We will use the following lemma, which is a known result in fair division theory. This result can be found, for example, as Lemma 7 in "Fair partitioning of " by Cardinal, Langerman, Palvolgyi (2021), or Theorem 2 in "Fair division of a discrete item" by Algaba et al. (2019). These papers attribute the proof technique to Gale (1993) and Scarf (1967).

Fair Partition Lemma: Let $v_1, \ldots, v_m \in \mathbb{R}^n_{\geq 0}$. Suppose that for each $i \in \{1, \ldots, n\}$, there exists a partition of $\{1, \ldots, m\}$ into n sets $J_{i,1}, \ldots, J_{i,n}$ such that $(\sum_{j \in J_{i,k}} v_j)_i \geq 1$ for all $k \in \{1, \ldots, n\}$. Then there exists a partition of $\{1, \ldots, m\}$ into n sets $K_1, \ldots, K_n$ such that $(\sum_{j \in K_i} v_j)_i \geq 1$ for all $i \in \{1, \ldots, n\}$.