Class 16 Conservation Laws

(03/18/2024)

Outline of Class 16

- Conservation of electric charge: Continuity Equation
- Review: Energy of electric & magnetic fields (PHYS 1250)
- Energy and momentum of electromagnetic (em) fields within framework of Maxwell's equations (in differential form).

    energy density of the em field

    Poynting Vector (em energy flux density)

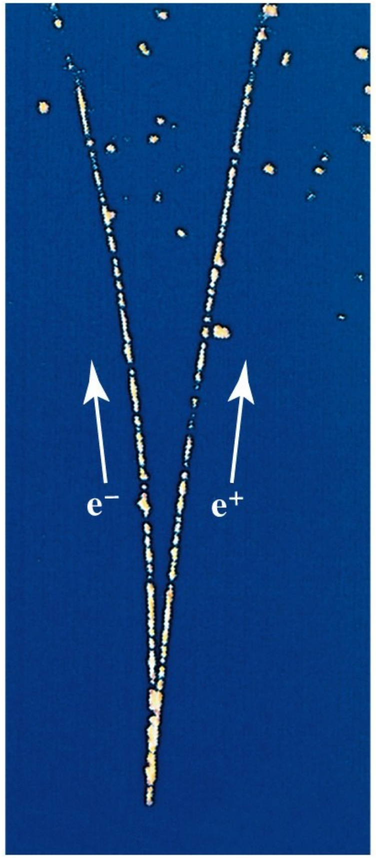
    Maxwell stress / strain tensor

- Angular momentum of the em fields

Charge conservation (PHYS 1250)

- Electric charge is quantized Q=Ne with N+0, 1, 2, .. , and ± e = 1.6x10^{-19}C.
- Electrical charge can be converted in physical processes but is overall conserved.

Example of charge conservation: Decay of γ-ray photon into electron (-e) and positron (+e)



Courtesy Lawrence Berkeley Laboratory halliday_9e_fig_21_11

$$\gamma \longrightarrow e^{-} + e^{+} \qquad \text{(pair production)}$$

Charge conservation (PHYS 4210)

- In em theory charge conservation is described by the continuity equation.

Continuity equation $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$

or $\frac{\partial \rho}{\partial t} = - \vec{\nabla} \cdot \vec{j} \Rightarrow \frac{\partial}{\partial t} \int_V \rho \, dV = - \int_V (\vec{\nabla} \cdot \vec{j}) \, dV = - \int_{a(V)} \vec{j} \cdot d\vec{a}$

volume integrals   surface integral

remember $Q = \int \rho \, dV \Rightarrow \frac{\partial Q_{in V}}{\partial t} = - \int_{a(V)} \vec{j} \cdot d\vec{a}$ (1)

interpretation of (1): amount of charge leaving volume V, i.e. $- \int_{a(V)} \vec{j} \cdot d\vec{a}$, is equal to the change of Q with time inside V

Conservation of electric and magnetic energy

Recall: The Coulomb force and Lorentz force act over distance.

This (major) property enables the introduction of the concept of el. & magnetic fields.

On top of it, it's possible to store energy in el. & magnetic fields (even in vacuum).

Electrical energy storage in a biased capacitor : $U_E = \frac{1}{2} C V^2$ with $E = \frac{V}{d}$.

For a parallel plate capacitor with $C = \varepsilon_0 \frac{A}{d}$

$$U_E = \frac{1}{2} \varepsilon_0 \frac{A}{d} E^2 d^2 = \frac{1}{2} \varepsilon_0 E^2 A \cdot d$$

$\Downarrow$ energy density $u_E = \frac{U}{A \cdot d} = \frac{1}{2} \varepsilon_0 E^2$

Magnetic energy storage in an inductor:

$U_B = \frac{1}{2} L i^2$ and energy density $u_B = \frac{1}{2} \frac{B^2}{\mu_0}$.

Conservation of electromagnetic field energy

$$\frac{d}{dt} E_{mech} + \frac{d}{dt} \int_V \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})\, dV + \int_{a(V)} \vec{S} \cdot d\vec{a} = 0$$

change in mechanical energy

change in energy densities of el. & mag. fields

Poynting vector $\vec{S}$ (energy flux density)

Lorentz force $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

mechanical work $W = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{v} \cdot dt$

mechanical power $\vec{F} \cdot \vec{v} = q\vec{v} \cdot \vec{E} = \int_V (\vec{J} \cdot \vec{E})\, dV$

mechanical power density $\vec{J} \cdot \vec{E}$

time change in mechanical energy

$$\frac{d}{dt} E_{mech} = \int_V (\vec{J} \cdot \vec{E})\, dV$$

In the textbook conservation laws are derived for vacuum where $\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$.

I derive the conservation laws more generally for linear dielectric and magnetic materials, i.e. $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$.

Let's consider the Lorentz force:

$\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$

Let's consider work done by the Lorentz force:

$W = \int \vec{F} \cdot d\vec{r}$ or $W = \int \vec{F} \cdot \vec{v} \cdot dt$ with $d\vec{r} = \vec{v}\,dt$ (1)

Let's consider power $= \dfrac{\text{energy}}{\text{time}}$ or $P = \dfrac{dE}{dt}$ (2)

From (1) and (2) power can be expressed as:

power $P = \vec{F} \cdot \vec{v}$ with Lorentz force

$\vec{F} \cdot \vec{v} = q\,\vec{v} \cdot \vec{E} + q\,\underbrace{\vec{v} \cdot (\vec{v} \times \vec{B})}_{=0}$ $\quad \vec{v} \cdot (\vec{v} \times \vec{B}) = 0$

because $(\vec{v} \times \vec{B}) \perp \vec{v}$

So, el. power is $\vec{F} \cdot \vec{v} = q\,\vec{v} \cdot \vec{E}$.

Power density is $\frac{\text{power}}{\text{volume}} \Rightarrow \frac{q\vec{v}\cdot\vec{E}}{\text{volume}}$.

The term $\frac{q\vec{v}}{\text{volume}} = \vec{J}$ (volume current density).

We can convince ourselves by looking at the units: $\left[\frac{q\vec{v}}{\text{volume}}\right] = \frac{C\cdot m/s}{m^3} = \frac{C}{s}\cdot\frac{1}{m^2} = \frac{A}{m^2} \Rightarrow |\vec{J}| = \frac{I}{A}$.

So, power density is $\vec{J}\cdot\vec{E}$, and power is also expressed as

$$\text{power} = \int_V \vec{J}\cdot\vec{E}\,dV.$$

time change in mechanical energy

$$\frac{d}{dt}E_{mech} = \int_V (\vec{J}\cdot\vec{E})\,dV$$

Now work with Maxwell's eqs., e.g. $\vec{\nabla}\times\vec{H} = \vec{J} + \dot{\vec{D}}$.

or $\vec{J} = \vec{\nabla}\times\vec{H} - \dot{\vec{D}}$

$$\int_V \vec{J}\cdot\vec{E}\,dV = \int_V \vec{E}(\vec{\nabla}\times\vec{H} - \dot{\vec{D}})\,dV \qquad (1)$$

with some vector calculus, e.g.

$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\,\vec{\nabla}\times\vec{E} - \vec{E}\,\vec{\nabla}\times\vec{H}$ or

$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{H}\,\vec{\nabla}\times\vec{E} - \vec{\nabla}\cdot(\vec{E}\times\vec{H})$ and $\vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$

we get $\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = -\vec{H}\cdot\dot{\vec{B}} - \vec{\nabla}(\vec{E}\times\vec{H})$. (2)

If we put (2) in (1) the result is:

$$\int_V \vec{J}\cdot\vec{E}\,dV = \int_V \left(-\vec{H}\cdot\dot{\vec{B}} - \vec{\nabla}\cdot(\vec{E}\times\vec{H}) - \vec{E}\cdot\dot{\vec{D}}\right)dV \qquad (3)$$

If $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ (3) converts to

$$\int_V \vec{j} \cdot \vec{E} \cdot dV = \int_V \left(-\mu \vec{H} \dot{\vec{H}} - \varepsilon \vec{E} \cdot \dot{\vec{E}} - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \right) dV$$

$$= -\frac{d}{dt} \int_V \left(\frac{1}{2} \varepsilon \vec{E}^2 + \frac{1}{2} \mu \vec{H}^2 \right) dV - \int_{a(V)} (\vec{E} \times \vec{H}) \cdot d\vec{a}$$

So in the end:

$$\frac{d}{dt} E_{mech} + \frac{d}{dt} \int_V \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \, dV + \int_{a(V)} \vec{S} \cdot d\vec{a} = 0$$

change in mechanical energy	change in energy densities of el. & mag. fields	Poynting vector $\vec{S}$ (energy flux density)

Momentum of the electromagnetic field

Recall from mechanics $\vec{F} = \frac{d\vec{p}_m}{dt}$ with

$\vec{p}_m = m\vec{v}$: momentum.

Let's look again at the Lorentz force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{d\vec{p}_m}{dt} \qquad (1)$$

When we introduce ρ and $\vec{j}$, eq. (1) is converted to:

$\frac{d\vec{p}_m}{dt} = \int_V (\rho\vec{E} + \vec{j} \times \vec{B})\, dV$ and with

Maxwell's eqs.: $\rho = \vec{\nabla}\cdot\vec{D}$ and $\vec{J} = \vec{\nabla}\times\vec{H} - \dot{\vec{D}}$, so

$$\frac{d\vec{p}_m}{dt} = \int_V \left(\vec{E}(\vec{\nabla}\cdot\vec{D}) + (\vec{\nabla}\times\vec{H} - \dot{\vec{D}})\times\vec{B}\right) dV$$ or

$$\frac{d\vec{p}_m}{dt} = \int_V \left(\vec{E}\,(\vec{\nabla}\cdot\vec{D}) + (\vec{\nabla}\times\vec{H})\times\vec{B} - \dot{\vec{D}}\times\vec{B}\right) dV$$

Now, let's consider $\dfrac{\partial}{\partial t}(\vec{D}\times\vec{B}) = \dot{\vec{D}}\times\vec{B} + \vec{D}\times\dot{\vec{B}}$

or $\dot{\vec{D}}\times\vec{B} = \dfrac{\partial}{\partial t}(\vec{D}\times\vec{B}) - \vec{D}\times\dot{\vec{B}} = \dfrac{\partial}{\partial t}(\vec{D}\times\vec{B}) - \vec{D}\times(-\vec{\nabla}\times\vec{E})$

Continue with

$$\frac{d\vec{p}_m}{dt} = \int_V \left((\vec{E}(\vec{\nabla}\cdot\vec{D}) + (\vec{\nabla}\times\vec{H})\times\vec{B} - (\frac{\partial}{\partial t}(\vec{D}\times\vec{B}) + \vec{D}\times(\vec{\nabla}\times\vec{E})) \right) dV$$

$$\frac{d\vec{p}_m}{dt} = \int_V \left(\vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times\vec{\nabla}\times\vec{E} + (\vec{\nabla}\times\vec{H})\times\vec{B} - \frac{\partial}{\partial t}(\vec{D}\times\vec{B}) \right) dV$$

now, I add zero to above expression as

$$0 = \vec{H}(\vec{\nabla}\cdot\vec{B})$$

$$\frac{d\vec{p}_m}{dt} = \int_V \left[(\vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times\vec{\nabla}\times\vec{E} + (\vec{\nabla}\times\vec{H})\times\vec{B} - \frac{\partial}{\partial t}(\vec{D}\times\vec{B}) + \vec{H}(\vec{\nabla}\cdot\vec{B}) \right] dV$$

Finally:

$$(1) \quad \frac{d\vec{p}_m}{dt} + \frac{d}{dt}\int_V (\vec{D}\times\vec{B})\, dV = \int_V \Big[(\vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E}) + \vec{H}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{H})\Big]\, dV$$

Here $(\vec{D}\times\vec{B})$ electromagnetic momentum density.

and $(\vec{E}(\vec{\nabla}\cdot\vec{D}) - \vec{D}\times(\vec{\nabla}\times\vec{E}) + \vec{H}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{H}))$

the Maxwell stress-strain tensor:

$$T_{ij} = D_i E_j - \frac{1}{2}\delta_{ij}(\vec{D}\cdot\vec{E}) + B_i H_j - \frac{1}{2}\delta_{ij}(\vec{B}\cdot\vec{H})$$

δ_{ij} is the Kronecker symbol, i.e. $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

(1) shorter $$\frac{dp_{im}}{dt} + \frac{d}{dt}\int (\vec{D}\times\vec{B})_i \, dV = \int_V \frac{\partial}{\partial x_j} T_{ij} \, dV$$

i-component of $\vec{p}_m$ with $i = 1, 2, 3$

Angular momentum $\vec{l} = \vec{r} \times \vec{p}$ (mechanics)

For the angular momentum density of the electromagnetic field we obtain

$$\vec{l} = \vec{r} \times (\vec{D} \times \vec{B})$$