Class 24 (4/18/24)

Radiation from Moving Electric Charges

Review Key Results Potentials

- Maxwell's Equations with sources ρ(**r**,t) and **j**(**r**,t)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}}, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\dot{\vec{E}}$$

- With $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$
- Inhomogeneous Wave Equations for V(**r**,t) and **A** (**r**,t)
- Solutions are the Retarded Potentials

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}dV', \quad A(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{V'}\frac{\vec{j}(\vec{r}\,',t_r)\,dV'}{|\vec{r}-\vec{r}\,'|}$$

Retarded Potentials
&
Radiation Fields
of an Oscillating Electric
Dipole

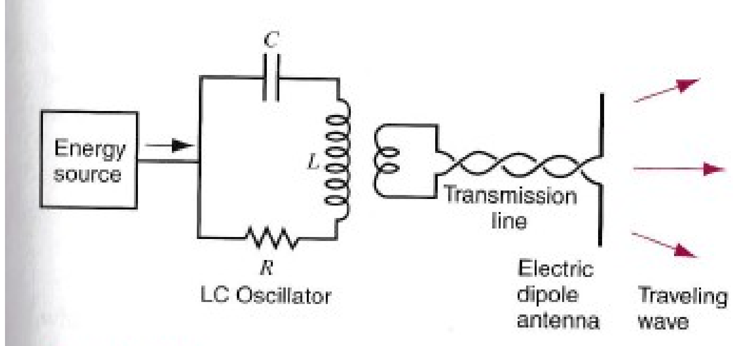


FIGURE 38-5. An arrangement for generating a traveling electromagnetic wave.

Electric dipole radiation

oscillating charge $q(t) = q_0 \cos(\omega t)$

oscillating current $i(t) = \frac{dq}{dt}$

Potentials:

$$V(r,\theta,t) = \underbrace{\frac{p_0\cos\theta}{4\pi\varepsilon_0}\left(-\frac{\omega}{c}\right)\frac{\sin[\omega(t-r/c)]}{r}}_{\text{radiation field potential } \propto \frac{1}{r}} + \underbrace{\frac{p_0\cos\theta}{4\pi\varepsilon_0}\,\frac{\cos[\omega(t-r/c)]}{r^2}}_{\text{quasi-static near field } (\propto \frac{1}{r^2}) \text{ potential}}$$

far away from the dipole

in the vicinity of the electric dipole

$$\vec{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi}\,\frac{\sin[\omega(t-r/c)]}{r}\,(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

Spherical waves

Radiation fields:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\theta}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\phi}$$

Calculate $\vec{S}(\vec{r},t) = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$, $\langle S\rangle$, $\langle P\rangle$

At large distances from the dipole, the observer of the approaching spherical wave “sees” a plane wave front because of the large radius of curvature.

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\,\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\right\}^2 \hat{r}$$

time average $\langle\vec{S}\rangle = \dfrac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\,\dfrac{\sin^2\theta}{r^2}$

$$\langle P\rangle = \int\langle\vec{S}\rangle\cdot da = \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$

Lienhard-Wiechert Potentials
&
Radiation Fields
of Moving Electric Charges

Radiation from moving electric charges is important because it is exploited to engineer sources of electromagnetic radiation with well-defined properties for the common purpose of life and research application.

- Generation of electric charges: Photoemission, Thermoemission
- Motion of electric charges: Acceleration by a bias voltage
- Frequency and power of the electromagnetic radiation controlled by amount of charge, type of acceleration (linear vs. centripetal) and speed of the charge (bias voltage controlled).

Moving (point) charges and coordinate systems

2 perspectives for $V(\vec{r},t)$:

origin of coordinate system not at point charge

origin of coordinate system (rest frame) of electron

electrostatic case: $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

Electric potential of point charge at origin **r**=0

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}$$

Electric potential of point charge not at origin but at position **r'**

analog

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}\cdot\frac{1}{R}$$

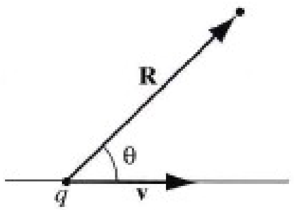


FIGURE 10.9

Recall problem 10.16

with θ being the angle between $\vec{v}$ and $\vec{R}$

where $\vec{R}$ is a vector from the moving charge at t

to the potential point $\vec{r}$ and $\vec{R} = \vec{r} - \vec{v}t$

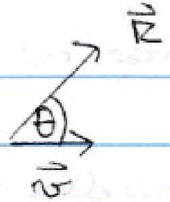
present time

$\theta = 0$ or π $\quad \vec{R}, \vec{v}$ $\uparrow\uparrow$ $\uparrow\downarrow$ $\quad \sin\theta = 0$ $\quad$ Coulomb pot

$\theta = \frac{\pi}{2}$ $\quad \vec{R} \perp \vec{v}$ $\quad \sin\theta = 1$ $\quad V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R\sqrt{1-v^2/c^2}}$

different perspective for $\vec{E}$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{1-v^2/c^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \frac{\hat{R}}{R^2}$$



$\theta = 0/\pi \quad \vec{v} \uparrow\downarrow \vec{R} \quad \sin\theta = 0$

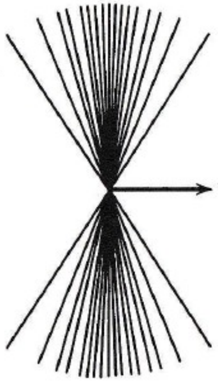
$$\vec{E}_{\parallel} = \frac{q}{4\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\right)\frac{\hat{R}}{R^2}$$

reduced compared to point charge at rest (reduction increases with v, in the relativistic case $\vec{E}_{\parallel} \to 0$)

$\theta = \frac{\pi}{2} \quad \vec{R} \perp \vec{v} \quad \sin\theta = 1$

$$\vec{E}_{\perp} = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{\hat{R}}{R^2}$$

enhanced compared to Coulomb field for charge at rest



Lienard-Wiechert potential for a moving point charge

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(|\vec{r}-\vec{r}'|c - (\vec{r}-\vec{r}')\cdot\vec{v})}$$

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

velocity field radiation field

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{|\vec{r}-\vec{r}'|}{(\vec{r}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + (\vec{r}-\vec{r}')\times(\vec{u}\times\vec{a})\right]$$

$$\vec{u} = c\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} - \vec{v}$$

Description of **E** if origin of the coordinate system is not at position of point charge q.

$$\vec{B}(\vec{r},t) = \frac{1}{c}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\times\vec{E}(\vec{r},t)$$

the time derivative of $\vec{A}$, i.e. $\frac{\partial \vec{A}}{\partial t}$ introduce $\vec{a} = \frac{d\vec{v}}{dt}$ into field expression, the spatial derivative of V, i.e. $\vec{\nabla}V$ contributes as well

keep in mind: linear acceleration: magnitude of $\vec{v}$ changes

centripetal acceleration: direction of $\vec{v}$ changes

Radiation Field

$$\vec{E}_{rad}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{|\vec{r}-\vec{r}'|}{\left((\vec{r}-\vec{r}')\cdot\vec{u}\right)^3}\left[(\vec{r}-\vec{r}')\times(\vec{u}\times\vec{a})\right]$$

$$\vec{B}_{rad}(\vec{r},t) = \frac{1}{c}\,\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\times\vec{E}(\vec{r},t)$$

$$\vec{S} = (\vec{E}_{rad}\times\vec{B}_{rad})\frac{1}{\mu_0} = \frac{1}{\mu_0 c}\left(\vec{E}\times\left(\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\times\vec{E}(\vec{r},t)\right)\right)$$

$$= \frac{1}{\mu_0 c}\left[\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\left(\vec{E}\cdot\vec{E}\right) - \vec{E}\,\underbrace{\left(\vec{E}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\right)}_{=0 \text{ because } \vec{E}\perp(\vec{r}-\vec{r}')}\right]$$

$$\vec{S} = \frac{1}{\mu_0 c}E^2_{rad}\,\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Next, the power radiated by the moving charge is calculated. This involves integration of the magnitude of the Poynting vector over a spherical surface. Mathematically this integration is the easiest if the origin of the coordinate system is a the position of the moving charge.

like in the case of the velocity field, we look at the fields radiated away from the charge from the perspective of the charge, i.e. $\vec{v} = 0$ and $\vec{u} = c\,\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$

$$(\vec{r}-\vec{r}')\cdot\vec{u} = c\,\frac{|\vec{r}-\vec{r}'|^2}{|\vec{r}-\vec{r}'|} = c\,|\vec{r}-\vec{r}'|$$

$$\vec{E}_{rad}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,|\vec{r}-\vec{r}'|\cdot\frac{1}{c^3}\,\frac{1}{|\vec{r}-\vec{r}'|^3}\left[(\vec{r}-\vec{r}')\times\left(c\,\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\times\vec{a}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{c^3}\,\frac{1}{|\vec{r}-\vec{r}'|^2}\left[c\,\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\left((\vec{r}-\vec{r}')\cdot\vec{a}\right) - \vec{a}\cdot\left(\frac{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')c}{|\vec{r}-\vec{r}'|}\right)\right]$$

factor out $|\vec{r}-\vec{r}'|$

$$\vec{E}_{rad}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{c^2}\cdot\left[\left(\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\cdot\vec{a}\right)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} - \vec{a}\right]\cdot\frac{1}{|\vec{r}-\vec{r}'|}$$

$$\vec{E}_{rad} \propto \frac{1}{|\vec{r}-\vec{r}'|}$$

$$\vec{S}_{rad} = \frac{1}{\mu_0 c} E^2_{rad} \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi}\right)^2 \frac{1}{|\vec{r}-\vec{r}'|^2}\left[a^2 - \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\cdot\vec{a}\right)^2\right]\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\cdot\vec{a} = \frac{|\vec{r}-\vec{r}'|}{|\vec{r}-\vec{r}'|}\cdot a\cdot \sin\theta$$

$$\vec{S}_{rad} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\,\frac{\sin^2\theta}{|\vec{r}-\vec{r}'|^2}\,\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$P_{tot} = \oint_{\text{sphere}} \vec{S}\cdot d\vec{A} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int_{\Omega}\frac{\sin^2\theta}{|\vec{r}-\vec{r}'|^2}\,|\vec{r}-\vec{r}'|^2 \sin\theta\, d\theta\, d\phi$$

$$P = \frac{\mu_0 q^2 a^2}{16\pi c}$$ Larmor formula

Velocity (Coulomb field)
of the moving charge

Moving charge

Accelerating electric charges to very high speed requires tremendous amounts of energy because the accelerating charge continuously loses energy in the form of radiation. From practical point of view, this makes it impossible to accelerate charges up to the value of the speed of light.

Power radiated away