Vector Spaces

A vector space V must satisfy:

1. there exists an additive identity (written 0) in such that $x + 0 = x$ for all $x \in V$.
2. for each $x \in V$, there exists an additive inverse (written $-x$) such that $x + (-x) = 0$.
3. there exists a multiplicative identity (written 1) in $\mathbb{R}$ such that $1x = x$ for all $x \in V$.
4. Commutative: $x + y = y + x$ for all $x, y \in V$
5. Associativity: $(x+y)+z = x+(y+z)$ and $\alpha(\beta x) = \alpha\beta(x)$ for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{R}$.
6. Distributivity: $\alpha(x+y) = \alpha x + \alpha y$ and $(\alpha+\beta)x = \alpha x + \beta x$ for all $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$

Polynomials in $\mathbb{R}$ are a vector space

- $\underbrace{\sum_{i=0}^{k} a_i x^i}_{p} = \underbrace{\sum_{i=0}^{L} a_i x^i}_{v} \iff k = L$ and $a_i = b_i \ \forall i$

- $p + v = \sum_{i=0}^{\max(k,L)} (a_i + b_i) x^i$ } add polynomials

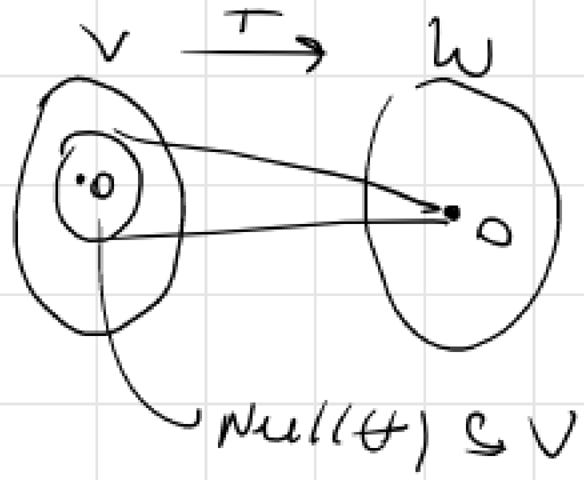
- $\lambda \in \mathbb{R} \rightarrow \lambda p = \sum_{i=0}^{k} (\lambda a_i) x^i$ } multiply by a scalar $\lambda \in \mathbb{R}$

Subspaces and Linear Maps

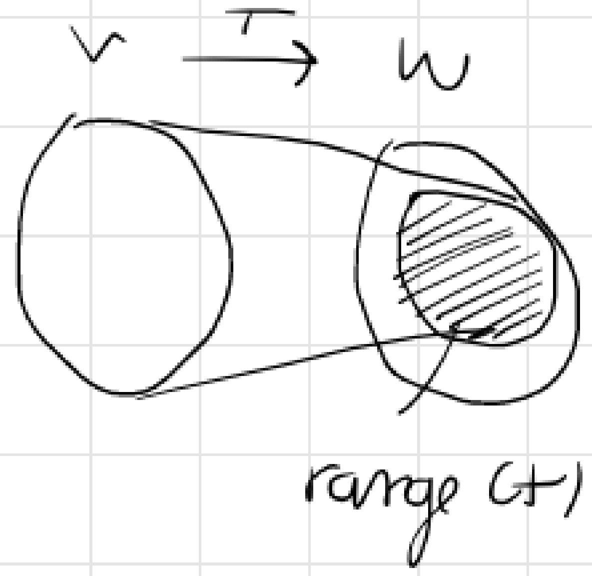
$T: V \to W$ is called a linear map, if:

$$\forall x, y \in V \quad \forall \lambda, \mu \in \mathbb{R}: T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$$

- Nullspace of T:
  $null(T) = \{x \in V \mid Tx = 0\}$



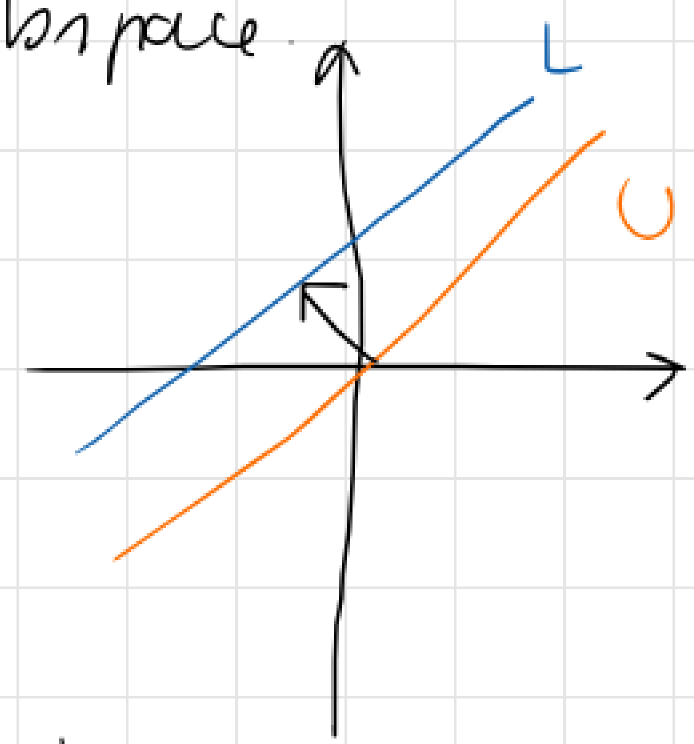
- range of T:
  $range(T) = \{y \in W \mid \exists x \in V \mid Tx = y\}$



$$\underbrace{\sum_{i=1}^{k} \lambda_i v_i = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_k = 0}_{\text{linear independent}}$$

- Affine subspace: Let V be a vector space, $x_0 \in V$ and $U \subseteq V$ a subspace.

$L = x_0 + U$



* in ML, affine subspaces define hyperplanes and are often described by parameters

* if $(b_1, \dots, b_k)$ is an ordered basis of U, then every element $x \in L$ can be uniquely described as:

$$x = x_0 + \lambda_1 b_1 + \dots + \lambda_k b_k$$