Random Walk and its Recurrence and Transience

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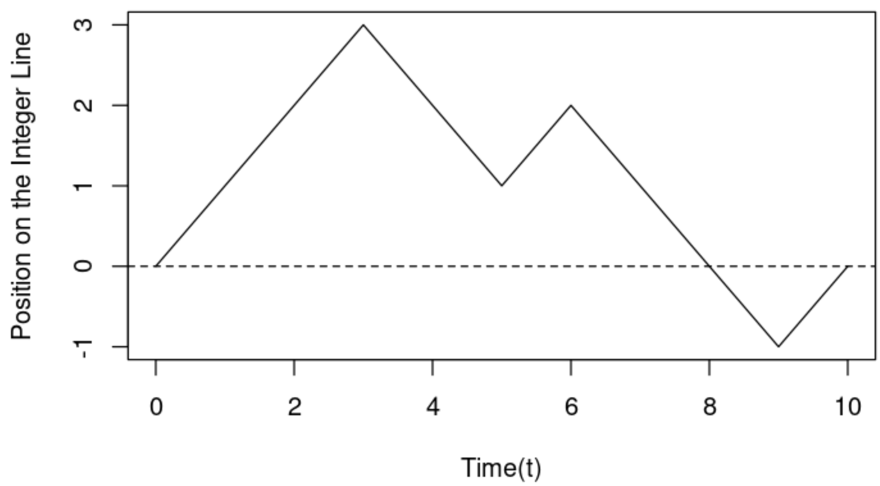
1 Basic Setup

We consider the line of integers, namely **Z**. Now, we consider a bug is walking on this line. Say, as an example, the bug starts at the origin, and moves to 1, then to 2, then to 3, then to 2, then to 1, then to 2 again, then to 1, then to origin, then to -1, then to origin again. Taking each stationary time increment as 1, we obtain the graph given below.



Now, we would like to look into a more formal definition of the *mess* hapenning. We fix $N \in \mathbb{N}$, and we say the *configuration space* is given by the following binary sequence of length N

$$\Omega_N = \left\{\omega = (\omega_1, \ldots, \omega_N) \in \{-1, +1\}^N\right\}$$

Now we write

$$X_k(\omega) = \omega_k$$

To denote the position of the random walk at time k. The position of the random walk after n steps (or after time n) is

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega), \quad 1 \leq n \leq N, \quad S_0(\omega) = 0$$

Thus $\forall \omega \in \Omega_N$, we obtain a trajectory $(S_n)_{n=0}^N$, which we call a *path*. As a probability distribution on Ω_N, we take the uniform distribution, i.e.

$$\mathbb{P}(A) = \frac{|A|}{2^N}, \quad A \subseteq \Omega_N$$

Now we give a formal definition.

Definition: The sequence of rankdom variables $(S_n)_{n=0}^N$ on finite probability space $(\Omega_N, \mathbb{P})$ is known as a *Simple Random Walk of length N starting at 0.* Note that it has independent increments.

2 Distribution of S_n and Probability of returning to the origin

We want to find out $\mathbb{P}(S_n = x)$. Say it takes k steps in $+1$ direction to achieve this. So we have

$$k + (-1)(n-k) = x \implies 2k - n = x \implies k = \frac{n+x}{2}$$

Hence

$$\mathbb{P}(S_n = x) = \frac{\binom{n}{k}}{2^n} = \binom{n}{\frac{n+x}{2}} 2^{-n}$$

Observe that the distribution is symmetric around origin. Moreover, the maximal probability is achieved at the *middle*, or when $x = 0$ and n is even. We have

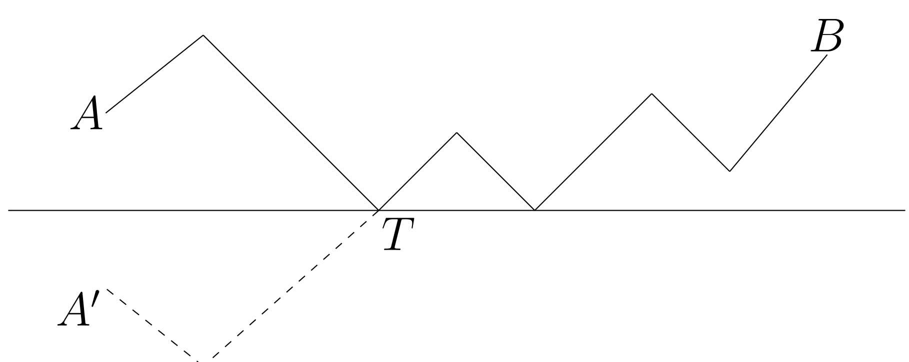
$$\mathbb{P}(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$$

Now we apply Stirling's Approximation $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ for large n, and obtain

$$\mathbb{P}(S_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}}$$

Now, is it really a coincidence that we obtain the mode when the random walk returns to the origin, or doees it have deeper significance? We will observe them at the following sections.

3 The Reflection Principle



Here we have a path which touches the time axis at point T, Now we reflect the path AT to obtain $A'T$, so there exist a one-to one correspondence between the paths from A to T and from A' to T. Hence number of paths from A to B is equal to number of paths from A' to B. We can restate this result in a more useful way.

Let $a \in \mathbb{N}$ and we define $\sigma_a = \min\{n \in \mathbb{N} \mid S_n = a\}$ is the first hitting time of a after time 0. Now for $a, c \in \mathbb{N}$, we have

$$\mathbb{P}(S_n = a - c, \sigma_n \leq n) = \mathbb{P}(S_n = a + c)$$

Now we compute the probabilityof two useful and interesting cases.

$$\begin{aligned}\mathbb{P}(\sigma_a \le n) &= \sum_{b\in\mathbb{Z}} \mathbb{P}(S_n = b, \sigma_a \le n)\\ &= \sum_{b\ge a} \mathbb{P}(S_n = b) + \sum_{b<a} \mathbb{P}(S_n = b, \sigma \le n)\\ &= \mathbb{P}(S_n \ge a) + \mathbb{P}(S_n > a)\\ &= \mathbb{P}(S_n \ge a) + \mathbb{P}(S_n < -a)\\ &= \mathbb{P}(S_n \notin [-a, a-1])\end{aligned}$$

Now, we use this to obtain

$$\mathbb{P}(\sigma_a = n) = \mathbb{P}(\sigma_a \le n) - \mathbb{P}(\sigma_a \le n-1) = \mathbb{P}(S_n \notin [-a, a-1]) - \mathbb{P}(S_{n-1} \notin [-a, a-1])???$$

4 Escape Time Distribution

For $n \in \mathbb{N}$, we want to calculate

$$\begin{aligned}\mathbb{P}(\sigma_0 > 2n) &= \mathbb{P}(S_1 \ne 0, \dots S_{2n} \ne 0)\\ &= 2\mathbb{P}(S_1 > 0, \dots S_{2n} > 0)\\ &= \frac{\text{number of } (2n-1) \text{ step paths starting at 1 which does not visit 0}}{2^{2n}}\\ &= \frac{\text{number of } (2n-1) \text{ step paths starting at 0 which does not visit -1}}{2^{2n}}\\ &= \mathbb{P}(\sigma_{-1} > 2n-1)\\ &= \mathbb{P}(\sigma_1 > 2n-1)\end{aligned}$$

Now we can say

$$\mathbb{P}(S_{2n-1} \in \{-1, 0\}) = \mathbb{P}(S_{2n-1} = -1) = \mathbb{P}(S_{2n-1} = 1) = \binom{2n-1}{n} 2^{-(2n-1)} = \binom{2n}{n} 2^{-2n} = \mathbb{P}(S_{2n} = 0)$$

Now we make two interesting observation.

$$\lim_{n\to\infty} \mathbb{P}(\sigma_0 > 2n) = \lim_{n\to\infty} \frac{1}{\sqrt{\pi n}} = 0$$

On the other hand

$$\mathbb{E}(\sigma_0) = \sum_{n=0}^{\infty} \mathbb{P}(\sigma_0 > n) = 2\sum_{n=0}^{\infty} \mathbb{P}(\sigma_0 > 2n) = 2\sum_{n=0}^{\infty} \mathbb{P}(S_{2n} = 0) \sim 2\sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$$

This suggest that one dimensional simple random walk is *recurrent*, i.e it returns to origin with probabilty 1, but we have to wait for a very long time for that return to happen. So, we call this a *null-recurrent* random walk.

5 Levy's Arcsine Law

Now we consider the *last visit to origin* before time $2N$.

$$L = \max\{0 \le n \le 2N \mid S_n = 0\}$$

Now when $L = 2n$, The random walk returns to the origin, and we have to find number of $2N - 2n$ length paths with $\sigma_0 > 2N - 2n$. So we obtain

$$\mathbb{P}(L = 2n) = \mathbb{P}(S_{2n} = 0)\mathbb{P}(\sigma_0 > 2N - 2n) = \mathbb{P}(S_{2n} = 0)\mathbb{P}(S_{2N-2n} = 0) = 2^{-2N}\binom{2n}{n}\binom{2N-2n}{N-n}$$

Now applying *Stirling's Approximation* again, we obtain

$$\mathbb{P}(L = 2n) \sim \frac{1}{\pi\sqrt{n(N-n)}} \sim \frac{1}{N} f\left(\frac{n}{N}\right), \quad n, N \to \infty, \quad f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad x \in [0,1]$$

Hence

$$\begin{aligned}
\mathbb{P}(\frac{L}{2N} \leq x) &\sim \int_0^x f(t)\,dt \\
&= \int_0^x \frac{1}{\pi\sqrt{t(1-t)}} dt \\
&= \frac{1}{\pi}\int_0^{\arcsin\sqrt{x}} \frac{2\sin\theta\cos\theta d\theta}{\sqrt{\sin^2\theta\cos^2\theta}} \qquad\qquad [\text{Putting } t = \sin^2\theta] \\
&= \frac{2}{\pi}\Big|\theta\Big|_0^{\arcsin\sqrt{x}} \\
&= \frac{2}{\pi}\arcsin\sqrt{x}
\end{aligned}$$

Hence it has distribution function which follows *arcsine* distribution.

6 Random Walk in Higher Dimensions

Now we extend the formal definition of teh random walk on $\mathbb{Z}$ to random walk on $\digamma^d$, $d \in \mathbb{N}$.
We fix $d \in \mathbb{N}$. For $x \in \mathbb{Z}^d$, we write

$$|x| = \sqrt{\left(\sum_{j=1}^{d} x_j^2\right)}$$

For given $N \in \mathbb{N}$, we have

$$\Omega_N = \left\{\omega = (\omega_1, \ldots, \omega_N) \mid \omega_k \in \mathbb{Z}^d, |\omega_k| = 1, \forall 1 \leq n \leq N\right\}$$

In the similar fashion as we did before, we can define $X_k(\omega) = \omega_k$, $1 \leq k \leq N$, and

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega), \quad 1 \leq n \leq N, \quad S_0(\omega) = 0.$$

We stil have an *uniform distribution*; i,e for $A \subseteq \Omega_N$, $\mathbb{P}(A) = \frac{|A|}{(2d)6[-N]}$. We also observe that S_n in this case is a d dimensional random vector.

$$S_n = \begin{pmatrix} S_n^{(1)} \\ \vdots \\ S_N^{(d)} \end{pmatrix}, \quad 0 \leq n \leq N, \quad S_n^{(j)} \in \mathbb{Z}, j = 1, 2, \ldots, d$$

7 Extension to infinite Trajectories

Here, we use a standard technique to define the probability space rigorously when $N \to \infty$.
Let $0 < N < M$. We define a projection $\pi_N : \Omega_M \to \Omega_N$ by

$$\pi_N(\omega_1, \dots, \omega_N, \omega_{N+1}, \dots, \omega_M) = (\omega_1, \dots, \omega_N)$$

Then we obtain a sequence of probability spaces $(\Omega_1, \mathbb{P}_1), (\Omega_2, \mathbb{P}_2), \dots$ satisfying the condition

$$\mathbb{P}_M(\{\omega \in \Omega_M \mid \pi_N\omega = \overline{\omega}\}) = \frac{(2d)^{M-n}}{(2d)^M} = \frac{1}{(2d)^N} = \mathbb{P}_N(\{\overline{\omega}\})$$

Now, we can use *Kolmogorov's Extension Theorem*, a unique probability measure $\mathbb{P}$ on the space of infinite sequence $\Omega = \Omega_\infty$.

$$\Omega = \left\{\omega = (\omega_1, \omega_2, \dots) \mid \omega_k \in \mathbb{Z}^d, |\omega_k| = 1, \forall\, k \in \mathbb{N}\right\}$$

$$\mathbb{P}\left(\{\omega \in \Omega \mid \pi_N\omega = \overline{\omega}\}\right) = \frac{1}{(2d)^N} = \mathbb{P}_N\left(\{\overline{\omega}\}\right)$$

8 SLLN and CLT

As per the definition we have already given; we calculate the expectation of the random vector X_1
Now;

$$X_1^{(j)} = \sum_{x \in \mathbb{Z}^d} x_j \mathbb{P}\left(X_1 = x\right) = 0 \qquad \text{[By Symmetry]}$$

And this holds for all $j = 1, 2, \dots, d$ Now; X_i if X_i 's are independent and identically distributed random variables.

$$\therefore \mathbb{E}\left(X_i\right) = \mathbb{E}\left(X_1\right) = \begin{pmatrix} \mathbb{E}\left(X_1^{(1)}\right) \\ \vdots \\ \mathbb{E}\left(X_1^{(d)}\right) \end{pmatrix} = 0$$

Again;

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{S_n}{n}$$

$\therefore$ By Strong Law of Large Numbers;

$$\lim_{n \to \infty} \frac{S_n}{n} = 0 \quad \textit{almost surely}$$

For a d dimensional normal distribution, the density function is given by

$$f_x(x) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)$$

We say; $X \sim \mathcal{N}_d(\mu, \Sigma)$. We have $X_1, \dots, X_n$ id random variables; $\mathbb{E}\left(x_i\right) = 0$ ai. Now, by Multivariate Central Limit Theorem

$$\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \mathbb{E}\left(x_i\right)\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i = \frac{S_n}{\sqrt{n}} \xrightarrow{\text{distribution}} \mathcal{N}_d\left(0, \Sigma\right)$$

Now, we evaluate this Σ. As the random variables are id; only the diagonal elements are non zero.

$$\therefore \operatorname{Var}(X_i) = \mathbb{E}\left(X_i^2\right) = \sum_{x \in \mathbb{Z}^d} x_i^2 \mathbb{P}(X_i = x) = \frac{1}{d}$$
$$\therefore \frac{S_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}_d\left(0, d^{-1}\mathbb{I}\right)$$

9 Recurrence and Transience

9.1 An Important Result

We start by claiming

$$\mathbb{P}(S_n = 0) = \sum_{i=1}^{n} \mathbb{P}(\sigma_0 = i)\mathbb{P}(S_{n-i} = 0) \quad , \quad n \in \mathbb{N}$$

Now we take $z \in [0,1]$ and consider two generating functions

$$G(0,z) = \sum_{n=0}^{\infty} z^n \mathbb{P}(S_n = 0) \quad F(0,z) = \sum_{n=0}^{\infty} z^n \mathbb{P}(\sigma_0 = n)$$

The motivation of the function G is obtained from *Random Walk Green's Function.* This is defined by

$$G(x,1) = \sum_{n=0}^{\infty} \mathbb{P}(S_n = x) = \sum_{n=0}^{\infty} \mathbb{E}\left(\mathbb{1}_{\{S_n = x\}}\right) = \mathbb{E}\left(\sum_{n=0}^{\infty} \mathbb{1}_{S_n = x}\right)$$

So it gives us expected number of visits to x. Observe $\{\sigma_0 = 0\} = \phi$ and $\mathbb{P}(S_0 = 0) = 1$. Now

$$\begin{aligned} G(0,z) &= 1 + \sum_{n \in \mathbb{N}} z^n \mathbb{P}(S_n = 0) = 1 + \sum_{n \in \mathbb{N}} \sum_{i=1}^{n} z^i \mathbb{P}(\sigma_0 = i) z^{n-i} \mathbb{P}(S_{n-i} = 0) \\ &= 1 + \sum_{i \in \mathbb{N}} z^i \mathbb{P}(\sigma_0 = i) \sum_{j \in \mathbb{N}_0} z^j \mathbb{P}(S_j = 0) = 1 + F(0,z)G(0,z) \end{aligned}$$

This can be rewritten as $F(0,z) = 1 - G(0,z)^{-1}$. Now we have some interesting observations.

$$\sum_{n=1}^{N} \mathbb{P}(\sigma_0 = n) = F(0,1) = \lim_{z \uparrow 1} F(0,z) = 1 - \lim_{z \uparrow 1} \frac{1}{G(0,z)}$$

If we have

$$G(0,1) = \sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) < \infty$$

Then clearly $F(0,1) < \infty$, i.e the walk is *transient.* On the other hand, if

$$G(0,1) = \sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) = \infty$$

Then we fix $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$. such that

$$\sum_{n=0}^{N} \mathbb{P}(S_n = 0) \geq \frac{2}{\varepsilon}$$

Then for z sufficiently close to 1, we have $z^n \geq \frac{1}{2}$, which gives

$$\sum_{n=0}^{N} z^n \mathbb{P}(S_n = 0) \geq \frac{1}{\varepsilon}$$

. So we have

$$\frac{1}{G(0,z)} \leq \frac{1}{\sum_{n=0}^{N} z^n \mathbb{P}(S_n = 0)} \leq \varepsilon$$

Since our ε is arbitray, we have

$$\lim_{z \uparrow 1} G(0,z)^{-1} = 0$$

So, $F(0,1) = 0$, and the walk is *recurrent*.

9.2 Recurrence in 1D

As we have already seen,

$$\mathbb{P}[S_{2n} = 0] \sim \frac{1}{\sqrt{\pi n}}$$

Now

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$$

Hence, the random walk is indeed *recurrent*.

9.3 Recurrence in 2D

We could have done this in the *traditional* way we have already seen, but here we introduce even *sleeker* proof for 2D. Let us consider a *corner walk*, in which each move adds with equal probability one of $\{(1,1), (-1,-1), (1,-1), (-1,1)\}$ to the current location. The advantage of this walk is it is independent in each coordinate and they resemble simple random walk in $\mathbb{Z}$. So

$$\mathbb{P}\{X_{2t} = (0,0)\} = \mathbb{P}_{(0,0)}\left\{X_{2t}^{(1)}\right\}\mathbb{P}_{(0,0)}\left\{X_{2t}^{(2)}\right\} \sim \frac{c}{n}$$

Now, the random walk in $\mathbb{Z}^2$ is the 45 degrees rotation of the *corner walk*. Now, if we consider $\mathbf{Y} = A\mathbf{X}$, then jacobian of the inverse transform has determinant 1 , and it is pretty easy to observe that $Y_{2t}^{(1)}$ and $Y_{2t}^{(2)}$ are indeed independent. As we know $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, the random walk is recurrent.

9.4 Transience in 3D

To cap this all off, we will now calculate u_{2n} for random walks on $\mathbb{Z}^3$ with a method suggested by [2]. We know that there are 6 possible directions to move in, one negative and one positive direction for each of the three axes, so there are a total of 6^{2n} different possible random walks. Of these walks, we are interested in the ones that return to 0 . Let

$$\begin{aligned} j = & \text{ the number of steps in direction } (1,0,0) \\ k = & \text{ the number of steps in the direction } (0,1,0) \end{aligned}$$

In order to return to 0 after the $2n$ steps, there must be an even number of moves in each axis direction, and an equal number in the positive and negative directions for each axis direction. Thus, there are $n-j-k$ steps in direction $(0,0,1)$. The other n steps will be in the respective opposite directions. Now, this just amounts to finding the number of ways to rearrange $2n$ different letters with repeats of $j,j,k,k,n-j-k,n-j-k$. Summing this over all possible j and k gives that

$$\begin{aligned}
u_{2n} &= 6^{-2n} \sum_{\substack{j,k\geq 0\\ j+k\leq n}} \frac{(2n)!}{(j!\cdot k!\cdot(n-j-k)!)^2} \\
&= 6^{-2n}\binom{2n}{n} \sum_{\substack{j,k\geq 0\\ j+k\leq n}} \left(\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right)^2 \\
&= 2^{-2n}\binom{2n}{n} \sum_{\substack{j,k\geq 0\\ j+k\leq n}} \left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right)^2
\end{aligned}$$

The easy part is done now, and what is left is to asymptotically estimate this. Now observe that

$$\sum_{\substack{j,k\geq 0\\ j+k\leq n}} 3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!} = 1$$

because there is a bijection between the number of rearrangements of j A 's, k B's, and $n-j-k$ C's and the number of n letter strings of A 's, B 's, or C 's. There are 3^n such n letter strings, from which we conclude our observation. Now it is possible to show that

$$\sum_{\substack{j,k\geq 0\\ j+k\leq n}} \left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right)^2 \leq \max\left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right)$$

from which it follows that

$$u_{2n} \leq 2^{-2n}\binom{2n}{n}\max\left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right)$$

Using Stirling's Formula again, we see that

$$\begin{aligned}
\frac{n!}{j!\cdot k!\cdot(n-j-k)!} &\sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi j}\left(\frac{j}{e}\right)^j\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\sqrt{2\pi(n-j-k)}\left(\frac{n-j-k}{e}\right)^{n-j-k}} \\
&= \frac{\sqrt{n}}{\sqrt{j\cdot k\cdot(n-j-k)}}\frac{n^n}{j^j k^k (n-j-k)^{n-j-k}}
\end{aligned}$$

It is obvious thatt $j=k=n-j-k\approx\frac{n}{3}$ maximizes this value. For ease of computation, we assume that $\frac{n}{3}$ is integer; we can do this as we are mainly interested in the asymptotics of the value. Thus, we get that

$$\max\left(3^{-n}\frac{n!}{j!\cdot k!\cdot(n-j-k)!}\right) = 3^{-n}\cdot\frac{\sqrt{n}}{\sqrt{\frac{n}{3}\cdot\frac{n}{3}\cdot\frac{n}{3}}}\cdot\frac{n^n}{\left(\frac{n}{3}\right)^{n/3}\left(\frac{n}{3}\right)^{n/3}\left(\frac{n}{3}\right)^{n/3}}$$
$$= 3^{-n}\cdot\frac{c\sqrt{n}}{n\sqrt{n}}\cdot\frac{n^n}{\left(\frac{n}{3}\right)^n}$$
$$= 3^{-n}\cdot\frac{c}{n}\cdot 3^n = \frac{c}{n}$$

where c is probably $3\sqrt{3}$, but for our purpose is simply some constant. Combining all of this, we see that

$$u_{2n} \le 2^{-2n}\binom{2n}{n}\cdot\frac{c}{n} \sim \frac{1}{\sqrt{\pi n}}\cdot\frac{c}{n} = \frac{c}{n^{3/2}} \implies \sum_{n=0}^{\infty} u_{2n} = \sum_{n=0}^{\infty}\frac{c}{n^{3/2}} < \infty$$

and hence by Lemma 2, the random walk in $\mathbb{Z}^3$ is transient.
We have found out that for $d = 3$, the walk is transient. Now for $d = 4$, this can be thought of as two walks, one in $\mathbb{Z}^3$, another in $\mathbb{Z}$, and therefore, we can claim the walk is transient again, and this claim can be further extended $\forall\, d > 3$. From this, we get an interesting result

$$\mathbb{P}(S_n = 0) \sim \Theta(n^{-\frac{d}{2}})$$

And this is another form of *Polya's Recurrence Theorem*. But below, we have an interesting arguement which helps us to formalise this claim rigorously.

9.5 Approaching the Problem using Fourier Analysis

We know that a finite measure μ on $\mathbb{Z}^d$ uniquely determines its characteristic function

$$\phi_\mu(k) = \mathbb{E}\left(e^{ikx}\right) = \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x}\mu(\{x\}), \quad x \in \mathbb{Z}^d$$

Here $k\cdot x = \sum_{i=1}^{d} k_i x_i$ is the standard inner product. Now, we are willing to retrieve $\mu(\{x\})$. So we use Fourier Inversion Formula

$$\mu(\{x\}) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi)^d} e^{-ik\cdot x}\phi_\mu(k)dk$$

Now, we already know that

$$\mu(x) = \frac{1}{2d}\mathbb{1}_{\{|x|=1\}}, \quad x \in \mathbb{Z}^d$$

Via symmetry: we derive an useful representation of $\phi_\mu(k)$

$$\phi_\mu(k) = \frac{1}{d}\sum_{j=1}^{d}\cos(k_j)$$

As $X_1, \ldots, X_n$ are iid random variables; we apply The convolution rule of characteristic function and obtain

$$\mathbb{P}\left(S_n = x\right) = \mathbb{P}\left(\sum X_i = x\right) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} e^{-ik\cdot x} (\phi(k))^n dk$$

$\therefore$ For $z \in [0,1]$; we have

$$\begin{aligned} G(0,z) &= \sum_{n=0}^{\infty} z^n \mathbb{P}\left(S_n = 0\right) = \sum_{n=0}^{\infty} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} z^n (\phi(k))^n dk \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} \frac{dk}{1 - z\phi(k)} \end{aligned}$$

As $\lim_{z\uparrow 1} G(0,z) = G(0,1)$; Hereby we com say

$$G(0,1) < \infty \Longleftrightarrow \int_{(-\pi,\pi)^d} \frac{dk}{1-\phi(k)} < \infty$$

Now, we use a bound for $(1 - \cos k)$ ts obtain our desired result.

$$\begin{aligned} &\frac{2k_j^2}{\pi^2} \leqslant 1 - \cos k_j \leqslant \frac{k_i^2}{2^2} \\ \Longrightarrow &\sum_{j=1}^{d} \frac{2k_j^2}{\pi^2} \leqslant d - \sum_{j=1}^{d} \cos k_j \leqslant \sum_{j=1}^{d} \frac{k_j^2}{2} \\ \Longrightarrow &\frac{2}{dx^2} \sum_{j=1}^{d} k_j^2 \leqslant 1 - \phi(k) \leqslant \frac{1}{2d} \sum_{j=1}^{d} \frac{k^2}{2} k_j^2 \end{aligned}$$

$$\begin{aligned} &\Longrightarrow \frac{d\pi^2}{2} \frac{1}{\|k\|^2} \geqslant \frac{1}{1-\phi(k)} \geqslant 2d \frac{1}{\|k\|^2} \\ &\Longrightarrow \frac{dA^2}{2} \int_{[-\pi,\pi]^d} \frac{dk}{\|k\|^2} \geqslant \int_{[-\pi,\pi)^d} \frac{d(k}{1-\phi(k)} \geqslant 2d \int_{[-\pi,\pi)^d} \frac{dk}{\|k\|^2} \end{aligned}$$

Now, $f(k) = \frac{1}{\|k\|^2}$ is a function on $\mathbb{R}^+$ and $f : \mathbb{R}^+ \to [0,\infty)$. So, we can write

$$\begin{aligned} &\int_{[-\pi,\pi)d} \frac{dk}{\|k\|^2} = 2\,\mathrm{Vol}\left(S^{d-1}\right) \int_0^{\pi} \frac{1}{r^2} r^{d-1} dr \\ &= 2\,\mathrm{Vol}\left(S^{d-1}\right)_0^{\pi} \int_0^{\pi} r^{d-3} dr \\ &\therefore \int_{[\pi,\pi)d} \frac{dk}{1-\phi(k)} \leqslant d\,\mathrm{Vol}\left(s^{d-1}\right) \pi^2 \int_0^{\pi} r^{d-3} dr \end{aligned}$$

This is enough to claim that $G(0,1) < \infty \iff d \geqslant 3$
On the other hand; we use.

$$\int_{[-\pi,\pi)^d} \frac{dk}{1-\phi(k)} \geqslant 4d \operatorname{Vol}\left(S^{d-1}\right) \int_0^\pi r^{d-3} dr$$

To claim $G(0,1) = \infty \iff d = 1, 2$

10 Conclusion

As we explored today, recurrence demonstrates how some systems, no matter how random, have a certain inevitability in their behavior, while transience highlights the unpredictable journey where some paths are taken once and never again. These concepts extend far beyond mathematics—they represent a way of thinking about dynamics, movement, and the probability of return in many real-world systems. So to conclude this talk, i just want to restate a famous quote by *Shizuo Kakutani*

"A drunk man will find his way home, but a drunk bird may get lost forever."