

## The Dynamics of Finite-Difference Models of the Shallow-Water Equations

ROBERT SADOURNY

*Laboratoire de Météorologie Dynamique du C.N.R.S., Paris, France*

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### ABSTRACT

Two simple numerical models of the shallow-water equations identical in all respects but for their conservation properties have been tested regarding their internal mixing processes. The experiments show that violation of enstrophy conservation results in a spurious accumulation of rotational energy in the smaller scales, reflected by an unrealistic increase of enstrophy, which ultimately produces a finite rate of energy dissipation in the zero viscosity limit, thus violating the well-known dynamics of two-dimensional flow. Further, the experiments show a tendency to equipartition of the kinetic energy of the divergent part of the flow in the inviscid limit, suggesting the possibility of a divergent energy cascade in the physical system, as well as a possible influence of the energy mixing on the process of adjustment toward balanced flow.

### 1. Introduction

Selecting a finite-difference scheme for the purpose of solving numerically the equations of atmospheric motion may involve some arbitrary choices. On the one hand, simple finite-difference schemes are desirable from the standpoint of computational economy; however, the simplest schemes poorly represent most properties of the original equations. On the other hand, the closest—but rather complicated—approximation is the spectral method which is equivalent to a “maximum” scheme involving the entire grid. An argument in favor of simpler finite-difference approximations, however, is the fact that in many kinds of problems all the properties of the original equations are not equally important: one may thus avoid the extreme complication of the spectral approach, and use simpler schemes only constrained to obey the more important requirements. For instance, if the purpose is short- or extended-time forecasting, local accuracy will play a prominent role, calling for higher order approximations in order to simulate more realistic phase velocities in the larger scales (e.g., Kreiss and Oliger, 1972), while better dispersion properties will be desirable in order to ensure better efficiency in the smaller scales (Winninghof and Arakawa, 1970). On the other hand, a very accurate simulation of the phase velocities of the transient motions is not essential in long-term general circulation experiments (or simulated climate experiments), which mainly require realistic long-term statistics. The principal quality of a finite-difference scheme designed for this type of problem will lie in its capacity to provide a correct representation of the nonlinear mixing of energy within a wide variety of scales. It is clear that the limitations of the finite-difference formulation can

be counterbalanced by its flexibility and ability to conform to a restricted number of carefully selected properties, while remaining simple and computationally efficient.

The adequacy of a finite-difference scheme to simulated climate experiments will mainly depend on its ability to reproduce the exchanges of spectral energy which occur in real flows. Now, atmospheric motion is essentially two-dimensional, at least for scales greater than  $\sim 100$  km. Charney's (1971) theory of quasi-geostrophic turbulence as well as reliable observational studies (Wiin-Nielsen, 1967; Morel and Necco, 1973; Morel and Larcheveque, 1974) indicate that this quasi-two-dimensional, quasi-nondivergent motion approximately follows the laws of two-dimensional turbulence. Most observational evidence, however, is based on the shape of the spectral repartition of energy—which results in fact from the underlying structure of the nonlinear interactions, at least in an inertial range. A review of most general circulation models using a wide variety of finite-difference approximations shows that all these models do produce realistic energy spectra in the “inertial” range; however, a correct energy spectrum for a numerical solution is not by itself a proof of the accuracy of the simulated energy transfers. In fact, it is always possible to force the energy distribution of any numerical solution to conform to a known spectral shape in the inertial range through ad-hoc assumptions, regarding, for instance, addition of artificial viscosity. However, if we are to trust numerical modeling as a method for providing better understanding of the real processes, we must then admit that a realistic energy spectrum should not be forced by artificial techniques, but should come instead as a by-product of the

first principles only, via correct treatment of the nonlinear interactions. More precisely, accurate long-term statistical distributions of kinetic energy should result from:

- 1) An accurate distribution of sources and sinks (outside the inertial range).
- 2) Accurate representation of the statistical transfers of energy within the resolved scales ("internal" transfers) in spite of the truncation error of the finite-difference scheme in the smaller scales.
- 3) An accurate parameterization of the statistical effects of nonlinear interactions with the subgrid-scale motions.

The fact that the atmospheric flow is quasi-two-dimensional is of uppermost importance in designing a finite-difference scheme suitable for very long-term integrations (Arakawa, 1966, 1970), since the spectral transfers of energy in two and three dimensions exhibit quite different structures. Dimensionality is not directly in question, but we are concerned with the existence of an additional invariant, enstrophy or potential enstrophy, in two dimensions. At least in the nondivergent case global conservation of the quadratic invariants is essential for the regulation of the nonlinear energy transfers, since conservation as a whole is equivalent to "local" conservation within a triad of wavevectors. Recognition of the peculiar dynamical effects of enstrophy conservation is an essential feature of the theory of turbulence in two dimensions, while its effects on the transfers of energy within a triad have long been recognized by Fjortoft (1953). It is clear that exact conservation of total enstrophy in the finite-difference formulation will play a central role in the ability of the model to represent correctly the exchanges of energy between modes.<sup>1</sup> In fact, the problem with finite differencing, as opposed to spectral methods, lies in the fact that truncation error in the smaller scales reaches a magnitude of the order of the derivative itself, so that the structure of a triad interaction involving smaller scales is poorly related to its characteristic structure in the exact equations, depending instead on the intrinsic conservation properties of the finite-difference scheme. Although in the larger scales an increase of accuracy will lead to more realistic energy exchanges in the absence of formal conservation, even these scales may be eventually contaminated by erroneous mixing of energy originating in higher wavenumbers.

The long-term statistical properties of the numerical solutions resulting from a finite-difference model constrained to the correct conservation properties should thus be equivalent to those of a spectral model of equivalent resolution. However, both will be limited to a finite number of degrees of freedom, which will

produce a distortion of the long-term statistics if an accurate parameterization of the statistical effect of the subgrid-scale motion is not included in the formulation. The problem here is that such a parameterization, ideally based on the first principles only, is not yet available. However, a comparison of finite-difference schemes regarding their internal mixing properties can be made without reference to the unresolved scales. Further, the mechanisms of the nonlinear exchanges of energy are best described by inviscid calculations. In the nondivergent case, for instance, the statistical properties of inviscid numerical models are now well known from both theoretical and experimental viewpoints (Fox and Orszag, 1973; Basdevant and Sadourny, 1975). These studies may provide a useful frame of reference for the study of more complicated models.

The purpose of the present paper is to analyze the importance of potential enstrophy conservation in finite-difference models of the shallow water equations, regarding their internal mixing processes: the experiments were performed in the inviscid limit, without energy sources. They could have included various classical schemes, but it looked perhaps more convincing to compare two simple schemes especially designed to be almost identical but for their conservation properties. The integrations were performed on a plane using doubly periodic boundary conditions; due to this somewhat abstract setting, the Coriolis parameter was taken equal to zero. The dynamics of more realistic motions on a rotating sphere will be the subject of a forthcoming paper.

## 2. Two simple finite-difference models of the shallow-water equations

Let us take the shallow water equations in the form

$$\left. \begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + \eta \mathbf{N} \times (P\mathbf{V}) + \text{grad}(P + \frac{1}{2} \mathbf{V} \cdot \mathbf{V}) &= 0 \\ \frac{\partial P}{\partial t} + \text{div}(P\mathbf{V}) &= 0 \end{aligned} \right\}, \quad (1)$$

where  $\mathbf{V}$  is the (two-dimensional) velocity vector,  $P$  the (two-dimensional) density or pressure,  $\eta$  the potential vorticity, i.e.,

$$\eta = \frac{\text{rot } \mathbf{V}}{P},$$

and  $\mathbf{N}$  the unit vector normal to the plane domain  $S$ , chosen as a square with periodic boundary conditions in both directions. The scale of the domain is somewhat arbitrary; consequently, the influence of an entrainment rotation is omitted, in order to avoid an arbitrary

<sup>1</sup> The importance of conforming to the formal invariants while departing from the exact Navier Stokes equations is also a characteristic feature of the theory of stochastic models of turbulent flows (Kraichnan, 1971; Frisch *et al.*, 1974).

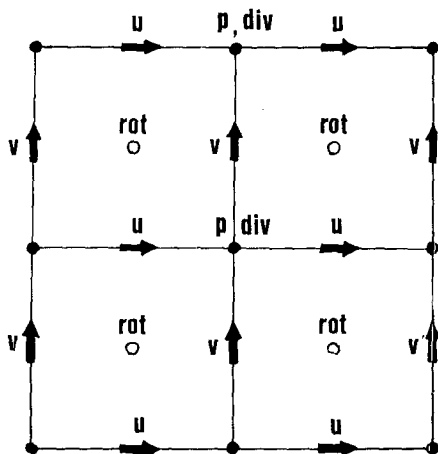


FIG. 1. Staggered arrays for the finite-difference models.

choice for the magnitude of the entrainment vorticity  $f$ . Only the slightly compressible case is considered, where divergence is small compared to vorticity. As usual, this relatively vague assumption is replaced by a balance condition for the initial fields:

$$\left. \begin{aligned} \text{div } \mathbf{V} &= 0 \\ \frac{\partial}{\partial t} \text{div } \mathbf{V} &= 0 \quad (t=0) \end{aligned} \right\}.$$

Among the formal properties of (1), one notices the existence of three positive invariant forms:

Total mass

$$M = \int_S P dS$$

Total energy

$$E = \int_S \frac{1}{2} (P + \mathbf{V} \cdot \mathbf{V}) P dS$$

Absolute potential enstrophy

$$Z = \int_S \frac{1}{2} \eta^2 P dS.$$

The importance of potential enstrophy conservation in the finite-difference formulation will be tested by considering two models almost identical but for this particular constraint. The grid disposition shown in Fig. 1 allows better dispersion properties and avoids two-grid interval noise. Together with this grid, one may choose the simplest derivation and averaging

operators:

$$\delta_x q(x, y) = \frac{1}{d} \left[ q\left(x + \frac{d}{2}, y\right) - q\left(x - \frac{d}{2}, y\right) \right]$$

$$\delta_y q(x, y) = \frac{1}{d} \left[ q\left(x, y + \frac{d}{2}\right) - q\left(x, y - \frac{d}{2}\right) \right]$$

$$\bar{q}^x(x, y) = \frac{1}{2} \left[ q\left(x + \frac{d}{2}, y\right) + q\left(x - \frac{d}{2}, y\right) \right]$$

$$\bar{q}^y(x, y) = \frac{1}{2} \left[ q\left(x, y + \frac{d}{2}\right) + q\left(x, y - \frac{d}{2}\right) \right].$$

The mass fluxes  $U$  and  $V$  are defined at the points where the velocity components  $u$  and  $v$  are located:

$$\left. \begin{aligned} U &= \bar{P}^x u \\ V &= \bar{P}^y v \end{aligned} \right\}.$$

The gradient operator will act on a quantity  $H$  defined at the locations where pressure is defined:

$$H = P + \frac{1}{2} (\bar{u}^2 + \bar{v}^2).$$

Potential vorticity is located at the mesh centers and defined as

$$\eta = \frac{\delta_x v - \delta_y u}{\bar{P}^x}.$$

Simple expressions are chosen for total mass, energy and absolute potential enstrophy:

$$\left. \begin{aligned} M &= \sum P \\ E &= \frac{1}{2} \sum (P^2 + P \bar{u}^2 + P \bar{v}^2) \\ Z &= \frac{1}{2} \sum \eta^2 \bar{P}^x \end{aligned} \right\},$$

the symbol  $\sum$  referring to a summation over all grid points of a same species. Note that the derivation operators  $\delta_x, \delta_y$  are skew-symmetric linear operators, and further, that the averaging operators  $(\bar{\phantom{x}}, \bar{\phantom{y}})$  are symmetric linear operators, i.e.,

$$\left. \begin{aligned} \sum a \bar{b} &= \sum \bar{b} a \\ \sum a \delta_x b &= - \sum b \delta_x a \end{aligned} \right\}.$$

It follows, for instance, that the time derivative of total energy reads

$$\frac{dE}{dt} = \sum \left( U \frac{\partial u}{\partial t} + V \frac{\partial v}{\partial t} + H \frac{\partial P}{\partial t} \right). \quad (2)$$

A simple energy-conserving model can be defined as

$$\left. \begin{aligned} \frac{\partial u}{\partial t} - \eta \overline{\overline{V}}^x + \delta_x H &= 0 \\ \frac{\partial v}{\partial t} + \eta \overline{\overline{U}}^y + \delta_y H &= 0 \\ \frac{\partial P}{\partial t} + \delta_x U + \delta_y V &= 0 \end{aligned} \right\}, \quad (3)$$

so that from (2) we have

$$\begin{aligned} \frac{dE}{dt} + \sum (V \eta \overline{\overline{U}}^y - U \eta \overline{\overline{V}}^x) \\ + \sum (U \delta_x H + H \delta_x U) \\ + \sum (V \delta_y H + H \delta_y V) = 0, \end{aligned}$$

where each of the three summations cancel due to the symmetry or skew symmetry properties of the operators.

On the other hand, a simple potential-entrophy-conserving model reads

$$\left. \begin{aligned} \frac{\partial u}{\partial t} - \eta \overline{\overline{V}}^x + \delta_x H &= 0 \\ \frac{\partial v}{\partial t} + \eta \overline{\overline{U}}^y + \delta_y H &= 0 \\ \frac{\partial P}{\partial t} + \delta_x U + \delta_y V &= 0 \end{aligned} \right\}. \quad (4)$$

In the corresponding vorticity equation, the discrete gradient vanishes ( $\delta_x \delta_y = \delta_y \delta_x$ ), so that

$$\frac{\partial}{\partial t} (\eta \overline{\overline{P}}^x) + \delta_x (\eta \overline{\overline{U}}^y) + \delta_y (\eta \overline{\overline{V}}^x) = 0,$$

which, when combined with the averaged continuity equation

$$\frac{\partial}{\partial t} (\overline{\overline{P}}^x) + \delta_x (\overline{\overline{U}}^y) + \delta_y (\overline{\overline{V}}^x) = 0,$$

yields the conservative potential entrophy equation

$$\frac{\partial}{\partial t} (\eta^2 \overline{\overline{P}}^x) + \delta_x (\eta^2 \overline{\overline{U}}^y) + \delta_y (\eta^2 \overline{\overline{V}}^x) = 0,$$

where the tilde in lieu of an overbar means a geometric, instead of an arithmetic, average. We may notice that (4) does not conserve energy except in the case of pure

rotational motion ( $\eta=0$ ). At this stage of simplicity, conservation of energy and conservation of potential entrophy are conflicting requirements. Eqs. (3) and (4) thus provide two quite similar numerical schemes based on the same staggered disposition of the same basic variables, differing only in the space averaging procedure for the rotation term. Eq. (3) conserves energy exactly but not potential entrophy, while Eq. (4) conserves potential entrophy exactly but not energy.

### 3. Nonlinear instability and energy catastrophe

The analysis of the long-range behavior of numerical models raises first the question of nonlinear instability. In the present experiments, the partial derivatives with respect to time were approximated by the well known "leapfrog" (three-level centered) scheme, in order to minimize dissipation. However, the slight instability inherent to this approximation necessitates some damping procedure; thus, the odd- and even-time solutions were averaged every  $N$  steps, i.e.,

$$\begin{aligned} q[(N+\frac{1}{2})\Delta t] &= \frac{1}{2}\{q[N\Delta t] + q[(N+1)\Delta t]\}, \\ q[(N-\frac{1}{2})\Delta t] &= \frac{1}{2}\{q[(N-1)\Delta t] + q[N\Delta t]\}, \end{aligned}$$

then starting afresh from the averaged fields on the left-hand side. This method amounts to a damping in higher frequencies mostly. The averaging frequency  $\nu=1/N$  can be considered a measure of the dissipativity of the time-differencing.

At first sight, energy conservation can be expected to act as a stronger constraint than potential entrophy conservation as far as nonlinear instability is concerned. In fact, instability would not occur if energy was exactly conserved in time, since all physical parameters are necessarily bounded as soon as energy is bounded. On the contrary, exact conservation of potential entrophy in time does not necessarily mean unconditional stability of the solution, since it cannot prevent possible instabilities in the divergent part of the flow. Nevertheless, in the case of the nondissipative leapfrog scheme ( $\nu=0$ ) where both models are unstable, the invariants being not exactly conserved in time, the potential-entrophy-conserving model was found more stable than its competitor, in the sense that instability occurred later for the same initial conditions. Concurrently, the energy in the divergent field remained small at all times.

Adding dissipativity to the models ( $\nu>0$ ) does not yield any drastic change in the evolution of the solution (only random fluctuations of the time of occurrence of instability around an average value, referred to as the "critical" time  $T_c$ ) until  $\nu$  reaches a critical value  $\nu_c$ . A slightly overcritical dissipativity  $\nu \gtrsim \nu_c$  stabilizes the solution far beyond  $T_c$ , practically for an infinite time. The stability properties of both models are summarized in the stability diagrams shown in Fig. 2. The potential

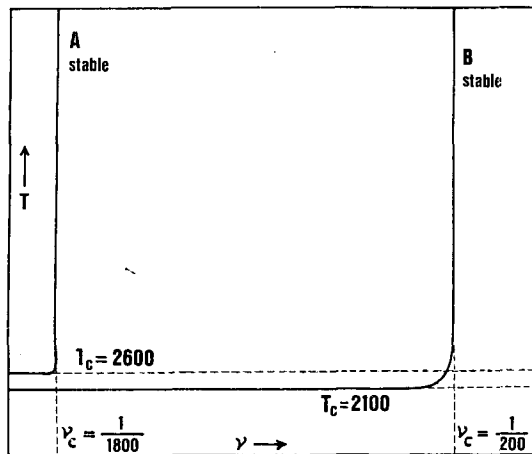


FIG. 2. Stability diagrams for the potential-entrophy-conserving model (A) and the energy-conserving model (B), showing the time of occurrence of instability  $T_c$  as a function of the damping frequency  $\nu$ .

entrophy-conserving model is significantly more stable than the energy-conserving model: not only is its critical time larger, but its critical dissipativity is an order of magnitude smaller. Ideally, the analysis of the internal mixing properties of each model should be performed in the purely inviscid case; however, the instability inherent to the leapfrog scheme makes this ideal situation impractical. The slightly overcritical case ( $\nu \gtrsim \nu_c$ ), where dissipativity is just large enough to control nonlinear instability, is the best practical approximation of the vanishing dissipation case. All experiments reported from now on have been performed around the critical dissipativity limit.

The simpler long-term statistics deal with the evolution of the two main invariants. The potential entrophy  $Z$  is not exactly conserved by the energy-conserving space differencing scheme. However, this model is an approximation of a system of equations which conserves  $Z$  exactly, so that, if one starts from sufficiently smooth initial conditions,  $Z$  remains stationary for a while. This stationarity approximately holds for  $t \ll T_c$ ; a net production of potential entrophy is observed later on, at a rate increasing with time. This increase of entrophy apparently triggers nonlinear instability in the energy-conserving model. Looking at energy only, one would not be able to predict the occurrence of instability: in fact, energy remains very stable, contrary to entrophy, and blows up suddenly at  $t = T_c$ . This sharp increase of entrophy while energy remains stationary in the close vicinity of the critical time reflects the fact that the unstable modes are to be found in the smaller scales. If dissipativity is slightly overcritical, the increase of entrophy occurs in the same manner, but stops suddenly at  $t = T_c$ ; thus,  $Z$  culminates at a value  $z_c$  independent of  $\nu$ . Energy dissipation is negligible as long as the critical time is not reached. The time  $t = T_c$  marks the beginning of a different

regime; as potential entrophy levels off, a strong dissipation of energy—an energy “catastrophe”—occurs suddenly (Fig. 3).

There is some analogy between the energy-conserving model and the Navier Stokes equations in three dimensions. The theory of three-dimensional turbulence shows that, if the initial fields are sufficiently smooth—in the sense that the initial entrophy  $Z_0$  is finite—then the nonlinear interactions force a net production of entrophy by transferring energy to the smaller scales. And the transfer is such that, in the inviscid limit, entrophy reaches infinity after a finite time  $T \sim Z_0^{-1}$ . If  $\nu$  here is the viscosity coefficient, one may show that in the limit  $\nu \rightarrow 0$ , energy is exactly conserved as long as the critical time is not reached. On the other hand, the rate of energy dissipation for  $t > T$  is independent of  $\nu$ . The critical time is the time of establishment of an energy cascade which produces a finite rate of energy dissipation even in the zero viscosity limit (Brissaud *et al.*, 1973). In the two-dimensional energy-conserving model, the rate of energy dissipation for  $t > T_c$  is also remarkably insensitive to the value of the dissipativity coefficient as long as it stays within an order of magnitude above its critical value.

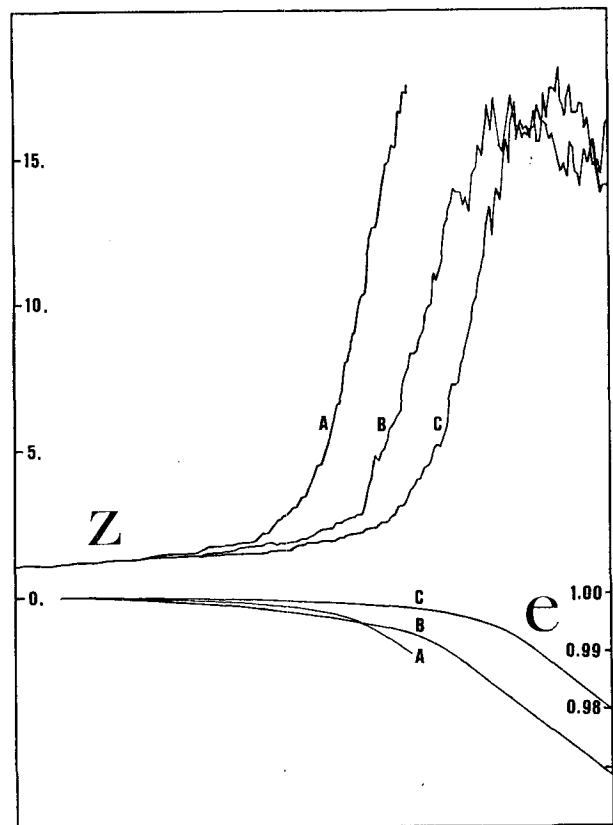


FIG. 3. Time evolution of potential entrophy ( $Z$ ) and energy ( $e$ ) in the case of the energy-conserving model, using quasi-critical dissipativity:  $\nu = 0.01$  (case A, where instability occurs),  $\nu = 0.02$  (case B),  $\nu = 0.005$  (case C).

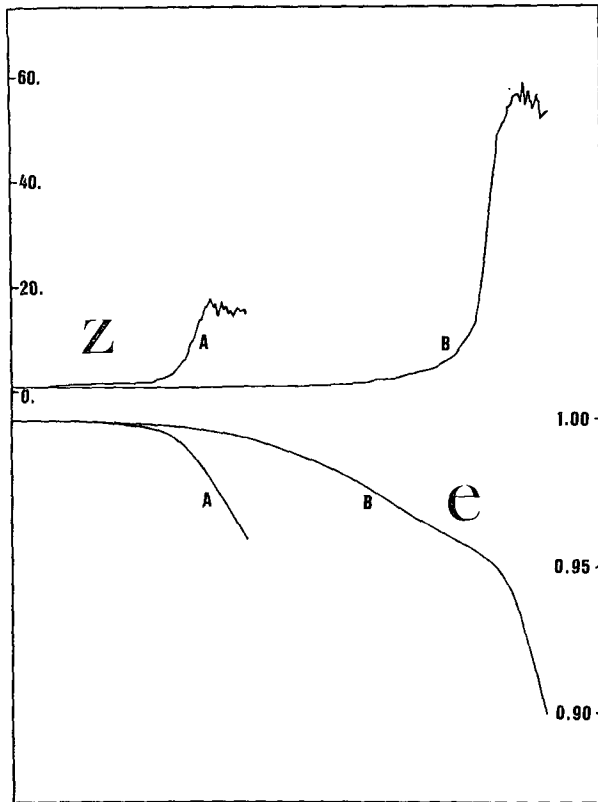


FIG. 4. Time evolution of potential enstrophy ( $Z$ ) and energy ( $e$ ) in the case of the energy-conserving model showing the influence of resolution: case A,  $16 \times 16$  grid; case B,  $32 \times 32$  grid (same initial conditions).

The analogy is by no means complete. If the modes initially excited belong to the larger scales of the model, potential enstrophy is quasi-conserved during a transient stage ( $t \ll T_c$ ). This shows in fact that the nonlinear exchanges of energy exhibit a quasi-two-dimensional structure as long as there is no significant excitation in the smaller scales. In other words, during this stage of the evolution, the nonlinear mixing of energy is mainly governed by the approximation properties. Increasing the accuracy in the larger scales by decreasing the mesh size delays the critical time and lengthens the period of time during which potential enstrophy remains almost stationary (Fig. 4). However, there is a slow buildup of energy in the smaller scales, which eventually produces a distortion of the nonlinear mixing. The triad interactions in the smaller scales are far from accurate, their structure being essentially governed by the intrinsic conservation properties of the finite-difference scheme; they produce three-dimension-like energy exchanges in the absence of formal enstrophy conservation. This effect of the smaller scales gets more and more pronounced as time goes on, and becomes predominant in the vicinity of the critical time, accounting for the enstrophy increase and the subsequent catastrophic dissipation of energy, reflecting the damping of the unstable modes.

A main characteristic of the energy-conserving model is its compulsion to dissipate energy. In the case of minimum (critical) dissipativity, energy is dissipated via the occurrence of a catastrophe at the critical time. Catastrophic energy dissipation—and the corresponding unrealistic increase of potential enstrophy—can be avoided by using higher values of dissipativity ( $\nu \gg \nu_c$ ) or by including artificial viscosity. In one way or another, energy dissipation will occur. However, assuming the motion is quasi-nondivergent and truncation occurs within a two-dimensional inertial range, any removal of rotational energy from the explicit scales is somewhat unrealistic; in two-dimensional turbulence, energy does not cascade through the inertial range into the dissipation range. The compulsion to energy dissipation is a pathological feature of the energy-conserving model due to the unrealistic mixing of energy in the smaller scales, and not related to the absence of subgrid-scale parameterization.

The behavior of the potential-enstrophy-conserving model is drastically different. An obvious defect of this very simple model is the lack of formal energy conservation by the rotation term. At the beginning of the integrations one may notice a very short transient phase during which energy decreases and adjusts to

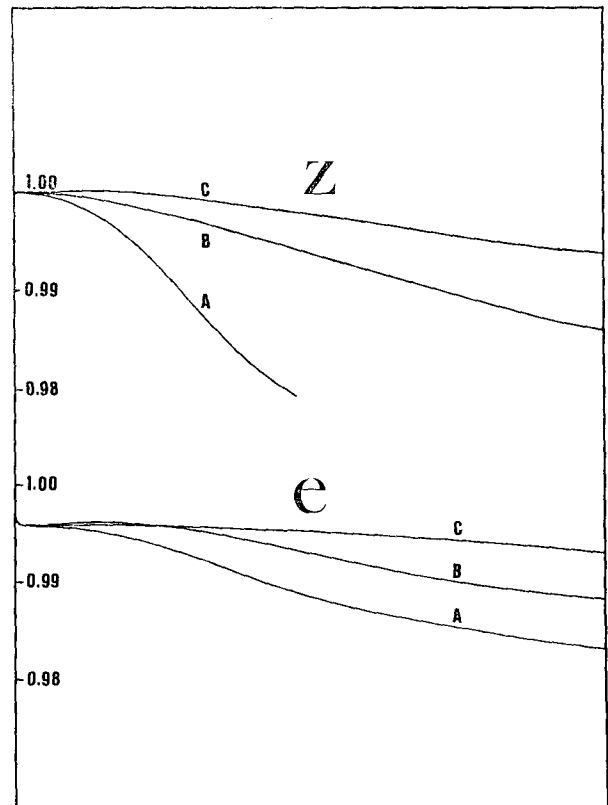


FIG. 5. Time evolution of potential enstrophy ( $Z$ ) and energy ( $e$ ) in the case of the potential-enstrophy-conserving model: case A,  $\nu=0.0025$ ; case B,  $\nu=0.001$ ; case C,  $\nu=0.00055$ .

some equilibrium value, remaining remarkably stationary afterward (Fig. 5). This initial loss of energy becomes extremely small as the grid size is reduced. As already noticed, nonlinear instability in this model is cured by a very small critical dissipativity. Contrary to what happens in the case of the energy-conserving model, instabilities develop among the intermediate scales, instabilities which are not related to a buildup of higher order moments of the spectral distribution of energy  $E(k)$ . For instance, the fourth-order moment  $\sum k^4 E(k)$  remains approximately stationary as instability begins to develop. Consequently, energy dissipation is extremely small when dissipativity is slightly overcritical, and there is not even an enstrophy catastrophe at the critical time, although enstrophy dissipation is more pronounced than energy dissipation (Fig. 5). The potential enstrophy-conserving model thus allows very long term integrations with negligible loss of energy in the explicit scales, in agreement with the theory of two-dimensional turbulence. Further, enstrophy conservation prevents any buildup of energy in the smaller scales: most of the energy remains in the larger scales, where good accuracy makes the necessity of formal energy conservation less stringent.

#### 4. Equilibrium spectra

More detailed statistics may be obtained in spectral space. Splitting the velocity field into its rotational and divergent parts  $\mathbf{V}_r$  and  $\mathbf{V}_d$ , and computing the Fourier transforms of  $P$ ,  $\mathbf{V}_r$ ,  $\mathbf{V}_d$  at every time step  $n$ , the long-range behavior of the model flows (in terms of spectral distributions of energy) may be analyzed by evaluating the time averages

$$ER(\mathbf{k}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N V_r^2(\mathbf{k}, n),$$

$$ED(\mathbf{k}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N V_d^2(\mathbf{k}, n),$$

$$EP(\mathbf{k}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N P^2(\mathbf{k}, n).$$

If we assume that truncated systems are ergodic, these expressions, computed on a single solution, are equivalent to the corresponding statistical ensemble averages.  $EP(\mathbf{k})$  is indeed the expected spectral distribution of potential energy, and  $ER(\mathbf{k})$  and  $ED(\mathbf{k})$  are the expected spectral distributions of rotational and divergent energy per unit mass. Theoretical expressions for expectations of this kind have been calculated for truncated incompressible model flows. In this case the related invariants, being quadratic, are easily expressed in Fourier space; the form of the equilibrium spectra is derived from macrocanonical ensemble averages (Fox and Orszag, 1973), or from microcanonical ensemble

averages (Basdevant and Sadourny, 1975). The ergodic hypothesis is well verified by numerical experiments, which yield equilibrium spectra in excellent agreement with the theory. In the case of the shallow water equations, however, the derivation of similar theories would be considerably more involved, and one has to rely on experimental results.

It is customary to define the energy distribution as a one-dimensional energy spectrum  $E(k)$ ,  $k = |\mathbf{k}|$ . In the finite-difference case, however, a more convenient variable is the pseudo wavenumber

$$K = \frac{2}{d} \left( \sin^2 \frac{k_x d}{2} + \sin^2 \frac{k_y d}{2} \right)^{1/2}$$

related to the eigenvalues of the finite-difference Laplacian:

$$-K^2 e^{i\mathbf{k} \cdot \mathbf{x}} = (\delta_x^2 + \delta_y^2) e^{i\mathbf{k} \cdot \mathbf{x}}.$$

Therefore, all energy spectra will be considered as functions of  $K$ , with  $EP(K)$ ,  $ED(K)$ ,  $ER(K)$  referring to averaged energies in any vector mode  $\mathbf{k}$  corresponding to  $K$ .

The spectral analysis shows that the critical time of the energy conserving model is the time needed by the nonlinear interactions to produce approximate equipartition of rotational energy between all Fourier modes. Assuming that divergent energy is zero initially and remains small, the value  $z_c$  at which enstrophy levels off is thus entirely determined by the initial kinetic energy ( $E_0$ ) and the number of grid points ( $n^2$ ). A simple estimate based on equipartition is

$$z_c \sim n^2 E_0$$

in good accordance with simulation results (Fig. 4). When resolution in both directions is doubled, the value at which potential enstrophy culminates is multiplied by a factor of 4. Smooth equipartitions of  $ER(K)$  are obtained when the model is integrated long enough (Fig. 6), looking quite similar to those observed in inviscid models of the Navier Stokes equations in three dimensions. The tendency toward equipartition of rotational energy is a more detailed illustration of the net energy transfer from lower to higher wavenumbers induced by the inaccurate structure of the nonlinear interactions in the smaller scales.

The potential-enstrophy-conserving model produces equilibrium spectra of a different kind. The long-range averaged distributions of rotational energy are characterized by an equipartition of enstrophy among higher wavenumbers [ $ER(K) \sim K^{-2}$ ], the shape of the spectrum in lower wavenumbers depending on initial conditions and containing most of the energy (Fig. 7). The small amount of energy reaching the smaller scales of the model, together with the low value of critical dissipativity  $\nu_c$ , account for the quasi-absence of energy dissipation in the slightly overcritical case. At least in

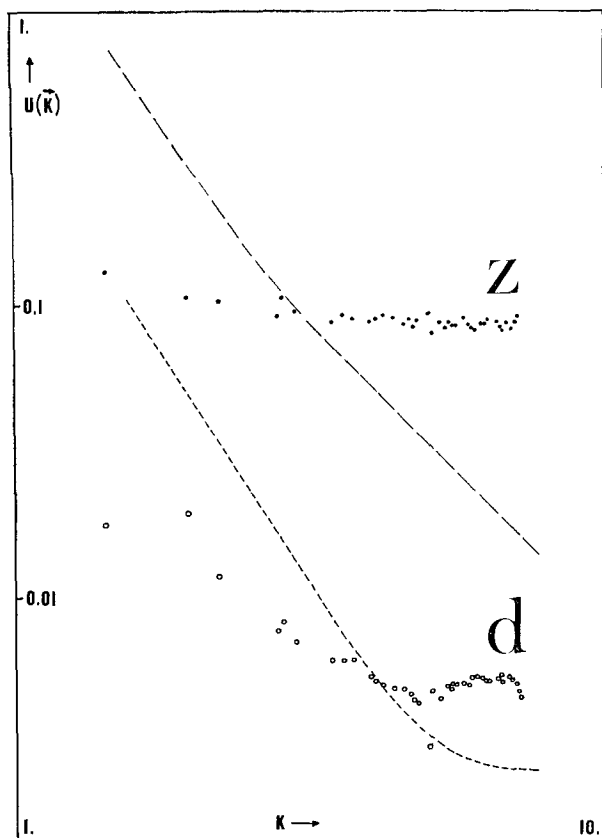


FIG. 6. Equilibrium energy spectra in the case of the quasi-inviscid energy-conserving model, showing average energy per vector mode as a function of the pseudo-wavenumber  $K$ .  $Z$ : rotational energy,  $d$ : divergent energy. Dashed lines indicate corresponding spectra in the case of the potential-estrophy-conserving model, using the same value of dissipativity ( $\nu=0.005$ ). Both coordinates are logarithmic.

the case of initially balanced fields, the shape of  $ER(K)$  is quite similar to what obtains in two-dimensional (enstrophy conserving) nondivergent model flows. This would mean that the interactions between the divergent and rotational parts of the velocity field have little effect on the long-term statistics of the rotational part. Extrapolating this property to the nontruncated equations, one can say that the rotational effect of the subgrid scales in a truncated model of the shallow-water equations should take the same form as in the nondivergent case—an accurate, energy-conserving parameterization of the enstrophy transfer toward the smaller scales.

There is no drastic change in the shape of the divergent energy distribution  $ED(K)$  as one goes from the energy-conserving model to the potential-estrophy-conserving model: as already noticed, the two models are identical in the case of pure divergent flow. The lower level of divergent energy in the first case can be explained by the higher value of critical dissipativity  $\nu_c$  (the time averaging procedure damping high fre-

quencies, such as gravity waves, mostly). In the case of the potential-estrophy-conserving model, where the distribution of rotational energy is more accurate, we observe a good correlation between  $ED(K)$  and  $ER(K)$  in the lower end of the spectrum;  $ED(K)$  is approximately an order of magnitude below  $ER(K)$ , which shows that the flow approximately remains in a state of balance, as far as the larger scales are concerned. On the other hand, there is a definite tendency toward an equipartition of divergent energy in the smaller scales where balance is no longer maintained due to the sharp decrease of rotational energy. In general, an equipartition in an inviscid truncated flow corresponds to a cascade in the original nontruncated system; if one refers to the well-known nondivergent case, truncation in two dimensions produces equipartitions of enstrophy in higher wavenumbers instead of enstrophy cascades, while truncation in three dimensions produces equipartitions of energy instead of energy cascades. Proceeding by analogy, one may conjecture that in nontruncated shallow-water flow, divergent energy does cascade into smaller and smaller scales, until it reaches the dissipation range.

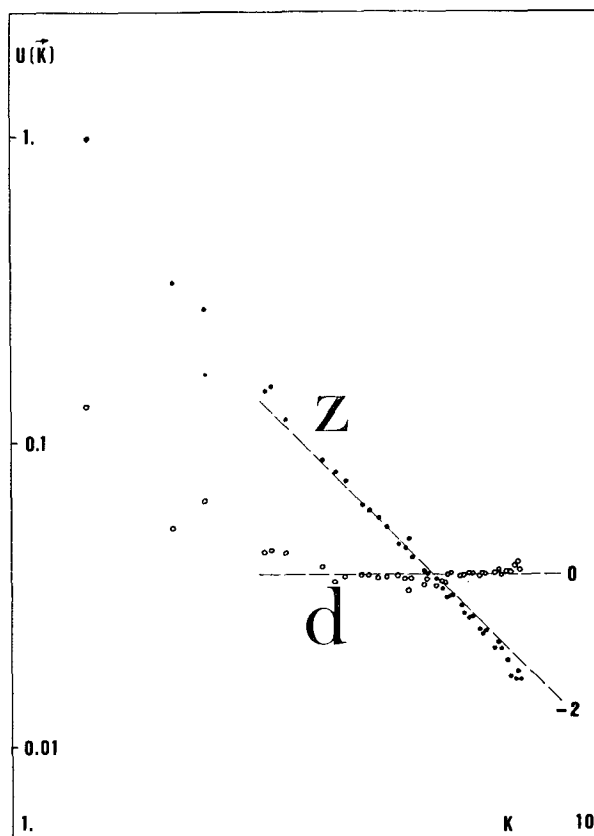


FIG. 7. Equilibrium energy spectra in the case of the quasi-inviscid potential-estrophy-conserving model, showing equipartition of enstrophy and divergent energy in higher wavenumbers. Same conventions as in Fig. 7.



Let us imagine that an unbalanced perturbation at wavenumber  $k$  is added at an arbitrary time to the system which produces the equilibrium spectra shown in Fig. 7. If the energy of the perturbation is small enough, it will not perturb the long-term averaged distributions. The ratio

$$a(k) = ER(K)/ED(K)$$

is thus a measure of the adjustment capability of the model flow to an unbalanced perturbation at wavenumber  $k$ . If the scale of the perturbation is small, the mechanism of this adjustment to balance is similar to a geostrophic adjustment (although less efficient), since the rotational motion in the larger scales yields an entrainment effect on the perturbation, similar to a Coriolis acceleration. It is clear from Fig. 7 that there can be no real adjustment in higher wavenumbers, where the flow will return to a statistical state of unbalance [ $a(k) \lesssim 1$ ], due to the nonlinear effect of divergent energy equipartition. This behavior does not contradict the linear theory by Cahn (1945) and Obukhov (1949), which calls for an infinite domain over all of which gravity wave energy may eventually disperse. It is not clear to which point adjustment occurs on a bounded domain (with zero outgoing fluxes or periodic boundary conditions) for the exact equations, as we do not know typical magnitudes of the ratio  $a(k)$  for real flows. The existence of a divergent energy cascade would produce a better adjustment, however, than that found in the inviscid truncated flow, by draining away the divergent energy which the truncation forces to accumulate.

The lack of correlation between rotational and divergent equilibrium spectra in the smaller scales of the model, as well as the different structure of nonlinear interactions in the divergent and rotational fields, call for separate parameterization of the rotational and divergent subgrid-scale motions.

The rotational part must be treated as in two-dimensional nondivergent flows. A possible method is Leith's (1968) early nonlinear viscosity approach which, however, leads to nonvanishing energy dissipation—although the nonlinear character strongly enhances enstrophy dissipation as compared to energy dissipation. Since the effect of the subgrid scales is to provide a net enstrophy transfer at the truncation, without dissipation of energy, the parameterization must essentially remove energy in the smaller scales and redistribute it in the middle scales of the model, where it should be slightly amplifying. A simple way of obtaining an effect of this sort is the adjunction of a linear term

$$\frac{\partial \mathbf{V}}{\partial t} + \dots = \nu_R (\mathbf{N} \times \text{grad rot})(-\alpha - \mathbf{N} \times \text{grad rot}) \mathbf{V},$$

which acts on the rotational field only. The differential

operators can be approximated in finite-difference form. The intensity of the simulated enstrophy transfer at the truncation is modulated through the choice of a coefficient  $\nu_R$ , while  $\alpha > 0$  is chosen in order to recover a vanishing dissipation of energy: in other words,  $\alpha$  has to be chosen as the maximum value which provides stability of the whole system, i.e., the value for which amplification in the middle scales is exactly balanced by the dynamical effects. Rotational energy is thus only redistributed in the spectrum, as in Leith's (1972) more accurate spectral method. However, a simple scheme of this kind is no more than one ad hoc linear technique among many others, in the absence of better grounded statistical theories.

On the other hand, a parameterization of the divergent energy cascade is necessary in order to recover more accurate spectral distributions for the divergent energy, and, consequently, better adjustment to balance in the smaller scales of the model. The formulation here should be closer to Smagorinsky's original method (Smagorinsky *et al.*, 1965), using a kind of nonlinear viscosity approach. More simply, a linear scheme such as

$$\frac{\partial \mathbf{V}}{\partial t} + \dots = \dots + \nu_D \text{grad div } \mathbf{V}$$

gives acceptable results (Sadourny, 1972), and has been applied successfully in four-dimensional assimilation experiments, where quick adjustment to balance in the smaller scales is of uppermost importance (Morel and Talagrand, 1974).

## 5. Conclusion

The truncation error of a finite-difference approximation in the smaller scales is so large that the nonlinear interactions involving spectral modes in the vicinity of truncation are no longer related to the exact equations, but are dependent on the space differencing only. The absence of formal conservation of potential enstrophy in the two-dimensional energy-conserving model thus yields a net transfer of rotational energy toward higher wavenumbers, as if the flow were indeed three-dimensional. This energy cascade is such that spurious dissipation of energy is unavoidable, taking the form of a sudden catastrophe in the critical dissipativity limit (quasi-inviscid case). An accurate parameterization of the subgrid-scale effects on the rotational field cannot solve this problem in any manner, since the transfer of rotational energy through truncation should vanish according to the theory of two-dimensional turbulence. Conservation of potential enstrophy in the nonlinear interactions involving triads of internal wavenumbers is thus an essential requirement for long-term numerical integrations. The fact that the equilibrium spectra of rotational energy in the case of the potential-enstrophy-conserving model are quite similar to equilibrium spectra ob-

tained in the pure non-divergent case would mean that the original Arakawa scheme which does not conserve potential enstrophy but conserves enstrophy in the case of vanishing divergence) should give equivalent results.

The equipartitions of enstrophy and divergent energy in the smaller scales observed in quasi-inviscid calculations should lead to the conclusion that the dynamics of the exact shallow-water equations involve a divergent energy cascade as well as an enstrophy cascade toward the dissipation range. If so, the subgrid-scale parameterization should be split into a rotational part simulating an enstrophy transfer at the truncation without dissipation of rotational energy, and a divergent part accounting for a divergent energy transfer. The latter should have the effect of providing a more realistic adjustment to balance in the smaller scales of the model, by preventing an accumulation of divergent energy. In the case of three-dimensional atmospheric flow, however, the relative importance of nonlinear effects as opposed to linear dispersion of gravity waves into outer space may be small, except for the Lamb wave which does not propagate vertically.

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