Discrete Kantorovich Formulation — Exercises

Exercise 1 (1D discrete $\Rightarrow$ optimal Kantorovich plan). Let $n = m = 3$ and consider two 1D discrete measures with sorted supports and weights $a = (1.1,\ 0.8,\ 1.1)$ and $b = (1,\ 1,\ 1)$. For the quadratic cost $c(x, y) = |x - y|^2$, compute the optimal coupling matrix $P \in \mathbb{R}_+^{3 \times 3}$.

Exercise 2 (Parametric 1D cases). (1) Two against two: $a = (1 + \varepsilon,\ 1 - \varepsilon)$ and $b = (1,\ 1)$ with $|\varepsilon| \leq 1$. Compute the optimal $P \in \mathbb{R}_+^{2 \times 2}$ as a function of the sign of ε.

(2) Three against three: $a = (1 + \varepsilon,\ 1 + \eta,\ 1 - \varepsilon - \eta)$ and $b = (1,\ 1,\ 1)$ with feasibility $1 + \varepsilon \geq 0$, $1 + \eta \geq 0$, $1 - \varepsilon - \eta \geq 0$. Compute the optimal monotone $P \in \mathbb{R}_+^{3 \times 3}$ in the two regimes (i) $\varepsilon > 0,\ \eta > 0$ and (ii) $\varepsilon < 0,\ \eta > 0$.

Exercise 3 (Couplings in basic cases).   (a) What are the couplings between δ_x and δ_y?

(b) What are the couplings between δ_x and $\frac{1}{2}(\delta_y + \delta_z)$, with $y \neq z$?

(c) Let α, β be finite *positive* measures with *different* total masses. What is $\Pi(\alpha, \beta)$, the set of couplings?

Exercise 4 (Discrete α_t from a coupling). Let $\alpha_0 = \sum_{i=1}^n a_i\, \delta_{x_i}$ and $\alpha_1 = \sum_{j=1}^m b_j\, \delta_{y_j}$ be probability measures on $\mathbb{R}^d$ ($a_i, b_j > 0$, $\sum_i a_i = \sum_j b_j = 1$). Let π be a (Kantorovich) optimal coupling between α_0 and α_1 and write it in discrete form

$$\pi = \sum_{i=1}^n \sum_{j=1}^m P_{ij}\, \delta_{(x_i, y_j)}, \qquad P \in \mathbb{R}_+^{n \times m},\ \ \sum_j P_{ij} = a_i,\ \ \sum_i P_{ij} = b_j.$$

For $t \in [0, 1]$ define $P_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by $P_t(x, y) = (1 - t)x + ty$ and set

$$\alpha_t := (P_t)_\# \pi.$$

(a) Show that α_t interpolates between α_0 and α_1, i.e. $\alpha_0 = (P_0)_\# \pi$ and $\alpha_1 = (P_1)_\# \pi$.

(b) Prove that α_t is discrete for every $t \in [0, 1]$. Give an upper bound on the number of Dirac masses of α_t in terms of P. Deduce the crude bound $\#\mathrm{supp}(\alpha_t) \leq nm$. (Bonus: explain why one can choose an *optimal* coupling with at most $n + m - 1$ nonzeros, hence an interpolation with at most $n + m - 1$ atoms.)

(c) Give the explicit expression of α_t in terms of the matrix $P \in \mathbb{R}_+^{n \times m}$.

(d) Assume $n = m$. Under which condition(s) can one choose an optimal coupling so that α_t has exactly n Dirac masses for all t? Give a concrete example where this happens and write α_t explicitly.

Optional – More difficult exercises

Exercise 5 (When does a finite-cost coupling exist?)**.** Let $n \in \mathbb{N}$ and consider the uniform discrete measures

$$\alpha = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}, \qquad \beta = \frac{1}{n} \sum_{j=1}^{n} \delta_{y_j}.$$

Let $C = (C_{ij}) \in (\mathbb{R} \cup \{+\infty\})^{n \times n}$ be a cost matrix; $+\infty$ means the pair (x_i, y_j) is forbidden. A coupling π has *finite cost* if it assigns zero mass to all $+\infty$ entries.

(i) 2×2 **by hand.** For

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

determine exactly when a finite-cost coupling exists in terms of the positions of $+\infty$. When it exists, write one explicit coupling matrix $P \in \mathbb{R}_+^{2 \times 2}$ with row/column sums $1/2$ supported only on finite entries.

(ii) **General uniform case.** Let $E = \{(i, j) : C_{ij} < +\infty\}$ be the set of allowed edges. Prove:

> There exists a finite-cost coupling $\iff$ there exists a permutation σ with $(i, \sigma(i)) \in E \ \forall i$.

(*Hint:* If P is a coupling, then $Q := nP$ is doubly stochastic. Recall Birkhoff–von Neumann: every doubly stochastic matrix is a convex combination of permutation matrices.)

Solution. **(i)** 2×2 **case.** There are two permutation supports:

$$\text{Id: } \{(1,1),(2,2)\}, \qquad \text{Swap: } \{(1,2),(2,1)\}.$$

A finite-cost coupling exists iff at least one support is fully finite:

$$(C_{11}, C_{22} \text{ finite}) \quad \text{or} \quad (C_{12}, C_{21} \text{ finite}).$$

Obstructions: any all-$+\infty$ row or column; or the pattern blocking both permutations (e.g. exactly one of $\{(1,1),(2,2)\}$ and exactly one of $\{(1,2),(2,1)\}$ is $+\infty$).

When feasible, an explicit coupling is the scaled permutation on the allowed support. If $(1,1)$ and $(2,2)$ are finite, take

$$P = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix};$$

if $(1,2)$ and $(2,1)$ are finite, take

$$P = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In each case rows and columns sum to $1/2$ and no mass sits on forbidden entries.

(ii) General uniform case.
($\Rightarrow$) If there is a permutation σ with $(i, \sigma(i)) \in E$ for all i, then

$$P_{i,\sigma(i)} = \frac{1}{n}, \qquad P_{ij} = 0 \text{ otherwise},$$

is a coupling with finite cost.

($\Leftarrow$) Suppose a finite-cost coupling exists. Let P be its matrix (row/column sums $1/n$, zeros on forbidden entries) and set $Q := nP$. Then Q is doubly stochastic with $Q_{ij} = 0$ on forbidden cells. By Birkhoff–von Neumann,

$$Q = \sum_{k=1}^{K} \lambda_k \, \Pi_{\sigma_k}, \qquad \lambda_k > 0, \ \sum_k \lambda_k = 1,$$

where Π_{σ_k} are permutation matrices. For any forbidden (i, j),

$$0 = Q_{ij} = \sum_k \lambda_k (\Pi_{\sigma_k})_{ij},$$

and since each term is nonnegative, we must have $(\Pi_{\sigma_k})_{ij} = 0$ for all k; thus every σ_k avoids forbidden entries. As some $\lambda_k > 0$, at least one allowed permutation exists. This proves the equivalence.