Quantized Fields

field operators: $|i\rangle$ single particle states , complete orthonorm
$\langle \bar{x} | i \rangle = \varphi_i(\bar{x})$

$$\boxed{\begin{aligned} \psi(\bar{x}) &= \sum_i \varphi_i(\bar{x})\, a_i \\ \psi^+(\bar{x}) &= \sum_i \varphi_i^*(\bar{x})\, a_i^+ \end{aligned}}$$

$$a_i = \int \varphi_i^*(\bar{x})\, \psi(\bar{x})\, d^3x$$

$$a_i^+ = \int \varphi_i(\bar{x})\, \psi^+(\bar{x})\, d^3x$$

Boson systems:

$$[\psi(\bar{x}), \psi^+(\bar{x}')] = \delta(\bar{x}-\bar{x}')$$

$$[\psi(\bar{x}), \psi(\bar{x}')] = [\psi^+(\bar{x}), \psi^+(\bar{x}')] = 0$$

Fermions

$$\{\psi(\bar{x}), \psi^+(\bar{x}')\} = \delta(\bar{x}-\bar{x}') \quad , \quad \{\psi(\bar{x}), \psi(\bar{x})\} = \{\psi^+(\bar{x}), \psi^+(\bar{x}')\} = 0$$

$$\hat{N} = \sum_i a_i^+ a_i = \sum_i \hat{n}_i$$

$$\int \psi^+(\bar{x})\, \psi(\bar{x})\, d^3x = \sum_{ij} \underbrace{\int \varphi_i^*(\bar{x})\, \varphi_j(\bar{x})\, d^3x}_{\delta_{ij}}\; a_i^+ a_j = \sum_i a_i^+ a_i = \sum_i \hat{n}_i = \hat{N}$$

$\psi^+(\bar{x})\psi(\bar{x})$ particle density operator $\int \rho(\bar{x})\, d\bar{x} = N$

particle density op. $\rho(\bar{x}) = \sum_{i\ (\text{particles})} \delta(\bar{x}-\bar{x}_i) = \psi^+(\bar{x})\, \psi(\bar{x})$

$$\psi(\bar{x}) = \sum_k \varphi_k(\bar{x})\, a_k$$

$$\psi^+(\bar{x}) = \sum_k \varphi_k^*(\bar{x})\, a_k^+$$

$$\hat{H} = \int d^3x\, \psi^+(\bar{x})\, \mathcal{H}(\bar{x})\, \psi(\bar{x}) + \frac{1}{2}\iint d^3x_1 d^3x_2\, \psi^+(\bar{x}_1)\psi^+(\bar{x}_2)\, V(\bar{x}_1,\bar{x}_2)\, \psi(\bar{x}_2)\psi(\bar{x}_1)$$

$$= \sum_{k,l} \langle k|\mathcal{H}|l\rangle\, a_k^+ a_l + \frac{1}{2}\sum_{k,l,m,n} \langle kl|V|mn\rangle\, a_k^+ a_l^+ a_m a_n$$

$$[\hat{N}, \hat{H}] = 0$$

$|k\rangle$ single particle states chosen $\mathcal{H}|k\rangle = \varepsilon_k |k\rangle$

eigenstates of the single-particle Hamiltonian

$$\hat{H} = \sum_k \varepsilon_k\, a_k^+ a_k + \frac{1}{2}\sum_{k,l,m,n} \langle kl|V|mn\rangle\, a_k^+ a_l^+ a_m a_n$$

Ideal (non-interacting) quantum gas

Grand canonical ensemble $\hat{N}, \hat{H}$ in part. num representation

$$\hat{\rho} = \frac{e^{-\beta(\hat{H} - \mu\hat{N})}}{Z_G} \qquad Z_G = \mathrm{Tr}\left(e^{-\beta(\hat{H} - \mu\hat{N})}\right)$$

$$\hat{H} = \sum_k \varepsilon_k n_k$$

ε_k: spectrum known in principle

$$\hat{N} = \sum_k n_k$$

Fermi and Bose Statistics

$$\sum_r n_r = N \qquad \sum_r n_r \varepsilon_r = E_{\{n_i\}} \qquad \text{based on one particle states}$$

$$Z_G = \mathrm{Tr}\, e^{-\beta(\hat{\mathcal{H}} - \mu \hat{N})} = \sum_{\{n_i\}} e^{-\beta(E_{\{n_i\}} - \mu N)} = \sum_{\{n_i\}} e^{-\beta \sum_r (\varepsilon_r - \mu) n_r}$$

$$= \sum_{\{n_i\}} \prod_r e^{-\beta(\varepsilon_r - \mu) n_r} = \boxed{\prod_r \sum_{n_r} e^{-\beta(\varepsilon_r - \mu) n_r}}$$

Fermi-Dirac: $\quad n_r = 0, 1$

Bose-Einstein: $\quad n_r = 0, 1, 2, \ldots$

F-D: $$Z_G = \prod_r \left(1 + e^{-\beta(\varepsilon_r - \mu)}\right)$$

$$\langle n_r \rangle = \frac{\partial}{\partial(-\beta\varepsilon_r)} \ln Z_G = \frac{e^{-\beta(\varepsilon_r - \mu)}}{e^{-\beta(\varepsilon_r - \mu)} + 1} = \boxed{\frac{1}{e^{\beta(\varepsilon_r - \mu)} + 1}}$$

B-E: $$Z_G = \prod_r \sum_{n_r = 0}^{\infty} e^{-\beta(\varepsilon_r - \mu) n_r} \overset{\boxed{\mu < \varepsilon_r \ \forall r}}{=} \prod_r \frac{1}{1 - e^{-\beta(\varepsilon_r - \mu)}}$$

convergence requirement!

$$\langle n_r \rangle = \frac{\partial}{\partial(-\beta\varepsilon_r)} \ln Z_G = \frac{e^{-\beta\varepsilon_r}}{1 - e^{-\beta(\varepsilon_r - \mu)}} = \boxed{\frac{1}{e^{\beta(\varepsilon_r - \mu)} - 1}}$$

$$\langle N \rangle = \sum_r \langle n_r \rangle = \sum_r \frac{1}{e^{\beta(\varepsilon_r - \mu)} \pm 1}$$

F-D $+$

B-E $-$

$$\langle E \rangle = \sum_r \varepsilon_r \langle n_r \rangle = \sum_r \frac{\varepsilon_r}{e^{\beta(\varepsilon_r - \mu)} \pm 1}$$

Free Fermi / Bose Gas

$k^{x,y,z} = \frac{2\pi}{L} n^{x,y,z}$ $\quad n^{x,y,z} = 0,1,2,\ldots$

$$\langle N \rangle = \sum_r \frac{1}{e^{\beta(\varepsilon_r - \mu)} \pm 1} \qquad \sum_r \to \frac{V}{(2\pi)^3} \int d^3k$$

spin s: $(2s+1)$ multiplicity

$$\sum_r \to (2s+1) \frac{V}{(2\pi)^3} \int d^3k = \qquad \boxed{r = \{k^x, k^y, k^z; s_z\}}$$

$s_z = -s, -s+1, \ldots, +s$

$$= (2s+1) \frac{V}{h^3} \int d^3p \qquad \bar{p} = \hbar \bar{k}$$

$(2s+1)$ values

$$\langle N \rangle = (2s+1) \frac{V}{h^3} \int \frac{d^3p}{e^{\beta(\varepsilon(\bar{p}) - \mu)} \pm 1} = \int g(\varepsilon) \langle n(\varepsilon) \rangle \, d\varepsilon$$

$$\langle E \rangle = (2s+1) \frac{V}{h^3} \int \frac{\varepsilon(\bar{p}) \, d^3p}{e^{\beta(\varepsilon(\bar{p}) - \mu)} \pm 1} = \int g(\varepsilon) \, \varepsilon \, \langle n(\varepsilon) \rangle \, d\varepsilon$$

non-relativistic quantum gas: $\varepsilon(\bar{p}) = \frac{p^2}{2m}$

extreme-relativistic quantum gas: $\varepsilon(\bar{p}) = c|\bar{p}| = cp$

relativistic quantum gas $\varepsilon(\bar{p}) = \sqrt{m^2c^4 + c^2p^2}$

$$\boxed{(2s+1) \frac{V}{h^3} d^3p = g(\varepsilon) \, d\varepsilon}$$

one-particle density of states

using $\varepsilon(\bar{p})$, one can obtain $g(\varepsilon)$

$$\Phi(T,V,\mu) = -kT \ln Z_G =$$

$$= -kT \ln \prod_r \left[\sum_{n_r} e^{-\beta(\varepsilon_r-\mu)n_r} \right] =$$

$$= -kT (\pm) \ln \prod_r \left[1 \pm e^{-\beta(\varepsilon_r-\mu)} \right] = \mp kT \ln \prod_r \left[1 \pm e^{-\beta(\varepsilon_r-\mu)} \right]$$

upper : F-D
lower : B-E

$$= \mp kT \sum_r \ln\left[1 \pm e^{-\beta(\varepsilon_r-\mu)}\right] = \mp kT (2s+1) \frac{V}{h^3} \int d^3p \, \ln\left[1 \pm e^{-\beta(\varepsilon(p)-\mu)}\right]$$

$\varepsilon(\vec{p}) = \varepsilon(p)$.

$$\boxed{\Phi(T,V,\mu) = \mp kT (2s+1) \frac{V}{h^3} \int_0^\infty 4\pi p^2 dp \, \ln\left[1 \pm e^{-\beta(\varepsilon(p)-\mu)}\right]}$$

$$\Phi(T,V,\mu) = -PV$$ grand thermodynamic potential

$$PV = \pm kT (2s+1) \frac{4\pi V}{h^3} \int_0^\infty p^2 dp \, \ln\left[1 \pm e^{-\beta(\varepsilon(p)-\mu)}\right] =$$ integrating by part

$$= \pm kT (2s+1) \frac{4\pi V}{h^3} \left\{ \underbrace{\frac{p^3}{3} \ln\left[1 \pm e^{-\beta(\varepsilon(p)-\mu)}\right] \Bigg|_0^\infty}_{\emptyset} - \frac{1}{3} \int_0^\infty \frac{p^3 dp (\pm)\left(-\beta \frac{d\varepsilon}{dp}\right) e^{-\beta(\varepsilon-\mu)}}{1 \pm e^{-\beta(\varepsilon(p)-\mu)}} \right.$$

$$= (2s+1) \frac{4\pi V}{h^3} \frac{1}{3} \int \frac{dp \, p^3 \frac{d\varepsilon}{dp}}{e^{\beta(\varepsilon(p)-\mu)} \pm 1} = (2s+1) \frac{4\pi V}{h^3} \frac{1}{3} \int \frac{dp \, p^2 \left(p \frac{d\varepsilon}{dp}\right)}{e^{\beta(\varepsilon(p)-\mu)} \pm 1}$$

non rel: $\varepsilon(p) = \frac{p^2}{2m}$ $\quad p \frac{d\varepsilon}{dp} = p \cdot \frac{p}{m} = \frac{p^2}{m} = 2\,\varepsilon(p)$

extrem rel: $\varepsilon(p) = cp$ $\quad p \frac{d\varepsilon}{dp} = p \cdot c = \varepsilon(p)$

non.-rel: $\boxed{PV = \frac{2}{3}E}$

extrem-rel. $\boxed{PV = \frac{E}{3}}$ independent of statistics

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The classical limit $s = 0$

$$\frac{\langle N\rangle}{V} = \frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \frac{1}{e^{\beta(\frac{p^2}{2m}-\mu)} \pm 1} =$$

$\frac{\beta p^2}{2m} \equiv x^2$

$p = \sqrt{2mkT}\, x$

$dp = \sqrt{2mkT}\, dx$

$$= \frac{4\pi}{h^3}(2mkT)^{3/2}\int_0^\infty dx \frac{x^2}{e^{x^2-\mu/kT} \pm 1}$$

classical limit:

$$\lambda_T = \sqrt{\frac{h^2}{2\pi mkT}}$$

$$\boxed{\frac{\langle N\rangle}{V}\cdot\lambda^3 = \frac{4\pi}{\pi^{3/2}}\int_0^\infty dx \frac{x^2}{e^{x^2-\mu/kT} \pm 1}}$$

$$\frac{\lambda^3}{V/N} \ll 1$$

monotonic decreasing function

$$\frac{\langle N\rangle}{V}\lambda^3 = f(-\mu/kT) \ll 1$$

$$\boxed{-\mu/kT \gg 1}$$

$$\frac{1}{e^{\beta(\frac{p^2}{2m}-\mu)}+1} \simeq e^{\beta\mu}\, e^{-\beta\frac{p^2}{2m}}$$