non.-rel: $\boxed{PV = \frac{2}{3}E}$

extrem-rel. $\boxed{PV = \frac{E}{3}}$ <u>independent</u> of statistics

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The classical limit $s=0$

$$\frac{\langle N\rangle}{V} = \frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \frac{1}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)} \pm 1} =$$

$\frac{\beta p^2}{2m} \equiv x^2$

$$= \frac{4\pi}{h^3}(2mkT)^{3/2}\int_0^\infty dx \frac{x^2}{e^{x^2-\mu/kT} \pm 1}$$

$p = \sqrt{2mkT}\, x$
$dp = \sqrt{2mkT}\, dx$

classical limit:

$$\lambda_T = \sqrt{\frac{h^2}{2\pi mkT}}$$

$$\boxed{\frac{\langle N\rangle}{V}\cdot\lambda^3 = \frac{4\pi}{\pi^{3/2}}\int_0^\infty dx \frac{x^2}{e^{x^2-\mu/kT} \pm 1}}$$

$$\frac{\lambda^3}{V/N} \ll 1$$

monotonic decreasing function

$$\frac{\langle N\rangle}{V}\lambda^3 = f\left(-\mu/kT\right) \ll 1$$

$$\boxed{-\mu/kT \gg 1}$$

$$\frac{1}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)}+1} \simeq e^{\beta\mu}\, e^{-\beta\frac{p^2}{2m}}$$

Quantum Corrections to classical ideal gas

$$\varepsilon(p) = \frac{p^2}{2m}$$

$$\Phi(T, V, \mu) = -kT \ln Z_G \qquad \Phi(T, V, \mu) = -PV$$

$$\Longrightarrow \quad PV = kT \ln Z_G \qquad \frac{PV}{kT} = \ln Z_G$$

$$\frac{PV}{kT} = \pm(2s+1)\frac{V}{h^3}\int_0^\infty 4\pi p^2\,dp\,\ln\left[1 \pm e^{-\beta\left(\frac{p^2}{2m}-\mu\right)}\right]$$

$$N = (2s+1)\frac{V}{h^3}\int_0^\infty 4\pi p^2\,dp\,\frac{1}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)} \pm 1}$$

$$E = (2s+1)\frac{V}{h^3}\int_0^\infty 4\pi p^2\,dp\,\frac{p^2/2m}{e^{\beta\left(\frac{p^2}{2m}-\mu\right)} \pm 1} \qquad \boxed{PV = \frac{2}{3}E}$$

$$x \equiv \frac{\beta p^2}{2m} = \frac{p^2}{2mkT} \qquad p = \sqrt{2mkT}\,x^{1/2} \qquad dp = \sqrt{2mkT}\,\frac{1}{2}x^{-1/2}$$

$$N = (2s+1)\frac{V}{h^3}\int_0^\infty 4\pi\,(2mkT)^{3/2}\,\frac{1}{2}\,\frac{x^{1/2}\,dx}{e^x e^{-\beta\mu} \pm 1} = (2s+1)\frac{V}{h^3}(2\pi)(2mkT)^{3/2} \times \int_0^\infty \frac{x^{1/2}\,dx}{z^{-1}e^x \pm 1}$$

$\boxed{z \equiv e^{+\beta\mu}}$ classical limit $-\frac{\mu}{kT} \gg 1$
$z \ll 1$

$$N = (2s+1)\,2\pi\frac{V}{h^3}(2mkT)^{3/2}\,\Gamma(3/2)\,f^{\pm}_{3/2}(z)$$

$$f^{\pm}_{\nu}(z) \equiv \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}\,dx}{z^{-1}e^x \pm 1}$$

$$\Gamma(1/2) = \sqrt{\pi} \qquad \Gamma(3/2) = \frac{\sqrt{\pi}}{2} \qquad \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$$

$$\boxed{N = (2s+1)\frac{V}{h^3}(2\pi mkT)^{3/2}\,f^{\pm}_{3/2}(z)}$$

$$E = (2s+1)\, 2\pi \frac{V}{h^3} (2mkT)^{3/2}\, kT \int_0^\infty \frac{x^{3/2}}{z^{-1}e^x \pm 1}\, dx$$

$$= (2s+1)\, 2\pi \frac{V}{h^3} (2mkT)^{3/2} kT\, \Gamma(5/2)\, f^{\pm}_{5/2}(z) =$$

$$= (2s+1) \frac{3}{2} \frac{V}{h^3} (2\pi m kT)^{3/2} kT\, f^{\pm}_{5/2}(z)$$

<u>In summary:</u>

$$f^{\pm}_{\nu}(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x \pm 1}\, dx$$

$$z = e^{\mu/kT}$$

$$\boxed{\begin{aligned} PV &= \frac{2}{3}E \\ N &= (2s+1)\frac{V}{\lambda^3} f^{\pm}_{3/2}(z) \\ E &= \frac{3}{2} kT\, (2s+1)\frac{V}{\lambda^3} f^{\pm}_{5/2}(z) \end{aligned}}$$

$$\boxed{\frac{E}{N} = \frac{3}{2} kT \frac{f^{\pm}_{5/2}(z)}{f^{\pm}_{3/2}(z)}}$$

$(z = e^{\mu/kT})$

Classical Limit: $\mu/kT \ll -1 \implies z \ll 1$

Need small-z expansion of $f^{\pm}_{\nu}(z)$

$$\boxed{f^{\pm}_{\nu}(z) = z \mp \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} \mp \ldots}$$

$$\frac{E}{N} = \frac{3}{2} kT \frac{f^{\pm}_{5/2}(z)}{f^{\pm}_{3/2}(z)} = \frac{3}{2} kT \frac{z \mp \frac{z^2}{2^{5/2}} + \ldots}{z \mp \frac{z^2}{2^{3/2}} + \ldots} \simeq$$

$$= \frac{3}{2} kT \left(1 \mp \frac{z}{2^{5/2}} + \ldots\right)\left(1 \pm \frac{z}{2^{3/2}} + \ldots\right) = \frac{3}{2} kT \left\{1 \pm \left(\frac{1}{2^{3/2}} - \frac{1}{2^{5/2}}\right) z + \ldots\right\}$$

$$= \frac{3}{2} kT \left\{1 \pm \frac{1}{2^{5/2}} z\right\}$$

$n = \frac{N}{V}$ $\quad \frac{\lambda^3 n}{2s+1} = f^{\pm}_{3/2}(z) = z \mp \frac{z^2}{2^{3/2}} + \dots$

$$u \equiv \frac{\lambda^3 n}{2s+1} \quad (\ll 1)$$

$$u = z \mp \frac{z^2}{2^{3/2}} + \dots$$

$$z = u \pm \frac{z^2}{2^{3/2}} + \dots = u \pm \frac{\left(u \pm z^2/2^{3/2}\right)^2}{2^{3/2}} + \dots = u \pm \frac{u^2}{2^{3/2}} + \dots$$

to lowest order: $z \simeq \frac{\lambda^3 N/V}{2s+1}$

$$\frac{E}{N} = \frac{3}{2}kT\left\{1 \pm \frac{\lambda^3 N/V}{2^{5/2}(2s+1)}\right\} \qquad \frac{\lambda^3 N/V}{2s+1} \ll 1$$

equation of state

$$\boxed{PV = NkT\left\{1 \pm \frac{\lambda^3 (N/V)}{2^{5/2}(2s+1)}\right\}} \qquad \lambda = \sqrt{\frac{h^2}{2\pi mkT}}$$

Specific Heat (maybe HW)

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{3}{2}Nk\left\{1 \pm \frac{\lambda^3(N/V)}{2^{5/2}(2s+1)}\right\} + \frac{3}{2}NkT \cdot \left(\mp \frac{3}{2}\frac{1}{T}\right)\frac{\lambda^3(N/V)}{2^{5/2}(2s+1)}$$

$$\boxed{= \frac{3}{2}Nk\left\{1 \mp \frac{1}{2^{7/2}}\frac{\lambda^3(N/V)}{(2s+1)} \dots\right\}}$$

Degenerate Quantum Gases

Degenerate Fermi gas - electron gas in metals

$s = 1/2$

$$\langle N \rangle \equiv N = 2 \frac{V}{h^3} \int_0^\infty \frac{4\pi p^2 \, dp}{e^{\beta(\varepsilon(p)-\mu)}+1} = \int_0^\infty g(\varepsilon) \langle n(\varepsilon) \rangle \, d\varepsilon$$

non-relativistic $\bar{e}$ gas: $\quad \varepsilon(p) = \frac{p^2}{2m}$

$$\langle n(\varepsilon) \rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

$$g(\varepsilon) \, d\varepsilon = 2 \frac{V}{h^3} 4\pi p^2 dp \qquad \varepsilon(p) = \frac{p^2}{2m}$$

$$p = \sqrt{2m} \, \varepsilon^{1/2}$$

$$g(\varepsilon) = 2 \frac{V}{h^3} 4\pi p^2 \frac{dp}{d\varepsilon} =$$

$$= \cancel{2} \frac{V}{h^3} 4\pi \, 2m \varepsilon \, (2m)^{1/2} \frac{1}{\cancel{2}} \varepsilon^{-1/2} = \frac{4\pi V}{h^3} (2m)^{3/2} \varepsilon^{1/2}$$

$$\boxed{g(\varepsilon) = \frac{4\pi V (2m)^{3/2}}{h^3} \varepsilon^{1/2}}$$

one-particle density of states
$\left(\begin{array}{c} d=3, \\ s=1/2 \end{array} \; \varepsilon(p) = p^2/2m \right)$

$$N = \int_0^\infty g(\varepsilon)\langle n(\varepsilon)\rangle\, d\varepsilon$$

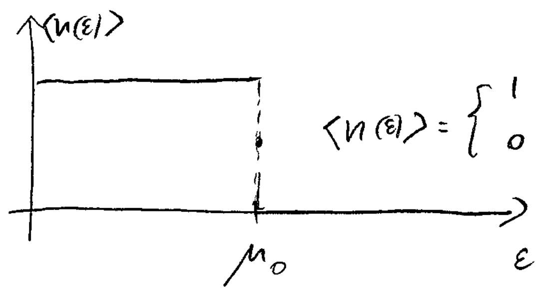
$$E = \int_0^\infty \varepsilon\, g(\varepsilon)\langle n(\varepsilon)\rangle$$

$$\langle n(\varepsilon)\rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

F-D. distribution

$T=0$

$$\beta = \frac{1}{kT}$$



$$\langle n(\varepsilon)\rangle = \begin{cases} 1 & \varepsilon < \mu_0 \\ 0 & \varepsilon > \mu_0 \end{cases}$$

"controll" param: $\dfrac{\lambda^3 N}{V}$

$$\lambda = \left(\frac{h^2}{2\pi m kT}\right)^{1/2}$$

$\mu_0 = \mu(T=0) = \varepsilon_F$ Fermi energy

$\varepsilon_F = \dfrac{p_F^2}{2m}$

$kT_F \equiv \varepsilon_F$ charact. mom/temp

$$T=0:\quad N = \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon = \int_0^{\varepsilon_F} \frac{4\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}\, d\varepsilon = \frac{4\pi V}{h^3}(2m)^{3/2}\int_0^{\varepsilon_F}\varepsilon^{1/2}\, d\varepsilon$$

$$= \frac{4\pi V}{h^3}(2m)^{3/2}\frac{2}{3}\varepsilon_F^{3/2} = \frac{8\pi V}{3h^3}(2m)^{3/2}\varepsilon_F^{3/2}$$

$$\varepsilon_F = \left(\frac{3h^3 N}{8\pi V}\right)^{2/3}\frac{1}{2m} = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{h^2}{2m}\left(\frac{3n_e}{8\pi}\right)^{2/3}$$

$n_e = \dfrac{N}{V}$ e^- density $\qquad p_F = \left(\dfrac{3n_e}{8\pi}\right)^{1/3} h$

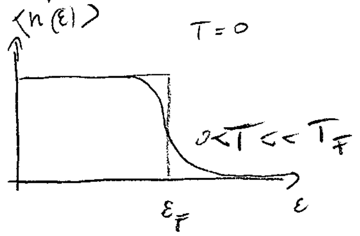
$$T=0:\quad E_0 = \int_0^\infty \varepsilon\, g(\varepsilon)\langle n(\varepsilon)\rangle\, d\varepsilon = \int_0^{\varepsilon_F}\varepsilon\, g(\varepsilon)\, d\varepsilon = \frac{2}{5}\frac{4\pi V}{h^3}(2m)^{3/2}\varepsilon_F^{5/2} =$$

$$\boxed{E_0 = \frac{3}{5}N\varepsilon_F}$$

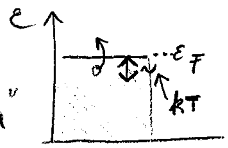
ground-state pressure $PV = \frac{2}{3}E \Rightarrow P_0 = \frac{2}{5}n_e\varepsilon_F \propto n_e^{5/3}$

Asymptotic Expansions for low temperatures
(the Sommerfeld expansion)

Simple considerations $T \ll T_F = \frac{\varepsilon_F}{k}$



$(\varepsilon-\mu)\sim kT$ width around μ.



only fraction $\sim \frac{kT}{\varepsilon_F}$ is excited $\quad N' \sim N\left(\frac{kT}{\varepsilon_F}\right)$

$$E \sim \text{const.}\, N' kT = NkT\left(\frac{kT}{\varepsilon_F}\right) \sim T^2$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} \sim T$$

} actually yield correct scaling with T, as we shall see in systematic expansion

---

$$N = \int_0^\infty g(\varepsilon)\langle n(\varepsilon)\rangle\, d\varepsilon \qquad \langle n(\varepsilon)\rangle = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1}$$

$$E = \int_0^\infty g(\varepsilon)\langle n(\varepsilon)\rangle\, \varepsilon\, d\varepsilon$$

Sommerfeld expansion: for any function $f(\varepsilon)$ $\quad T \ll T_F$

$$\int_0^\infty f(\varepsilon)\langle n(\varepsilon)\rangle d\varepsilon = \int_0^\mu f(\varepsilon)\,d\varepsilon + (kT)^2\frac{\pi^2}{6}\left.\frac{df}{d\varepsilon}\right|_{\varepsilon=\mu} + (kT)^4\frac{7\pi^4}{360}\left.\frac{d^3f}{d\varepsilon^3}\right|_{\varepsilon=\mu} + \dots$$

$$N = \int_0^\infty g(\varepsilon)\langle n(\varepsilon)\rangle d\varepsilon \simeq \int_0^\mu g(\varepsilon)\,d\varepsilon + (kT)^2\frac{\pi^2}{6}g'(\mu) + o(T^4)$$

$\mu \neq \varepsilon_F$ on the other hand:

$$N = \int_0^{\varepsilon_F} g(\varepsilon)\,d\varepsilon \qquad \text{and} \qquad \underline{N = \text{fixed}}$$

$T=0$: $\mu = \varepsilon_F$ $\qquad 0 < T << T_F \qquad \boxed{\mu \lesssim \varepsilon_F}$

$$\cancel{N} \simeq \int_0^\mu g(\varepsilon)\,d\varepsilon + (kT)^2\frac{\pi^2}{6}g'(\mu) \simeq \int_0^{\varepsilon_F} g(\varepsilon)\,d\varepsilon + \int_{\varepsilon_F}^{\mu} g(\varepsilon)\,d\varepsilon + (kT)^2\frac{\pi^2}{6}g'$$

$$\simeq \cancel{N} + (\mu - \varepsilon_F)\,g(\varepsilon_F) + (kT)^2\frac{\pi^2}{6}g'(\varepsilon_F)$$

$$\boxed{\mu \simeq \varepsilon_F - (kT)^2\frac{\pi^2}{6}\frac{g'(\varepsilon_F)}{g(\varepsilon_F)}} \qquad g(\varepsilon) \propto \varepsilon^{1/2} \qquad \frac{g'(\varepsilon_F)}{g(\varepsilon_F)} = \frac{1}{2\varepsilon_F}$$

$$\boxed{\mu \simeq \varepsilon_F - \frac{\pi^2}{12}\frac{(kT)^2}{\varepsilon_F}} = \varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right) = \varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right)$$

$$E = \int_0^\infty \varepsilon g(\varepsilon)\langle n(\varepsilon)\rangle d\varepsilon \cong \int_0^\mu \varepsilon g(\varepsilon)\,d\varepsilon + (kT)^2\frac{\pi^2}{6}\left[\varepsilon g(\varepsilon)\right]'_{\varepsilon=\mu}$$

$$\simeq \frac{2}{5}\frac{4\pi V}{h^3}(2m)^{3/2}\mu^{5/2} + (kT)^2\frac{\pi^2}{6}\frac{4\pi V}{h^3}(2m)^{3/2}\frac{3}{2}\mu^{1/2} =$$

$$\boxed{= \frac{2}{5}\frac{4\pi V}{h^3}(2m)^{3/2}\mu^{5/2}\left\{1 + \frac{5}{8}\pi^2\left(\frac{kT}{\mu}\right)^2\right\} = E(T,V,\mu)} \quad (*)$$

$$N = \int_0^\infty g(\varepsilon)\langle n(\varepsilon)\rangle \, d\varepsilon \simeq \int_0^\mu g(\varepsilon)\, d\varepsilon + (kT)^2 \frac{\pi^2}{6} g'(\mu)$$

$$= \frac{2}{3}\frac{4\pi V}{h^3}(2m)^{3/2}\mu^{3/2} + (kT)^2\frac{\pi^2}{6}\,\frac{1}{2}\,\frac{4\pi V}{h^3}(2m)^{3/2}\mu^{-1/2}$$

$$= \frac{2}{3}\frac{4\pi V}{h^3}(2m)^{3/2}\mu^{3/2}\left\{1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right\} = N(T,\mu,V)$$

$$\frac{E}{N} \simeq \frac{3}{5}\mu\,\frac{1+\frac{5}{8}\pi^2\left(\frac{kT}{\mu}\right)^2}{1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2} \qquad \frac{kT}{\mu} \ll 1$$

$$\approx \frac{3}{5}\mu\left(1+\frac{5}{8}\pi^2\left(\frac{kT}{\mu}\right)^2\right)\left(1-\frac{1}{8}\pi^2\left(\frac{kT}{\mu}\right)^2\right)$$

$$\approx \frac{3}{5}\mu\left\{1+\frac{1}{2}\pi^2\left(\frac{kT}{\mu}\right)^2\right\}$$

and using

$$\mu \simeq \varepsilon_F\left\{1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right\}$$

$$\frac{E}{N} \simeq \frac{3}{5}\varepsilon_F\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)\left(1+\frac{1}{2}\pi^2\left(\frac{kT}{\varepsilon_F}\right)\right)$$

$$\boxed{\simeq \frac{3}{5}\varepsilon_F\left(1+\frac{5}{12}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2\right)} \qquad \boxed{C_V = \frac{\pi^2}{2} N k \frac{kT}{\varepsilon_F}}$$

using $PV = \frac{2}{3}E$:

$$PV = \frac{2}{5} N \varepsilon_F\left\{1+\frac{5}{12}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2\right\} \qquad \text{equation of state.}$$