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Solving Shallow Water Equations by a Mixed Implicit Finite Element Method

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In this paper the shallow water equations are solved by using a numerical scheme implicit in time, and finite elements of Raviart and Thomas for the space discretization.

The nonlinear discretized problem is solved by a duality iterative algorithm. Numerical results for the dam break test problem and for simulation of real tidal currents are given.

1. Introduction

THE Saint Venant's or shallow water equations are an important mathematical model for a variety of problems in coastal engineering.

Particular applications such as tidal flow computations enable us to study, for instance, the thermal impact of an electric power plant or the transport of a pollutant in estuaries.

Many articles deal with the numerical solution of shallow water equations using finite differences and finite element methods. See, for instance, Glaister (1988), Goussebaile *et al.* (1984), Kawahara *et al.* (1978), Lynch & Gray (1979), Peraire *et al.* (1986), Taylor & Davies (1975), Zienkiewicz & Heinrich (1979).

From a mathematical point of view the shallow water equations are a system of nonlinear hyperbolic partial differential equations. Then they present some difficulties similar to those appearing in, for example, the Euler equations modelling compressible flows. In particular discontinuities or shocks can develop.

Signals propagate along characteristics with speeds $u \cdot v$ and $u \cdot v \pm c$, where $c = \sqrt{(gh)}$, h is the depth of water, and v is a unit vector. If the Froude number

$$\mathrm{Fr} = |u|/c$$

is less than 1, some of these are positive and some are negative so that propagation takes place in all directions (subcritical flow). If $\mathrm{Fr} > 1$ (supercritical flow), the information is propagated only in certain directions. In particular, it is not propagated upstream.

It is a well-known result that explicit centred schemes for linear hyperbolic problems are unconditionally unstable and thus two main possibilities exist for numerical solution: using upwinding for explicit schemes or using implicit schemes. In the first case stability requires a Courant Friedrichs–Lewy condition which usually imposes very small time steps. Notice that, for instance, in tidal flow computations while u is small c can be two orders of magnitude higher.

In this paper we use a method of transport-diffusion (or Lagrange–Galerkin) to upwind the convective term and an implicit centred discretization of the 'pressure term' in the momentum equations. In this way we obtain an unconditionally stable method in the sense of the classical linear Fourier stability analysis.

Space discretization is done by Raviart and Thomas mixed finite elements on a triangular mesh. They consist of discontinuous functions of degree one for the flow rate and piecewise constant functions for the height.

The nonlinear system arising per time step is solved by an iterative procedure involving linear systems with constant matrix.

Numerical results are presented for a one-dimensional test example as well as an application to the simulation of tidal currents in the 'Ría de Pontevedra' (Galicia, Spain).

2. The shallow water equations

Suppose we have an incompressible Newtonian fluid occupying a three-dimensional domain. Consider the cartesian axis given in Fig. 1.

Let $h(x_1, x_2, t)$ be the height of the fluid at point (x_1, x_2) at time t and $H(x_1, x_2)$ the depth of the point (x_1, x_2) from a reference level. Notice that H can be negative at some regions.

The shallow water equations present an average model which can be obtained from the Navier–Stokes equations by depth integration supposing hydrostatic pressure only and neglecting all dynamic effects in the vertical direction.

Let Ω be the x_1, x_2 projection of the domain filled by the fluid and $u = (u_1, u_2)$ the averaged horizontal velocity. Then the shallow water equations can be written

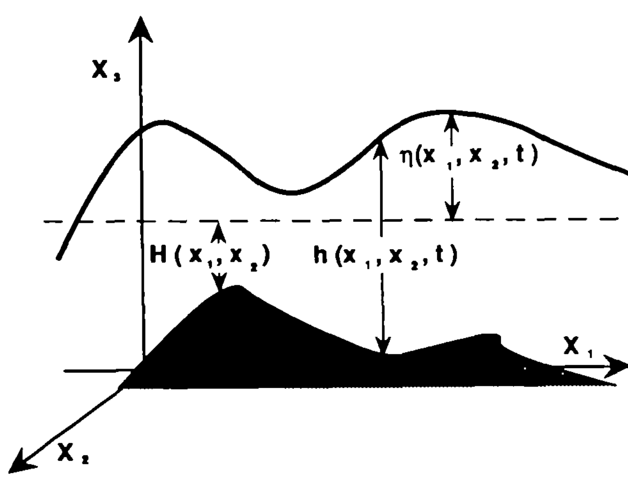

FIG. 1.

in a conservative form as follows (see, for instance, Peraire *et al.*, 1986):

$$\frac{\partial h}{\partial t}+\nabla\cdot Q=0, \tag{2.1}$$

$$\frac{\partial Q}{\partial t}+\nabla\cdot[u\otimes Q+\tfrac{1}{2}g(h^2-H^2)\delta]=g(h-H)\nabla H+F+\frac{1}{\rho}(\tau_w-\tau_f), \tag{2.2}$$

where

$Q=uh$,
H is the depth of water from a reference level,
g is gravity,
δ is the unit tensor,
$F=2\Omega\sin\phi(Q_2,-Q_1)$ (Coriolis effect),
ρ is density,

$$\tau_w=\gamma_{10}v\cdot|v| \text{ (wind stress)}, \tag{2.3}$$

$$\tau_f=\rho g u\,|u|/c^2 \text{ (bottom friction stress)}, \tag{2.4}$$

Ω is the angular velocity of the earth,
ϕ is the north latitude,
v is the velocity of wind at 10 m above water surface,
c is the Chézy coefficient,
$|u|=(u_1^2+u_2^2)^{\frac{1}{2}}$.
Boundary conditions are of the following two types:
(i) Coast or effluent (Γ_0)

$$Q\cdot\nu=f \quad \nu \text{ unit normal vector, } (f=0 \text{ on the coast}). \tag{2.5}$$

(ii) Open sea (Γ_1)

$$h=\varphi+H \quad (\varphi \text{ a given function}). \tag{2.6}$$

Finally, initial conditions are given:

$$h(x,0)=h_0(x), \tag{2.7}$$

$$u(x,0)=u_0(x). \tag{2.8}$$

For certain reasons related to the numerical methods to be given later, it is more convenient to rewrite (2.1)–(2.2) in the following way:

$$\frac{\partial \eta}{\partial t}+\nabla\cdot Q=0 \tag{2.9}$$

$$\frac{\partial Q}{\partial t}+\nabla\cdot[(u\otimes Q)+\tfrac{1}{2}g(\eta^2+2\eta H)\delta]=g\eta\nabla H+F+\frac{1}{\rho}(\tau_w-\tau_f), \tag{2.10}$$

where $\eta=h-H$.

The existence of a solution of (2.9)–(2.10) with (2.5), (2.6), (2.7), and (2.8) is an open question. However, some results for related problems can be found in Matsamura & Nishida (1982).

3. Variational formulation

In order to use Galerkin finite element methods we begin by giving a weak formulation of (2.9) and (2.10).

As the height of water h cannot be negative, we are interested in solutions satisfying $\eta > -H$. Then it is evident that in this case we can replace equation (2.10) by

$$\frac{\partial Q}{\partial t} + \nabla \cdot (u \otimes Q) + \tfrac{1}{2} g \nabla G(\eta) - g\eta \nabla H = F + \frac{1}{\rho}(\tau_w - \tau_f), \tag{3.1}$$

where G denotes the maximal monotone operator given by $G = \partial\psi$ (see Fig. 2) with

$$\psi(\eta) = \begin{cases} \tfrac{1}{3}\eta^3 + \eta^2 H & \text{if } \eta + H \geqslant 0, \\ \infty & \text{otherwise.} \end{cases} \tag{3.2}$$

Recall that $\partial\psi$ denotes the subdifferential of the function ψ (see, for instance, Ekeland & Temam, 1976).

Q and η satisfy

$$\int_\Omega \frac{\partial \eta}{\partial t} y \, dx + \int_\Omega \nabla \cdot Q y \, dx = 0 \quad \forall y, \tag{3.3}$$

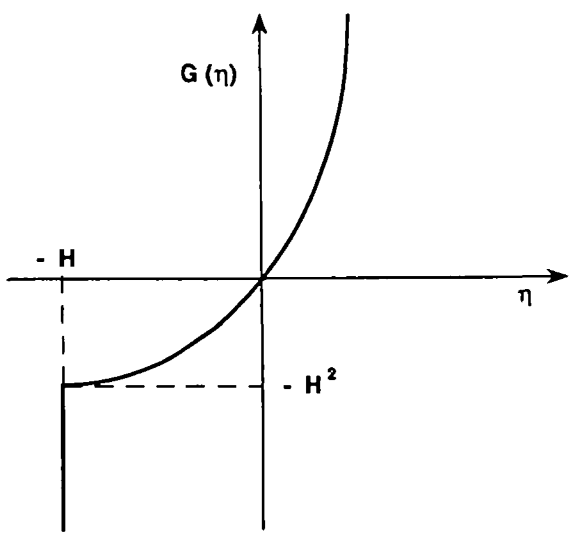

FIG. 2.

$$\int_{\Omega} \frac{\partial Q}{\partial t} \cdot z \, dx + \int_{\Omega} \nabla \cdot (u \otimes Q) \cdot z \, dx - \tfrac{1}{2} g \int_{\Omega} G(\eta) \nabla \cdot z \, dx - g \int_{\Omega} \eta \nabla H \cdot z \, dx$$

$$= \int_{\Omega} F \cdot z \, dx + \int_{\Omega} \frac{1}{\rho} [\tau_w(Q, \eta) - \tau_f(Q, \eta)] \cdot z \, dx$$

$$- g \int_{\Gamma_1} (\tfrac{1}{2}\varphi^2 + \varphi H) z \cdot \nu \, d\Gamma \quad \forall z / z \cdot \nu \restriction_{\Gamma_0} = 0, \tag{3.4}$$

$$Q \cdot \nu = f \quad \text{on } \Gamma_0 \times (0, T), \tag{3.5}$$

$$\eta(x, 0) = \eta_0(x) \quad \text{in } \Omega, \tag{3.6}$$

$$Q(x, 0) = \big(\eta_0(x) + H(x)\big) u_0(x) \quad \text{in } \Omega. \tag{3.7}$$

4. Time discretization: characteristics method

In this section we give an implicit discretization in time which deals with the convective term $\nabla \cdot (u \otimes Q)$ by using a method of characteristics (see Pironneau, 1982; Bercovier *et al.*, 1983; Morton *et al.*, 1988; Bermúdez & Durany, 1987).

For this, recall that the following equality holds:

$$\frac{DJQ}{Dt}(x, t) = \frac{\partial Q}{\partial t}(x, t) + \nabla \cdot (u \otimes Q)(x, t), \tag{4.1}$$

where in general Dy/Dt denotes the total derivative of y with respect to t, i.e.

$$\frac{Dy}{Dt}(x, t) = \frac{\partial}{\partial \tau} y\big(X(x, t; \tau), \tau\big) \restriction_{\tau = t}, \tag{4.2}$$

$\tau \to X(x, t; \tau)$, is the trajectory of the particle being at point x at instant t, i.e.

$$\left.\begin{aligned} &\frac{d}{d\tau} X(x, t; \tau) = u\big(X(x, t; \tau), \tau\big), \\ &X(x, t; t) = x, \end{aligned}\right\} \tag{4.3}$$

and $J(x, t; \tau)$ represents the evolution of the element of volume which is the solution of the following ordinary differential equation:

$$\left.\begin{aligned} &\frac{d}{d\tau} J(x, t; \tau) = \nabla \cdot u\big(X(x, t; \tau), \tau\big), \\ &J(x, t; t) = 1. \end{aligned}\right\} \tag{4.4}$$

Let Δt be a time step, $t_n = n \cdot \Delta t$ $(n = 0, 1, \ldots)$. Denote $X^n(x) = X(x, t_{n+1}; t_n)$ then we approximate $(DJQ/Dt)(x, t)$ at $t = t_{n+1}$ by

$$\frac{DJQ}{Dt}(x, t_{n+1}) \simeq \frac{Q^{n+1}(x) - Q^n\big(X^n(x)\big) J^n(x)}{\Delta t} \tag{4.5}$$

(observe that $J(x, t_{n+1}; t_{n+1}) = 1$).

This formula leads to the following semidiscrete problem:

$$\frac{\eta^{n+1}-\eta^n}{\Delta t}+\nabla\cdot Q^{n+1}=0, \tag{4.6}$$

$$\int_\Omega \frac{Q^{n+1}-Q^n\circ X^nJ^n}{\Delta t}\cdot z\,\mathrm{d}x-\tfrac{1}{2}g\int_\Omega G(\eta^{n+1})\nabla\cdot z\,\mathrm{d}x-g\int_\Omega \eta^{n+1}\nabla H\cdot z\,\mathrm{d}x$$
$$=\int_\Omega F^n\cdot z\,\mathrm{d}x+\frac{1}{\rho}\int_\Omega[\tau_w(Q^n,\eta^n)-\tau_f(Q^n,\eta^n)]\cdot z\,\mathrm{d}x$$
$$-g\int_{\Gamma_1}[\tfrac{1}{2}\varphi(t_{n+1})^2+\varphi(t_{n+1})H]z\cdot\nu\,\mathrm{d}\Gamma. \tag{4.7}$$

Now we can eliminate η^{n+1} to obtain

$$\int_\Omega \frac{Q^{n+1}-Q^n\circ X^nJ^n}{\Delta t}\cdot z\,\mathrm{d}x-\tfrac{1}{2}g\int_\Omega G(\eta^n-\Delta t\nabla\cdot Q^{n+1})\nabla\cdot z\,\mathrm{d}x$$
$$-g\int_\Omega(\eta^n-\Delta t\nabla\cdot Q^{n+1})\nabla H\cdot z\,\mathrm{d}x$$
$$=\int_\Omega F^n\cdot z\,\mathrm{d}x+\frac{1}{\rho}\int_\Omega[\tau_w(Q^n,\eta^n)-\tau_f(Q^n,\eta^n)]\cdot z\,\mathrm{d}x$$
$$-g\int_{\Gamma_1}[\tfrac{1}{2}\varphi(t_{n+1})^2+\varphi(t_{n+1})H]z\cdot\nu\,\mathrm{d}\Gamma\quad\forall z/z\cdot\nu=0\quad\text{on }\Gamma_0, \tag{4.8}$$

$$Q^{n+1}\cdot\nu=f^{n+1}\quad\text{on }\Gamma_0, \tag{4.9}$$

$$\eta^{n+1}=\eta^n-\Delta t\nabla\cdot Q^{n+1}, \tag{4.10}$$

$$\eta^0=\eta_0,\qquad Q^0=u_0(\eta_0+H). \tag{4.11}$$

The following proposition gives the existence of a solution for this semi-discretized problem.

PROPOSITION *Assume* $\eta^n\in L^3(\Omega)$,

$$Q^n\circ X^nJ^n,\ F^n,\ \tau_w^n,\ \tau_f^n\in(L^2(\Omega))^2,$$
$$H\in C^1(\Omega),\qquad \varphi\in C(\Gamma_1 x(0,T)).$$

Then there exists Q^{n+1}, η^{n+1} *such that*

$$Q^{n+1}\in V=\{Q\in(L^2(\Omega))^2\colon \nabla\cdot Q\in L^3(\Omega)\},\qquad \eta^{n+1}\in L^3(\Omega)$$

is a unique solution of the problem (4.8)–(4.11).

Proof. Notice first that if $Q\in V$ then $Q\cdot\nu$ is defined in $H^{-\frac{1}{2}}(\Gamma_0)$. Now the problem (4.8)–(4.9) can be written as the variational inequality

$$\frac{1}{\Delta t}\int_\Omega Q^{n+1}\cdot(z-Q^{n+1})\,\mathrm{d}x-g\,\Delta t\int_\Omega \nabla\cdot Q^{n+1}\nabla H\cdot(z-Q^{n+1})\,\mathrm{d}x$$
$$+\frac{g}{2\,\Delta t}[\Psi(\eta^n-\Delta t\nabla\cdot z)-\Psi(\eta^n-\Delta t\nabla\cdot Q^{n+1})]$$
$$\geqslant(L^n,z-Q^{n+1})\quad\forall z\in V,\quad z\cdot\nu=0\quad\text{on }\Gamma_0, \tag{4.12}$$

$$Q^{n+1}\cdot\nu=f^{n+1}\quad\text{on }\Gamma_0, \tag{4.13}$$

where

$$\Psi(y)=\int_\Omega \psi(y)\,\mathrm{d}x \quad \text{for } y\in \mathrm{L}^3(\Omega), \tag{4.14}$$

$$L^n(z)=\frac{1}{\Delta t}\int Q^n\circ X^n J^n\cdot z\,\mathrm{d}x+g\int_\Omega \eta^n \nabla H\cdot z\,\mathrm{d}x+\int_\Omega F^n\cdot z\,\mathrm{d}x$$
$$+\frac{1}{\rho}\int_\Omega[\tau_w(Q^n,\eta^n)-\tau_f(Q^n,\eta^n)]\cdot z\,\mathrm{d}x-g\int_{\Gamma_1}[\tfrac{1}{2}\varphi(t_{n+1})^2+\varphi(t_{n+1})H]z\cdot \nu\,\mathrm{d}\Gamma. \tag{4.15}$$

In order to show the existence of a solution of this variational inequality we shall apply a result from Lions (1969: Thm 8.5).

Let A be the linear bounded operator from V to V' defined by

$$(AQ,z)_{V'V}=\frac{1}{\Delta t}\int_\Omega Q\cdot z\,\mathrm{d}x-g\,\Delta t\int_\Omega \nabla\cdot Q\nabla H\cdot z\,\mathrm{d}x. \tag{4.16}$$

Then (4.12)–(4.13) satisfy the assumptions in the above-mentioned theorem. Indeed, A is a linear bounded operator and then pseudomonotonic from V into V' and Ψ is a proper lower semicontinuous functional. Furthermore, taking $\bar{Q}\in V'$ such that

$$\eta^n-\Delta t\nabla\cdot\bar{Q}\geqslant -H \quad \text{a.e. in } \Omega,$$

then $\Psi(\eta^n-\Delta t\bar{Q})<\infty$ and we have

$$\lim_{\|Q\|\to\infty}\frac{(AQ,Q-\bar{Q})+\Psi(\eta^n-\Delta tQ)}{\|Q\|}=\infty \quad \square \tag{4.17}$$

5. Discretization in space: finite elements

Mixed finite elements of Raviart and Thomas (1977) have been used to discretize in space the problem (4.8), (4.11).

Let τ_h be a family of triangulations of Ω. Associated with it we consider two finite-dimensional vector spaces. The first one called V_h consists of discontinuous vector functions which are polynomials of degree one on each triangle. More precisely V_h is defined by

$$V_h=\{Q_h\in H(\mathrm{div},\Omega)\colon Q_h\restriction_K\in V_K,\ \forall\, K\in\tau_h\}, \tag{5.1}$$

where $H(\mathrm{div},\Omega)$ is the functional space defined by

$$H(\mathrm{div},\Omega)=\{q\in \mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Omega)\colon \mathrm{div}\,q\in\mathrm{L}^2(\Omega)\},$$

and, for $K\in\tau_h$, V_K is defined by

$$V_K=\left\{q=\frac{1}{J_K}B_K\hat{q}\circ F_K^{-1},\ \forall\,\hat{q}\in\hat{V}\right\}, \tag{5.2}$$

with $\hat{V}$ the three-dimensional vector space spanned by the functions

$$\left.\begin{aligned}\mu_1(x_1,x_2)&=(x_1,-1+x_2),\\ \mu_2(x_1,x_2)&=(\sqrt{2}x_1,\sqrt{2}x_2),\\ \mu_3(x_1,x_2)&=(-1+x_1,x_2),\end{aligned}\right\} \tag{5.3}$$

In (5.2), F_K denotes the unique affine invertible mapping from $\hat{K}$ to K, $\hat{K}$ being the triangle with vertices (0, 0), (1, 0), and (0, 1), B_K is the linear part of F_K and $J_K = \det(B_K)$.

Recall that while functions in V_h are discontinuous on the edges of the triangles their normal components are continuous and constant on them. In fact they are taken as degrees of freedom.

Let M_h be the space of constant per triangle functions, i.e.

$$M_h = \{\eta_h : \eta_{h\restriction K} \in P_0,\ \forall\, K \in \tau_h\}. \tag{5.4}$$

Then the discretized problem consists in finding $Q_h^{n+1} \in V_h$ and $\eta_h^{n+1} \in M_h$ such that

$$\begin{aligned}
&\int_\Omega \frac{Q_h^{n+1} - Q_h^n \circ X_h^n J_h^n}{\Delta t} \cdot z_h \,\mathrm{d}x - \tfrac{1}{2} g \int_\Omega G(\eta_h^n - \Delta t \cdot \nabla \cdot Q_h^{n+1}) \nabla \cdot z_h \,\mathrm{d}x \\
&\quad - g \int_\Omega (\eta_h^n - \Delta t \nabla \cdot Q_h^{n+1}) \nabla H \cdot z_h \,\mathrm{d}x \\
&\quad = \int_\Omega F_h^n \cdot z_h \,\mathrm{d}x + \frac{1}{\rho} \int_\Omega [\tau_w(Q_h^n, \eta_h^n) - \tau_f(Q_h^n, \eta_h^n)] \cdot z_h \,\mathrm{d}x \\
&\quad - g \int_{\Gamma_1} [\tfrac{1}{2}\varphi(t_{n+1})^2 + \varphi(t_{n+1}) H] z_h \cdot \nu \,\mathrm{d}\Gamma \quad \forall\, z_h \in V_h, \quad z_h \cdot \nu = 0 \quad \text{on } \Gamma_0,
\end{aligned} \tag{5.5}$$

$$Q_h^{n+1} \cdot \nu = f_h^{n+1} \quad \text{on } \Gamma_0, \tag{5.6}$$

$$\eta_h^{n+1} = \eta_h^n - \Delta t \nabla \cdot Q_h^{n+1}. \tag{5.7}$$

In order to get a numerical scheme preserving conservativity of mass we have slightly modified the essential boundary condition (5.6) as follows:

We consider first the discrete continuity equation and integrate it in Ω. We have

$$\int_\Omega \eta_h^{n+1} \,\mathrm{d}x = \int_\Omega \eta_h^n \,\mathrm{d}x - \Delta t \int_\Omega \nabla \cdot Q_h^{n+1} \,\mathrm{d}x, \tag{5.8}$$

and, using Gauss' theorem,

$$\int_\Omega \eta_h^{n+1} \,\mathrm{d}x = \int_\Omega \eta_h^n \,\mathrm{d}x - \Delta t \int_{\Gamma_0} Q_h^{n+1} \cdot \nu \,\mathrm{d}\Gamma - \Delta t \int_{\Gamma_1} Q_h^{n+1} \cdot \nu \,\mathrm{d}\Gamma, \tag{5.9}$$

because the normal component of Q_h^{n+1} is continuous across edges.

This equality, in good agreement with physics, says that the variation of mass is due to flow across boundaries. This flow has two parts: one corresponding to effluents, which is known (f is given on Γ_0), and another one corresponding to the flow across the open sea boundary, which is unknown. Then we are interested in a discrete problem satisfying

$$\int_{\Gamma_0} Q_h^{n+1} \cdot \nu \,\mathrm{d}\Gamma = \int_{\Gamma_0} f_h^{n+1} \,\mathrm{d}\Gamma, \tag{5.10}$$

that is having a global mass conservation property. For these reasons condition

(5.6) has been replaced by

$$Q_e^{n+1} \cdot \nu_e = \frac{1}{\text{length}\,(e)} \int_e f_h^{n+1}\, d\Gamma \quad \text{for all edge } e \text{ on } \Gamma_0, \tag{5.11}$$

where ν_e denotes the unit vector normal to edge e. Notice that, since the degrees of freedom are the normal components of Q on edges, it is very easy to take into account these boundary conditions as well as the ones on the coast.

Linear stability analysis of a one-dimensional version of this numerical scheme is shown in the Appendix. The discrete Fourier analysis predicts unconditional stability.

6. Solving the discretized problem

The first step consists in computing the transport term, i.e.

$$\int_\Omega Q_h^n \circ X_h^n J_h^n \cdot z_h \, dx. \tag{6.1}$$

Notice that, while Q_h^n is a polynomial on each triangle of the mesh, $Q_h^n \circ X_h^n$ is not so. Hence, (6.1) has to be calculated by numerical quadrature. This approximation causes the nonconservativity of the scheme in terms of momentum. On the other hand the effect of using nonexact integration on the stability of Lagrange–Galerkin methods has been extensively analysed in Morton *et al.* (1988).

Here we have taken the formula involving the nodes which are the midpoints of the edges. All these nodes are moved back along the characteristics at each time step. This is done by solving the ordinary differential equation in (4.3) with an explicit Euler scheme.

On the other hand observe that the discretized problem is nonlinear due to the presence of the operator G. To solve it we have used an iterative algorithm given in Bermúdez & Moreno (1981).

The idea is as follows. For $w > 0$ let G^w be the operator $G^w = G - wI$, I being the identity operator. Then (5.5) can be written (we drop index h for simplicity)

$$\begin{aligned}
&\frac{1}{\Delta t}\int_\Omega Q^{n+1} \cdot z \, dx + \Delta t w(\tfrac{1}{2}g)\int_\Omega \nabla \cdot Q^{n+1} \nabla \cdot z \, dx + \Delta t g \int_\Omega \nabla \cdot Q^{n+1} \nabla H \cdot z \, dx \\
&= \tfrac{1}{2}g \int_\Omega p^{n+1} \nabla \cdot z \, dx + \frac{1}{\Delta t}\int_\Omega Q^n \circ X^n J^n \cdot z \, dx + \tfrac{1}{2} g w \int_\Omega \eta^n \nabla \cdot z \, dx \\
&\quad + g \int_\Omega \eta^n \nabla H \cdot z \, dx + \int_\Omega F^n \cdot z \, dx + \frac{1}{\rho}\int_\Omega [\tau_w(Q^n, \eta^n) - \tau_f(Q^n, \eta^n)] \cdot z \, dx \\
&\quad - g \int_{\Gamma_1} [\tfrac{1}{2}\varphi(t_{n+1})^2 + \varphi(t_{n+1})H] z \cdot \nu \, d\Gamma,
\end{aligned} \tag{6.2}$$

with

$$p^{n+1} = G(\eta^n - \Delta t \nabla \cdot Q^{n+1}) - w(\eta^n - \Delta t \nabla \cdot Q^{n+1}). \tag{6.3}$$

Bermúdez and Moreno (1981) show that the previous equality is equivalent to

$$p^{n+1} = G_\lambda^w(\eta^n - \Delta t \nabla \cdot Q^{n+1} + \lambda p^{n+1}) \quad \text{for } \lambda w < 1, \tag{6.4}$$

where G_λ^w is the so-called Yosida regularization of G^w given in this case by

$$G_\lambda^w(x) = \begin{cases} \{2\lambda x + 1 - \lambda w + 2\lambda H - \sqrt{[(1 - \lambda\omega + 2\lambda H)^2 + 4\lambda x]}\}/2\lambda^2 \\ \qquad\qquad\qquad\qquad\qquad \text{if } x \geqslant -(1-\lambda\omega)H - \lambda H^2 \\ (x+H)/\lambda \qquad\qquad\qquad\qquad \text{otherwise} \end{cases} \tag{6.5}$$

The iterative algorithm is as follows:
Initial step: $p^{n+1,0}$ is arbitrarily given.
Step r: Compute $Q^{n+1,r}$ solution of the linear problem

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega Q^{n+1,r} \cdot z \,\mathrm{d}x + \Delta t w(\tfrac{1}{2}g)\int_\Omega \nabla \cdot Q^{n+1,r} \nabla \cdot z \,\mathrm{d}x + \Delta t g \int_\Omega \nabla \cdot Q^{n+1,r} \nabla H \cdot z \,\mathrm{d}x \\ &= \tfrac{1}{2}g\int_\Omega p^{n+1,r}\nabla \cdot z \,\mathrm{d}x + \frac{1}{\Delta t}\int_\Omega Q^n \circ X^n J^n \cdot z \,\mathrm{d}x + \tfrac{1}{2}gw\int_\Omega \eta^n \nabla \cdot z \,\mathrm{d}x \\ &\quad + g\int_\Omega \eta^n \nabla H \cdot z \,\mathrm{d}x + \int_\Omega F^n \cdot z \,\mathrm{d}x + \frac{1}{\rho}\int_\Omega [\tau_w(Q^n, \eta^n) - \tau_f(Q^n, \eta^n)] \cdot z \,\mathrm{d}x \\ &\quad - g\int_{\Gamma_1} [\tfrac{1}{2}\varphi(t_{n+1})^2 + \varphi(t_{n+1})H] z \cdot \nu \,\mathrm{d}\Gamma. \end{aligned} \tag{6.6}$$

Update $p^{n+1,r}$ by

$$p^{n+1,r+1} = G_\lambda^w(\eta^n - \Delta t \nabla \cdot Q^{n+1,r} + \lambda p^{n+1,r}). \tag{6.7}$$

Notice that problems (6.6) have matrices independent of n and r. They have been solved by using preconditioned biconjugate gradient methods. Preconditioning is done by incomplete Gauss factorization which is computed only once.

7. Numerical results

In this section some numerical results corresponding to two test examples are given. First the well-known dam break or hydraulic jump problem is considered. Then we present some results for tidal currents at the 'ría' of Pontevedra (Galicia, Spain). All figures have been drawn using the Modulef library.

7.1 *Dam Break Problem*

This concerns a one-dimensional version of problem (2.9)–(2.10) without terms F, τ_w, and τ_f corresponding to an initial condition

$$\eta_0(x) = \begin{cases} \eta_1 & \text{if } x < 0, \\ \eta_2 & \text{if } x > 0, \end{cases} \tag{7.1}$$

$$u_0(x) = 0. \tag{7.2}$$

The analytical solution of this problem is given in Stoker (1957). It includes a rarefaction wave and a shock discontinuity. Assuming $H \equiv 0$, it is given by

$$\eta(x,t)=\begin{cases}\eta_1 & \text{if } x/t \leqslant -\surd(g\eta_1),\\ (1/9g)[2\surd(g\eta_1)-x/t]^2 & \text{if } -\surd(g\eta_1)\leqslant x/t \leqslant [u_m-\surd(g\eta_m)],\\ \eta_m & \text{if } [u_m-\surd(g\eta_m)]\leqslant x/t\leqslant s,\\ \eta_2 & \text{if } s\leqslant x/t<\infty,\end{cases} \tag{7.3}$$

$$u(x,t)=\begin{cases}0 & \text{if } x/t \leqslant -\surd(g\eta_1),\\ \tfrac{2}{3}[2\surd(g\eta_1)-x/t] & \text{if } -\surd(g\eta_1)\leqslant x/t \leqslant [u_m-\surd(g\eta_m)],\\ u_m & \text{if } [u_m-\surd(g\eta_m)]\leqslant x/t\leqslant s,\\ 0 & \text{if } s\leqslant x/t<\infty,\end{cases} \tag{7.4}$$

where η_m and u_m are given in terms of velocity of propagation of the shock s by

$$\eta_m=\tfrac{1}{2}\left[\sqrt{\left(1+\frac{8s^2}{g\eta_2}\right)}-1\right]\eta_0, \tag{7.5}$$

$$u_m=s-\frac{g\eta_2}{4s}\left[1+\sqrt{\left(1+\frac{8s^2}{g\eta_2}\right)}\right], \tag{7.6}$$

and s is the positive real solution of the equation

$$u_m+2\surd(g\eta_m)-2\surd(g\eta_1)=0. \tag{7.7}$$

We have solved this problem using the one-dimensional version of the numerical scheme (5.3)–(5.5) that is

$$\frac{Q_j^{n+1}-(Q^n\circ X^n)_j J_j^n}{\Delta t}+\tfrac{1}{2}g\frac{(\eta_{j+1}^{n+1})^2-(\eta_j^{n+1})^2}{\Delta x}=0, \tag{7.8}$$

$$\frac{\eta_j^{n+1}-\eta_j^n}{\Delta t}+\frac{Q_{j+1}^{n+1}-Q_j^{n+1}}{\Delta x}=0. \tag{7.9}$$

In (7.8) we have computed $X^n(x_j)$ by using the explicit Euler method to solve the ordinary differential equations of trajectories. We have also used an Euler-trapezoidal rule predictor–corrector scheme as suggested by one of the referees. Numerical results for both methods are very similar and we only present here those corresponding to the first one.

Figures 3 to 8 show exact and approximated solutions at time $t = 0{\cdot}1$ computed with $\Delta x = 10^{-2}$ and time steps equal to 10^{-2}, 10^{-3}, and 10^{-4}. They correspond to $\eta_1 = 1$ and $\eta_2 = 0{\cdot}5$ in (7.1).

Units for x and t are metres and seconds respectively. Therefore, as the characteristic velocities are $u \pm \surd(gh)$, Courant numbers are about 4, 0·4, and 0·04 respectively. From the Fourier analysis given in the Appendix, we can deduce that diffusion of the scheme is quite important for high frequencies except if Δt is taken small enough. The analytical solution being discontinuous explains the behaviour observed for $\Delta t = 10^{-2}$ in Figs. 3 and 4.

On the other hand, for $\Delta t = 10^{-3}$, results are quite similar to the ones obtained by using explicit flux vector and flux difference splitting in Glaister (1988) or

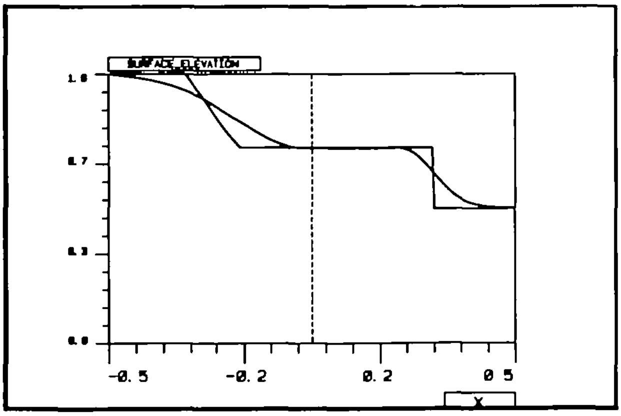

FIG. 3. Dam break problem $\Delta t = 10^{-2}$, $t = 0{\cdot}1$; surface elevation.

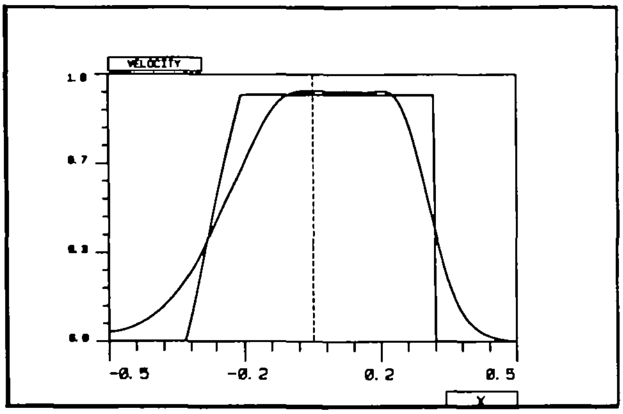

FIG. 4. Dam break problem $\Delta t = 10^{-2}$, $t = 0{\cdot}1$; velocity.

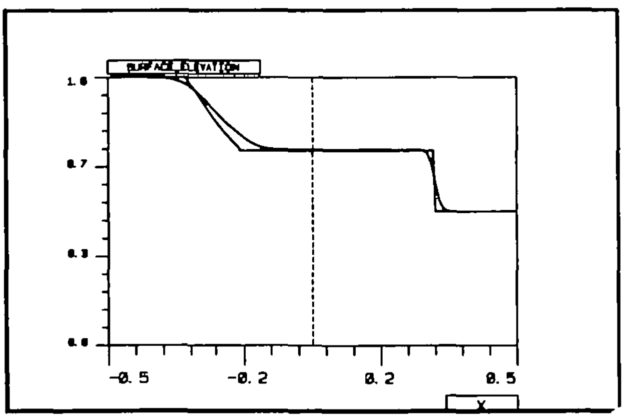

FIG. 5. Dam break problem $\Delta t = 10^{-3}$, $t = 0{\cdot}1$; surface elevation.

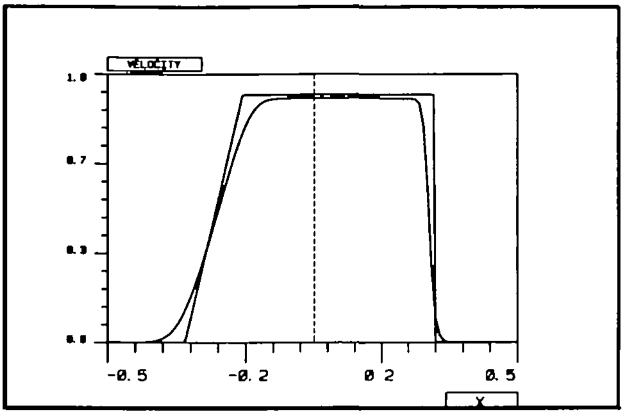

FIG. 6. Dam break problem $\Delta t = 10^{-3}$, $t = 0{\cdot}1$; velocity.

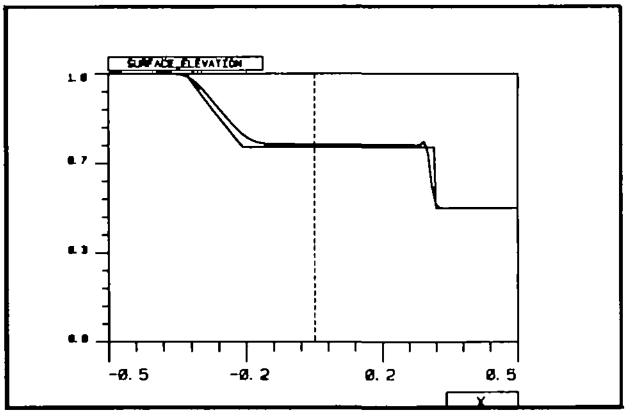

FIG. 7. Dam break problem $\Delta t = 10^{-4}$, $t = 0{\cdot}1$; surface elevation.

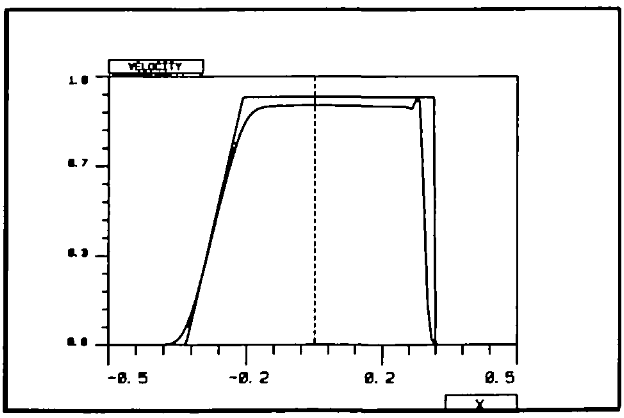

FIG. 8. Dam break problem $\Delta t = 10^{-4}$, $t = 0{\cdot}1$; velocity.

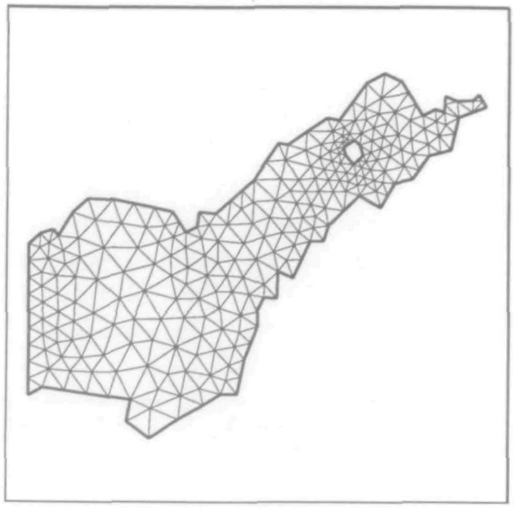

FIG. 9. Mesh.

Bermúdez & Vázquez (1989). However, the latter gives a better approximation of the state constant zone and, for $\Delta t = 10^{-4}$, oscillations behind the shock are very weak. The advantage of the method we present in this paper is unconditional stability which allows the use of large time steps specially in applications to long-wave problems like tidal current computations.

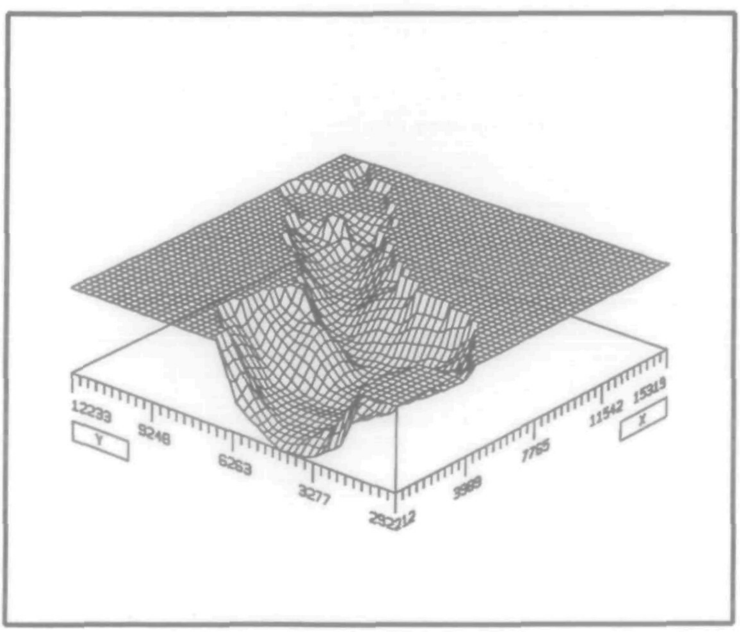

FIG. 10. Depth.

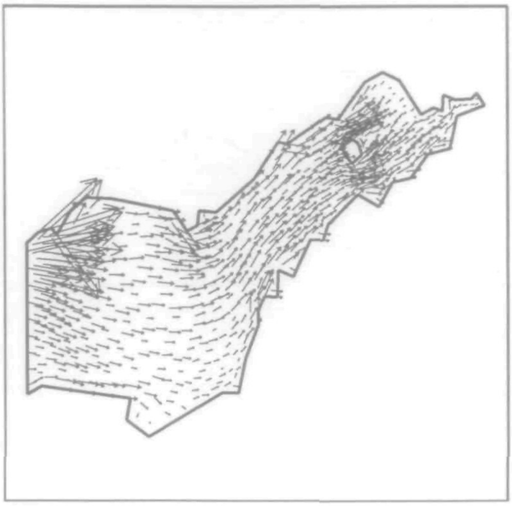

FIG. 11. Tidal current in the ría of Pontevedra $t = 10\,800$ s.

7.2 *Tidal Currents in the 'Ría' of Pontevedra*

This test concerns tidal currents in the 'ría' of Pontevedra. A tidal wave is simulated at the open sea boundary of the domain by the following function (see (2.6))

$$\varphi(t) = 1 + \sin\left(\frac{4\pi t}{86\,400} - \tfrac{1}{2}\pi\right).$$

Parameters appearing in the Coriolis term are taken to be

$$\Omega = 2\pi/86\,400, \qquad \phi = 40°.$$

FIG. 12. Tidal current in the ría of Pontevedra $t = 21\,600$ s.

FIG. 13. Tidal current in the ría of Pontevedra $t = 32\,400$ s.

The coefficient γ_{10} in the wind stress term is given by

$$\gamma_{10} = \rho_{\text{air}}(0{\cdot}75 + 0{\cdot}067\,|v|)10^{-3},$$

where ρ_{air} is the density of air ($1{\cdot}28\,\text{kg/m}^3$) and the wind velocity is $4{\cdot}16$ m/s from the NE direction. Finally, the Chézy coefficient is taken to be equal to $64\,\text{m}^{\frac{1}{2}}/\text{s}$.

In order to get the stationary cycle, we have taken zero initial conditions and integrated the evolutionary problem. Convergence is achieved after two or three tidal cycles.

FIG. 14. Tidal current in the ría of Pontevedra $t = 43\,200$ s.

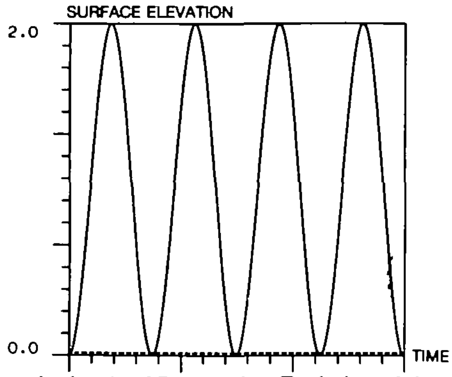

FIG. 15. Tidal current in the ría of Pontevedra. Evolution of the surface elevation.

Figure 9 shows the finite element mesh having 454 triangles, 274 vertices, and 728 nodes. The depth of the ría of Pontevedra is represented in Fig. 10. Finally, Figs. 11 to 14 show the velocity field for several times. Computations have been done using a time step $\Delta t = 360$ s which corresponds to a Courant number of about 20.

We emphasize the good behaviour of the method in terms of conservation of mass and propagation of the tidal wave. This last fact can be observed in Fig. 15 where the evolution of surface elevation at a triangle placed behind the island to a distance of about 14 km from the open sea boundary is represented.

On the other hand the boundary condition on the coast is exactly imposed along edges which guarantee no flux across it.

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Appendix

Fourier Stability Analysis

Consider the linear one-dimensional version of the scheme given in this paper. In order to do a linear stability analysis, assume positive constant velocity u and for the sake of simplicity take Δx and Δt such that

$$\alpha = \frac{u\,\Delta t}{\Delta x} \leqslant 1.$$

Then we have

$$\frac{\eta^{n+1}_{j+\frac12} - \eta^{n}_{j+\frac12}}{\Delta t} + \frac{Q^{n+1}_{j+1} - Q^{n+1}_{j}}{\Delta x} = 0,$$

$$\frac{Q^{n+1}_{j} - (\alpha Q^{n}_{j-1} + (1-\alpha)Q^{n}_{j})}{\Delta t} + g\bar{h}\,\frac{\eta^{n+1}_{j+\frac12} - \eta^{n+1}_{j-\frac12}}{\Delta x} = 0\,,$$

or in vector form

$$\begin{bmatrix} 0 & 0 \\ -g\bar{h}\,\Delta t/\Delta x & 0 \end{bmatrix}\begin{bmatrix} \eta^{n+1}_{j-\frac12} \\ Q^{n+1}_{j-1} \end{bmatrix} + \begin{bmatrix} 1 & -\Delta t/\Delta x \\ g\bar{h}\,\Delta t/\Delta x & 1 \end{bmatrix}\begin{bmatrix} \eta^{n+1}_{j+\frac12} \\ Q^{n+1}_{j} \end{bmatrix} + \begin{bmatrix} 0 & \Delta t/\Delta x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \eta^{n+1}_{j+\frac32} \\ Q^{n+1}_{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} \eta^{n}_{j-\frac12} \\ Q^{n}_{j-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha \end{bmatrix}\begin{bmatrix} \eta^{n}_{j+\frac12} \\ Q^{n}_{j} \end{bmatrix}.$$

In order to analyse stability we try a solution in separate variables of the form

$$\begin{bmatrix} \eta^n_{j+\frac{1}{2}} \\ Q^n_j \end{bmatrix} = \begin{bmatrix} v^n \\ W^n \end{bmatrix} e^{ij\omega\,\Delta x},$$

with $\omega \in (-\pi/\Delta x, \pi/\Delta x)$.

Then we obtain

$$A\begin{bmatrix} V^{n+1} \\ W^{n+1} \end{bmatrix} = B\begin{bmatrix} V^n \\ W^n \end{bmatrix},$$

where matrices A and B are given by

$$A = \begin{bmatrix} 1 & \Delta t/\Delta x(e^{i\omega\,\Delta x} - 1) \\ g\bar{h}\,\Delta t/\Delta x(1 - e^{-i\omega\,\Delta x}) & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & \alpha e^{-i\omega\,\Delta x} + 1 - \alpha \end{bmatrix}.$$

The characteristic equation of matrix $A^{-1}B$ is

$$a\lambda^2 + b\lambda + c = 0,$$

with

$$a = 1 + 4g\bar{h}(\Delta t^2/\Delta x^2)\sin^2(\tfrac{1}{2}\omega\,\Delta x),$$

$$b = -2 - \alpha(e^{-i\omega\,\Delta x} - 1), \qquad c = 1 + \alpha(e^{-i\omega\,\Delta x} - 1).$$

Schur's conditions for the roots of this equation to have modulus less than or equal to one are

$$|c| \leqslant |a|, \qquad |b| \leqslant |a + c|,$$

which become in our case

$$|1 + \alpha(e^{-i\omega\,\Delta x} - 1) \leqslant 1 + \delta,$$

$$|2 + \alpha(e^{-i\omega\,\Delta x} - 1)| \leqslant |2 + \delta + \alpha(e^{-i\omega\,\Delta x} - 1)|,$$

where δ is given by

$$\delta = 4g\bar{h}(\Delta t^2/\Delta x^2)\sin^2(\tfrac{1}{2}\omega\,\Delta x).$$

It is easy to check that these conditions hold because $0 \leqslant \alpha \leqslant 1$ and $\delta > 0$.