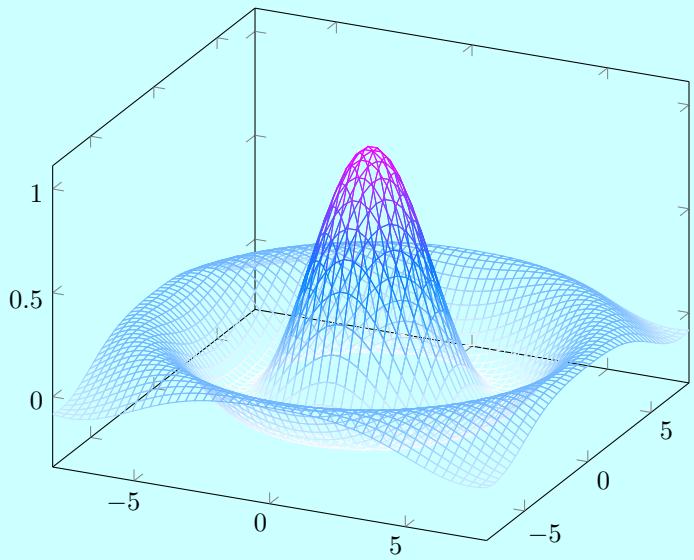
Stability Analysis of Ordinary Differential Equations

Undergraduate Directed Group Reading Project
(Winter) at Indian Statistical Institute, Bangalore.

November 3, 2024



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1 Linear Systems

1.1 Autonomous System

An autonomous System of Ordinary Differential Equations is a system which does NOT explicitly depend on the independent variables. It is of the form

$$\frac{d}{dt}x(t) = f(x(t)) \ ; \quad x \in \mathbb{R}^n$$

Solutions are invariant under horizontal translations.

Proof: Say, $x_1(t)$ is a solution of the ODE $\frac{dx}{dt} = f(x)$, $x(0) = x_0$.
Then $x_2(t) = x_1(t - t_0)$ solves $\frac{dx}{dt} = f(x)$, $x(t_0) = x_0$.
Now we set $s = t - t_0$ which essentially gives $x_2(t) = x_1(s)$ and $ds = dt$. Thus,

$$\frac{d}{dt}x_2(t) = \frac{d}{ds}x_1(s) = f(x_1(s)) = f(x_2(t))$$

And for the initial condition, we have $x_2(t_0) = x_1(t_0 - t_0) = x_0$

An autonomous system of two first order differential equations has the form

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

If the system is linear, we can express it in the given format

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

For which we can write

$$\dot{\mathbf{x}} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\mathbf{x} \ \ ; \ \ (a, b, c, d) \in \mathbb{R}^4$$

1.2 Uncoupled System

An uncoupled system of Ordinary Differential Equations is a system in which differential equation of one of the dependent variables is independent of the others. Clearly in this

case, the matrix A is (NOT always) be diagonal.

$$\frac{dx}{dt} = ax \implies x = c_1 e^{at}$$

$$\frac{dy}{dt} = by \implies y = c_2 e^{bt}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mathbf{x} \implies \mathbf{x} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{C}e^{At}$$

After a bit careful examination, it is evident that the solutions of this differential equation lies on $\mathbb{R}^2$ and they have the form $y = kx^{\frac{b}{a}}$, where $k = \frac{c_2^{\frac{1}{b}}}{c_1^{\frac{1}{a}}}$.

Phase Plane: While trying to describe the motion of the particle governed by the provided differential equations, we can draw the solution curves in the plane $\mathbb{R}^n$, and this is known as the *Phase Plane*. Clearly, in the above uncoupled system, $\mathbb{R}^2$ is the *Phase Plane*.

Phase Portrait: The set of all solution curves drawn in the Phase space is known as *Phase Portrait*.

Dynamical Systems: A dynamical system governed by $\dot{\mathbf{x}} = A\mathbf{x}$ is a function $\phi : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ and it is given by $\phi(\mathbf{C}, t) = \mathbf{C}e^{At}$. Geometrically, it describes motion of points in phase plane along the solution curves.

Equilibrium Point: For $c_1 = c_2 = 0$, $\mathbf{x}(t) = 0 \ \forall t \in \mathbb{R}$ and the origin is referred to as an *equilibrium point* of a system of Differential Equations.

The function $f(\mathbf{x}) = A\mathbf{x}$ defines a mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^n$; which defines a vector field on $\mathbb{R}^n$. If we draw each vector along with its initial points, then we get a pictorial representation of the vector field. It is an interesting observation that at each point in the *phase space*, the solution curve is tangent to the vectors in the vector field. Actually, it is pretty obvious, as at time t, the velocity vector $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ is tangent to the solution curve.

We observe this for $\dot{\mathbf{x}} = I\mathbf{x}$

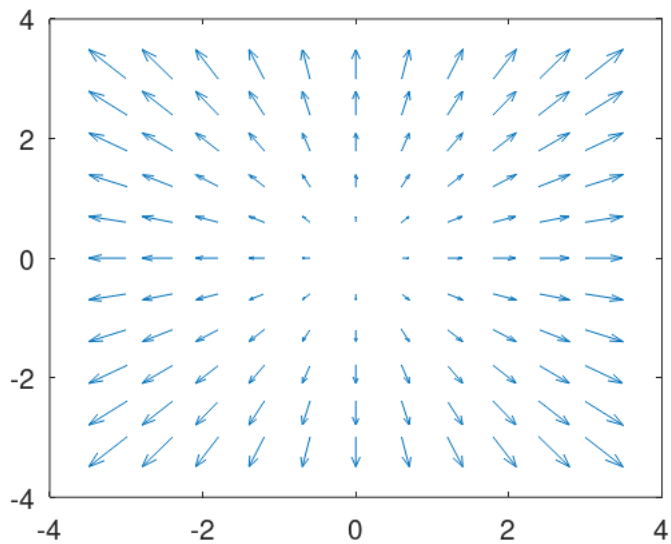


Figure 1: Vector field representation for $\dot{\mathbf{x}} = I\mathbf{x}$

Asymptotic Stability of Origin: Here, we look at

$$\lim_{t\to\infty}(x(t), y(t)) = \lim_{t\to\infty}(c_1 e^{at}, c_2 e^{bt})$$

If $a < 0$ and $b < 0$, then this limit goes to $(0, 0)$. Otherwise, most of the solutions diverge to infinity,
Roughly speaking, an equilibrium (x_0, y_0) is asymptotically stable if every trajectory $(x(t), y(t))$ beginning from an initial condition near (x_0, y_0) stays near (x_0, y_0) for $t > 0$, and

$$\lim_{t\to\infty}(x(t), y(t)) = (x_0, y_0)$$

The equilibrium is unstable if there are trajectories with initial conditions arbitrarily close to the equilibrium that move far away from that equilibrium.
Later on, we will discuss about this in greater detail.

Invariance of the Axes: There is another observation that we can make for uncoupled systems. Suppose that the initial condition for an uncoupled system lies on the x axis; that is, suppose $y_0 = 0$, then the solution $(x(t), y(t)) = (x_0 e^{at}, 0)$ also lies on the x axis for all time. Similarly, if the initial condition lies on the y axis, then the solution $(0, y_0 e^{bt})$ lies on the y axis for all time.

1.3 Diagonalization

Theorem: If *eigenvalues* $\lambda_1, \lambda_2, ..., \lambda_n$ of a matrix A are *real* and *distinct*, then any set of corresponding *eigenvectors* $\{v_1, v_2, ...v_n\}$ forms a basis of $\mathbb{R}^n$. The matrix $P = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix}$ is invertible and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

This *theorem* can be used to reduce the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ to an uncoupled linear system. To do so, we first define the change of coordinates $\mathbf{x} = P\mathbf{y}$. So we have,

$$\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y}$$

$$\implies \dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \mathbf{y}$$

$$\implies \mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \mathbf{y}(0)$$

$$\implies P^{-1}\mathbf{x}(\mathbf{t}) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}\mathbf{x}(0)$$

$$\implies \mathbf{x}(t) = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}\mathbf{x}(0)$$

Stable, Unstable and Center Subspace

It is evident that the solution is stable $\forall\, t \in \mathbb{R}$ iff all *eigenvalues* are negative. Keeping this in mind, we consider $\{v_1, \ldots, v_k\}$ to be the *eigenvectors* corresponding to *negative eigenvalues*, and $\{v_{k+1}, \ldots, v_n\}$ to be the *eigenvectors* corresponding to *positive eigenvalues*.
Then we denote the *stable subspace of the Linear System* by

$$E^S = span\{v_1, \ldots, v_k\}$$

and the *unstable subspace of the Linear System* by

$$E^U = span\{v_{k+1}, \ldots, v_n\}$$

If we have *pure imaginary eigenvalues*, then we also get a *center subspace*, namely E^C.

1.4 Matrix Norm

Here, while performing all the calculations, we consider L^2 norm.
We define the *norm* of a matrix A to be

$$||A|| = \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{||Ax||}{||x||} = \max_{||x|| \leq 1} ||Ax||$$

Some Properties:

- $||A|| \geq 0$; $||A|| = 0 \iff A = 0$.
- $||\lambda A|| \leq |\lambda| \cdot ||A||$, $\lambda \in \mathbb{R}, A \in \mathbb{R}^n$.
- $||A + B|| \leq ||A|| + ||B||$.
- $||Ax|| \leq ||A|| \cdot ||x||$.
- $||AB|| \leq ||A|| \cdot ||B||$.
- $||A^k|| \leq ||A||^k$, $k \in \mathbb{N} \cup \{0\}$
- $\|T^{-1}\| \geq \frac{1}{\|T\|}$

Now, $||Ax|| \geq 0$ for all $||x|| \leq 1$; hence $||A|| \geq 0$.
Now $A = 0 \implies ||Ax|| = 0 \;\; \forall x \in \mathbb{R}^n \implies ||A|| = 0$.
Say, the $(i,j)^{\text{th}}$ entry of A is non zero. Hence $||Ax|| = \sqrt{a_{ij}^2 x_j^2} > 0$. $(x_j \neq 0)$ Similarly by induction, we can show that if k elements of A are non-zero, then $||Ax|| > 0$. Hence if

$||A|| = 0$, then $A = 0$.
$\therefore ||A|| \geq 0$ and $||A|| = 0 \iff A = 0$. ■

On the other hand,

$$||\lambda A|| = \max_{||x||\leq 1} ||\lambda A x|| = \max_{||x||\leq 1} |\lambda| \cdot ||Ax|| = |\lambda| \max_{||x||\leq 1} ||Ax|| = |\lambda| \cdot ||A|| \;\; ; \;\; \lambda \in \mathbb{R} \;\blacksquare$$

Again,

$$||A+B|| = \max_{||x||\leq 1} ||(A+B)x|| \leq \max_{||x||\leq 1} (||Ax||+||Bx||) \leq \left(\max_{||x||\leq 1} ||Ax|| + \max_{||x||\leq 1} ||Bx|| \right) = ||A||+||B|| \;\blacksquare$$

Again,

$$||A|| = \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{||Ax||}{||x||} \implies \frac{||Ax||}{||x||} \leq ||A|| \implies ||Ax|| \leq ||A|| \cdot ||x|| \;\blacksquare$$

Moreover,

$$||AB|| = \max_{||x||\leq 1} ||ABx|| \leq ||A|| \max_{||x||\leq 1} ||Bx|| = ||A|| \cdot ||B|| \;\blacksquare$$

We also observe

$$||A^k|| \leq ||A|| \cdot ||A^{k-1}|| \leq \cdots \leq ||A||^k \;\blacksquare$$

And lastly,

$$1 = \|TT^{-1}\| \leq \|T\| \cdot \|T^{-1}\| \implies \|T^{-1}\| \geq \frac{1}{\|T\|} \;\blacksquare$$

Limit of a Linear Operator: A sequence of linear operators $\{T_k\}_{k\geq 1} \subseteq \mathcal{L}(\mathbb{R}^n)$ is said to converge to a limiting linear operator $T \in \mathcal{L}(\mathbb{R}^n)$ as $k \to \infty$ if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall k \geq N$, $||T_k - T|| < \varepsilon$.

Show that for each $t \in \mathbb{R}$, the solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is a continuous function of the initial condition.

Proof: Say, the solution obtained is given by $\phi(t, \mathbf{x_0}) = \mathbf{x_0} e^{At}$. For a fixed t, we take the matrix norm to be defined analogously as L^2 norm. We also define $\delta := \frac{\varepsilon}{||e^{At}||}$. Now for $||\mathbf{y_0} - \mathbf{x_0}|| < \delta$, we have $||\phi(t, \mathbf{y_0}) - \phi(t, \mathbf{x_0})|| \leq ||e^{At}|| \cdot ||\mathbf{y_0} - \mathbf{x_0}|| < \varepsilon$ ■

1.5 Exponentials of Operators

Say we have a given $T \in \mathcal{L}(\mathbb{R}^n)$ and a given $t_0 \in \mathbb{R}$. Say $||T|| = a$

$$\left\| \frac{T^k t^k}{k!} \right\| \leq \frac{||T^k|| \cdot |t|^k}{k!} \leq \frac{||T||^k \cdot t_0^k}{k!} = \frac{a^k \cdot t_0^k}{k!}$$

for all $|t| < t_0$. Now

$$\sum_{k=0}^{\infty} \frac{(at_0)^k}{k!} = e^{at_0}$$

So, by *Weierstrass M test*, the sum $\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$ converges *uniformly* and *absolutely*. So, now we define the matrix exponential as

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \quad ; \quad t \in \mathbb{R}$$

Note that $\|e^{At}\| \leq e^{\|T\| \cdot |t|}$

Theorem 1: If $P, T \in \mathcal{L}(\mathbb{R}^n)$ and $S = PTP^{-1}$, then $e^S = Pe^T P^{-1}$.

Proof: According to the definition,

$$e^S = \sum_{k=1}^{\infty} \frac{S^k}{k!} = \sum_{k=1}^{\infty} \frac{(PTP^{-1})^k}{k!} = P \sum_{k=1}^{\infty} \frac{T^k}{k!} P^{-1} = Pe^T P^{-1} \quad \blacksquare$$

Theorem 2: If $S, T \in \mathcal{L}(\mathbb{R}^n)$ and they *commute*, then $e^{S+T} = e^S e^T$.

Proof: If S and T *commute*, then

$$(S+T)^n = \sum_{k=0}^{n} \binom{n}{k} S^k T^{n-k}$$

Therefore

$$e^{S+T} = \sum_{n=0}^{\infty} \frac{(S+T)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+j=n} \frac{n!}{k!j!} S^k T^j = \left(\sum_{k=0}^{\infty} \frac{S^k}{k!} \right) \cdot \left(\sum_{j=0}^{\infty} \frac{T^j}{j!} \right) = e^S e^T \quad \blacksquare$$

Theorem 3: if $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then $e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$

Proof: we take $z = a + ib = re^{i\theta}$; under standard notations. Now we can write

$$A^2 = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = \begin{bmatrix} r^2\cos 2\theta & -r^2\sin 2\theta \\ r^2\sin 2\theta & r^2\cos 2\theta \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(z^2) & -\mathrm{Im}(z^2) \\ \mathrm{Im}(z^2) & \mathrm{Re}(z^2) \end{bmatrix}$$

Thus by induction,$A^k = \begin{bmatrix} \mathrm{Re}(z^k) & -\mathrm{Im}(z^k) \\ \mathrm{Im}(z^k) & \mathrm{Re}(z^k) \end{bmatrix}$. Now we have

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \begin{bmatrix} \mathrm{Re}(\frac{z^k}{k!}) & -\mathrm{Im}(\frac{z^k}{k!}) \\ \mathrm{Im}(\frac{z^k}{k!}) & \mathrm{Re}(\frac{z^k}{k!}) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(e^z) & -\mathrm{Im}(e^z) \\ \mathrm{Im}(e^z) & \mathrm{Re}(e^z) \end{bmatrix}$$

Now $e^z = e^{a+ib} = e^a(\cos b + i \sin b)$, so we have $\mathrm{Re}(e^z) = e^a \cos b$ and $\mathrm{Im}(e^z) = e^a \sin b$.

$$\therefore e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix} \quad \blacksquare$$

Note: *If $a = 0$, this matrix represents anticlockwise rotation by b degrees.*

Theorem 4: If $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, then $e^A = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

Proof: $A = aI + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = aI + B$. Clearly aI and B commute. Moreover $B^k = 0 \, \forall k \geq 2 \implies e^B = I + B$. So we can hereby conclude

$$e^A = e^{aI+B} = e^a e^B = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \blacksquare$$

Theorem 5: If $A = PDP^{-1}$, where D is *diagonal*, then $\det(e^A) = e^{\mathrm{trace}(D)}$.

Proof: $e^A = Pe^D P^{-1} \implies \det(e^A) = \det(Pe^D P^{-1}) = \det(e^D)$ As D is *diagonal* we can write,

$$e^A = e^{\mathrm{trace}(D)} = e^{\mathrm{trace}(P^{-1}AP)} = e^{\mathrm{trace}(P^{-1}PA)} = e^{\mathrm{trace}(A)} \quad \blacksquare$$

Theorem 6: If $\mathbf{x}$ is an *eigenvector* of T with *eigenvalue* λ, then $\mathbf{x}$ is also an *eigenvector* of e^T with *eigenvalue* e^λ

Proof: $T^2\mathbf{x} = T(\lambda\mathbf{x}) = \lambda(T\mathbf{x}) = \lambda^2\mathbf{x}$. Thus by induction, $T^k\mathbf{x} = \lambda^k\mathbf{x}$. Now we have

$$e^T\mathbf{x} = \left(\sum_{k=0}^{\infty} \frac{T^k}{k!}\right)\mathbf{x} = \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right)\mathbf{x} = e^\lambda\mathbf{x} \quad \blacksquare$$

Theorem 7: $T \in \mathcal{L}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$ is T invariant; then show E is also e^T invariant.

Proof: Clearly if $\mathbf{v} \in E$, then it is a linear combination of the basis vectors, so is $\frac{\mathbf{v}}{k!}$, $k \in \mathbb{N} \cup \{0\}$.
On the other hand, it is trivial that $E \supseteq T(E) \supseteq T^2(E) \supseteq \cdots \supseteq T^k(E) \supseteq \ldots$
Say, we have

$$\mathbf{v} = \bigcap_{k \in \mathbb{N} \cup \{0\}} T^k(E) \qquad \text{(We define } T^0(E) = E\text{)}$$

Now,

$$e^T(\mathbf{v}) = \left(\sum_{k=0}^{\infty} \frac{T^k}{k!} \right)(\mathbf{v}) = \sum_{k=0}^{\infty} T^k \left(\frac{\mathbf{v}}{k!} \right)$$

We know

$$T^k \left(\frac{\mathbf{v}}{k!} \right) \in E \ \forall k \in \mathbb{N} \cup \{0\}$$

These altogether concludes

$$e^T(\mathbf{v}) \in E \implies e^T(E) \subseteq E \ \blacksquare$$

1.6 The Fundamental Theorem for Linear Systems

Here, our aim is to establish the fact that for $\mathbf{x_0} \in \mathbb{R}^n$, the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$\mathbf{x}(\mathbf{0}) = \mathbf{x_0}$$

has a unique solution $\forall t \in \mathbb{R}$ which is given by

$$\mathbf{x}(t) = \mathbf{x_0} e^{At}$$

Lemma: Let A be a square matrix. Then

$$\frac{d}{dt} e^{At} = A e^{At}$$

Proof:

$$\frac{d}{dt} e^{At} = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h} = e^{At} \lim_{h \to 0} \frac{e^{Ah} - I}{h} = e^{At} \lim_{h \to 0} \left(A + \sum_{k=1}^{\infty} \frac{A^{k+1} h^k}{(k+1)!} \right) = A e^{At}$$

Note: *Here, we can place the limit inside the summation because* $|h| \leq 1$

If $\mathbf{x}(t)$ has the mentioned form, then we can easily observe

$$\mathbf{x}'(t) = \frac{d}{dt}\mathbf{x_0}e^{At} = \mathbf{x_0}Ae^{At} = A\mathbf{x}(t)$$

Now to show that this is the only solution, we consider $\mathbf{x}(t)$ to be any solution of the provided initial value problem. Now we fix $\mathbf{y}(t) = e^{-At}\mathbf{x}(t)$. Now we differentiate both side to obtain

$$\mathbf{y}'(t) = -Ae^{-At}\mathbf{x_0} + e^{-At}\mathbf{x}'(t) = -Ae^{-At}\mathbf{x_0} + e^{-At}A\mathbf{x}(t) = 0$$

Setting $t = 0$, we obtain $\mathbf{y}(0) = \mathbf{x_0}$, and this suffices the proof of uniqueness. ■

1.7 Linear Systems in $\mathbb{R}^2$

In this section, we describe various *phase portraits* of the equation

$$\dot{\mathbf{x}} = A\mathbf{x} \quad , \quad \mathbf{x} \in \mathbb{R}^2$$

Say, $\mathbf{v}$ is an *eigenvector* of A with *eigenvalue* λ. Now, $\mathbf{x} = A\mathbf{v}$, where a is a *scalar*. Hence

$$\dot{\mathbf{x}} = A(a\mathbf{v}) = a\lambda v$$

The derivative is a multiple of $\mathbf{v}$ and hence points along the line determined by $\mathbf{v}$. As $\lambda > 0$, the derivative points in the direction of $\mathbf{v}$ when a is positive and in the opposite direction when a is negative.
We consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and we draw the *vector field* and a couple of solutions *(go to next page)*. Notice that the picture looks like a ***source*** with arrows coming out from the origin. Hence we call this type of picture a source or sometimes an ***unstable node***.

If $A = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$, then both the *eigenvalues* are *negative*. We call this kind of picture a ***sink*** or sometimes a ***stable node***.

If $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$, then one *eigenvalue* is positive, and the other is negative. Then, we reverse the arrows on one line (corresponding to the negative eigenvalue) in **Figure 2**. This is known as a ***Saddle***

Suppose the eigenvalues are purely imaginary. That is, suppose the eigenvalues are $\pm ib$. For example, let $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$. Consider the eigenvalue $2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$. The real and imaginary parts of $\vec{v}e^{i2t}$ are

$$\text{Re}\begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{i2t} = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}, \quad \text{Im}\begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{i2t} = \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$

We can take any linear combination of them to get other solutions, which one we take depends on the initial conditions. Now note that the real part is a *parametric equation* for an *ellipse*. Same with the imaginary part and in fact any linear combination of the two. This is what happens in general when the *eigenvalues* are *purely imaginary*. So when the eigenvalues are purely imaginary, we get ellipses for the solutions. This type of picture is sometimes called a ***center***.

Now suppose the complex eigenvalues have a positive real part. For example, let $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$. We take $1 + 2i$ and its eigenvector $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$, and find the real and imaginary of $\vec{v}e^{(1+2i)t}$ are

$$\mathrm{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} \quad \mathrm{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$

Note the e^t in front of the solutions. This means that the solutions grow in magnitude while spinning around the origin. Hence we get a ***spiral source.***

Finally suppose the complex eigenvalues have a negative real part. Here we get a e^{-t} in front of the solution. This means that the solutions shrink in magnitude while spinning around the origin. Hence we get a ***spiral sink***.

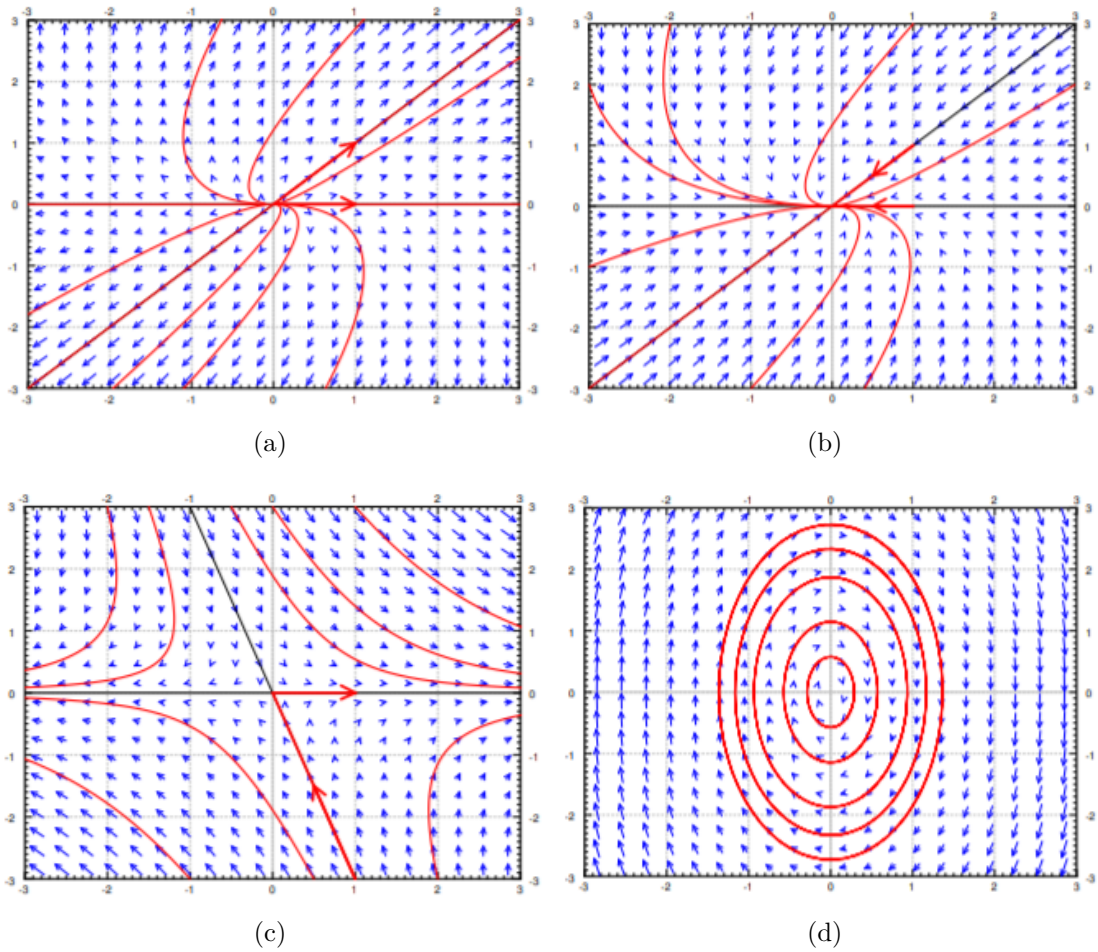


Figure 2: (a) Source (b) Sink (c) Saddle (d) Center

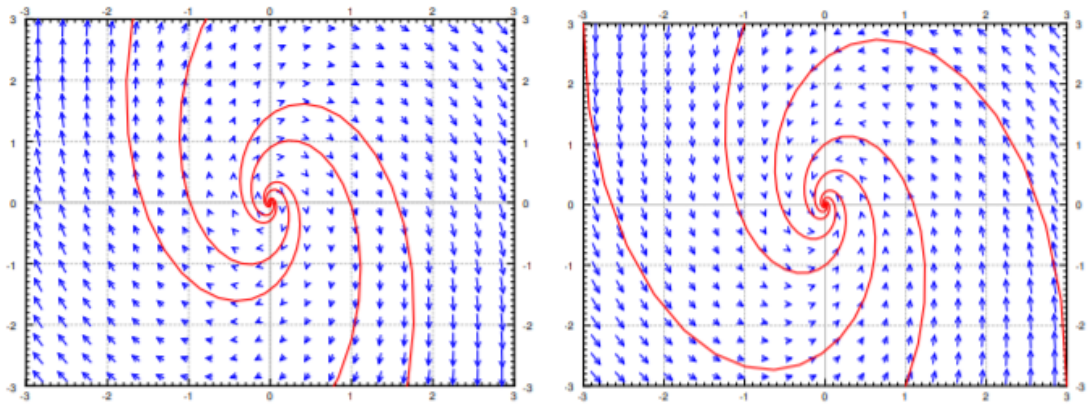
1.8 System with Complex *eigenvalues*

If $A \in GL_{2n}(\mathbb{R})$ and has complex *eigenvalues*, they occur as *conjugate pairs.* The following Theorem gives us an insight about this.

Theorem: If $A \in GL_{2n}(\mathbb{R})$ has $2n$ distinct *complex eigenvalues*, $\lambda_j = a_j + ib_j$ and $\overline{\lambda_j} = a_j - ib_j$, $\forall j = 1(1)n$ with corresponding *eigenvectors* $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\overline{\mathbf{w}_j} = \mathbf{u}_j - i\mathbf{v}_j$; then $\{\mathbf{u_1}, \mathbf{v_1}, \ldots, \mathbf{u_n}, \mathbf{v_n}\}$ forms a *basis* for $\mathbb{R}^{2n}$. Moreover the matrix $P = \begin{bmatrix}\mathbf{v_1} & \mathbf{u_1} & \ldots & \mathbf{v_n} & \mathbf{u_n}\end{bmatrix}$ is *invertible* and

$$P^{-1}AP = \operatorname{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

is a $2n \times 2n$ matrix with 2×2 blocks across the diagonal.



(a) (b)

Figure 3: (a) Spiral Source (b)Spiral Sink

Proof: We say, if V is a real vector space, its *complexification* $V^{\mathbb{C}}$ is the complex vector space consisting of elements $x + iy$ where $x, y \in V$. If $T : V \to W$, its *complexification* $T^{\mathbb{C}} : V^{\mathbb{C}} \to W^{\mathbb{C}}$ is defined by

$$T^{\mathbb{C}}(x + iy) = Tx + iTy$$

Clearly $T^{\mathbb{C}}$ has same *eigenvalues* as of T. So we have $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ and $\overline{\mathbf{w}} = \mathbf{u} - i\mathbf{v}$ in $V^{\mathbb{C}}$ with *eigenvalues* λ and $\overline{\lambda}$. Clearly

$$\mathbf{u} = \frac{\mathbf{w} + \overline{\mathbf{w}}}{2} \quad , \quad \mathbf{v} = \frac{\mathbf{w} - \overline{\mathbf{w}}}{2i}$$

Clearly, $\mathbf{u}$ and $\mathbf{v}$ are *linearly independent*, and they form a *basis* for V. Now we want to compute the matrix of T with respect to this new *basis*. So we compute

$$T^{\mathbb{C}}(\mathbf{w}) = \lambda\mathbf{w} = (a + ib)(\mathbf{u} + i\mathbf{v}) = (a\mathbf{u} - b\mathbf{v}) + i(a\mathbf{v} + b\mathbf{u})$$

Moreover we also have

$$T^{\mathbb{C}}(\mathbf{w}) = T\mathbf{u} + iT\mathbf{v}$$

So, on comparison, we have

$$T\mathbf{v} = a\mathbf{v} + b\mathbf{u} = \begin{bmatrix} \mathbf{v} & \mathbf{u} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad , \quad T\mathbf{u} = a\mathbf{u} - b\mathbf{v} = \begin{bmatrix} \mathbf{v} & \mathbf{u} \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix}$$

So, clearly in the *basis* $\{\mathbf{v}, \mathbf{u}\}$, the matrix of T is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and hereby we can conclude the matrix $P = \begin{bmatrix} \mathbf{v_1} & \mathbf{u_1} & \dots & \mathbf{v_n} & \mathbf{u_n} \end{bmatrix}$ is *invertible* and

$$P^{-1}AP = \operatorname{diag} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

is a 2×2 matrix with blocks across the diagonal. ■

If we use $P = \begin{bmatrix} \mathbf{u_1} & \mathbf{v_1} & \dots & \mathbf{u_n} & \mathbf{v_n} \end{bmatrix}$, then we have

$$P^{-1}AP = \text{diag} \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$$

So, now we have the solution of the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x} \quad , \quad \mathbf{x}(\mathbf{0}) = \mathbf{x_0}$$

as

$$\mathbf{x}(t) = P \text{ diag } e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1}\mathbf{x_0}$$

1.9 Multiple *eigenvalues*

Till now, we have only dealt with those systems which have *distinct eigenvalues*. Now, we want to solve the system where A has *multiple eigenvalues*.

Definition:

Let λ be a *eigenvalue* of a $n \times n$ matrix A with multiplicity $m \leq n$. Then for $k = 1(1)m$, any non-zero solution $\mathbf{v}$ of

$$(A - \lambda I)^k \mathbf{v} = 0$$

is known as a *generalised eigenvector* of A.

Theorem: If $T \in \mathcal{L}(V)$ with *real eigenvalues*, then there is *only one way* of writing T as $S + N$, where S is *diagonalizable*, N is *nilpotent*, and $SN = NS$.

Proof: Let E_k be the *generalised eigenspace* of T, $\forall k = 1(1)m$. We define $T_k = T|_{E_k}$. Now we have

$$V = \bigoplus_{k=1}^{m} E_k \quad , \quad T = \bigoplus_{k=1}^{m} T_k$$

Note that S and N both commute with S and N, hence both of them commute with $T = S + N$ as well. So we have E_k is *invariant* under S and N. Now we say $S_k = \lambda_k I \in \mathcal{L}(E_k)$ and $N_k = T_k - S_k$. If we can show $S|_{E_k} = S_k$, it will then conclude $N|_{E_k} = N_k$, and thus we can show the *uniqueness*.
Enough to show $S|_{E_k} - S_k = \mathbf{0}$.
Now, it is given that S is *diagonalizable*, so is $S|_{E_k}$, then $S|_{E_k} - \lambda_k I$ is also *diagonalizable*.

Hence $S|_{E_k} - S_k$ is diagonalizable.
Now, on the other hand, $S|_{E_k} - S_k = T|_{E_k} - N|_{E_k} - T_k + N_k = N_k - N|_{E_k}$. Here, $N|_{E_k}$ commutes with T_k and $\lambda_k I$; so it also commutes with N_k. Using *Binomial Theorem*, we can hereby conclude that $N_k - N|_{E_k}$ is *nilpotent*.
So, $S|_{E_k} - S_k$ is a *nilpotent* and *diagonal* matrix, i.e $S|_{E_k} - S_k = \mathbf{0}$ ■

So, now we have the solution of the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x} \quad , \quad \mathbf{x}(\mathbf{0}) = \mathbf{x_0}$$

as

$$\mathbf{x}(t) = P \operatorname{diag}\left[e^{\lambda_j t}\right] P^{-1} \left[I + Nt + \cdots + \frac{N^k t^k}{k!}\right] \mathbf{x_0}$$

If λ is an*eigenvalue* with multiplicity n, then the solution of the *initial value problem* is

$$\mathbf{x}(t) = e^{\lambda t} \left[I + Nt + \cdots + \frac{N^k t^k}{k!}\right] \mathbf{x_0}$$

Under the light of this theorem, we can right the theorem discussed in the previous section in a newly tailored way.

Theorem: If $A \in GL_{2n}(\mathbb{R})$ has $2n$ *complex eigenvalues*, $\lambda_j = a_j + ib_j$ and $\overline{\lambda_j} = a_j - ib_j$, $\forall j = 1(1)n$ with corresponding *eigenvectors* $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\overline{\mathbf{w}_j} = \mathbf{u}_j - i\mathbf{v}_j$; then $\{\mathbf{u_1}, \mathbf{v_1}, \ldots, \mathbf{u_n}, \mathbf{v_n}\}$ forms a *basis* for $\mathbb{R}^{2n}$. Moreover the matrix $P = \begin{bmatrix}\mathbf{v_1} & \mathbf{u_1} & \ldots & \mathbf{v_n} & \mathbf{u_n}\end{bmatrix}$ is *invertible*, $A = S + N$ and

$$P^{-1}SP = \operatorname{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

is a $2n \times 2n$ matrix with 2×2 blocks across the diagonal, the matrix N is *nilpotent* of order $k \leq 2n$.

So, now we have the solution of the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x} \quad , \quad \mathbf{x}(\mathbf{0}) = \mathbf{x_0}$$

as

$$\mathbf{x}(t) = P \operatorname{diag} e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1} \left[I + Nt + \cdots + \frac{N^k t^k}{k!}\right] \mathbf{x_0}$$

1.10 Jordan Form

1.11 Stability Theory

Say, for am matrix A has *generalised eigenvalues* $\lambda_j = a_j + ib_j$ and *generalised eigenvector* $\mathbf{v}_j = \mathbf{u}_j + i\mathbf{w}_j$. Then the *stable subspace*, *unstable subspace* and *central subspace* is given

by

$$E^S = \text{Span}\{\mathbf{u}_j, \mathbf{v}_j \mid a_j < 0\}$$
$$E^U = \text{Span}\{\mathbf{u}_j, \mathbf{v}_j \mid a_j > 0\}$$
$$E^C = \text{Span}\{\mathbf{u}_j, \mathbf{v}_j \mid a_j = 0\}$$

Solutions in E^S tend to approach $\mathbf{x}(\mathbf{0})$ as $t \to \infty$; and solutions in E^U tend to approach $\mathbf{x}(\mathbf{0})$ as $t \to -\infty$.
The set of mappings $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ may be regarded as the movement of points $\mathbf{x_0} \in \mathbb{R}^n$ along the trajectories.

Hyperbolic flow: If all *eigenvalues* of A has non-zero real parts, then the *flow* $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic flow*, and the corresponding linear system is known as *hyperbolic linear system.*

A subspace $E \subset \mathbb{R}^n$ is said to be *invariant with respect to the flow* if $e^{At}E \subset E$, $\forall t \in \mathbb{R}$.

Lemma: Let E be a *generalised eigenspace* of matrix A with respect to its *generalised eigenvalue* λ. Show that $AE \subset E$.

Proof: Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be basis of *generalised eigenvectors* for E. Then for $\mathbf{v} \in E$

$$\mathbf{v} = \sum_{k=1}^{n} c_k \mathbf{v}_k \implies A\mathbf{v} = \sum_{k=1}^{n} c_k A\mathbf{v}_k$$

Now each of the $\mathbf{v}_k$s being *generalised eigenvectors*, we say

$$\mathbf{V}_k = (A - \lambda I) \quad \mathbf{V}_k \in \text{Ker}(A - \lambda I)^{j-1} \subset E$$

Thus, by induction, $A\mathbf{v}_k = \lambda \mathbf{v}_k + \mathbf{V}_k \in E$, so does their linear combination. Hence $AE \subset E$ ■

Clearly, according to the definition $\mathbb{R}^n = E^S \oplus E^U \oplus E^C$.
For $\mathbf{x_0} \in E^S$,

$$\mathbf{x_0} = \sum_{k=1}^{n_s} c_k \mathbf{V}_k \quad \{\mathbf{V}_k\}_{k=1}^{n_s} \subset B \text{ is a } \textit{basis} \text{ for the stable subspace } E^S$$

Now,

$$e^{At}\mathbf{x_0} = \sum_{k=1}^{n_s} c_k e^{At} \mathbf{V}_k$$

As $A^k \mathbf{V}_j \in E^S$, then $e^{At}\mathbf{x_0} \in E^S$, $\forall t \in \mathbb{R}$.

So, E^S is *invariant* with respect to the *flow*, so is E^U and E^C. ■

Sink (or Source): If all *eigenvalues* has negative (or positive) real part, then the *origin* is known as *sink (or source)* of the linear system.

Theorem: The following statements are equivalent

(a) For all $\mathbf{x_0} \in \mathbb{R}^n$ $\lim_{t\to\infty} e^{At}\mathbf{x_0} = 0$ and for $\mathbf{x_0} \neq \mathbf{0}$, $\lim_{t\to-\infty} e^{At}\mathbf{x_0} = \infty$

(b) All *eigenvalues* of A has negative real part.

(c) There are positive constants $a, c, m, M \in \mathbb{R}$ such that $\forall \mathbf{x_0} \in \mathbb{R}^n$

$$|e^{At}\mathbf{x_0}| \leq Me^{-ct}\mathbf{x_0} \quad t \geq 0$$

$$|e^{At}\mathbf{x_0}| \geq me^{-at}\mathbf{x_0} \quad t \leq 0$$

Proof: Here we use the fact that any solution of the linear system is the linear combination of functions of the form $t^k e^{at}\cos bt$ or $t^k e^{at}\sin bt$.
Say, one of the *eigenvalues* has positive real part. For that particular *eigenvalue* $\forall \mathbf{x_0} \neq 0$ $\lim_{t\to\infty} e^{At}\mathbf{x_0} = \infty$ and for $\mathbf{x_0} \in \mathbb{R}^n$, $\lim_{t\to-\infty} e^{At}\mathbf{x_0} = 0$, contradicting (a). If one of the *eigenvalues* has a zero real part, then the solution is of the form $t^k \cos bt$ or $t^k \sin bt$, and again clearly $\forall \mathbf{x_0} \in \mathbb{R}^n$ $\lim_{t\to\infty} e^{At}\mathbf{x_0} \neq 0$ So, we can say (a) $\Longrightarrow$ (b). ■
sin and cos being periodic function, for *eigenvalues* with a negative *real* part, We can give them a bound as described in (c). So (b) $\Longrightarrow$ (c). ■
Using squeeze theorem on the relation obtained at (c) and by taking $t \to \infty$, we get $\forall \mathbf{x_0} \in \mathbb{R}^n$ $\lim_{t\to\infty} e^{At}\mathbf{x_0} = 0$, and the second inequality in part (c) gives us $\mathbf{x_0} \neq \mathbf{0}$, $\lim_{t\to-\infty} e^{At}\mathbf{x_0} = \infty$. Hence (c) $\Longrightarrow$ (a). ■

In the similar fashion, we can devise another theorem, with similar proof.

Theorem: The following statements are equivalent

(a) For all $\mathbf{x_0} \in \mathbb{R}^n$ $\lim_{t\to-\infty} e^{At}\mathbf{x_0} = 0$ and for $\mathbf{x_0} \neq \mathbf{0}$, $\lim_{t\to\infty} e^{At}\mathbf{x_0} = \infty$

(b) All *eigenvalues* of A has positive real part.

(c) There are positive constants $a, c, m, M \in \mathbb{R}$ such that $\forall \mathbf{x_0} \in \mathbb{R}^n$

$$|e^{At}\mathbf{x_0}| \leq Me^{ct}\mathbf{x_0} \quad t \leq 0$$

$$|e^{At}\mathbf{x_0}| \geq me^{at}\mathbf{x_0} \quad t \geq 0$$

1.12 Nonhomogeneous Linear Systems

In this section, we are concerned about differential equations of the type

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$$

Where A is a $n \times n$ matrix and $\mathbf{b}(t)$ is a vector valued function.

Fundamental Matrix Solution: A fundamental matrix solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is any *nonsingular* $n \times n$ matrix function $\mathbf{\Phi}(t)$ that satisfies $\mathbf{\Phi}'(t) = A\mathbf{\Phi}(t)$, $\forall t \in \mathbb{R}$.

Once we find a *Fundamental Matrix Solution* for the *homogeneous* system , we can find the solution to the corresponding *nonhomogeneous* system.

Theorem: If $\mathbf{\Phi}(t)$ is a *fundamental matrix solution*, then the solution of the *nonhomogeneous* system and the initial condition $\mathbf{x}(\mathbf{0}) = x_0$ is *unique*, and is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\mathbf{0})\mathbf{x_0} + \int_0^t \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\tau)\mathbf{b}(\tau)\, d\tau$$

Proof: We differentiate $\mathbf{x}(t)$ as defined above.

$$\dot{\mathbf{x}} = \mathbf{\Phi}'(t)\mathbf{\Phi}^{-1}(\mathbf{0})\mathbf{x_0} + \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t)\mathbf{b}(t) + \int_0^t \mathbf{\Phi}'(t)\mathbf{\Phi}^{-1}(\tau)\mathbf{b}(\tau)\, d\tau$$

$$\implies \dot{\mathbf{x}} = A\left(\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\mathbf{0})\mathbf{x_0} + \int_0^t \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(\tau)\mathbf{b}(\tau)\, d\tau\right) + \mathbf{b}(t)$$

$$\therefore \dot{\mathbf{x}} = A\mathbf{x}(t) + \mathbf{b}(t) \ \blacksquare$$

With $\mathbf{\Phi}(t) = e^{At}$, the solution of the *nonhomogeneous* linear system looks like

$$\mathbf{x}(t) = e^{At}\mathbf{x_0} + e^{At}\int_0^t e^{-A\tau}\mathbf{b}(\tau)\, d\tau$$

2 Nonlinear Systems: Local Theory

2.1 Some Preliminary Concepts and Definitions

Differentiability: The function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *differentiable* at $\mathbf{x_0} \in \mathbb{R}^n$ if there exists a linear transformation $D\mathbf{f}(\mathbf{x_0}) \in \mathcal{L}(\mathbb{R}^n)$ that satisfies

$$\lim_{||h||\to 0} \frac{||\mathbf{f}(\mathbf{x_0}+\mathbf{h}) - \mathbf{f}(\mathbf{x_0}) - D\mathbf{f}(\mathbf{x_0})\mathbf{h}||}{||\mathbf{h}||} = 0$$

The linear transformation $D\mathbf{f}(\mathbf{x_0})$ is the derrivative of $\mathbf{f}$ at $\mathbf{x_0}$. Now we look into a theorem that enables us to compute derrivative in coordinates.

Theorem: Consider a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ differentiable at $a \in \mathbb{R}^n$. Then all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist at a. In particular, for f differentiable at a, we have,

$$(Df)(a) = J_f(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right)_{m\times n}$$

Proof: Without loss of generality, we take $m = 1$, and let $a = (a_1, a_2, \ldots, a_n)$. Fix an arbitrary index $i \in \{1, 2, \ldots, n\}$. We define $\eta_i : [a_i - \epsilon, a_i + \epsilon] \to \mathbb{R}^n$, defined by

$$\eta_i(t) = (a_1, \ldots, a_{i-1}, t, a_{t+1}, \ldots, a_n) = a + (t - a_i)e_i$$

As $\mathcal{O}_n$ is open and η_i is continuous, we can find ϵ small such that $f([[a_i - \epsilon, a_i + \epsilon]]) \subseteq \mathbb{R}^n$. Evidently, η_i is differentiable and $(D\eta_i) = [0, \ldots, 1, \ldots, 0]^t = e_i^t$ over $[a_i - \epsilon, a_i + \epsilon]$. Now, by the definition of partial derivatives, $D(f \circ \eta_i)(a_i) = f_{x_i}(a)$.

Again, by chain rule, as f is differentiable at a, $D(f \circ \eta_i)(a_i) = f_{x_i}(a)$ exists, and

$$\begin{aligned} D(f \circ \eta_i)(a_i) &= Df(\eta_i(a_i)) \cdot D\eta_i(a_i) \\ \implies f_{x_i}(a) &= Df(a) \cdot e_i^t = [Df(a)]_i \end{aligned}$$

As the index i was arbitrary to begin with, this completes the proof. ■

Continuity: Suppose V_1 and V_2 be two *normed* linear spaces with respective norms $||\cdots||_1$ and $||.||_2$. Then $\mathbf{f} : V_1 \to V_2$ is continuous at $\mathbf{x_0} \in V_1$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\mathbf{x} \in V_1$ and $||\mathbf{x} - \mathbf{x_0}||_1 \delta$ implies $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x_0})||_2 < \varepsilon$. $\mathbf{f}$ is said to be continuous on $E \subseteq V_1$ if it is continuous for all points in E, and we write $\mathbf{f} \in \mathcal{C}(E)$.

$\mathcal{C}^1(E)$ **Functions:** If the function $\mathbf{f} : E \to \mathbb{R}^n$ is differentiable on E, then we say $\mathbf{f} \in \mathcal{C}^1(E)$.
The following theorem, almost analogous to the previous one, helps us to decide whether a function belongs to $\mathcal{C}^1(E)$.

Theorem: Suppose E is an open subset of $\mathbb{R}^n$ and $\mathbf{f} : E \to \mathbb{R}^n$. Then $\mathbf{f} \in \mathcal{C}^1(E)$ *iff* $\frac{\partial f_i}{\partial x_j}$ exists $\forall i, j = 1(1)n$, and they are *continuous*.

Remarks: Higher order derrivatives can be defined in a similar fashion. And the similar notion holds for the conditionof $f \in \mathcal{C}^k$.
A function $\mathbf{f} : R \to \mathbb{R}^n$is said to be *analytic*, if each of its components are *analytic*, i.e for $j = 1(1)n$ and $\mathbf{x_0} \in E$, $f_j(x)$ has a *taylor series* which converges to $f_j(x)$ in some neighborhood of $\mathbf{x_0}$ in E.

2.2 The Fundamental Existence Uniqueness Theorem

In this section, our primary focus will revolve around *Piccard's Classical Method of Successive Approximations*. We will establish the *existence*, *uniqueness*, *Continuity* and *Differentiability* of the soution of the intial value problem for given *intial condition* and *parameters* under the hypothesis that $f \in \mathcal{C}^1(E)$.

Definition: Suppose $\mathbf{f} \in \mathcal{C}(E)$, where E is an *open* subset of $\mathbb{R}^n$. Then $\mathbf{x}(t)$ is a solution of the differential equation on an interval I if $\mathbf{x}(t)$ is differentiable on I and if for all $t \in I$, $\mathbf{x}(t) \in E$ and $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$.

Locally Lipschitz: The function $\mathbf{f}$ is said to be *locally lipschitz* on E if for each $\mathbf{x_0} \in E$, there is an ε-neighborhood of $\mathbf{x_0}$, $N_\varepsilon(\mathbf{x_0}) \subset E$ and a constant $K > 0$ such that $\mathbf{x}, \mathbf{y} \in N_\varepsilon(\mathbf{x_0})$

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| \leq K||\mathbf{x} - \mathbf{y}||$$

Lemma: If $\mathbf{f} : E \to \mathbb{R}^n$ where E is an open subset of $\mathbb{R}^n$ and $\mathbf{f} \in \mathcal{C}^1(E)$, then $\mathbf{f}$ is *locally lipschitz* on E.

Proof: Since E is open, $\exists \varepsilon > 0$ for given $\mathbf{x_0} \in E$, such that $N_\varepsilon(\mathbf{x_0}) \subset E$. Now we define

$$K = \max_{||\mathbf{x}-\mathbf{x_0}||<\varepsilon/2} ||D\mathbf{f}(\mathbf{x})||$$

We say that the $\varepsilon/2$ neighborhood around $\mathbf{x_0}$ as N_0. Now, for $\mathbf{x}, \mathbf{y} \in N_0$, we set $\mathbf{u} = \mathbf{x} - \mathbf{y}$. So for $0 \leq s \leq 1$, we have $\mathbf{x} + s\mathbf{u} \in N_0$, since N_0 is *convex*. We define $\mathbf{F} : [0, 1] \mapsto \mathbb{R}^n$ by

$$\mathbf{F}(s) = \mathbf{f}(\mathbf{x} + s\mathbf{u}) \implies \mathbf{F}'(s) = D\mathbf{f}(\mathbf{x} + s\mathbf{u})\mathbf{u}$$

Therefore, now we have

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| = |\mathbf{F}(1) - \mathbf{F}(0)| = \left| \int_0^1 \mathbf{F}'(s)\, ds \right| \leq \int_0^1 |D\mathbf{f}(\mathbf{x} + s\mathbf{u})\mathbf{u}|\, ds$$

$$\leq \int_0^1 ||D\mathbf{f}(\mathbf{x} + s\mathbf{u})\mathbf{u}||\, |\mathbf{u}|\, ds \leq K|\mathbf{u}| = K|\mathbf{y} - \mathbf{x}|$$

Complete Space: Let V be a normed linear space. Then a sequence $\mathbf{u}_k \subset V$ is called a *Cauchy Sequence* if for all $\varepsilon > 0$ there is an N such that $k, m \geq N$ implies that

$$||\mathbf{u}_k - \mathbf{u}_m|| < \varepsilon$$

The space V is caleed *Complete* if every Cauchy Sequence in V converges to some element in V.
The space $\mathcal{C}(I)$ is *complete normed linear space*, as a sequence of functions is *uniformly convergent* if and only if it is a *Cauchy Sequence*.

Theorem: Let E be an *open subset* of $\mathbb{R}^n$ containing $\mathbf{x_0}$ and assume $\mathbf{f} \in \mathcal{C}^1(E)$. Then $\exists a > 0$ such that the *initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(\mathbf{0}) = \mathbf{x_0}$$

has a *unique solution* in the interval $[-a, a]$.

Proof: Since $\mathbf{f} \in \mathcal{C}^1(E)$, it follows from the lemma proven above, we can say, $\exists \varepsilon > 0$ such that $N_\varepsilon(\mathbf{x_0}) \subset E$ and a constant $K > 0$ such that $\forall \mathbf{x}, \mathbf{y} \in N_\varepsilon(\mathbf{x_0})$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|$$

We set $b = \varepsilon/2$. Then the continuous function $\mathbf{f}(x)$ is bounded on the *compact set*

$$N_0 = \{x \in \mathbb{R}^n \colon |\mathbf{x} - \mathbf{x_0}| \leq b\}$$

Let

$$M = \max_{\mathbf{x} \in N_0} |\mathbf{f}(\mathbf{x})|$$

Now, we use Piccard's Successive Approximations. We assume $\exists a \in \mathbb{R}^+$ such that $\mathbf{u_k}(t)$ is defined and *continuous* on $[-a, a]$, as well as it satisfies

$$\max_{[-a,a]} |\mathbf{u_k}(t) - \mathbf{x_0}| \leq b$$

It certainly follow that $\mathbf{f}(\mathbf{u_k}(t))$ is *defined* and *continuous* on $[-a, a]$ and therefore that

$$\mathbf{u_{k+1}}(t) = \mathbf{x_0} + \int_0^t \mathbf{f}(\mathbf{u_k})(s)\, ds$$

is defined and continuous on $[a, -a]$. aand satisfies

$$|\mathbf{u_{k+1}}(t) - \mathbf{x_0}| \leq \int_0^t |\mathbf{f}(\mathbf{u_k}(s))|\, ds \leq Ma, \quad , \quad \forall t \in [-a, a]$$

Thus, by choosing $0 < a < b/M$, it follows from *induction* that $\mathbf{u_k}(t)$ is defined and continiuous.
Now, since we have