Group Theory HW 4 Paul Len

N.IV.1 #s 2, 8, 10, 11, 12 N.IV.2 #1

2 Prove the Jacobi identity

$$[[A,B],C]+[[B,C],A]+[[C,A],B]=0$$

$$[AB-BA,C]+[BC-CB,A]+[CA-AC,B]=0$$

$$((AB-BA)C-C(AB-BA))+((BC-CB)A-A(BC-CB))+((CA-AC)B-B(CA-AC))$$

$$\cancel{ABC}-\cancel{BAC}-\cancel{CAB}+\cancel{CBA}+\cancel{BCA}-\cancel{CBA}-\cancel{ABC}+\cancel{ACB}+\cancel{CAB}-\cancel{ACB}-\cancel{BCA}+\cancel{BAC}$$

All terms cancel out.

<u>Jacobi identity confirmed</u>

8. Show $\varepsilon^{ijk\dots n} R^{ip} R^{jq} = \varepsilon^{pqr\dots s} R^{kr} \dots R^{ns}$

for $D=2$

$$\varepsilon^{ij} R^{ip} R^{jq} = \varepsilon^{12} \underbrace{R^{kr} R^{ns}}_{\mathrm{Det}(R)}$$

$$\cancelto{0}{\varepsilon^{11} R^{1p} R^{1q}} + \varepsilon^{12} R^{11} R^{22} + \varepsilon^{21} R^{21} R^{12} + \cancelto{0}{\varepsilon^{22} R^{2p} R^{2q}}$$

$$= \varepsilon^{12} \mathrm{Det}(R)$$

$$\Leftrightarrow \varepsilon^{12} R^{11} R^{22} - \varepsilon^{12} R^{21} R^{12} = \varepsilon^{12} \mathrm{Det}(R)$$

$$\varepsilon^{12} \underbrace{\left(R^{11} R^{22} - R^{21} R^{12} \right)}_{\mathrm{Det}(R)} = \varepsilon^{12} \mathrm{Det}(R)$$

$$\varepsilon^{12} \mathrm{Det}(R) = \varepsilon^{12} \mathrm{Det}(R)$$

True!

10: for $SO(3)$ show $\varepsilon^{ijk} A^i B^j = C^k$
defines $\vec{C} = \vec{A} \otimes \vec{B}$ using
$\varepsilon^{ijk\ldots n} R^{ip} R^{jq} R^{kr} \ldots R^{ns} = \varepsilon^{pqr\ldots s}$

$$\varepsilon^{ijk} A^i B^j \equiv C^k \Rightarrow \varepsilon^{ijk} \underbrace{R^{mi} R^{nj}}_{\text{a rotation } R^{kl}} A^i B^j = R^{kl} C^k$$

$$A^i = R^{mi} A^i \qquad \hookrightarrow \varepsilon^{mnl} R^{kl} A^m B^n = R^{kl} C^l$$

$$B^j = R^{nj} B^j$$

C^k transforms into
$R^{kl} C^l$, the cross-product

11:

$$\varepsilon^{ijk} \varepsilon^{lnk} = \delta^{il} \delta^{jn} - \delta^{in} \delta^{jl} \qquad \varepsilon^{ijk} = \det(e_i e_j e_k)$$

A^j, B^l, C^n

$$\vec{A} \otimes (\vec{B} \otimes \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\det(e_i e_j e_k) \det(e_l e_n e_k) = \delta^{il} \delta^{jn} - \delta^{in} \delta^{jl}$$

$$\hookrightarrow \det \begin{pmatrix} e_l^T \\ e_n^T \\ e_k^T \end{pmatrix}$$

$$\det \begin{pmatrix} e_1^T \\ e_n^T \\ e_k^T \end{pmatrix} \begin{array}{c} (e_i\ e_j\ e_k) \\ \begin{bmatrix} e_1^T e_i, & e_1^T e_j, & e_1^T e_k \\ e_n^T e_i, & e_n^T e_j, & e_n^T e_k \\ \cancel{e_k^T e_i}\,0, & \cancel{e_k^T e_j}\,0, & \cancel{e_k^T e_k}\,1 \end{bmatrix} \end{array}$$

$$= \det \begin{bmatrix} e_1^T e_i, & e_1^T e_j, & e_1^T e_k \\ e_n^T e_i, & e_n^T e_j, & e_n^T e_k \\ 0, & 0, & 1 \end{bmatrix} = \delta^{il}\delta^{jn} - \delta^{in}\delta^{jl}$$

$$= e_1^T e_i \begin{vmatrix} e_n^T e_j, & e_n^T e_k \\ 0, & 1 \end{vmatrix} - e_1^T e_j \begin{vmatrix} e_n^T e_i, & e_n^T e_k \\ 0, & 1 \end{vmatrix}$$

$$+ \cancel{e_1^T e_k \begin{vmatrix} e_n^T e_i, & e_n^T e_j \\ 0, & 0 \end{vmatrix}}^{\,0} = \delta^{il}\delta^{jn} - \delta^{in}\delta^{jl}$$

given $\delta^{ab} = e_a^T e_b$

$$\delta_{1i}\delta_{nj} - \delta_{1j}\delta_{ni} = \delta_{il}\delta_{jn} - \delta_{in}\delta_{jl} \checkmark$$

A^i, B^j, C^n Prove

$$\vec{A} \otimes (\vec{B} \otimes \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$a = a_i e_i$, $b \times c = (b \times c)_j e_j$

$e_i \times e_j = \varepsilon_{ijk} e_k$

$b \times c = \varepsilon_{mnj} b_m c_n e_j$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (a_i e_i \times ((b \times c)_j e_j)$$

$\to a_i (b \times c)_j (e_i \times e_j)$

$\to a_i (b \times c)_j \varepsilon_{ijk} e_k$

$\to \underbrace{\varepsilon_{mnj} \varepsilon_{ijk}}_{\text{using identity from above}} a_i b_m c_n e_k$

$\to \left(\delta^{il} \delta^{jn} - \delta^{in} \delta^{jl}\right) a_i b_m c_n e_k$

$= a_i b_m c_n \delta^{il} \delta^{jn} - a_i b_m c_n \delta^{in} \delta^{jl}$

$= a_n b_m c_n \vec{e}_m - a_m b_m c_n e_n$

$= b(a \cdot c) - c(a \cdot b)$

12 SO(5) — Credit to alyssa b.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{matrix}$$

$$SO(3) = \underset{\text{adjunct}}{30} \oplus \underset{\text{traceless}}{14} \oplus \underset{\text{asymmetric}}{10} \oplus \underset{\text{fundamental}}{5} \oplus \overset{\text{identity}}{1}$$

<u>5 smallest reps</u>

2.4.1 Write J_x, J_y & J_z in the $j = \frac{1}{2}, 1, 2$

~~A~~ Credit <u>Mugs Martin</u>

$$J_x = \frac{1}{2}(J_+ + J_-) \qquad j=\tfrac{1}{2},\ J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ J_- = J_+^T$$

$$J_y = \frac{-1}{2}(J_+ - J_-) \qquad j=1,\ J_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ J_- = J_+^T$$

$$J_z = \frac{1}{2}[J_+, J_-] \qquad j=2,\ J_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ J_- = J_+^T$$

J_x in $j = 1/2$

$$\frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in $j=1$

$$\frac{1}{2}\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in $j=2$

$$\frac{1}{2}\left[\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

J_y in $j = 1/2$

$$-\frac{1}{2}\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]=-\frac{1}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

in $j=1$

$$-\frac{1}{2}\left[\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}-\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\right]=-\frac{1}{2}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$$

in $j=2$

$$-\frac{1}{2}\left[\begin{pmatrix}0&2&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&\sqrt{6}&0\\0&0&0&0&2\\0&0&0&0&0\end{pmatrix}-\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\0&\sqrt{6}&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&2&0\end{pmatrix}\right]=-\frac{1}{2}\begin{pmatrix}0&2&0&0&0\\2&0&\sqrt{6}&0&0\\0,&-\sqrt{6}&0&\sqrt{6}&0\\0,&-&\sqrt{6}&0&0\\0&0&0,&-2&0\end{pmatrix}$$

J_z in $j = 1/2$ $\qquad AB - BA$

$$\frac{1}{2}\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}\overset{\begin{pmatrix}0&0\\1&0\end{pmatrix}}{\begin{pmatrix}1&0\\0&0\end{pmatrix}} - \begin{pmatrix}0&0\\1&0\end{pmatrix}\overset{\begin{pmatrix}0&1\\0&0\end{pmatrix}}{\begin{pmatrix}0&0\\0&1\end{pmatrix}}\right] = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

in $j = 1$

$$\frac{1}{2}\left[\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\overset{\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}} - \begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\overset{\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}}{\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}}\right] = \frac{1}{2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

in $j = 2$

$$\frac{1}{2}\left[\begin{pmatrix}0&2&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&\sqrt{6}&0\\0&0&0&0&2\\0&0&0&0&0\end{pmatrix}\overset{\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\0&\sqrt{6}&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&2&0\end{pmatrix}}{\begin{pmatrix}4&0&0&0&0\\0&6&0&0&0\\0&0&6&0&0\\0&0&0&4&0\\0&0&0&0&0\end{pmatrix}} - \begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\0&\sqrt{6}&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&2&0\end{pmatrix}\overset{\begin{pmatrix}0&2&0&0&0\\0&0&\sqrt{6}&0&0\\0&0&0&\sqrt{6}&0\\0&0&0&0&2\\0&0&0&0&0\end{pmatrix}}{\begin{pmatrix}0&0&0&0&0\\0&4&0&0&0\\0&0&6&0&0\\0&0&0&6&0\\0&0&0&0&4\end{pmatrix}}\right]$$

$$= \frac{1}{2}\begin{pmatrix}4&0&0&0&0\\0&2&0&0&0\\0&0&0&0&0\\0&0&0&-2&0\\0&0&0&0&-4\end{pmatrix}$$

$$[J_x, J_y] = J_z$$

$$\left[\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]$$

$$\frac{1}{4}\left(\overset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} - \overset{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} \right) = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= J_z$$

$$\left[\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right]$$

$$\frac{1}{4}\left(\overset{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}} - \overset{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \right) = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= J_z$$

$$\left[\frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, -\frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0, & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0, & - & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0, & -2 & 0 \end{pmatrix}\right]$$

$$\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ -2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{matrix} -4, & 0, & -2\sqrt{6}, & 0, & 0 \\ 0, & 2 & 0, & -6, & 0 \\ -2\sqrt{6}, & 0, & 0, & 0, & 0 \\ 0, & 6, & 0, & -6, & 0 \\ 0, & 0, & 2\sqrt{6}, & 0, & 0 \end{matrix} - \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ -2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & -2\sqrt{6} & 0 & 0 \\ 0 & 6 & 0 & -6 & 0 \\ 2\sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -6 & 0 \\ 0 & 0 & 2\sqrt{6} & 0 & -4 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -8 \end{pmatrix} = J_2$$

$$[J_y, J_z] = iJ_x$$

$$\left[-\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= \frac{1}{4}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}} \right) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = iJ_x$$

$$\left[-\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right]$$

$$\frac{1}{4}\left(\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \overset{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}} \right) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= iJ_x$$

$$\left[\frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}\right]$$

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{6} & 0 & -2\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} 0 & 8 & 0 & 0 & 0 \\ 4 & 0 & 2\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & -2\sqrt{6} & 0 & 0 \\ 0 & 2\sqrt{6} & 0 & -2\sqrt{6} & 0 \\ 0 & 0 & 2\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix} = J_x$$