1) Properties of MGF:

i) $M_{x-a}(t) = e^{-at} M_x(t)$

ii) The MGF of the sum of a number of independent Random Variable is equal to the product of their respective moment generative function.

i.e, $M_{x_1+x_2+\cdots+x_n}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$

iii) If $M_x(t) = E[e^{tx}]$ then $M_{cx}(t) = M_x(ct)$

iv) If $y = ax+b$ then $M_y(t) = e^{bt} \cdot M_x(at)$ where $M_x(t)$ = mgf of x.

v) If $Y = \frac{x-a}{h}$ then $M_y(t) = e^{-at/h} \cdot M_x(t/h)$

2. Distribution:

1) Binomial Distribution:

A discrete random variable x is said to follow binomial distribution if its probability mass function is given by

$P[X=x] = n_{C_x} p^x \cdot q^{n-x}, \quad x = 0, 1, 2, \ldots n.$ and $p+q=1$ where n and p are the parameters.

MGF (Moment Generating Function]:

$M_x(t) = E[e^{tx}]$

$$= \sum_{x=0}^{n} e^{tx} \cdot p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \cdot {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} {}^{n}C_{x} \, (pe^{t})^{x} \cdot q^{n-x}$$

$$M_x(t) = (q + pe^{t})^{n}$$

$$M_x'(t) = n\,(q + pe^{t})^{n-1} \cdot pe^{t}$$

$$M_x'(0) = np = \mu_1'$$

$$M_x''(t) = np\left[(n-1)(q+pe^{t})^{n-2} \cdot pe^{t} \cdot e^{t} + (q+pe^{t})^{n-1} \cdot e^{t}\right]$$

$$M_x''(0) = np\left[(n-1)p + 1\right]$$

$$= np\left[np - p + 1\right]$$

$$= np\left[np + q\right]$$

$$\therefore \mu_2' = n^2p^2 + npq$$

$$\text{Variance} = \mu_2' - \mu_1^2$$

$$= n^2p^2 + npq - (np)^2$$

$$= npq$$

Thus $M.G.F = (q + pe^{t})^{n}$

Mean $= np$

Variance $= npq$

& standard deviation $= \sqrt{var} = \sqrt{npq}$

2) POISSON DISTRIBUTION:

A discrete random variable x is said to follow poisson Distribution if its PDF is given by.

$$P[X=x] = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}, \quad x = 0, 1, 2, 3, \ldots, \infty$$

where, λ is the parameter.

Poisson Distribution is the limiting form of binomial Distribution as $n \to \infty$ and $p \to 0$ in such a way that $np = \lambda$ (A finite constant).

PROOF:

In the case of binomial distribution, the probability of r success is given by $P[X=r] = {}^{n}C_{r} \cdot p^{r} \cdot q^{n-r}$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \cdot q^{n-r} \cdot p^{r}$$

Putting $np = \lambda$, ie, $p = \frac{\lambda}{n}$ and $q = 1 - \frac{\lambda}{n}$

$$P[X=r] = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\frac{\lambda^{r}}{r!} \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\frac{\lambda^{r}}{r!} \cdot \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{r}}$$

Taking limit as $n \to \infty$
(Keeping r fixed)

$$P(X=r) = \frac{e^{-\lambda} \cdot \lambda^{r}}{r!} \text{ as } \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}$$

Mean = variance = λ.

3) Geometry Distribution:

A Random variable X is said to follow geometric Distribution if its pdf is defined by $P[X=x] = q^{x} \cdot p$, $x = 0, 1, 2, 3, \ldots$ where $0 \le p \le 1$ and $p+q=1$

Here $q^{x} \cdot p$ denotes the probability that there x failures preceeding the first success p.

$$\text{Mean} = E(X) = \frac{q}{p}$$

$$E(X^2) = \frac{2q^2}{p^2} + \frac{q}{p}$$

$$\text{Variance}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{q}{p^2}$$

$$\text{M.G.F}: E[e^{tx}] = \frac{p}{1-qe^{t}}$$

Another form of Geometry Distribution:

$P[X=r] = q^{r-1} \cdot p$, $r = 1, 2, 3, \ldots$

4) Uniform Distribution:

A Random Variable X is said to follow uniform Distribution if its pdf is given by $f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$

where a and b are the parameters

Distribution function:

$$F(x) = \begin{cases} 0, & x < a \\ x-a/b-a, & a < x < b \\ 1, & x > b \end{cases}$$

$$\text{Mean} = b+a/2$$

$$E(x^2) = \frac{b^2+ab+a^2}{3}$$

$$\text{Var} = E(x^2) - [E(x)]^2$$

$$= \frac{(b-a)^2}{12}$$

$$M.G.F = M_x(t)$$

$$= E[e^{tx}]$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)}$$

$$\text{Moments}, \mu_r' = \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}, b > a$$

5) Exponential distribution :

The continuous random variable, X is said to have an exponential distribution, if it has the following probability density function:

$$f_X(x \mid \lambda) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}$$

Mean:

Mean of the exponential distribution is calculated using the integration by parts.

$$\text{Mean} = E[X] = \int_0^\infty x \lambda e^{-\lambda x} . dx$$

$$= \lambda \left[\left[\frac{-x e^{-\lambda x}}{\lambda} \right]_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} . dx \right]$$

$$= \lambda \left[0 + \frac{1}{\lambda} . \frac{e^{-\lambda x}}{\lambda} \right]_0^\infty$$

$$= \lambda . \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

Variance:

$$E[X^2] = \int_0^\infty x^2 . \lambda e^{-\lambda x} = \frac{2}{\lambda^2}$$

$$Var(x) = E(x^2) - E(x)^2$$

$$Var(x) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Mgf:

$$M_x(t) = \frac{\lambda}{\lambda - t}.$$

6) Gamma Distribution:

The gamma distribution is a continuous probability distribution,

$$f(x) = \frac{\lambda^{\alpha} \cdot x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, 0 < x < \infty$$

$$M_x(t) = E[e^{tx}]$$

$$M_x(t) = \int_0^{\infty} e^{tx} \cdot f(x) \cdot dx$$

$$= \int_0^{\infty} \frac{e^{tx} \cdot \lambda^{\alpha} \cdot x^{\alpha-1} \cdot e^{-\lambda x}}{\Gamma(\alpha)} \cdot dx$$

$$= \int_0^{\infty} \frac{e^{-x(\lambda-t)} \cdot \lambda^{\alpha} \cdot x^{\alpha-1}}{\Gamma(\alpha)} \cdot dx$$

$$\frac{\lambda^{\alpha}}{(\lambda-t)^{\alpha}} \int_0^{\infty} \frac{(\lambda-t)^{\alpha} \cdot x^{\alpha-1} \cdot e^{-x(\lambda-t)}}{\Gamma(\alpha)} \cdot dx$$

we know that $\int_0^\infty f(x).dx = 1$

$$\therefore \int_0^\infty \frac{(\lambda-t)^\alpha . x^{\alpha-1} . e^{-x(\lambda-t)}}{\Gamma(\alpha)} ..dx = 1$$

$$= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \times 1$$

$$M_X(t) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha}$$

$$\therefore \text{Mean} = M_x'(0)$$

$$M_X(t) = \lambda^\alpha (\lambda-t)^{-\alpha}$$

$$M_X'(t) = \lambda^\alpha (-\alpha)(\lambda-t)^{-\alpha-1} . (-1)$$

$$= \alpha . \lambda^\alpha . (\lambda-t)^{-\alpha-1}$$

$$M_X'(0) = \alpha . \lambda^\alpha . (\lambda-0)^{-\alpha-1}$$

$$M_X'(0) = \alpha . \lambda^\alpha (\lambda)^{-\alpha-1}$$

$$= \frac{\alpha\lambda^\alpha}{\lambda^{\alpha+1}} = \frac{\alpha \cancel{\lambda^\alpha}}{\cancel{\lambda^\alpha} . \lambda}$$

$$\therefore \text{Mean} = \frac{\alpha}{\lambda}$$

$$\therefore Var(X) = E[X] - [E(X)]^2 = \frac{\alpha^2+\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\alpha}{\lambda^2}$$

$$\therefore Mgf = M_X(t) = \left[\frac{1}{1-\left(\frac{t}{\lambda}\right)}\right]^\alpha, \text{ for } t < \lambda.$$