Name:

PHYS 4330 Theoretical Mechanics

Exam #1

14 February 2023, 8 am to 9:50 am

Please put your name on all pages you hand in for grading and number all pages in sequential order.

Staple this sheet as the front page to the pages you are handing in.

The exam is paper and pencil only. No book, notes, calculators or equation sheets are allowed.

Write down all steps of your solutions neatly. This will make it easy for us to recognize the excellence of your work and give you full or partial credit.

Name:

Problem	Points	Score
1	13	11.5
2	5	5
3	9	8.5
4	13	13
Total	40	38
Percentage		

1. Non relativistic Newtonian Mechanics

A mass m is constrained to move along the x axis subject to a force $F(v) = -F_0 e^{v/B}$, where F_0 and B are positive constants and v is the velocity.

a) Find the expression for the speed $v(t)$. Assume that the initial velocity $v_0 > 0$ for time $t = 0$ s.

b) At what time will the mass first come to rest?

c) Using the equation you found in part (a), find the equation of motion for the mass [$x(t)$]. Find out how far the mass travels when it comes to rest (time found in part (b)). Assume the mass starts at $x = 0$ at $t = 0$.

Useful Integral: $\int \ln(ax+b)dx = \frac{ax+b}{a}\ln(ax+b) - x$

2. Relativistic Mechanics

A robber's getaway vehicle, which can travel at an impressive $0.9c$, is pursued by a cop, whose vehicle can travel at a mere $0.75c$. Realizing that he cannot catch up with the robber, the cop tries to shoot him with a tracking dart that travels at $0.75c$ (relative to the cop). Will the dart reach its mark? Assume that all vehicles and projectiles travel in a straight line.

*You **MUST** provide a numeric answer and reasoning. Simply guessing will result in 0 credit.*

Lorentz Transformation Relations for one dimensional boost: $x' = \gamma(x - vt)$, $t' = \gamma\left(t - \frac{xv}{c^2}\right)$

Name:

3. Relativistic Lagrangian

The Lagrangian of the relativistic harmonic oscillator is:

$$L(x,\dot{x}) = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} - \frac{1}{2}kx^2 \qquad (1).$$

In (1) m and k stand for the mass and spring constant of the oscillator, c is the speed of light.

Find Lagrange's equation of motion. Show that Lagrange's equation of motion for the relativistic harmonic oscillator reduces to the expected non-relativistic result for $\dot{x} << c$.

You do not need to solve the differential equations.

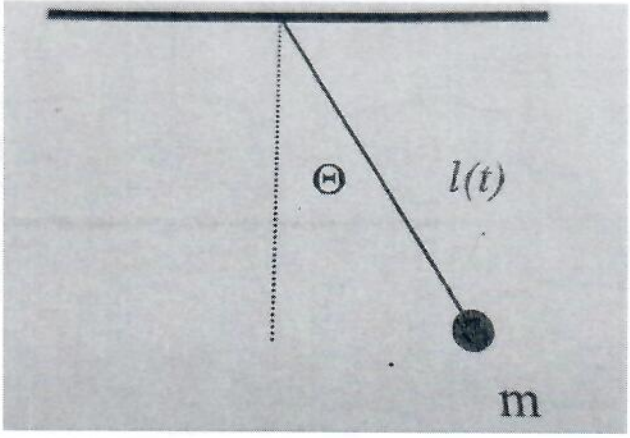
4. Lagrangian Mechanics

A pendulum of length l and mass bob m is oscillating at **small angles** when the length of the pendulum string is shortened at a constant velocity of α ($dl/dt = -\alpha$).

a) Find the Lagrangian of the system.

b) Find the Lagrange's equation of motion for θ, of the bob.

You do not need to solve the resulting differential equation.



$\cancel{F(v) = ma = m\frac{dv}{dt}}$

$\cancel{F(v) = m\frac{dv}{dt}\frac{dx}{dx}}$

$\cancel{= m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}}$ $\longrightarrow$ $\cancel{F(v)dx = mv\,dv}$

$\cancel{\int F(v)dx = \int mv\,dv}$

$\cancel{F(v)x = \frac{1}{2}mv^2}$

11.5/13

$F(v) = m\frac{dv}{dt} = -F_0 e^{v/B}$ ✓

$dt = -m\frac{1}{F_0}e^{-v/B}\,dv$ ✓

$t + C_1 = -\frac{m}{F_0}\int e^{-\frac{v}{B}}dv = -\frac{m}{F_0}\left(-Be^{-\frac{v}{B}}\right) = \frac{mB}{F_0}e^{-\frac{v}{B}} + C_2$ ✓ ✓ $\rightarrow C_3 = C_1 - C_2$

$\frac{F_0 t}{mB} + C = e^{-\frac{v}{B}} \rightarrow \ln\left(\frac{F_0 t}{mB} + C\right) = -\frac{v}{B} \rightarrow \boxed{v = -B\ln\left(\frac{F_0 t}{mB} + C\right)}$ $\quad C = \frac{F_0 C_3}{mB}$

$v(0) > 0 = -B\ln(C)$ so $0 < C < 1$ need to solve for C -1

b) $v(t) = 0 = -B\ln\left(\frac{F_0 t}{mB} + C\right)$ ✓

$e^0 = e^{\ln\left(\frac{F_0 t}{mB} + C\right)}$

$1 = \frac{F_0 t}{mB} + C \rightarrow 1 - C = \frac{F_0 t}{mB} \rightarrow \boxed{t = \frac{mB}{F_0}(1 - C)}$ ok

c) $v(t) = \frac{dx}{dt}$ ✓ $\rightarrow dx = -B\ln\left(\frac{F_0 t}{mB} + C\right)dt$ ✓

$x = -B\int \ln\left(\frac{F_0 t}{mB} + C\right)dt$ $\qquad a = \frac{F_0}{mB} \quad b = C$

$= -B\;\frac{at + b}{a}\ln(at + b) - t$

$= -B\;\frac{\frac{F_0 t}{mB} + C}{\frac{F_0}{mB}}\ln\left(\frac{F_0 t}{mB} + C\right) - t$ didn't multiply by "-B" -0.5

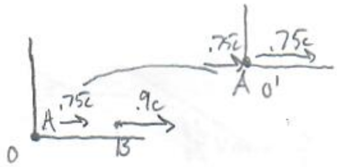
$\boxed{x(t) = -\frac{mB^2}{F_0}\left(\frac{F_0 t}{mB} + C\right)\ln\left(\frac{F_0 t}{mB} + C\right) - t + C_2}$ $\quad x(0) = 0 \quad \boxed{C_2 = \frac{mB^2}{F_0}C\ln C}$ need to solve for these

$x\left(\frac{mB}{F_0}(1 - C)\right) = -\frac{mB^2}{F_0}\left[(1 - C + C)\ln(1 - C + C)\right] - \frac{mB}{F_0}(1 - C) + C_2$ $\qquad \frac{F_0\left(\frac{mB}{F_0}(1 - C)\right)}{mB} = 1 - C$

$= -\frac{mB^2}{F_0}\underbrace{\ln(1)}_{0} - \frac{mB}{F_0} + \frac{mBC}{F_0} + \frac{mB^2}{F_0}C\ln C$

$= \boxed{\frac{mB^2 C}{F_0}\ln C + \frac{mBC}{F_0} - \frac{mB}{F_0}}$ ok

2. need velocity of projectile in ref. frame O ... v in O' is .75c



$\frac{dx'}{dt'} = u'$ $dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial t}dt$

$dt' = \frac{\partial t'}{\partial x}dx + \frac{\partial t'}{\partial t}dt$

$dx' = \gamma dx - \gamma v dt$
$dt' = -\gamma v/c^2\, dx + \gamma dt$ $\Rightarrow$ $\frac{dx'}{dt'} = \frac{\gamma dx - \gamma v dt}{\gamma dt - \gamma \frac{v}{c^2} dx} \cdot \frac{\frac{1}{dt}}{\frac{1}{dt}}$

$= \frac{\gamma \frac{dx}{dt} - \gamma v}{\gamma - \gamma \frac{v}{c^2} \cdot \frac{dx}{dt}}$

5/5

$u' = \frac{u - v}{1 - \frac{vu}{c^2}}$

$.75c = \frac{u - .75c}{1 - \frac{.75cu}{c^2}} = \frac{u - .75c}{1 - .75u/c}$

$.75c\left(1 - .75\frac{u}{c}\right) = u - .75c$

$.75c - .75^2 u - u = -.75c$

$u(.75^2 + 1) = 2(.75c) = 1.5c$

$u = \frac{1.5c}{(.75^2+1)} = \frac{3/2}{\frac{25}{16}}c = \left(\frac{3}{2}\right)\left(\frac{16}{25}\right)c = \boxed{\frac{24}{25}c} > .9c$

$.75^2 = \frac{9}{16}$
$.75^2 + 1 = \frac{25}{16}$

so yes the projectile will catch up

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

8.5/9

$\frac{\partial L}{\partial \dot{x}} = -\frac{1}{2}mc^2\left(1-\frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}}\left(-2\frac{\dot{x}}{c^2}\right) = mc^2\frac{\dot{x}}{c^2}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}} = \frac{m\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}$ ✓

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}} - \frac{1}{2}m\dot{x}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-3/2}(-2\dot{x})(\ddot{x}) = m\ddot{x}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}} + \frac{m\dot{x}^2\ddot{x}}{c^2}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-3/2}$

$\frac{\partial L}{\partial x} = -kx$ ✓

math error
−0.5

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$ ✓

$\boxed{m\ddot{x}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}} + m\dot{x}^2\ddot{x}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-3/2} + kx = 0}$ ✓

$(1+x)^n \simeq 1+nx$ for small x

$\left(1-\frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}} \simeq 1+\frac{1}{2}\frac{\dot{x}^2}{c^2}$ $\quad\left(1-\frac{\dot{x}^2}{c^2}\right)^{-3/2} \simeq 1+\frac{3\dot{x}^2}{2c^2}$

~~$m\ddot{x} + \frac{m\ddot{x}\dot{x}^2}{2c^2} + m\dot{x}^2\ddot{x} + \frac{3m\dot{x}^4\ddot{x}}{2c^2} + kx = 0$~~

$m\ddot{x}\left(1+\cancel{\frac{\dot{x}^2}{2c^2}}^{0}\right) + \cancel{m\dot{x}^2}^{0}\ddot{x}\left(1+\cancel{\frac{3\dot{x}^2}{2c^2}}^{0}\right) + kx = 0$ ✓

if $\dot{x} \ll c$, $\dot{x}^2 \lll c$

$\boxed{m\ddot{x} + kx = 0}$ ✓

$K = \frac{1}{2}m\dot{x}^2 \quad U = \frac{1}{2}kx^2$

$\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

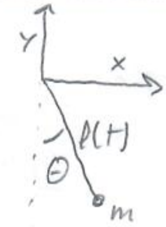
$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = m\ddot{x}$

$\frac{\partial L}{\partial x} = -kx$

$m\ddot{x} + kx = 0$

4

$^{13}/_{13}$



4.a) $U = mgy$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ so $x = l(t)\sin\theta$ $y = -l(t)\cos\theta$

$U = -mgl(t)\cos\theta$

$\dot{x} = \frac{dl}{dt}\sin\theta + l(t)\cos\theta\dot{\theta}$ $\dot{y} = -\frac{dl}{dt}\cos\theta + l(t)\sin\theta\dot{\theta}$

$= -\alpha\sin\theta + l\cos\theta\dot{\theta}$ $= \alpha\cos\theta + l\sin\theta\dot{\theta}$

$\dot{x}^2 = \alpha^2\sin^2\theta + l^2\dot{\theta}^2\cos^2\theta - 2\alpha l\dot{\theta}\sin\theta\cos\theta$

$\dot{y}^2 = \alpha^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta + 2\alpha l\dot{\theta}\sin\theta\cos\theta$

$T = \frac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2)$

$$\boxed{\begin{aligned} L = T - U &= \tfrac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) + mgl\cos\theta \\ &\simeq \tfrac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) + mgl \end{aligned}}$$

but, for small angles, $\cos\theta \simeq 1$

b) $\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{d}{dt}ml^2\dot{\theta} = ml^2\ddot{\theta} + 2ml\dot{\theta}(-a)$

$\frac{\partial L}{\partial\theta} = -mgl\sin\theta$

$ml^2\ddot{\theta} - 2mla\dot{\theta} + mgl\sin\theta = 0$

but, for small angles, $\sin\theta \sim \theta$

$\cancel{m}l^2\ddot{\theta} - 2\cancel{m}la\dot{\theta} + \cancel{m}gl\theta = 0$

$$\boxed{l\ddot{\theta} - 2\alpha\dot{\theta} + g\theta = 0}$$