Sliced Optimal Transport: Exercises

Optimal Transport for Machine Learning
October 29, 2025

Exercise 1 (Comparing isotropic centered Gaussians). Consider $\mu = \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_d)$ and $\nu = \mathcal{N}(\mathbf{0}, \tau^2 \mathbf{I}_d)$, with $\sigma \geq 0$, $\tau \geq 0$.

(a) Compute the Sliced-Wasserstein distance of order 2 between μ and ν.

(b) Compare the obtained expression with the Wasserstein distance of order 2 between μ and ν.

Exercise 2 (Projected optimal transport plans). Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ (measures on $\mathbb{R}^d$ with finite second moments). Denote by γ an optimal transport plan between μ and ν. For any $\theta \in \mathbb{S}^{d-1}$, define $\gamma_\theta = (p_\theta \otimes p_\theta)_\sharp \gamma$.

(a) Show that γ_θ is a transport plan between $(p_\theta)_\sharp \mu$ and $(p_\theta)_\sharp \nu$.

(b) Is γ_θ an optimal transport plan between $(p_\theta)_\sharp \mu$ and $(p_\theta)_\sharp \nu$?

Exercise 3 (Translation decomposition for $\mathbf{SW}_2$). Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ (measures on $\mathbb{R}^d$ with finite second moments), and denote by μ_0, ν_0 the respective *centered* versions.

(a) Prove that for any $\theta \sim \mathcal{U}_{\mathbb{S}^{d-1}}$, $\mathbb{E}_{s \sim (p_\theta)_\sharp \mu}[s] = \langle \theta, \mathbb{E}_{x \sim \mu}[x] \rangle$ and $\mathbb{E}_{t \sim (p_\theta)_\sharp \nu}[t] = \langle \theta, \mathbb{E}_{y \sim \nu}[y] \rangle$.

(b) Prove that $((p_\theta)_\sharp \mu)_0 = (p_\theta)_\sharp \mu_0$ and $((p_\theta)_\sharp \nu)_0 = (p_\theta)_\sharp \nu_0$, with $\mu_0 = (T_{-u})_\sharp \mu$ and $\nu_0 = (T_{-v})_\sharp \nu$ for a unique pair $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$ (whose values must be precised).

(c) Show that $\mathbf{SW}_2(\mu, \nu)$ admits the following decomposition,

$$\mathbf{SW}_2(\mu, \nu)^2 = \frac{1}{d}\|u - v\|^2 + \mathbf{SW}_2(\mu_0, \nu_0)^2$$

(Hint: $\mathbb{E}_{\theta \sim \mathcal{U}_{\mathbb{S}^{d-1}}}[\theta \theta^T] = (1/d)\,\mathbf{I}_d$.)