Class 2 (1/11/24)

Electrostatic Fields

Outline

- Maxwell’s equation in differential form
- Maxwell’s equation in integral form
- Some vector calculus: Divergence & Stokes (Curl) theorem
- Fundamental definitions of electrostatic fields
- Generalized description of the electric field of point charges and extended electric charge distribution
- Some useful math which we will use frequently throughout the course

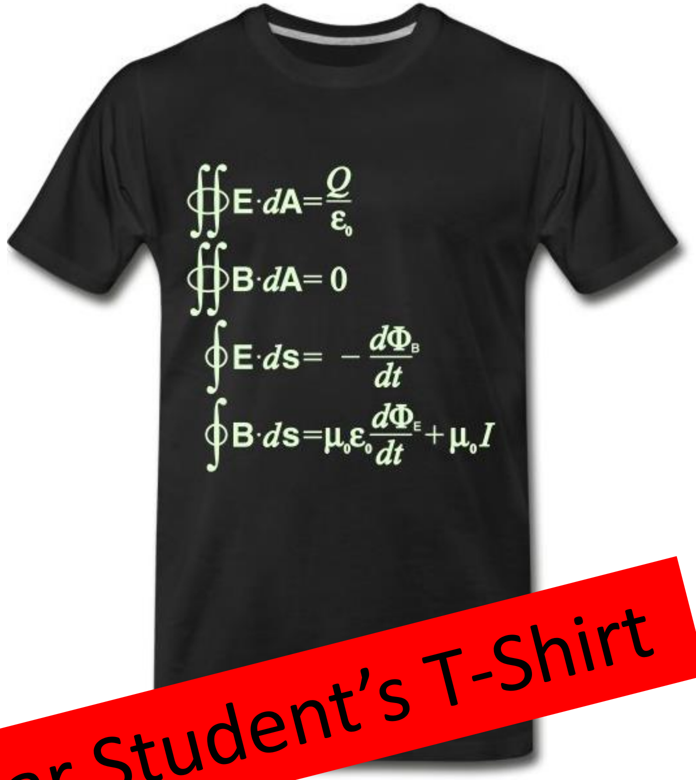
From Introductory Physics: Maxwell's equation in integral form

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}} \quad \text{(Gauss' law)}$$

$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0 \quad \text{(Gauss' law for magnetic fields)}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law of induction)}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{\text{enc}} \quad \text{(Ampere–Maxwell law)}$$



First Year Student's T-Shirt

The learning outcome of electromagnetics theory is for students to be able to analyze electrical and magnetic phenomena in nature or engineering using Maxwell's equation in differential form.

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Vector Calculus : Divergence & Stokes (Curl) Theorem

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

A good approach to working with the theorems is simply to internalize those relationship and apply them if it is appropriate. Mathematical Sciences has proven that the theorems are correct.

Deriving Maxwell’s equation in differential form from Maxwell’s equation in integral form using the Divergence Theorem:

$$\oint \vec{E}\cdot d\vec{A} = \int (\vec{\nabla}\cdot\vec{E})\cdot dV = \frac{Q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho \, dV$$

$$\int \left(\vec{\nabla}\cdot\vec{E} - \frac{1}{\varepsilon_0}\rho\right) dV = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint \vec{B}\cdot d\vec{A} = \int (\vec{\nabla}\cdot\vec{B})\, dV = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

Physical meaning of Gauss law for static electric fields: Electric charges are sources of static electric field ! If there is an electric charge then there will be a static electric field. If we measure a static electric field somewhere in space, then there are somewhere electric charges which give rise to this field.

Physical meaning of Gauss law for magnetic fields: Magnetic monopoles where never observed existing in nature, or has it been possible to engineer them !

Deriving Maxwell's equation in differential form from Maxwell's equation in integral form using the Curl Theorem:

$$\oint \vec{E}\cdot d\vec{s} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

$$\int \left(\vec{\nabla}\times\vec{E} + \frac{d\vec{B}}{dt}\right) d\vec{A} = 0 \quad \Rightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\oint \vec{B}\cdot d\vec{s} = \int (\vec{\nabla}\times\vec{B})\cdot d\vec{A} = \mu_0 \int \vec{J}\cdot d\vec{A} + \mu_0\varepsilon_0 \frac{d}{dt}\int \vec{E}\cdot d\vec{A}$$

$$\int \left(\vec{\nabla}\times\vec{B} - \mu_0\vec{J} - \mu_0\varepsilon_0\frac{d\vec{E}}{dt}\right)\cdot d\vec{A} = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}$$

Breaking the physics down:

- Electrodynamics

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- Electrostatics & Magnetostatics

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} = 0$$

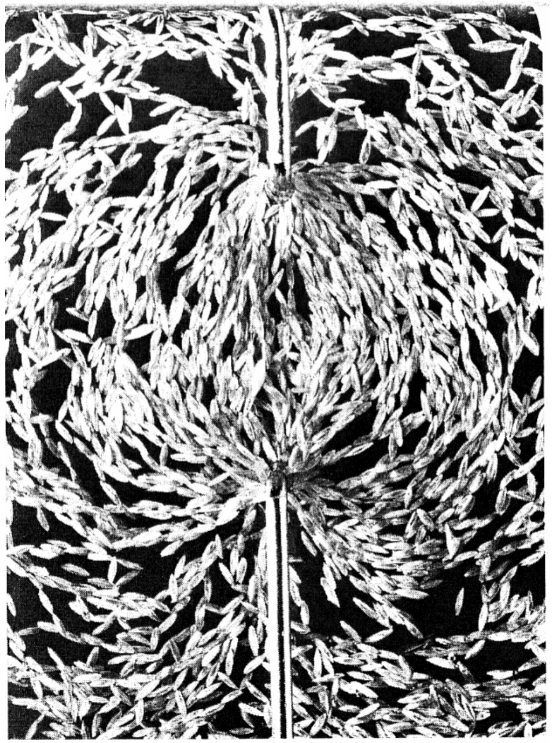
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Breaking it down further: Definition of Electrostatic fields

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} = 0$$

Gauss law and Faraday's law for static electric fields in differential form are literally the mathematical description of electric field lines as we observe them in nature and as we visualize them using certain experimental techniques as illustrated on the next slides.



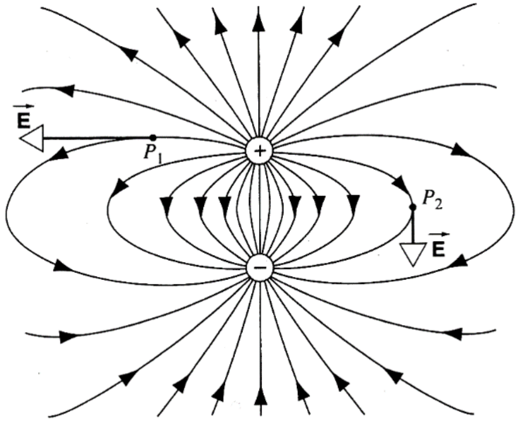
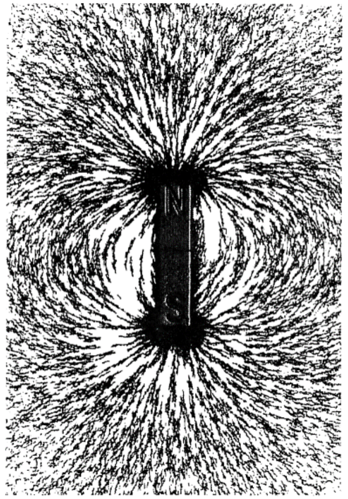


FIGURE 26-12. Electric field lines for an electric dipole.



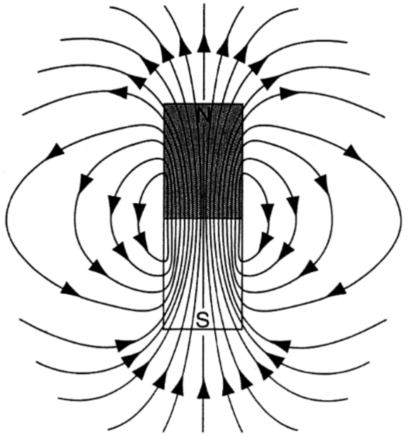
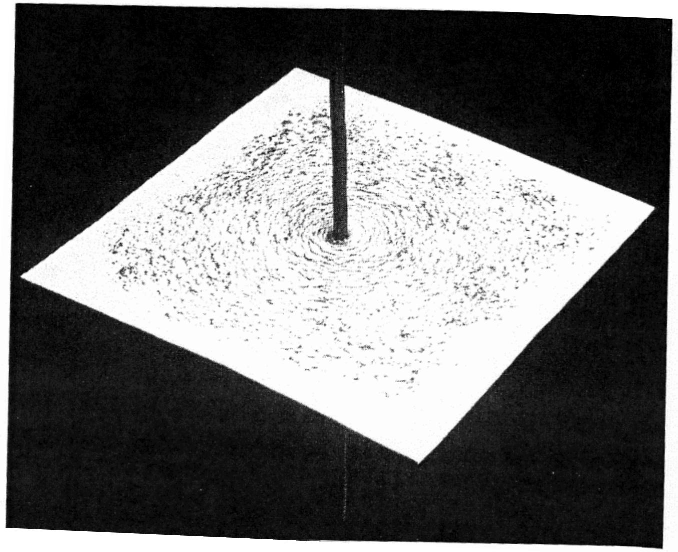


FIGURE 32-6. (*a*) The magnetic field lines for a bar magnet. The lines form closed loops, leaving the magnet at its north pole and entering at its south pole. (*b*) The field lines can be made visible by sprinkling iron filings on a sheet of paper covering a bar magnet.



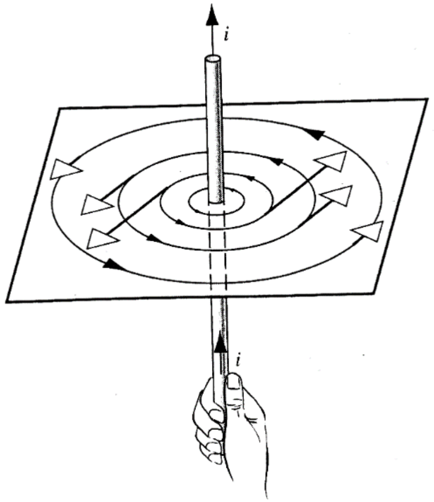
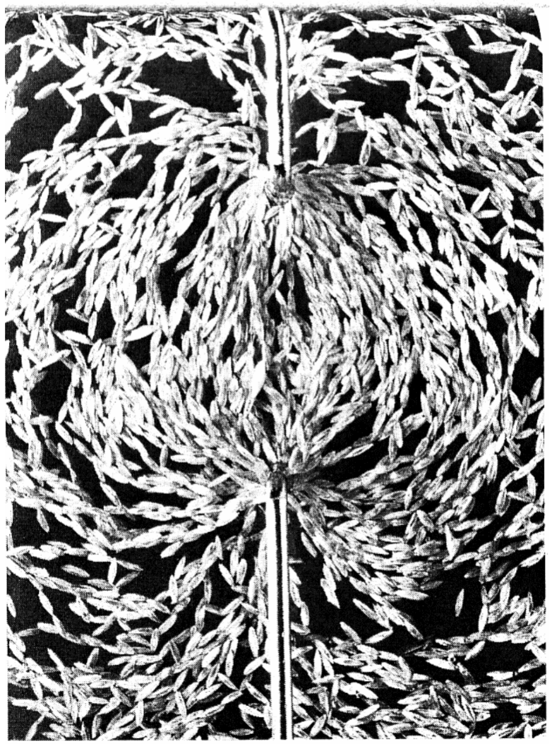
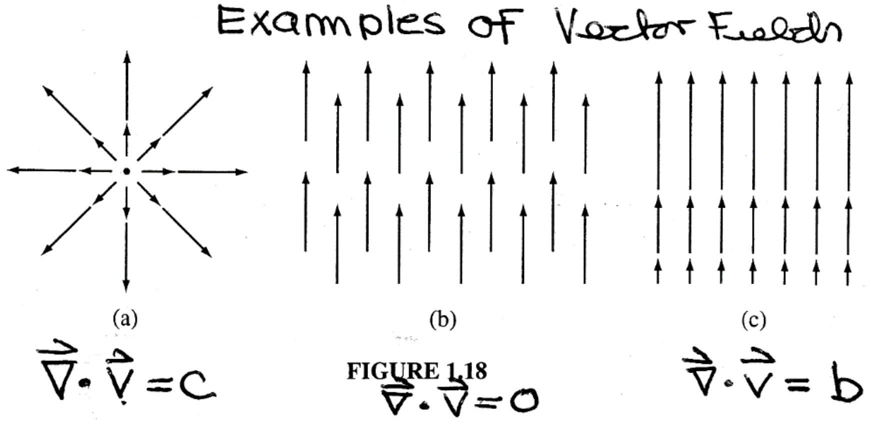


FIGURE 33-8. The lines of the magnetic field are concentric circles for a long, straight, current-carrying wire. Their direction is given by the right-hand rule.

Source: Halliday, Resnick, Krane: Physics Vol.2, 5th edition, John Wile & Sons, Inc.



Visualization of the electric fields of an electric dipole: A positive and a negative electric charge separated by a fixed distance.



Source: Griffiths: Introduction To Electrodynamics 4th Edition, Cambridge University Press

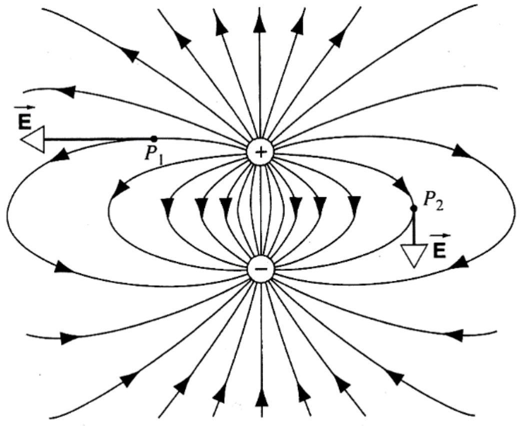
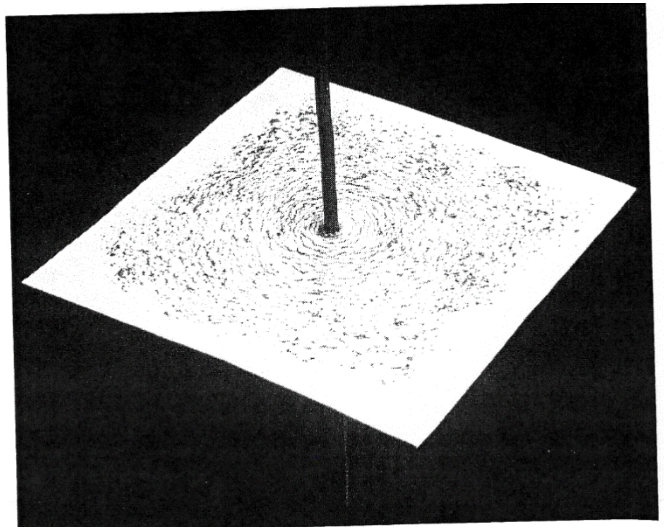
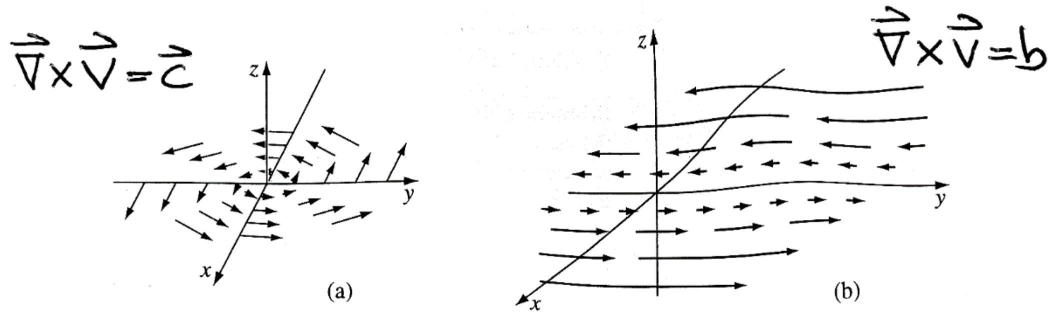


FIGURE 26-12. Electric field lines for an electric dipole.

Source: Halliday, Resnick, Krane: Physics Vol.2, 5th edition, John Wile & Sons, Inc.



Visualization of the magnetic field lines around a conducting wire carrying an electric current.



Source: Halliday, Resnick, Krane: Physics Vol.2, 5th edition, John Wile & Sons, Inc.

Source: Griffiths: Introduction To Electrodynamics 4th Edition, Cambridge University Press

Some scientific facts and thoughts on electric charge, force, and field:

Electrical charges can be positive or negative.

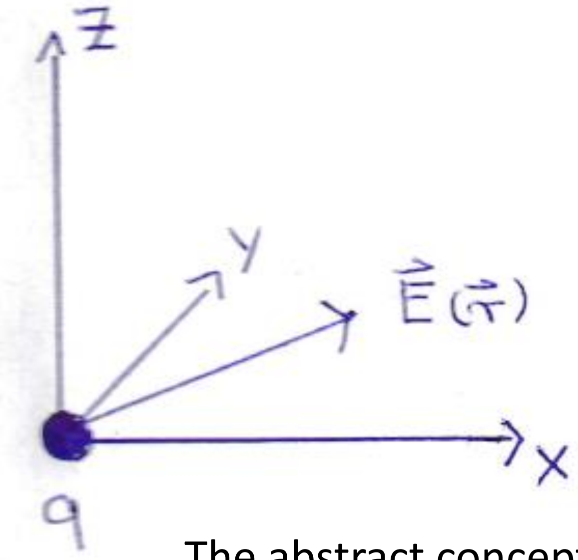
The force between electrical charges is the Coulomb force. It's a fundamental force of nature.

The Coulomb force is a force which acts over distance. It is not required that the two electrical charges touch for the force acting. This particular property of the Coulomb force: Acting over distance allows the introduction of the wonderful concept of electric field. The advantage of using electric fields is that the electric field is just the property of 1 charge (this simplifies calculations). The Coulomb force is determined by the property of two charges.

If multiple charges are present then the net electric field E_{net} is obtained from the superposition principle. The electric fields of individual charges add up at vectors: $\mathbf{E}_{net}=\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_3+\ldots$.

Electric field **E** is a property of the space around an electric charge. If a charge Q is placed somewhere in this space it experiences the electric force $\mathbf{F}=Q\mathbf{E}$.

Electric Field of a point charge at the origin of a coordinate system (special case)

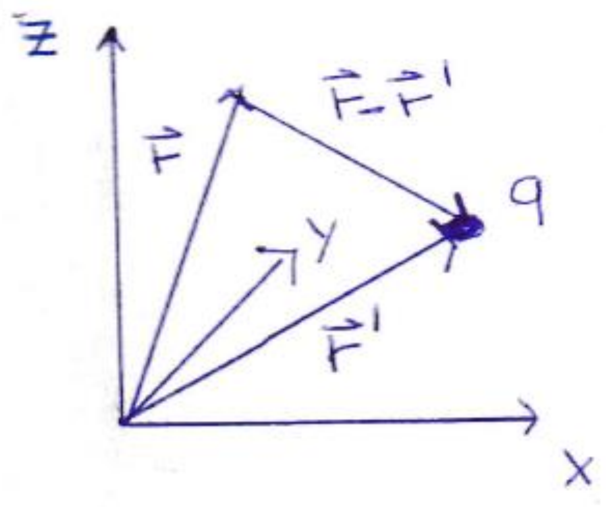


$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

with unit vector $\hat{r} = \frac{\vec{r}}{r}$

The abstract concept of a point charge assumes that the volume over which the charge spreads is really, really small compared to all other distances, e.g. **r,** involved in the situation. Therefore, we can safely assume that the charge is concentrated in a very, very tiny point.

Description of the electric field of a point charge **<u>not at</u>** the origin of a coordinate system (general case):



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}'|^2}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

Extended electric charge distribution

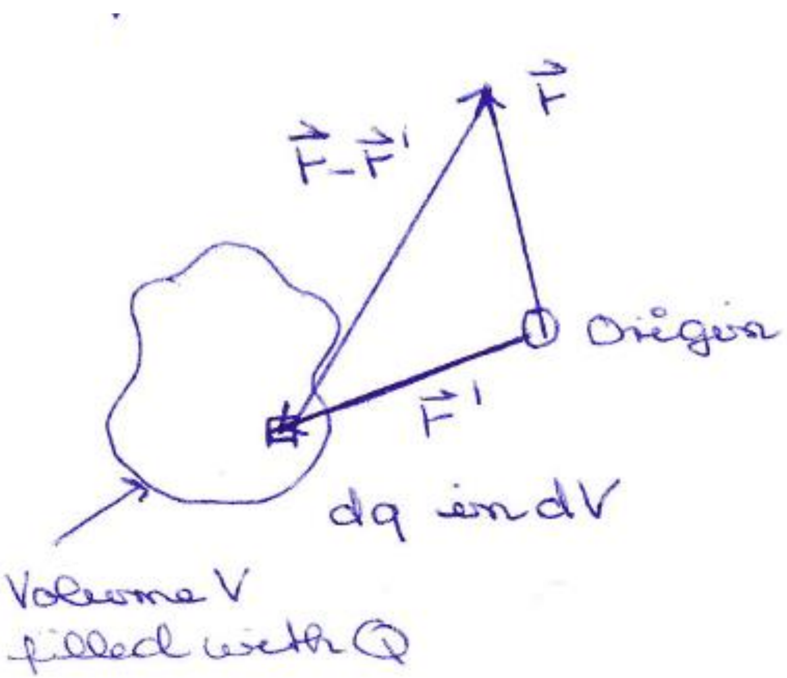
An extended charge distribution is electrical charge Q which is spread out over a finite volume V or over a surface area A or along the OR along a line of length L.

electric volume charge density $\rho = \dfrac{Q}{V}$

electric surface charge density $\sigma = \dfrac{Q}{A}$

electric line charge density $\lambda = \dfrac{Q}{L}$

Description of the electric field of an extended electric charge distribution **<u>not at</u>** the origin of a coordinate system (general case):



$$d\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{dq}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,dV'$$

Some more useful math you will use a lot:

$$\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$\vec{\nabla}\times(\vec{\nabla}F)=0$ for any scalar function F

$$\vec{\nabla}\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = 4\pi\,\delta(\vec{r}-\vec{r}')$$

δ- function : $\delta(x)=\begin{cases}0 & x\neq 0\\ \infty & x=0\end{cases}$

$$\int_{-\infty}^{+\infty} F(x)\,\delta(x-a)\,dx = F(a)$$

3-dim δ- function $\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$

$$\int_V F(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,dV = F(\vec{a})$$

Let's check if the general description of **E**(**r**) fulfills the fundamental description of electrostatic fields (and practice and appreciate some of the useful math):

$\vec{\nabla}_{\vec{r}} \cdot \vec{E}(\vec{r}) \Rightarrow$ $\vec{\nabla}_{\vec{r}}$ operates on $\vec{E}(\vec{r})$ not $\vec{r}\,'$

$$\vec{\nabla}_{\vec{r}} \cdot \vec{E} = \vec{\nabla}_{\vec{r}} \left(\frac{1}{4\pi\varepsilon_0} \int_{V'} \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} dV' \right)$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{V'} \rho(\vec{r}\,') \underbrace{\left(\vec{\nabla}_{\vec{r}} \cdot \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \right)} dV'$$

$$\vec{\nabla}_{\vec{r}} \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{V'} \rho(\vec{r}\,') 4\pi\, \delta^3(\vec{r}-\vec{r}\,')\, dV' = \frac{\rho(\vec{r})}{\varepsilon_0}$$

First rewrite **E**(**r**), next calculate the curl of **E**(**r**):

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, dV'$$

$$\Updownarrow$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho(\vec{r}\,') \left[-\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|}\right] dV'$$

$$\vec{\nabla}_{\vec{r}} \times \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho(\vec{r}\,') \Big[- \underbrace{\vec{\nabla}_{\vec{r}} \times \Big(\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|}\Big)}_{\vec{\nabla}\times\vec{\nabla}f = 0}\Big] dV' = 0$$

That’s the end for today.

- There’s a superhero named Static Shocks who derives his super powers from electrostatics:

Source: https://dcau.fandom.com/wiki/Static