Lecture 20 - The nature of light

In empty space (i.e., no charge, no current), Maxwell's equations are:

$$\oint \vec{E} \cdot d\vec{a} = 0 \qquad \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{a} = 0 \qquad \oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

The divergence theorem, tells us that

$$\oint_s \vec{A} \cdot d\vec{a} = \int_v (\vec{\nabla} \cdot \vec{A})\, d\tau$$

And the curl (Stokes) theorem says:

$$\oint \vec{A} \cdot d\vec{l} = \int_s (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$

for any vector function $\vec{A}(\vec{r})$

If we apply these identities to Maxwell's equations, we can rewrite them in differential form:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's equations can be combined to yield wave equations for the electric and magnetic fields:

- Take the curl of the curl equations:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

- Use Ampere and Faraday's laws, and the product rule

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} . \vec{A}) - \nabla^2 \vec{A}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} . \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} . \vec{B}) - \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$\boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$

These results mean that each cartesian coordinate of each field in vacuum with no sources obeys the 3D wave equation $\vec{\nabla}^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$!

Maxwell's equations thus imply that empty space supports the propagation of electromagnetic waves traveling at velocity $v = 1/\sqrt{\epsilon_0 \mu_0} = 2.997\,924\,58 \times 10^8 \text{ m.s}^{-1}$

The implication is amazing! Perhaps light is an electromagnetic wave. Of course, this surprises no one today, but imagine the revelation it was in Maxwell's time. Remember that ϵ_0 and μ_0 are constants measured with charged balls and wires - nothing to do with light. And yet, from Maxwell's theory you can calculate c from these two numbers

Maxwell wrote: "This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws."

The simplest solutions to the wave equation are the harmonic plane waves. In 1D:

$$\vec{E} = \vec{E}_0 \cos\left[\frac{2\pi}{\lambda}\left(x - \frac{1}{\sqrt{\mu_0 \epsilon_0}} t\right)\right] = \vec{E}_0 \cos\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

$$\vec{B} = \vec{B}_0 \cos\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

Review : the harmonic <u>travelling</u> waves

$$y(t) = y_m \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) = y_m \sin\left[\frac{2\pi}{\lambda}\left(x - \frac{\lambda}{T}t\right)\right]$$

wavelength period

represents a wave of amplitude y_m travelling in the $+x$- direction.

The wave velocity : $v_{phase} = \frac{\lambda}{T} = \lambda f$, so : frequency

$$y(t) = y_m \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

Let $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi f$, then $v = \lambda f = \frac{\omega}{k}$, so

angular frequency

$$y(t) = y_m \sin(kx - \omega t)$$

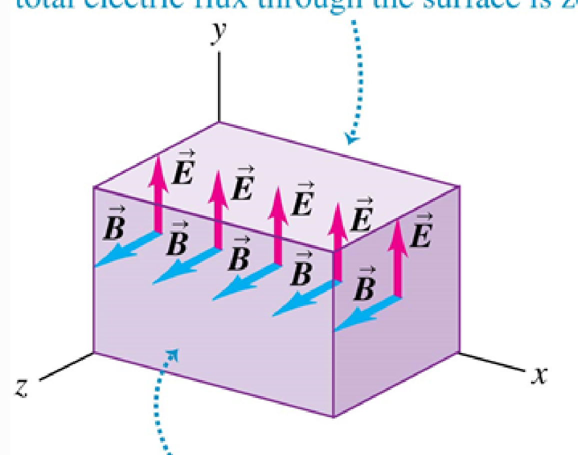
wavenumber

Maxwell's equations give us information about $\vec{E}$ and $\vec{B}$ in an electromagnetic wave :

- Gauss' laws

Take a Gaussian surface through which the wave is travelling. No charge is enclosed so Gauss' law tells us that $\vec{E}$ & $\vec{B}$ must be perpendicular to propagation direction : they are transverse.

The electric field is the same on the top and bottom sides of the Gaussian surface, so the total electric flux through the surface is zero.



The magnetic field is the same on the left and right sides of the Gaussian surface, so the total magnetic flux through the surface is zero.

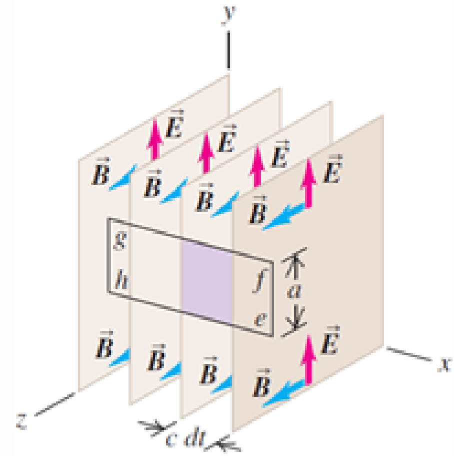
- Faraday's law

In a time dt; Φ_B through the purple rectangle increases by $d\Phi_B$. The rectangle has an area $A = a(c\,dt)$ so $d\Phi_B = Bac\,dt$ and $\frac{d\Phi_B}{dt} = Bac = Ea$ from Faraday's law. Thus:

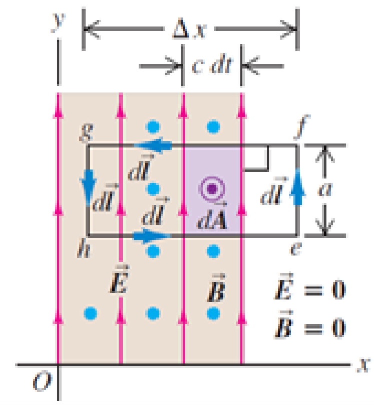
$$\boxed{E = cB}$$

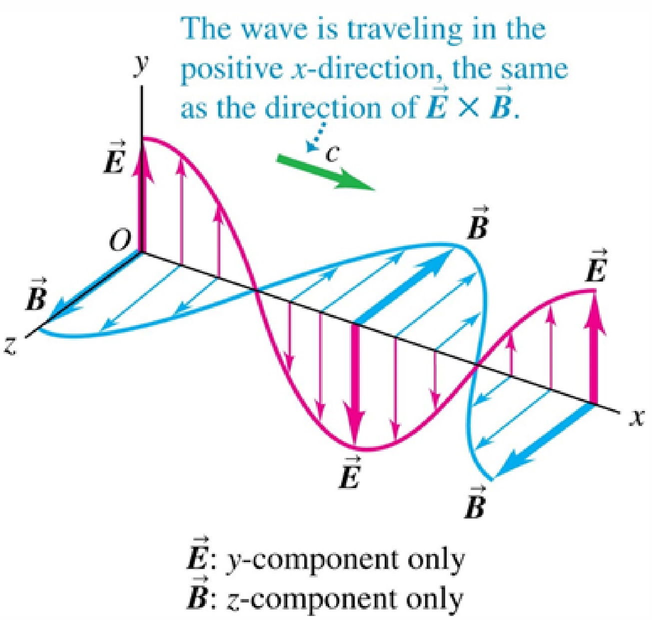
E: electric field magnitude; c: speed of light in vacuum; B: magnetic field magnitude.

(a) In time dt, the wave front moves a distance $c\,dt$ in the $+x$-direction.



(b) Side view of situation in (a)



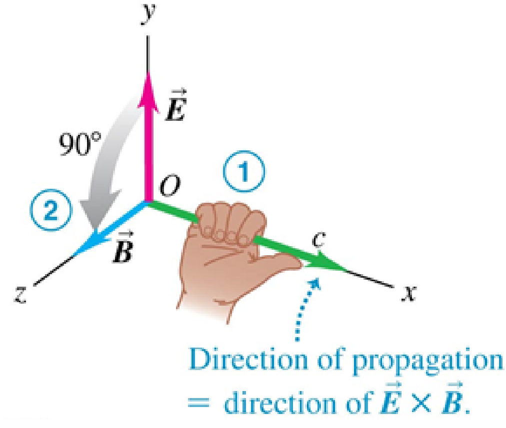


The direction of propagation of an electromagnetic wave is given by the Poynting vector

$$\boxed{\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})}$$

Right-hand rule for an electromagnetic wave:

1. Point the thumb of your right hand in the wave's direction of propagation.
2. Imagine rotating the $\vec{E}$-field vector 90° in the sense your fingers curl. That is the direction of the $\vec{B}$ field.



The intensity of an electromagnetic wave is defined as the average power per unit area transported by the wave (units : $W.m^{-2}$), it is given by the time-average of the Poynting vector.

$$\boxed{I = \bar{S} = \frac{1}{\mu_0}\overline{EB} = \frac{1}{2\mu_0}E_0 B_0}$$

Since $E_0 = cB_0$, intensity can also be written

$$\boxed{I = \frac{E_0^2}{2\mu_0 c} = \frac{cB_0^2}{2\mu_0}}$$

An electromagnetic wave incident on a surface carries energy (Poynting vector) and momentum. Radiation pressure (force per unit area) on a surface is given by:

$$\boxed{P = \epsilon_0 \frac{E_0^2}{2} = \frac{\bar{S}}{c} = \frac{I}{c}}$$ (for perfect absorber)

What is special about light?

- the speed limit for information transfer
- directly detected by our senses.
- source of many beautiful and interesting effects in nature.
- the carrier for all of the energy we use from the sun.
- massless particles with energy and momentum
- the historical source of our problems with wave/particle duality.

There are three important and useful ways to think about light:

1) Geometrical optics deals with rays. It is useful to study the behavior of systems with lengthscales much larger than the wavelength.

2) Wave optics focuses on interference and diffraction effects, as well as nonlinear optics. Useful when both amplitude and phase should be considered.

3) Quantum optics deals with the smallest lengthscales, when individual quanta of light (photons) should be considered.

Properties of light in the particle and wave picture.

- Speed
  - In the wave picture, the wavefront moves at $c = 1/\sqrt{\epsilon_0 \mu_0} = 3 \times 10^8 \ m.s^{-1}$
  - In the particle picture, energy is carried by photons moving at the speed of light. (special relativity demands that these particles be massless.)
- Intensity
  - In the wave picture, intensity is the time-avg of the Poynting vector:
$$I = \bar{S} = \frac{1}{2}\frac{1}{\mu_0} EB = \frac{1}{2}\epsilon_0 c E^2 = \frac{1}{2}\frac{c}{\mu_0} B^2$$
  - In the photon picture, each photon carries energy

$$\boxed{E_p = h\nu = \frac{hc}{\lambda} = \hbar\omega}$$

h = Planck's constant $= 6.6 \times 10^{-34}$ J.s

Intensity is the due to addition of many photons :

$$\boxed{I = F h\nu} \qquad \text{with} \quad F = \underset{\text{photon density}}{\rho} c = \text{photon flux}/(s.m^2)$$

▸ Momentum

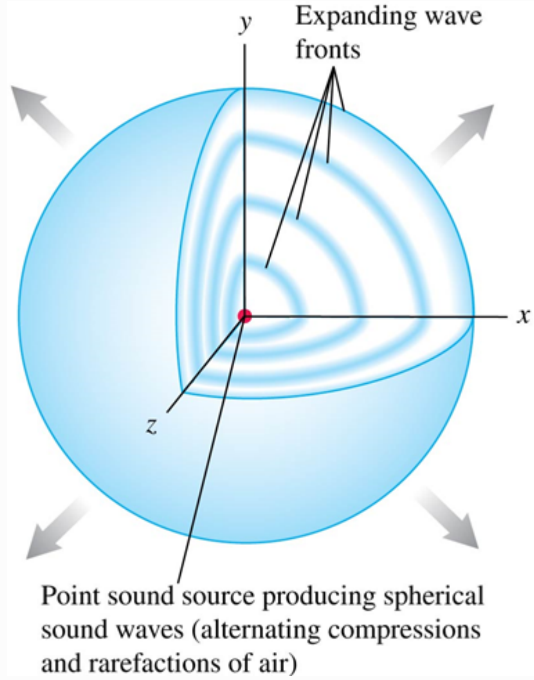
- In the wave picture, light's momentum can be observed through the radiation pressure :

$$P_{rad} = \frac{I}{c} = \frac{\epsilon_0}{2} E^2$$

- In the photon picture, each particle carries momentum :

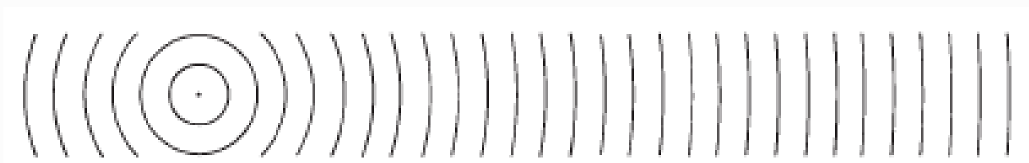
$$\boxed{p = \frac{h}{\lambda} = \frac{E_p}{c}}$$

Waves, wave fronts, and rays

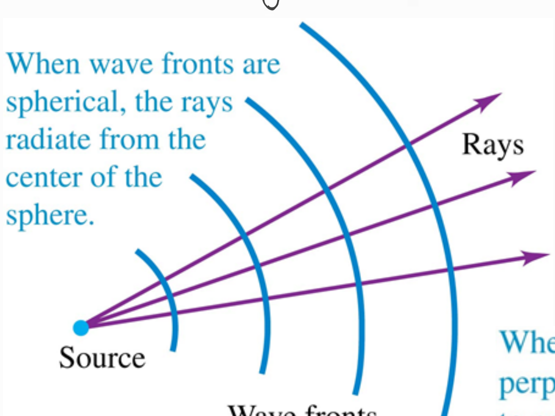


A wave front is a surface connecting points of equal phase. In the case of plane waves, these are parallel planes perpendicular to the propagation direction. Spherical waves originate from a point source and propagate radially at cst speed: the wave fronts are spheres.

Far from the source, spherical waves approach plane waves:

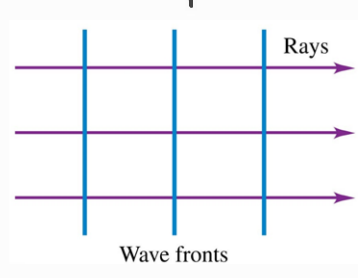


Often, it's more convenient to represent a light wave by rays rather than wave fronts.



A ray is an imaginary line parallel to the propagation direction and orthogonal to the wave front.

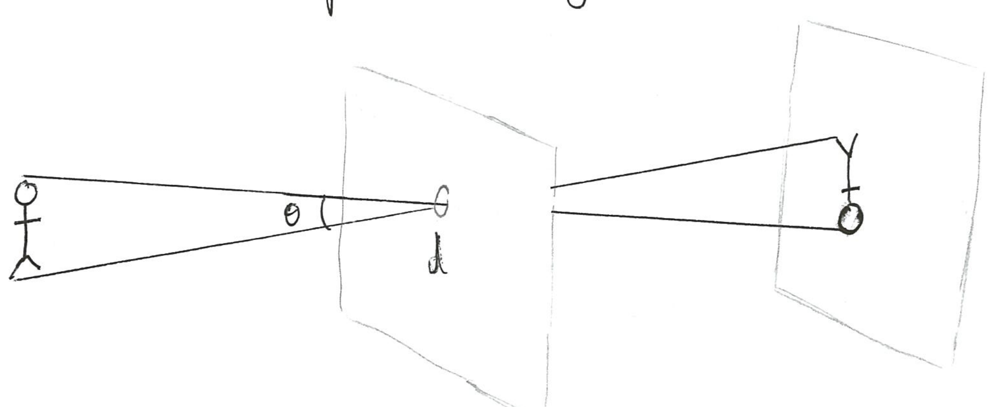
When wave fronts are planar, the rays are perpendicular to the wave fronts and parallel to each other.



In geometrical (ray) optics, we use rays to describe how light travels through media. Rays can

- travel in a straight line from point to point.
- experience specular reflection from a plane surface.
- experience diffuse reflection from a rough surface
- refract when passing from one material (index of refraction) to another.

Ray optics is a useful approximation for imaging as long as the smallest angle we are interested in is large compared to the wavelength divided by the smallest aperture size: $\theta \gg \lambda/d$



Rays travelling from one point to another follow a path such that, compared with nearby paths, the time required is either a min or a max or remains unchanged.

This is known as Fermat's principle.

A version of this idea is known as the principle of least action in quantum mechanics.