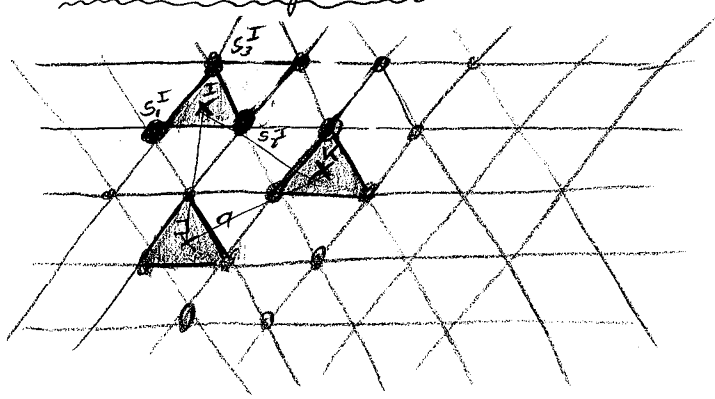
RG for the 2-d Ising Model with no external field

$$\hat{\mathcal{H}} = K \sum_{\langle ij \rangle} s_i s_j$$

$$Z_N(K) = \sum_{\{s_i\}} e^{K \sum_{\langle ij \rangle} s_i s_j}$$

Block - Spin construction / Coarse - graining

d=2 triangular lattice



$$a' = \frac{1}{3} \cdot \frac{\sqrt{3}}{2} a + \frac{\sqrt{3}}{2} a + \frac{2}{3} \frac{\sqrt{3}}{2} a$$
$$= \sqrt{3} a$$
$$\Downarrow$$
$$\boxed{l = \sqrt{3}}$$

$$S_I = \text{sign}(s_1^I + s_2^I + s_3^I)$$ "majority" rule

- lattice remains the same $(a \to \sqrt{3} a)$
- range of variables remains the same

$$\boxed{\{s_i^I\} = \{s_1^I, s_2^I, s_3^I\}}$$ within the I'th plaquette

e.g. $S_I = +1$ $\quad \{s_i^I\} =$

s_1^I	s_2^I	s_3^I
+	+	+
+	+	−
+	−	+
−	+	+

$S_I = -1$ $\quad \{s_i^I\} =$

−	−	−
+	−	−
−	+	−
−	−	+

8 configs in total

$$e^{\hat{\mathcal{H}}'[K',\{S_I\}]} = \sum_{\substack{\{S_i\} \\ \mathrm{sign}(S_1^I+S_2^I+S_3^I)=S_I}}' e^{\hat{\mathcal{H}}[K,\{S_i\}]}$$

- split interaction "within" and "between" blocks

$$\hat{\mathcal{H}} = K\sum_{\langle ij\rangle} S_iS_j = \overbrace{K\sum_{I}\sum_{(ij)\in I} S_iS_j}^{\mathcal{H}_0} + \overbrace{K\sum_{I\neq J}\sum_{\substack{i\in I\\ j\in J}} S_iS_j}^{V}$$

$$\boxed{\sum_{\{S_i\}}' e^{\hat{\mathcal{H}}[K,\{S_i\}]} = \sum_{\{S_i\}}' e^{\mathcal{H}_0+V} =}$$

$\mathcal{H}[K, S_i^I]$

$A(S_i^I)$

$$= \sum_{\{S_i\}}' e^{\mathcal{H}_0} e^{V} = \sum_{\{S_i\}}'\left\{ e^{\mathcal{H}_0} \sum_{n=0}^{\infty} \frac{V^n}{n!}\right\}$$

$$\langle A(S_i)\rangle_0 \equiv \frac{\sum_{\{S_i\}}' e^{\mathcal{H}_0(} A(S_I,\{S_i^I\})}{\sum_{\{S_i\}}' e^{\mathcal{H}_0(S_I,\{S_i^I\})}} = \frac{\sum_{\{S_i\}}' e^{\mathcal{H}_0} A(S_I,\{S_i^I\})}{[Z_0(K,S_I)]^M}$$

do not contain "inter" block interaction

M is the total number of block in the system

$$e^{\hat{\mathcal{H}}'[K',\{S_I\}]} = [Z_0(K,S_I)]^M \underbrace{\sum_{n=0}^{\infty} \frac{\langle V^n(S_I,\{S_i^I\})\rangle_0}{n!}}_{\langle e^V\rangle_0}$$

$$\hat{\mathcal{H}}[K',\{S_I\}] = M\log Z_0(K,S_I) + \log\sum_{n=0}^{\infty}\frac{\langle V^n(S_I,\{S_i^I\})\rangle_0}{n!}$$

$x=V$ (Cumulant expansion:) $\log\left[1+\langle x\rangle+\frac{\langle x^2\rangle}{2!}+o(\langle x^3\rangle)\right] = \ldots$

$$\langle x\rangle + \frac{\langle x^2\rangle}{2!} + o\langle x^3\rangle - \frac{1}{2}\left(\langle x\rangle + \frac{\langle x^2\rangle}{2!} + o\langle x^3\rangle\right)^2 = \langle x\rangle + \frac{1}{2}\left(\langle x^2\rangle - \langle x\rangle^2\right)$$
$$+\,o\langle x^3\rangle$$

$$\boxed{\hat{\mathcal{H}}'[K', S_I] = M \log Z_0(K, S_I) + \langle V \rangle_0 + \frac{1}{2}\left(\langle V^2\rangle_0 - \langle V\rangle_0^2\right)}$$

$$+\,\sigma(V^3)$$

$$e^{\hat{\mathcal{H}}[K', S_I]} = e^{M \log Z_0(K, S_I) + \langle V \rangle_0 + \frac{1}{2}\left(\langle V^2\rangle_0 - \langle V\rangle_0^2\right) + \dots}$$

(i) $Z_0(K, S_I)$: (within a block) 

$S_I = +1$

$$Z_0(K, S_I = +1) = \sum_{\{S_i^I\}}{}' e^{K(S_1^I S_2^I + S_2^I S_3^I + S_3^I S_1^I)}$$

$$\left(\sum_{i \in I} S_i^I\right) = +1$$

+ + +
+ + −
+ − +
− + +

$$= e^{3K} + 3e^{-K}$$

− − −
− − +
− + −
+ − −

$S_I = -1$ $\quad Z_0(K, S_I = -1) = e^{3K} + 3e^{-K}$

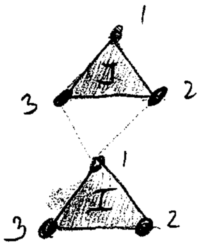
independent of S_I!

$$\boxed{Z_0(K) = e^{3K} + 3e^{-K}}$$

$$\hat{\mathcal{H}}[K', S_I] = M \log Z_0(K) + \langle V(S_I, \{S_i^I\})\rangle_0$$

(ii) $$\langle V(S_I,\{S_i\})\rangle_0 = \left\langle K\sum_{I\neq J}\sum_{\substack{i\in I\\ j\in J}} S_iS_j\right\rangle = \left\langle K\sum_{I\neq J}\sum_{ij} S_i^I S_j^J\right\rangle_0$$

couples spins in nearest-neighbor blocks



$$= K\sum_{\langle I,J\rangle}\underbrace{\left\langle\sum_{ij} S_i^I S_j^J\right\rangle_0}_{V_{IJ}}$$

$$\langle V_{IJ}\rangle = \left\langle\sum_{ij} S_i^I S_j^J\right\rangle_0 = \langle S_1^I S_2^J + S_1^I S_3^J\rangle_0 \overset{\text{(by symmetry)}}{=} 2\langle S_1^I S_2^J\rangle_0$$

however $\mathcal{H}_0$ separates contributions from different blocks, thus ($e^{\mathcal{H}_0} = \prod_I e^{K\sum_{ij\in I} S_iS_j}$, factorizes for different blocks)

$$\langle V_{IJ}\rangle_0 = 2\langle S_1^I\rangle_0\langle S_2^J\rangle_0$$

$$\langle S_1^I\rangle_0 = \frac{\sum'_{\{S_i\}} e^{\mathcal{H}_0(K,S_I,\{S_i^I\})}S_1}{\sum'_{\{S_i\}} e^{\mathcal{H}_0}} = \frac{\sum'_{\{S_i^I\}} e^{K(S_1^IS_2^I + S_2^IS_3^I + S_3^IS_1^I)}S_1^I}{Z_0(K)}$$

$$S_I = +1 \;:\; \frac{e^{3K} + 2e^{-K} - e^{-K}}{e^{3K}+3e^{-K}} = \frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}$$

$$S_I = -1 \;:\; \frac{-e^{3K} - 2e^{-K} + e^{-K}}{e^{3K}+3e^{-K}} = -\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}$$

$$\langle S_1^I\rangle_0 = \frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\,S_I$$

$$K\langle V_{IJ}\rangle_0 = 2K\langle S_1^I\rangle_0\langle S_2^J\rangle_0 = 2K\left[\frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\right]^2 S_IS_J$$

$$\hat{\mathcal{H}}' = M \log Z_0(K) + K' \sum_{\langle I,J \rangle} S_I S_J + \dots$$

where $$K' = 2K \left[\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right]^2 = 2K\,\phi^2(K)$$

$$\phi(K) = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} = \frac{e^{4K} + 1}{e^{4K} + 3}$$

$$\boxed{K' = 2K\,\phi^2(K)}$$ ← RG tr

<u>Fixed Points of the RG. transformation</u>

$$K^* = 2K^* \phi^2(K^*)$$ K^* is the fixed point

$K = \frac{J}{kT}$

(i) $K^* = 0$

(ii) $K^* = \infty$

(iii) $\phi(K^*) = 1/\sqrt{2}$

$$\boxed{K^* = \frac{1}{4}\log\left(1 + 2\sqrt{2}\right) \simeq 0.34}$$

note: exact Onsager: 0.27 (on triangular lattice)

meanfield: $K_c q = 1$ $K_c = 1/6 = 0.17$

$(q = 6)$

<u>Stability</u>

$$K' = K^* + \delta K' = K'(K^* + \delta K) = \underbrace{K'(K^*)}_{K^*} + \left.\frac{\partial K'}{\partial K}\right|_{K=K^*} \delta K$$

$$\delta K' = \left.\frac{\partial K'}{\partial K}\right|_{K=K^*} \delta K = \Lambda_t\, \delta K$$

$$K - K^* = \frac{J}{k}\left(\frac{1}{T} - \frac{1}{T^*}\right) = \frac{J}{kT^*}\,\frac{T^* - T}{T}$$

$$\approx \frac{J}{kT^*}(-t) \qquad t = \frac{T - T^*}{T^*}$$

$$\left.\frac{\partial K'}{\partial K}\right|_{K=K^*} = \Lambda_t \simeq 1.62 : (>1)$$

critical point ($\xi = $ "∞")

$\ell = \sqrt{3}$ RG: $$\boxed{\Lambda_t = \ell^{y_t}}$$

$$y_t = \frac{\log \Lambda_t}{\log \ell} \simeq 0.88 > 0 \qquad \nu \simeq 1/y_t = 1.132 > 0$$

$\nu_{exact} = 1$, $\nu_{MF} = 1/2$

calculations for RG fixed points

(A) $\phi^2(K^*) = 1/2 \qquad \phi(K^*) = 1/\sqrt{2}$

$$\boxed{\phi(K) = \frac{e^{4K}+1}{e^{4K}+3}}$$

$$\sqrt{2}\,(e^{4K}+1) = (e^{4K}+3)$$

$$(\sqrt{2}-1)\,e^{4K} = 3-\sqrt{2} \quad \Rightarrow \quad e^{4K} = \frac{3-\sqrt{2}}{\sqrt{2}-1} = \frac{(3-\sqrt{2})(\sqrt{2}+1)}{1} = 1+2\sqrt{2}$$

$$K^* = \frac{1}{4}\log(1+2\sqrt{2}) \simeq 0.34 \qquad \text{or} \qquad \boxed{e^{4K^*} = 1+2\sqrt{2}}$$

$\simeq 3.8284$

(B)

$$K' = 2K\phi^2(K)$$

$$\frac{\partial K'}{\partial K} = 2\phi^2(K) + 4K\phi(K)\phi'(K) = 2\phi^2(K) + 4K\phi(K)\frac{(e^{4K}+3)-(e^{4K}+1)}{(e^{4K}+3)^2}\,4e^{4K}$$

$$= 2\phi^2(K) + 4K\phi(K)\frac{8e^{4K}}{(e^{4K}+3)^2} = 2\phi^2(K) + 2K\phi^2(K)\frac{2}{\phi(K)}\frac{8e^{4K}}{(e^{4K}+3)^2}$$

$$\left.\frac{\partial K'}{\partial K}\right|_{K=K^*} = 1 + K^*16\frac{e^{4K}}{1+e^{4K}}\cdot\frac{1}{3+e^{4K}} \simeq 1.62 \qquad \Rightarrow \Lambda_t = 1.62$$

$$\delta K' = \Lambda_t\,\delta K$$

$$\boxed{\Lambda_t = e^{y_t}}$$

Table 3.1 CRITICAL EXPONENTS FOR THE ISING UNIVERSALITY CLASS

Exponent	Mean Field	Experiment	Ising ($d = 2$)	Ising ($d = 3$)
α	0 (disc.)	0.110 – 0.116	0 (log)	0.110(5)
β	1/2	0.316 – 0.327	1/8	0.325±0.0015
γ	1	1.23 – 1.25	7/4	1.2405±0.0015
δ	3	4.6 – 4.9	15	4.82(4)
ν	1/2	0.625±0.010	1	0.630(2)
η	0	0.016 – 0.06	1/4	0.032±0.003

3.7.3 How Good is Mean Field Theory?

Table 3.1 compares critical exponents calculated in mean field theory with those measured in experiment or deduced from theory for the Ising model in two and three dimensions. The table is essentially illustrative: the values given are not necessarily the most accurate known at the time of writing. In addition the experimental values are just given approximately, with a range reflecting inevitable experimental uncertainty. The values for ν and δ are not independent from the other values, obtained using **scaling laws**. The experimental values quoted are actually obtained from experiments on fluid systems.[3] Our discussion of the lattice gas model implies that these fluid systems should be in the universality class of the Ising model. Indeed, this expectation is borne out by the comparison of the experimental values and those from the three dimensional Ising model. The latter values are in some cases given with the number in brackets representing the uncertainty in the last digit quoted.[4]

The numerical values of the critical exponents calculated from mean field theory are in reasonable agreement with those given by experiment and the Ising model in three dimensions, although there are clearly systematic differences. First of all, the mean field theory exponents here do not depend on dimension, whereas it is clear that the exact critical exponents do. It is possible for mean field theory to exhibit exponents with a value dependent upon dimension. An example is the mean field theory

[3] J.V. Sengers in *Phase Transitions*, Proceedings of the Cargèse Summer School 1980 (Plenum, New York, 1982).

[4] J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev. B* **21**, 3976 (1980); numerical values and details of the calculational techniques used to obtain these estimates are given by J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989), Chapter 25.