Wasserstein Distances and duality

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Exercise 1 (Total variation and 0/1 cost). Let $d(x, y) = 1_{x \neq y}$ be the discrete 0/1 distance. We wish to show that for any $p \geq 1$, and probability measures α and β,

$$W_p^p(\alpha, \beta) = \frac{1}{2} \|\alpha - \beta\|_{\mathrm{TV}}.$$

Start with two discrete probability measures $\alpha = \sum_i a_i \, \delta_{x_i}$ and $\beta = \sum_i b_i \, \delta_{x_i}$ on the same finite set $\{x_i\}_i$ with weights $a = (a_i)$ and $b = (b_i)$ on the same support.

Exercise 2 (Equivalence of topologies on a finite discrete metric space).
Let (X, d) be a finite discrete metric space. Define

$$d_{\min} := \min_{x \neq y} d(x, y), \qquad d_{\max} := \max_{x,y} d(x, y).$$

Show that for all probability measures α, β on X,

$$\frac{d_{\min}}{2} \|\alpha - \beta\|_{\mathrm{TV}} \;\leq\; W_1(\alpha, \beta) \;\leq\; \frac{d_{\max}}{2} \|\alpha - \beta\|_{\mathrm{TV}}.$$

Conclude that the W_1–topology and the total variation topology coincide on $\mathcal{P}(X)$.

Exercise 3 (c-transforms). Show the following identities

- $f \leq f' \to f^c \geq f'^c$
- $f^{c\bar{c}} \geq f$
- $g^{\bar{c}c} \geq g$
- $f^{c\bar{c}c} = f^c$