Flow Matching and Dynamic Optimal Transport

Exercise 1 . Starting from a coupling (X_0, X_1), we define the linear interpolation $X_t = tX_1 + (1-t)X_0$. We remind that the velocity field defined in flow matching writes

$$v_t(x) = \mathbb{E}[X_1 - X_0|X_t = x] = \frac{1}{1-t}\mathbb{E}[X_1 - X_t|X_t = x] = \frac{1}{1-t}\left(\mathbb{E}[X_1|X_t = x] - x\right). \quad (1)$$

Let $A \in \mathbb{R}^{d\times d}$ be invertible, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}_+^*$. Show the following properties:

(i) The velocity $v_t^{A,b}$ defined from the coupling $(AX_0 + b, AX_1 + b)$ is given by

$$v_t^{A,b}(x) = Av_t(A^{-1}(x-b)).$$

(ii) The velocity v_t^b from the coupling $(X_0, X_1 + b)$ is given by $v_t^b(x) = v_t(x - tb) + b$.

(iii) The velocity v_t^c defined from the coupling (X_0, cX_1) is given by

$$v_t^c = \frac{c}{1-t+tc} v_r\left(\frac{x}{1-t+tc}\right) + \frac{c-1}{1-t+tc}x, \quad \text{with} \quad r = \frac{tc}{1-t+tc}.$$

Exercise 2 . Assume that α is an absolutely continuous probability measure with respect to the Lebesgue measure, and β is a probability measure on $\mathbb{R}^d$. Write T^* the optimal transport map between α and β for the squared Euclidean cost. Can we say something on what T^* becomes in the cases considered in Exercise 1?

Exercise 3 . Assume that α and β are probability measures on $\mathbb{R}$. Let (X_0, X_1) be a coupling between α and β such that the velocity field $v_t(x) = \mathbb{E}[X_1 - X_0|X_t = x]$ has a unique flow on $[0, 1]$. Show that the flow φ_t associated to v_t is such that φ_1 is an optimal transport map between α and β.

Exercise 4 . Assume that $(X_0, X_1) \sim \mathcal{N}(0, \Sigma)$ with $\Sigma = \begin{pmatrix} \Sigma_0 & \Sigma_{10} \\ \Sigma_{01} & \Sigma_1 \end{pmatrix}$, for positive definite Σ_0 and Σ_1. Remind that if (Z, T) is a Gaussian vector, then $\mathbb{E}[Z|T] = \mathbb{E}[Z] + \mathrm{Cov}(Z, T)\mathrm{Cov}(T, T)^{-1}(T - \mathbb{E}[T])$.

(i) Show that the velocity field $v_t(x) := \mathbb{E}[X_1 - X_0|X_t = x]$ is given by

$$v_t(x) = \frac{1}{1-t}\left(((1-t)\Sigma_{01} + t\Sigma_1)\Sigma_t^{-1} - \mathrm{Id}\right)x, \quad (2)$$

where $\Sigma_t = \mathrm{Cov}(X_t) = (1-t)^2\Sigma_0 + (1-t)t(\Sigma_{01} + \Sigma_{10}) + t^2\Sigma_1$.

(ii) Let $\Sigma_{01} = \Sigma_{10} = 0$ and assume that Σ_0 and Σ_1 can be jointly diagonalized. Show that the flow φ_t associated to v_t is such that φ_1 is an optimal transport map between α and β.

Exercise 5 .

Assume that $(X_0, X_1) \sim \sum_{k=1}^K \pi_k \mathcal{N}(m^k, \Sigma^k)$ with $m^k = \begin{pmatrix} m_0^k \\ m_1^k \end{pmatrix}$ and $\Sigma^k = \begin{pmatrix} \Sigma_0^k & \Sigma_{10}^k \\ \Sigma_{01}^k & \Sigma_1^k \end{pmatrix}$ for positive definite Σ_0^k and Σ_1^k. Write v_t^k the velocity field (2) for the covariance matrix Σ^k and write $w_t^k(x) = v_t^k(x - tm_1^k - (1-t)m_0^k) + m_1^k - m_0^k$. Show that the velocity field $v_t(x) := \mathbb{E}[X_1 - X_0 | X_t = x]$ is given by

$$v_t(x) = \sum_{k=1}^K \alpha^k(x) w_t^k(x), \tag{3}$$

where $\alpha^k(x) = \frac{\pi_k p_t^k(x)}{\sum_{j=1}^K \pi_j p_t^j(x)}$, with p_t^j is the Gaussian density of $\mathcal{N}(m_t^j, \Sigma_t^j)$ with $m_t^j = tm_1^j + (1-t)m_0^j$ and $\Sigma_t^j = t^2\Sigma_1^j + (1-t)^2\Sigma_0^j + t(1-t)(\Sigma_{10}^j + \Sigma_{01}^j)$. What happens if X_0 is a standard centered Gaussian and X_1 follows a discrete distribution of K equal Dirac masses?