1. Calculating the matrix elements $\langle \mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_N | e^{-\beta H} | \mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_N \rangle$, in class we formally obtained the partition function in the canonical ensemble for an N-particle non-interacting non-relativistic quantum system

$$Z_N = \mathrm{Tr}(e^{-\beta H}) = \int d^{3N}x \, \langle \mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_N | e^{-\beta H} | \mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_N \rangle,$$

where

$$\langle \mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_N | e^{-\beta H} | \mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_N \rangle = \frac{1}{N!\lambda^{3N}} \sum_P \delta_P f(P\mathbf{x}_1 - \mathbf{x}_1) f(P\mathbf{x}_2 - \mathbf{x}_2) \ldots f(P\mathbf{x}_N - \mathbf{x}_N)$$

$f(u) = e^{-\frac{\pi u^2}{\lambda^2}}$, $\lambda = \left(\frac{h^2}{2\pi mkT}\right)^{1/2}$, $\delta_P \equiv 1$ for Bosons and $\delta_P = (-1)^{[P]}$ for Fermions.

In order to study lowest-order quantum corrections to classical non-interacting systems ($\lambda^3\left(\frac{N}{V}\right) << 1$), it is sufficient to consider only the trivial permutation ("no permutation") and two-particle permutations in the above expression, i.e.,

$$\langle \boldsymbol{x}_1\boldsymbol{x}_2 \ldots \boldsymbol{x}_N | e^{-\beta H} | \boldsymbol{x}_1\boldsymbol{x}_2 \ldots \boldsymbol{x}_N \rangle = \frac{1}{N!\lambda^{3N}} \left\{ 1 \pm \sum_{i<j} f^2(r_{ij}) + \ldots \right\}$$

where $r_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$, and the $\pm$ sign are for Bosons/Fermions).

(a) Find the equation of state in this first order approximation.
(b) Find $\langle E \rangle$ as a function of T, V, and N in the same approximation.

2. We defined in class the Fermi-Dirac (+) and Bose-Einstein (-) integrals

$$f_\nu^\pm(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{dx \, x^{\nu-1}}{z^{-1}e^x \pm 1}.$$

Prove that for $z << 1$

$$f_\nu^\pm = \sum_{l=1}^\infty (\mp 1)^{l-1} \frac{z^l}{l^\nu} = z \mp \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} \mp \ldots$$

3. Working in the grand-canonical ensemble, we obtained in class for the ideal non-relativistic Fermi (+) and Bose (-) systems with spin s:

$$N = (2s+1)\frac{V}{\lambda^3} f_{3/2}^{\pm}(z),$$

$$E = \frac{3}{2}kT\,(2s+1)\frac{V}{\lambda^3} f_{5/2}^{\pm}(z),$$

where $z = e^{\mu/kT}$ and $\lambda = \left(\frac{h^2}{2\pi mkT}\right)^{1/2}$.

(a) Obtain the energy E as a function of T, V, N, and the equation of state up to *first order* in $\frac{\lambda^3 N}{V}$. Note that after using the small z approximation for $f_\nu^{\pm}(z)$, z must be eliminated from the equations in favor of $\frac{\lambda^3 N}{V}$. To this end you should use the first equation above for N. How do your final results compare with those of Problem 1.(a) and (b)?

(b) Using $E(T,V,N)$, obtain the specific heat of the quantum gas up to the same order in $\frac{\lambda^3 N}{V}$.

4. Consider He gas at room temperature and atmospheric pressure, and determine whether the classical approximation for the equation of state ($PV = NkT$) is justified or not. Repeat the above considerations for the electron "gas" at room temperature.