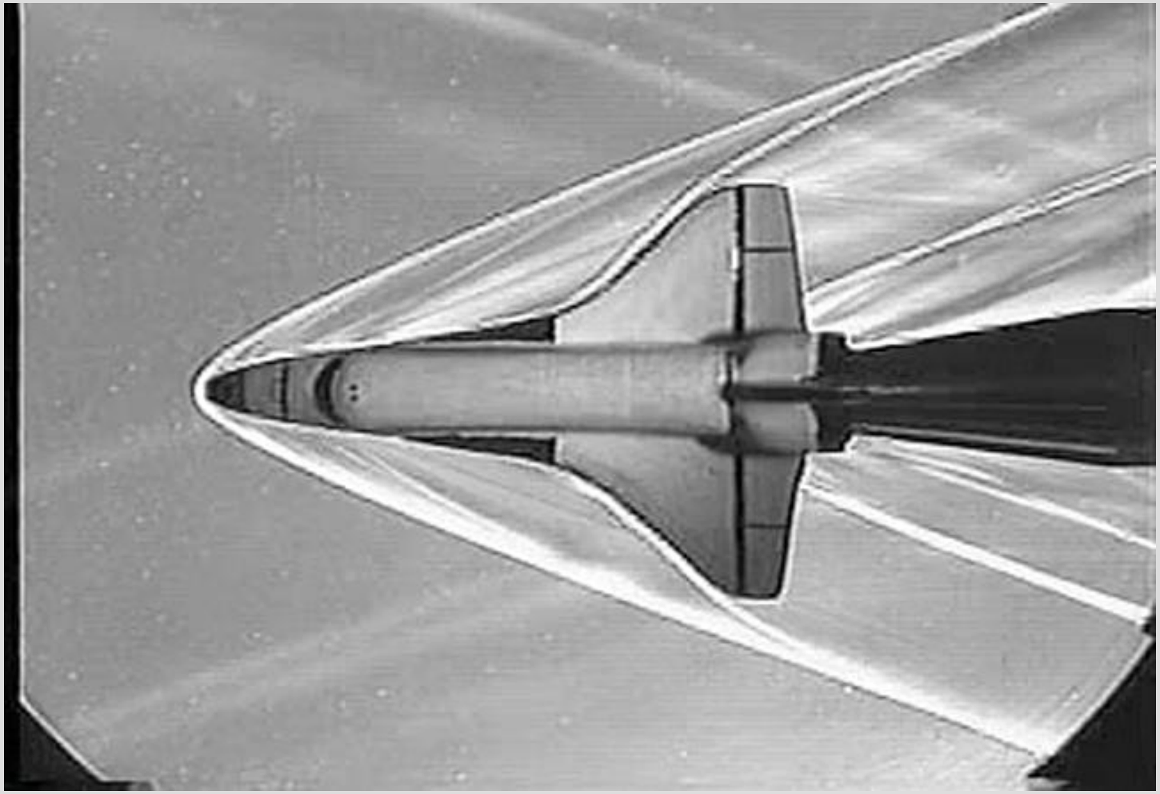
$$\dot{m} = \mathbf{2.86\ kg/s}$$

Discussion This result can easily be verified by using property values at the nozzle exit after the shock at all Mach numbers significantly greater than unity.

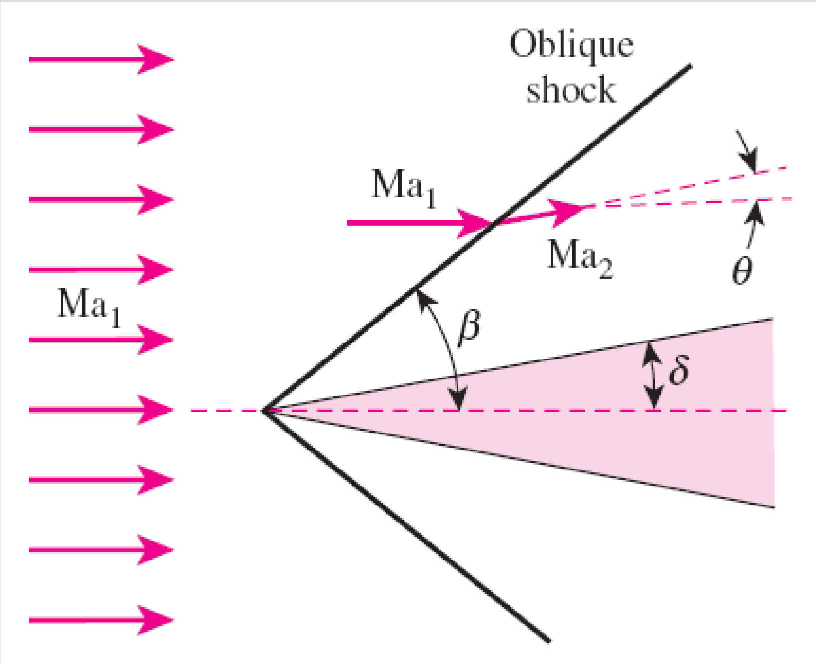
Oblique Shocks

When the space shuttle travels at supersonic speeds through the atmosphere, it produces a complicated shock pattern consisting of inclined shock waves called **oblique shocks**.

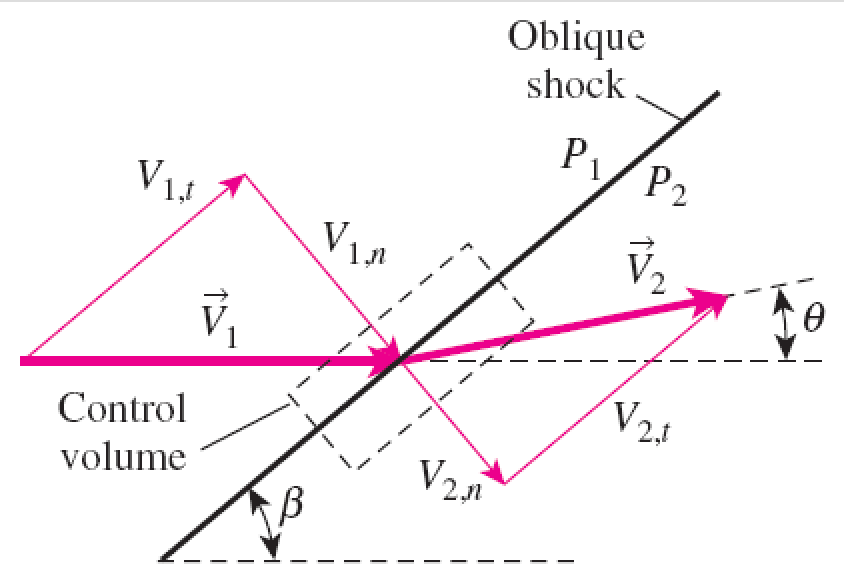
Some portions of an oblique shock are curved, while other portions are straight.



Schlieren image of a small model of the space shuttle *Orbiter* being tested at Mach 3 in the supersonic wind tunnel of the Penn State Gas Dynamics Lab. Several *oblique shocks* are seen in the air surrounding the spacecraft.

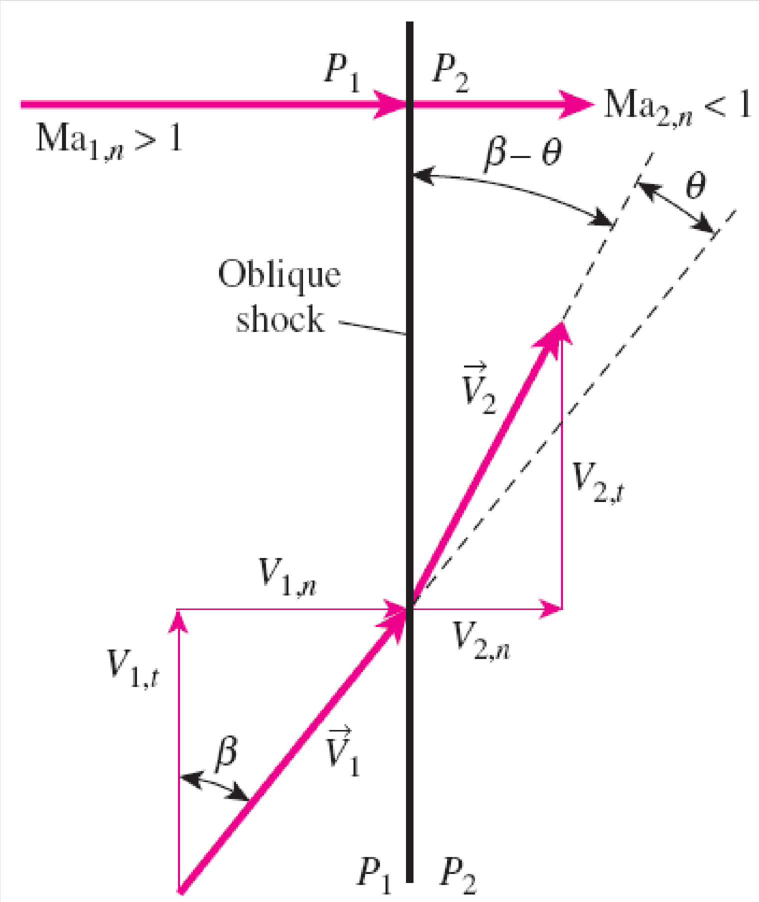


An oblique shock of ***shock angle*** **(*wave angle*)** β formed by a slender, two-dimensional wedge of half-angle δ. The flow is turned by ***deflection angle*** **(*turning angle*)** θ downstream of the shock, and the Mach number decreases.



Velocity vectors through an oblique shock of shock angle β and deflection angle θ.

Unlike normal shocks, in which the downstream Mach number is always subsonic, Ma_2 downstream of an oblique shock can be subsonic, sonic, or supersonic, depending on the upstream Mach number Ma_1 and the turning angle.



The same velocity vectors of Fig. 17–38, but rotated by angle $\pi/2-\beta$, so that the oblique shock is vertical. Normal Mach numbers $\mathrm{Ma}_{1,n}$ and $\mathrm{Ma}_{2,n}$ are also defined.

$$h_{01} = h_{02} \quad \rightarrow \quad T_{01} = T_{02}$$

$$\mathrm{Ma}_{2,n} = \sqrt{\frac{(k-1)\mathrm{Ma}_{1,n}^2 + 2}{2k\mathrm{Ma}_{1,n}^2 - k + 1}}$$

$$\frac{P_2}{P_1} = \frac{2k\mathrm{Ma}_{1,n}^2 - k + 1}{k+1}$$

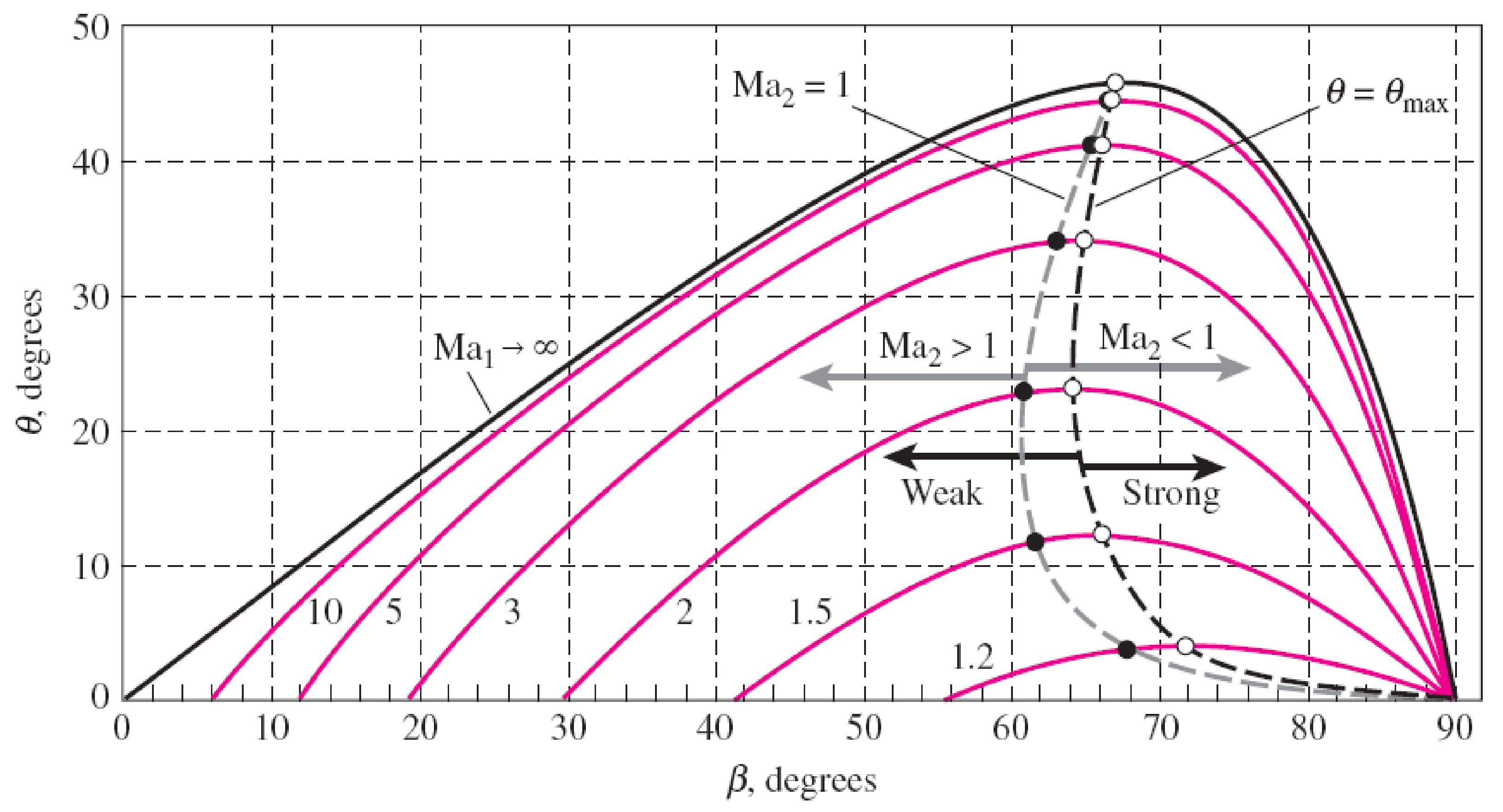
$$\frac{\rho_2}{\rho_1} = \frac{V_{1,n}}{V_{2,n}} = \frac{(k+1)\mathrm{Ma}_{1,n}^2}{2 + (k+1)\mathrm{Ma}_{1,n}^2}$$

$$\frac{T_2}{T_1} = [2 + (k+1)\mathrm{Ma}_{1,n}^2]\frac{2k\mathrm{Ma}_{1,n}^2 - k + 1}{(k+1)^2\mathrm{Ma}_{1,n}^2}$$

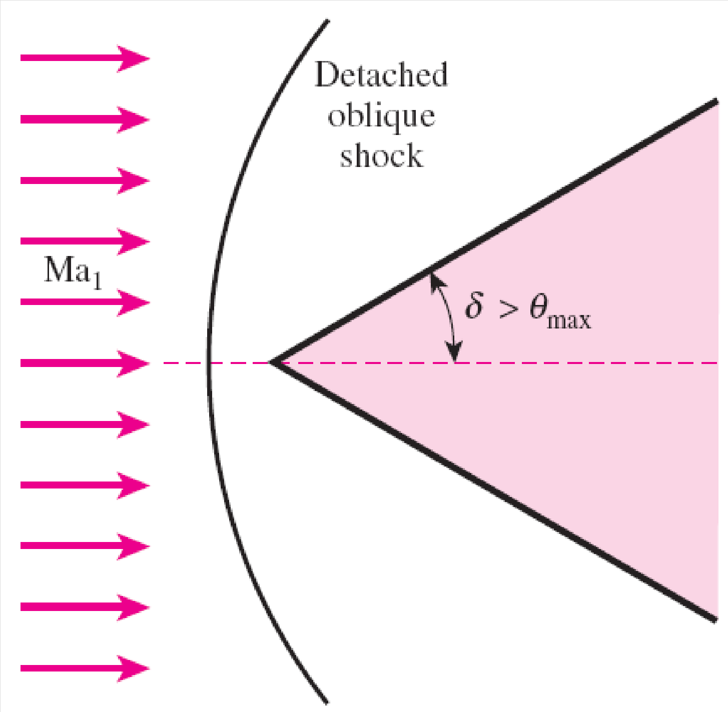
$$\frac{P_{02}}{P_{01}} = \left[\frac{(k+1)\mathrm{Ma}_{1,n}^2}{2 + (k+1)\mathrm{Ma}_{1,n}^2}\right]^{k/(k-1)} \left[\frac{(k+1)}{2k\mathrm{Ma}_{1,n}^2 - k + 1}\right]^{1/(k-1)}$$

Relationships across an oblique shock for an ideal gas in terms of the normal component of upstream Mach number $\mathrm{Ma}_{1,n}$.

All the equations, shock tables, etc., for normal shocks apply to oblique shocks as well, provided that we use only the **normal** components of the Mach number.



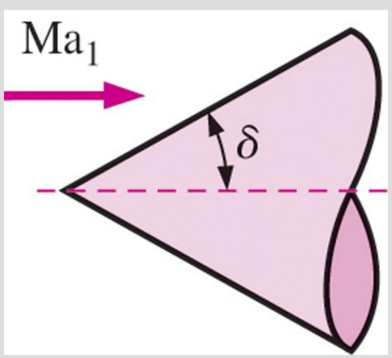
The dependence of straight oblique shock deflection angle θ on shock angle β for several values of upstream Mach number Ma_1. Calculations are for an ideal gas with k = 1.4. The dashed black line connects points of maximum deflection angle ($\theta = \theta_{max}$). *Weak oblique shocks* are to the left of this line, while *strong oblique shocks* are to the right of this line. The dashed gray line connects points where the downstream Mach number is *sonic* ($Ma_2 = 1$). *Supersonic downstream flow* ($Ma_2 > 1$) is to the left of this line, while *subsonic downstream flow* ($Ma_2 < 1$) is to the right of this line.

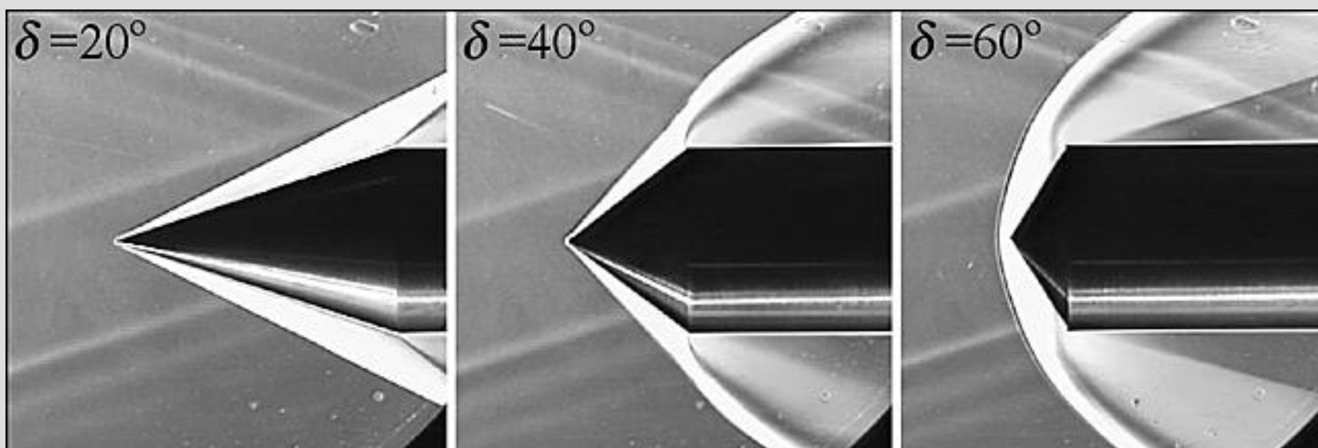


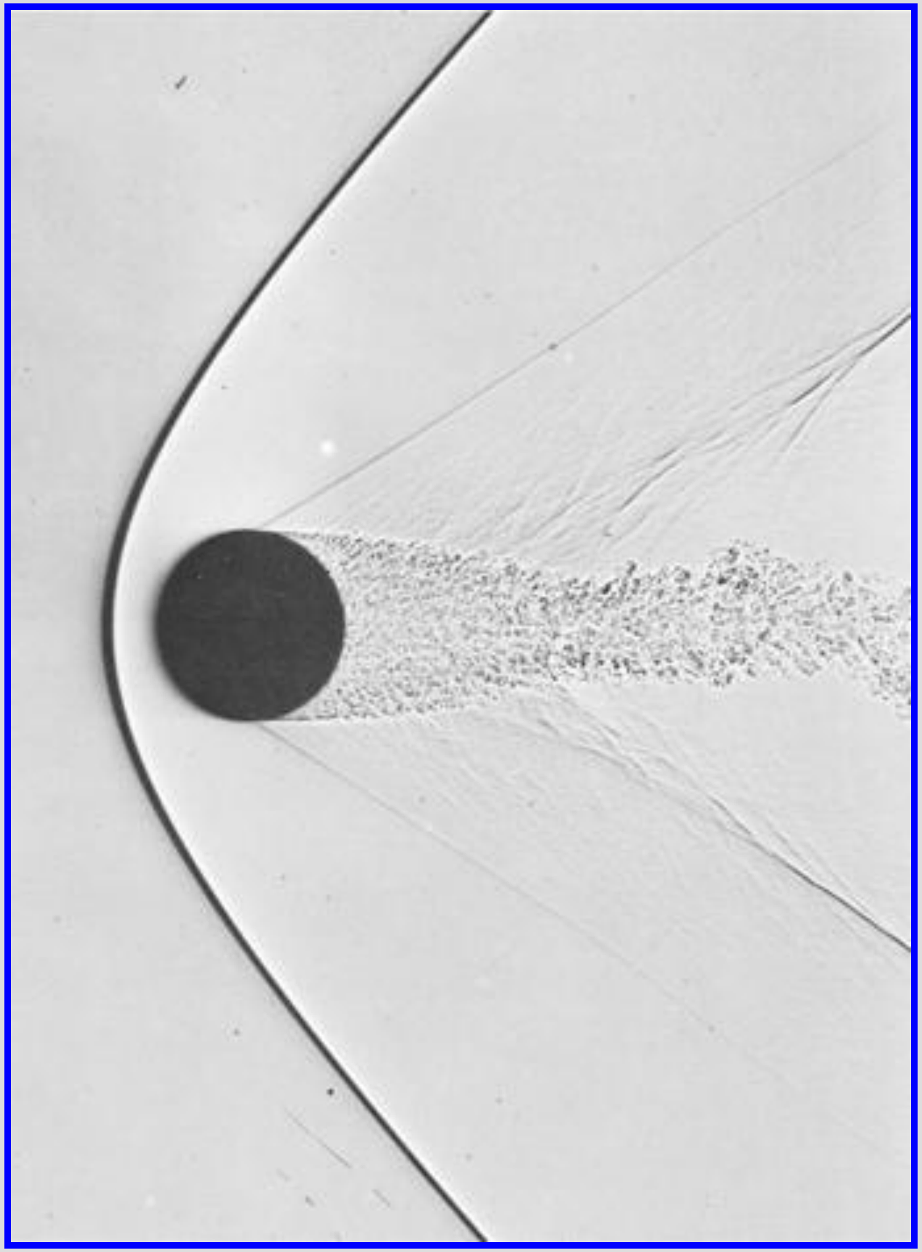
A *detached oblique shock* occurs upstream of a two-dimensional wedge of half-angle δ when δ is greater than the maximum possible deflection angle θ. A shock of this kind is called a *bow wave* because of its resemblance to the water wave that forms at the bow of a ship.

Still frames from schlieren videography illustrating the detachment of an oblique shock from a cone with increasing cone half-angle δ in air at Mach 3. At (*a*) δ=20° and (*b*) δ=40°, the oblique shock remains attached, but by (*c*) δ=60°, the oblique shock has detached, forming a bow wave.

Mach angle $$\mu = \sin^{-1}(1/Ma_1)$$







Shadowgram of a one-half-in diameter sphere in free flight through air at Ma = 1.53. The flow is subsonic behind the part of the bow wave that is ahead of the sphere and over its surface back to about 45°. At about 90° the laminar boundary layer separates through an oblique shock wave and quickly becomes turbulent. The fluctuating wake generates a system of weak disturbances that merge into the second “recompression” shock wave.

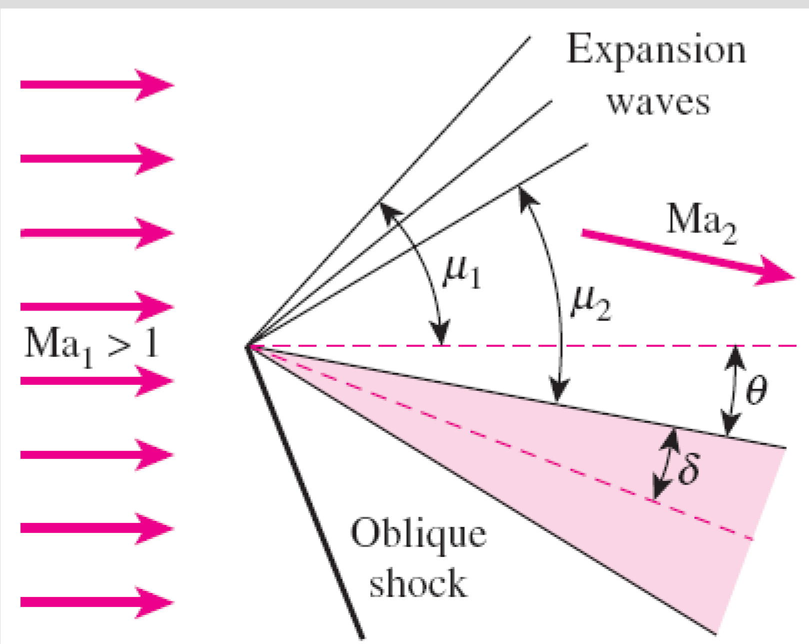
Prandtl–Meyer Expansion Waves

We now address situations where supersonic flow is turned in the *opposite* direction, such as in the upper portion of a two-dimensional wedge at an angle of attack greater than its half-angle δ.

We refer to this type of flow as an **expanding flow**, whereas a flow that produces an oblique shock may be called a **compressing flow**.

As previously, the flow changes direction to conserve mass. However, unlike a compressing flow, an expanding flow does *not* result in a shock wave.

Rather, a continuous expanding region called an **expansion fan** appears, composed of an infinite number of Mach waves called **Prandtl–Meyer expansion waves**.

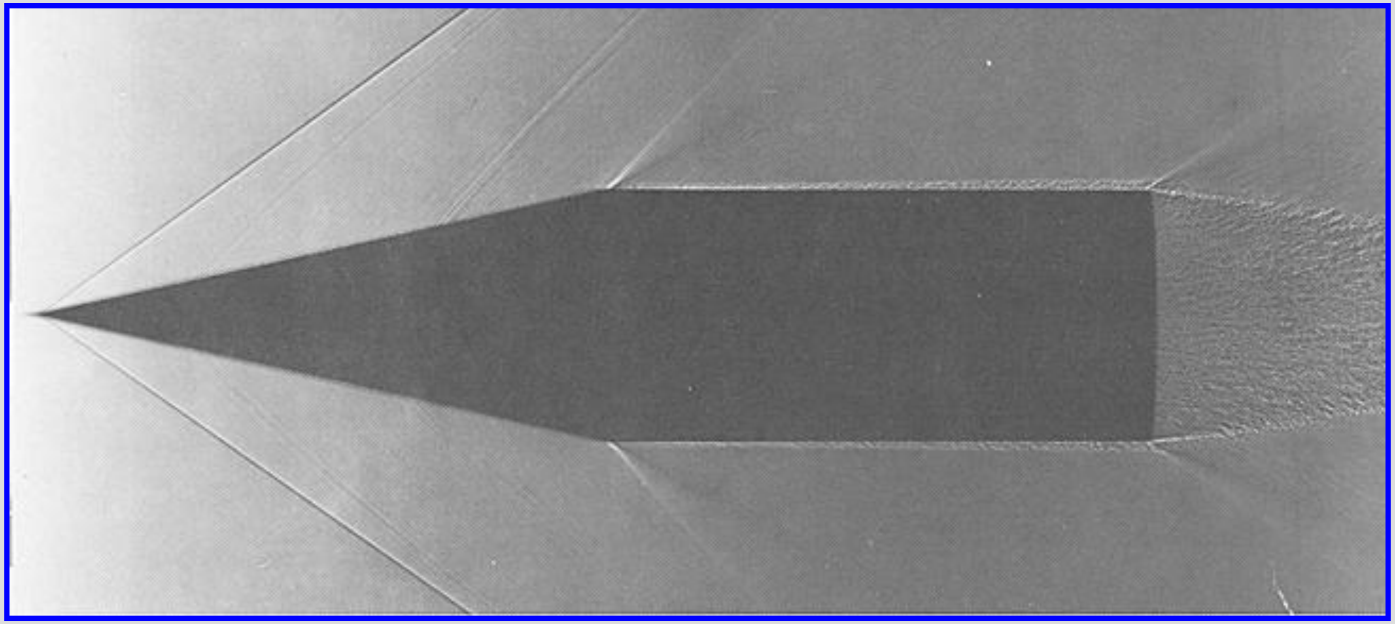


An expansion fan in the upper portion of the flow formed by a two-dimensional wedge at the angle of attack in a supersonic flow. The flow is turned by angle u, and the Mach number increases across the expansion fan. Mach angles upstream and downstream of the expansion fan are indicated. Only three expansion waves are shown for simplicity, but in fact, there are an infinite number of them. (An oblique shock is present in the bottom portion of this flow.)

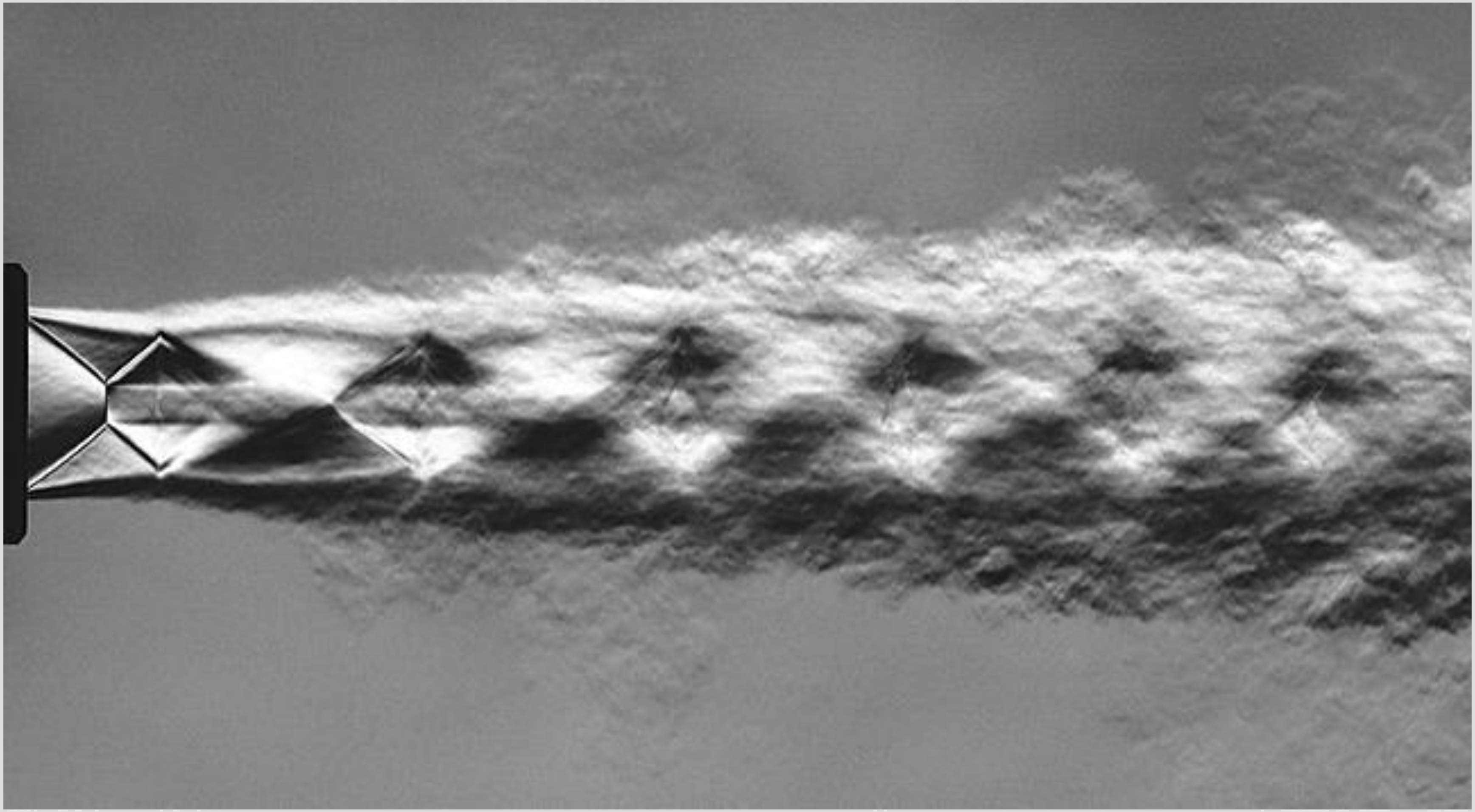
Turning angle across an expansion fan: $\theta = \nu(\mathrm{Ma}_2) - \nu(\mathrm{Ma}_1)$

Prandtl–Meyer function

$$\nu(\mathrm{Ma}) = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left[\sqrt{\frac{k-1}{k+1}(\mathrm{Ma}^2-1)}\right] - \tan^{-1}\left(\sqrt{\mathrm{Ma}^2-1}\right)$$



A cone-cylinder of 12.5° half-angle in a Mach number 1.84 flow. The boundary layer becomes turbulent shortly downstream of the nose, generating Mach waves that are visible in this shadowgraph. Expansion waves are seen at the corners and at the trailing edge of the cone.



The complex interactions between shock waves and expansion waves in an “overexpanded” supersonic jet. The flow is visualized by a schlierenlike differential interferogram.

DUCT FLOW WITH HEAT TRANSFER AND NEGLIGIBLE FRICTION (RAYLEIGH FLOW)

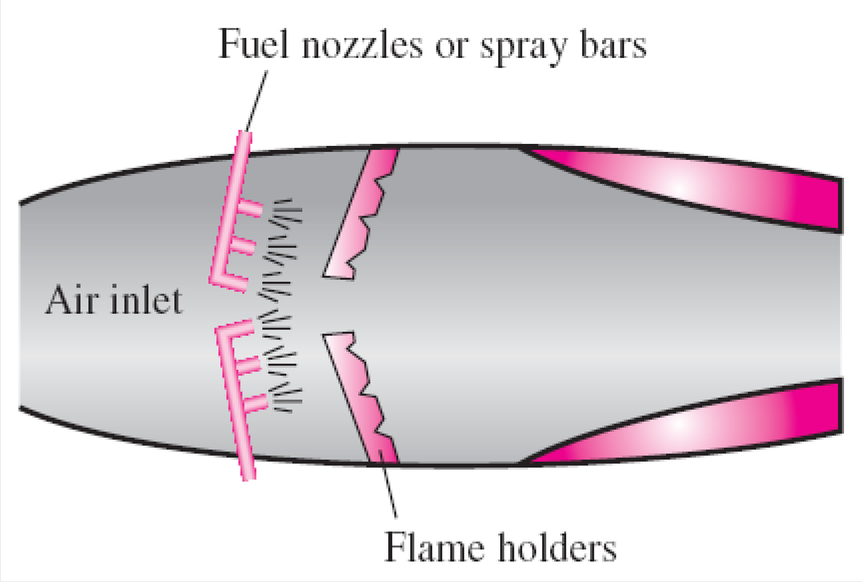
So far we have limited our consideration mostly to *isentropic flow* (no heat transfer and no irreversibilities such as friction).

Many compressible flow problems encountered in practice involve chemical reactions such as combustion, nuclear reactions, evaporation, and condensation as well as heat gain or heat loss through the duct wall.

Such problems are difficult to analyze exactly since they may involve significant changes in chemical composition during flow, and the conversion of latent, chemical, and nuclear energies to thermal energy.

A simplified model is Rayleigh flow.

Rayleigh flows: Steady one-dimensional flow of an ideal gas with constant specific heats through a constant-area duct with heat transfer, but with negligible friction.



Many practical compressible flow problems involve combustion, which may be modeled as heat gain through the duct wall.

$$\rho_1 V_1 = \rho_2 V_2$$ Mass equation

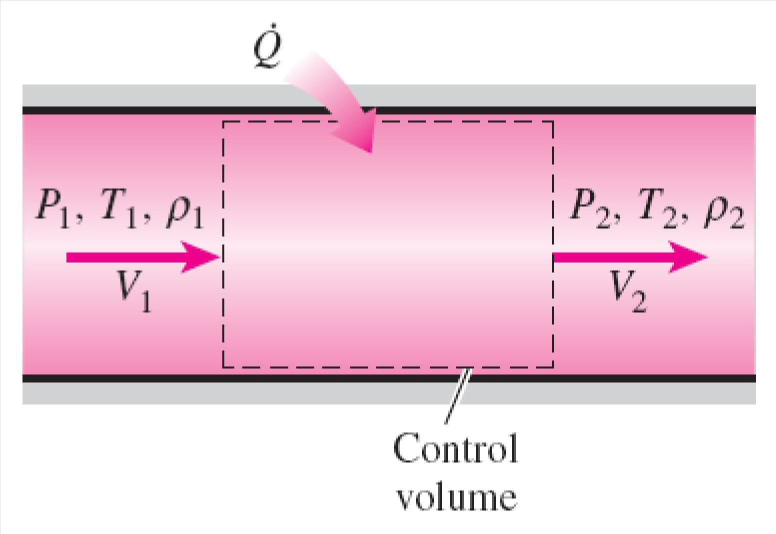
$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2$$ *x*-Momentum equation

$$q = c_p(T_2 - T_1) + \frac{V_2^2 - V_1^2}{2}$$ Energy equation

$$q = h_{02} - h_{01} = c_p(T_{02} - T_{01})$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$ Entropy change

$$\frac{P_1}{\rho_1 T_1} = \frac{P_2}{\rho_2 T_2}$$ Equation of state

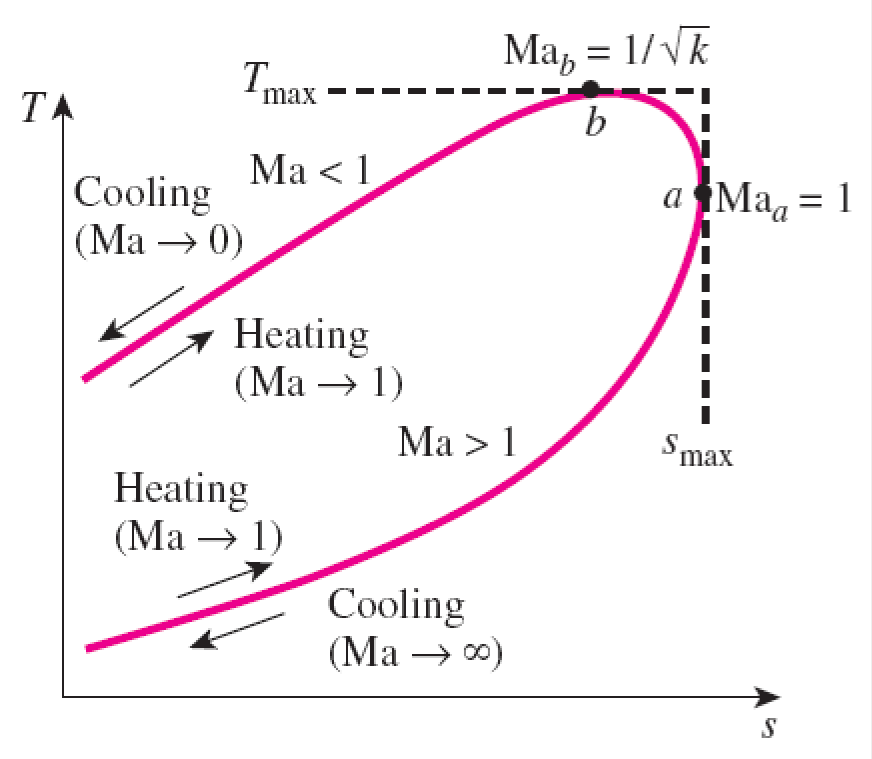


Control volume for flow in a constant-area duct with heat transfer and negligible friction.

Consider a gas with known properties R, k, and c_p. For a specified inlet state 1, the inlet properties P_1, T_1, ρ_1, V_1, and s_1 are known. The five exit properties P_2, T_2, ρ_2, V_2, and s_2 can be determined from the above equations for any specified value of heat transfer q.

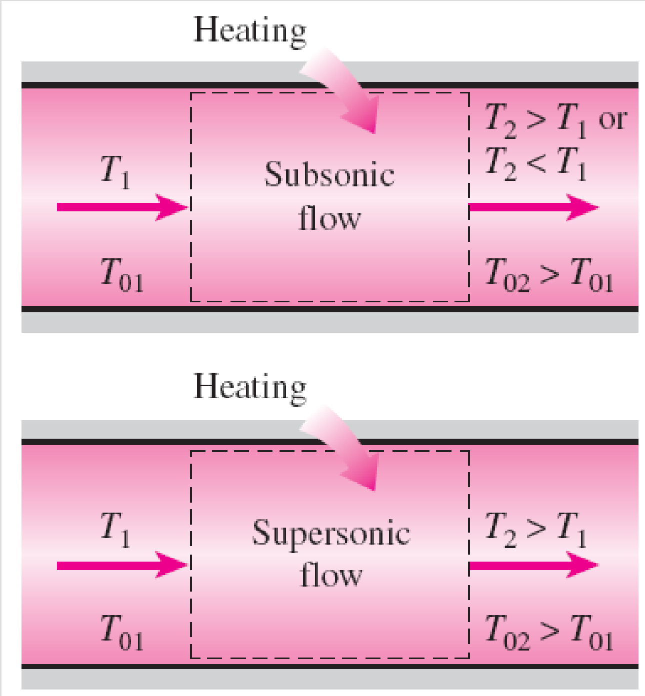
From the Rayleigh line and the equations

1. All the states that satisfy the conservation of mass, momentum, and energy equations as well as the property relations are on the Rayleigh line.
2. Entropy increases with heat gain, and thus we proceed to the right on the Rayleigh line as heat is transferred to the fluid.
3. Heating increases the Mach number for subsonic flow, but decreases it for supersonic flow.
4. Heating increases the stagnation temperature T_0 for both subsonic and supersonic flows, and cooling decreases it.
5. Velocity and static pressure have opposite trends.
6. Density and velocity are inversely proportional.



T-s diagram for flow in a constant-area duct with heat transfer and negligible friction (Rayleigh flow). This line is called **Rayleigh line**.

7. The entropy change corresponding to a specified temperature change (and thus a given amount of heat transfer) is larger in supersonic flow.



During heating, fluid temperature always increases if the Rayleigh flow is supersonic, but the temperature may actually drop if the flow is subsonic.

Heating or cooling has opposite effects on most properties. Also, the stagnation pressure decreases during heating and increases during cooling regardless of whether the flow is subsonic or supersonic.

TABLE 17–3

The effects of heating and cooling on the properties of Rayleigh flow

	Heating		Cooling	
Property	Subsonic	Supersonic	Subsonic	Supersonic
Velocity, V	Increase	Decrease	Decrease	Increase
Mach number, Ma	Increase	Decrease	Decrease	Increase
Stagnation temperature, T_0	Increase	Increase	Decrease	Decrease
Temperature, T	Increase for $\mathrm{Ma} < 1/k^{1/2}$ Decrease for $\mathrm{Ma} > 1/k^{1/2}$	Increase	Decrease for $\mathrm{Ma} < 1/k^{1/2}$ Increase for $\mathrm{Ma} > 1/k^{1/2}$	Decrease
Density, ρ	Decrease	Increase	Increase	Decrease
Stagnation pressure, P_0	Decrease	Decrease	Increase	Increase
Pressure, P	Decrease	Increase	Increase	Decrease
Entropy, s	Increase	Increase	Decrease	Decrease

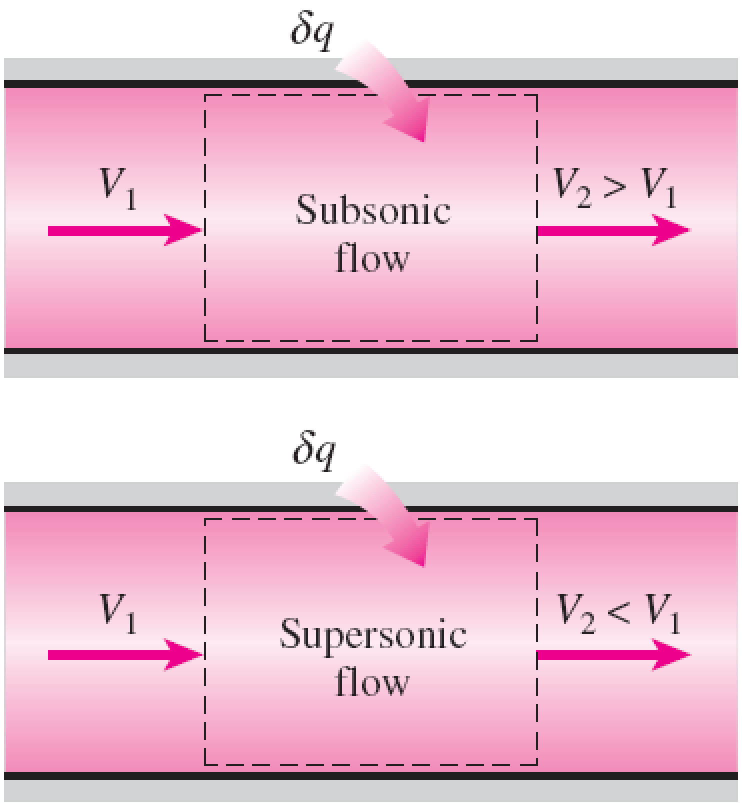


FIGURE 17–57
Heating increases the flow velocity in subsonic flow, but decreases it in supersonic flow.

Property Relations for Rayleigh Flow

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2} \qquad \frac{T_2}{T_1} = \left[\frac{\mathrm{Ma}_2(1 + k\mathrm{Ma}_1^2)}{\mathrm{Ma}_1(1 + k\mathrm{Ma}_2^2)}\right]^2 \qquad \frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{\mathrm{Ma}_1^2(1 + k\mathrm{Ma}_2^2)}{\mathrm{Ma}_2^2(1 + k\mathrm{Ma}_1^2)}$$

$$\frac{P}{P^*} = \frac{1 + k}{1 + k\mathrm{Ma}^2} \qquad \frac{T}{T^*} = \left[\frac{\mathrm{Ma}(1 + k)}{1 + k\mathrm{Ma}^2}\right]^2 \quad \text{and} \quad \frac{V}{V^*} = \frac{\rho^*}{\rho} = \frac{(1 + k)\mathrm{Ma}^2}{1 + k\mathrm{Ma}^2}$$

$$\frac{T_0}{T_0^*} = \frac{(k + 1)\mathrm{Ma}^2[2 + (k - 1)\mathrm{Ma}^2]}{(1 + k\mathrm{Ma}^2)^2}$$

$$\frac{P_0}{P_0^*} = \frac{k + 1}{1 + k\mathrm{Ma}^2}\left[\frac{2 + (k - 1)\mathrm{Ma}^2}{k + 1}\right]^{k/(k-1)}$$

Representative results are given in Table A–34.

$$\frac{T_0}{T_0^*} = \frac{(k + 1)\mathrm{Ma}^2[2 + (k - 1)\mathrm{Ma}^2]}{(1 + k\mathrm{Ma}^2)^2}$$

$$\frac{P_0}{P_0^*} = \frac{k + 1}{1 + k\mathrm{Ma}^2}\left[\frac{2 + (k - 1)\mathrm{Ma}^2}{k + 1}\right]^{k/(k+1)}$$

$$\frac{T}{T^*} = \left[\frac{\mathrm{Ma}(1 + k)}{1 + k\mathrm{Ma}^2}\right]^2$$

$$\frac{P}{P^*} = \frac{1 + k}{1 + k\mathrm{Ma}^2}$$

$$\frac{V}{V^*} = \frac{\rho^*}{\rho} = \frac{(1 + k)\mathrm{Ma}^2}{1 + k\mathrm{Ma}^2}$$

FIGURE 17–58
Summary of relations for Rayleigh flow.

Choked Rayleigh Flow

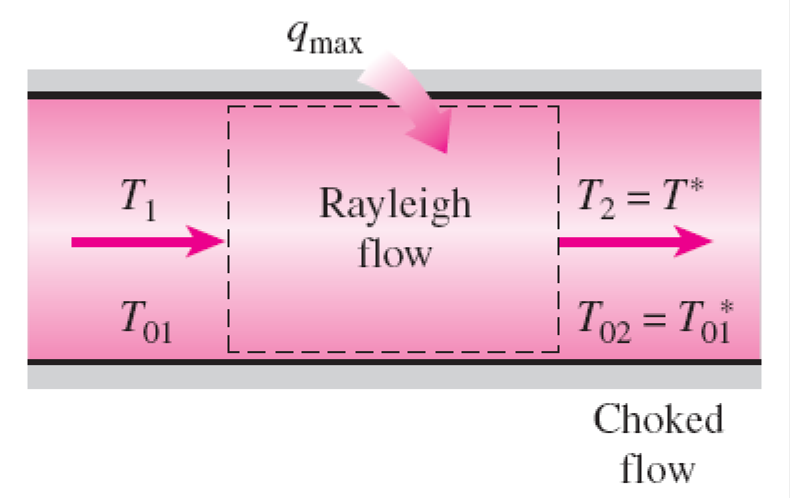
The fluid at the critical state of Ma =1 cannot be accelerated to supersonic velocities by heating. Therefore, the flow is ***choked***.

For a given inlet state, the corresponding critical state fixes the maximum possible heat transfer for steady flow:

$$q_{max} = h_0^* - h_{01} = c_p(T_0^* - T_{01})$$

$$\text{Lim}_{\text{Ma}\to\infty} \frac{T_0}{T_0^*} = 1 - \frac{1}{k^2}$$

which yields $T_0/T^*_0 = 0.49$ for $k = 1.4$. Therefore, if the critical stagnation temperature is 1000 K, air cannot be cooled below 490 K in Rayleigh flow. Physically this means that the flow velocity reaches infinity by the time the temperature reaches 490 K—a physical impossibility. When supersonic flow cannot be sustained, the flow undergoes a normal shock wave and becomes subsonic.



For a given inlet state, the maximum possible heat transfer occurs when sonic conditions are reached at the exit state.

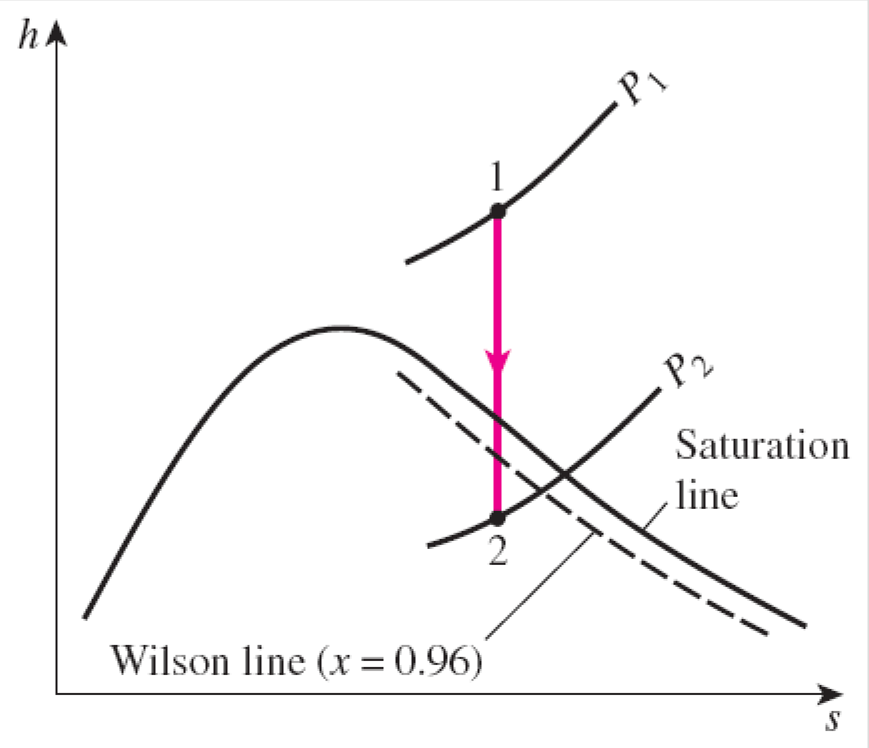
STEAM NOZZLES

Water vapor at moderate or high pressures deviates considerably from ideal-gas behavior, and thus most of the relations developed in this chapter are not applicable to the flow of steam through the nozzles or blade passages encountered in steam turbines.

Supersaturated steam: The steam that exists in the wet region without containing any liquid.

Supersaturation states are nonequilibrium (or metastable) states.

Wilson line: The locus of points where condensation takes place regardless of the initial temperatureand pressure at the nozzle entrance.



The *h-s* diagram for the isentropic expansion of steam in a nozzle.

When steam is assumed ideal gas with $k = 1.3$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} = 0.546$$

Summary

- Stagnation properties
- Speed of sound and Mach number
- One-dimensional isentropic flow
  - ✓ Variation of fluid velocity with flow area
  - ✓ Property relations for isentropic flow of ideal gases
- Isentropic flow through nozzles
  - ✓ Converging nozzles
  - ✓ Converging–diverging nozzles
- Shock waves and expansion waves
  - ✓ Normal shocks
  - ✓ Oblique shocks
  - ✓ Prandtl–Meyer expansion waves
- Duct flow with heat transfer and negligible friction (Rayleigh flow)
  - ✓ Property relations for Rayleigh flow
  - ✓ Choked Rayleigh flow
- Steam nozzles

Isentropic Flow

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Mach = M

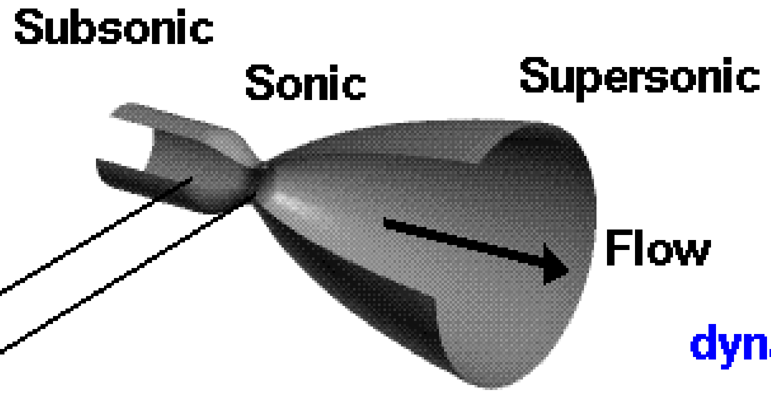
speed of sound = a

gas constant = R

specific heat ratio = γ

t = total conditions

* = sonic conditions



velocity = v

pressure = p

temperature = T

density = ρ

area = A

dynamic pressure = q

(1) $$M = \frac{v}{a}$$

(2) $$a = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma R T}$$

(3) $$\frac{p}{\rho^{\gamma}} = \text{Constant} = \frac{p_t}{\rho_t^{\gamma}}$$

(4) $$\frac{p}{p_t} = \left(\frac{\rho}{\rho_t}\right)^{\gamma} = \left(\frac{T}{T_t}\right)^{\frac{\gamma}{\gamma-1}}$$

(5) $$q = \frac{1}{2}\rho v^2 = \frac{\gamma}{2} p M^2$$

(6) $$\frac{p}{p_t} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma}{\gamma-1}}$$

(7) $$\frac{T}{T_t} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1}$$

(8) $$\frac{\rho}{\rho_t} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{-1}{\gamma-1}}$$

(9) $$\frac{A}{A^*} = \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \frac{\left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{M}$$

https://www.grc.nasa.gov/www/k-12/airplane/isentrop.html

Mass Flow Choking

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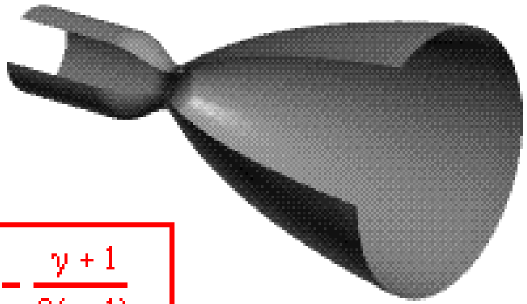
A = Area R = Gas Constant V = Velocity T_t = Total Temperature

r = Density γ = Specific Heat Ratio M = Mach p_t = Total Pressure



Mass Flow Rate: $\dot{m} = r\,V\,A$

For an ideal compressible gas:

$$\boxed{\dot{m} = \frac{A\,p_t}{\sqrt{T_t}}\sqrt{\frac{\gamma}{R}}\,M\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}}$$

Mass Flow Rate is a maximum when $M = 1$

At these conditions, flow is choked.

$$\dot{m} = \frac{A\,p_t}{\sqrt{T_t}}\sqrt{\frac{\gamma}{R}}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

https://www.grc.nasa.gov/www/k-12/airplane/mflchk.html

Alguns parâmetros de desempenho úteis

Parâmetros usados em cálculos de Nozzles, nomeadamente, apresentados como parâmetros de desempenho no software NASA CEA:

- Velocidade característica de um nozzle, c^*:

$$c^* = \frac{p_0 A_t}{\dot{m}}$$

- Permite relacionar caudal mássico de gases, $\dot{m}$, a pressão da câmara de combustão, p_0, e a área da garganta do Nozzle, A_t.
- Este parâmetro é uma medida da capacidade da câmara de combustão produzir gases com alta pressão de estagnação.

- Coeficiente de força de propulsão (Força de propulsão adimensional):

$$C_F = \frac{F_{thrust}}{p_0 A_t}$$

Este parâmetro traduz a capacidade do nozzle converter a pressão gerada na câmara de combustão em força efetiva de propulsão

Alguns parâmetros de desempenho

- Podemos representar a força propulsão em função destes dois parâmetros:

$$F_{thrust} = C_F p_0 A_t = \dot{m} C_F c^*$$

- Tendo em conta que

$$F_{thrust} \approx \dot{m} C$$

- Então,

$$C = C_F c^*$$

- Nesta equação vemos que o principal parâmetro associado à força de propulsão (v_e) depende fundamentalmente da capacidade da câmara de combustão em gerar pressão (c^*) e da capacidade do nozzle em traduzir essa pressão em velocidade de saída (C_F)

- O impulso específico também pode ser representado em função da velocidade característica, c^* e o coeficiente de força de propulsão C_F:

$$I_{sp} = \frac{v_e}{g} = \frac{C_F c^*}{g}$$

Exercício 3.2.1

Usar o NASA CEA para dimensionar um Nozzle que

- funcione com metano líquido e oxigénio líquido
- que funcione com uma pressão de combustão de 250 bar
- Analisar para pressão ambiente à saída e pressão correspondente à altitude de 50 km
- Analise graficamente a influência da razão oxigénio combustível entre 1 e 10, verificando qual maximiza o desempenho e qual a respetiva razão oxigénio - combustível
- Calcule a geometria do Nozzle e o caudal para que este produza uma força de propulsão de cerca de 500 kN para a razão O/F ideal