1. Linear Algebra

1. $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$EM = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} = \begin{pmatrix} c, & d \\ a, & b \end{pmatrix}$$

* The first & second rows are swapped.

2. $E = \begin{pmatrix} S_1, & 0 \\ 0, & S_2 \end{pmatrix}$ $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$EM = \begin{pmatrix} S_1, & 0 \\ 0, & S_2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} S_1 a, & S_1 b \\ S_2 c, & S_2 d \end{pmatrix}$$

top row multiplied by S_1

bottom row multiplied by S_2

3. $E = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ $M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$EM = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} a, & b \\ c+sa, & d+sb \end{pmatrix} \leftarrow$$ 1st row is added to second row

for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$E = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a+sc, & b+sd \\ c & d \end{pmatrix} \leftarrow$$ vice versa

4.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$ME = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}\begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

etc. for operations $\begin{pmatrix} s_1, & 0 \\ 0, & s_2 \end{pmatrix}$ & $\begin{pmatrix} 1, & 0 \\ s, & 1 \end{pmatrix}$

$$\begin{pmatrix} s_1, & 0 \\ 0, & s_2 \end{pmatrix} \qquad \begin{pmatrix} 1, & 0 \\ s, & 1 \end{pmatrix}$$

$$ME = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}\begin{pmatrix} s_1 a, & s_2 b \\ s_1 c, & s_2 d \end{pmatrix} \quad \& \quad ME = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}\begin{pmatrix} a+sb, & b \\ c+sd, & d \end{pmatrix}$$

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$$e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} a, & b, & c \\ d, & e, & f \\ g, & h, & i \end{bmatrix}$$

from the left

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$eM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

rows swapped

from the right

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$eM = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a & c & b \\ d & f & e \\ g & i & h \end{bmatrix}$$

columns swapped

6. Swapping rows a & b in an $n \times n$ matrix "M"

Matrix E w/ elements

$$e_{ij} = \begin{cases} 0 & \text{when } ij = aa \text{ or } bb \\ 1 & \text{when } ij = ab \text{ or } ba \\ \delta_{ij} & \text{else} \end{cases}$$

& EM = modified matrix

for columns, take the transpose of E first

Multiplying each row "i" by S_i

Matrix E w/ elements:

$e_{ij} = \delta_{ij} S_i$ where S_i is the factor row i is multiplied by.

for columns, take transpose

EM = modified matrix

To multiply row "a" by "S" & add to row "b"

Matrix E w/ elements $e_{ij} = \begin{cases} e_{bb} = 1 & \\ e_{ba} = r & \\ e_{bj} = 0 & \text{when } j \neq a, b \\ \delta_{ij} & \text{when } h \neq j \end{cases}$

EM = modified matrix.

Like before, take transpose for columns

7.

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad d = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$C^{-1} C = I$$

$$\begin{array}{cc} & C \\ & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ C^{-1} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; I \end{array}$$

$$\begin{array}{ccc} & d & C \\ & \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ C^{-1} & & \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0 \end{bmatrix} & \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & a \end{bmatrix} \\ & \uparrow & \uparrow \\ & C^{-1}d & \text{diagonally permuted} \end{array}$$

8. Anti-Symmetric matrix

$$A = -A^T \; ; \; a_{ij} = -a_{ji}$$

$$\det(A^T) = \det(-A) \rightarrow (-1)^n \det(A)$$

$\therefore$ for odd n

$\det(-A) = -\det(A)$ which has to be 0

a.

$$M = \begin{pmatrix} a, & 1 \\ 0, & b \end{pmatrix}, \quad \det\left|\begin{pmatrix} a, & 1 \\ 0, & b \end{pmatrix} - \begin{pmatrix} \lambda, & 0 \\ 0, & \lambda \end{pmatrix}\right| = 0$$

$$\det\begin{vmatrix} a-\lambda, & 1 \\ 0, & b-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(b-\lambda) \cancel{-0} = 0$$

$$\lambda_1 = a$$

$$(M - \lambda I)\vec{\psi} = 0 \qquad \lambda_2 = b$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \text{for } \lambda = a \quad \begin{bmatrix} x \\ y \end{bmatrix} \quad y = 0,\ x = 1$$

$$\begin{bmatrix} a-\lambda, & 1 \\ 0, & b-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0, & 1 \\ 0, & b-a \end{bmatrix}\begin{bmatrix} y+0=0 \\ y(b-a)+0=0 \end{bmatrix}$$

$$\psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{for } \lambda = b \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\psi_2 = \begin{bmatrix} -1/(a-b) \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a-b & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y + x(a-b) = 0 \\ 0 = 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & -1/(a-b) \\ 0 & 1 \end{bmatrix} \qquad y = -x(a-b)$$

$$y = 1$$

$$1 = -x(a-b)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad c = 0 \quad b = \frac{1}{a-b} \qquad x = \frac{-1}{a-b},\ y = 1$$

$$d = 1 \quad a = 1$$

$$\begin{bmatrix} 1 & -1/a-b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \downarrow$$

$$S^{-1} = \begin{bmatrix} 1, & \frac{1}{a-b} \\ 0, & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$S\Lambda S^{-1} = M \quad \begin{bmatrix} a, & 0 \\ 0, & b \end{bmatrix} \begin{bmatrix} 1, & \frac{1}{a-b} \\ 0, & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1, & \frac{-1}{a-b} \\ 0, & 1 \end{bmatrix} \begin{bmatrix} a & -\frac{b}{a-b} \\ 0 & b \end{bmatrix} \begin{bmatrix} a, & \cancel{\frac{a}{a-b} - \frac{b}{a-b}}\ 1 \\ 0, & b \end{bmatrix} = \begin{bmatrix} a, & 1 \\ 0, & b \end{bmatrix} = M \checkmark$$

Λ is valid.

If $A = B$, then SDS^{-1} can't exist b/c dividing by 0. Therefore, it cannot be diagonalized.

10 $M = \begin{bmatrix} a, b \\ c, d \end{bmatrix}$, $\det(M - \lambda I) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$

$M^2 - (a+d)M + (ad-bc)I = 0$

$\rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

$\begin{bmatrix} a^2+bc-a^2-ad+ad-bc & ad+bd-ad-bd \\ ac+cd-ac-cd & bc+d^2-ad-d^2+ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$n \times n$ matrix w/ characteristic polynomial $\lambda^n + \sum_{k=1}^{n} c_k \lambda \lambda^{n-k}$ has n eigenvalues & can be diagonalized

Plugging this in gives $(SDS^{-1})^n + \sum_{k=1}^{n} c_k (SDS^{-1})^{n-k}$

$(SDS^{-1})^n = SD^nS^{-1}$ so $S(D^n + \sum_{k=1}^{n}(kD^{n-k}))S^{-1}$

$= S(0)S^{-1}$

11. $M = \begin{pmatrix} c, & a-b \\ a+b, & -c \end{pmatrix}$

$$\det\left(\begin{bmatrix} c, & a-b \\ a,+b, & -c \end{bmatrix} - \begin{bmatrix} \lambda, & 0 \\ 0, & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{vmatrix} c-\lambda, & a-b \\ a+b, & -c-\lambda \end{vmatrix} = 0$$

$$(c-\lambda)(-c-\lambda) - (a-b)(a+b) = 0$$

$$-(c-\lambda)(c+\lambda) - (a-b)(a+b) = 0$$

$$-(c^2-\lambda^2) - (a^2-b^2)$$

$$-c^2+\lambda^2-a^2+b^2$$

$$\lambda^2 - a^2+b^2-c^2 = 0$$

$$\lambda^2 = a^2-b^2+c^2$$

$$\boxed{\lambda = \pm\sqrt{a^2-b^2+c^2}}$$

λ is imaginary if $\boxed{b^2 > a^2+c^2}$

This result verrifies the fundamental theorem of algebra, an $n \times n$ matrix has n solutions for λ

12. for complex a, b, c

$$\lambda = \pm\sqrt{(a+bi)^2-(c+di)^2+(e+fi)^2}$$

$$\pm\sqrt{a^2+2abi-b^2-c^2-2cdi+d^2+e^2+2efi-f^2}$$

$$\pm\sqrt{\underbrace{2i(ab-cd+ef)+a^2-b^2-c^2+d^2+e^2-f^2}_{}}$$

If this is greater then 0, then it is real

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$A = m \times n$ $\det(I_m + AB) = \det(I_n + BA)$

$B = n \times m$

for A & B square & invertible

$A \Rightarrow n \times n$ $AB \Rightarrow n \times n$

$B \Rightarrow n \times n$ Prove:

$\det(I_n + AB) = \det(I_n + BA)$

$\parallel$

$II + ABI$

$AA^{-1}AA^{-1} + ABAA^{-1}$

Distributive property

$A(\cancel{A^{-1}A}^{I} + BA)A^{-1} = I + AB$

$A(I + BA)A^{-1} = I + AB$

$\det|A(I+BA)A^{-1}| = \det(I + BA)$

$= \det|I + AB|$

Proving Sylvester's theorem for square, invertible matrices

14 Show $(M^{-1})^T = (M^T)^{-1}$ & $(M^{-1})^* = (M^*)^{-1}$

$$M^{-1}M = I$$
$$(M^{-1}M)^T = I^T = I$$
$$M^T(M^{-1})^T = I = M^T M^{T^{-1}}$$
$$\cancel{M^T}(M^{-1})^T = \cancel{M^T}(M^T)^{-1}$$
$$\underline{(M^{-1})^T = (M^T)^{-1}}$$
$$\therefore \underline{(M^{-1})^T = (M^T)^{-1}}$$

$$M^{-1}M = I$$
$$(M^{-1}M)^* = I^*$$
$$(M^{-1})^* M^* = I$$
$$(M^{-1})^* M^* = I = (M^*)^{-1} M^*$$
$$(M^{-1})^* \cancel{M^*} = (M^*)^{-1} \cancel{M^*}$$
$$\underline{(M^{-1})^* = (M^*)^{-1}}$$

15. $M_{ij} = X_j^{i-1}$

$$\det|M| = \prod_{0 \le i < j \le n} (X_i - X_j)$$

$n=2$: $\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \end{bmatrix}$ $n=3$: $\begin{bmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{bmatrix}$

$\det|M_2| = X_2 - X_1 =$

$\det|M_3| = 1(X_2 X_3^2 - X_2^2 X_3) + X_1(X_3^2 - X_2^2) + X_1^2(X_2 X_3^2 - X_2^2 X_3)$

... This matrix represents a polynomial w/ roots are $X_1, X_2 \dots$

determinant goes $\Rightarrow 0$ if $X_n = X_m$, corresponding to matrix w/ 2 identical rows

16 Area $= \frac{1}{2} \det \begin{vmatrix} X_1, & X_2, & X_3 \\ Y_1, & Y_2, & Y_3 \\ 1 & 1 & 1 \end{vmatrix}$

Set 1 corner @ the origin $(X_1, Y_1 = 0)$

$\begin{vmatrix} 0 & X_2 & X_3 \\ 0 & Y_2 & X_3 \\ 1 & 1 & 1 \end{vmatrix}$ if we put one edge on the x-axis we get $X_2 = b$, $Y_3 = h$, $Y_2 = 0$

$\downarrow$

$\det \begin{vmatrix} 0 & b & X_3 \\ 0 & 0 & h \\ 1 & 1 & 1 \end{vmatrix} = \cancel{0(0-0)} + b(h-0) + \cancel{X_3(0-0)}$

$A = \boxed{\frac{1}{2} bh}$

2N 1.1 #3 : Show

$\mathbb{Z}_2 \otimes \mathbb{Z}_4 \ncong \mathbb{Z}_8$

$\mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_{nm}$ if $\gcd(m,n)=1$, or relatively prime (1)

$2 \cdot 4 = 8 \therefore \gcd(2,4) \neq 1$, and $\underline{\mathbb{Z}_2 \otimes \mathbb{Z}_4 \ncong \mathbb{Z}_8}$

Proof of (1)

if $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ & $\gcd = d > 1$,

$(a,b) \in \mathbb{Z}_m \times \mathbb{Z}_n$

$(a,b) + \ldots + (a,b) = \left(\frac{mn}{d}a, \frac{mn}{d}b\right)$

$= \left(m\underbrace{\frac{n}{d}a}, n\underbrace{\frac{m}{d}b}\right)$ whole #'s

$|(a,b)| < m,n$

for all (a,b) in $\mathbb{Z}_n \times \mathbb{Z}_m$

$m\frac{n}{d}a \bmod m = 0$, $\rightarrow = \underline{(0,0)}$

not cyclic b/c order is incorrect.

3. Show that $\mathbb{Z}_4 \otimes \mathbb{Z}_9 = \mathbb{Z}_{36}$

from above rule ($\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ if $\gcd(n,m)=1$),

we can see that the $\gcd(4,9)$ is 1,

they are relatively prime.

gcd

factors of 4 : 1, 2, 4

9 : 1, 3, 9

4 N 1.2 #7

Show that Q forms a group

Check 4 conditions

1. Closure ✓

w/ $1, -1, i, -i, j, -j, k, -k$

$i^2 = j^2 = k^2 = -1$ & $ij = -ji = k$ $\quad -1(1) = -1$

$jk = -kj = i \quad 1(1) = 1$ ✓

Closed! $\quad ki = -ik = j$

✓

2. Associativity

Multiplication is associative

3. Identity

Identity $= 1$, b/c binary operation is multiplication

4. The negative of any of the generalized imaginary numbers is the inverse, and the inverse of -1 is -1.

Satisfying all these requirements, Q is a group

5. N1.2 #12

Given elements f & g, show fg and gf are in the same equivalance class even though $fg \neq gf$

$$gf = f^{-1}(fg)f$$

&

$$fg = g^{-1}(gf)g$$

$\therefore$ There exists an element for both fg & gf that allows them to satisfy the equivalency relation.

6. Verify that $Z_2 \subset Z_4$, but $Z_2 \not\subset Z_5$

Using lagranges theorem, we know that any group w/ n elements has to have subgroups of size m, where $n = (\text{int})\, m$. Clearly, $2 \cdot 2 = 4$, and 2 is an integer #, but 2.5 is not an integer & $\therefore Z_2$ is not a subgroup of $\mathbb{Z}_5$