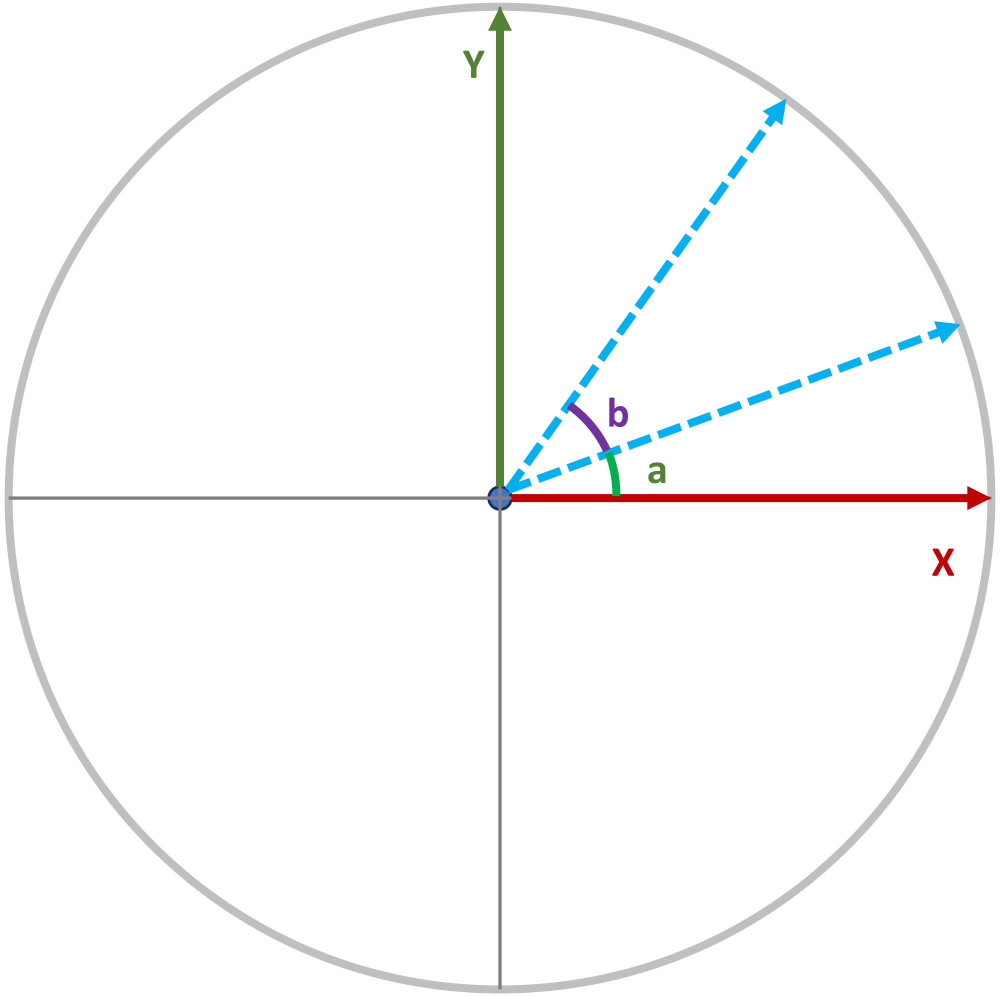
Proof and explanation for:

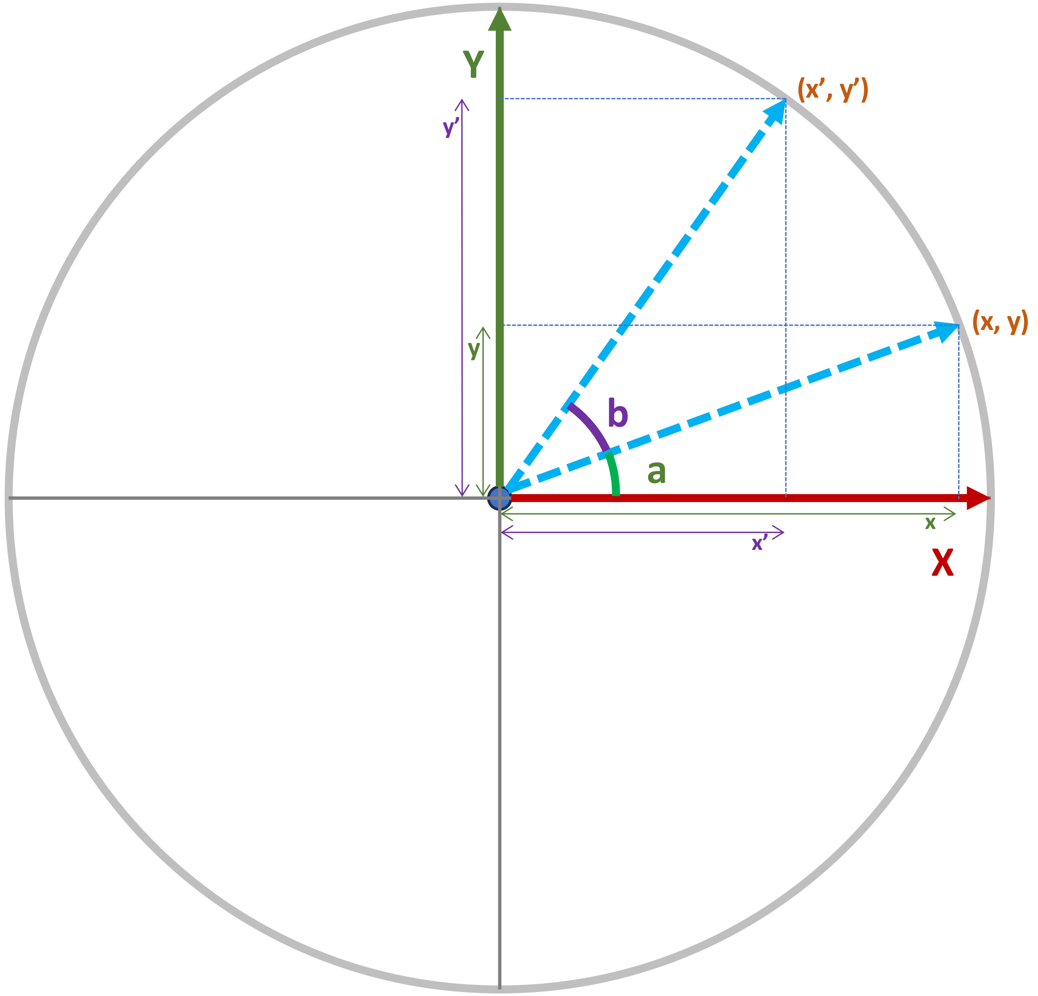
$$\cos(a + b) = \cos(a) . \cos(b) - \sin(a) . \sin(b)$$

and

$$\sin(a + b) = \sin(a) . \cos(b) + \cos(a) . \sin(b)$$



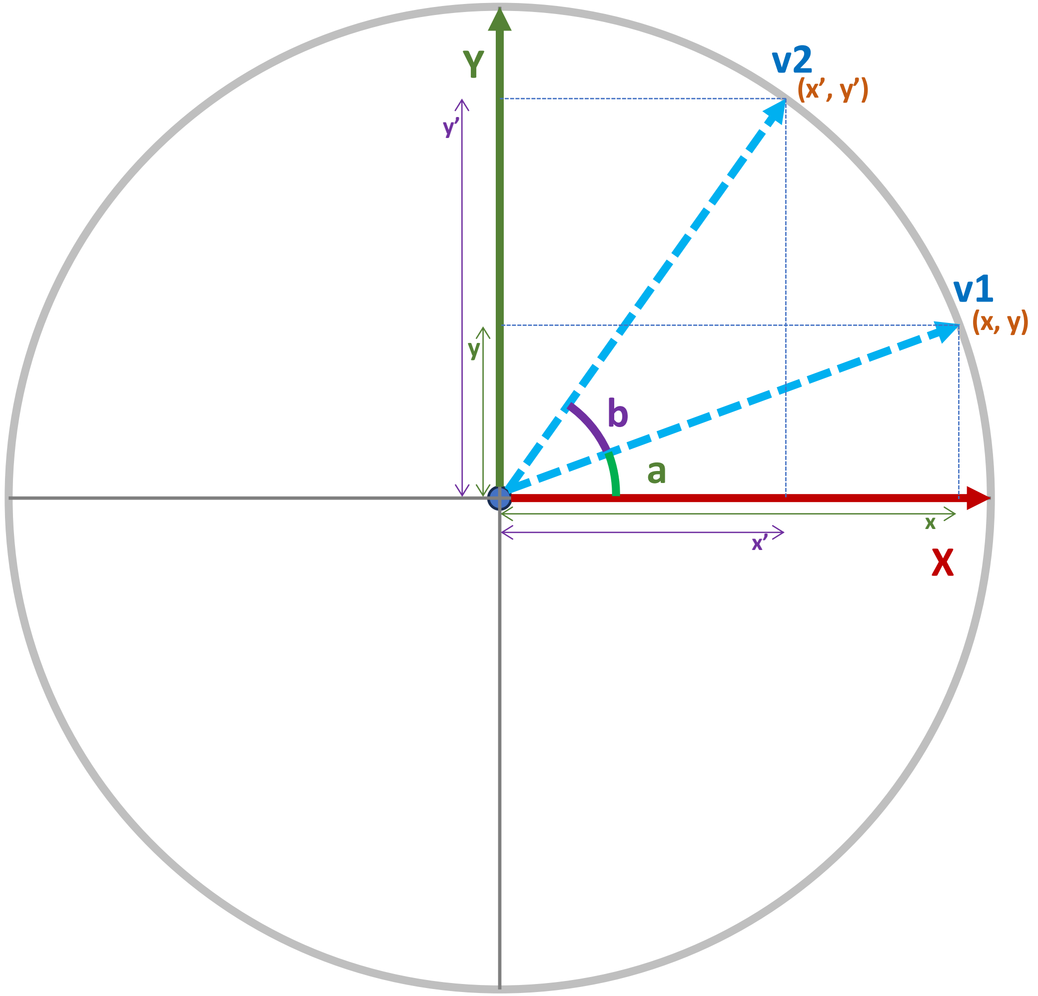
Here we have angles ‘a’ and ‘b’



And here we have the coordinates (x,y) for angle ‘a’

And coordinates (x’,y’) for angle ‘b’

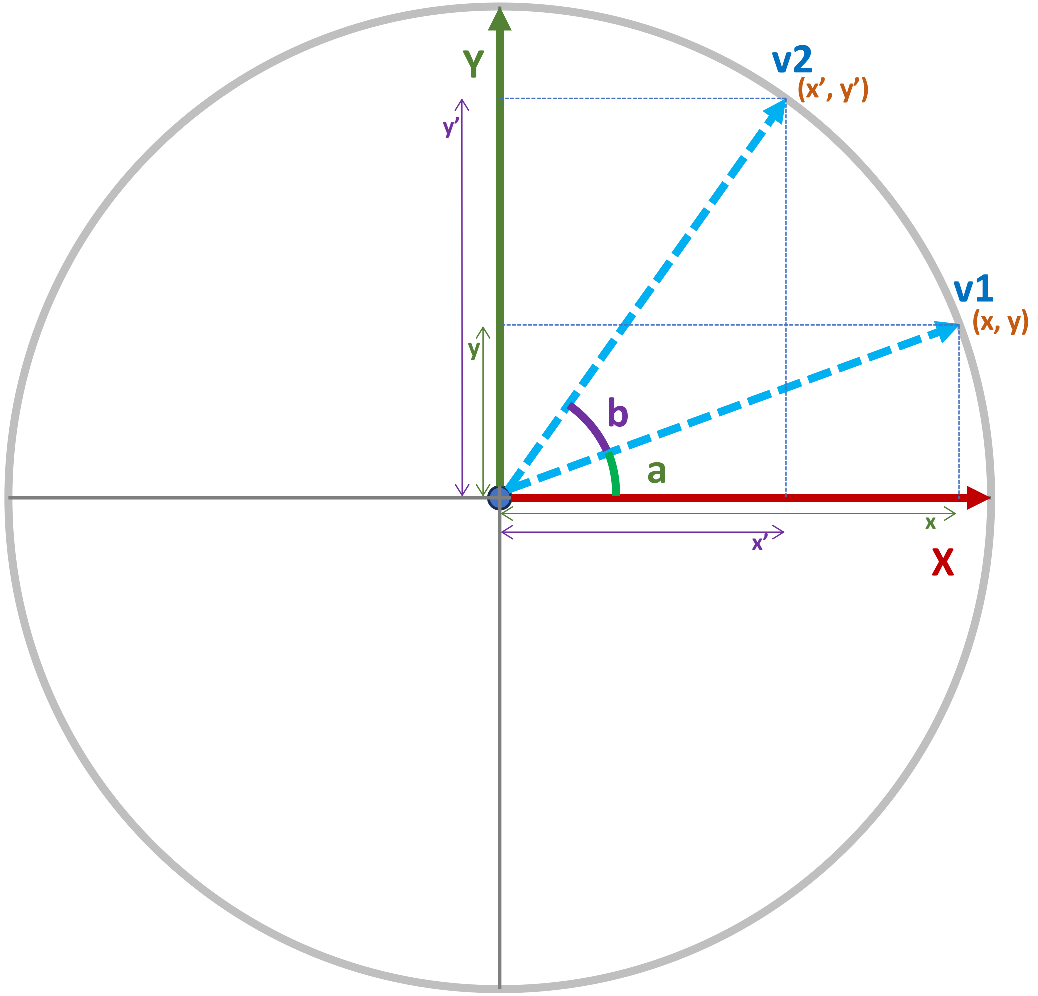
Note we are also highlighting the length of x, x', y, and y' against their respective axis



(x,y) can also be interpreted as a vector 'v1'.
Vector 'v1' has its origin at (0,0) and end at (x,y)

Similarly, we can draw the same assumptions with (x',y'), making it vector 'v2'

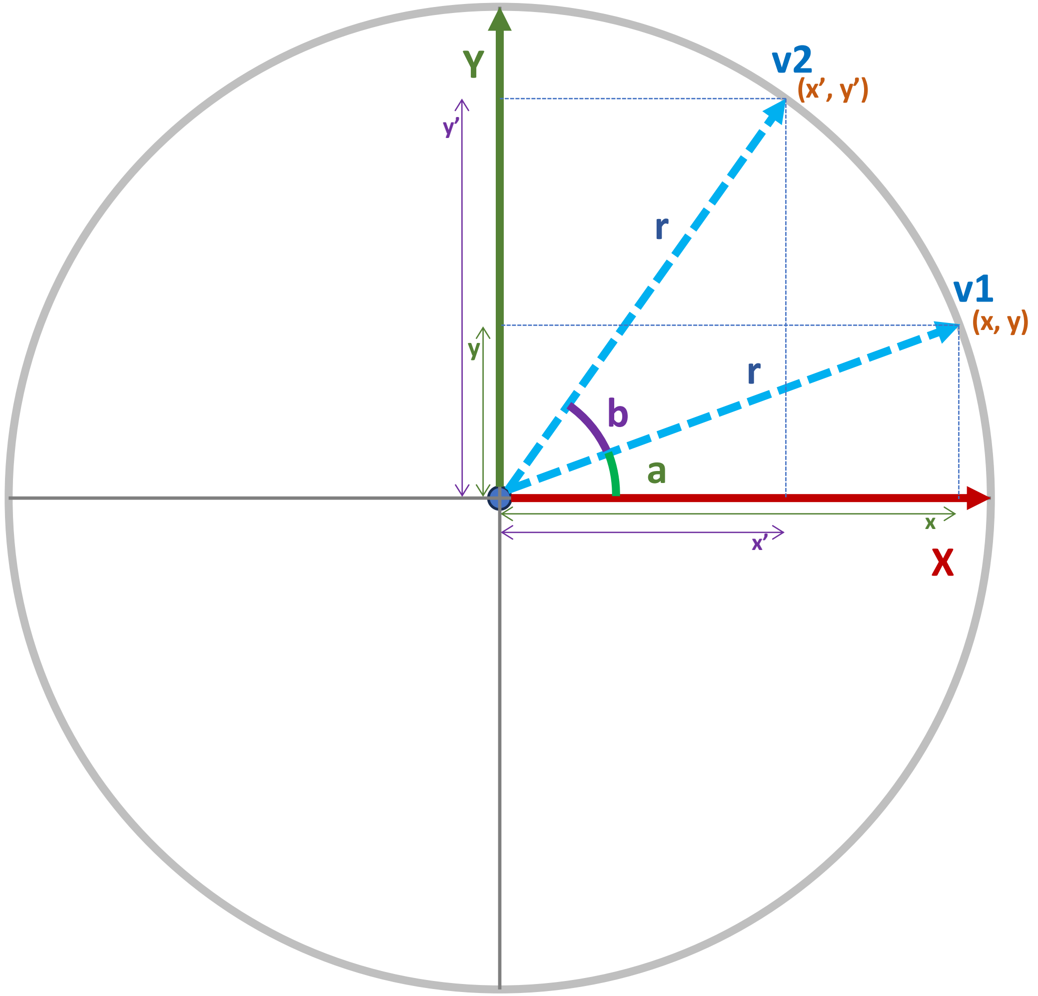
And they all have the same length (magnitude).



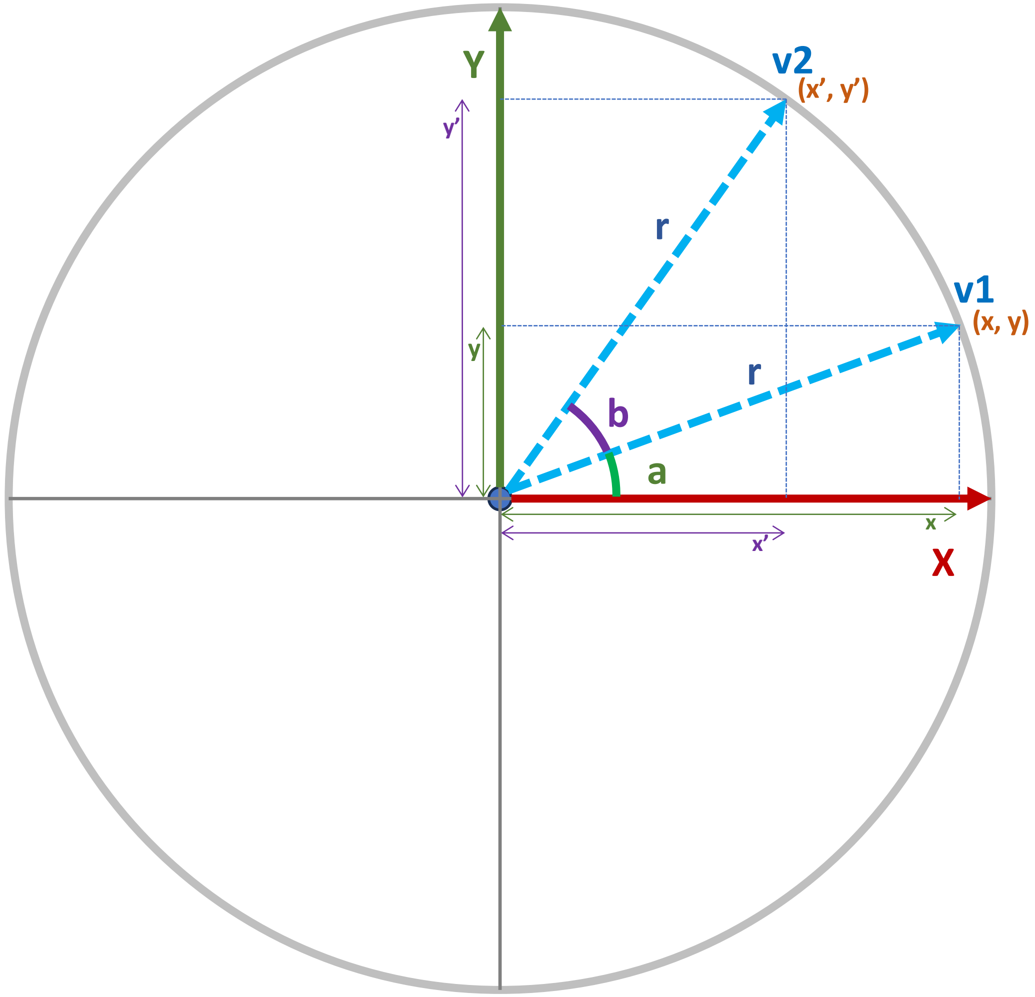
'v1' is a vector in which a rotation of 'a' degrees was applied

'v2' can be thought of 'v1' but applying an additional 'b' degrees of rotation

Note that this transforms the coordinates of (x,y) to (x',y')



And since they are vectors of same magnitude, their radius 'r' is the same



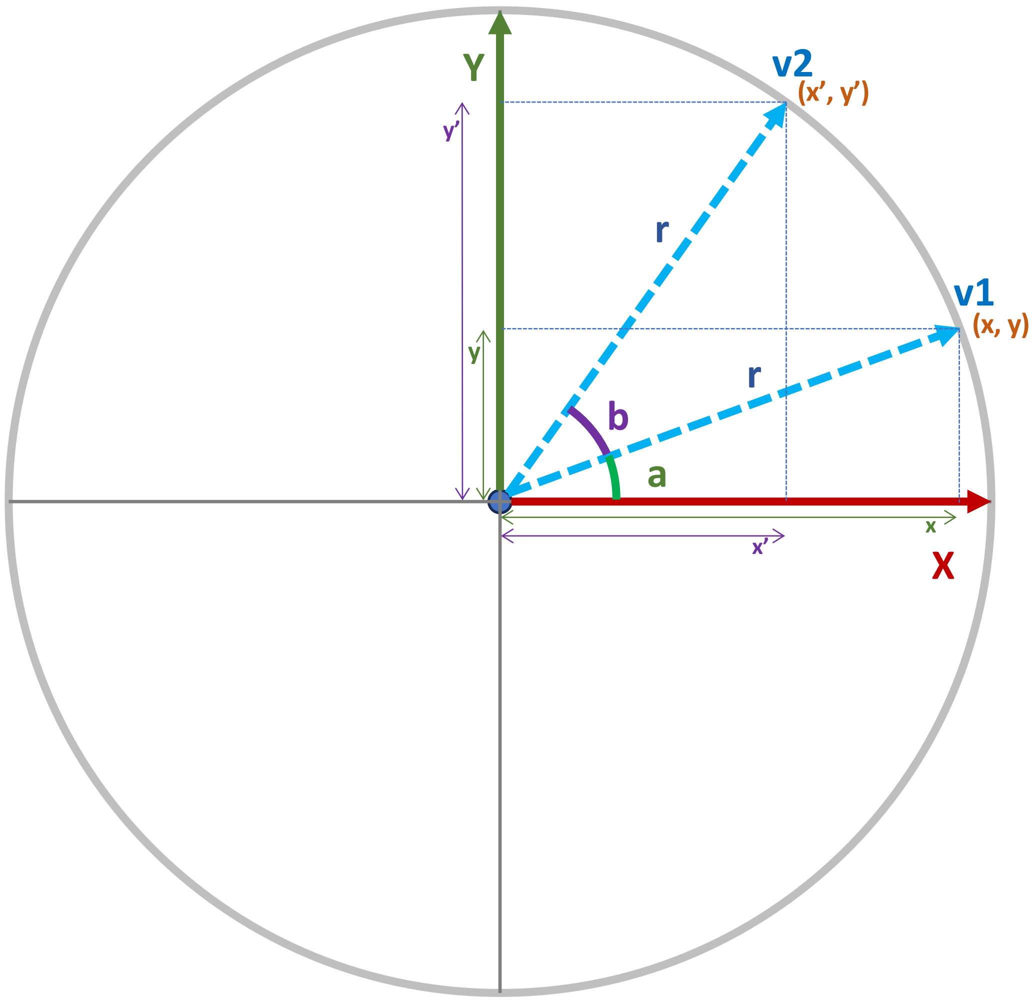
Considering the angle 'a' and the radius 'r' we have:

cos(a) = adjacent / hypotenuse
-> cos(a) = x / y

sin(a) = opposite / hypotenuse
-> sin(a) = y / r

And...

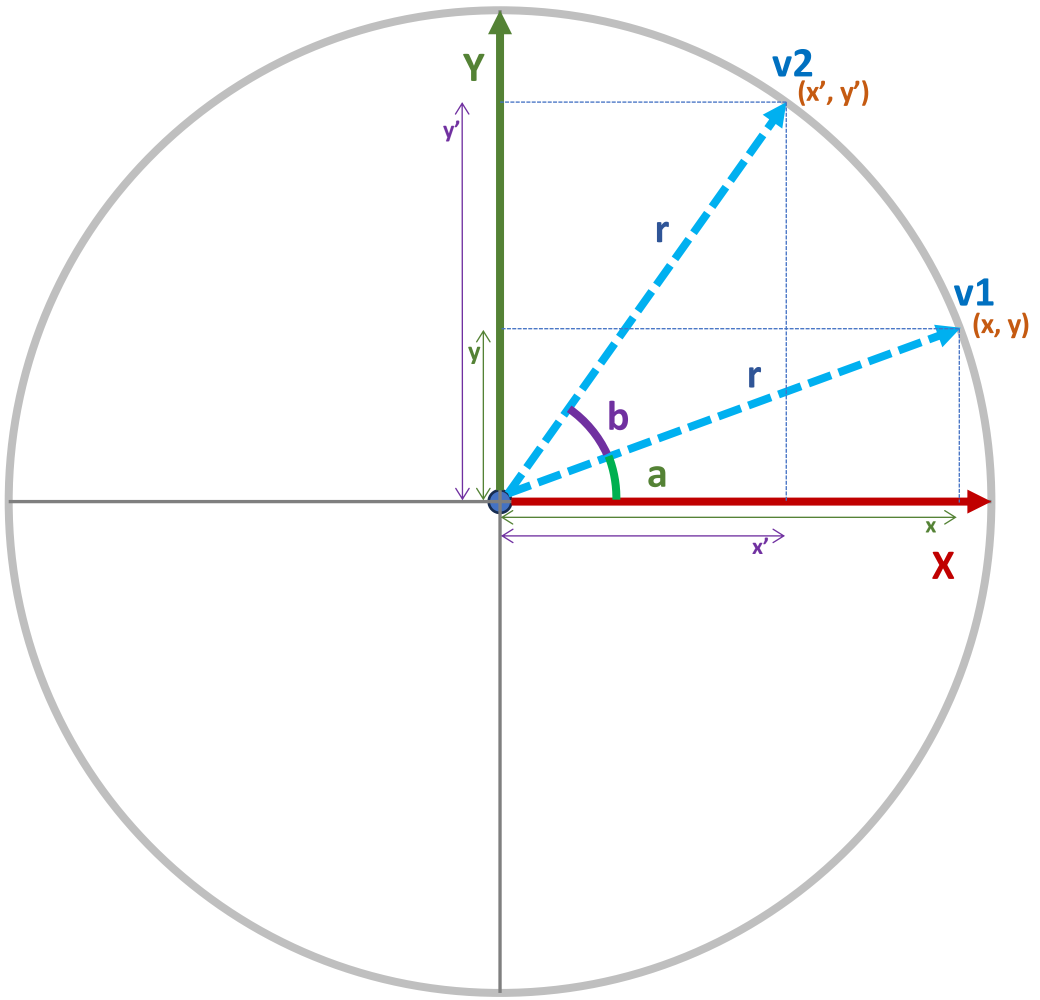
cos(a) = x / r -> x = cos(a) * r
sin(a) = y / r -> y = sin(a) * r



And then we apply the same procedure to find what is the result of applying a rotation of 'a' degrees and then an additional rotation of 'b' degrees

$\cos(a + b) = x' / r \quad \rightarrow \quad x' = r * \cos(a + b)$

$\sin(a + b) = y' / r \quad \rightarrow \quad y' = r * \sin(a + b)$



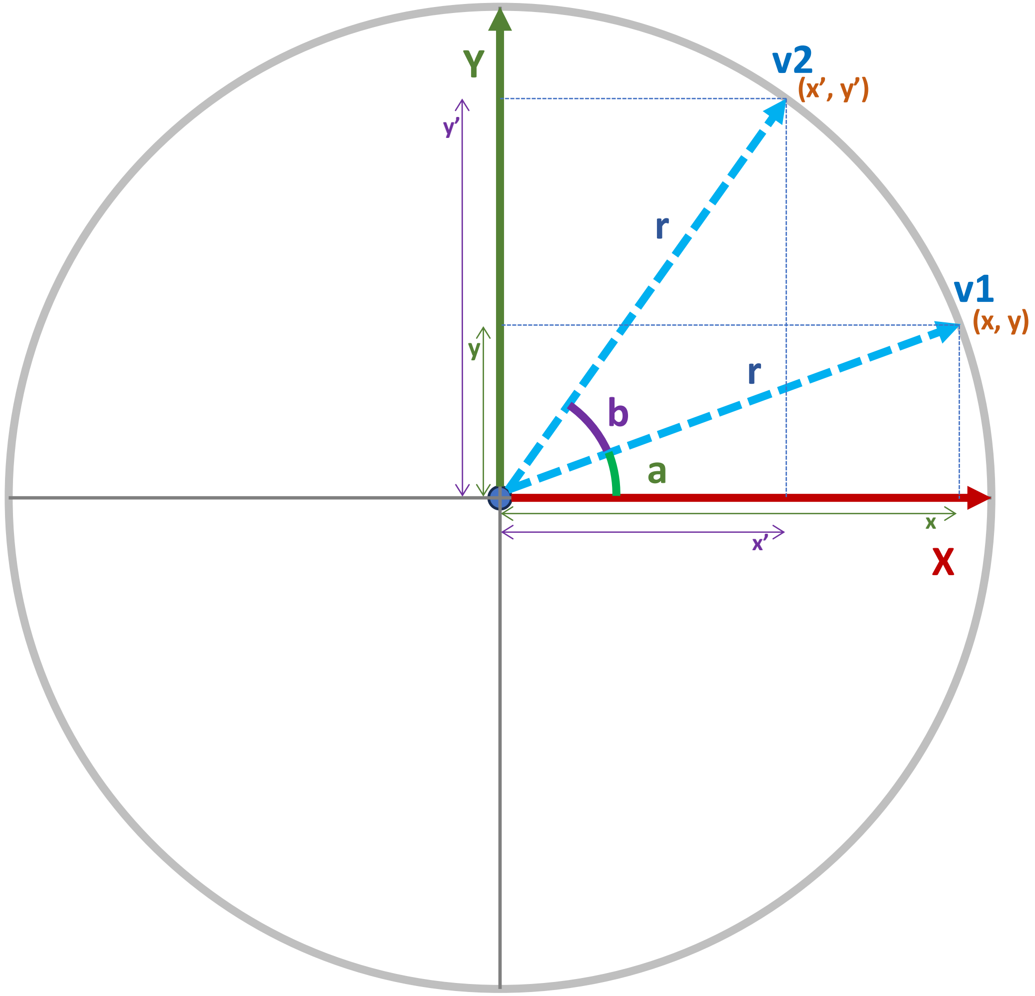
Since we have :

x' = r * cos(a + b)

and

y' = r * sin(a + b)

We will apply the 'angle addition formula for cosine' and 'angle addition formula for sine'



Angle addition formula for cosine:

x' = r * cos(a + b)   ->

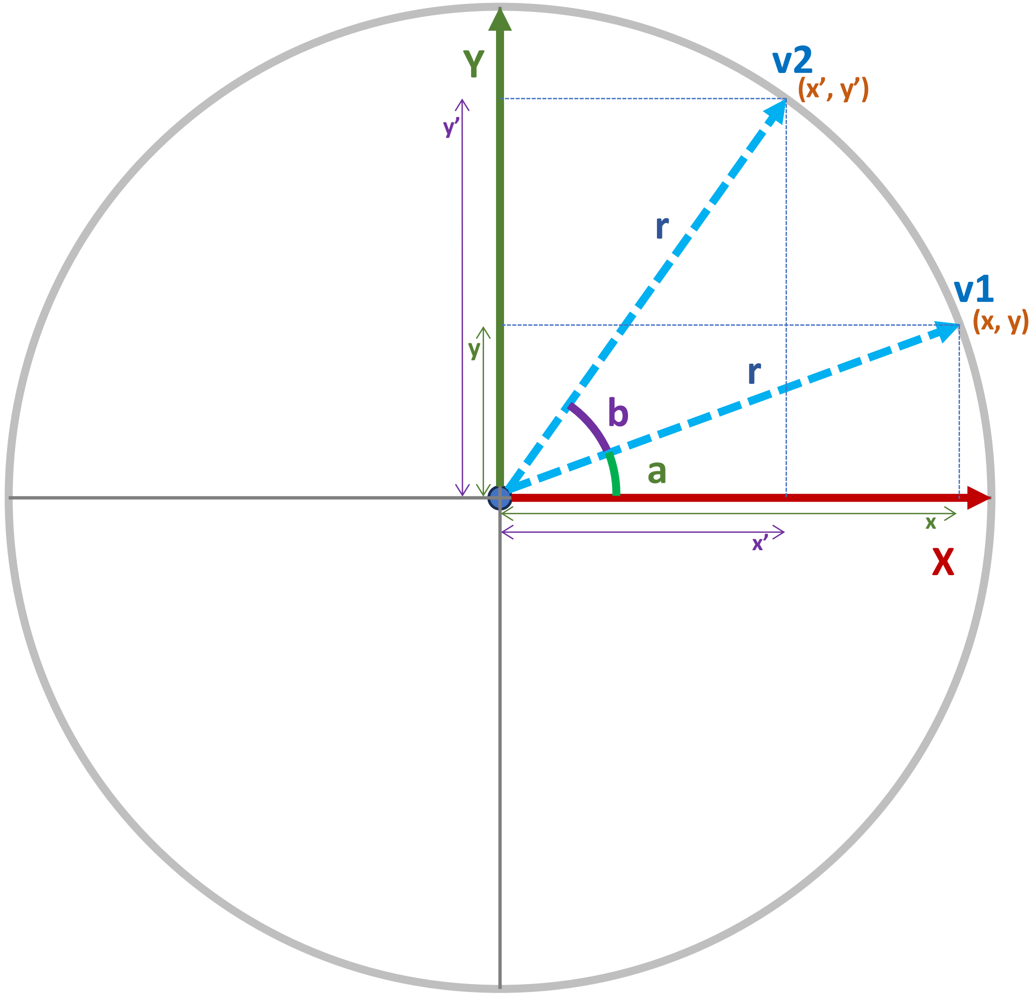
x' = r * (cos(a)*cos(b) - sin(a)*sin(b))

x' = r * cos(a) * cos(b) - r * sin(a) * sin(b)

But since: x = r * cos(a)   and   y = r * sin(a)

We replace these parts in the formula and we will end up with:

x' = x * cos(b) - y * sin(b)



Similarly, we will do the same for the Angle addition formula for sine:

y' = r * sin(a + b) ->

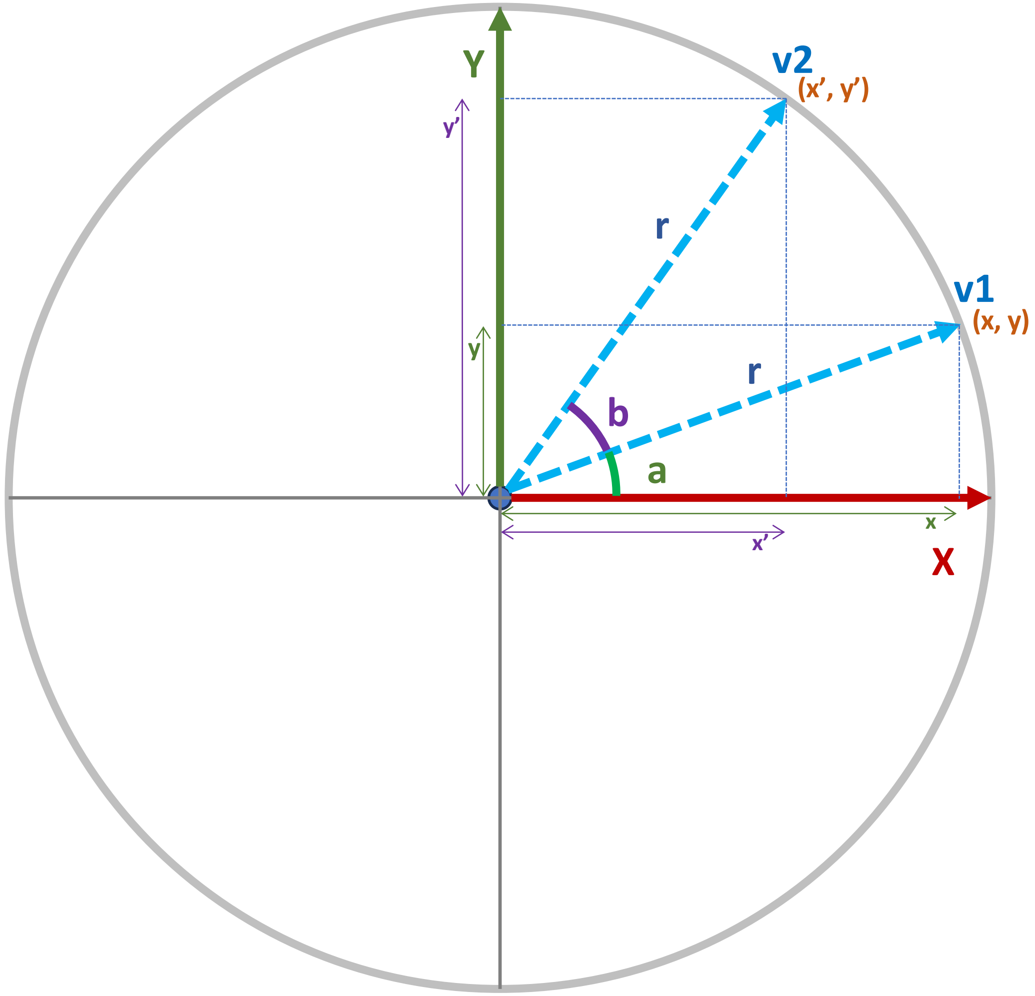
y' = r * (sin(a) * cos(b) + cos(a) * sin(b))

y' = r * sin(a) * cos(b) + r * cos(a) * sin(b)

But since: x = r * cos(a) and y = r * sin(a)

We replace these parts in the formula and we will end up with:

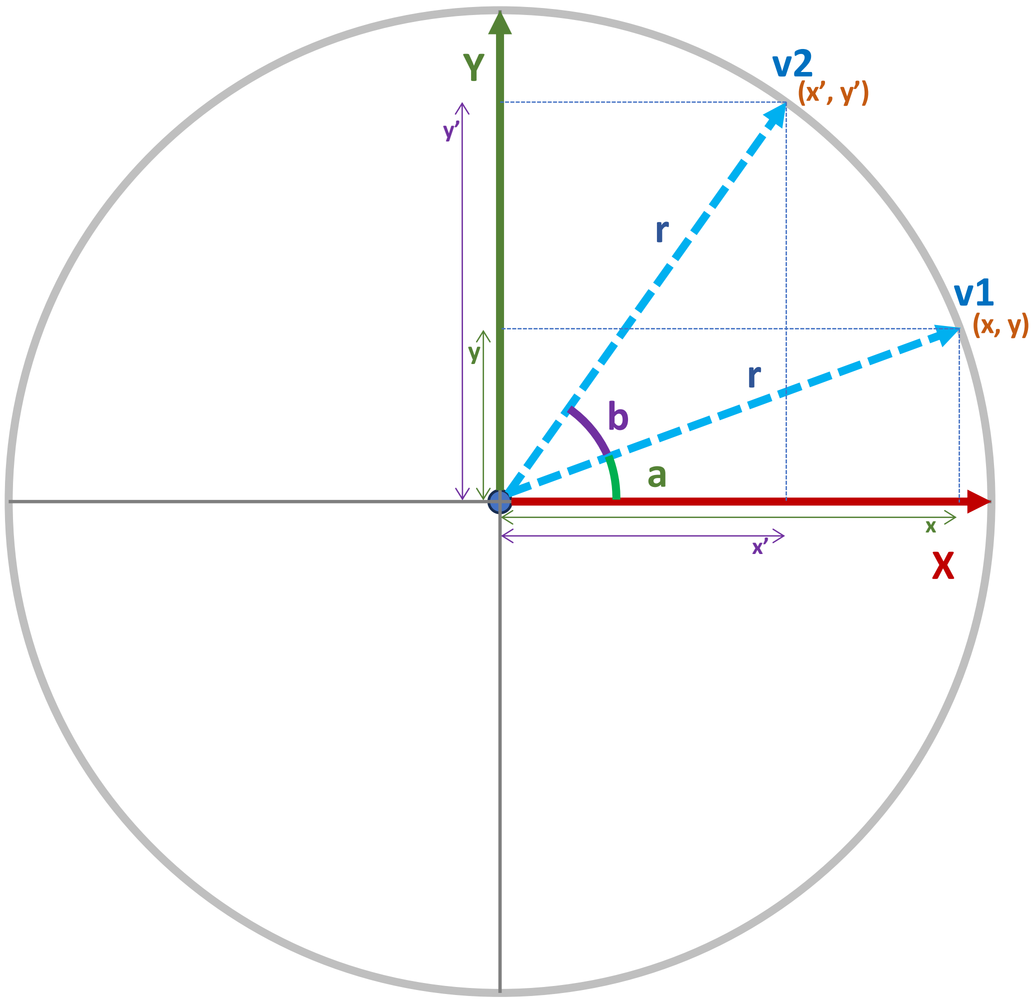
y' = y * cos(b) + x * sin(b)



Therefore:

$$x' = x * \cos(b) \quad - \quad y * \sin(b)$$

$$y' = y * \cos(b) \ + \ x * \sin(b)$$



Now, if we want to rotate it in 3D we follow this:

Z-axis rotation:

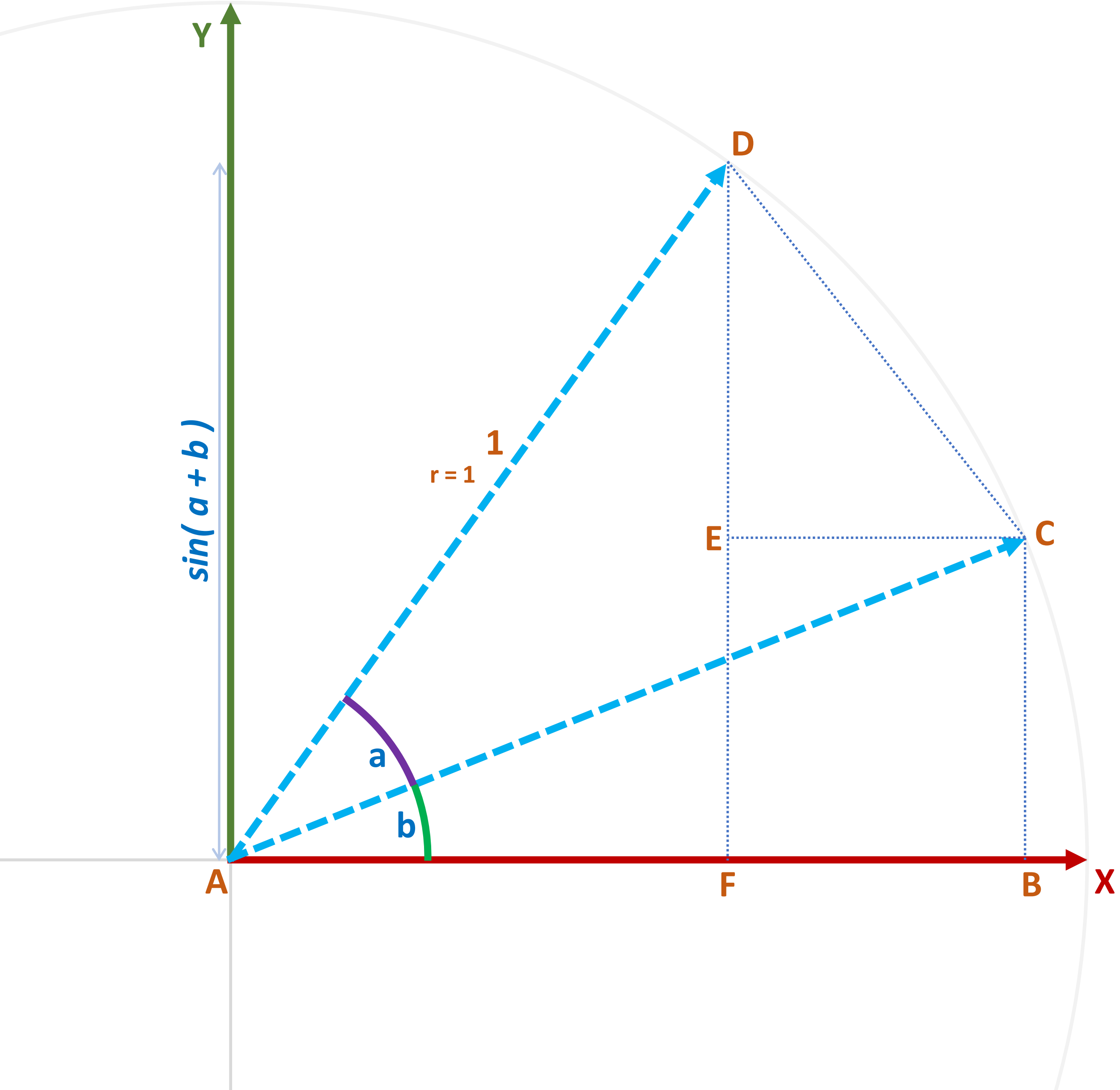
x' = x * cos(b) - y * sin(b)
y' = x * sin(b) + y * cos(b)
z' = *no change*

Y-axis rotation:

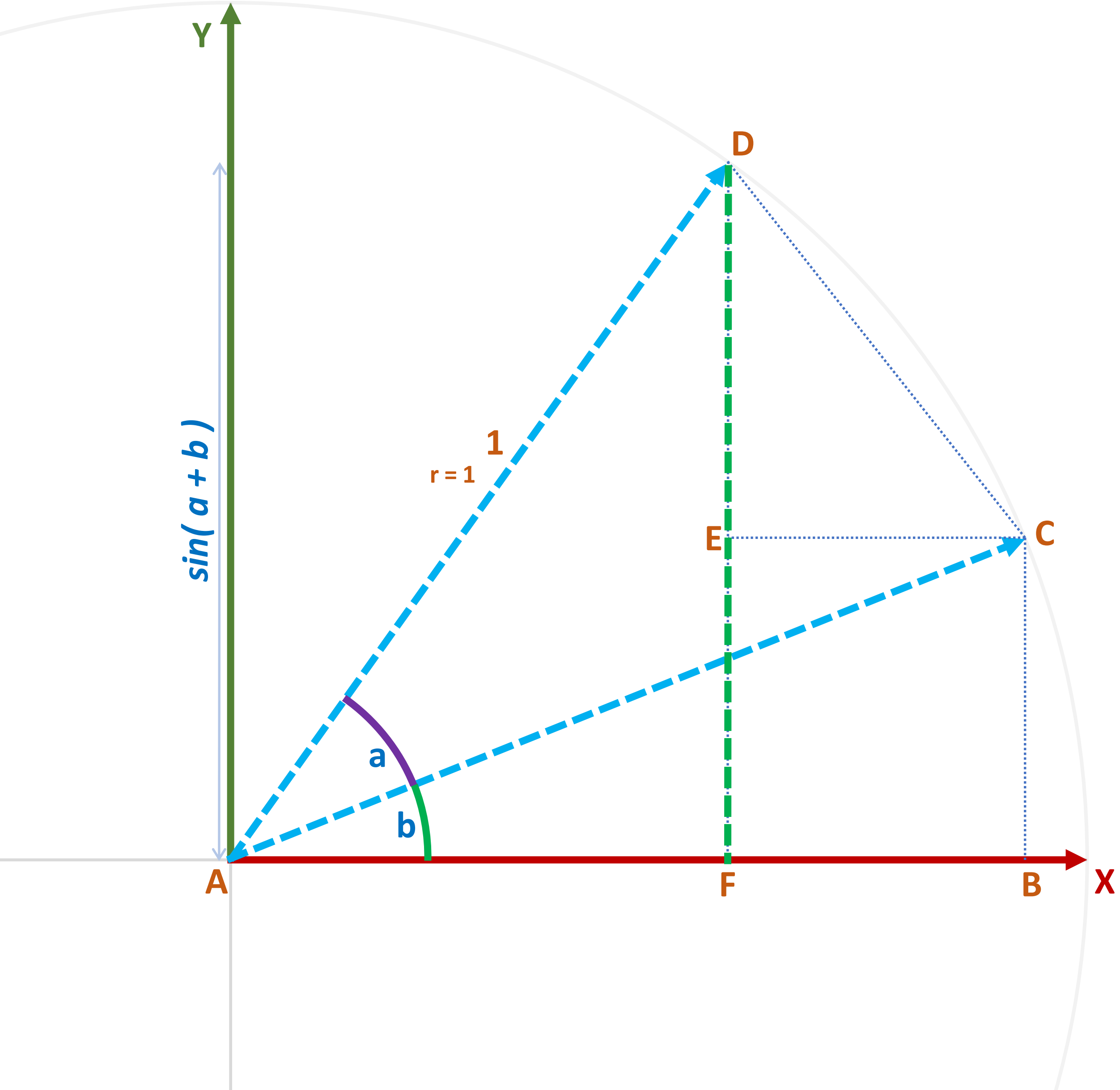
x' = x * cos(b) - z * sin(b)
y' = *no change*
z' = x * sin(b) + z * cos(b)

X-axis rotation:

x' = *no change*
y' = y * cos(b) - z * sin(b)
z' = y * sin(b) + z * cos(b)



Proof that:
sin(a + b) = sin(a) * cos(b) + cos(a) * sin(b)



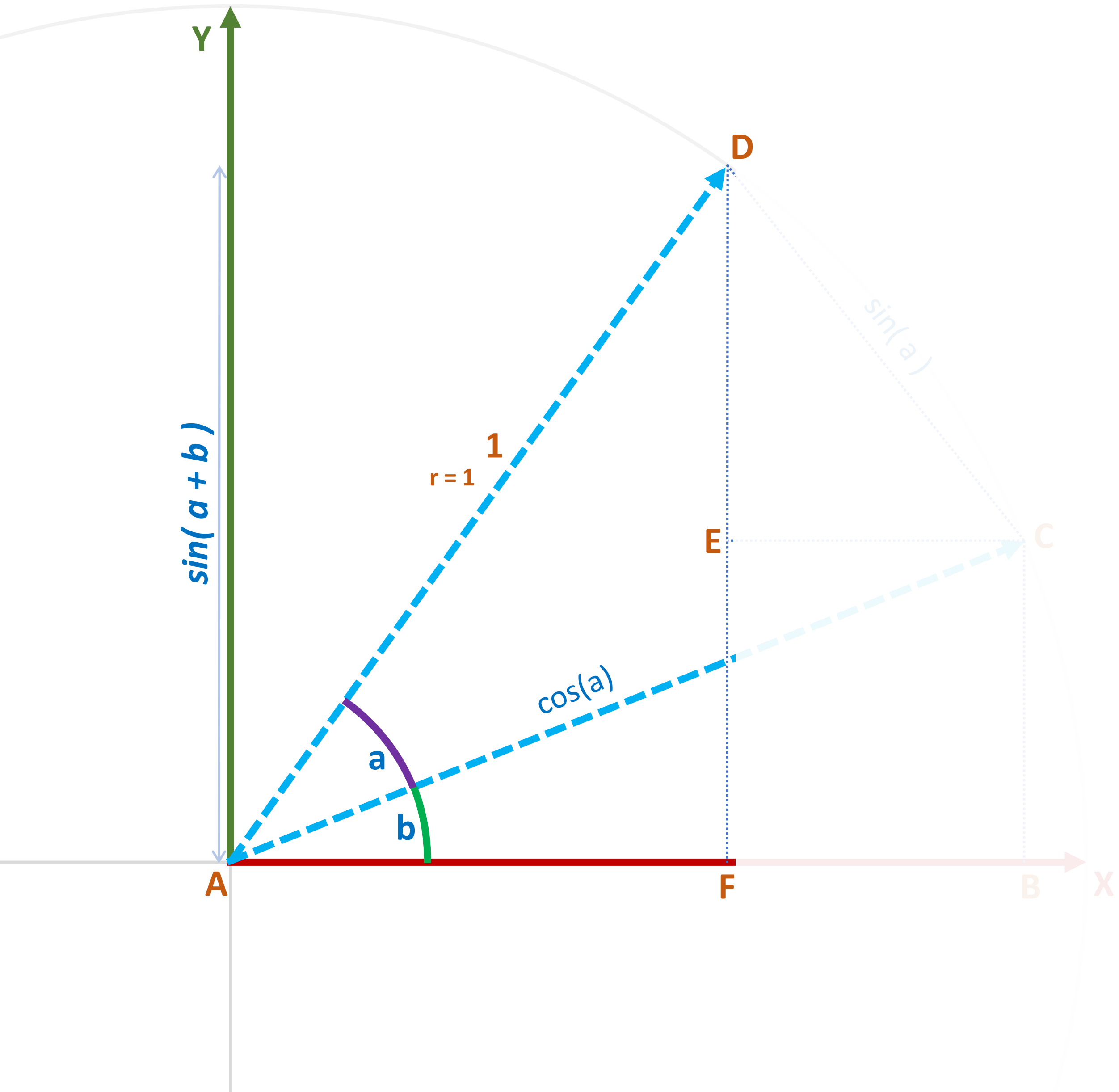
Note that we have the triangles:

A,B,C for angle ‘b’

A,C,D for angle ‘a’

A,F,D for angle (a + b)

and we are going to try to proof that the side FD is the sin(a + b)

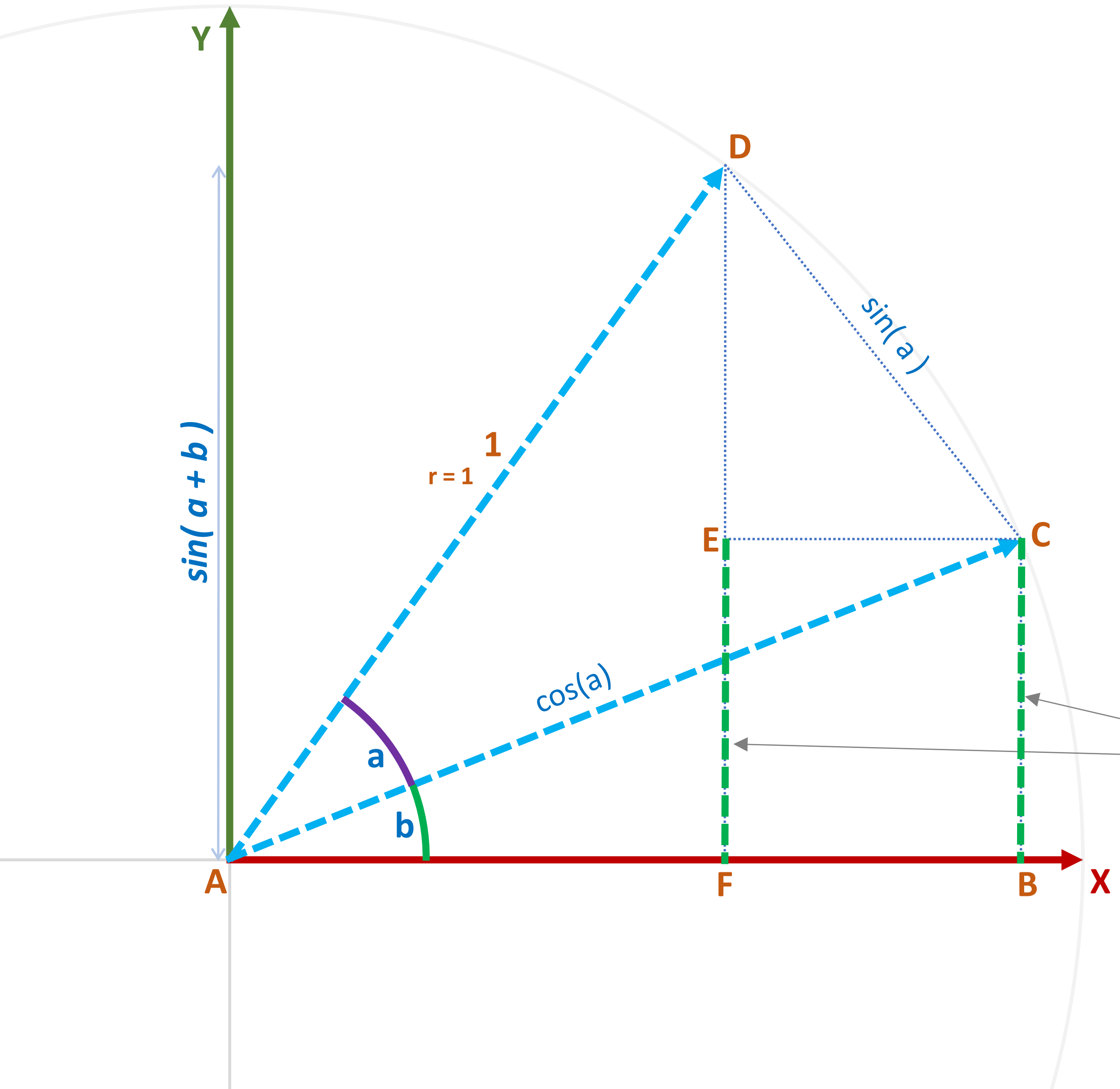


Knowing that:

sin(x) = opposite side / hypotenuse

In our example we have that

$$\sin(a+b) = \frac{DE + EF}{1}$$



Knowing that:

sin(x) = opposite side / hypotenuse

In our example we have that

$$\sin(a+b) = \frac{DE + EF}{1}$$

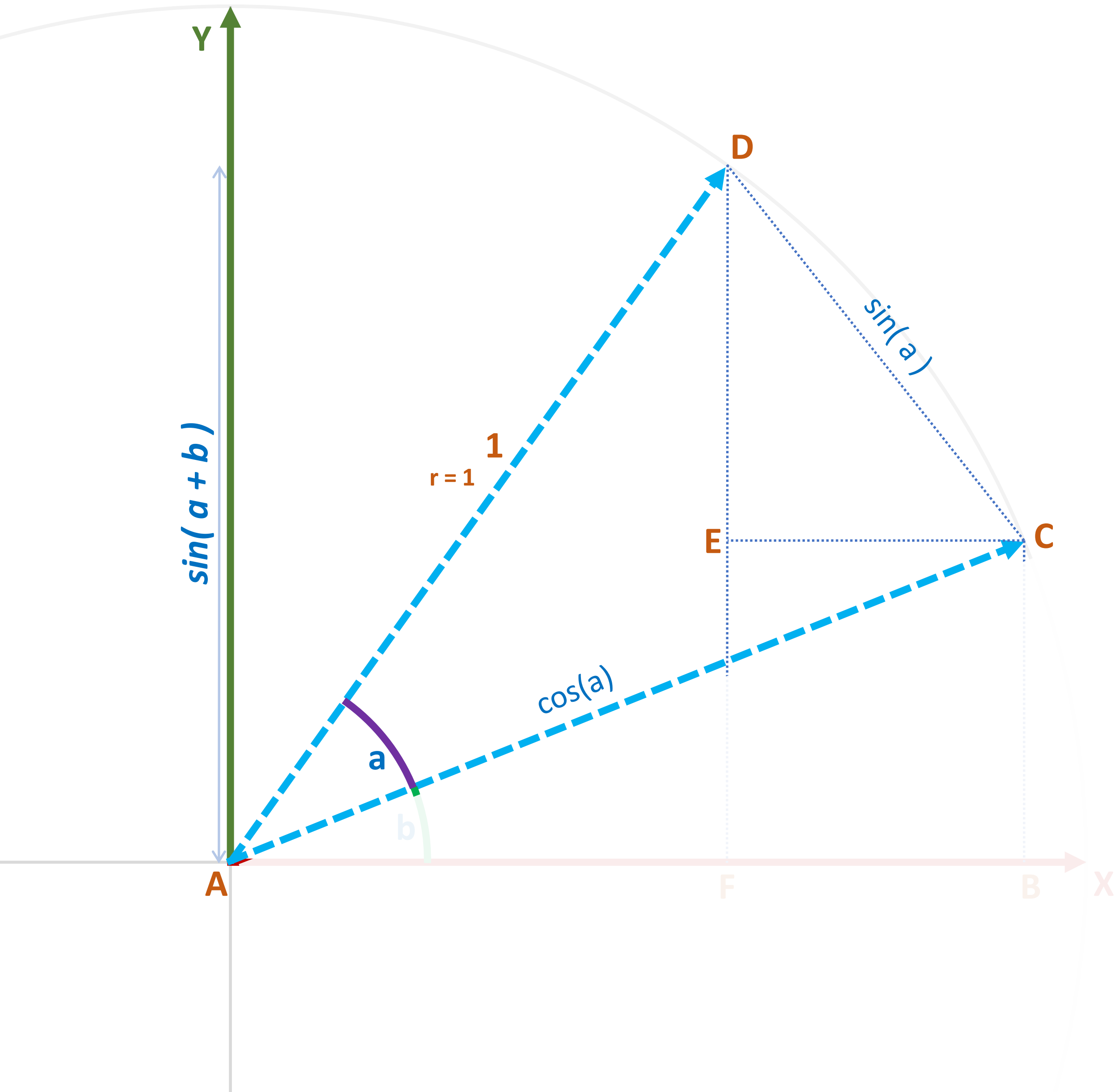
And since the denominator is 1 (r=1) we simplify to:

$$\sin(a+b) = DE + EF$$

And we know that

EF = CB, so:

$$\sin(a+b) = DE + CB$$

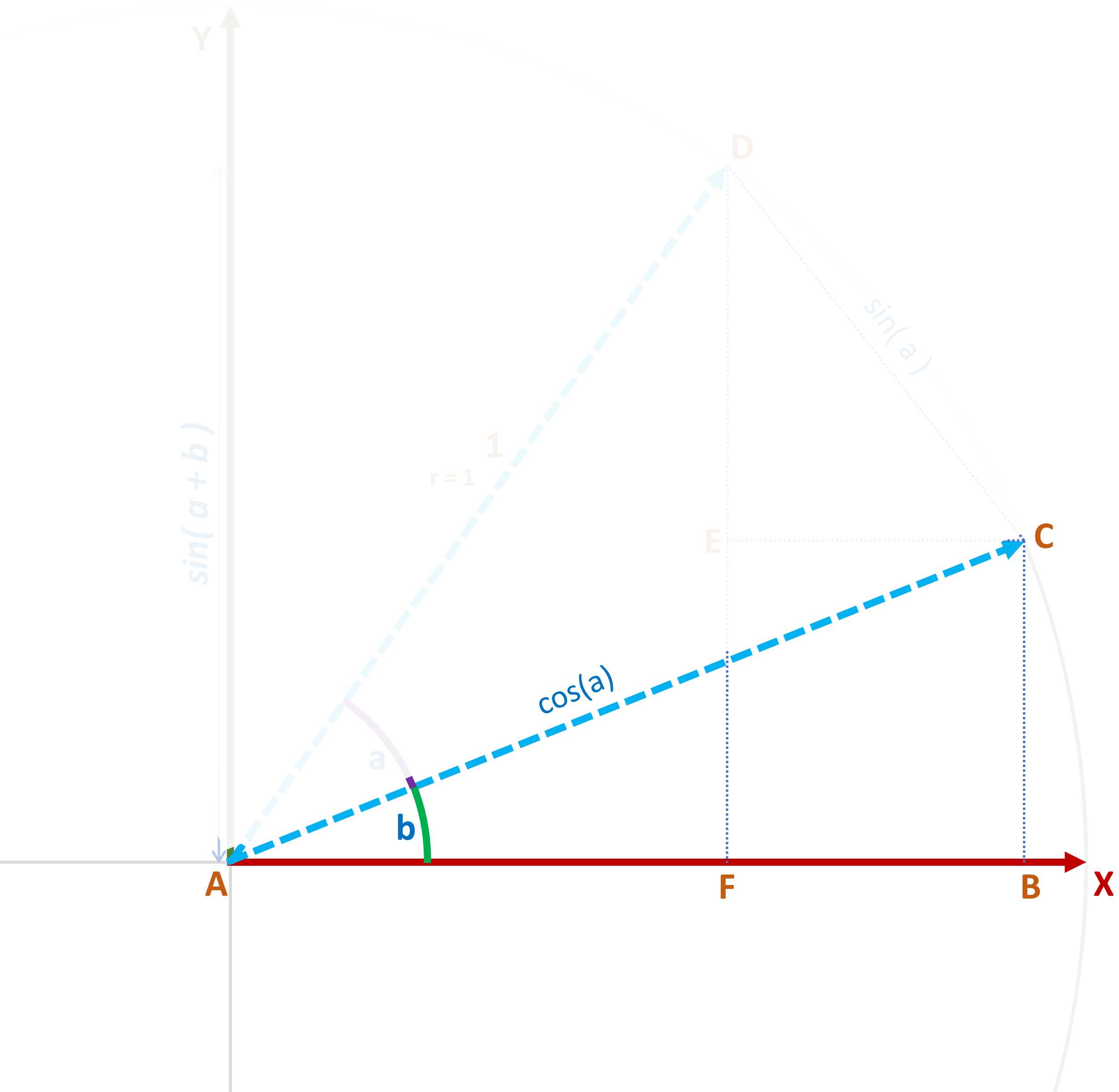


And we also know from trigonometry that

CD = sin(a)

and that

AC = cos(a)



And we also know from trigonometry that

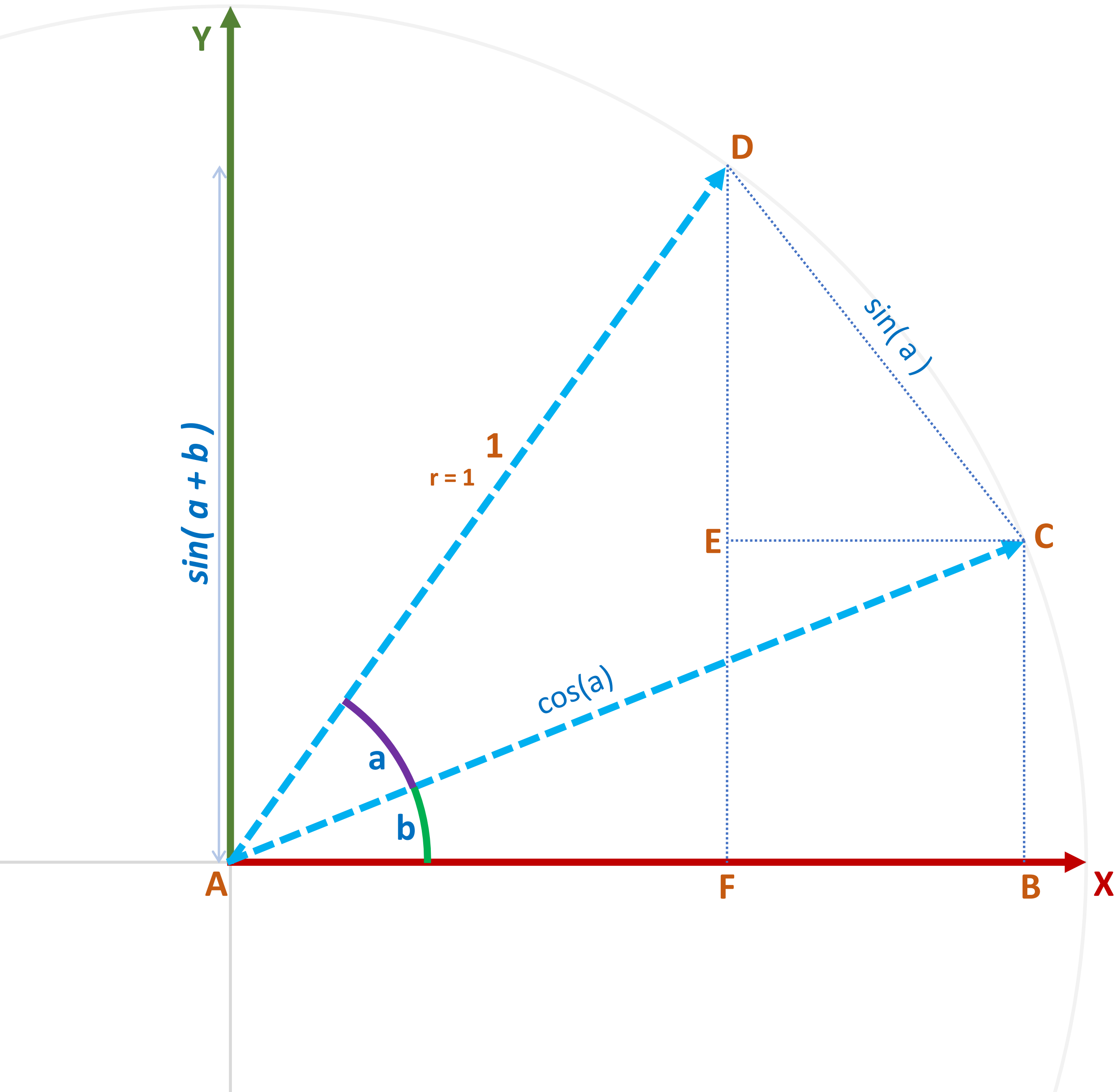
CD = sin(a)

and that

AC = cos(a)

We can also define:

$$\sin(b) = \frac{CB}{\cos(a)}$$



And we also know from trigonometry that

CD = sin(a)

and that

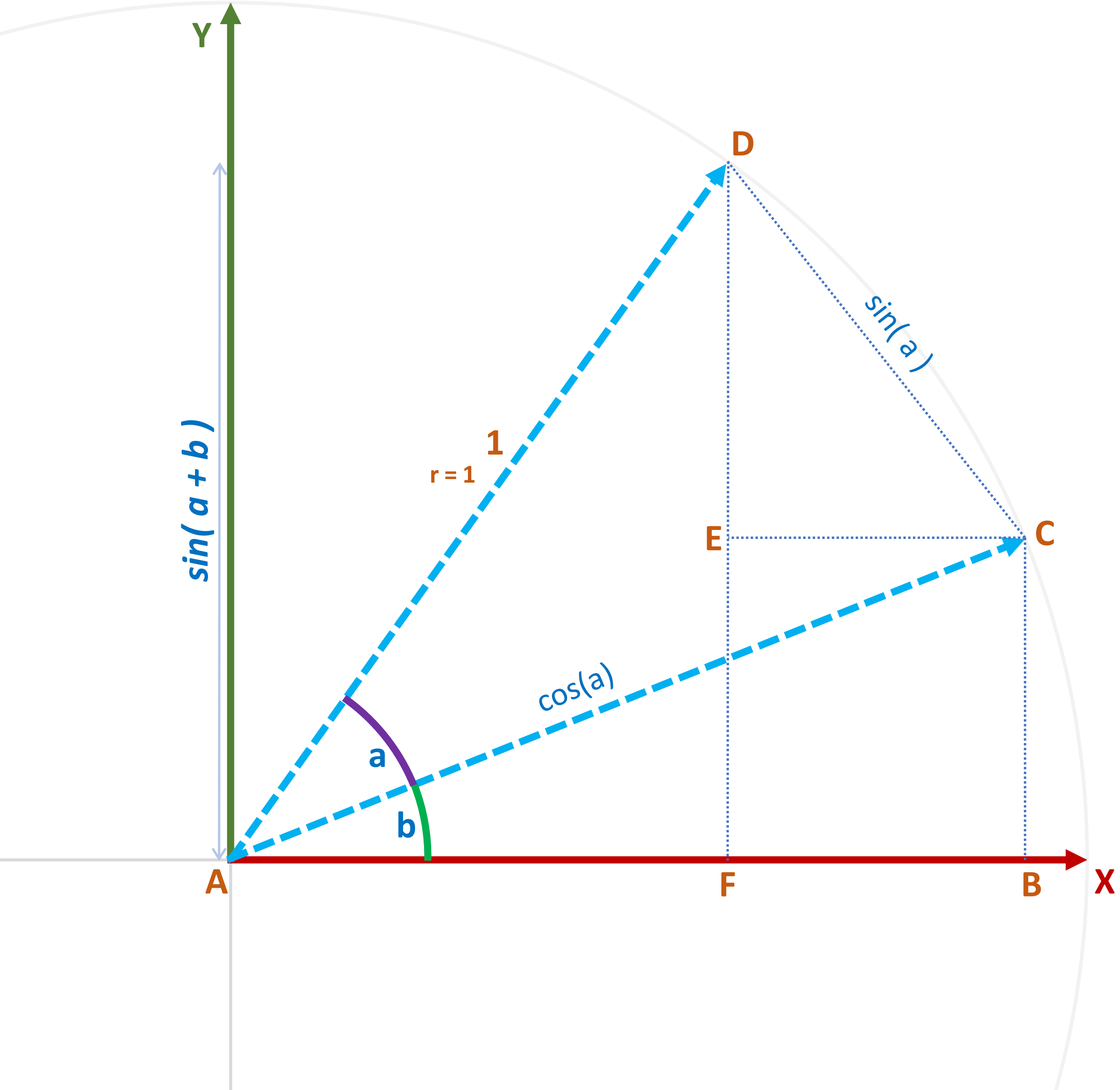
AC = cos(a)

We can also define:

$$\sin(b) = \frac{CB}{\cos(a)}$$

By manipulating the equation, we have:

CB = cos(a) * sin(b)



Now let's recap what we know:

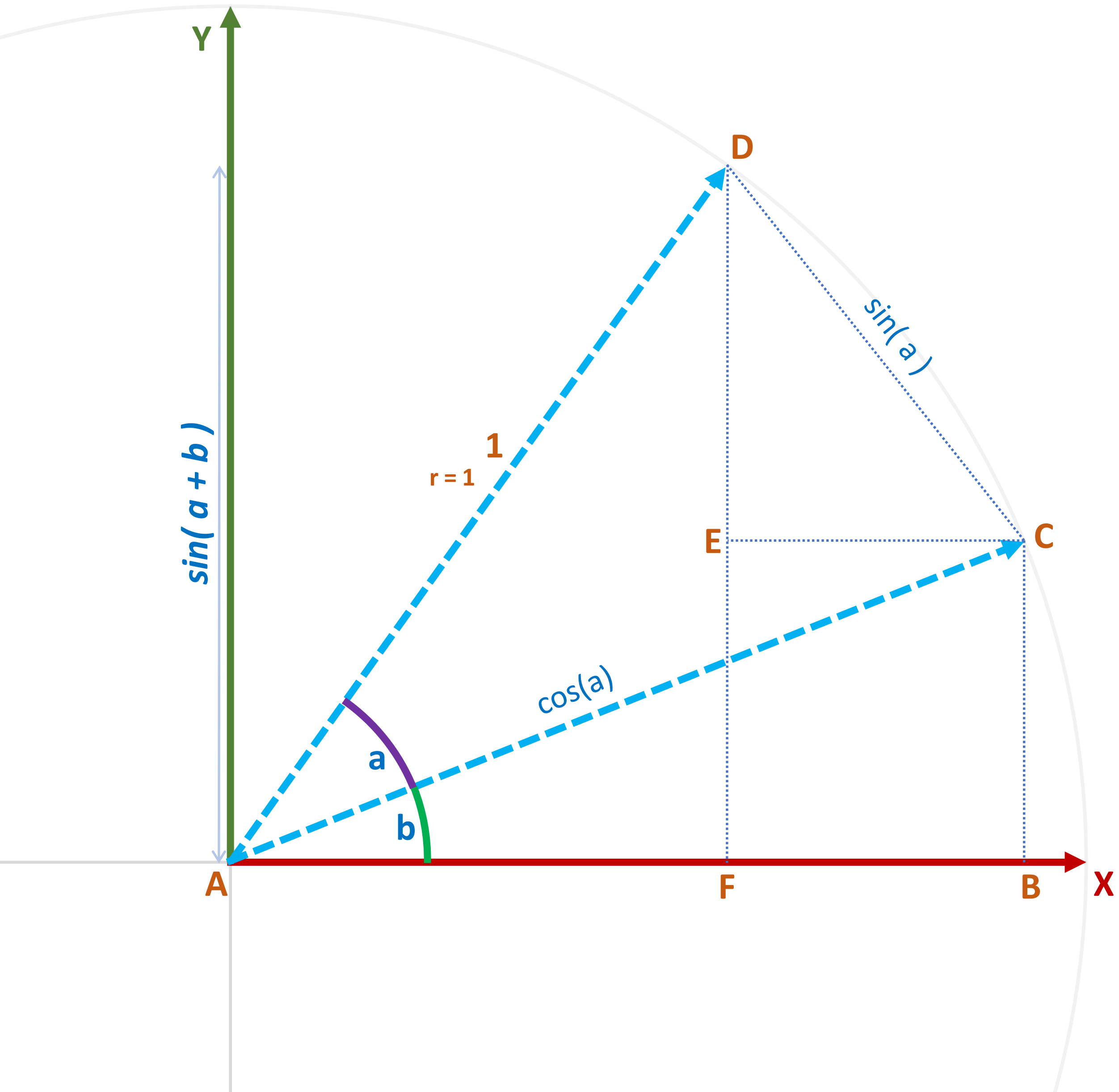
$$\sin(a+b) = \frac{DE + EF}{1}$$

$$\sin(a+b) = DE + EF$$

$$\sin(a+b) = DE + CB$$

$$\sin(b) = \frac{CB}{\cos(a)}$$

CB = cos(a) * sin(b)



Now let's recap what we know:

$$CB = \cos(a) * \sin(b)$$

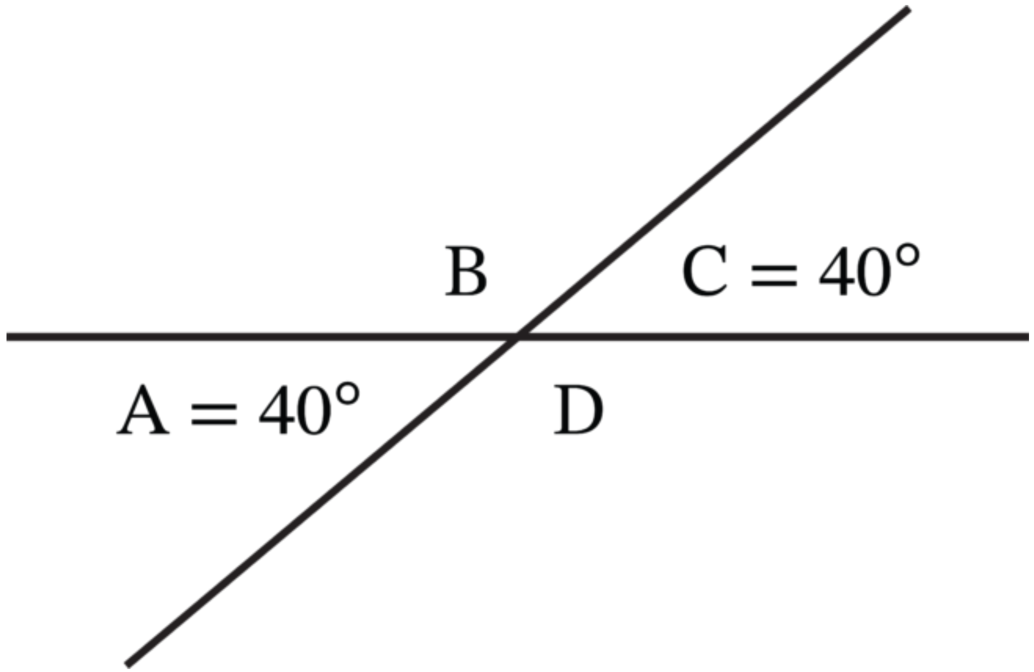
And considering:

$$\sin(a + b) = \sin(a) * \cos(b) + \cos(a) + \sin(b)$$

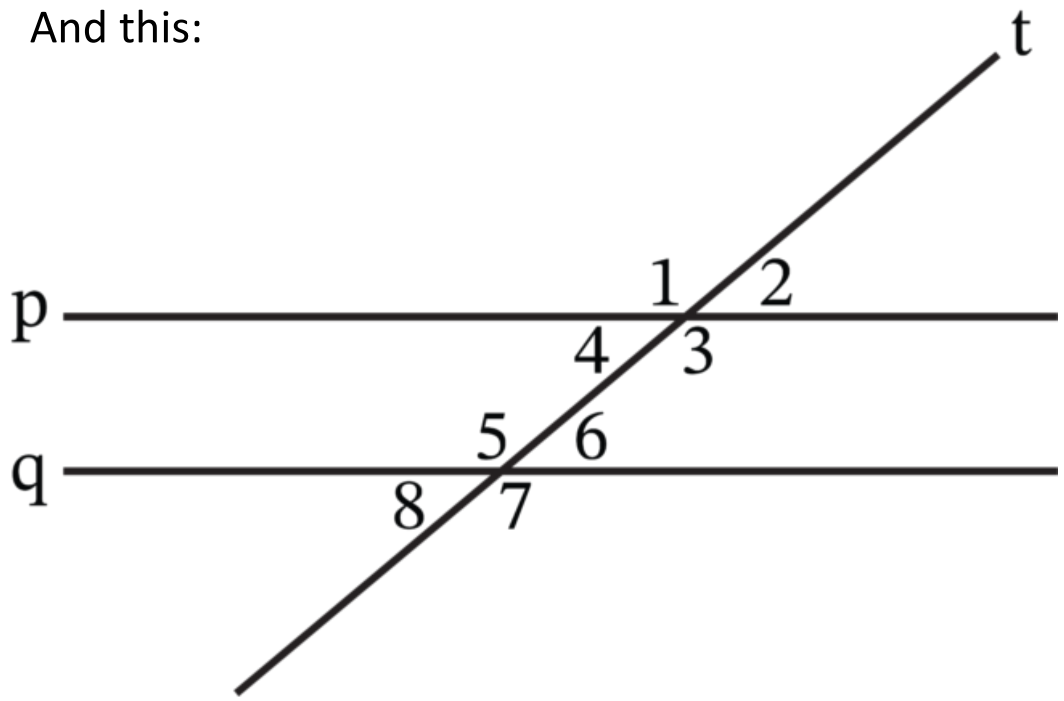
We have:

$$\sin(a + b) = \sin(a) * \cos(b) + CB$$

Now consider this:

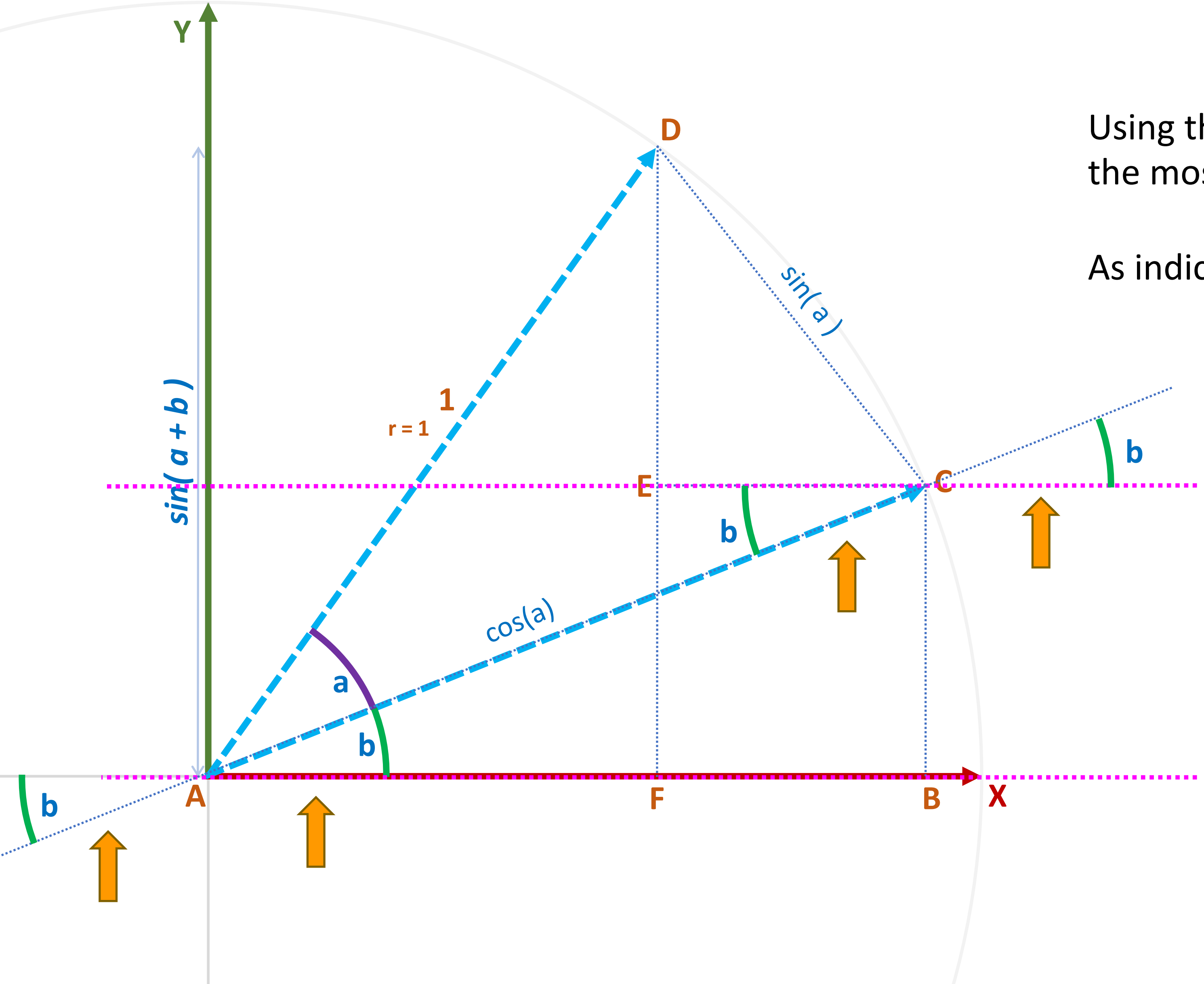


And this:



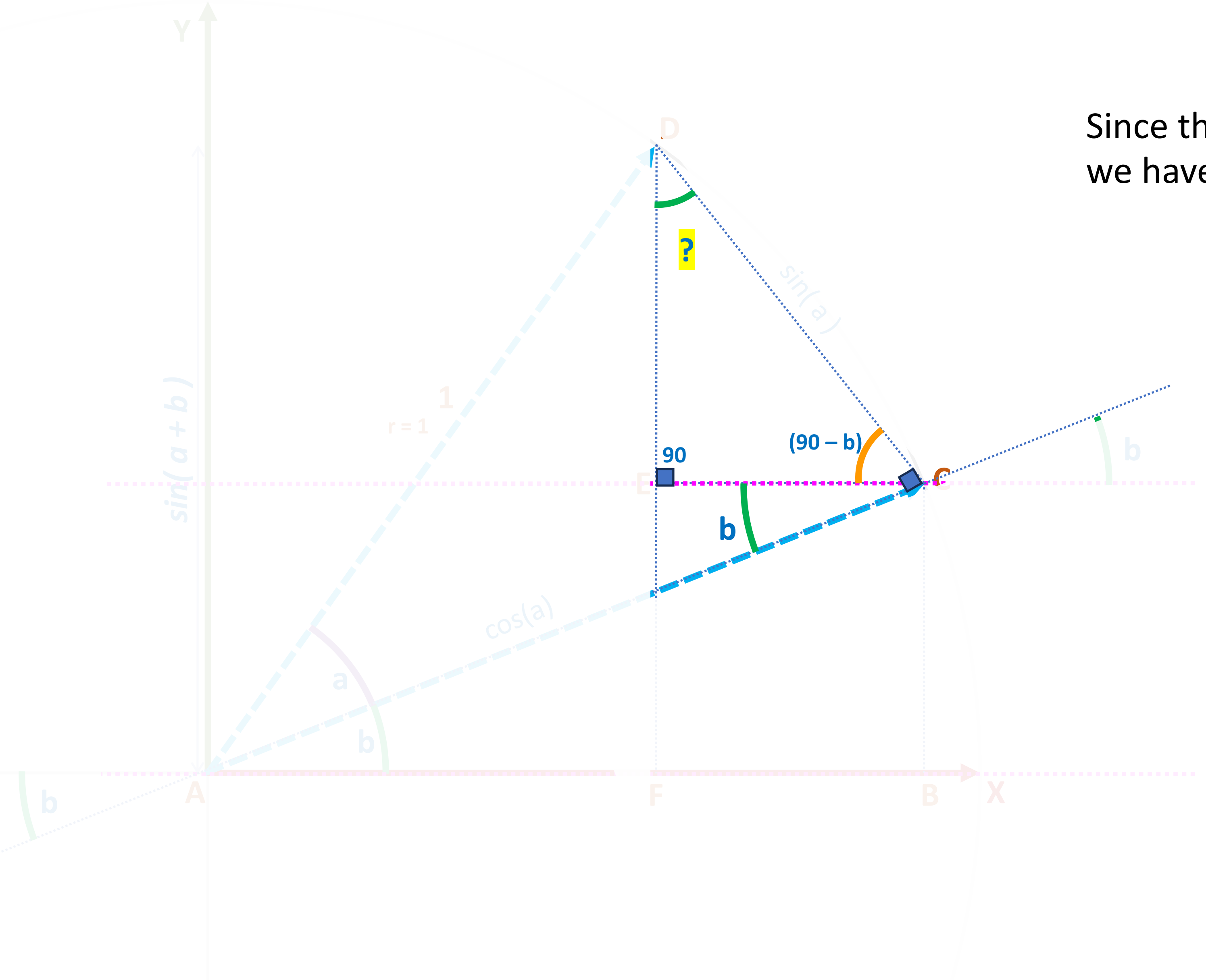
Note that: *"A close look at these intersections reveals that they are identical to each other. Because p and q are parallel, their intersections with t form congruent angles. So, not only are angles 1 and 3 congruent, but angles 5 and 7 are also included in this same congruence. Likewise, angles 2, 4, 6, and 8 are all congruent."*

- https://www.mometrix.com/academy/congruent-angles/



Using the same properties listed before we can now find the most obvious congruent angles.

As indicated below



Since the sum of all internal angles of a triangle is 180°
we have:

$$90 + (90 - b) + ? = 180$$

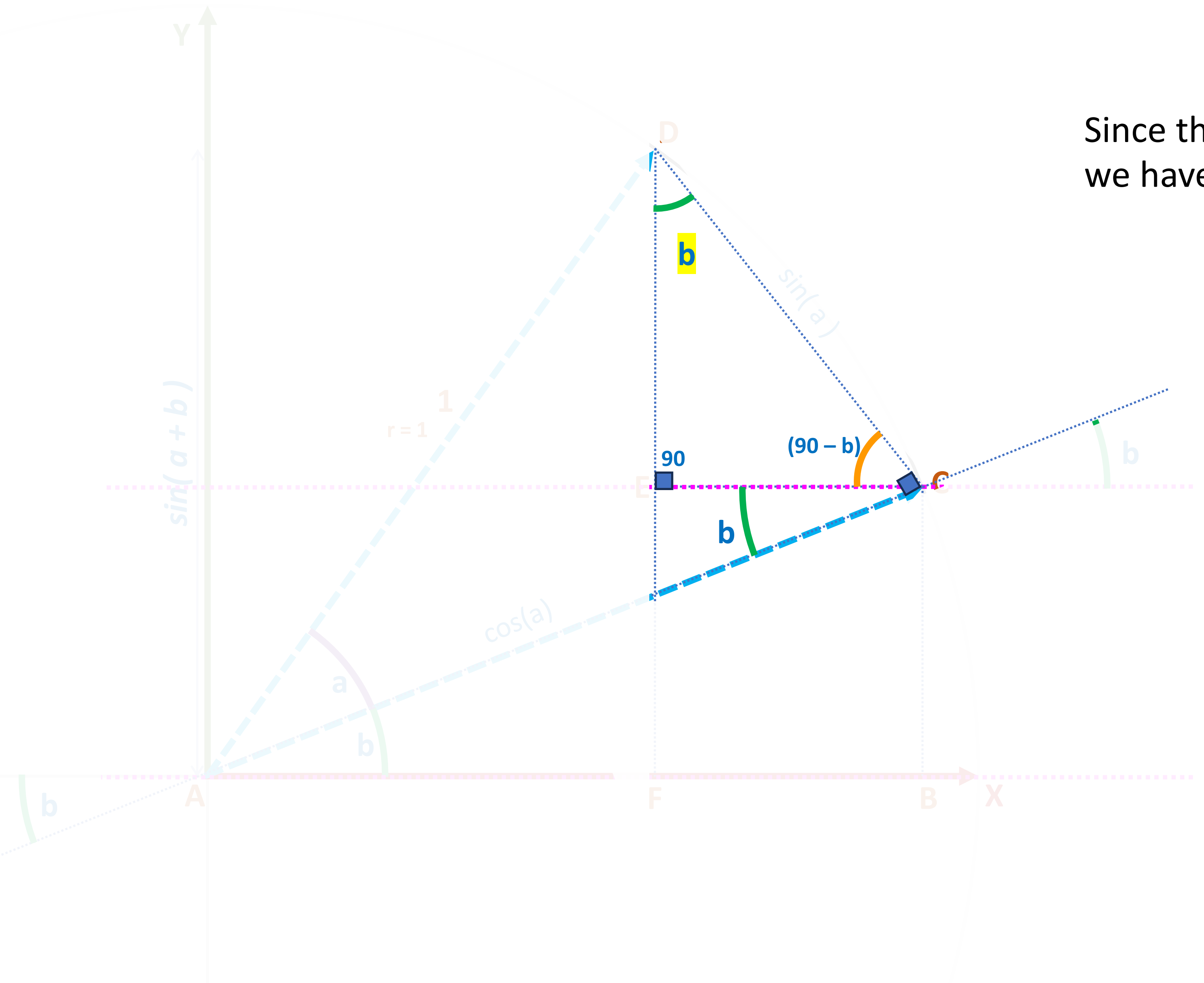
$$90 + 90 – b + ? = 180$$

$$180 – b + ? = 180$$

$$-b + ? = 180 – 180$$

$$-b + ? = 0$$

? = b



Since the sum of all internal angles of a triangle is 180° we have:

$$90 + (90 - b) + ? = 180$$

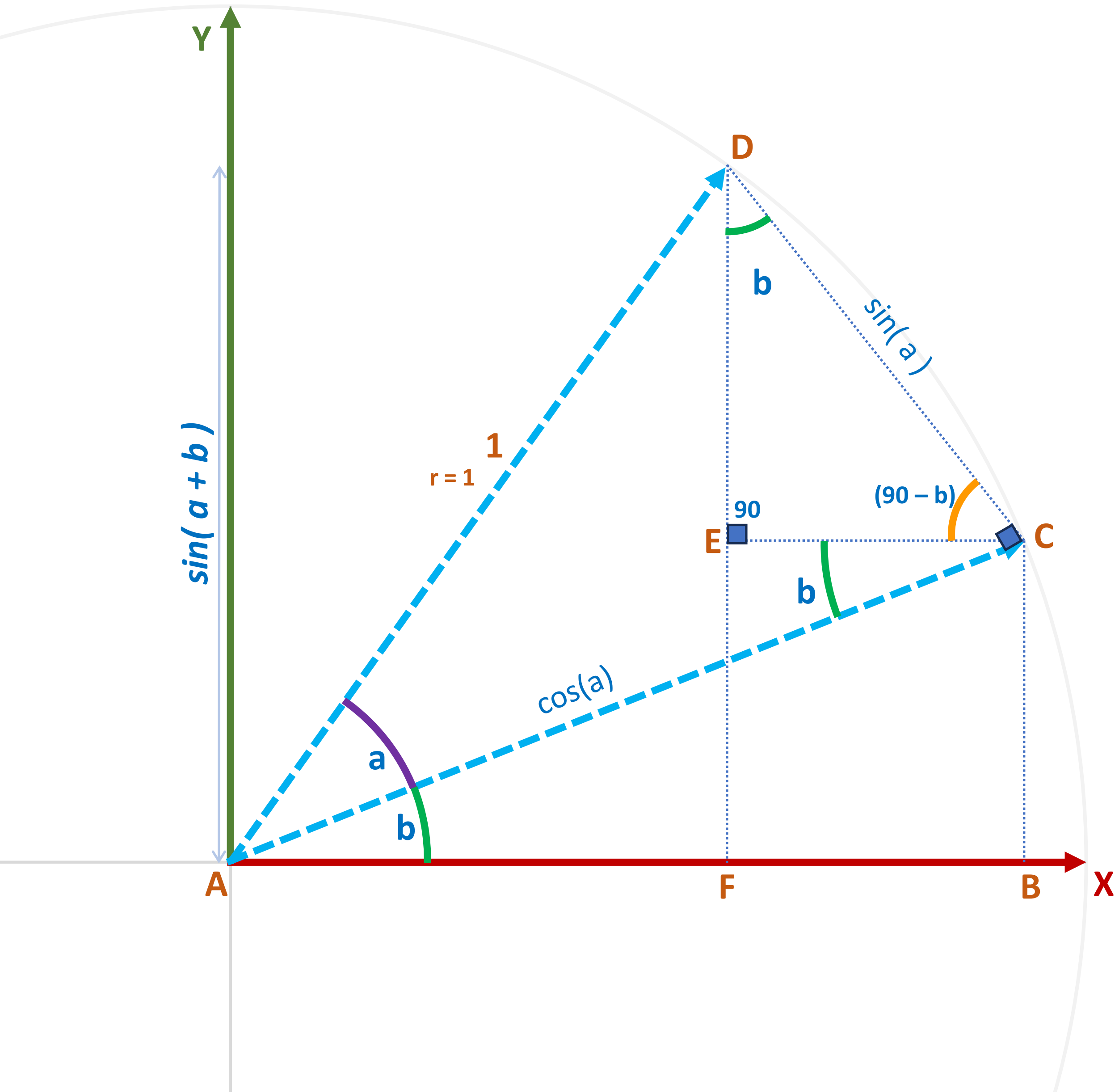
$$90 + 90 - b + ? = 180$$

$$180 - b + ? = 180$$

$$-b + ? = 180 - 180$$

$$-b + ? = 0$$

? = b

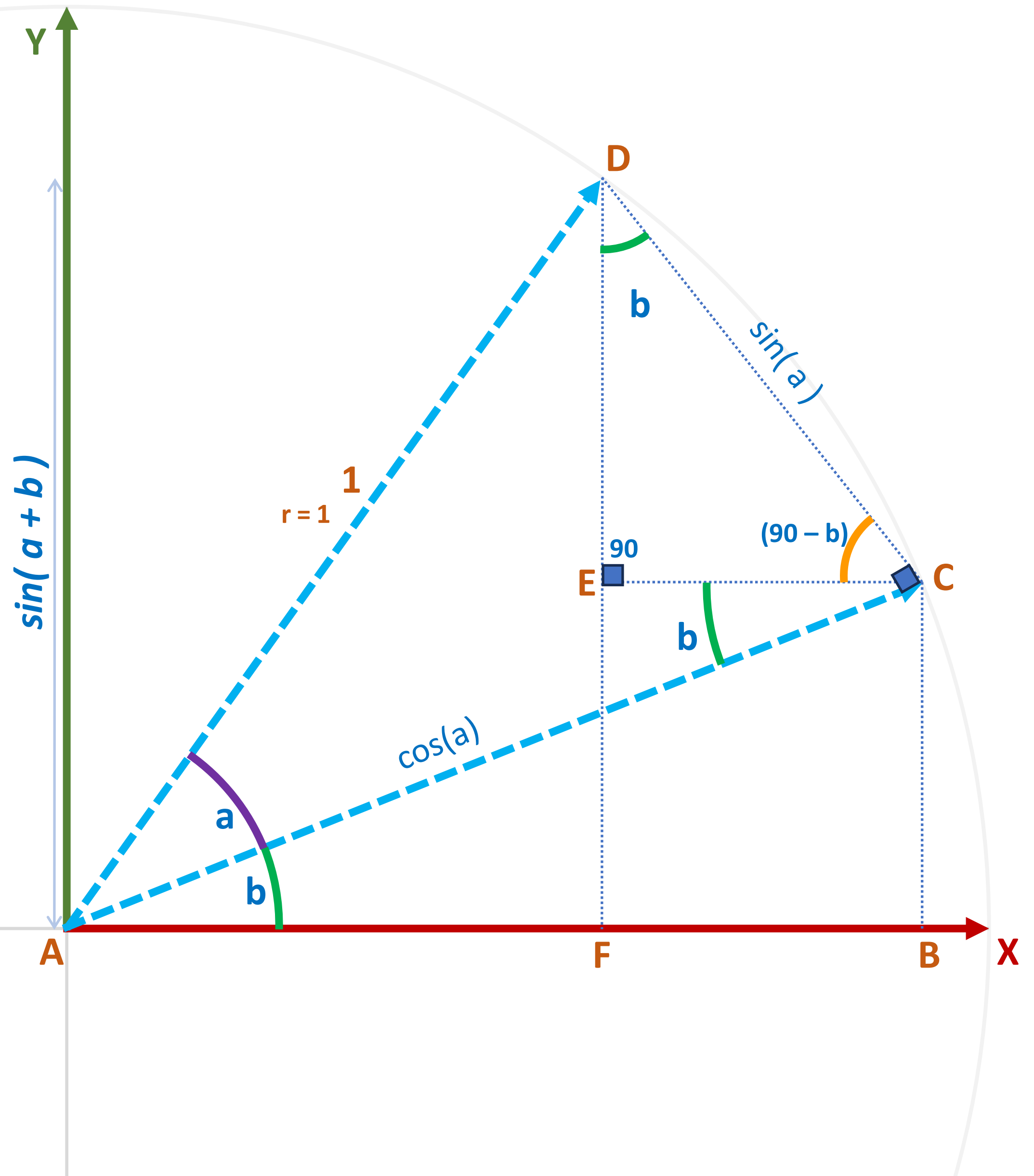


Now we can draw further conclusions:

$$\cos(b) = \frac{DE}{sin(a)}$$

Transforming the equation we get:

DE = sin(a) . cos(b)



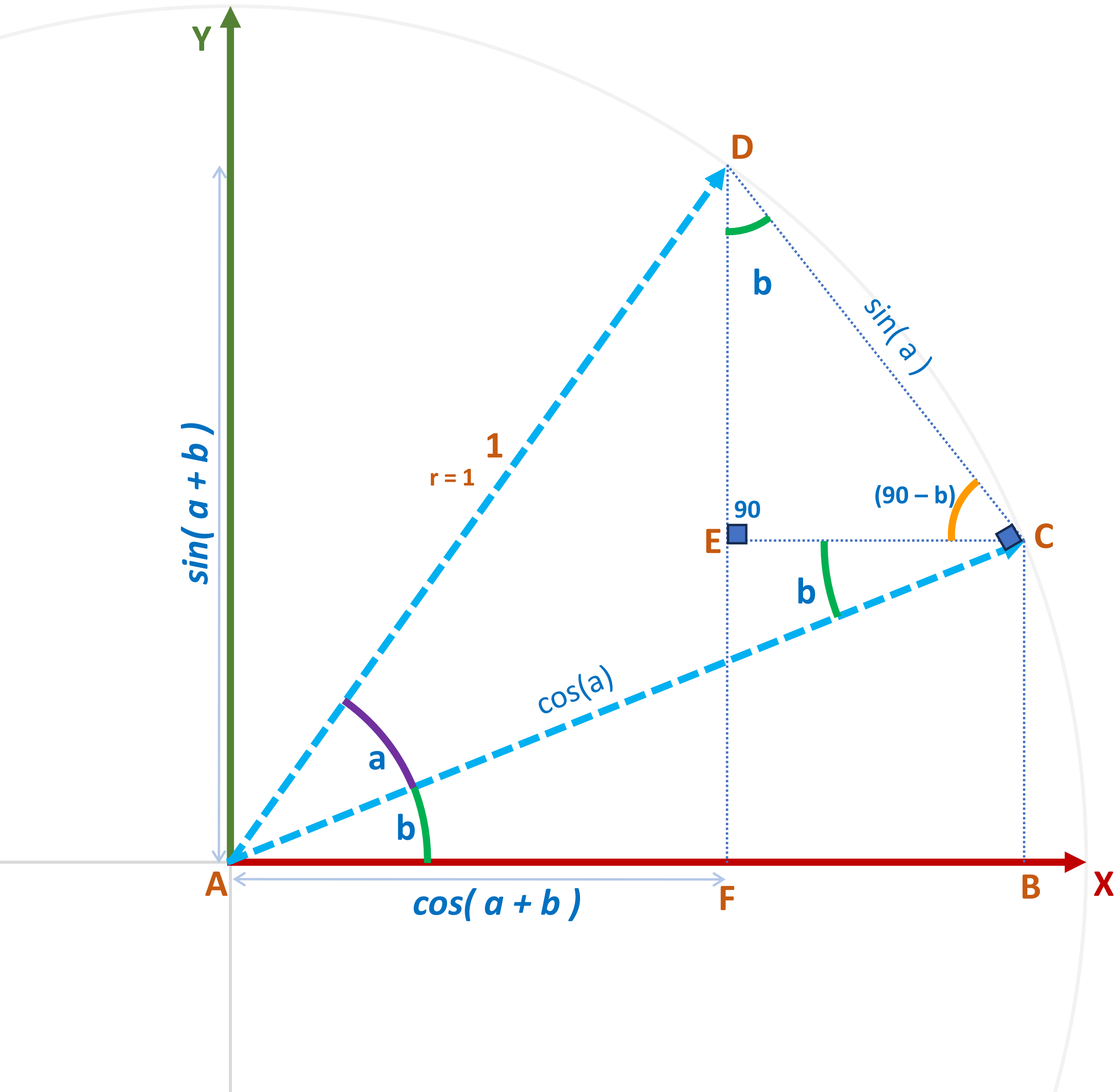
And by extension:

DE = sin(a) . cos(b)

sin(a + b) = sin(a) * cos(b) + CB

sin(a + b) = DE + CB

And with this we finalized the sin(a + b) proof



And now to the proof that:

cos(a + b) = cos(a) . cos(b) - sin(a) . cos(b)

And remember that:

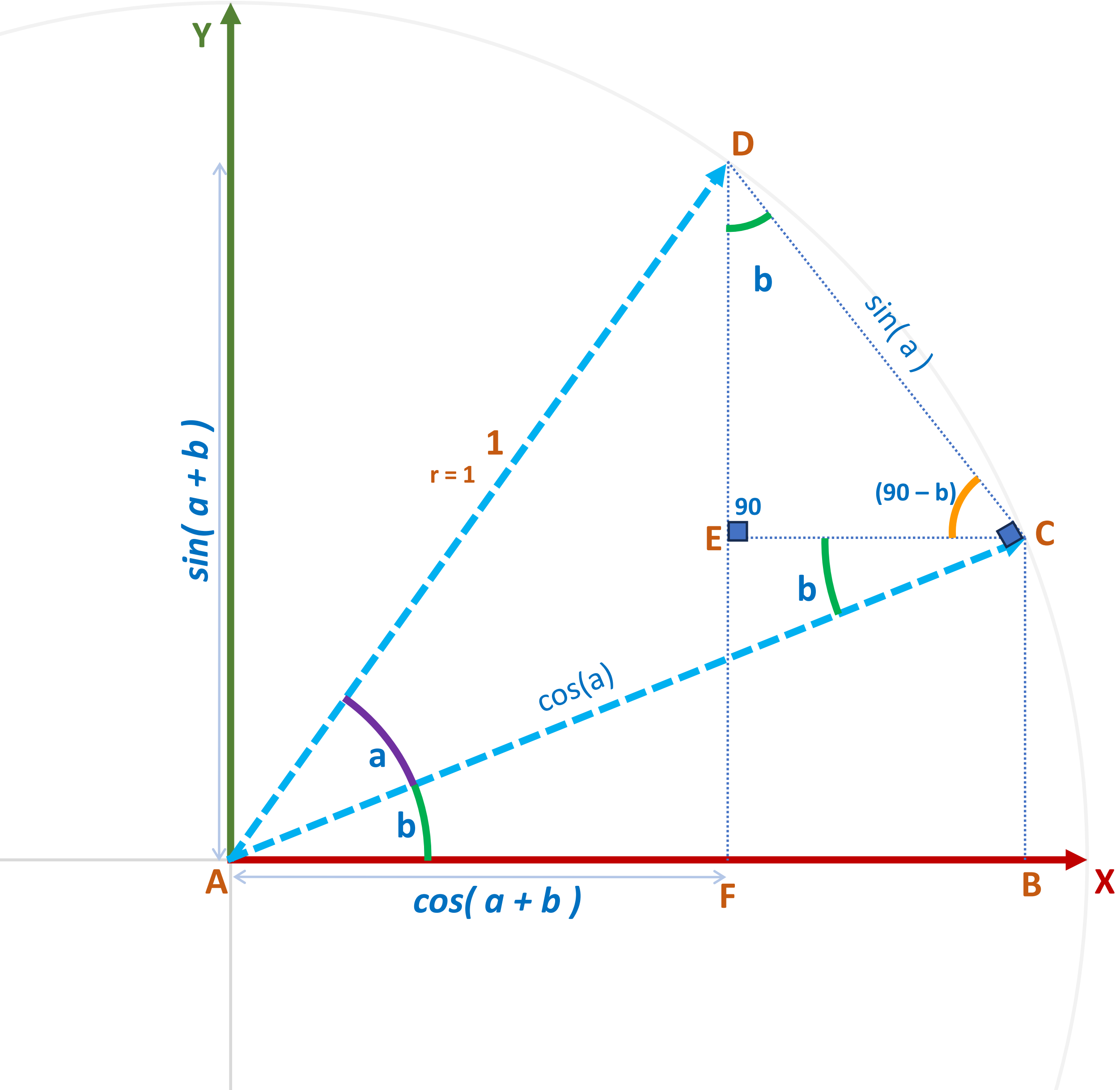
$$\cos(x) = \frac{adj}{\mathrm{hyp}}$$

The cos(a + b) can also be written as:

$$\cos(a + b) = \frac{AF}{1}$$

Or simply:

cos(a + b) = AF



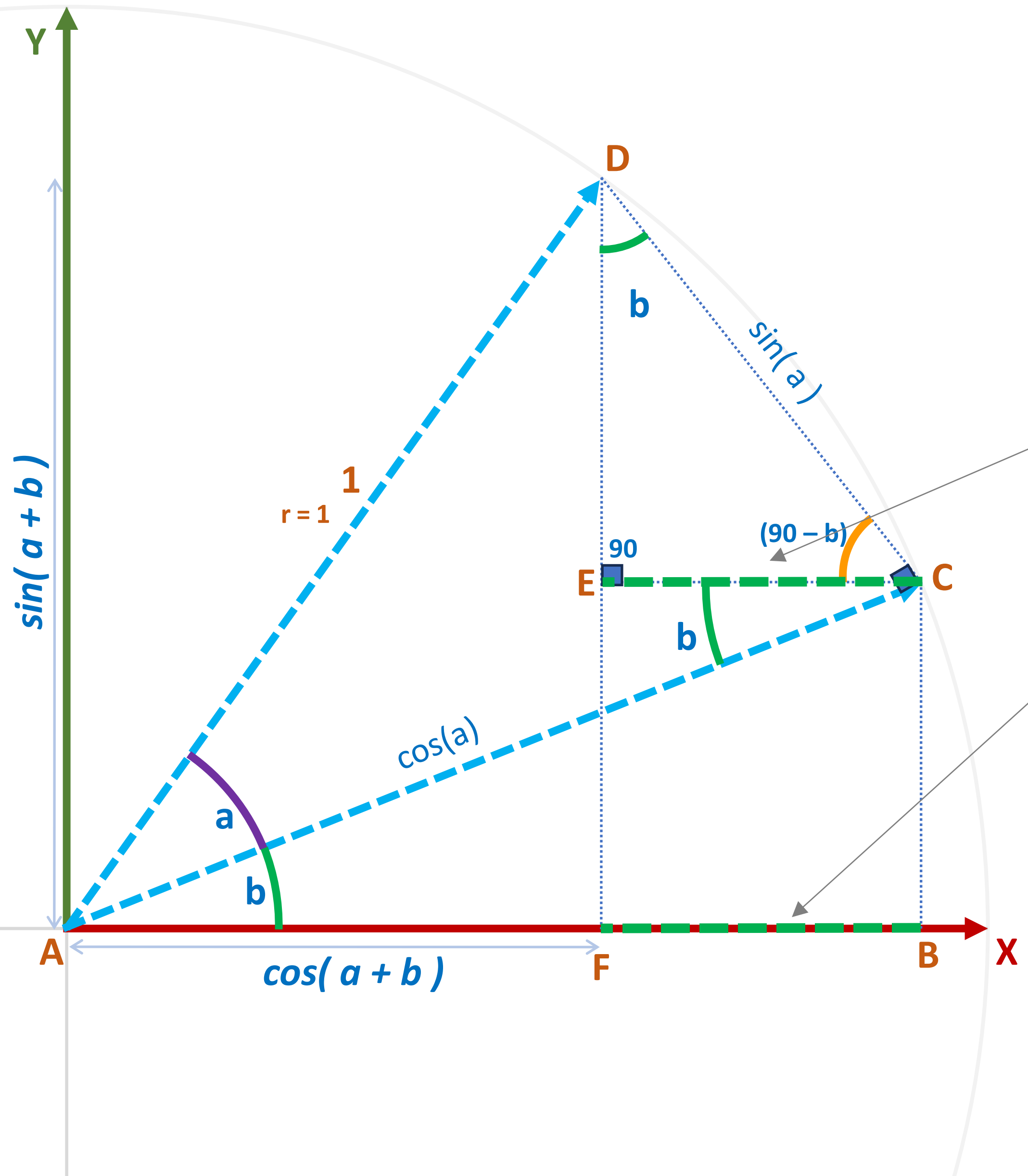
Considering that

cos(a + b) = AF

We can explore other ways to tackle this problem:

$$\cos(a+b) = \frac{AB\ - FB}{1}$$

cos(a + b) = AB - FB

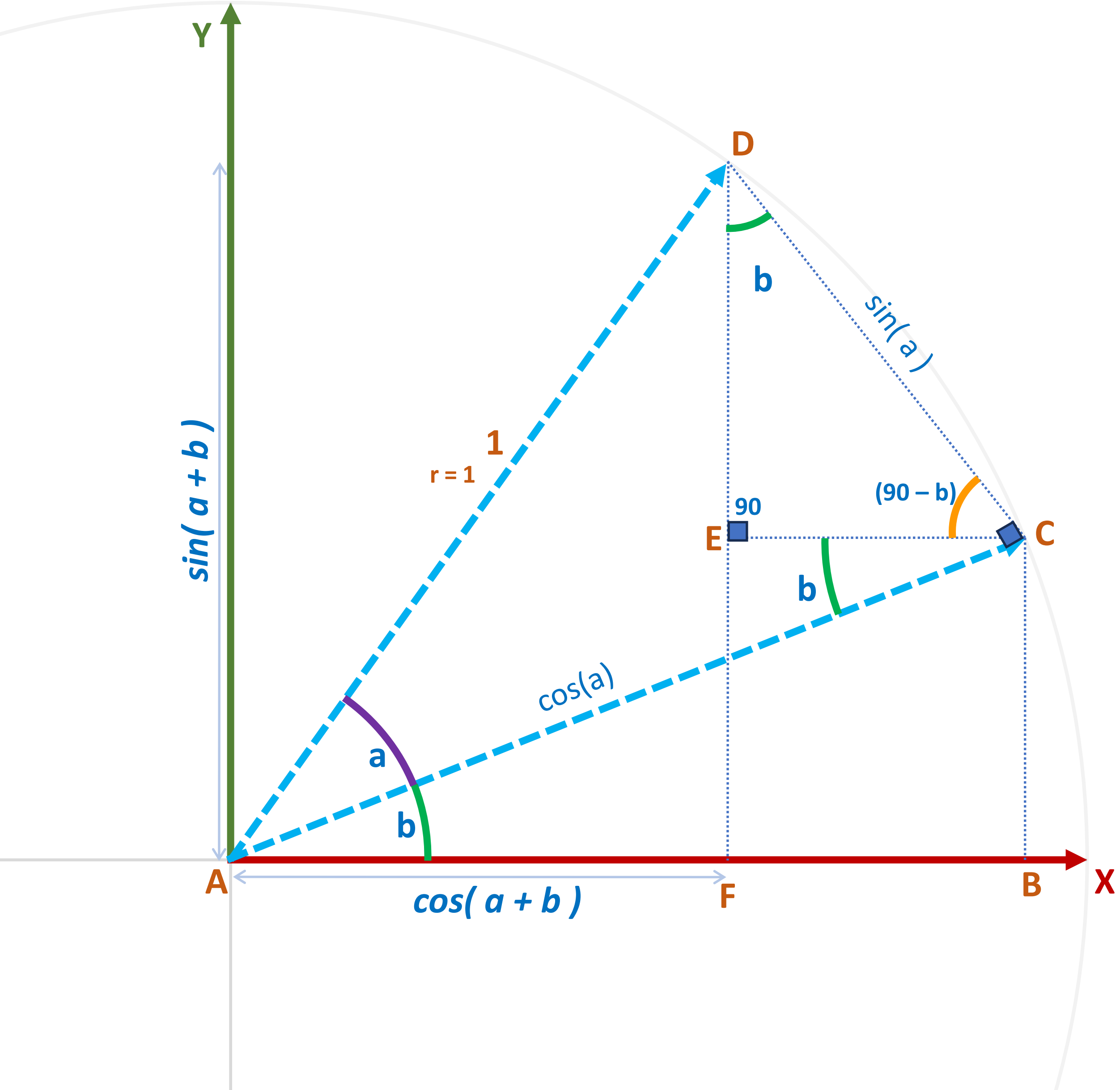


Considering that

FB = EC

We can write:

cos(a + b) = AB - EC



Now we know that

$$\cos(x) = \frac{adj}{\text{hyp}} \quad \text{->} \quad \cos(b) = \frac{AB}{\cos(a)}$$

We can write:

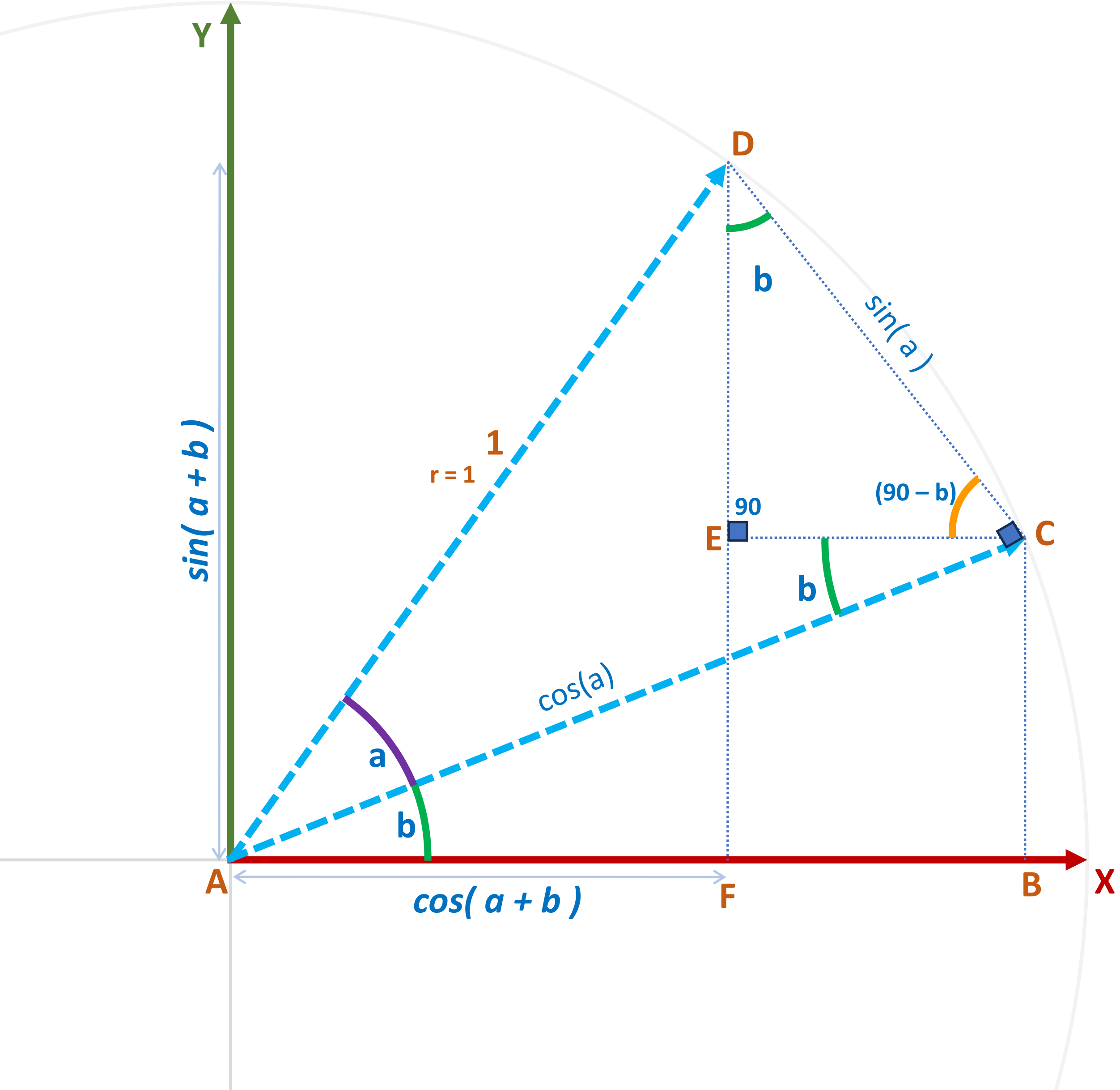
AB = cos(a) . cos(b)

And considering:

cos(a + b) = cos(a) . cos(b) - sin(a) . sin(b)

cos(a + b) = AB - sin(a) . sin(b)

We just proof half of the equation



Considering our last proof:

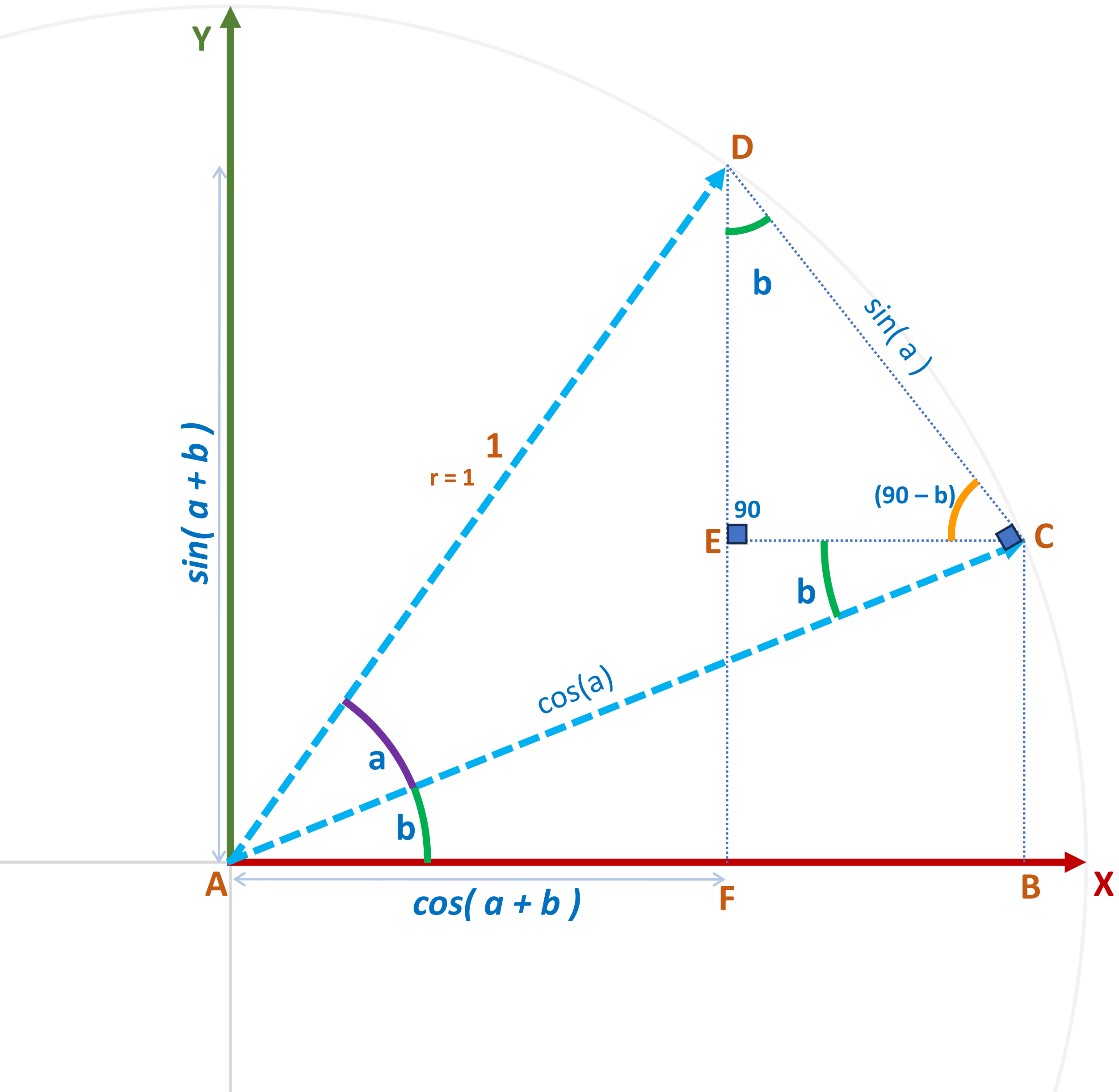
$$\cos(a + b) = AB - \sin(a) \cdot \sin(b)$$

And we want to prove that:

$$\cos(a + b) = AB - EC$$

We need to find evidence that:

$$EC = \sin(a) \cdot \sin(b)$$



Now take the triangle D,E,C. We want to find the length of EC.

And let's take the sine formula:

$$\sin(x) = \frac{opp}{hyp} \quad \rightarrow \quad \sin(b) = \frac{EC}{\sin(a)}$$

Therefore:

EC = sin(a) . sin(b)

And

cos(a + b) = AB - FB

FB = EC

cos(a + b) = AB - EC