Université de Paris

INFLATORS AND DIFFEO-VALUED FIELDS

Paulo Andrés Soto Moreno

Advisor: Sylvy Anscombe

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1 Introduction

A first order theory is *dependent*, also known as NIP, if it is not possible to definably codify the relation $\{(i, S) \subseteq \omega \times \mathcal{P}(\omega) : i \in S\}$. More precisely, an $\mathscr{L}$-theory T is dependent if it is not possible to find some $\mathscr{L}$-formula $\varphi(x; y)$ and some sequences $(a_i : i < \omega), (b_S : S \subseteq \omega)$ of tuples from a fixed monster model of T such that

$$\models \varphi(a_i; b_S) \iff i \in S.$$

Once the combinatorial property is fixed, it is common to find some notion of rank or dimension in order to measure the complexity of types and definable sets, such as Morley rank in ω-stable theories, D-rank in simple theories or SU-rank in supersimple theories. One suitable rank notion for types in dependent theories T is the *dp-rank*. Dependent theories are exactly those in which dp-rank is always bounded above by $|T|^+$.

Also, Shelah-style model-theoretic hypotheses on fields have proved to reveal additional structure on them. For example, it is well known that any infinite ω-stable field is algebraically closed [MacIntyre, 1971], and that an infinite dp-minimal – i.e. of dp-rank 1 – field is either separably closed, real closed or admits a non-trivial definable henselian valuation [Johnson, 2015].

First attempts to classify dp-finite fields were motivated by a conjecture of Shelah: any infinite NIP field is either separably closed, real closed or admits a non-trivial definable henselian valuation. In a remarkable series of articles culminating with [Johnson, 2020b], Johnson proved this conjecture for dp-finite fields. Throughout, he made extensive use of the so called *directories* and *inflators*. The goal of this document is to present the background and the algebraic theory thereof, in preparation for future work, namely aiming to generalize his techniques to *strongly dependent* fields.

This document is organized as follows. In Chapter 2 we present the definitions of the objects we are going to be dealing with, such as k_0-linear Abelian Categories, Directories and Inflators, and we present a summary of the basic techniques for computing inflators, which we call *Inflator Calculus*, following Chapter 5 of [Johnson, 2019a]. There is a number of Category Theory facts that we will use without proof, such as Mitchel's Embedding Theorem 2.2.2. We follow Chapter 8 of [Kashiwara and Schapira, 2006] as a reference for these Category Theory arguments. In Chapter 3 we present how to define a multi-valuation ring structure on a field K whenever it admits an inflator. The key operation of this chapter is *mutation*, which induces the desired multi-valued structure on K after a suitable iteration process. For this chapter we follow Chapter 10 of [Johnson, 2019a]. Chapter 4 deals with a full classification of 1-inflators on a field K, following Theorem 5.20 of [Johnson, 2019a]. The next natural step in the development of the theory of inflators would be to classify 2-inflators, hoping that such classification could be related with multi-valued structure on a field K. We shall see in Chapter 5 that we can define 2-inflators on a field admitting a valuation and a *mock* derivation. We call these fields *Diffeo-valued Fields*, and we shall study its algebraic properties. We will follow Chapter 8 of [Johnson, 2020a] to attempt to approach a partial characterization of 2-inflators, and we will use some results on κ-resplendent models as presented in Section 9.3 of [Poizat, 2000].

2 Preliminaries

Throughout the document K will denote a field and k_0 will be an infinite subfield of K. When we endow K with a valuation ring $\mathcal{O} = \mathcal{O}_v$ for a valuation $v : K \to \Gamma \cup \{\infty\}$, the valuation is always intended to be trivial in k_0, i.e. $k_0^\times \subseteq \mathcal{O}^\times$, so that the residue field $k = k_v$ is a field extension of k_0 via $\mathrm{res} : \mathcal{O} \to k$, i.e. res is a morphism of k_0-algebras.

> *All vectors from subspaces of a finite power of K are intended to be read as column vectors.*

A matrix $[\vec{c}_1 \ \dots \ \vec{c}_n] \in K^{m\times n}$, where each column $\vec{c}_i$ represents the vector $(c_{i1}, \dots, c_{im}) \in K^m$, is supposed to be read as the vector in $K^{mn} = (K^m)^n$ given by the column vector

$$\begin{bmatrix} \vec{c}_1 \\ \vdots \\ \vec{c}_n \end{bmatrix}.$$

We will use the notation $[n] := \{1, \dots, n\}$ for any $n < \omega$.

2.1 Kronecker Product

Let $V \leq K^n$ and $W \leq K^m$ be two vector spaces over K of dimensions ν and μ respectively. Suppose $\{\vec{v}_1, \dots, \vec{v}_\nu\}$ is a basis for V and $\{\vec{w}_1, \dots, \vec{w}_\mu\}$ is a basis for W. Define

$$\Phi : V \otimes W \to K^{mn}$$

as the K-linear extension of the map sending $\vec{v}_i \otimes \vec{w}_j$ to the vector[1] $\vec{w}_j \cdot (\vec{v}_i)^t$, where $\cdot$ represents matrix multiplication and $(-)^t$ denotes transpose. If $\vec{v}_i = (v_{i1}, \dots, v_{in})$ and $\vec{w}_j = (w_{j1}, \dots, w_{jm})$, then $\Phi(\vec{v}_i \otimes \vec{w}_j) = \begin{bmatrix} v_{i1}\vec{w}_j & \dots & v_{in}\vec{w}_j \end{bmatrix} = \begin{bmatrix} w_{j1}\vec{v}_i & \dots & w_{jm}\vec{v}_i \end{bmatrix}^t$.

We define the *Kronecker product* as $\Phi(V \otimes W)$. The following lemma shows that Φ is in fact injective, so as an abuse of notation we may use the tensor product notation $V \otimes W$ to refer to the Kronecker product of V and W.

Lemma 2.1.1. *Φ is injective.*

Proof. Let $\lambda_{ij} \in K$ be such that $\sum_{ij} \lambda_{ij} \vec{w}_j \cdot (\vec{v}_i)^t = 0$. Then

$$\begin{bmatrix} \sum_{ij} \lambda_{ij} v_{i1} \vec{w}_j & \dots & \sum_{ij} \lambda_{ij} v_{in} \vec{w}_j \end{bmatrix} = 0,$$

so for any $k \in [n]$, the k'th column would satisfy the equation

$$\sum_{ij} \lambda_{ij} v_{ik} \vec{w}_j = \sum_{j=1}^{\mu} \left(\sum_{i=1}^{\nu} \lambda_{ij} v_{ik} \right) \vec{w}_j = 0 \in K^m.$$

Since $\{\vec{w}_1, \dots, \vec{w}_\mu\}$ is linearly independent, it follows that $\sum_{i=1}^{\nu} \lambda_{ij} v_{ik} = 0 \in K$ for each $k \in [n]$ and each $j \in [\mu]$. But then

$$\sum_{i=1}^{\nu} \lambda_{ij} \vec{v}_i = \begin{bmatrix} \sum_{i=1}^{\nu} \lambda_{ij} v_{i1} \\ \vdots \\ \sum_{i=1}^{\nu} \lambda_{ij} v_{in} \end{bmatrix} = 0 \in K^n$$

for each $j \in [\mu]$, so by linear independence of $\{\vec{v}_1, \dots, \vec{v}_\nu\}$, we have that $\lambda_{ij} = 0$ for each $i \in [\nu]$ and each $j \in [\mu]$. $\square$

[1] Recall that we fixed a way of reading matrices in $K^{m\times n}$ as (column) vectors in K^{mn}.

Lemma 2.1.2. *Let $\xi \in \mathrm{GL}_n(k_0)$ and $\chi \in \mathrm{GL}_m(k_0)$. Then $(\xi \cdot V) \otimes (\chi \cdot W) = (\xi \otimes \chi) \cdot (V \otimes W)$, where $\xi \otimes \chi \in \mathrm{GL}_{mn}(k_0)$ is the block matrix given by*

$$\begin{bmatrix} \xi_{11}\chi & \dots & \xi_{1n}\chi \\ & \vdots & \\ \xi_{n1}\chi & \dots & \xi_{nn}\chi \end{bmatrix}.$$

Proof. Let $\vec{v} = (v_1, \dots, v_n) \in V$ and $\vec{w} = (w_1, \dots, w_m) \in W$ be basic elements of V and W respectively. It follows that $\xi \cdot \vec{v} = (\sum_{i=1}^n \xi_{1i} v_i, \dots, \sum_{i=1}^n \xi_{ni} v_i)$, and $\vec{v} \otimes \vec{w} = (v_1 \vec{w}, \dots, v_n \vec{w})$, which implies that

$$\begin{aligned} \xi \cdot \vec{v} \otimes \chi \cdot \vec{w} &= \left(\sum_{i=1}^n \xi_{1i} v_i (\chi \cdot \vec{w}), \dots, \sum_{i=1}^n \xi_{ni} v_i (\chi \cdot \vec{w}) \right) \\ &= \left(\sum_{i=1}^n \xi_{1i} \chi (v_i \vec{w}), \dots, \sum_{i=1}^n \xi_{ni} \chi (v_i \vec{w}) \right) \\ &= (\xi \otimes \chi) \cdot (\vec{v} \otimes \vec{w}). \end{aligned}$$

The result follows by K-linearity. □

2.2 Abelian Categories and Directories

Definition 2.2.1. We call an abelian category $\mathscr{C}$ *k_0-linear* if for any pair of objects A, B of $\mathscr{C}$, the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a vector space over k_0.

Let $\mathscr{C}$ be a k_0-linear abelian category, and let A be an object thereof. Two monomorphisms $X \hookrightarrow A, Y \hookrightarrow A$ are *equivalent* if there is an isomorphism $X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & A \\ & Y \arrow[u, hook] \end{tikzcd}$$

commute. A *subobject* of A is the equivalence class of a monomorphism $X \hookrightarrow A$. There is a partial order on subobjects of A given by $[X \hookrightarrow A] \leq [Y \hookrightarrow A]$ if and only if there is a monomorphism $X \hookrightarrow Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, hook] & A \\ & Y \arrow[u, hook] \end{tikzcd}$$

commutes.

The poset $\mathrm{Sub}_{\mathscr{C}}(A)$ of subobjects of A admits a structure of bounded lattice, with bottom element $[0 \hookrightarrow A]$, top element $[\mathrm{Id} : A \to A]$, meet defined by $[X \hookrightarrow A] \wedge [Y \hookrightarrow A] = [X \times_A Y \hookrightarrow A]$ and join defined by $[X \hookrightarrow A] \vee [Y \hookrightarrow A] = [X \oplus Y \hookrightarrow A]$. Note that if $\mathscr{C}$ is the category $S\mathrm{Mod}$ of modules over a ring S, then subobjects of A correspond exactly to S-submodules of A, and the lattice operations $(\perp, \top, \vee, \wedge)$ correspond exactly to $(0, A, +, \cap)$.

For a given $n > 0$, the poset $\mathrm{Sub}_{\mathscr{C}}(A^n)$ also admits an action of $\mathrm{GL}_n(k_0)$ defined as follows. Let $\phi : X \to A^n$ be a monomorphism, and let $\pi_i : A^n \to A$ be the projection to the i^{th} coordinate. Then there are morphisms $\vec{\pi} = (\pi_1 \circ \phi, \ldots, \pi_n \circ \phi) \in \mathrm{Hom}_{\mathscr{C}}(X, A)^n = \mathrm{Hom}_{\mathscr{C}}(X, A^n)$. By k_0-linearity, $\mu \cdot \vec{\pi} \in \mathrm{Hom}_{\mathscr{C}}(X, A^n)$, so we may define $\mu \cdot [X \hookrightarrow A^n]$ as the equivalence class of the image of $\mu \cdot \vec{\pi}$.

The following fact implies that the lattice structure on the subobjects of A is in fact *modular,* meaning that for any pair of elements x, y of the lattice, the maps $- \vee y : [x \wedge y, x] \to [y, x \vee y]$ and $x \wedge - : [y, x \vee y] \to [x \wedge y, x]$ are isomorphisms of bounded lattices. We will refer to this fact as *Mithchel's Embedding Theorem.*

Fact 2.2.2 (Cf. Theorem 9.6.10 of [Kashiwara and Schapira, 2006])**.** *Let $\mathscr{C}$ be a k_0-linear abelian category. Then there is a k_0-algebra S and a fully faithful exact functor $F : \mathscr{C} \to S\,\mathrm{Mod}$. Equivalently, $\mathscr{C}$ is a full subcategory of $S\,\mathrm{Mod}$ for some k_0-algebra S.*

Modularity is now seen to be a consequence of the so called *diamond* isomorphism theorem: if N_1, N_2 are two submodules of a module M, then $(N_1+N_2)/N_1 \cong N_2/(N_1 \cap N_2)$.

Recall that the *length* of a lower bounded modular lattice M, when seen as a poset, is the maximal $n \in \mathbb{N}$ such that there is a chain of elements

$$x_0 = \perp < x_1 < \ldots < x_n$$

in M, or ∞ if there is no such maximum. An element a of M has *finite length* if the lattice $[\perp, a] = \{b \in M : \perp \leq b \leq a\}$ has finite length.

Therefore it makes sense to talk about the *length* $l(A)$ of an object A of a k_0-linear abelian category $\mathscr{C}$, defined as the length of the subobject lattice $\mathrm{Sub}_{\mathscr{C}}(A)$.

We say that A has *finite length* if $\mathrm{Sub}_{\mathscr{C}}(A)$ has finite length, and call A *semi-simple* if A is a *finite* direct sum of *simple* objects, i.e. of objects of length 1. We do not call A semi-simple if the summands defining it form an infinite family[2], implying that semi-simple objects are always of finite length.

Definition 2.2.3. Let $(M, \vee, \wedge, \perp)$ be a lower-bounded modular lattice and let $\lambda \leq \omega$ be a cardinal. We say that a family $\{a_i : i < \lambda\} \subseteq M$ is *lattice-independent* if

$$\left(\bigvee_{i<n} a_i \right) \wedge a_n = \perp$$

for any $n < \lambda$. We say that the family $\{a_i : i < \lambda\}$ is *lattice-independent over* $b \in M$ if $b < a_n$ for all $n < \lambda$ and

$$\left(\bigvee_{i<n} a_i \right) \wedge a_n = b$$

for any $n < \lambda$.

Note that a family $\{a_i : i < \lambda\}$ is lattice-independent over b if and only if it is lattice independent in $\{x \in M : b \leq x\}$.

Lemma 2.2.4. *Let M be a modular lattice and let $\{a_i : i < \omega\} \subseteq M$ be a lattice-independent family of elements of M of finite length. Then*

$$l\left(\bigvee_{i=0}^{n} a_i \right) = \sum_{i=0}^{n} l(a_i).$$

[2] As opposed to the usual definition of semi-simple, which does not require a finite amount of summands.

Proof. By induction on n. Since $a_0 \wedge a_1 = \perp$ implies $l(a_0 \wedge a_1) = 0$, modularity ensures that

$$l(a_0 \vee a_1) = l(a_0) + l(a_1) - l(a_0 \wedge a_1) = l(a_0) + l(a_1).$$

The inductive step is similar:

$$\begin{aligned} l\left(\bigvee_{i=0}^{n+1} a_i\right) &= l\left(\bigvee_{i=0}^{n} a_i \vee a_{n+1}\right) \\ &= l\left(\bigvee_{i=0}^{n} a_i\right) + l(a_{n+1}) - l\left(\left(\bigvee_{i=0}^{n} a_i\right) \wedge a_{n+1}\right) \\ &= l\left(\bigvee_{i=0}^{n} a_i\right) + l(a_{n+1}) \\ &= \sum_{i=0}^{n} l(a_i) + l(a_{n+1}) \\ &= \sum_{i=0}^{n+1} l(a_i), \end{aligned}$$

where the second equality holds by modularity, the third holds because of the hypothesis $(\bigvee_{i=0}^{n} a_i) \wedge a_{n+1} = \perp$, and the fourth holds by the inductive hypothesis. $\square$

Lemma 2.2.5. *Let A be an S-module and let V be an S-submodule of $A^2 = A \oplus A$ of length $l(V) = l(A) < \infty$. The following statements are equivalent:*

1. *V is the graph of an endomorphism of A.*
2. *$V \cap (0 \oplus A) = 0 \oplus 0$.*
3. *$V + (0 \oplus A) = A \oplus A$.*

Proof. It is clear that (1) implies both (2) and (3). Items (2) and (3) are equivalent via the modular equality

$$l(V + (0 \oplus A)) + l(V \cap (0 \oplus A)) = l(V) + l(0 \oplus A).$$

Indeed, if (2) holds, then $l(V \cap (0 \oplus A)) = 0$ and

$$\begin{aligned} l(V + (0 \oplus A)) &= l(V + (0 \oplus A)) + l(V \cap (0 \oplus A)) \\ &= l(V) + l(0 \oplus A) \\ &= l(V) + l(A) \\ &= 2l(A) \\ &= l(A \oplus A), \end{aligned}$$

so that $V + (0 \oplus A) = A \oplus A$. Also, if (3) holds, then $l(V + (0 \oplus A)) = 2l(A)$ and

$$\begin{aligned} l(V \cap (0 \oplus A)) &= l(V) + l(0 \oplus A) - l(V + (0 \oplus A)) \\ &= l(V) + l(A) - l(V + (0 \oplus A)) \\ &= 2l(A) - 2l(A) \\ &= 0 \\ &= l(0 \oplus 0), \end{aligned}$$

so that $V \cap (0 \oplus A) = 0 \oplus 0$. Now assume items (2) and (3). Then (1) holds if and only if the projection to the first coordinate $\pi : V \to A$ is an isomorphism, for in this case we may put $\psi = \pi^{-1}$. To this end, let $x \in A$. By item (3) we know that $(x, 0) = (a, b) + (0, y) = (a, b+y)$ for some $(a, b) \in V$ and some $y \in A$. Then $\pi(a, b) = a = x$ as wanted. Also, if $\pi(a, b) = 0$ for $(a, b) \in V$ then $a = 0$, so $(0, b) \in V \cap (0 \oplus A) = 0 \oplus 0$ by item (2). Therefore $(a, b) = (0, 0)$ and π is injective. $\square$

The latter lemma can be stated in terms of an abelian category $\mathscr{C}$, an object A of $\mathscr{C}$ of finite length and a subobject V of $A \oplus A$ having the same length of A. We may recover the lemma using Mitchel's Embedding Theorem 2.2.2.

Definition 2.2.6. We define the *directory of* A as the collection of modular lattices

$$\mathrm{Dir}_{\mathscr{C}}(A) = (\mathrm{Sub}_{\mathscr{C}}(A), \mathrm{Sub}_{\mathscr{C}}(A^2), \mathrm{Sub}_{\mathscr{C}}(A^3), \ldots)$$

together with the lattice structure on each $\mathrm{Sub}_{\mathscr{C}}(A^n)$, the operation

$$\oplus : \mathrm{Sub}_{\mathscr{C}}(A^n) \times \mathrm{Sub}_{\mathscr{C}}(A^m) \to \mathrm{Sub}_{\mathscr{C}}(A^{n+m})$$

given by $[X \hookrightarrow A^n] \oplus [Y \hookrightarrow A^m] := [X \oplus Y \hookrightarrow A^{n+m}]$, and the action of $\mathrm{GL}_n(k_0)$ on each $\mathrm{Sub}_{\mathscr{C}}(A^n)$.

If $\mathscr{C}$ is the category KVect of vector spaces over K, or SMod of modules over a ring S, we write $\mathrm{Dir}_K(A)$ and $\mathrm{Dir}_S(A)$ respectively. We say that $\mathrm{Dir}_{\mathscr{C}}(A)$ is *semi-simple* if A is semi-simple.

Fact 2.2.7 (Artin-Wedderburn, cf. Theorem 2.7 of [Johnson, 2019a])**.** *Let A be an object of an abelian category $\mathscr{C}$. The following statements are equivalent:*

1. $\mathrm{Dir}_{\mathscr{C}}(A)$ *is semi-simple.*
2. $\mathrm{Dir}_{\mathscr{C}}(A) \cong \mathrm{Dir}_S(M)$ *for some semi-simple ring S and some finitely generated S-module M.*
3. $\mathrm{Dir}_{\mathscr{C}}(A) \cong \mathrm{Dir}_S(S)$ *for some semi-simple ring S.*

Definition 2.2.8. Let $\mathscr{C}, \mathscr{D}$ be two k_0-linear abelian categories and let A, B be two objects of $\mathscr{C}$ and $\mathscr{D}$ respectively. A system of functions $\varsigma : \mathrm{Dir}_{\mathscr{C}}(A) \to \mathrm{Dir}_{\mathscr{D}}(B)$ given by $\varsigma_n : \mathrm{Sub}_{\mathscr{C}}(A^n) \to \mathrm{Sub}_{\mathscr{D}}(B^n)$ for each $n > 0$, is called a *directory morphism* if:

1. For any $n > 0$, the map ς_n is monotone:

$$X \leq Y \implies \varsigma_n(X) \leq \varsigma_n(Y)$$

   for any pair of subobjects $X, Y \in \mathrm{Sub}_{\mathscr{C}}(A^n)$.

2. For any pair $n, m > 0$ and any pair of subobjects $X \in \mathrm{Sub}_{\mathscr{C}}(A^n)$ and $Y \in \mathrm{Sub}_{\mathscr{C}}(A^m)$,

$$\varsigma_{n+m}(X \oplus Y) = \varsigma_n(X) \oplus \varsigma_m(Y).$$

3. For any $n > 0$, any $\mu \in \mathrm{GL}_n(k_0)$ and any $X \in \mathrm{Sub}_{\mathscr{C}}(A^n)$,

$$\varsigma_n(\mu \cdot X) = \mu \cdot \varsigma_n(X).$$

We say that two directory morphisms $\varsigma : \mathrm{Dir}_{\mathscr{C}}(A) \to \mathrm{Dir}_{\mathscr{D}}(B)$, $\varsigma' : \mathrm{Dir}_{\mathscr{C}}(A) \to \mathrm{Dir}_{\mathscr{D}'}(B')$ are *equivalent* if there is a directory isomorphism $\mathrm{Dir}_{\mathscr{D}}(B) \to \mathrm{Dir}_{\mathscr{D}'}(B')$ making the diagram

$$\begin{tikzcd} \mathrm{Dir}_{\mathscr{C}}(A) \arrow[r, "\varsigma"] \arrow[dr, "\varsigma'"'] & \mathrm{Dir}_{\mathscr{D}}(B) \arrow[d] \\ & \mathrm{Dir}_{\mathscr{D}'}(B') \end{tikzcd}$$

commute.

Fact 2.2.9 (Cf. Examples 2.9, 2.20 and 2.11 of [Johnson, 2019a])**.** *Let $\mathscr{C}, \mathscr{D}$ be two k_0-linear abelian categories and let A, B be a pair of objects from $\mathscr{C}$.*

1. *Any morphism $\phi : A \to B$ induces the pullback and pushforward morphisms of directories*

$$\phi^* : \mathrm{Dir}_{\mathscr{C}}(B) \to \mathrm{Dir}_{\mathscr{C}}(A),$$
$$\phi_* : \mathrm{Dir}_{\mathscr{C}}(A) \to \mathrm{Dir}_{\mathscr{C}}(B)$$

defined by inverse and direct image along the coordinatewise map $\varphi^{\oplus n} : A^n \to B^n$.

2. *Any left-exact k_0-linear functor $F : \mathscr{C} \to \mathscr{D}$ induces a directory morphism*

$$F_* : \mathrm{Dir}_{\mathscr{C}}(A) \to \mathrm{Dir}_{\mathscr{D}}(F(A)),$$

which at each level is defined by $F_*([X \overset{\varphi}{\hookrightarrow} A^n]) = [F(x) \xhookrightarrow{F(\varphi)} F(A)^n]$, which is well defined because $F(A^n) \cong F(A)^n$.

3. *Any k_0-linear equivalence of categories $F : \mathscr{C} \to \mathscr{D}$ induces a directory isomorphism*

$$F_* : \mathrm{Dir}_{\mathscr{C}}(A) \to \mathrm{Dir}_{\mathscr{D}}(F(A)),$$

when defined as in the latter item.

Definition 2.2.10. Let A be a semi-simple object of an abelian category $\mathscr{C}$ and let $d > 0$. A directory morphism $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_{\mathscr{C}}(A)$ is called a *d-inflator* if

$$l(\varsigma_n(V)) = d \cdot \dim_K(V)$$

for any $V \leq K^n$, $n > 0$. We say that ς is an *inflator* if it is a d-inflator for some $d > 0$.

As an example, fix a field extension $k_0 \subseteq K_1 \subseteq K_2$. Then there are two left-exact k_0-linear functors

$$F : K_1\mathrm{Vect} \to K_2\mathrm{Vect},\ F(V) = K_2 \otimes_{K_1} V,$$
$$G : K_2\mathrm{Vect} \to K_1\mathrm{Vect},\ G(V) = V.$$

The functor F induces a directory morphism $\varsigma : \mathrm{Dir}_{K_1}(K_1) \to \mathrm{Dir}_{K_2}(K_2)$ which is in fact a 1-inflator. This follows from the fact that if $\{\vec{v_1}, \ldots, \vec{v_l}\}$ is a basis for V, then $\{1 \otimes \vec{v_1}, \ldots, 1 \otimes \vec{v_l}\}$ is a basis for $K_2 \otimes_{K_1} V$ as a K_2-vector space. This means that $\dim_{K_2}(K_2 \otimes_{K_1} V) = \dim_{K_1}(V) = l$. The same proof works in the case where K_2 is a division k_0-algebra extending a field extension K_1 of k_0.

Also, the forgetful functor G induces a directory morphism $\varsigma : \mathrm{Dir}_{K_2}(K_2) \to \mathrm{Dir}_{K_1}(K_2)$ given by $\varsigma_n(V) = V$ for any $V \leq K_2^n$. If the degree $[K_2 : K_1]$ is finite, then ς is in fact a $[K_2 : K_1]$-inflator. Indeed, $\dim_{K_1}(V) = \dim_{K_2}(V) \cdot \dim_{K_1}(K_2) = [K_2 : K_1] \cdot \dim_{K_2}(V)$.

There is another way to construct 1-inflators, which is by means of valuation rings on a field.

Proposition 2.2.11. *Let K be a valued field with valuation ring $\mathcal{O}$, maximal ideal $\mathscr{M}$, residue field k and residue map $\operatorname{res} : \mathcal{O} \to k$. For a given $n > 0$, let $r : \mathcal{O}^n \to k^n$ represent coordinatewise* res.

1. *Let $V \in \operatorname{Sub}_K(K^n)$. Then there is a matrix $\mu \in \operatorname{GL}_n(\mathcal{O})$ such that $\mu \cdot V = K^l \oplus 0^{n-l}$, where $l = \dim_k(r(V \cap \mathcal{O}^n))$.*

2. *There is a 1-inflator $\varsigma : \operatorname{Dir}_K(K) \to \operatorname{Dir}_k(k)$ given by $\varsigma_n(V) = r(V \cap \mathcal{O}^n)$.*

Proof. 1. Let $\{\vec{w_1}, \dots, \vec{w_l}\}$ be a basis for $r(V \cap \mathcal{O}^n) \leq k^n$, and let $\vec{a_i} \in V \cap \mathcal{O}^n$ be such that $r(\vec{a_i}) = \vec{w_i}$ for each $i \in [l]$. Complete $\{\vec{w_1}, \dots, \vec{w_l}\}$ with $\vec{w_{l+1}}, \dots, \vec{w_n} \in k^n$ so that $\{\vec{w_1}, \dots, \vec{w_n}\}$ is a basis for k^n, and define $\vec{a_i} \in \mathcal{O}^n$ so that $r(\vec{a_i}) = \vec{w_i}$ for each $i \in \{l+1, \dots, n\}$. Consider the matrix $\nu = \begin{bmatrix} \vec{a_1} & \dots & \vec{a_n} \end{bmatrix} \in \mathcal{O}^{n \times n}$. Then ν is invertible in $\mathcal{O}$, because otherwise $\det(\nu) \in \mathscr{M} = \mathcal{O} \setminus \mathcal{O}^\times$ and thus

$$\det\left(\begin{bmatrix} \vec{w_1} & \dots & \vec{w_n} \end{bmatrix}\right) = \det(\operatorname{res}(\nu)) = \operatorname{res}(\det(\nu)) = 0,$$

which is absurd.

Now we prove that $V = \nu \cdot (K^l \oplus 0^{n-l})$, which yields the desired result by putting $\mu = \nu^{-1}$. The right hand side is the set of all K-linear combinations of $\vec{a_1}, \dots, \vec{a_l}$, which is contained in V. Now, if $\vec{v} \in V$ and $\vec{v} \notin \nu \cdot (K^l \oplus 0^{n-l})$, then there is some coordinate $m \in \{l+1, \dots, n\}$ such that $\vec{v} = \nu \cdot \vec{x}$ for some $\vec{x} \in K^n$ with $x_m \neq 0$. Thus $\sum_{i=1}^n x_i \vec{a_i} = \vec{v} \in V$. By substracting $x_1\vec{a_1} + \dots + x_l\vec{a_l} \in V$, we may assume that $\vec{x} = (0^l, x_{l+1}, \dots, x_n)$. Let $j \in \{l+1, \dots, n\}$ be such that x_j is of minimal valuation. Therefore $\sum_{i=l+1}^n \frac{x_i}{x_j}\vec{a_i} = \frac{1}{x_j}\vec{v} \in V \cap \mathcal{O}^n$ by choice of x_j. Hence

$$\begin{aligned} r\left(\sum_{i=l+1}^n \frac{x_i}{x_j}\vec{a_i}\right) &= \sum_{i=l+1}^n \operatorname{res}\left(\frac{x_i}{x_j}\right)\vec{w_i} \\ &= \sum_{i=1}^l y_i \vec{w_i} \end{aligned}$$

for some $y_1, \dots, y_l \in k$. But then, by linear k-independence, $y_1 = \dots = y_l = \operatorname{res}\left(\frac{x_{l+1}}{x_j}\right) = \dots = \operatorname{res}\left(\frac{x_n}{x_j}\right) = 0$, which is absurd because $\operatorname{res}\left(\frac{x_j}{x_j}\right) = \operatorname{res}(1) \neq 0$.

2. The map ς is a directory morphism because it can be expressed as the composition of the directory morphisms

$$\operatorname{Dir}_K(K) \to \operatorname{Dir}_{\mathcal{O}}(K) \to \operatorname{Dir}_{\mathcal{O}}(\mathcal{O}) \to \operatorname{Dir}_k(k)$$

where the first arrow is induced by the forgetful functor $K\mathrm{Vect} \to \mathcal{O}\mathrm{Mod}$, the second arrow is the pullback along the monomorphism $\mathcal{O} \hookrightarrow K$ and the third one is the pushforward along the epimorphism $\operatorname{res} : \mathcal{O} \to k$.

Now we check the length condition. Let $V \leq K^n$, and use the latter item to find a matrix $\mu \in \operatorname{GL}_n(\mathcal{O})$ such that $\mu \cdot V = K^l \oplus 0^{n-l}$ with $l = \dim_k(r(V \cap \mathcal{O}^n))$. Then $\mu \in \operatorname{GL}_n(K)$, so $\dim_K(V) = \dim_K(\mu \cdot V) = l$, as wanted.

□

We shall see in Chapter 4 that the only way of obtaining 1-inflators on a field K is by composing the 1-inflator arising from some valuation ring on K, and by the 1-inflator obtained by (skew) field extensions of the residue field of the corresponding valuation ring.

2.3 Inflator Calculus

There is a number of results that we will repeatedly use throughout the document when computing inflators. This section is devoted to prove all these results.

Fix a d-inflator $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_S(M)$, so that M is a semi-simple S-module of length d and S is a k_0-algebra.

Lemma 2.3.1. *For any $n > 0$, $\varsigma_n(0^n) = 0^n$ and $\varsigma_n(K^n) = M^n$.*

Proof. By the length scaling condition, $l(\varsigma_n(0^n)) = d{\cdot}\dim_K(0^n) = 0 = l(0^n)$ and $l(\varsigma_n(K^n)) = d \cdot \dim_K(K^n) = d \cdot n = l(M^n)$. But the only submodule of length 0 in M^n is 0^n and the only submodule of length $d \cdot n$ in M^n is M^n itself, so $\varsigma_n(0^n) = 0^n$ and $\varsigma_n(K^n) = M^n$ as desired. □

Lemma 2.3.2. *Fix some $n > 0$ and let $V, W \leq K^n$. Then*

$$l(\varsigma_n(V) \cap \varsigma_n(W)) = d \cdot \dim(V \cap W) \implies \varsigma_n(V) \cap \varsigma_n(W) = \varsigma_n(V \cap W)$$

and

$$l(\varsigma_n(V) + \varsigma_n(W)) = d \cdot \dim(V + W) \implies \varsigma_n(V) + \varsigma_n(W) = \varsigma_n(V + W).$$

Proof. For the first statement, since $V \cap W \subseteq V$ and $V \cap W \subseteq W$, by monotonicity of ς_n we get that $\varsigma_n(V \cap W) \subseteq \varsigma_n(V)$ and $\varsigma_n(V \cap W) \subseteq \varsigma_n(W)$, so that

$$\varsigma_n(V \cap W) \subseteq \varsigma_n(V) \cap \varsigma_n(W).$$

The hypothesis implies that $l(\varsigma_n(V) \cap \varsigma_n(W)) = d \cdot \dim_K(V \cap W) = l(\varsigma_n(V \cap W))$, which means that $\varsigma_n(V \cap W)$ is a submodule of $\varsigma_n(V) \cap \varsigma_n(W)$ of full length. Hence

$$\varsigma_n(V) \cap \varsigma_n(W) = \varsigma_n(V \cap W).$$

The second statement can be proved analogously. Indeed, since $V \subseteq V + W$ and $W \subseteq V + W$, by monotonicity of ς_n we get that $\varsigma_n(V) \subseteq \varsigma_n(V + W)$ and $\varsigma_n(W) \subseteq \varsigma_n(V + W)$, so that

$$\varsigma_n(V) + \varsigma_n(W) \subseteq \varsigma_n(V + W).$$

The hypothesis implies that $l(\varsigma_n(V) + \varsigma_n(W)) = d \cdot \dim_K(V + W) = l(\varsigma_n(V + W))$, which means that $\varsigma_n(V) + \varsigma_n(W)$ is a submodule of $\varsigma_n(V + W)$ of full length. Hence

$$\varsigma_n(V) + \varsigma_n(W) = \varsigma_n(V + W).$$

□

Throughout the document, if $\alpha \in K$, then Θ_α is defined to be the line of slope α in K^2, i.e.

$$\Theta_\alpha := \{(x, \alpha x) : x \in K\}.$$

Analogously, if $\varphi \in \mathrm{End}_S(M)$ for some S-module M, then Θ_φ will denote the graph of the endomorphism φ, i.e.

$$\Theta_\varphi := \{(x, \varphi(x)) : x \in M\}.$$

Definition 2.3.3. Let $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_S(M)$ be an inflator, and let $\varphi \in \mathrm{End}_S(M)$. We say that $a \in K$ *specializes* to φ (with respect to ς) if $\varsigma_2(\Theta_a) = \Theta_\varphi$.

Proposition 2.3.4. *Let $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_S(M)$ be a d-inflator. Then the set*

$$R = \{a \in K : \varsigma_2(\Theta_a) = \Theta_\varphi \textit{ for some } \varphi \in \mathrm{End}_S(M)\}$$

is a k_0-subalgebra of K. Define $\widehat{\mathrm{res}} : R \to \mathrm{End}_S(M)$ by

$$\widehat{\mathrm{res}}(a) = \varphi \textit{ if and only if } \varsigma_2(\Theta_a) = \Theta_\varphi.$$

Then $\widehat{\mathrm{res}}$ is a k_0-algebra morphism whose kernel

$$I = \{a \in K : \varsigma_2(\Theta_a) = M \oplus 0\}$$

satisfies $1 + I \subseteq R^\times$. Therefore I is contained in the Jacobson radical of R.

Proof. We shall prove the following statements, for any $\alpha, \beta \in K$, $\varphi, \psi \in \mathrm{End}_S(M)$ and $q \in k_0$.

1. If α specializes to φ and β specializes to ψ, then $\alpha + \beta$ specializes to $\varphi + \psi$. Indeed, note that
$$(\Theta_\alpha \oplus K) \cap \mu \cdot (\Theta_\beta \oplus K) = \{(x, \alpha x, \beta x) : x \in K\},$$
where $\mu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{GL}_3(k_0)$, is a 1-dimensional K-vector space. Its counterpart
$$(\Theta_\varphi \oplus M) \cap \mu \cdot (\Theta_\psi \oplus M) = \{(x, \varphi(x), \psi(x)) : x \in M\}$$
is a submodule of M^3 of length d. Therefore, by Lemma 2.3.2,
$$\begin{aligned} \varsigma_3(\{(x, \alpha x, \beta x) : x \in K\}) &= \varsigma_3\big((\Theta_\alpha \oplus K) \cap \mu \cdot (\Theta_\beta \oplus K)\big) \\ &= \varsigma_3(\Theta_\alpha \oplus K) \cap \mu \cdot \varsigma_3(\Theta_\beta \oplus K) \\ &= (\Theta_\varphi \oplus M) \cap \mu \cdot (\Theta_\psi \oplus M) \\ &= \{(x, \varphi(x), \psi(x)) : x \in M\}. \end{aligned}$$
Therefore
$$\begin{aligned} \varsigma_4(\{(x, \alpha x, \beta x, 0) : x \in K\}) &= \varsigma_3\big(((\Theta_\alpha \oplus K) \cap \mu \cdot (\Theta_\beta \oplus K)) \oplus 0\big) \\ &= ((\Theta_\varphi \oplus M) \cap \mu \cdot (\Theta_\psi \oplus M)) \oplus 0 \\ &= \{(x, \varphi(x), \psi(x), 0) : x \in M\}. \end{aligned}$$
Upon acting with the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \in \mathrm{GL}_4(k_0)$, we get that
$$\varsigma_4(\{(x, \alpha x, \beta x, \alpha x + \beta x) : x \in K\}) = \{(x, \varphi(x), \psi(x), (\varphi + \psi)(x)) : x \in M\}.$$
Now note that
$$\{(x, y, z, \alpha x + \beta x) : x, y, z \in K\} = \{(x, \alpha x, \beta x, \alpha x + \beta x) : x \in K\} + (0 \oplus K \oplus K \oplus 0)$$
is a K-vector space of dimension 3, and that its counterpart
$$\{(x, y, z, (\varphi+\psi)x) : x, y, z \in M\} = \{(x, \varphi(x), \psi(x), (\varphi+\psi)(x)) : x \in M\} + (0 \oplus M \oplus M \oplus 0)$$

is a submodule of M^4 of length $3d$. Again, by Lemma 2.3.2,

$$\begin{aligned}
\varsigma_4\left(\{(x,y,z,\alpha x+\beta x):x,y,z\in K\}\right) &= \varsigma_4\left(\{(x,\alpha x,\beta x,\alpha x+\beta x):x\in K\}+(0\oplus K\oplus K\oplus 0)\right)\\
&= \varsigma_4\left(\{(x,\alpha x,\beta x,\alpha x+\beta x):x\in K\}\right)+\varsigma_4\left(0\oplus K\oplus K\oplus 0\right)\\
&= \{(x,\varphi(x),\psi(x),(\varphi+\psi)(x)):x\in M\}+(0\oplus M\oplus M\oplus 0)\\
&= \{(x,y,z,(\varphi+\psi)x):x,y,z\in M\}.
\end{aligned}$$

Upon acting with the matrix $\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\in \mathrm{GL}_4(k_0)$, we get that

$$\varsigma_4\left(\{(x,\alpha x+\beta x,z,y):x,y,z\in K\}\right)=\{(x,(\varphi+\psi)x,z,y):x,y,z\in M\}.$$

The left hand side is equal to $\varsigma_4(\Theta_{\alpha+\beta}\oplus K\oplus K)=\varsigma_2(\Theta_{\alpha+\beta})\oplus M\oplus M$ and the right hand side is equal to $\Theta_{\varphi+\psi}\oplus M\oplus M$. Upon projecting to the first two coordinates, we get that $\varsigma_2(\Theta_{\alpha+\beta})=\Theta_{\varphi+\psi}$ as wanted.

2. If α specializes to φ and β specializes to ψ, then $\alpha\beta$ specializes to $\varphi\circ\psi$. Indeed, consider the space $(K\oplus\Theta_\alpha)\cap(\Theta_\beta\oplus K)=\{(x,\beta x,\alpha\beta x):x\in K\}$, which has dimension 1. Its counterpart $(M\oplus\Theta_\varphi)\cap(\Theta_\psi\oplus M)=\{(x,\psi(x),\varphi(\psi(x))):x\in M\}$ has length equal to $l(M)=d$, so by Lemma 2.3.2,

$$\begin{aligned}
\varsigma_3(\{(x,\beta x,\alpha\beta x):x\in K\}) &= \varsigma_3((K\oplus\Theta_\alpha)\cap(\Theta_\beta\oplus K))\\
&= \varsigma_3(K\oplus\Theta_\alpha)\cap\varsigma_3(\Theta_\beta\oplus K)\\
&= (M\oplus\Theta_\varphi)\cap(\Theta_\psi\oplus M)\\
&= \{(x,\psi(x),\varphi(\psi(x))):x\in M\}.
\end{aligned}$$

Now note that $\big((K\oplus\Theta_\alpha)\cap(\Theta_\beta\oplus K)\big)+0\oplus K\oplus 0=\{(x,y,\alpha\beta x):x,y\in K\}$, which has dimension 2. Its counterpart $\big((M\oplus\Theta_\varphi)\cap(\Theta_\psi\oplus M)\big)+0\oplus M\oplus 0=\{(x,y,\varphi(\psi(x))):x,y\in M\}$ has length $2d$, so again by Lemma 2.3.2,

$$\begin{aligned}
\varsigma_3(\{(x,y,\alpha\beta x):x,y\in K\}) &= \varsigma_3\left(\big((K\oplus\Theta_\alpha)\cap(\Theta_\beta\oplus K)\big)+0\oplus K\oplus 0\right)\\
&= \varsigma_3\big((K\oplus\Theta_\alpha)\cap(\Theta_\beta\oplus K)\big)+\varsigma_3(0\oplus K\oplus 0)\\
&= \big((M\oplus\Theta_\varphi)\cap(\Theta_\psi\oplus M)\big)+0\oplus M\oplus 0,\\
&= \{(x,y,\varphi(\psi(x))):x,y\in M\}.
\end{aligned}$$

Upon acting with the matrix $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\in\mathrm{GL}_3(k_0)$, we get that

$$\varsigma_3(\{(x,\alpha\beta x,y):x,y\in K\})=\{(x,\varphi(\psi(x)),y):x,y\in M\}.$$

The left hand side is equal to $\varsigma_3(\Theta_{\alpha\beta}\oplus K)=\varsigma_2(\Theta_{\alpha\beta})\oplus M$ and the right hand side is equal to $\Theta_{\varphi\circ\psi}\oplus M$. Upon projecting to the first two coordinates, we get that $\varsigma_2(\Theta_{\alpha\beta})=\Theta_{\varphi\circ\psi}$ as wanted.

3. q specializes to $q\cdot\mathrm{Id_M}$. Indeed, upon acting with $\mu=\begin{bmatrix}1&0\\q&1\end{bmatrix}\in\mathrm{GL}_2(k_0)$ on $K\oplus 0$, we get that

$$\begin{aligned}
\varsigma_2(K\oplus 0)&=M\oplus 0,\\
\varsigma_2(\mu\cdot(K\oplus 0))&=\mu\cdot(M\oplus 0),
\end{aligned}$$

which is what we needed because $\Theta_q=\mu\cdot(K\oplus 0)$ and $\Theta_{q\cdot\mathrm{Id_M}}=\mu\cdot(M\oplus 0)$.

4. If α specializes to an automorphism $\varphi \in \mathrm{Aut}_S(M)$, then α^{-1} specializes to φ^{-1}. Indeed, if $\mu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{GL}_2(k_0)$, then

$$\begin{aligned} \Theta_{\alpha^{-1}} &= \{(x, \alpha^{-1}x) : x \in K\} \\ &= \{(\alpha x, x) : x \in K\} \\ &= \mu \cdot \Theta_\alpha \end{aligned}$$

and

$$\begin{aligned} \varsigma_2(\Theta_{\alpha^{-1}}) &= \varsigma_2(\mu \cdot \Theta_\alpha) \\ &= \mu \cdot \Theta_\varphi \\ &= \{(\varphi(x), x) : x \in M\} \\ &= \{(x, \varphi^{-1}(x)) : x \in M\} \\ &= \Theta_{\varphi^{-1}} \end{aligned}$$

as desired.

As a general fact, $1 + I \subseteq R^\times$ implies $I \subseteq \bigcap_{\mathscr{M}} \mathscr{M}$. Indeed, if $x \in I$ is such that $x \notin \mathscr{M}$ for some maximal ideal $\mathscr{M}$ of R, then maximality implies that $Rx + \mathscr{M} = R$, so there are some $y \in R$ and $m \in \mathscr{M}$ such that $yx + m = 1$, so $m = 1 - yx \in (1 + I) \cap \mathscr{M} \subseteq R^\times \cap \mathscr{M}$, which is absurd as maximal ideals are proper. Now, if $a \in I$, then $\widehat{\mathrm{res}}(a) = 0$. Since $\widehat{\mathrm{res}}(1) = \mathrm{Id}_M$, then $\widehat{\mathrm{res}}(1 + a) = \mathrm{Id}_M + 0 = \mathrm{Id}_M$ is invertible, so $1 + a \in R^\times$. $\square$

Definition 2.3.5. We call R the *fundamental ring* and I the *fundamental ideal* of ς. The map $\widehat{\mathrm{res}} : R \to \mathrm{End}_S(M)$ is called the *generalized residue map* associated to ς.

The fundamental ring and the fundamental ideal of an inflator ς determine intrinsic structure on K. For example, we shall see in Proposition 4.0.2 that R is a valuation ring of K whose maximal ideal is I, whenever ς is a 1-inflator.

3 From Inflators to Valuation Rings

3.1 Bézout Domains and Multi-valued Fields

Lemma 3.1.1 (Cf. Remark 10.27 of [Johnson, 2019b]). *Let K be a field extending a subfield k_0, and R a k_0-subalgebra of K. Then R is a multi-valuation ring on K, trivial on k_0, whenever the following conditions hold:*

1. $\mathrm{Frac}(R) = K$.
2. *R is a Bézout domain.*
3. *R has finitely many maximal ideals.*

Proof. We will first prove that if $\mathscr{M}$ is a maximal ideal of R, then the localization $R_{\mathscr{M}}$ is a valuation ring of K. Indeed, let $x \in K^\times$ be such that $x \notin R_{\mathscr{M}}$. The hypothesis $\mathrm{Frac}(R) = K$ yields the existence of some nonzero $a, b \in R$ such that $x = \frac{a}{b}$. We must argue that $x^{-1} = \frac{b}{a} \in R_{\mathscr{M}}$. Suppose otherwise, and let $c \in R$ be such that $Ra + Rb = Rc$. Therefore

$$\begin{cases} a = \alpha c, \\ b = \beta c, \\ \gamma a + \delta b = c \end{cases}$$

for some $\alpha, \beta, \gamma, \delta \in R$. Then $\gamma\alpha + \delta\beta = 1$, so $R\alpha + R\beta = R$. But $R\alpha + R\beta \subseteq \mathscr{M}$ because $\frac{a}{b} = \frac{\alpha}{\beta}, \frac{b}{a} = \frac{\beta}{\alpha} \notin R_{\mathscr{M}}$ imply that α and β are elements of $\mathscr{M}$, which contradicts maximality of $\mathscr{M}$.

These valuations are trivial on k_0. Indeed, if $\mathscr{M}$ is any maximal ideal of R and if $q \in k_0^\times \cap \mathscr{M}_{\mathscr{M}}$, then there would be some $a, b \in R$ such that $q = \dfrac{a}{b}$, $a \in \mathscr{M}$ and $b \notin \mathscr{M}$. Then $qb = a \in \mathscr{M}$, so as R is a k_0-algebra, $b = q^{-1}qb \in \mathscr{M}$, which is absurd.

If there are finitely many distinct maximal ideals $\mathscr{M}_1, \ldots, \mathscr{M}_d$ of R, then we claim that $R = R_{\mathscr{M}_1} \cap \ldots \cap R_{\mathscr{M}_d}$. Indeed, if $a, b \in R$ are not zero and $\frac{a}{b} \in R_{\mathscr{M}_1} \cap \ldots \cap R_{\mathscr{M}_d}$, then $b \notin \mathscr{M}_i$ for any $i \in [d]$, so $Rb = R$, i.e., b is a unit. Therefore $\frac{a}{b} = b^{-1}a \in R$ as wanted. □

In fact, if R has d maximal ideals, then the proof shows that R is the intersection of at most d valuation rings.

Lemma 3.1.2. *Let K be a field, k_0 a subfield, and let R be a k_0-subalgebra. The following statements are equivalent:*

1. *R is an intersection of n or fewer valuation rings $\mathcal{O}$ on K satisfying $k_0^\times \subseteq \mathcal{O}^\times$.*
2. *For any $\alpha \in K$ and any n distinct elements $q_1, \ldots, q_n$ of k_0, at least one of*

$$\alpha, \frac{1}{\alpha - q_1}, \ldots, \frac{1}{\alpha - q_n}$$

   is in R.

Proof. Assume first that $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_d$ with $d \leq n$, and each $\mathcal{O}_i$ is a valuation ring with maximal ideal $\mathscr{M}_i$. If none of the listed elements are in R, then

$$\begin{cases} \alpha^{-1} \in \mathscr{M}_{i_0} \text{ for some } i_0 \in [d], \\ \alpha - q_1 \in \mathscr{M}_{i_1} \text{ for some } i_1 \in [d], \\ \vdots \\ \alpha - q_n \in \mathscr{M}_{i_n} \text{ for some } i_n \in [d]. \end{cases}$$

Since $n+1>d$, there should be some pair of different indices $l, m \in \{0, \dots, n\}$ such that $i_l = i_m =: j$. Therefore, if $\mathscr{M} = \mathscr{M}_j$, we would either have that $\alpha - q_l, \alpha - q_m \in \mathscr{M}$ or $\alpha^{-1}, \alpha - q_m \in \mathscr{M}$ if, say, $l = 0$.

In the first case we may substract and get $q_l - q_m \in \mathscr{M}$, which is absurd since $k_0 \cap \mathscr{M} = \{0\}$. In the second case we see immediately that $q_m \neq 0$. We may multiply both terms and get that $1 - q_m\alpha^{-1} \in \mathscr{M}$. Since $\mathscr{M}$ is an ideal of $\mathcal{O}$, we may multiply by $q_m^{-1} \in \mathcal{O}$ and get $q_m^{-1} - a^{-1} \in \mathscr{M}$. Adding $a^{-1} \in \mathscr{M}$ we get $q_m^{-1} \in \mathscr{M}$, which is again absurd.

Now fix some distinct $q_1, \dots, q_n \in k_0$ and suppose that one of

$$\alpha, \frac{1}{\alpha - q_1}, \dots, \frac{1}{\alpha - q_n}$$

is in R. We may verify the conditions of Lemma 3.1.1 to conclude that R is the intersection of at most n valuation rings, trivial on k_0.

1. $\mathrm{Frac}(R) = K$. Indeed, if $\alpha \in K$ then either $\alpha \in R$ or $\dfrac{1}{\alpha - q_i} \in R$ for some $i \in [n]$. Therefore, since R extends k_0,

$$\alpha = \frac{1}{1/(\alpha - q_i)} + q_i \in \mathrm{Frac}(R).$$

2. R is a Bézout domain. Indeed, let $a, b \in R$. Then one of

$$\frac{a}{b}, \frac{b}{a - q_1 b}, \dots, \frac{b}{a - q_n b}$$

is in R.

If $\dfrac{a}{b} \in R$, then $a = \dfrac{a}{b}b \in Rb$ implies that $Ra \subseteq Rb$ and $Ra + Rb = Rb$. If $\dfrac{b}{a - qb} \in R$ for some $q \in k_0$, then $b = \dfrac{b}{a - qb}(a - qb) \in R(a - qb)$ and $Rb \subseteq R(a - qb)$. Also $a = a - qb + qb \in R(a - qb) + Rb \subseteq R(a - qb)$, so $Ra + Rb \subseteq R(a - qb)$. Clearly $R(a - qb) \subseteq Ra + Rb$, so $Ra + Rb$ is singly generated by $a - qb$.

3. R has at most n maximal ideals. Indeed, if $\mathscr{M}_0, \dots, \mathscr{M}_n$ were $n+1$ different maximal ideals of R, we could use the Chinese Reminder Theorem to find some $a, b \in R$ such that

$$\begin{cases} a \equiv 1 & \mod \mathscr{M}_0, \\ b \equiv 0 & \mod \mathscr{M}_0, \\ a \equiv q_i & \mod \mathscr{M}_i, \\ b \equiv 1 & \mod \mathscr{M}_i \end{cases}$$

for any $i \in [n]$. Then $\dfrac{a}{b} \notin R$, otherwise $a = \dfrac{a}{b}b \equiv 0 \mod \mathscr{M}_0$. Also $\dfrac{b}{a - q_i b} \notin R$ for any $i \in [n]$. Otherwise, modulo $\mathscr{M}_i$, we have that $a \equiv q_i, b \equiv 1$ imply $a \equiv q_i b$, i.e., $a - q_i b \equiv 0$. Thus $b = \dfrac{b}{a - q_i b}(a - q_i b) \equiv 0$, which is absurd.

□

3.2 Tame Locus

Fix a d-inflator $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_S(M)$ with fundamental ring R and fundamental ideal I. Recall that M is a semi-simple S-module of length d, and S is a k_0-algebra.

Lemma 3.2.1. *Let $\alpha \in K$, and define $S_\alpha = \{\alpha\} \cup \left\{ \frac{1}{\alpha - q} : q \in k_0 \right\}$. If $S_\alpha \cap R \neq \emptyset$, then all but at most d elements of S_α are elements of R.*

Proof. First we treat the case in which $\alpha \in R$. Let $\varphi \in \mathrm{End}_S(M)$ be the specialization of α, so that $\widehat{\mathrm{res}}(\alpha) = \varphi$. If $q \in k_0$, we know that $\widehat{\mathrm{res}}(q) = q \cdot \mathrm{Id}$. Therefore $\widehat{\mathrm{res}}(\alpha - q) = \varphi - q \cdot \mathrm{Id}$, i.e.,

$$\varsigma_2(\{(x, (\alpha - q)x) : x \in K\}) = \{(x, (\varphi - q)x) : x \in K\}.$$

Acting with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{GL}_2(k_0)$, we get that

$$\begin{aligned} \varsigma_2 \left(\left\{ \left(x, \frac{1}{\alpha - q} x \right) : x \in K \right\} \right) &= \varsigma_2(\{((\alpha - q)x, x) : x \in M\}) \\ &= \{((\varphi - q)x, x) : x \in M\} \\ &=: V_q \end{aligned}$$

Therefore $\frac{1}{\alpha - q} \in R$ if and only if $V_q = \Theta_\psi$ for some $\psi \in \mathrm{End}_S(M)$, which by Lemma 2.2.5 is equivalent to $V_q \cap (0 \oplus M) = 0 \oplus 0$. This means that $\frac{1}{\alpha - q} \notin R$ if and only if there is some nonzero $b \in M$ such that $(0, b) \in V_q$, i.e. such that $\varphi(b) = qb$, meaning that b is some nonzero eigenvector of φ of eigenvalue q. If E_q is the eigenspace of φ of eigenvalue q, this is equivalent to saying that E_q is not trivial.

If the family of submodules $\{E_q : q \in k_0\}$ is lattice-theoretically independent, then there are at most $d = l(M)$ eigenvalues $q \in K$ such that $E_q \neq 0$, for otherwise if $E_{q_i} \neq 0$ for $i \in [d+1]$, then

$$\sum_{i=1}^{d+1} E_{q_i}$$

is a submodule of M of length at least $d + 1$, which is absurd. Hence there are at most d elements $q \in k_0$ such that $\frac{1}{\alpha - q}$ are not in R.

The hypothesis of the last paragraph holds. Otherwise, if $n > 0$ is minimal such that

$$\left(\sum_{i=1}^{n} E_{q_i} \right) \cap E_{q_{n+1}} \neq 0$$

then there is a nonzero $x \in M$ and some $y_1, \ldots, y_n \in M$ such that

$$\begin{cases} \varphi(x) = q_{n+1} x, \\ \varphi(y_1) = q_1 y_1, \\ \vdots \\ \varphi(y_n) = q_n y_n, \\ x = y_1 + \ldots + y_n. \end{cases}$$

It follows that there is some $j \in [n]$ such that $y_j \neq 0$, and that $\sum_{i=1}^{n} q_i y_i = \sum_{i=1}^{n} q_{n+1} y_i$. Thus

$$\sum_{[n]\setminus\{j\}} (q_{n+1} - q_i) y_i = (q_j - q_{n+1}) y_j \in \left(\sum_{[n]\setminus\{j\}} E_{q_i} \right) \cap E_{q_j}$$

is a nonzero element of M, which contradicts the minimality of n. This proves that the family $\{E_q : q \in k_0\}$ is lattice-theoretically independent. Then Lemma 2.2.4 proves that

$$l\left(\sum_{i=1}^{d+1} E_{q_i}\right) = \sum_{i=1}^{d+1} l(E_{q_i}) \geq d+1$$

whenever the E_{q_i} are not trivial, as wanted.

Now we treat the case in which $\dfrac{1}{\alpha - q_0} \in R$ for some fix $q_0 \in k_0$. By the latter argument,

$$\frac{1}{\left(\dfrac{1}{\alpha - q_0}\right) - q} = \frac{a - q_0}{-q\alpha + 1 + qq_0} \in R$$

for all but finitely many $q \in k_0$. Multiplying by $\dfrac{1}{\alpha - q_0} \in R$ we get that

$$\frac{1}{-q\alpha + 1 + qq_0} \in R$$

for all but finitely many $q \in k_0$. Multiplying by $-q \in k_0 \subseteq R$, we get that

$$\frac{1}{\alpha - \left(\dfrac{1}{q} + q_0\right)} \in R$$

for all but finitely many $q \in k_0$. We may dismiss the case in which $q = 0$, so that the function $k_0^\times \to k_0 \setminus \{q_0\}$ given by $x \mapsto \dfrac{1}{x} + q_0$ is a bijection, meaning that

$$\frac{1}{\alpha - q} \in R$$

for all but finitely many $q \in k_0$ as wanted. □

Definition 3.2.2. An element $\alpha \in K$ is said to be *tame* (with respect to ς) if all but finitely many elements of S_α are elements of R. We call α *wild* otherwise.

The proof of Lemma 3.2.1 shows that the elements of R are all tame.

Proposition 3.2.3. *The following statements are equivalent:*

1. *R is a multi-valuation ring on K, trivial on k_0.*

2. *R is an intersection of d or fewer valuation rings, trivial on k_0.*

3. *Every $\alpha \in K$ is tame.*

Proof. (2) $\implies$ (1) is trivial. For (1) $\implies$ (3), put $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_n$ for some valuation rings $\mathcal{O}_i$, trivial on k_0, and let $\alpha \in K$. Pick any set of pairwise different elements $q_1, \ldots, q_n \in k_0$, and use Lemma 3.1.2 to find some element in $S_\alpha \cap R$. Then Lemma 3.2.1 assures that α is indeed tame.

Finally, for (3) $\implies$ (2), we may prove that for any set of pairwise different elements $q_1, \ldots, q_d \in k_0$ and any $\alpha \in K$, one of

$$\alpha, \frac{1}{\alpha - q_1}, \ldots, \frac{1}{\alpha - q_d}$$

is in R, and then conclude using Lemma 3.1.2. But this is the case because there can be at most d elements of this list that may not be elements of R, in light of Lemma 3.2.1, $\square$

Definition 3.2.4. We say that ς is *of multi-valuation type* if R is a multi-valuation ring. We say ς is *weakly of multi-valuation type* if R contains a non-zero ideal of some multi-valuation ring on K.

3.3 Mutation

Let $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_S(M)$ be a d-inflator, where S is a k_0-algebra and M is a semi-simple S-module of length d. Denote by R its fundamental ring and by I its fundamental ideal.

Let $L = K \cdot (a_1, \ldots, a_n)$ be a line in K^n. Then the Kronecker product $K^m \otimes L$ is a subobject of K^{nm}, as defined in the preliminary section. If $V \leq K^m$, then $V \otimes L \leq K^m \otimes L \leq K^{nm}$. We may define a new map $\varsigma' : \mathrm{Dir}_K(K) \to \mathrm{Dir}_S(\varsigma_n(L))$ by putting

$$\varsigma'_m(V) = \varsigma_{nm}(V \otimes L).$$

Note that for any $m > 0$, $K^m \otimes L = L^m$ as subspaces of K^{nm}, so that $\varsigma'_m(V) = \varsigma_{nm}(V \otimes L) \leq \varsigma_{nm}(K^m \otimes L) = \varsigma_{nm}(L^m) = (\varsigma_n(L))^m$, so the target directory is the adequate one. Since both $- \otimes L$ and ς preserve $\leq$, then ς' preserves $\leq$ as well. Also, $- \otimes L$ distributes over $\oplus$, so as ς distributes over $\oplus$ as well, it follows that ς' too. Finally, let $\xi \in \mathrm{GL}_m(k_0)$. Then

$$\begin{aligned}
\varsigma'_m(\xi \cdot V) &= \varsigma_{nm}((\xi \cdot V) \otimes L) \\
&= \varsigma_{nm}((\xi \otimes \mathrm{Id}_n) \cdot (V \otimes L)) \\
&= (\xi \otimes \mathrm{Id}_n) \cdot \varsigma_{nm}(V \otimes L) \\
&= \xi \cdot \varsigma'_m(V).
\end{aligned}$$

The last equality follows because the action of $\xi \otimes \mathrm{Id}_n$ in M^{nm} is equal to the action of ξ on $(M^n)^m$, which can be checked by tracking the action of both matrices in a vector of $M^{nm} = (M^n)^m$ respectively.

To sum up, ς' is a well-defined directory morphism. It is in fact a d-inflator. Indeed, if $V \leq K^m$,

$$\begin{aligned}
l(\varsigma'_m(V)) &= l(\varsigma_{nm}(V \otimes L)) \\
&= d \cdot \dim_K(V \otimes L) \\
&= d \cdot \dim_K(V) \cdot \dim_K(L) \\
&= d \cdot \dim_K(V).
\end{aligned}$$

Definition 3.3.1. The *mutation along* L of ς is the d-inflator $\varsigma' : \mathrm{Dir}_K(K) \to \mathrm{Dir}_S(\varsigma_n(L))$ given by

$$\varsigma'_m(V) = \varsigma_{nm}(V \otimes L)$$

for any $V \leq K^m$. We denote by R_L and I_L the fundamental ring and the fundamental ideal of ς'.

Lemma 3.3.2. *Let ς' be the mutation of ς along a line $L = K \cdot (b_1, \ldots, b_n) \leq K^n$. If $\alpha \in K$ specializes to $\varphi \in \mathrm{End}_S(M)$ with respect to ς, then α specializes to $\varphi' \in \mathrm{End}_S(\varsigma_n(L))$ with respect to ς', where φ' is the restriction of $\varphi^{\oplus n}$ to $\varsigma_n(L)$.*[3]

Proof. We must argue that $\varphi' = \varphi^{\oplus n} \restriction \varsigma_n(L) \in \mathrm{End}_S(\varsigma_n(L))$ and that $\varsigma'_2(\Theta_a) = \varsigma_{2n}(\Theta_a \otimes L) = \Theta_{\varphi'}$. In order to compute $\varsigma_{2n}(\Theta_a \otimes L)$, we have to describe $\Theta_a \otimes L$ in terms of objects suitable for inflator calculus. Since $(1, a)$ generates Θ_a and $(b_1, \ldots, b_n)$ generates L, then $\Theta_a \otimes L$ is generated by their Kronecker product

$$\begin{bmatrix} b_1 & ab_1 \\ \vdots & \vdots \\ b_n & ab_n \end{bmatrix},$$

i.e., $\Theta_a \otimes L = \{(b_1 x, \ldots, b_n x, ab_1 x, \ldots, ab_n x) : x \in K\}$. In fact,

$$\Theta_a \otimes L = (\mu \cdot \Theta_a^{\oplus n}) \cap (L \oplus L) = (\mu \cdot \Theta_a^{\oplus n}) \cap (L \oplus K^n) \tag{3.3.1}$$

where $\mu \in \mathrm{GL}_{2n}(k_0)$ is the permutation matrix associated to the permutation

$$\begin{pmatrix} 1 & n+1 & 2 & n+2 & \ldots & n & 2n \\ 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \end{pmatrix}.$$

Indeed,

$$\begin{aligned} \Theta_a^{\oplus n} &= \{(x_1, ax_1, \ldots, x_n, ax_n) : x_1, \ldots, x_n \in K\}, \\ \mu \cdot \Theta_a^{\oplus n} &= \{(x_1, , \ldots, x_n, ax_1, \ldots, ax_n) : x_1, \ldots, x_n \in K\}, \\ (\mu \cdot \Theta_a^{\oplus n}) \cap (L \oplus L) &= \{(x_1, , \ldots, x_n, ax_1, \ldots, ax_n) : x_i = b_i x \text{ and } ax_i = b_i y \text{ for some } x, y \in K\} \\ &= \{(b_1 x, \ldots, b_n x, ab_1 x, \ldots, ab_n x) : x \in K\} \\ &= \Theta_a \otimes L, \end{aligned}$$

where the fourth equality holds as $x_i = b_i x$ and $ax_i = b_i y$ imply that $y = ax$. Moreover, if $(x_1, \ldots, x_n, ax_1, \ldots, ax_n)$ is such that $x_1, \ldots, x_n \in L$ (i.e., this is a vector of $(\mu \cdot \Theta_a^{\oplus n}) \cap (L \oplus K^n)$, then it is equal to $(b_1 x, \ldots, b_n x, ab_1 x, \ldots, ab_n x)$ for some $x \in K$, i.e.

$$(\mu \cdot \Theta_a^{\oplus n}) \cap (L \oplus K^n) \subseteq (\mu \cdot \Theta_a^{\oplus n}) \cap (L \oplus L),$$

so equation 3.3.1 holds.

Therefore, since $\varsigma_2(\Theta_a) = \Theta_\varphi$, we get that

$$\varsigma_{2n}(\Theta_a \otimes L) = (\mu \cdot \Theta_\varphi^{\oplus n}) \cap (\varsigma_n(L) \oplus \varsigma_n(L)) = (\mu \cdot \Theta_\varphi^{\oplus n}) \cap (\varsigma_n(L) \oplus M^n).$$

The second equality shows that

$$\begin{aligned} \vec{x} \in \varsigma_n(L) &\implies (\vec{x}, \varphi^{\oplus n}(\vec{x})) \in (\mu \cdot \Theta_\varphi^{\oplus n}) \cap (\varsigma_n(L) \oplus M^n) \\ &\implies (\vec{x}, \varphi^{\oplus n}(\vec{x})) \in (\mu \cdot \Theta_\varphi^{\oplus n}) \cap (\varsigma_n(L) \oplus \varsigma_n(L)) \\ &\implies \varphi^{\oplus n}(\vec{x}) \in \varsigma_n(L), \end{aligned}$$

[3] Here we define $\varphi^{\oplus n}(x_1, \ldots, x_n) := (\varphi(x_1), \ldots, \varphi(x_n))$.

so indeed $\varphi' = \varphi^{\oplus n} \restriction \varsigma_n(L) \in \mathrm{End}_S(\varsigma_n(L))$. Also, the first equality shows that

$$\begin{aligned}\varsigma_{2n}(\Theta_a \otimes L) &= (\mu \cdot \Theta_\varphi^{\oplus n}) \cap (\varsigma_n(L) \oplus \varsigma_n(L)) \\ &= \{(\vec{x}, \varphi^{\oplus n}(\vec{x})) : \vec{x} \in \varsigma_n(L) \text{ and } \varphi^{\oplus n}(\vec{x}) \in \varsigma_n(L)\} \\ &= \{(\vec{x}, \varphi^{\oplus n}(\vec{x})) : \vec{x} \in \varsigma_n(L)\} \\ &= \Theta_{\varphi'},\end{aligned}$$

as wanted. ☐

We get immediately the following corollary.

Corollary 3.3.3. *$R \subseteq R_L$ and $I \subseteq I_L$ for any line $L \leq K^n$, $n > 0$.*

Lemma 3.3.4. *Suppose ς' is the mutation of ς along a line $L_1 \leq K^{n_1}$, and let ς'' be the mutation of ς' along a line $L_2 \leq K^{n_2}$. Then ς'' is equivalent to the mutation of ς along the line $L_2 \otimes L_1 \leq K^{n_1 n_2}$.*

Proof. Let $V \leq K^m$. Then

$$\begin{aligned}\varsigma''_m(V) &= \varsigma'_{n_2 m}(V \otimes L_2) \\ &= \varsigma_{n_1 n_2 m}((V \otimes L_2) \otimes L_1) \\ &= \varsigma_{n_1 n_2 m}(V \otimes (L_2 \otimes L_1))\end{aligned}$$

because $(V \otimes L_2) \otimes L_1 = V \otimes (L_2 \otimes L_1)$ as subspaces of $K^{n_1 n_2 m}$. ☐

Lemma 3.3.5. *Suppose ς', ς'' are the mutations of ς along $L_1 \leq K^n$ and $L_2 \leq K^n$ respectively. If $L_1 = \xi \cdot L_2$ for some $\xi \in \mathrm{GL}_n(k_0)$, then ς' and ς'' are equivalent.*

Proof. The fact that $L_2 = \xi \cdot L_1$ implies that $\varsigma_n(L_2) = \varsigma_n(\xi \cdot L_1) = \xi \cdot \varsigma_n(L_1)$, so ξ induces a morphism $\mathrm{Dir}_S(\varsigma_n(L_1)) \to \mathrm{Dir}_S(\varsigma_n(L_2))$ in which $N \mapsto \xi^{\oplus n} \cdot N$ for any $N \in \mathrm{Sub}_S(\varsigma_n(L_1)^n)$. Now, if $V \leq K^m$, then

$$\begin{aligned}\varsigma''_m(V) &= \varsigma_{nm}(V \otimes L_2) \\ &= \varsigma_{nm}(V \otimes \xi \cdot L_1) \\ &= \varsigma_{nm}((\mathrm{Id} \otimes \xi) \cdot (V \otimes L_1)) \\ &= (\mathrm{Id} \otimes \xi) \cdot \varsigma_{nm}(V \otimes L_1) \\ &= \xi^{\oplus m} \cdot \varsigma'_m(V),\end{aligned}$$

where the last equality can be checked by tracking the action of $(\mathrm{Id} \otimes \xi)$ on a vector of M^{nm} and of $\xi^{\oplus m}$ on a vector of $(M^n)^m$. Therefore the diagram

$$\begin{tikzcd} \mathrm{Dir}_K(K) \arrow[r, "\varsigma'"] \arrow[dr, "\varsigma''"'] & \mathrm{Dir}_S(\varsigma_n(L_1)) \arrow[d] \\ & \mathrm{Dir}_S(\varsigma_n(L_2)) \end{tikzcd}$$

commutes, as wanted. ☐

Corollary 3.3.6. *Under the conditions of the latter lemma, $R_{L_1} = R_{L_2}$ and $I_{L_1} = I_{L_2}$.*

Proof. It is clear that if $\varphi \in \mathrm{End}_S(\varsigma_n(L_2))$, then $\xi^{-1}\varphi\xi \in \mathrm{End}_S(\varsigma_n(L_1))$. We have to prove that $(\xi^{-1})^{\oplus 2}\Theta_\varphi = \Theta_{\xi^{-1}\varphi\xi}$. Indeed,

$$\begin{aligned}(\xi^{-1})^{\oplus 2} \cdot \Theta_\varphi &= (\xi^{-1})^{\oplus 2} \cdot \{(x, \varphi(x)) : x \in \varsigma_n(L_2)\} \\ &= (\xi^{-1})^{\oplus 2} \cdot \{(x, \varphi(x)) : x \in \xi \cdot \varsigma_n(L_1)\} \\ &= (\xi^{-1})^{\oplus 2} \cdot \{(\xi y, \varphi(\xi y)) : y \in \varsigma_n(L_1)\} \\ &= \{(\xi^{-1}\xi y, \xi^{-1}\varphi(\xi y)) : y \in \varsigma_n(L_1)\} \\ &= \{(y, (\xi^{-1}\varphi\xi)(y)) : y \in \varsigma_n(L_1)\} \\ &= \Theta_{\xi^{-1}\varphi\xi}.\end{aligned}$$

Now, if $a \in K$,

$$\begin{aligned} a \in R_{L_2} &\implies \varsigma_2''(\Theta_a) = \Theta_\varphi \text{ for some } \varphi \in \mathrm{End}_S(\varsigma_n(L_2)) \\ &\implies \xi^{\oplus 2} \cdot \varsigma_2'(\Theta_a) = \Theta_\varphi \\ &\implies \varsigma_2'(\Theta_a) = (\xi^{-1})^{\oplus 2} \cdot \Theta_\varphi \\ &\implies \varsigma_2'(\Theta_a) = \Theta_{\xi^{-1}\varphi\xi} \\ &\implies a \in R_{L_1}.\end{aligned}$$

The reverse inclusion follows similarly, with ξ^{-1} instead of ξ. If a specializes to 0 with respect to ς'', then it specializes to $\xi^{-1}0\xi = 0$ with respect to ς', so $I_{L_2} \subseteq I_{L_1}$. Equality follows again by symmetry. $\square$

In particular, if $L_1 \leq K^{n_1}$ and $L_2 \leq K^{n_2}$ are lines, then $R_{L_1 \otimes L_2} = R_{L_2 \otimes L_1}$ and $I_{L_1 \otimes L_2} = I_{L_2 \otimes L_1}$. This holds because $L_1 \otimes L_2 = \mu \cdot (L_2 \otimes L_1)$ where $\mu \in \mathrm{GL}_{n^2}(k_0)$ is the matrix corresponding to the permutation induced by transposing the vectors, when read as matrices, of $L_2 \otimes L_1$.

Definition 3.3.7. The *limiting ring* is the set $R_\infty = \bigcup\{R_L : L \text{ is a line in } K^n, n > 0\}$ and the *limiting ideal* is the set $I_\infty = \bigcup\{I_L : L \text{ is a line in } K^n, n > 0\}$.

Corollary 3.3.8. *The limiting ring R_∞ is a k_0-subalgebra of K extending R and I_∞ is an ideal of R_∞ extending I. Moreover, $1 + I_\infty \subseteq R_\infty^\times$, so I_∞ is contained in the Jacobson radical of R_∞.*

Proof. It is clear that $\varsigma = \varsigma'$ where ς' is the mutation of ς along $L = K$, so $R \subseteq R_\infty$ (resp. $I \subseteq I_\infty$). The union defining R_∞ (resp. I_∞) is directed: if L_1, L_2 are lines in some powers of K, then $R_{L_1 \otimes L_2} = R_{L_2 \otimes L_1}$ extends both R_{L_1} and R_{L_2} (resp. $I_{L_1 \otimes L_2} = I_{L_2 \otimes L_1}$ extends both I_{L_1} and I_{L_2}), by Corollary 3.3.3 and the comment after Corollary 3.3.6. Now $1 + I_\infty \subseteq R_\infty^\times$ because if $a \in I_\infty$, then $a \in I_L$ for some line L, so

$$1 + a \in 1 + I_L \subseteq R_L^\times \subseteq R_\infty^\times$$

by Proposition 2.3.4. $\square$

Lemma 3.3.9. *Let $\alpha \in K$ and let $q \in k_0$ be such that $a \neq q$. Let ς' be the mutation of ς along the line $L = K \cdot (1, \alpha, \ldots, \alpha^{d-1})$. If $\dfrac{1}{\alpha - q} \notin R_L$, then there is some nonzero $\varepsilon \in M$ such that*

$$(\varepsilon, q\varepsilon, \ldots, q^d\varepsilon) \in \varsigma_{d+1}(\{(x, \alpha x, \ldots, \alpha^{d-1}x, \alpha^d x) : x \in K\}).$$

Proof. By definition of the fundamental ring and by Lemma 2.2.5,

$$\frac{1}{\alpha - q} \notin R_L \iff \varsigma_2'(K \cdot (\alpha - q, 1)) \text{ is not the graph of any } \varphi \in \mathrm{End}_S(\varsigma_d(L))$$
$$\iff \varsigma_2'(K \cdot (\alpha - q, 1)) \cap (0^d \oplus \varsigma_d(L)) \neq 0^d \oplus 0^d,$$

so there is a nontrivial vector $\vec{x} \in \varsigma_d(L) \subseteq M^d$ such that

$$(0^d, \vec{x}) \in \varsigma_2'(K \cdot (\alpha - q, 1)). \tag{3.3.2}$$

Note that $\varsigma_2'(K\cdot(\alpha-q,1)) = \varsigma_{2n}\big(K\cdot(\alpha-q,1)\otimes K\cdot(1,\alpha,\dots,\alpha^{d-1})\big)$ and that the Kronecker product of $(\alpha - q, 1)$ and $(1, \alpha, \dots, \alpha^{d-1})$ equals

$$\begin{bmatrix} \alpha - q & 1 \\ \alpha^2 - q\alpha & \alpha \\ \vdots & \vdots \\ \alpha^d - q\alpha^{d-1} & \alpha^{d-1} \end{bmatrix},$$

this is, $K \cdot (\alpha - q, 1) \otimes K \cdot (1, \alpha, \dots, \alpha^{d-1})$ is equal to

$$\{(\alpha x - qx, \alpha^2 x - q\alpha x, \dots, \alpha^d x - q\alpha^{d-1}x, x, \alpha x, \dots, \alpha^{d-1}x) : x \in K\}.$$

If $N = \{(x, \alpha x, \dots, \alpha^{d-1}x, \alpha^d x) : x \in K\}$, we can write $K \cdot (\alpha - q, 1) \otimes K \cdot (1, \alpha, \dots, \alpha^{d-1})$ as

$$\{(x_1 - qx_0, x_2 - qx_1, \dots, x_d - qx_{d-1}, x_0, \dots, x_{d-1}) : (x_0, \dots, x_d) \in N\}.$$

This subspace can be recovered as the image of $N \oplus 0^{d-1}$ under the matrix $\mu \in \mathrm{GL}_{2n}(k_0)$ that represents the transformation $T : N \oplus 0^{d-1} \to K^{2d}$ given by

$$T(x_0, \dots, x_d, 0, \dots, 0) = (x_1 - qx_0, \dots, x_d - qx_{d-1}, x_0, \dots, x_{d-1}).$$

This transformation is clearly injective, so its matrix representation μ is indeed in $\mathrm{GL}_{2n}(k_0)$. Therefore $\varsigma_2'(K \cdot (\alpha - q)) = \mu \cdot (\varsigma_{d+1}(N) \oplus 0^{d-1})$, which is in turn equal to

$$\{(x_1 - qx_0, x_2 - qx_1, \dots, x_d - qx_{d-1}, x_0, \dots, x_{d-1}) : (x_0, \dots, x_d) \in \varsigma_{d+1}(N)\}.$$

Equation 3.3.2 shows that if $\vec{x} = (x_0, \dots, x_{d-1})$, then $x_i = qx_{i-1}$ for every $i \in [d]$ and that

$$(x_0, \dots, x_d) = (x_0, qx_0, \dots, q^d x_0) \in \varsigma_{d+1}(N)$$

for some $x_0 =: \varepsilon \in M$. Since $\vec{x}$ is not trivial, ε cannot be zero, as wanted. □

Proposition 3.3.10. *Let $\alpha \in K$, and let $L = K(1, \alpha, \alpha^2, \dots, \alpha^{d-1})$. Then α is tame with respect to the mutation of ς along L.*

Proof. If α is not tame with respect to ς', then there are infinitely many $q \in k_0$ such that $\dfrac{1}{\alpha - q} \notin R_L$. Choose $d + 1$ many of such q's, say, $q_0, \dots, q_d$. By Lemma 3.3.9, there are $d + 1$ nontrivial elements $\varepsilon_0, \dots, \varepsilon_d$ of M such that

$$(\varepsilon_i, q_i\varepsilon_i, \dots, q_i^d \varepsilon_i) \in \varsigma_{d+1}(\{(x, \alpha x, \dots, \alpha^d x) : x \in K\})$$

for each $i \in \{0, \ldots, d\}$. Note that $(\varepsilon_i, q_i\varepsilon_i, \ldots, q_i^d\varepsilon_i) = \mu \cdot \varepsilon_i\mathbf{e}_i$ where $\mu \in \mathrm{GL}_{d+1}(k_0)$ is the Vandermonde matrix

$$\begin{bmatrix} 1 & q_0 & q_0^2 & \cdots & q_0^d \\ & & \vdots & & \\ 1 & q_d & q_d^2 & \cdots & q_d^d \end{bmatrix}$$

and $\varepsilon_i\mathbf{e}_i$ is the vector $(0^{i-1}, \varepsilon_i, 0^{d+1-i})$. Therefore, if $E := \mu^{-1} \cdot \{(x, \alpha x, \ldots, \alpha^d x) : x \in K\}$, we have that

$$\begin{aligned} \varepsilon_i\mathbf{e}_i \in \mu^{-1} \cdot \varsigma_{d+1}(\{(x, \alpha x, \ldots, \alpha^d x) : x \in K\}) \\ = \varsigma_{d+1}(\mu^{-1} \cdot \{(x, \alpha x, \ldots, \alpha^d x) : x \in K\}) \\ = \varsigma_{d+1}(E) \end{aligned}$$

for each $i \in \{0, \ldots, d\}$, which is impossible because $l(\varsigma_{d+1}(E)) = d \cdot \dim_K(E) = d$ and the family $\{\varepsilon_i\mathbf{e}_i : i \in \{0, \ldots, d\}\}$ generates a submodule of $\varsigma_{d+1}(E)$ of length $d + 1$. □

Proposition 3.3.11. *R_∞ is a multi-valuation ring of K. In fact, it is an intersection of at most d-valuation rings. If I is non-trivial, then R_∞ is a non-trivial multi-valuation ring, i.e. R_∞ is a proper subring of K.*

Proof. For the first statement we apply the criterion given by Lemma 3.1.2 to R_∞. To this end, let $q_1, \ldots, q_d$ be distinct elements of k_0, and let $\alpha \in K$. Let $L = K \cdot (1, \alpha, \alpha^2, \ldots, \alpha^{d-1})$. Then α is tame with respect with the mutation of ς along L, so all but finitely many of the elements of $S_\alpha = \{\alpha\} \cup \left\{\frac{1}{\alpha - q} : q \in k_0\right\}$ are in R_L. In particular $S_\alpha \cap R_L \neq \emptyset$, so by Lemma 3.2.1, all but at most d elements of S_α are in R_L. Therefore one of

$$\alpha, \frac{1}{\alpha - q_1}, \ldots, \frac{1}{\alpha - q_d}$$

is in $R_L \subseteq R_\infty$ because there can be at most d exceptions, hence the conditions of the criterion are fulfilled.

If I is not trivial, then none of the ideals $I \subseteq I_\infty \subseteq J_\infty$ are trivial, where J_∞ is the Jacobson radical of R_∞. If $R_\infty = K$, then $J_\infty = \{0\}$, which is impossible. □

4 Classification of 1-Inflators

This section is devoted to prove Johnson's classification of 1-inflators given in Theorem 5.20 of [Johnson, 2019a].

Lemma 4.0.1. *Let $L = (L, +, \cdot, 0, 1)$ be a ring. If $l(L) = 1$ when computed as a module over itself, then $\mathrm{End}_L(L) \cong L^{op}$, where L^{op} is defined as the ring $(L, +, \cdot^{op}, 0, 1)$ having $x \cdot^{op} y = y \cdot x$ for any $x, y \in L$. Moreover, L^{op} is a division ring.*

Proof. The rings L^{op} and $\mathrm{End}_L(L)$ are isomorphic via the morphisms $\lambda \mapsto (x \mapsto x\lambda)$ for $\lambda \in L$ and $\varphi \mapsto \varphi(1)$ for $\varphi \in \mathrm{End}_L(L)$. Since $l(L) = 1$, all nonzero elements in $\mathrm{End}_L(L)$ are invertible. Indeed, if $\varphi \in \mathrm{End}_L(L)$ is not zero, then the inclusions

$$\{0\} \subseteq \ker\varphi \subsetneq L,$$
$$\{0\} \subsetneq \operatorname{im}\varphi \subseteq L$$

yield that $\ker\varphi = \{0\}$ and $\operatorname{im}\varphi = L$, i.e. φ is an isomorphism. Thus L^{op} itself is a division ring. □

Proposition 4.0.2 (cf. Proposition 5.19 of [Johnson, 2019a]). *Let $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_L(L)$ be a 1-inflator. Then*

$$\mathcal{O} = \{a \in K : \varsigma_2(\Theta_a) = \Theta_\varphi \textit{ for some } \varphi \in \mathrm{End}_L(L)\}$$

is a valuation k_0-subalgebra of K with maximal ideal

$$\mathscr{M} = \ker(\widehat{\mathrm{res}}) = \{a \in K : \varsigma_2(\Theta_a) = L \oplus 0\}.$$

Proof. Let $a \in K^\times$ be such that $a \notin \mathcal{O}$, i.e., $\varsigma_2(\Theta_a)$ is not the graph of any endomorphism of L. Since ς is a 1-inflator, $l(\varsigma_2(\Theta_a)) = \dim_K(\Theta_a) = 1 = l(L)$. Thus, by Lemma 2.2.5, there is some nonzero $b \in L$ such that $(0, b) \in \varsigma_2(\Theta_a)$. The same argument shows that if $a^{-1} \notin \mathcal{O}$, then there is some nonzero $c \in L$ such that $(0, c) \in \varsigma_2(\Theta_{a^{-1}})$, yielding that $(c, 0) \in \varsigma_2(\{(a^{-1}x, x) : x \in K\}) = \varsigma_2(\Theta_a)$ upon swaping coordinates[4]. But then $Lc \oplus Lb$ is an L-submodule of $\varsigma_2(\Theta_a)$ of length 2, which is absurd. This shows that $\mathcal{O}$ is indeed a valuation ring. By Proposition 2.3.4, $\mathcal{O}$ is also a k_0-subalgebra of K and $\mathscr{M}$ is an ideal of $\mathcal{O}$.

In order to see that $\mathscr{M}$ is maximal, it is enough to see that $a \in \mathcal{O}^\times$ for any $a \in \mathcal{O} \setminus \mathscr{M}$. Indeed, if a specializes to some non-zero φ, then Lemma 4.0.1 assures that $\varphi \in \mathrm{Aut}_L(L)$, so again by Proposition 2.3.4 we see that $\alpha^{-1} \in \mathcal{O}$. □

Thus the generalized residue map $\widehat{\mathrm{res}} : \mathcal{O} \to \mathrm{End}_L(L) \cong L^{op}$ factors through $k := \mathcal{O}/\mathscr{M}$.

$$\begin{tikzcd}
\mathcal{O} \arrow[rr, "\widehat{\mathrm{res}}"] \arrow[dr, two heads, "\mathrm{res}"'] & & L^{op} \\
& k \arrow[ur, hook] &
\end{tikzcd}$$

[4]This is achieved by acting with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{GL}_2(k_0)$.

Since $k = k^{op}$, we may embed k into L, i.e., L is a (skew) field extension of k. The latter factorization induces a 1-inflator ς' defined as the composition

$$\mathrm{Dir}_K(K) \xrightarrow{\varsigma''} \mathrm{Dir}_k(k) \xrightarrow{L\otimes_k -} \mathrm{Dir}_L(L)$$

where ς'' is given by the valuation structure in K and $L \otimes_k -$ is given by the extension L/k.

We must argue that $\varsigma(V) = \varsigma'(V)$ for every $V \in \mathrm{Sub}(K^n)$. Let $l = \dim_K V$. We know that there are some $\vec{a_1}, \dots, \vec{a_l} \in \mathcal{O}^n$ such that

- ✓ $\{\vec{a_1}, \dots, \vec{a_l}\}$ generates V as a K-vector space,
- ✓ $\{\vec{a_1}, \dots, \vec{a_l}\}$ freely generates $V \cap \mathcal{O}^n$ as an $\mathcal{O}$-module, and
- ✓ $\{r(\vec{a_1}), \dots, r(\vec{a_l})\}$ is a basis for $r(V \cap \mathcal{O}^n)$, where $r : \mathcal{O}^n \to k^n$ denotes coordinatewise $\mathrm{res} : \mathcal{O} \to k$.

Then $\{1 \otimes r(\vec{a_1}), \dots, 1 \otimes r(\vec{a_l})\}$ generates $\varsigma'(V) = L \otimes_k r(V \cap \mathcal{O}^n)$ as an L-module. Let ρ denote $\iota \circ \mathrm{res}$, where ι denotes the embedding of k into L. Since $\otimes$ represents the Kronecker product, we may write $1 \otimes r(\vec{a_i}) = (\rho(a_{i1}), \dots, \rho(a_{in}))$ and define $b_{ij} := \rho(a_{ij})$, so that $(\rho(a_{i1}), \dots, \rho(a_{in})) = (b_{i1}, \dots, b_{in}) =: \vec{b_i} \in L^n$ and $\vec{b_1}, \dots, \vec{b_l}$ generate $\varsigma'(V)$ as an L-module.

In order to compute $\varsigma(V)$ via "inflator calculus", we need to note the following. Let $\vec{x} \in K^n$. Then

$$\begin{aligned}
\vec{x} \in V &\iff \exists \lambda_1, \dots, \lambda_l \in K : \vec{x} = \lambda_1 \vec{a_1} + \ldots + \lambda_l \vec{a_l} \\
&\iff \exists \lambda_1, \dots, \lambda_l \in K : \bigwedge_{j=1}^{n} x_j = \lambda_1 a_{1j} + \ldots + \lambda_l a_{lj} \\
&\iff \exists (\lambda_1, \dots, \lambda_l) \in K^l, (y_{ij}) \in K^{ln} : \bigwedge_{j=1}^{n} x_j = y_{1j} + \ldots + y_{lj} \wedge \bigwedge_{i=1}^{l} \bigwedge_{j=1}^{n} y_{ij} = \lambda_i a_{ij}.
\end{aligned}$$

For each $j \in [n]$ and $i \in [l]$, if we define $W_{ij} = \{(\vec{\lambda}, \vec{y}, \vec{x}) : y_{ij} = \lambda_i a_{ij}\}$ and $U_j = \{(\vec{\lambda}, \vec{y}, \vec{x}) : x_j = y_{1j} + \ldots + y_{lj}\}$ as subspaces of $K^l \times K^{ln} \times K^n$, we may recover V via the equality

$$K^l \times K^{ln} \times V = \bigcap_{j=1}^{n} U_j \cap \bigcap_{i=1}^{l} \bigcap_{j=1}^{n} W_{ij}.$$

We thus get

$$\begin{aligned}
L^l \times L^{ln} \times \varsigma(V) &= \varsigma(K^l \times K^{ln} \times V) \\
&= \varsigma\left(\bigcap_{j=1}^{n} U_j \cap \bigcap_{i=1}^{l} \bigcap_{j=1}^{n} W_{ij} \right) \\
&\subseteq \bigcap_{j=1}^{n} \varsigma(U_j) \cap \bigcap_{i=1}^{l} \bigcap_{j=1}^{n} \varsigma(W_{ij}).
\end{aligned}$$

If we aim to describe $\bigcap_{j=1}^{n} \varsigma(U_j) \cap \bigcap_{i=1}^{l} \bigcap_{j=1}^{n} \varsigma(W_{ij})$, we shall compute $\varsigma(U_j)$ and $\varsigma(W_{ij})$ for any $i \in [l], j \in [n]$. Since $\widehat{\mathrm{res}}$ factors through k as $\iota \circ \mathrm{res} = \rho$, we have that $\widehat{\mathrm{res}}(a_{ij}) =$

$\rho(a_{ij}) = b_{ij} \in L^{op}$ and that

$$\begin{aligned}\varsigma(\Theta_{a_{ij}}) &= \{(x, \varphi(x)) : x \in L\}\\ &= \{(x, x \cdot \varphi(1)) : x \in L\}\\ &= \{(x, x \cdot b_{ij}) : x \in L\}.\end{aligned}$$

Note that

$$W_{ij} = \mu \cdot \left(\Theta_{a_{ij}} \oplus K^{l+nl+n-2}\right)$$

where $\mu \in \mathrm{GL}_{l+nl+n}(K_0)$ is the permutation matrix associated to $1 \leftrightarrow i$, (the i^{th} coordinate in the $\vec{\lambda}$-section) and $2 \leftrightarrow ij$ (the ij^{th} coordinate in the $\vec{y}$-section) and the identity elsewhere. Therefore, as an L-submodule of $L^l \times L^{nl} \times L^n$,

$$\begin{aligned}\varsigma(W_{ij}) &= \mu \cdot \left(\varsigma(\Theta_{a_{ij}}) \oplus L^{l+nl+n-2}\right)\\ &= \{(\vec{\lambda}, \vec{y}, \vec{x}) : y_{ij} = \lambda_i b_{ij}\}.\end{aligned}$$

Also note that

$$U_j = \nu \cdot \left(K^l \oplus 0\right)$$

where $\nu \in \mathrm{GL}_{l+1}(K_0)$ is the matrix associated to the transformation

$$(x_1, \ldots, x_l, x_{l+1}) \mapsto (x_1, \ldots, x_l, x_1 + \ldots + x_{l+1}).$$

Therefore, again as an L-submodule of $L^l \times L^{nl} \times L^n$, we get that

$$\begin{aligned}\varsigma(U_j) &= \nu \cdot \left(L^l \oplus 0\right)\\ &= \{(\vec{\lambda}, \vec{y}, \vec{x}) : x_j = y_{1j} + \ldots + y_{lj}\}.\end{aligned}$$

It follows that

$$\bigcap_{j=1}^{n} \varsigma(U_j) \cap \bigcap_{i=1}^{l} \bigcap_{j=1}^{n} \varsigma(W_{ij}) = L^l \times L^{nl} \times \varsigma'(V)$$

is an L-module of length at most $l+nl+l = l(n+2)$ because $\varsigma'(V)$ is generated by l elements. Since the length of $\varsigma\left(\bigcap_{j=1}^{n} U_j \cap \bigcap_{i=1}^{l} \bigcap_{j=1}^{n} W_{ij}\right)$ coincides with $\dim_K(K^l \times K^{nl} \times V) = l(n+2)$, we get the equality

$$L^l \times L^{nl} \times \varsigma(V) = L^l \times L^{nl} \times \varsigma'(V)$$

and thus $\varsigma(V) = \varsigma'(V)$, as wanted.

5 Diffeovaluation Inflators

This section is devoted to introduce dense diffeovalued fields, and to define a 2-inflator out of the valuation and the derivation structure thereof. We will follow Chapter 8 of [Johnson, 2020a].

5.1 Mock $K/\mathscr{M}$'s

Let $(K, \mathcal{O})$ be a valued field with maximal ideal $\mathscr{M}$ and residue field k.

Definition 5.1.1. A *mock* $K/\mathscr{M}$ is a divisible $\mathcal{O}$-module D extending k satisfying that either $x \in \mathcal{O} \cdot y$ or $y \in \mathcal{O} \cdot x$ for any $x, y \in D$.

Note that the theory of mock $K/\mathscr{M}$'s is first-order axiomatizable in the two sorted language (K, D) where $(K, \mathcal{O})$ has the valued field structure and D has the $\mathcal{O}$-module structure. This theory is consistent because the $\mathcal{O}$-module $K/\mathscr{M}$ is itself a mock $K/\mathscr{M}$. In fact, whenever the value group Γ has countable cofinality, this is the only choice for a mock $K/\mathscr{M}$.

Lemma 5.1.2 (Cf. Proposition 8.2 of [Johnson, 2019a])**.** *Let D be a mock $K/\mathscr{M}$. If the value group Γ has countable cofinality, then $D \cong K/\mathscr{M}$.*

Proof. Let $(\gamma_i : i < \omega)$ be an increasing cofinal sequence in Γ. Then $(\delta_i : i < \omega) := (-\gamma_i : i < \omega)$ is a decreasing sequence with no lower bound in Γ. Let $a_i \in K$ be such that $v(a_i) = \delta_i$ for any $i < \omega$. We may assume that $a_0 = 1$, so that $\delta_0 = 0$. This way we can write K as the union of the chain $\mathcal{O} \subseteq \mathcal{O} \cdot a_1 \subseteq \mathcal{O} \cdot a_2 \subseteq \ldots$. Use divisibility of D to define the sequence

$$\begin{cases} b_0 = \phi(1 + \mathscr{M}), \\ b_{i+1} = \dfrac{a_{i+1}}{a_i} b_i, \end{cases}$$

where $\phi : k \to D$ is the embedding of k into D. One can see by induction that none of the b_i are zero. Define $f_i : \mathcal{O} \cdot a_i \to D$ by $f_i(x) = \dfrac{x}{a_i} b_i$. Then $f_i(a_i) = b_i$ and if $x \in \mathcal{O} \cdot a_i$ then $f_i(x) = \dfrac{x}{a_i} b_i = \dfrac{x}{a_i} \dfrac{a_{i+1}}{a_{i+1}} b_i = \dfrac{x}{a_{i+1}} b_{i+1} = f_{i+1}(x)$. Therefore $f := \bigcup_{i<\omega} f_i : K \to D$ is a well defined morphism of $\mathcal{O}$-modules.

We may show that $\ker(f) = \mathscr{M}$ and that f is surjective, yielding the desired isomorphism. Let $x \in K$. If $x \in \mathcal{O}$, then

$$\begin{aligned} f(x) = 0 &\iff x f(1) = 0 \\ &\iff x f_0(1) = 0 \\ &\iff x \frac{1}{a_0} b_0 = 0 \\ &\iff x \phi(1 + \mathscr{M}) = 0 \\ &\iff \phi(x + \mathscr{M}) = 0 \\ &\iff x + \mathscr{M} = \mathscr{M} \\ &\iff x \in \mathscr{M}. \end{aligned}$$

If $x \notin \mathcal{O}$ then $\dfrac{1}{x} \in \mathcal{O}$, and $b_0 = f(1) = f(\dfrac{1}{x} x) = \dfrac{1}{x} f(x)$, i.e. $f(x) = x b_0$. Therefore $f(x) = 0 \iff x b_0 = 0 \iff b_0 = 0$ by divisibility, which is absurd. Therefore $f(x) = 0 \iff x \in \mathscr{M}$ even in this case.

Finally, if $y \in D$ is not in the image of f, then it is not in the image of any of the f_i. By $\mathcal{O}$-linearity, $y \notin \mathcal{O} \cdot b_i$ for every $i < \omega$, so by the condition on mock $K/\mathscr{M}'s$, we have that $b_i \in \mathcal{O} \cdot y$ for all $i < \omega$. Let $c_i \in \mathcal{O}$ be such that $b_i = c_i y$. If i is large enough so that $v(a_i) < v(c_0^{-1})$, then $\frac{1}{a_i c_0} \in \mathscr{M}$ and $0 = \frac{1}{a_i c_0} b_0 = \frac{1}{a_i} y$. But $b_0 = f_i(a_0) = f_i(1) = \frac{1}{a_i} b_i = \frac{c_i}{a_i} y = c_i \cdot 0 = 0$, which is absurd. □

Corollary 5.1.3. *1. If (K, D) is countable, then $D \cong K/\mathscr{M}$.*

2. If (K, D) is an $\aleph_1$-resplendent model of the theory of mock $K/\mathscr{M}$'s, then $D \cong K/\mathscr{M}$.

3. There is a map $\mathrm{val} : D \to \Gamma \cup \{\infty\}$ satisfying the following properties:

   (a) $\mathrm{val}(x) \leq 0$ or $\mathrm{val}(x) = \infty$ for any $x \in D$.

   (b) $\mathrm{val}(x) = \infty$ if and only if $x = 0$.

   (c) $\mathrm{val}(x) \geq 0$ if and only if $x \in \mathrm{im}(k \hookrightarrow D)$.

   (d) For any $a \in \mathcal{O}$ and any $x \in D$,

$$\mathrm{val}(ax) = \begin{cases} v(a) + \mathrm{val}(x) & \text{if } v(a) + \mathrm{val}(x) \leq 0, \\ \infty & \text{if } v(a) + \mathrm{val}(x) > 0. \end{cases}$$

   (e) $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$ for any $x, y \in D$.

   (f) $y \in \mathcal{O} \cdot x$ whenever $\mathrm{val}(y) \geq \mathrm{val}(x)$.

   (g) For any $\gamma \in \Gamma$ there is some $x \in D$ such that $\mathrm{val}(x) \leq \gamma$.

Proof. 1. If (K, D) is countable, then Γ has countable cofinality because it is itself countable. Since D is also countable, the result follows from Lemma 5.1.2.

2. Let $f : K \to D$ be a new function symbol, and let $\varphi(f)$ be the sentence stating that f is a surjective morphism of $\mathcal{O}$-modules whose kernel is equal to $\mathscr{M}$. Then $\varphi(f)$ is consistent with $\mathrm{Th}(K, D)$. Indeed, take a countable elementary substructure $(K_0, D_0) \preceq (K, D)$ and use the latter item to construct the desired $f : K \to D$. Therefore $(K_0, D_0, f) \models \{\varphi(f)\} \cup \mathrm{Th}(K, D)$. The result follows by $\aleph_1$-resplendence.

3. Let (K, D) be a mock $K/\mathscr{M}$, and let $D_0 = \mathrm{im}(k \hookrightarrow D)$. Let $\mathrm{val} : D \to \Gamma \cup \{\infty\}$ be a new function symbol, and let $\varphi(\mathrm{val})$ be the sentence stating the following:

   ✓ For every $x \in D \setminus \{0\}$ there is some $a \in \mathcal{O}$ such that $ax \in D_0 \setminus \{0\}$,

   ✓ For any $a, b \in \mathcal{O}$, if $ax, bx \in D_0 \setminus \{0\}$, then $v(a) = v(b)$,

   ✓ For all $x \in D$,
$$\mathrm{val}(x) = \begin{cases} \infty & \text{if } x = 0, \\ -v(a) & \text{if } ax \in D_0 \setminus \{0\} \end{cases}.$$

Let (K^*, D^*) be a sufficiently resplendent elementary extension of (K, D). By the latter item, there is an isomorphism $D^* \cong K^*/\mathscr{M}^*$. We may define val as the restriction of v^* to D, so that it satisfies $\varphi(\mathrm{val})$. Indeed, if $x + \mathscr{M} \in K/\mathscr{M}$ and $x \notin \mathscr{M}$, we have that $x \neq 0$ and thus either $x + \mathscr{M} \in k \setminus \{0\}$ or $1 + \mathscr{M} = x^{-1}(x + \mathscr{M}) \in k \setminus \{0\}$. Also, if $v(a) < v(b)$ and $ax + \mathscr{M}, bx + \mathscr{M} \in k \setminus \{0\}$, we would have that

$$\begin{aligned} v(bx - ba^{-1}) &= v(b) + v(x - a^{-1}) \\ &= v(b) - v(a) + v(ax - 1) \\ &> v(ax - 1) \\ &\geq \min\{v(ax), 0\} \\ &\geq 0, \end{aligned}$$

so $bx + \mathscr{M} = ba^{-1} + \mathscr{M} = \mathscr{M} = 0$, which is absurd.

□

5.2 Diffeovalued Fields

Definition 5.2.1. Let $(K, \mathcal{O})$ be a valued field with maximal ideal $\mathscr{M}$ and residue field k, and let D be a mock $K/\mathscr{M}$. A *derivation* $\partial : \mathcal{O} \to D$ is an additive map satisfying Leibniz rule: $\partial(xy) = x\,\partial\,y + y\,\partial\,x$ for all $x, y \in \mathcal{O}$.

We say that a field K is a *diffeovalued field* if there is a valuation ring $\mathcal{O}$ of K and a derivation $\partial : \mathcal{O} \to D$, where D is a mock $K/\mathscr{M}$.

In the language $(K, \mathcal{O}, D, \partial)$, the theory of diffeovalued fields is first-order axiomatizable.

Definition 5.2.2. A diffeovalued field $(K, \mathcal{O}, D, \partial)$ is said to be *dense* if every fiber of ∂ is dense in $\mathcal{O}$ with respect to the valuation topology, i.e. if for every $y \in D$ and every pair $z, w \in \mathcal{O}$ there is some $x \in \mathcal{O}$ such that

$$\begin{cases} v(x - z) > v(w), \\ \partial\,x = y. \end{cases}$$

In the latter language, being dense is first-order axiomatizable.

Lemma 5.2.3. *Let $(K, \mathcal{O}, D, \partial)$ be a dense diffeovalued field. Define*

$$\begin{aligned} R &= \{x \in \mathcal{O} : \mathrm{val}(\partial\,x) \geq 0\}, \\ Q &= \{x \in \mathcal{O} : \mathrm{val}(\partial\,x) > 0\}, \\ I &= \{x \in \mathscr{M} : \mathrm{val}(\partial\,x) > 0\}. \end{aligned}$$

1. *R and Q are proper subrings of K.*
2. *I is a proper ideal in R and in Q.*
3. $\mathrm{Frac}(Q) = \mathrm{Frac}(R) = K$.
4. *Q is a local ring having I as maximal ideal.*
5. *$I \neq 0$.*

Proof. 1. Q is a subring of R and R is a subring of $\mathcal{O}$ by item 3 of Corollary 5.1.3. Denseness implies that $\mathcal{O}$ is a proper subring of K. Otherwise, the valuation would be trivial and thus the valuation topology would be discrete. Hence all fibers of ∂ would be equal to $\mathcal{O}$, so D must be a singleton. But D extends $k \cong K$.

2. I is a subring of Q, therefore of R. Since $1 \in Q$, We have that $I \subseteq QI \subseteq RI \subseteq I$. Indeed, if $r \in R$ and $i \in I$, it follows that

$$\begin{aligned} v(ri) &= v(r) + v(i) \\ &\geq v(i) \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{val}(\partial(ri)) &= \mathrm{val}(r\,\partial\, i + i\,\partial\, r) \\ &= \mathrm{val}(i\,\partial\, r) \\ &= \begin{cases} \infty \\ v(i) + \mathrm{val}(\partial\, r) \end{cases} \\ &\geq \begin{cases} \infty \\ v(i) \end{cases} \\ &> 0. \end{aligned}$$

Properness follows because $1 \notin I$.

3. The result follows from $\mathrm{Frac}(Q) \subseteq \mathrm{Frac}(R) \subseteq K \subseteq \mathrm{Frac}(Q)$. Let $x \in K$. We may assume that $x \in \mathcal{O}$ by replacing x with x^{-1}. If $x \in Q$, we are done. Otherwise, $\mathrm{val}(\partial\, x) \leq 0$. Use density to find some $y \in \mathcal{O} \setminus \{0\}$ such that $v(y) > -\mathrm{val}(\partial\, x)$ and $\partial\, y = 0$. Then $xy \in \mathcal{O}$ and

$$\begin{aligned} \mathrm{val}(\partial(xy)) &= \mathrm{val}(x\,\partial\, y + y\,\partial\, x) \\ &= \mathrm{val}(x\,\partial\, y) \\ &= \mathrm{val}(0) \\ &= \infty \\ &> 0. \end{aligned}$$

Since $\mathrm{val}(\partial\, y) = \infty > 0$, we have that $y \in Q$ and the result follows.

4. We may show that any element $x \in Q \setminus I$ is invertible in Q. Any such element satisfies that $v(x) = 0$ and $\partial\, x = 0$, so $\partial\, x^{-1} = -x^{-2}\,\partial\, x = 0$ and thus $x^{-1} \in Q$.

5. If $I = 0$ then $Q \cong Q/I$ is a field, so $Q = \mathrm{Frac}(Q) = K$, contradicting properness of Q.

□

Lemma 5.2.4. *Let $(K, \mathcal{O}, D, \partial)$ be a dense diffeovalued field with residue field k. Put $D_0 = \mathrm{im}(k \hookrightarrow D)$, and let $\mathrm{res}' : D_0 \to k$ be the inverse of the embedding $k \hookrightarrow D$. Let R, Q and I be as before. Then*

1. *$\mathrm{res}'(x\,\partial\, y) = \mathrm{res}(x)\,\mathrm{res}'(\partial\, y)$ for any $x \in \mathcal{O}$ and $y \in R$.*

2. *$Q/I \cong k$ as rings via res.*

3. *R and I are Q-submodules of K.*

4. *Define* $\widehat{\mathrm{res}} : R \to k^2$ *to be* $\widehat{\mathrm{res}}(x) = (\mathrm{res}(x), \mathrm{res}'(\partial\, x))$. *Then* $\widehat{\mathrm{res}}$ *induces an isomorphism of* Q*-modules between* R/I *and* k^2. *In particular* R/I *is a semi-simple* Q*-module of length 2.*

Proof. 1. Let $\phi : k \to D$ be the embedding. It is an $\mathcal{O}$-linear ring morphism. Then $\mathrm{res}'(x\,\partial\, y) = \mathrm{res}(x)\,\mathrm{res}'(\partial\, y)$ if and only if $x\,\partial\, y = \phi(\mathrm{res}(x)\,\mathrm{res}'(\partial\, y))$. Indeed, for any $x \in \mathcal{O}$,

$$\begin{aligned}\phi(\mathrm{res}(x)) &= \phi(x + \mathscr{M})\\ &= \phi(x(1 + \mathscr{M}))\\ &= x \cdot \phi(1 + \mathscr{M})\\ &= x \cdot 1\\ &= x\end{aligned}$$

and

$$\begin{aligned}\phi(\mathrm{res}(x)\,\mathrm{res}'(\partial\, y)) &= \phi(\mathrm{res}(x))\phi(\mathrm{res}'(\partial\, y))\\ &= x \cdot \partial\, y.\end{aligned}$$

2. Since $I = Q \cap \mathscr{M}$, it is clear that $I = \ker(\mathrm{res} \restriction Q)$. We must show that $\mathrm{res} \restriction Q$ is surjective. Let $y \in \mathcal{O}$. Use denseness to find some $x \in \mathcal{O}$ such that $v(x - y) > 0$ and $\partial\, x = 0$. Then $x \in Q$ and $x + \mathscr{M} = y + \mathscr{M}$.

3. Q is a subring of R and I is an ideal of Q.

4. Note that $\ker(\widehat{\mathrm{res}}) = \{x \in R : v(x) > 0, \mathrm{val}(\partial\, x) > 0\} = I$, because $\mathrm{val}(\partial\, x) > 0$ if and only if $\partial\, x = 0$, since D is a mock $K/\mathscr{M}$. We now show that $\widehat{\mathrm{res}}$ is surjective. Given any pair $(y + \mathscr{M}, y' + \mathscr{M}) \in k^2$, use density to find a pair $x, x' \in \mathcal{O}$ such that $v(x - y) > 0, \partial\, x = 0, \partial(x') = \phi(y' + \mathscr{M})$. Then $x, x' \in R$, $x + \mathscr{M} = y + \mathscr{M}$ and $\mathrm{res}'(\partial(x')) = y' + \mathscr{M}$. Finally, we prove Q-linearity. Let $x \in Q$ and $y \in R$. Then $xy \in \mathcal{O}$ and $\mathrm{res}(xy) = \mathrm{res}(x)\,\mathrm{res}(y)$. Also, since $\partial\, x = 0$,

$$\begin{aligned}\mathrm{res}'(\partial(xy)) &= \mathrm{res}'(x\,\partial\, y + y\,\partial\, x)\\ &= \mathrm{res}'(x\,\partial\, y)\\ &= \mathrm{res}(x)\,\mathrm{res}'(\partial\, y)\end{aligned}$$

as wanted.

□

We have defined all the ingredients necessary to state (without proof) the following Theorem.

Theorem 5.2.5 (Cf. Theorem 8.24 of [Johnson, 2020a]). *Let* $(K, \mathcal{O}, D, \partial)$ *be a dense diffeovalued field. Suppose* Q *is a* k_0*-algebra for some infinite field* k_0*. There is a* k_0*-linear 2-inflator* $\varsigma : \mathrm{Dir}_K(K) \to \mathrm{Dir}_k(k^2)$ *given by*

$$\varsigma(V) = \{(\widehat{\mathrm{res}}(x_1), \ldots, \widehat{\mathrm{res}}(x_n)) : (x_1, \ldots, x_n) \in V \cap R^n\},$$

where $V \in \mathrm{Sub}_K(K^n)$ *and* $\widehat{\mathrm{res}}$ *is as in Lemma 5.2.4.*

$$
\begin{tikzcd}
 & K \arrow[r, "v"] & \Gamma \cup \{\infty\} \\
 & \mathcal{O} \arrow[u, hook] \arrow[r, "\partial"] \arrow[ddl, "\mathrm{res}"'] & D \arrow[u, "\mathrm{val}"'] \\
 & R \arrow[u, hook] \arrow[r, "\partial"] & D_0 \arrow[u, hook] \arrow[dll, "\mathrm{res}'"] \\
k & &
\end{tikzcd}
$$

Figure 1: A Diffeo-valued field and its intrinsic structure.

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