



Faculty of Health Sciences



Day 7: Multiple linear regression, confounding, interaction

Paul Blanche

Section of Biostatistics, University of Copenhagen

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Outline

The multiple linear model

The simplest example: one binary variable

Why multiple regression?

ANCOVA and model checking

Digression: Table-I and the statistical analysis plan (SAP)

Prediction interval vs confidence intervals

Interaction

Interaction or subgroup analysis?

When, why and where to seek statistical help?



Regression

The type of outcome determines which kind of model is relevant:

Quantitative (continuous) outcome

- ▶ **Linear** regression.
 - ▶ To model **means**.
 - ▶ Association parameters: **differences** between **mean** values¹

0-1 (binary) outcome

- ▶ **Logistic** regression.
 - ▶ To model **probabilities**.
 - ▶ Association parameters: **odds ratio** (OR) or equivalently **differences** between **log(odds)**.



Case: vitamin D data

Data, $n=412$:

country	vitd	age	bmi	vitdintake
1	22.4	11.888	19.254	7.188
1	37.0	12.441	17.567	1.186
1	12.9	13.025	17.700	1.480
1	13.6	13.501	16.953	1.612
1	9.1	12.474	20.806	3.940
1	13.4	12.973	18.242	8.152



(also data on sun exposure: sunexp)

Outcome: vitamin D measured in morning blood samples, after an overnight fast (nmol/l).

Reference:

- ▶ Andersen and Skovgaard. *Regression with linear predictors*. Springer, 2010.
- ▶ Andersen et al., Eur. J. Clin. Nutr. (2005)

Note: the slides of today borrow many examples and explanations presented in more details in the above textbook reference.

Remarks on the case study and log-transformation

- ▶ It is common, and often sensible, to study the log of a concentration, instead of directly the **concentration** itself, when using linear regression. This is because:
 - ▶ **concentration cannot be negative.**
 - ▶ the **variability** between observations is **often higher for higher concentrations.**
- ▶ We will log-transform in our case study:

$$\text{outcome} = \log_{10}(\text{vitamin D concentration}) \ .$$



Remarks on the case study and log-transformation

- ▶ It is common, and often sensible, to study the log of a concentration, instead of directly the **concentration** itself, when using linear regression. This is because:

- ▶ **concentration cannot be negative.**
- ▶ the **variability** between observations is **often higher for higher concentrations.**

- ▶ We will log-transform in our case study:

$$\text{outcome} = \log_{10}(\text{vitamin D concentration}) \quad .$$

- ▶ But, it is not always needed and important to log-transform! **Do not systematically log-transform!**
- ▶ Usually it is interesting to present and interpret the results on the **original scale**, using the back-transformation (\exp).
- ▶ We will not transform back in this lecture (although we could), just to keep everything as simple as possible and focus on different statistical considerations.



The multiple linear model

The i -th observation (e.g. from subject i) of the outcome Y is described as:

$$Y_i = \alpha + \beta_1 x_i + \beta_2 z_i + \dots + \varepsilon_i$$

- ▶ x_i, z_i, \dots are the predictor (i.e. explanatory) variables / covariates.
- ▶ the linear predictor $\alpha + \beta_1 x_i + \beta_2 z_i + \dots$ is the **mean outcome for any subject i having covariate values x_i, z_i, \dots**
- ▶ ε_i 's are individual '**error**' **terms** ("random/unexplained deviation from the mean") assumed normally distributed with **zero mean** and the same variance σ_ε^2 regardless of the values x_i, z_i, \dots

Model assumptions (1-2 important, 3 not always):

1. Individual observations are independent.
2. The variance of 'error' terms is the same for all groups (**homogeneity**).
3. 'Error' terms are normally distributed.



The multiple model generalizes simpler models

Many simple settings can be thought as a **special case** of the multiple linear model.

Which and why?

- ▶ **t-test** (Lecture 2)
 - ▶ one binary predictor variable
- ▶ **univariate linear model** (Lecture 3)
 - ▶ one quantitative predictor variable
- ▶ **ANOVA** (Lecture 4)
 - ▶ one categorical predictor variable (one-way ANOVA)
 - ▶ two categorical predictor variables (two-way ANOVA)
- ▶ **ANCOVA** (Today's Lecture)
 - ▶ one categorical and one quantitative predictor variable

Note: when using t-test and ANOVA that assume the same standard deviation for all groups.



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Case: one binary variable

- ▶ **Research question:** is the mean (log) vitamin D different between elderly women (> 69) having a “normal” weight and those being “overweight”?
- ▶ **Predictor variable(s):** body mass index “normal” (18.5-25) or “overweight” (>25).
- ▶ **Data example:** Irish women, $n = 42$ ($16 + 25$).
- ▶ **Linear model:**

$$Y_i = \alpha + \beta z_i + \varepsilon_i$$

with

$$z_i = \begin{cases} 1 & \text{if } i \text{ is "overweight"} \\ 0 & \text{if } i \text{ has a "normal" weight} \end{cases}$$

- ▶ α : **mean** for “normal” weight
- ▶ $\alpha + \beta$: **mean** for “overweight”
- ▶ β : **difference in mean** between “overweight” and “normal”



R code & default output

R code:

```
lm1 <- lm(log10(vitd)~bmigroup,data=irlwomen)  
summary(lm1)
```

Output:

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	1.71987	0.04554	37.765	<2e-16 ***
bmigroup1	-0.12682	0.05832	-2.175	0.0358 *



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R code:

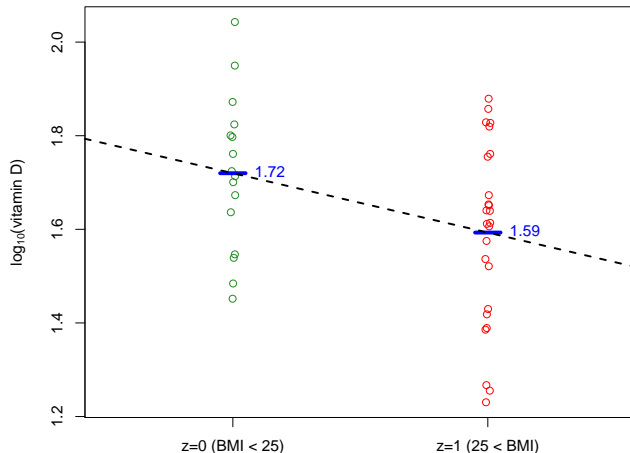
```
tapply(log10(irlwomen$vitd), irlwomen$bmigroup, mean)  
diff(tapply(log10(irlwomen$vitd), irlwomen$bmigroup, mean))
```

Output:

```
1.719873 1.593053  
-0.1268206
```



Visualizing the raw data & results



The regression line passes through the sample means, i.e. the two **estimated means** corresponds to the sample means in each group.



Formatted results & 95% CIs

R code:

```
publish(lm1)
```

Output:

Variable	Units	Coefficient	CI.95	p-value
(Intercept)		1.72	[1.63;1.81]	<1e-04
bmigroup	0	Ref		
	1	-0.13	[-0.24;-0.01]	0.0358

R code:

```
t.test(log10(irlwomen$vitd) ~ irlwomen$bmigroup, var.equal=TRUE)
```

Output:

Two Sample t-test

```
data: log10(irlwomen$vitd) by irlwomen$bmigroup
t = 2.1745, df = 39, p-value = 0.0358
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 0.008853898 0.244787253
sample estimates:
mean in group 0 mean in group 1
1.719873        1.593053
```



Conclusions with only one binary variable

- ▶ Model estimates **match** the observed means in each group.
- ▶ The estimated regression coefficient (slope) is **identical** to the difference between the sample means.
- ▶ The p-values computed by the linear model and the t-test (assuming equal variances in the two groups) are **identical**.
- ▶ The confidence interval for the regression coefficient (slope) is **identical** to that computed along the t-test to complement the p-value (assuming equal variances in the two groups are identical).

Furthermore: similar remarks about identical results for the ANOVA case. That is why we were already using the `lm()` function of R in the ANOVA case (although R and other software have also specific function for ANOVA analyses, which are different from the function to estimate linear models).



Digression: median and back-transformation

R code:

```
rbind(tapply(log10(irlwomen$vitd), irlwomen$bmigroup, mean),
      tapply(log10(irlwomen$vitd), irlwomen$bmigroup, median))
```

Output:

```
      0      1
[1,] 1.719873 1.593053
[2,] 1.718883 1.613842
```

Here, because the “model” for the mean is a good model for the median (M) and because $\text{median}(\log(Y)) = \log(\text{median}(Y))$ we have:

$$\log_{10}(\widehat{M}_1) - \log_{10}(\widehat{M}_0) = \log_{10} \left(\frac{\widehat{M}_1}{\widehat{M}_0} \right) = -0.12682$$

and $\widehat{M}_1/\widehat{M}_0 = 10^{-0.12682} = 0.75$; hence we conclude that we estimate that the overweight women to have a 25% lower vitamin D concentration compared to the normal weight women.



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Why multiple regression?

- ▶ better **adjusting** / explaining (main focus in this course)
- ▶ better **predict or gain power** (more advanced topic)

Same reasons as for why logistic regression can be more useful than simpler 2x2 tables analyses.

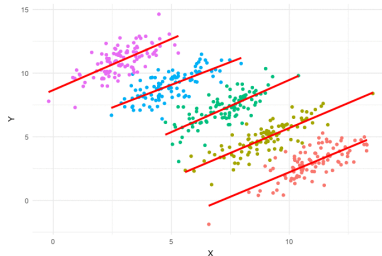
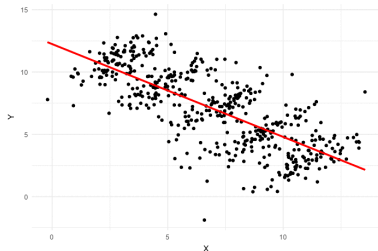
- ▶ Useful when we want to make **comparisons** with respect to one factor/variable (e.g. treatment or exposure) **among individuals otherwise similar with respect to other variables** that we adjust for (e.g. age, sex, comorbidity...).
- ▶ Multiple regression is a tool to deal with **confounding and unbalanced designs**.
- ▶ Multiple regression offers an **alternative to stratification** (i.e. subgroup analysis) when the data are not very large or/and we can assume that some differences are “similar” within different subgroups.



Multiple regression to limit confounding

We often compare two groups with the aim to get a “tentative” causal interpretation of the statistical association that we can show. To do so, we adjust on some variables to make a comparison among subjects as similar as possible with respect to some relevant variables.

Extreme hypothetical example of confounding:



Case: comparing countries

Initial research question: *“Is the average log-vitamin D different in the Irish and Polish population of elderly women?”*

Quick **look at** the collected data via a typical **“Table I”**:

		Ireland (n=41)	Poland (n=65)
Age	median [iqr]	72[70.8, 73.3]	71.7[70.4, 72.6]
BMI	18.5-25	16(39%)	12(19%)
	> 25	25(61%)	53(81%)
Sun exposure	avoid	16(39%)	26(40%)
	sometimes	21(51%)	34(52%)
	prefer	4(10%)	5(8%)
Vitamin D intake	median [iqr]	5.5[3.2, 12.1]	5.2[3.0, 11.9]

Updated research question: *“Is there a difference in average log-vitamin D between Irish and Polish elderly women having the same BMI group?”*



Case: comparing countries while “adjusting” for BMI group

- ▶ **Research question:** is there a difference in average log-vitamin D between Irish and Polish elderly women having the **same BMI group**?
- ▶ **Predictor variable(s):**
 - ▶ BMI “normal” (18.5-25) / “overweight” (>25).
 - ▶ Country Ireland / Poland
- ▶ **Data example:** Irish and Poland women, $n = 106$ (41 + 65).
- ▶ **Linear model:** $Y_i = \alpha + \beta_1 x_i + \beta_2 z_i + \varepsilon_i$

$$x_i = \begin{cases} 1 & \text{if } i \text{ is Polish} \\ 0 & \text{if } i \text{ is Irish} \end{cases} \quad z_i = \begin{cases} 1 & \text{if } i \text{ is "overweight"} \\ 0 & \text{if } i \text{ has a "normal" weight} \end{cases}$$

This is a two-way ANOVA model! (without interaction)



Parameters interpretation

According to the model, the means of log-vitamin D are:

BMI \ Country	Ireland	Poland
"Normal"	α	$\alpha + \beta_1$
"Overweight"	$\alpha + \beta_2$	$\alpha + \beta_1 + \beta_2$

- ▶ α : mean outcome for Irish with "normal" BMI (reference group).
- ▶ β_1 : difference in mean outcome between Irish and Polish among women of the same BMI group (whatever it is).
- ▶ β_2 : difference in mean outcome between women with "overweight" and those having a "normal" BMI, among women of the same country (whatever it is).



R code & default output

R code:

```
lm2 <- lm(log10(vitd) ~ Country + bmigroup, data = irlpolwomen)
summary(lm2)
```

Output:

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	1.72854	0.04016	43.040	< 2e-16	***
CountryPoland	-0.14164	0.03947	-3.589	0.000511	***
bmigroup1	-0.14103	0.04360	-3.235	0.001638	**



Formatted results & 95% CIs

R code:

```
publish(lm2)
```

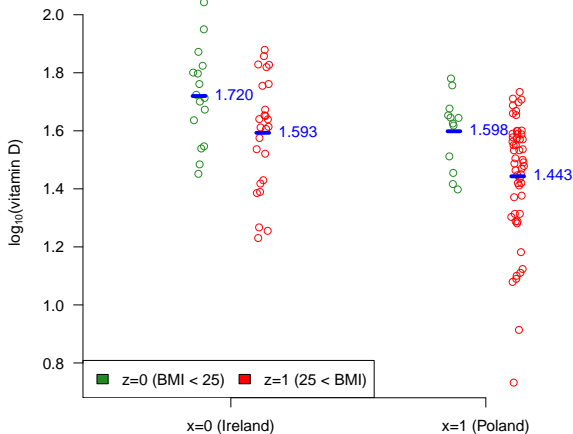
Output:

Variable	Units	Coefficient	CI.95	p-value
(Intercept)		1.73	[1.65;1.81]	< 1e-04
Country	Ireland	Ref		
	Poland	-0.14	[-0.22;-0.06]	0.0005108
bmigroup	0	Ref		
	1	-0.14	[-0.23;-0.06]	0.0016379

Conclusions?

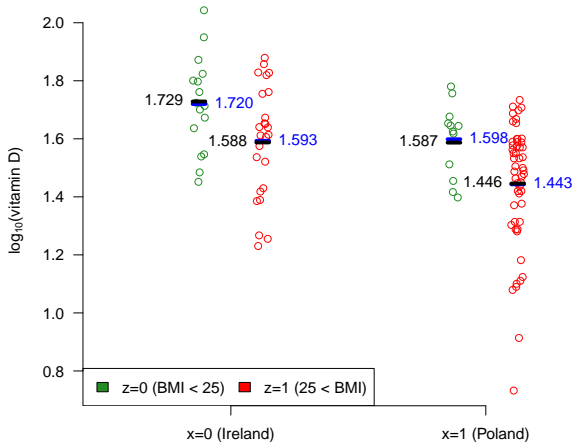


Visualizing the raw data



► Observed (sample) means: blue

Visualizing the raw data & results



- ▶ Observed (sample) means: blue
- ▶ Estimated means (from the model): black



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Case: comparing countries while “adjusting” for BMI

- ▶ **Research question:** is there a difference in mean log-vitamin D between Irish and Polish elderly women having the **same BMI**?
- ▶ **Predictor variable(s):**
 - ▶ BMI as a **quantitative** (continuous) variable
 - ▶ Country Ireland / Poland
- ▶ **Data example:** Irish and Poland women, $n = 106$ (41 + 65).
- ▶ **Linear model:** $Y_i = \alpha + \beta_1 x_i + \beta_2 z_i + \varepsilon_i$

$$x_i = \begin{cases} 1 & \text{if } i \text{ is Polish} \\ 0 & \text{if } i \text{ is Irish} \end{cases} \quad z_i = \text{BMI of subject } i.$$

This is called an **ANCOVA** model (ANalysis of COVariance).



- ▶ α : mean outcome for Irish ($x=0$) with BMI=0 ($z=0$) (meaningless!)
- ▶ β_1 : difference in mean outcome between Irish and Polish among women having the same BMI (whatever it is).
- ▶ β_2 : difference in mean outcome between two women, one having a BMI one unit higher than the other ($z+1$ versus z), among women of the same country (whatever it is).
Note: whatever the two BMI values being compared, as long as there is a one unit difference between the two.



R code:

```
irlpolwomen$bmi5 <- irlpolwomen$bmi/5  
lm3 <- lm(log10(vitd) ~ bmi5 + Country, data = irlpolwomen)  
summary(lm3)
```

Output:

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	2.04273	0.12291	16.620	< 2e-16	***
bmi5	-0.07593	0.02262	-3.357	0.00110	**
CountryPoland	-0.13135	0.04005	-3.280	0.00142	**



R code:

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irlpolwomen$bmi5 <- irlpolwomen$bmi/5
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R code:

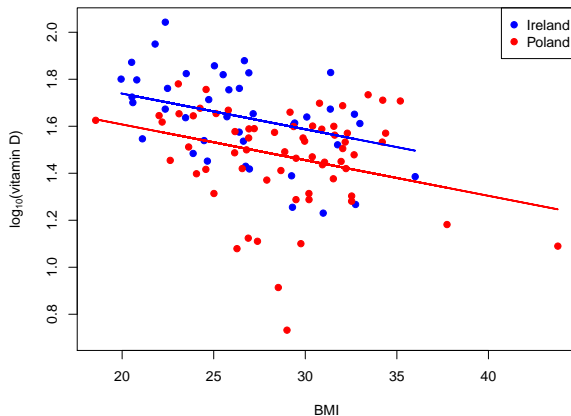
```
publish(lm3)
```

Output:

Variable	Units	Coefficient	CI.95	p-value
(Intercept)		2.04	[1.80;2.28]	< 1e-04
bmi5		-0.08	[-0.12;-0.03]	0.001103
Country	Ireland	Ref		
	Poland	-0.13	[-0.21;-0.05]	0.001418



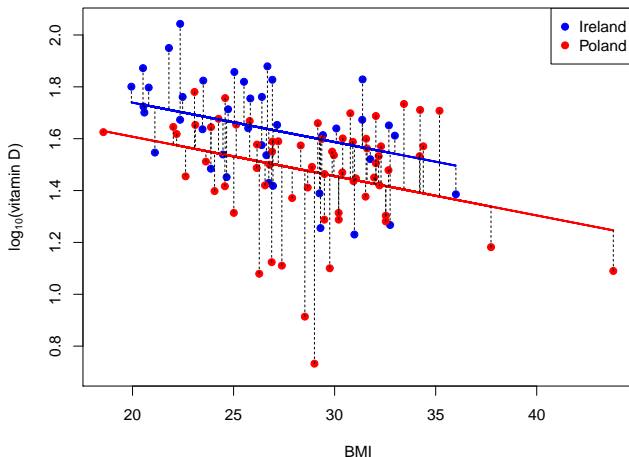
Visualizing the raw data & results



Note: an ANCOVA model is simply a regression model for parallel regression lines.

- ▶ β_2 : is the common slope of the two lines.
- ▶ β_1 : is the size of the vertical “shift” between the two lines.

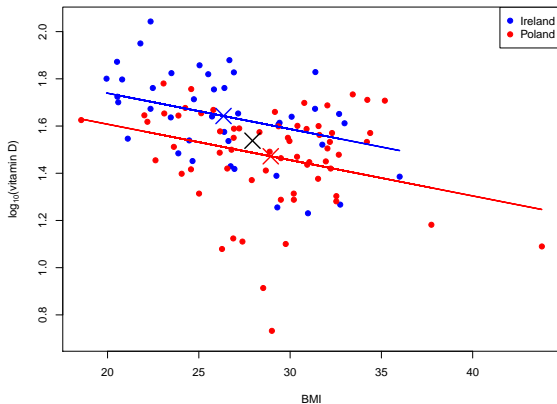
Estimate of σ_{ε} (standard deviation of the 'error' terms)



In the output of the `summary()`, “Residual standard error: 0.1921” is the estimate of σ_{ε} . It is computed “nearly” as the **standard deviation** of the **residuals** represented by the horizontal black dashed bars. It quantifies the **vertical “spread”** of the individual observations below/above the corresponding regression lines.

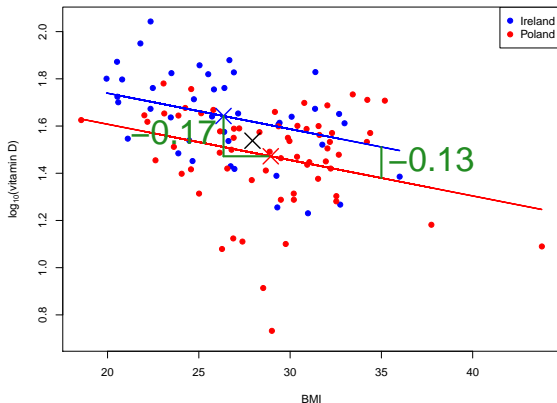


Comparing adjusted and unadjusted results



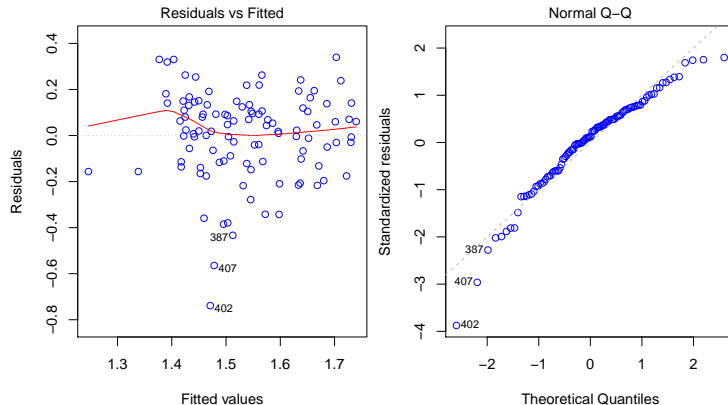
- crosses represent the mean of BMI (x-axis) and outcome (y-axis) of the entire sample (black) and for each country.

Comparing adjusted and unadjusted results



- ▶ crosses represent the mean of BMI (x-axis) and outcome (y-axis) of the entire sample (black) and for each country.
- ▶ Because the mean BMI is not the same in the two countries and because BMI is associated to the level of vitamin D, the adjusted and non-adjusted results are different.
- ▶ unadjusted difference between countries is -0.17, adjusted is -0.13.

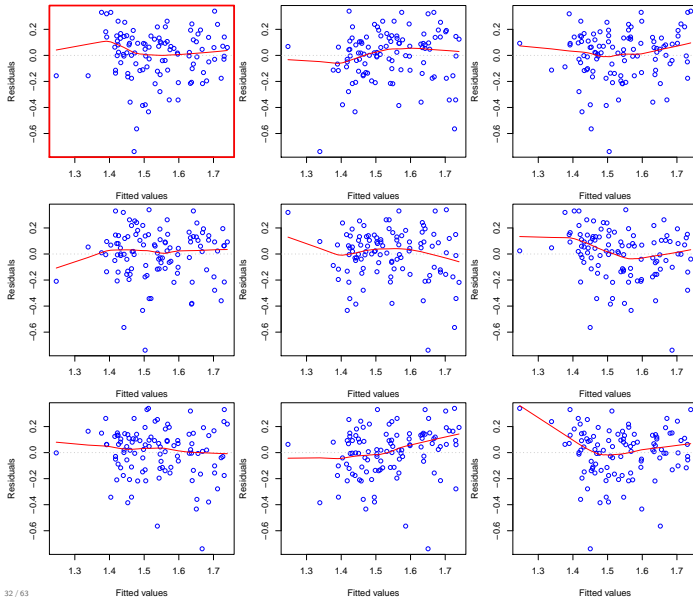
Model checking (default) plots



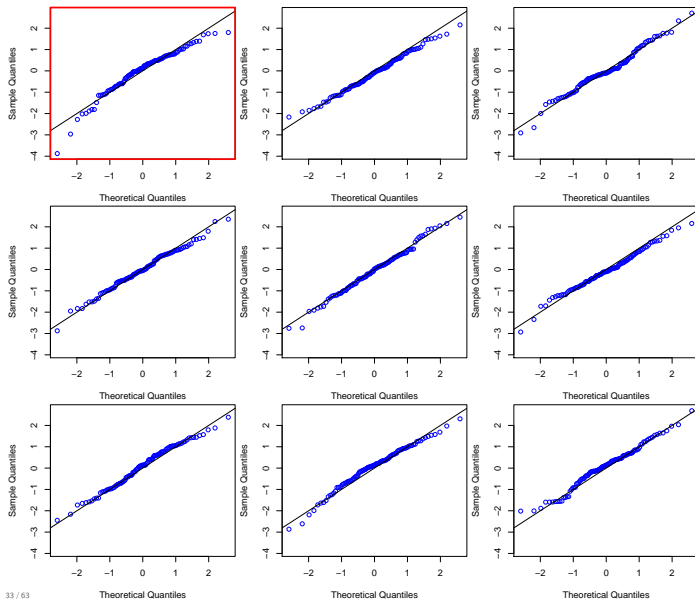
- ▶ Residual plot: always “important”.
- ▶ QQplot: mostly for small samples and when computing **prediction intervals**.
- ▶ Similar importance, for similar reasons, as in univariate linear regression (Lecture 3) and ANOVA model (Lecture 4).



Wally residual plot



Wally QQplot



ANCOVA with more than two categories

- ▶ **Research question:** is there a difference in mean log-vitamin D between **Danish, Finnish, Irish and Polish** elderly women having the **same BMI**?
- ▶ **Predictor variable(s):**
 - ▶ BMI as a **quantitative** (continuous) variable
 - ▶ Country Denmark / Finland / Ireland / Poland
- ▶ **Data example:** all elderly women, $n = 213$ ($53 + 54 + 41 + 65$).
- ▶ **Linear model:** $Y_i = \alpha + \beta_1 x_i + \beta_2 z_i + \beta_3 v_i + \beta_4 w_i + \varepsilon_i$

$$x_i = \text{BMI of subject } i \quad z_i = \begin{cases} 1 & \text{if } i \text{ is Finnish} \\ 0 & \text{otherwise} \end{cases}$$

$$v_i = \begin{cases} 1 & \text{if } i \text{ is Irish} \\ 0 & \text{otherwise} \end{cases} \quad w_i = \begin{cases} 1 & \text{if } i \text{ is Polish} \\ 0 & \text{otherwise} \end{cases}$$

Note: $z_i = v_i = w_i = 0$ for Danish women (reference group).



- ▶ β_2 : difference in mean outcome between Finnish and Danish among women having the same BMI (whatever it is).
- ▶ β_3 & β_4 : same but between Irish and Danish & Polish and Danish.



- ▶ β_2 : difference in mean outcome between Finnish and Danish among women having the same BMI (whatever it is).
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As in the simpler ANOVA context, we can test the global null hypothesis

" H_0 : there is no difference in mean log-vitamin level between women of the four countries, when comparing women of the same BMI",

that is

$$H_0 : \beta_2 = \beta_3 = \beta_4 = 0$$

Similarly as in the (simpler) ANOVA context, we can use either:

- ▶ F-test
- ▶ min-P method

Pros and cons are similar to those in the ANOVA context (Lecture 4).



R code & default output

R code:

```
lm4 <- lm(log10(vitd) ~ bmi5 + Country, data = dwomen)
summary(lm4)
```

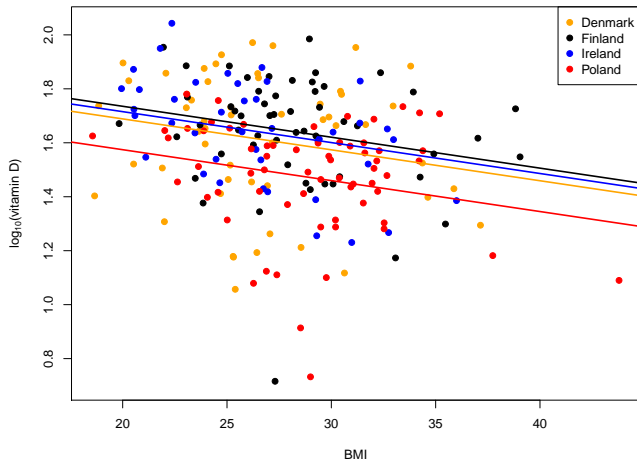
Output:

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	1.91780	0.09710	19.750	< 2e-16	***
bmi5	-0.05732	0.01746	-3.282	0.00121	**
CountryFinland	0.04674	0.04128	1.132	0.25891	
CountryIreland	0.02683	0.04390	0.611	0.54170	
CountryPoland	-0.11415	0.03995	-2.857	0.00471	**



Visualizing the raw data & results



R code:

```
anova(lm(log10(vitd) ~ bmi5, data = dwomen),
      lm(log10(vitd) ~ bmi5 + Country, data = dwomen))
```

Output:

Analysis of Variance Table

Model 1: log10(vitd) ~ bmi5

Model 2: log10(vitd) ~ bmi5 + Country

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	211	10.1690				
2	208	9.2635	3	0.9055	6.7773	0.0002212 ***

Comments:

- ▶ **F-test** p-value=0.0002212 is significant: there is a difference between countries, for the average outcome, when comparing women of the same BMI. **But which differences?**
- ▶ To **avoid coding mistakes and misunderstandings** of R output do compare the two models: do not use “anova(lm4)”.



Recommended analysis (see R-demo for code)

Statistical methods:

Comparisons between countries were made with a multiple linear model to adjust on BMI (ANCOVA). P-values and 95% confidence intervals were adjusted for multiple testing using the min-P method as implemented in the multcomp-package [ref.²] of the statistical software R [ref.³] and described in [ref.⁴].

Results (adjusted for multiple testing):

Comparison	Est. Diff	95% CI	p-value
Finland - Denmark	0.05	[-0.06; 0.15]	0.6695
Ireland - Denmark	0.03	[-0.09; 0.14]	0.9282
Poland - Denmark	-0.11	[-0.22;-0.01]	0.0239
Ireland - Finland	-0.02	[-0.13; 0.09]	0.9695
Poland - Finland	-0.16	[-0.26;-0.06]	0.0003
Poland - Ireland	-0.14	[-0.25;-0.03]	0.0069

Note:

- ▶ Significant **association** between countries and log vitamin D after adjusting on BMI, p-value= **0.0003** (i.e. the **minimum**)
- ▶ Similarly, we can use the method for the **“many-to-one”** setting (as in Lecture 4).
- ▶ This method **works with any linear model**, not just an ANCOVA (and does the F-test).

² Hothorn, Bretz & Westfall (2008). Simultaneous Inference in General Parametric Models. Biometrical Journal 50(3), 346–363.

³ R Core Team (2020). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL <https://www.R-project.org/>.

⁴ Bretz, Hothorn, & Westfall (2016). Multiple comparisons using R. CRC Press.



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When, why and where to seek statistical help?



Digression: How to adjust?

There is **usually no unique “best” way** to choose the variables **to adjust** on, but several interesting options, all with pros and cons. But, the choice should be **supported** by:

- ▶ **Research question**
 - ▶ Which groups do we want to compare? In which population? Among subjects similar with respect to what?
- ▶ **Background knowledge**
 - ▶ Why these groups? Why these population and comparisons among these “similar” subjects?
- ▶ **Available data** (variables & sample size)
 - ▶ What is the best compromise between what we ideally want to do and what we can do?

Note: several models may be needed when there are several research questions.⁵

⁵See e.g. Westreich & Greenland, Am. J. Epidemiol, 177.4 (2013): 292-298.



Digression: Table 1

- ▶ Using a simple descriptive “Table 1” to compare the distribution of all variables (except the outcome) between the groups that we want to compare often helps to choose how we should adjust.
- ▶ It is often useful to adjust on age, gender, baseline comorbidities etc or any variable which is not equally distributed between the groups⁶.
- ▶ This is fine and not “cheating” (i.e. not “data snooping” or “p-hacking”) as long we do not look at any association between the outcome and any variable before we make the choice on how to adjust.
- ▶ The aim of this descriptive “Table 1” is only to describe the population of each group, hence it is usually recommended that it does not include p-values.⁷

⁶when it is not a consequence of the treatments or exposures being compared

⁷See e.g. STROBE or CONSORT statements endorsed by most medical journals.



Digression: efficient “Table 1” making in R

R code:

```
Tab1ex <- univariateTable(Country~age+Q(bmi)+Q(vitdintake)+sunexp,
                           data=IrFinPo,
                           compare.groups = FALSE,
                           show.totals = FALSE)
```

Tab1ex

Output:

Variable	Level	Denmark (n=78)	Finland (n=88)	Ireland (n=54)
age	mean (sd)	52.7 (27.7)	49.1 (29)	57.8 (25.9)
bmi	median [iqr]	24.9 [20.9, 27.5]	25.5 [20.8, 28.8]	24.9 [22.4, 28.9]
vitdintake	median [iqr]	6.1 [2.7, 11.6]	7.9 [5.0, 15.2]	5.3 [2.9, 10.5]
sunexp	avoid	14 (17.9)	15 (17.0)	18 (33.3)
	sometimes	43 (55.1)	42 (47.7)	25 (46.3)
	prefer	21 (26.9)	31 (35.2)	11 (20.4)



Digression: statistical analysis plan (SAP)

- ▶ It is recommended to make a **statistical analysis plan** (SAP) **before** starting any analysis on the outcome data.
- ▶ It consists of a **list of** research **questions and corresponding analyses**, ideally with a few comments to explain their rationale.

It helps to:

- ▶ better discuss with your collaborators and supervisors.
- ▶ anticipate challenges.
- ▶ rigorously **prespecify** your analyses and therefore increase the trust that you can have in your results.

It is completely **fine to make revisions** to the statistical plan and perform post hoc analyses **as long as conclusions** based on these additional analyses **are suitably calibrated**.



Outline

The multiple linear model

The simplest example: one binary variable

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Digression: Table-I and the statistical analysis plan (SAP)

Prediction interval vs confidence intervals

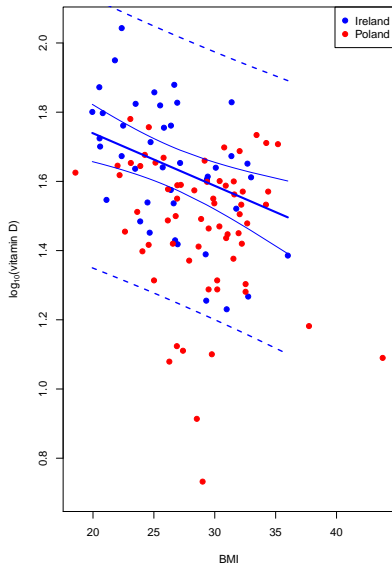
Interaction

Interaction or subgroup analysis?

When, why and where to seek statistical help?



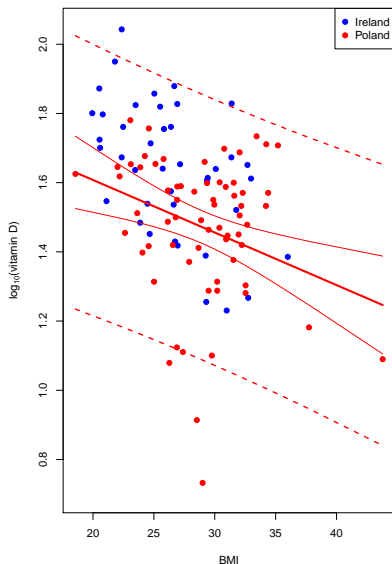
Prediction interval vs confidence intervals



- **Confidence interval** (of the estimated mean value): it quantifies the **uncertainty in the estimation** of the population mean. It tells us where we are “confident” that the population mean is (plain lines).
- **Prediction interval**: it tells us the **range of values** that include most (95%) of the observations in the **entire population** (dashed lines). Its width essentially depends on the estimated standard error of the “error term” σ_{ϵ} . It relies strongly on the normal distribution assumption of the “error term”.



Prediction interval vs confidence intervals



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Case: a (first) model with an interaction

► Research questions:

- Is BMI associated with log-vitamin D in both Irish and Polish elderly women?
- Is there a difference in the strength of the association between log-vitamin D and BMI between Irish and Polish elderly women?

► Predictor variable(s):

- BMI as a quantitative (continuous) variable
- Country Ireland / Poland

► Data example: Irish and Poland women, $n = 106$ ($41 + 65$).

► Linear model: $Y_i = \alpha + \beta_1 x_i + \beta_2 z_i + \beta_3 x_i \cdot z_i + \varepsilon_i$

$$x_i = \begin{cases} 1 & \text{if } i \text{ is Polish} \\ 0 & \text{if } i \text{ is Irish} \end{cases} \quad z_i = \text{BMI of subject } i.$$

The term $\beta_3 x_i \cdot z_i$ models an **interaction** between x and z .



R code & default output

R code:

```
lm5 <- lm(log10(vitd) ~ Country * bmi5, data = irlpolwomen)
summary(irlpolwomen)
```

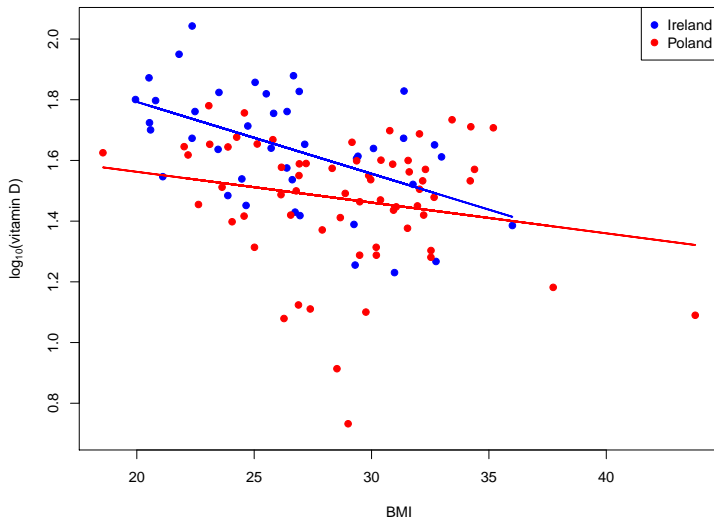
Output:

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	2.26626	0.19630	11.545	< 2e-16	***
CountryPoland	-0.50113	0.25719	-1.948	0.05410	.
bmi5	-0.11834	0.03681	-3.215	0.00175	**
CountryPoland:bmi5	0.06768	0.04650	1.455	0.14865	

Conclusions?





- ▶ Blue slope (Ireland) $\hat{\beta}_2 = -0.11834/5 \approx -0.12/5$
- ▶ Red slope (Poland) $\hat{\beta}_2 + \hat{\beta}_3 = (-0.11834 + 0.06768)/5 \approx -0.05/5$
- ▶ Difference in slope (Poland - Ireland) $\hat{\beta}_3 = 0.06768/5$

Formatted results & 95% CIs

R code:

```
publish(lm5)
```

Output:

	Variable	Units	Coefficient	CI.95	p-value
	(Intercept)		2.27	[1.88;2.65]	< 1e-04
bmi5:	Country(Ireland)		-0.12	[-0.19;-0.05]	0.001749
bmi5:	Country(Poland)		-0.05	[-0.11;0.01]	0.077570

Note: the effect of BMI on log-vitamin D is not significant among Polish elderly women (p-value=0.078). This **could not be read from the default R output** (unless you refit the model after changing the reference level for the country).

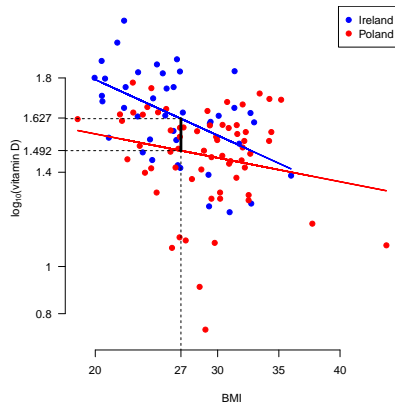


Trick: re-parametrization (to better understand all parameters)

We refit the same model after subtracting 27 to the BMI variable.

R code:

```
irlpolwomen$bmi5b <- (irlpolwomen$bmi-27)/5
lm5b <- lm(log10(vitd) ~ Country * bmi5b,
           data = irlpolwomen)
summary(lm5b)
```



Output:

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	1.62722	0.03021	53.862	< 2e-16 ***
CountryPoland	-0.13567	0.03995	-3.396	0.000975 ***
bmi5b	-0.11834	0.03681	-3.215	0.001749 **
CountryPoland:bmi5b	0.06768	0.04650	1.455	0.148649



Two-way ANOVA with interaction

- ▶ **Research question:** is there a difference in average log-vitamin D between Irish and Polish elderly women having the **same BMI group**?
- ▶ **Predictor variable(s):**
 - ▶ BMI "normal" (18.5-25) / "overweight" (>25).
 - ▶ Country Ireland / Poland
- ▶ **Data example:** Irish and Poland women, $n = 106$ (41 + 65).
- ▶ **Linear model:** $Y_i = \alpha + \beta_1 x_i + \beta_2 z_i + \beta_3 x_i \cdot z_i + \varepsilon_i$

$$x_i = \begin{cases} 1 & \text{if } i \text{ is Polish} \\ 0 & \text{if } i \text{ is Irish} \end{cases} \quad z_i = \begin{cases} 1 & \text{if } i \text{ is "overweight"} \\ 0 & \text{if } i \text{ has a "normal" weight} \end{cases}$$

This is a **two-way ANOVA model with interaction**.



Parameters interpretation

According to the model, the means of log-vitamin D are:

BMI \ Country	Ireland	Poland
"Normal"	α	$\alpha + \beta_1$
"Overweight"	$\alpha + \beta_2$	$\alpha + \beta_1 + \beta_2 + \beta_3$

- ▶ α : **mean** outcome for Irish with "normal" BMI (reference group).
- ▶ β_1 : **difference in mean** outcome between Irish and Polish among women with "normal" BMI.
- ▶ β_2 : **difference in mean** outcome between women with "overweight" and those having a "normal" BMI, among Irish women.
- ▶ $\beta_1 + \beta_3$: **difference in mean** outcome between Irish and Polish among women with "overweight".
- ▶ $\beta_2 + \beta_3$: **difference in mean** outcome between women with "overweight" and those having a "normal" BMI, among Polish women.



R code:

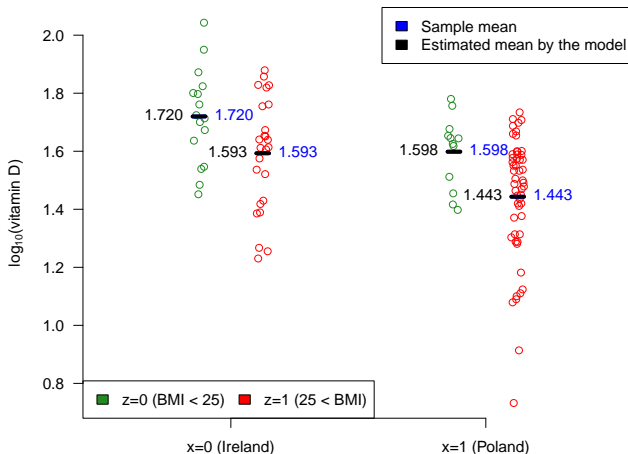
```
lm6 <- lm(log10(vitd) ~ Country * bmigroup, data = irlpolwomen)
summary(lm6)
```

Output:

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	1.71987	0.04840	35.538	<2e-16	***
CountryPoland	-0.12142	0.07393	-1.643	0.1036	
bmigroup1	-0.12682	0.06198	-2.046	0.0433	*
CountryPoland:bmigroup1	-0.02838	0.08758	-0.324	0.7466	





- Here the **estimated means** are equal to the **sample means**. We say that the model for the mean is **saturated** (because we have 4 parameters to estimate 4 means).



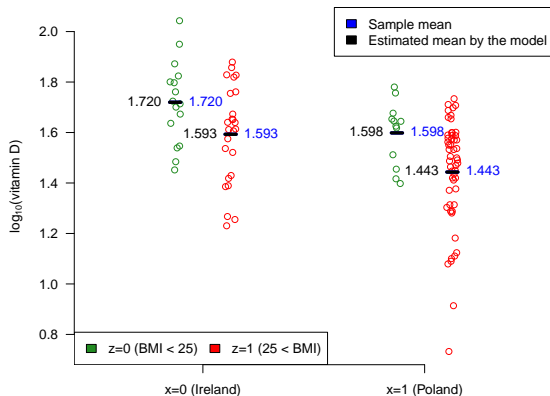
- ▶ $1.593 - 1.720 = -0.127 \approx -0.13$
- ▶ $1.598 - 1.443 = 0.155 \approx 0.16$
- ▶ $1.598 - 1.720 = -0.122 \approx -0.12$
- ▶ $1.443 - 1.593 = -0.15$

R code:

```
publish(lm6)
```

Output:

Variable	Units	Coefficient	CI.95	p-value
(Intercept)		1.72	[1.63;1.81]	< 1e-04
Country(Ireland): bmgroupp(1 vs 0)		-0.13	[-0.25;-0.00]	0.043303
Country(Poland): bmgroupp(1 vs 0)		-0.16	[-0.28;-0.03]	0.013723
bmgroupp(0): Country(Poland vs Ireland)		-0.12	[-0.27;0.03]	0.103562
bmgroupp(1): Country(Poland vs Ireland)		-0.15	[-0.24;-0.06]	0.00188



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Interaction versus subgroup analysis

- ▶ In the two previous examples, the only difference in the model assumptions between using a model with interaction and performing subgroup analysis (one per country) is the way we model the standard deviation of the error term σ_ε : we would model two different values with the subgroup analysis, whereas only one with the interaction model.
- ▶ If we had adjusted on more variables, then the difference would be more important, because the subgroup analysis would implicitly also model interactions with all these other variables.



Case: stratifying vs adjusting with interaction

Comparing estimated parameters:

	Adjust + inter		Subgroup	
	Poland	Ireland	Poland	Ireland
BMI (by 5)	-0.126	-0.050	-0.103	-0.047
Sun: sometimes vs avoid	0.020	0.020	-0.068	0.073
Sun: prefer vs avoid	0.054	0.054	-0.117	0.159

From the three models:

1. `lm(log10(vitd) ~ Country * bmi5b + sunexp, data = irlpolwomen)`
2. `lm(log10(vitd) ~ bmi5b + sunexp, data = poland)`
3. `lm(log10(vitd) ~ bmi5b + sunexp, data = ireland)`

Note: in model 1 (adjust + interaction), we assume that the “effect” of sun exposure is similar in Poland and Ireland, which is not the case with the subgroup analysis.



Final words on modeling

Many topics discussed today and on day 6 are important beyond the linear and logistic model.

Most of the reasoning about modeling choices, including:

- ▶ which variables to include?
- ▶ how? (with or without interaction, categorical version or not...)
- ▶ why does it matter?

applies for more complicated model that you may encounter/study/need during your research career.



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When, why, where to seek statistical help?

When? if you are not sure about how to...

- ▶ plan your experiment or clinical trial
- ▶ analyze your data
- ▶ answer reviewers or collaborators concerns

It is often more complicated than initially thought...

Why?

- ▶ you might get quick help and simple advice that make a big difference.
- ▶ to minimize the risk of “wasting” your precious research time and work by inappropriately analysis your data.
- ▶ why not? it can sometimes be free of charge :-)

Where?

- ▶ Section of Biostatistics (<https://publichealth.ku.dk/about-the-department/biostat/>)
 - ▶ by phone (quick & simple questions): free
 - ▶ short meeting (20 mins): free
 - ▶ new collaborations: sometimes free
- ▶ private consultants also exist.

