Class 8 (02/05/24)

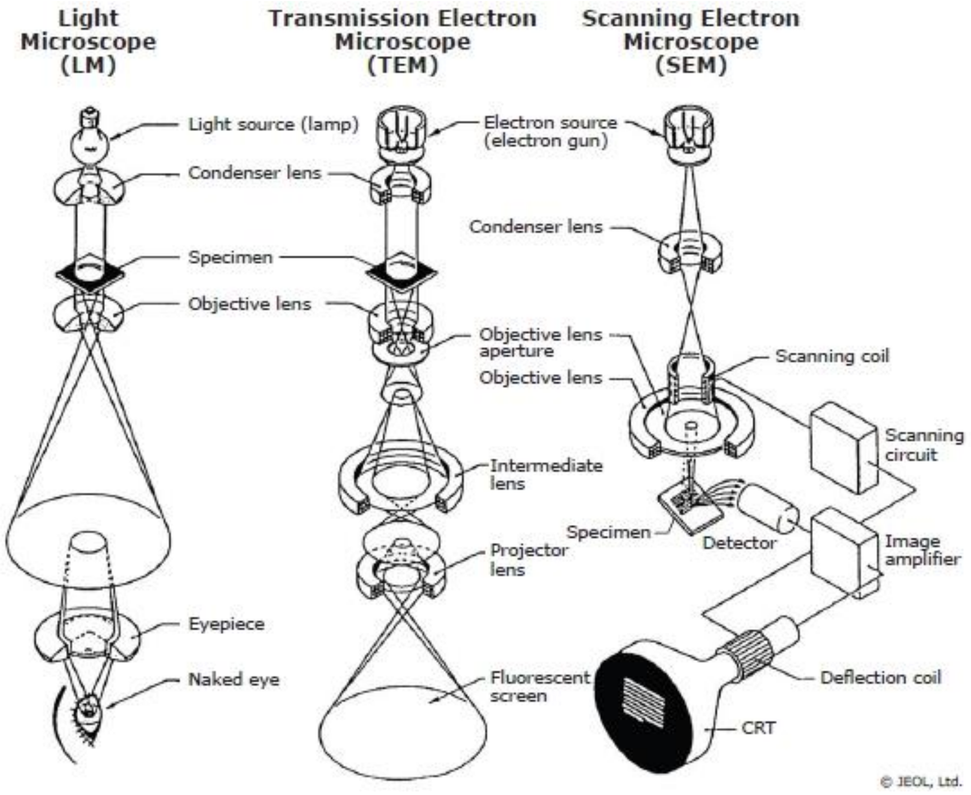
Laplace's Equation: Separation of variables in spherical coordinates r, ϕ, θ

Outline

- Summary: Solving Laplace's equation by separation of variables in cartesian coordinates x,y,z.
- Today: Solving Laplace equation by separation of variables in r, θ, ϕ.

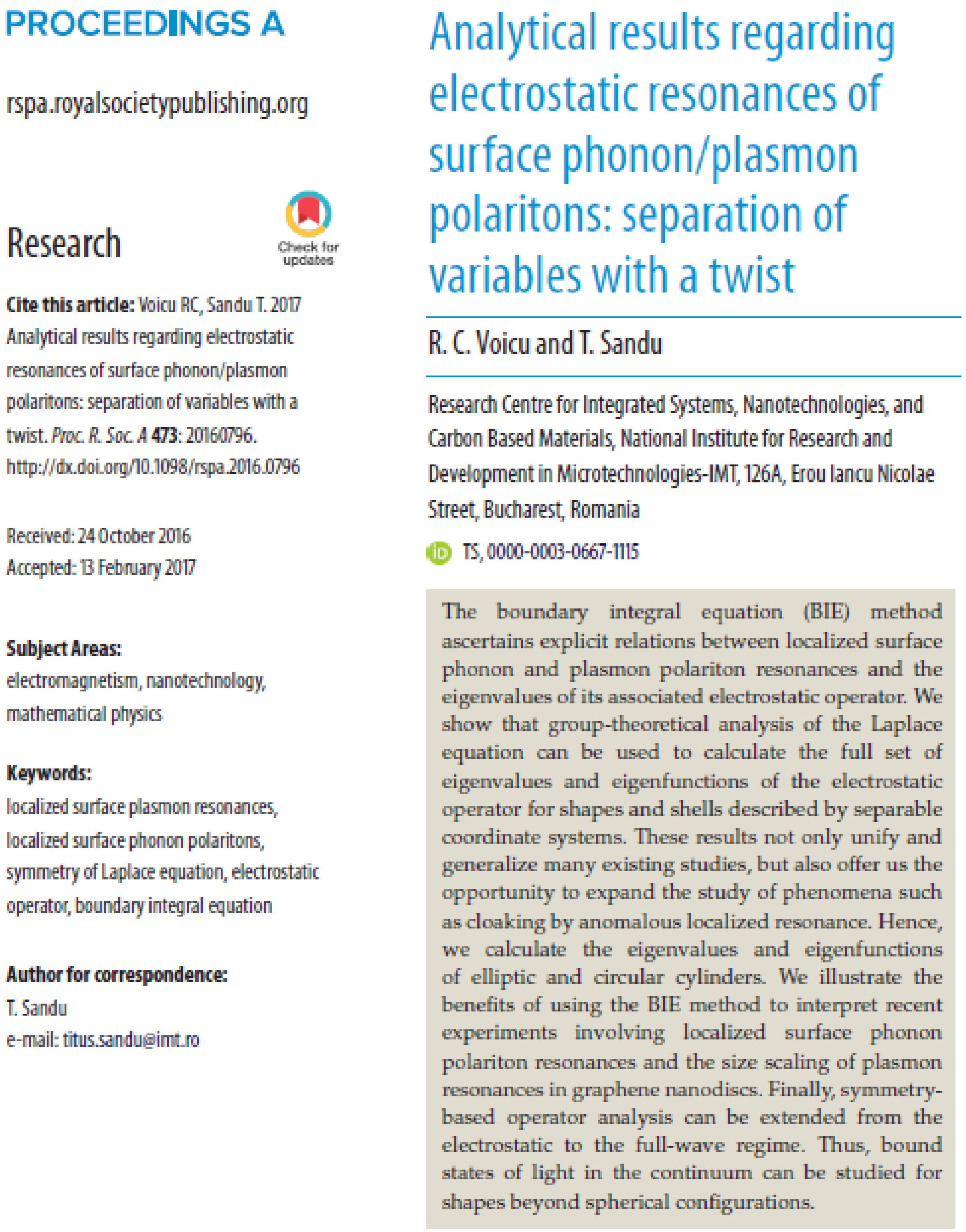
When do we use separation of variables in x, y, z (or 10 other coordinate systems) to solve $\vec{\nabla}^2V(\mathbf{r})=0$?

- There is excitement about knowing V(**r**) in a region of space which is free of electric charge, bounded by conducting surfaces and where the geometry of the problem is best described in Cartesian coordinates (or any of other 10 coordinate systems).
- One exemplary area of application are “electrostatic lenses” used for managing accelerating electron beams in e.g. electron microscopes (TEM/SEM). Electrostatic lenses consist of conducting metal electrodes at electric potential for manipulation of electrons.



Physics BS, MS, and PhD, make a living solving Laplace’s (& Poisson’s) equation in industry!

$\vec{E}=-\vec{\nabla}V, \quad \vec{F}=qE$



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Research

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Analytical results regarding electrostatic resonances of surface phonon/plasmon polaritons: separation of variables with a twist

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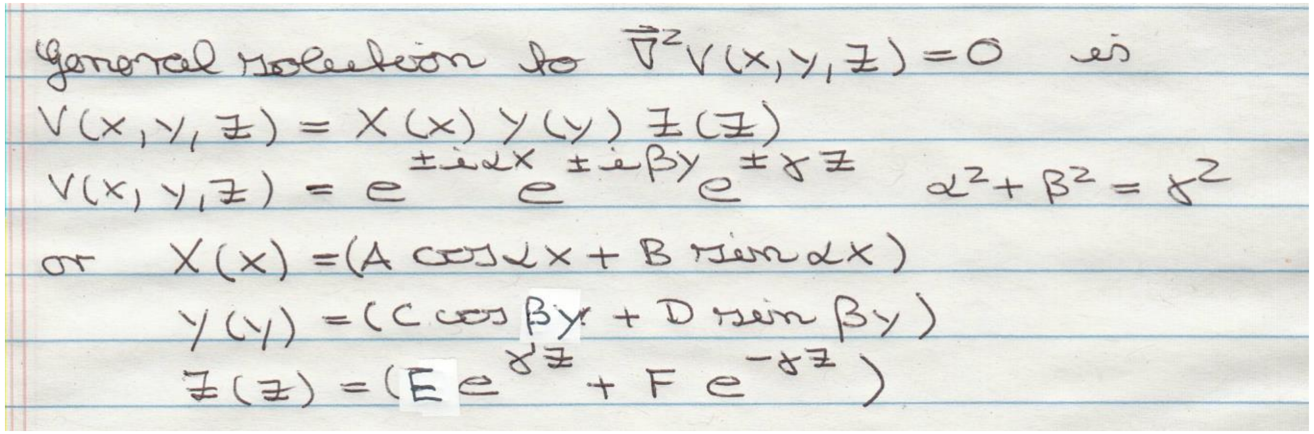
The boundary integral equation (BIE) method ascertains explicit relations between localized surface phonon and plasmon polariton resonances and the eigenvalues of its associated electrostatic operator. We show that group-theoretical analysis of the Laplace equation can be used to calculate the full set of eigenvalues and eigenfunctions of the electrostatic operator for shapes and shells described by separable coordinate systems. These results not only unify and generalize many existing studies, but also offer us the opportunity to expand the study of phenomena such as cloaking by anomalous localized resonance. Hence, we calculate the eigenvalues and eigenfunctions of elliptic and circular cylinders. We illustrate the benefits of using the BIE method to interpret recent experiments involving localized surface phonon polariton resonances and the size scaling of plasmon resonances in graphene nanodiscs. Finally, symmetry-based operator analysis can be extended from the electrostatic to the full-wave regime. Thus, bound states of light in the continuum can be studied for shapes beyond spherical configurations.

For more examples from research go to the RPI's Library Website, "Databases", "S", "Scopus", and search in "Articles, Abstracts, Keywords" for "electrostatics AND Laplace AND separation".

Solution of the Math Problem



General solution to $\vec{\nabla}^2 V(x,y,z)=0$ is

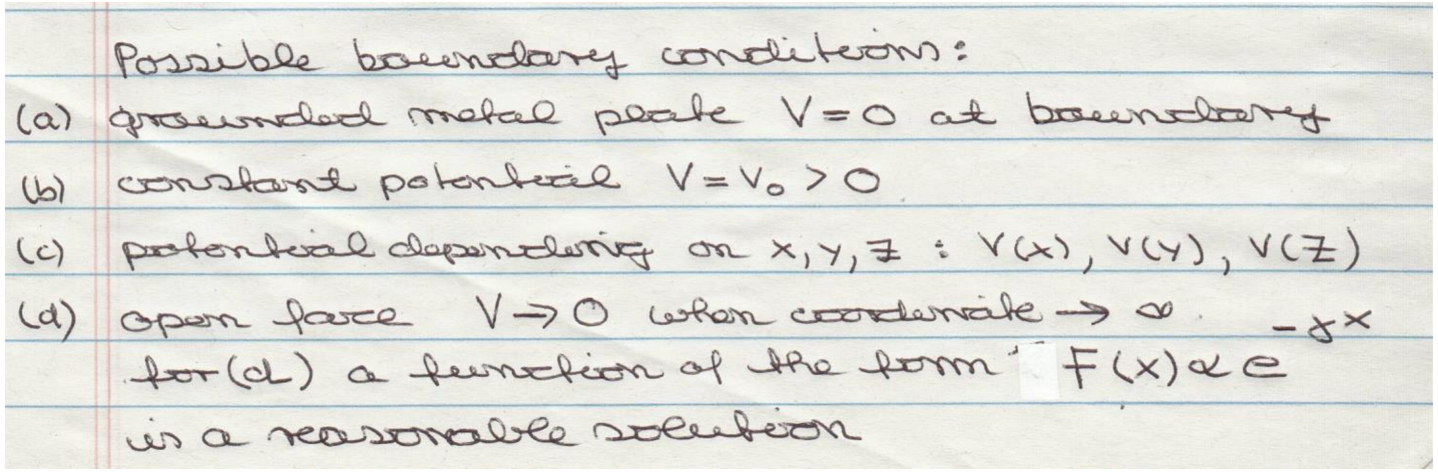
$V(x,y,z) = X(x)\,Y(y)\,Z(z)$

$V(x,y,z) = e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm \gamma z}$ $\quad \alpha^2+\beta^2=\gamma^2$

or $X(x) = (A\cos\alpha x + B\sin\alpha x)$

$Y(y) = (C\cos\beta y + D\sin\beta y)$

$Z(z) = (E e^{\gamma z} + F e^{-\gamma z})$



Possible boundary conditions:

(a) grounded metal plate $V=0$ at boundary

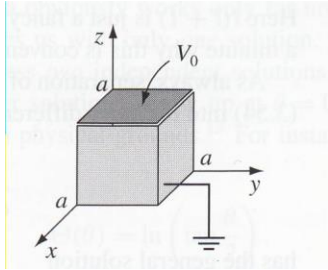
(b) constant potential $V=V_0>0$

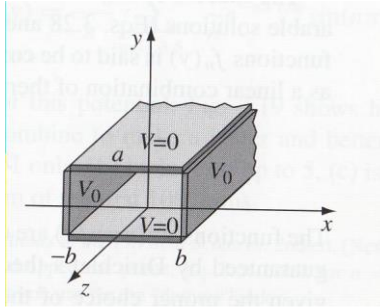
(c) potential depending on x,y,z : $V(x)$, $V(y)$, $V(z)$

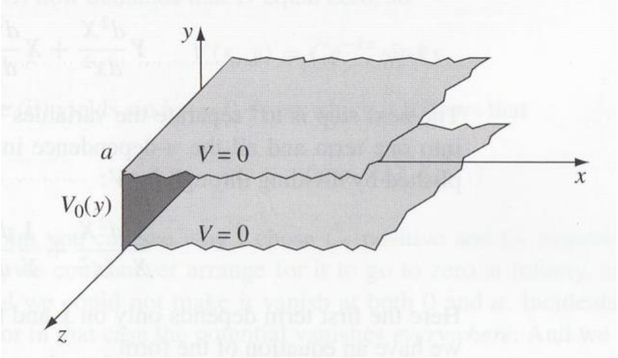
(d) open face $V\to 0$ when coordinate $\to\infty$

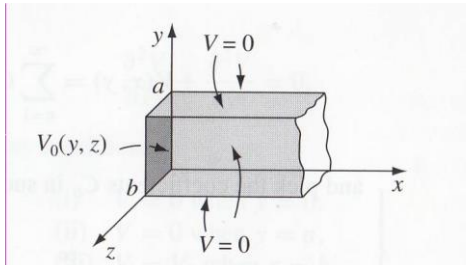
for (d) a function of the form $F(x)\propto e^{-\gamma x}$ is a reasonable solution

Possible Geometries

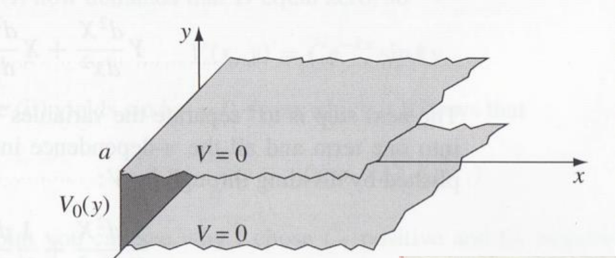


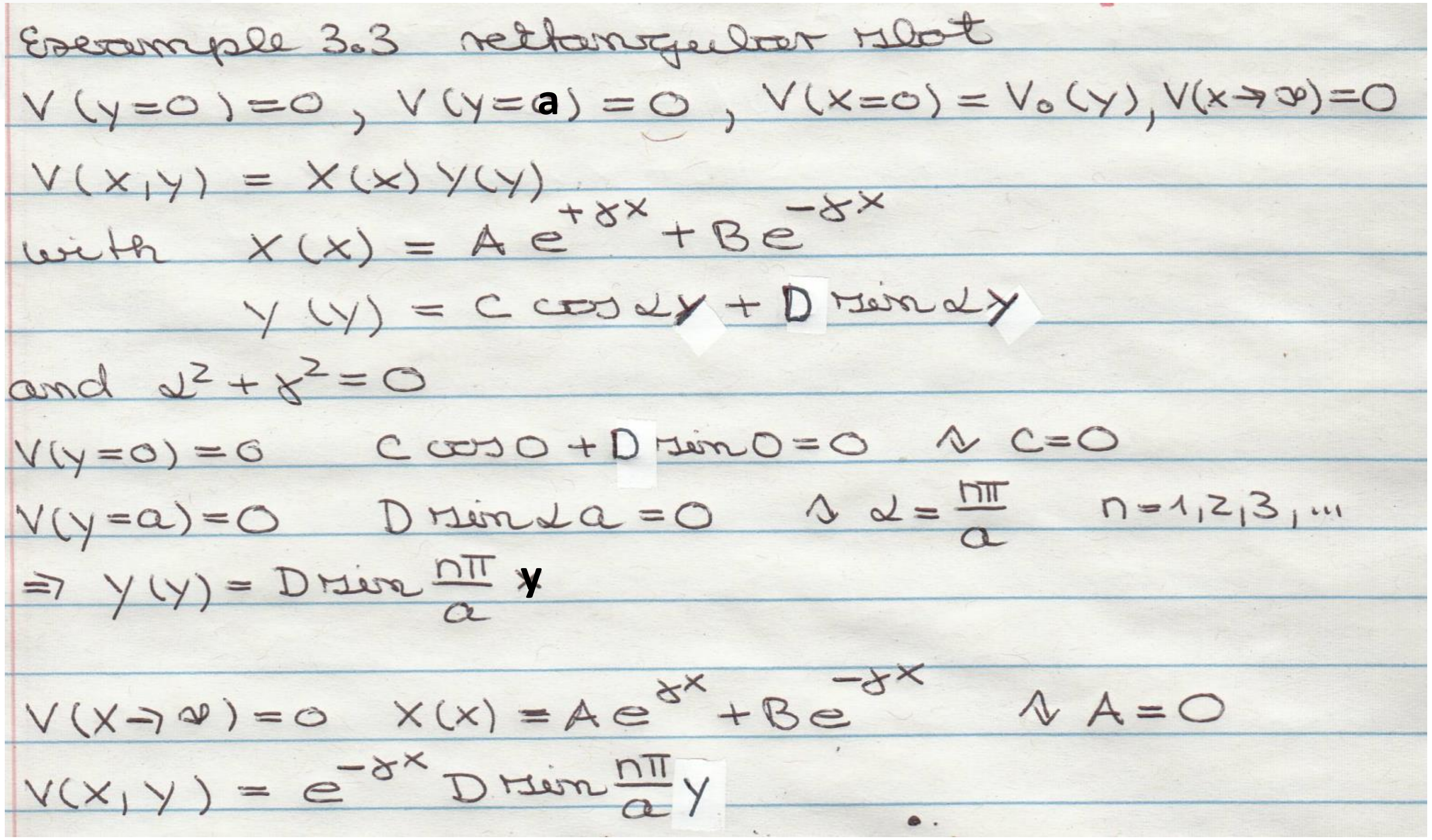






Taking the Solutions of the Math Problem and Applying them in an intelligent approach to the Physics Problem





Example 3.3 rectangular slot

$V(y=0)=0$, $V(y=a)=0$, $V(x=0)=V_0(y)$, $V(x\to\infty)=0$

$V(x,y) = X(x)\,Y(y)$

with $X(x) = A e^{+\gamma x} + B e^{-\gamma x}$

$Y(y) = C \cos \alpha y + D \sin \alpha y$

and $\alpha^2 + \gamma^2 = 0$

$V(y=0)=0$ $\quad C \cos 0 + D \sin 0 = 0 \quad \Rightarrow C=0$

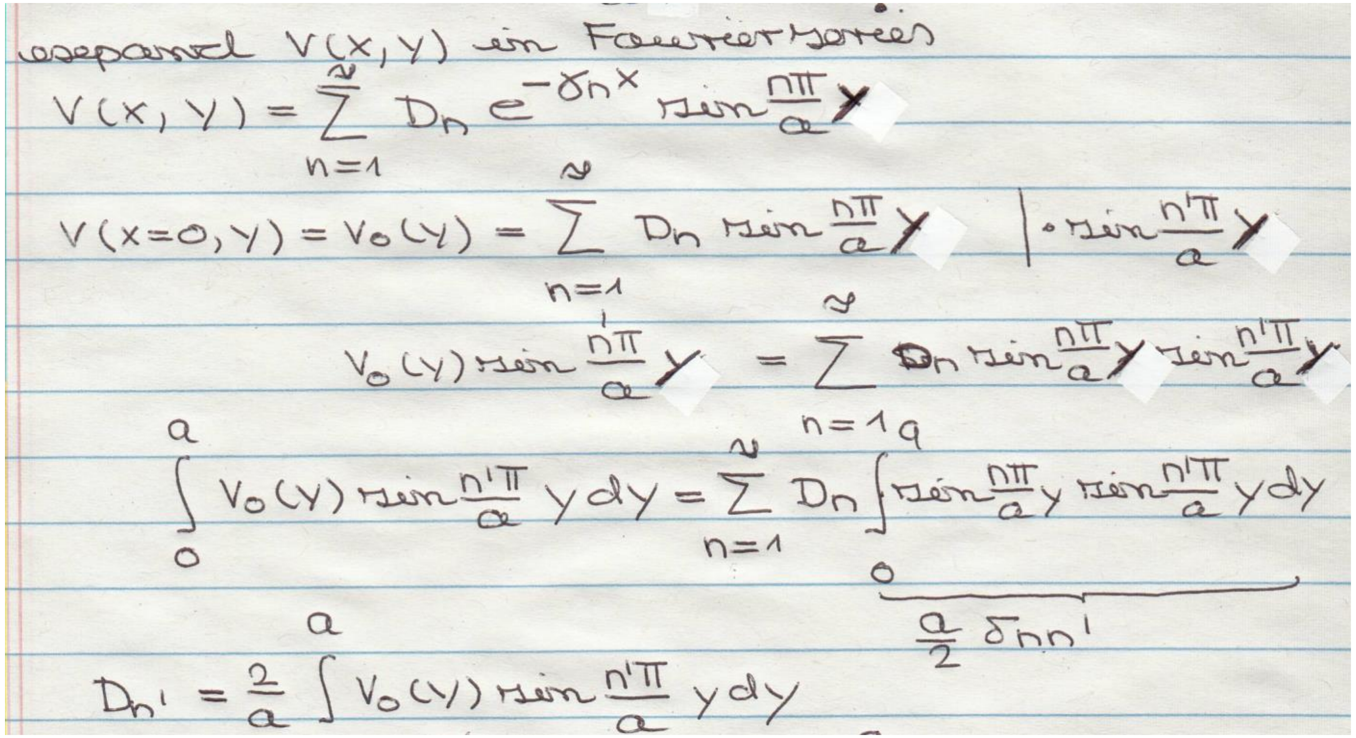
$V(y=a)=0$ $\quad D \sin \alpha a = 0 \quad \Rightarrow \alpha = \frac{n\pi}{a} \quad n=1,2,3,\ldots$

$\Rightarrow Y(y) = D \sin \frac{n\pi}{a} y$

$V(x\to\infty)=0$ $\quad X(x) = A e^{\gamma x} + B e^{-\gamma x} \quad \Rightarrow A=0$

$V(x,y) = e^{-\gamma x} D \sin \frac{n\pi}{a} y$.

Fourier's Trick



expand $V(x,y)$ in Fourier series

$$V(x,y) = \sum_{n=1}^{\infty} D_n e^{-\gamma_n x} \sin\frac{n\pi}{a}y$$

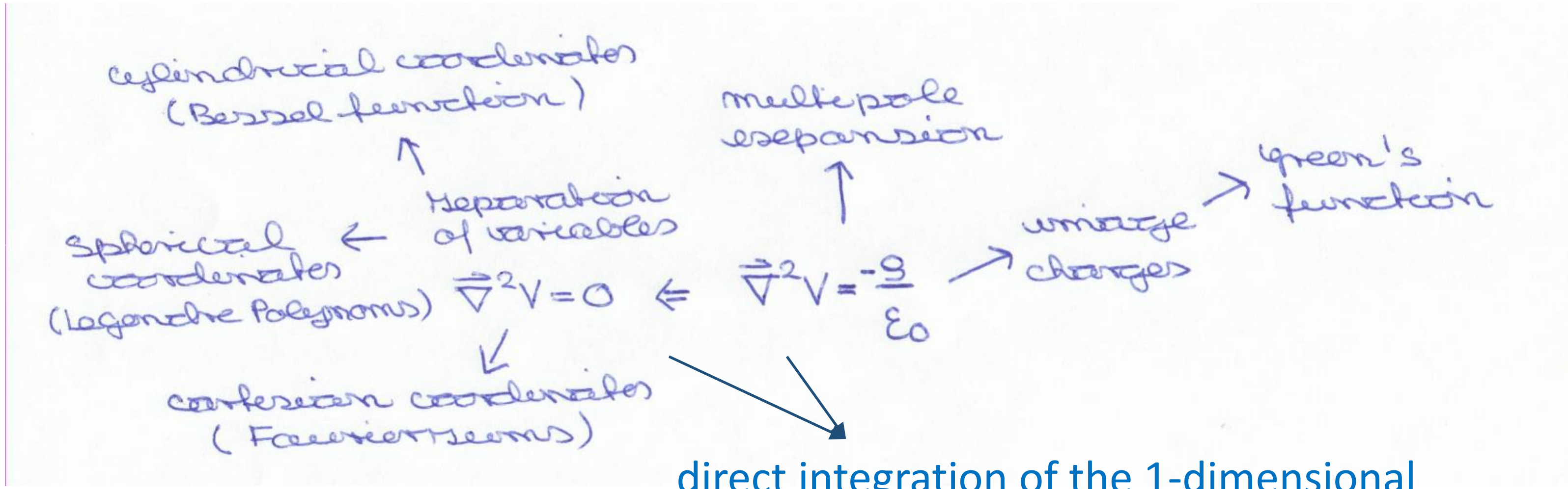
$$V(x=0,y) = V_0(y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{a}y \quad \Big| \cdot \sin\frac{n'\pi}{a}y$$

$$V_0(y) \sin\frac{n'\pi}{a}y = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{a}y \sin\frac{n'\pi}{a}y$$

$$\int_0^a V_0(y) \sin\frac{n'\pi}{a}y\,dy = \sum_{n=1}^{\infty} D_n \underbrace{\int_0^a \sin\frac{n\pi}{a}y \sin\frac{n'\pi}{a}y\,dy}_{\frac{a}{2}\delta_{nn'}}$$

$$D_{n'} = \frac{2}{a}\int_0^a V_0(y) \sin\frac{n'\pi}{a}y\,dy$$

Methods for solving Laplace’s & Poisson’s equations



direct integration of the 1-dimensional Laplace’s and Poisson’s equations

$$\vec{\nabla}^2 V(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

PHYS 4210 : Discussion limited to scenarios with azimuthal symmetry $\frac{\partial V}{\partial \varphi} = 0$.

Separation in r, θ : $V(r,\theta) = R(r)\,\Theta(\theta)$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - n(n+1)R = 0 \qquad (1)$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + n(n+1)\sin\theta = 0 \qquad (2)$$

Solution to (1) $\Rightarrow$ $R(r) = A_n r^n + B_n r^{-n-1}$ (try it!)

for (2) substitution $x = \cos\theta$, $dx = -\sin\theta\, d\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + n(n+1) = 0$$

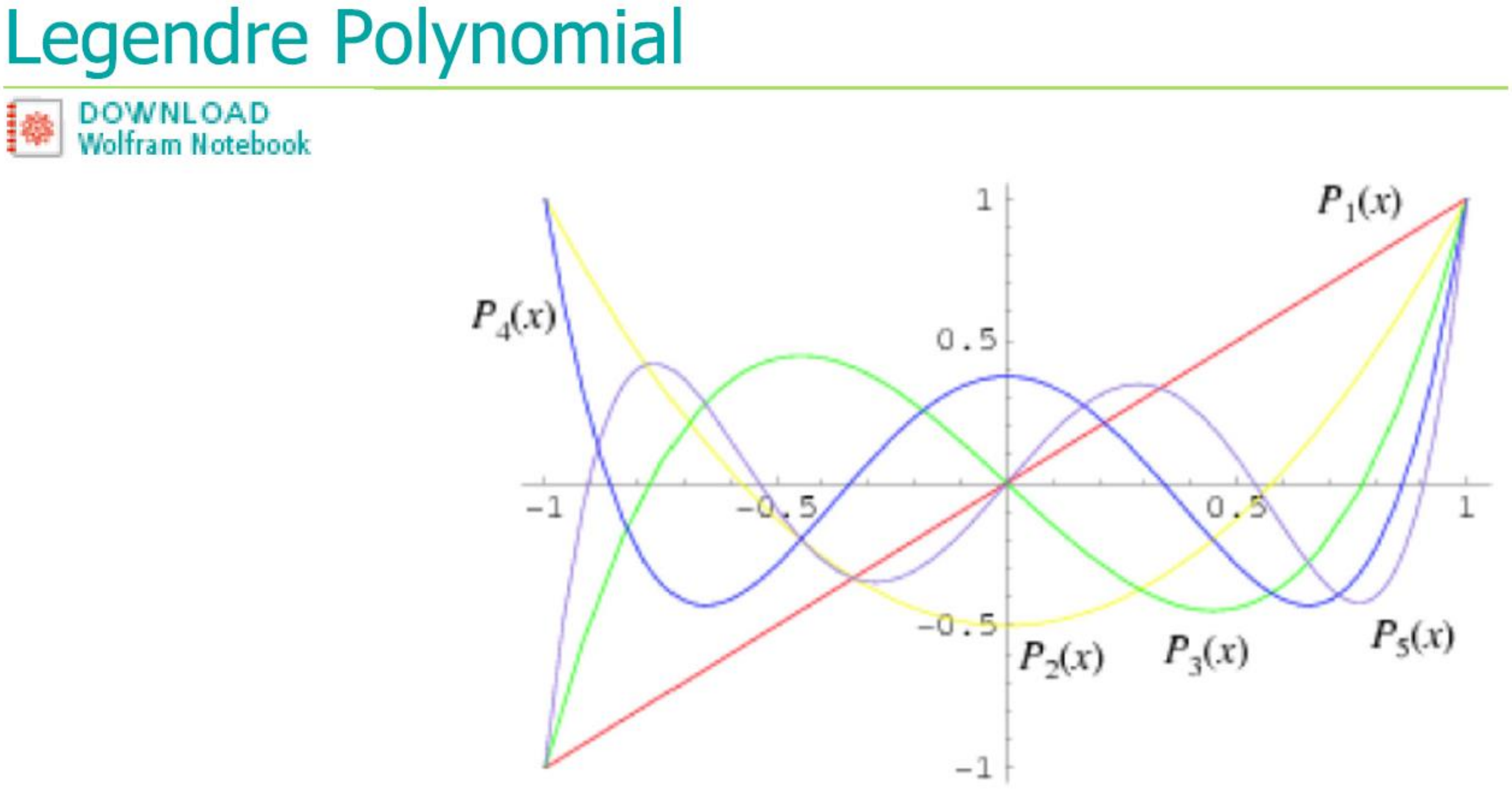
solutions Legendre Polynoms $P_n(\cos\theta)$

Solving Laplace's eq. in spherical coordinates is about solving a 2nd order partial differential equation, it's a math problem.

Legendre Differential Equation.

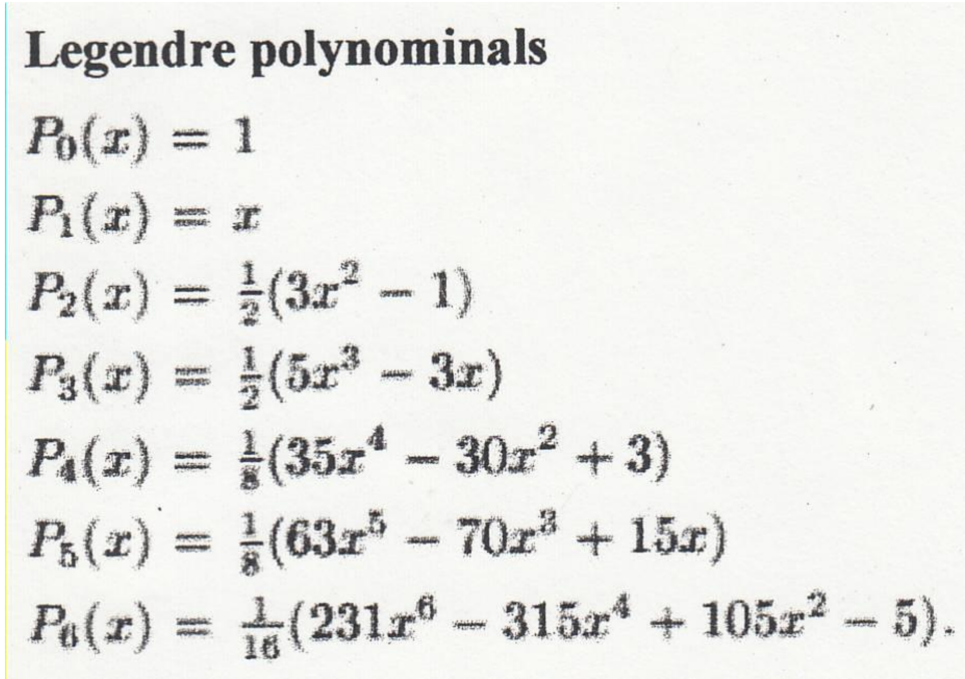
Let’s approach Legendre Polynominals:

https://mathworld.wolfram.com/LegendrePolynomial.html



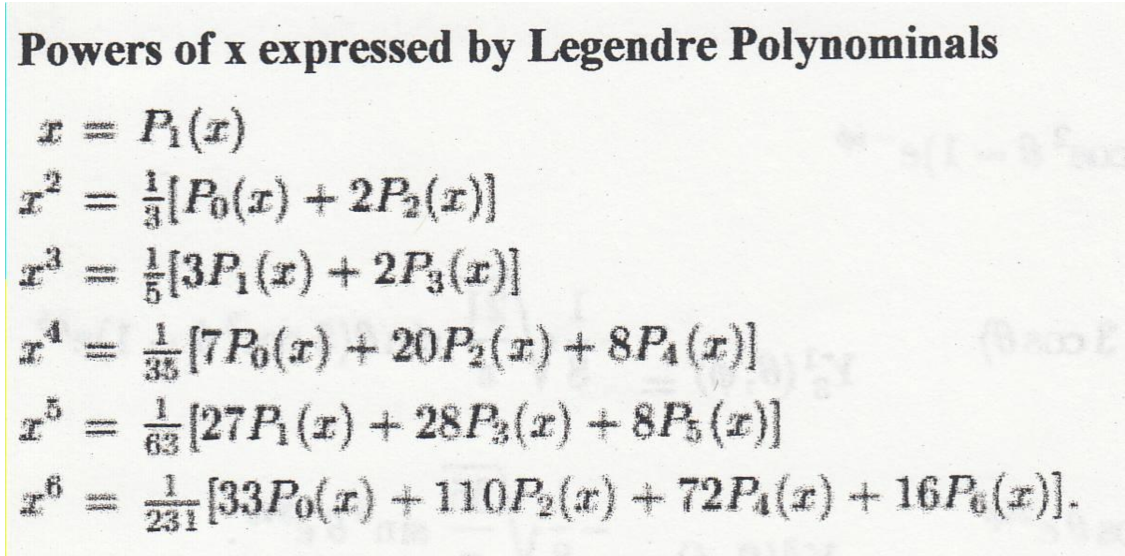
Legendre Polynomial

DOWNLOAD
Wolfram Notebook



Legendre polynominals

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$
$$P_6(x) = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).$$



Powers of x expressed by Legendre Polynominals

$$x = P_1(x)$$
$$x^2 = \tfrac{1}{3}[P_0(x) + 2P_2(x)]$$
$$x^3 = \tfrac{1}{5}[3P_1(x) + 2P_3(x)]$$
$$x^4 = \tfrac{1}{35}[7P_0(x) + 20P_2(x) + 8P_4(x)]$$
$$x^5 = \tfrac{1}{63}[27P_1(x) + 28P_3(x) + 8P_5(x)]$$
$$x^6 = \tfrac{1}{231}[33P_0(x) + 110P_2(x) + 72P_4(x) + 16P_6(x)].$$

Important properties of Legendre Polynominals.

Rodriguez Formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

e.g. $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, ...

Orthogonality relation: $\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1} \delta_{mn}$

Legendre Series $F(x) = \sum_{n=0}^{\infty} A_n P_n(x)$

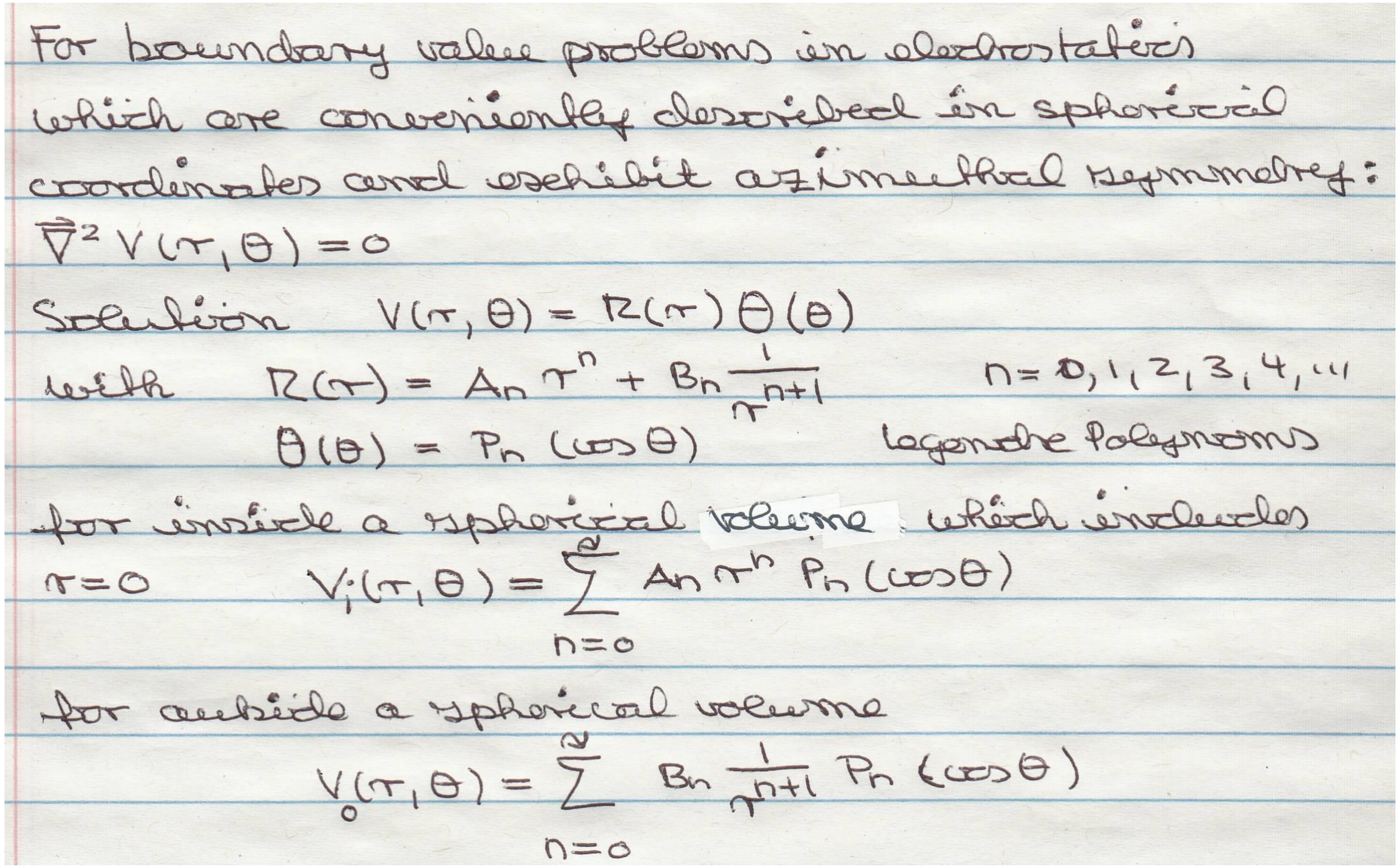
$$F(x) P_m(x) = \sum_{n=0}^{\infty} A_n P_m(x) P_n(x)$$

$$\int_{-1}^{1} F(x) P_m(x)\, dx = \sum_{n=0}^{\infty} A_n \int_{-1}^{1} P_m(x) P_n(x)\, dx$$

$$\int_{-1}^{1} F(x) P_m(x)\, dx = \sum_{n=0}^{\infty} A_n \frac{2}{2n+1} \delta_{mn}$$

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} F(x) P_n(x)\, dx$$

Summary of Solutions to Laplace Equation in Spherical Coordinates



For boundary value problems in electrostatics which are conveniently described in spherical coordinates and exhibit azimuthal symmetry:

$$\vec{\nabla}^2 V(r,\theta) = 0$$

Solution $V(r,\theta) = R(r)\,\Theta(\theta)$

with $R(r) = A_n r^n + B_n \frac{1}{r^{n+1}}$ $\quad n = 0, 1, 2, 3, 4, \ldots$

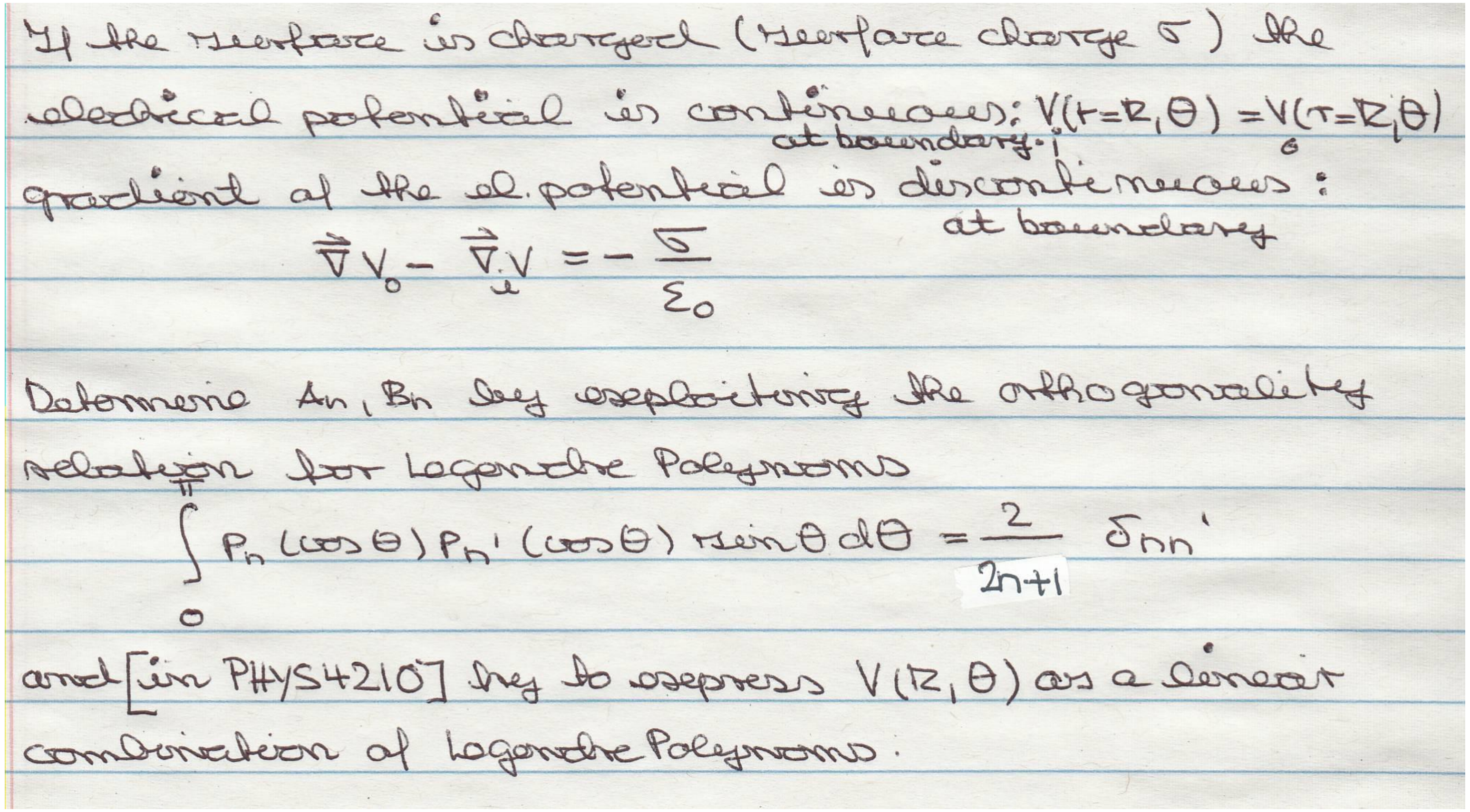
$\Theta(\theta) = P_n(\cos\theta)$ $\quad$ Legendre Polynoms

for inside a spherical volume which includes $r=0$

$$V_i(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

for outside a spherical volume

$$V_o(r,\theta) = \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P_n(\cos\theta)$$



If the surface is charged (surface charge σ) the electrical potential is continuous: $V_i(r=R,\theta) = V_o(r=R,\theta)$

at boundary.

gradient of the el. potential is discontinuous:

at boundary

$$\vec{\nabla} V_o - \vec{\nabla} V_i = -\frac{\sigma}{\varepsilon_0}$$

Determine A_n, B_n by exploiting the orthogonality relation for Legendre Polynoms

$$\int_0^\pi P_n(\cos\theta)\, P_{n'}(\cos\theta) \sin\theta \, d\theta = \frac{2}{2n+1}\,\delta_{nn'}$$

and [in PHYS4210] try to express $V(R,\theta)$ as a linear combination of Legendre Polynoms.