Class 21 (04/08/24)

Potential & Fields II

- Retarded Potentials -

Potentials
Radiation Fields

Example: Electromagnetic Dipole Radiation

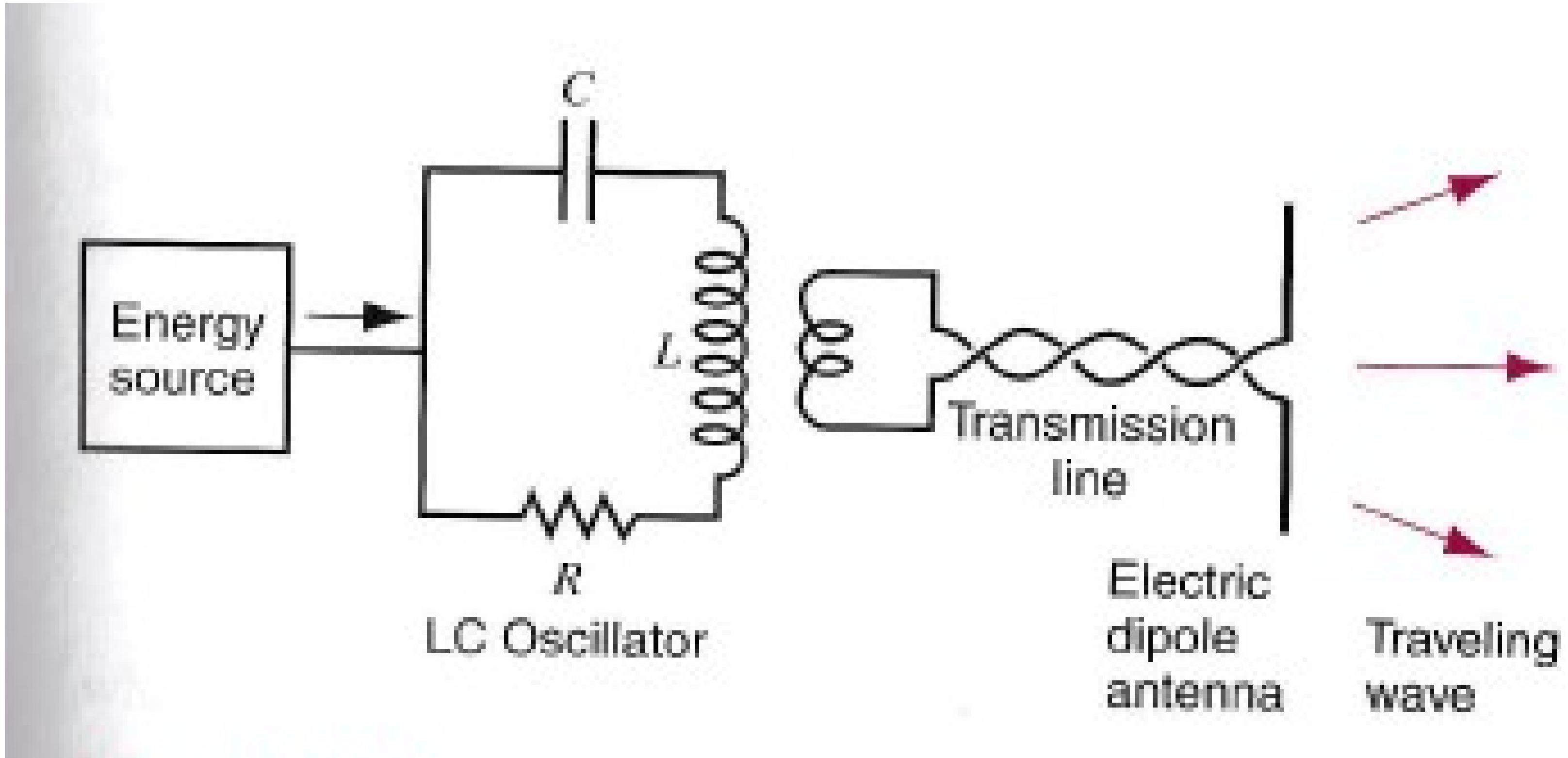


FIGURE 38-5. An arrangement for generating a traveling electromagnetic wave.

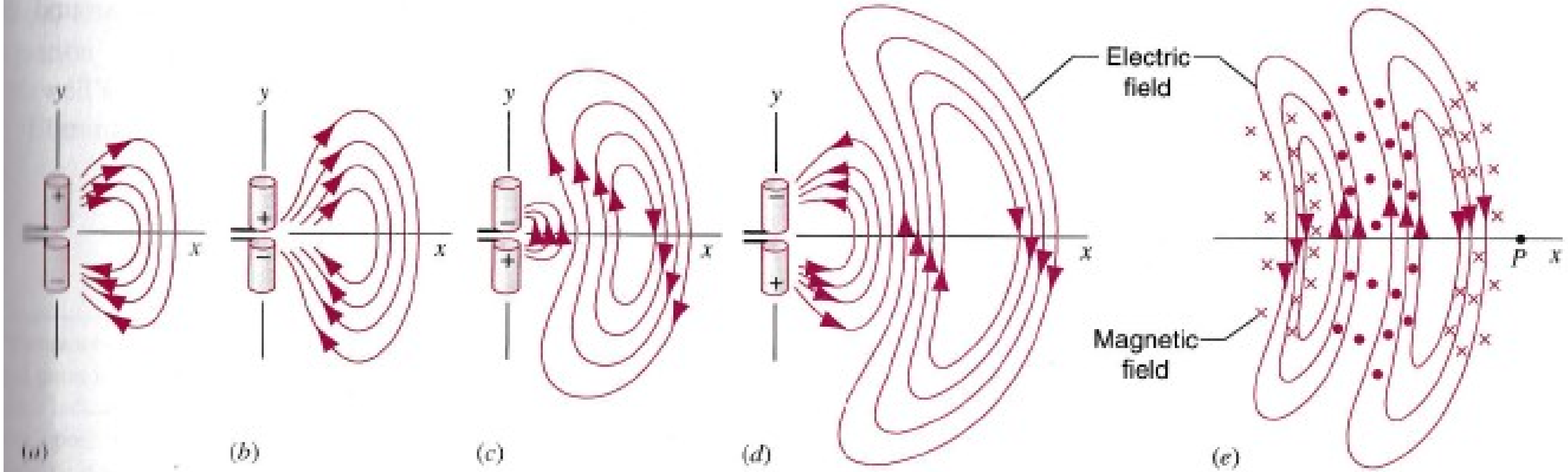


FIGURE 38-6. Successive stages in the emission of an electromagnetic wave from a dipole antenna. In (*a*)–(*d*), only the electric field patterns are shown. In (*e*), the magnetic field is shown as perpendicular to the plane of the page.

Maxwell's equation with sources ρ, $\vec{j}$ in vacuum

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}}, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\dot{\vec{E}}$$

with $\vec{E} = -\vec{\nabla}V - \dfrac{\partial\vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla}\times\vec{A}$ (1), (2)

Lorentz gauge $\vec{\nabla}\cdot\vec{A} = -\dfrac{1}{\mu_0\varepsilon_0}\dfrac{\partial V}{\partial t}$

inhomogeneous wave equations for V, $\vec{A}$

$$\vec{\nabla}^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{j} \quad (3)$$

$$\vec{\nabla}^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \quad (4)$$

electromagnetic radiation pattern originating from time-dependent (e.g. oscillating current) $\vec{j}(\vec{x},t)$ or moving (point) charges $\rho(\vec{x},t)$

solve (3) and (4) for given $\vec{j}$, ρ, calculate $\vec{E}$, $\vec{B}$ from $\vec{A}$, V using (1), (2) inhomogeneous wave equation for $\vec{A}, V$

$$\vec{\nabla}^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\vec{\nabla}^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

Retarded Potentials

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} dV'$$

with

$$t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|} dV'$$

The retarded potentials described above will be used to calculate dipole radiation (class 22)

Retarded Potentials

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\,dV'$$

with

$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\,dV'$$

j (**r**, t_r = t - (**r-r'**)/c) **E** (**r**, t), **B** (**r**,t)

Origin of Coordinate System O

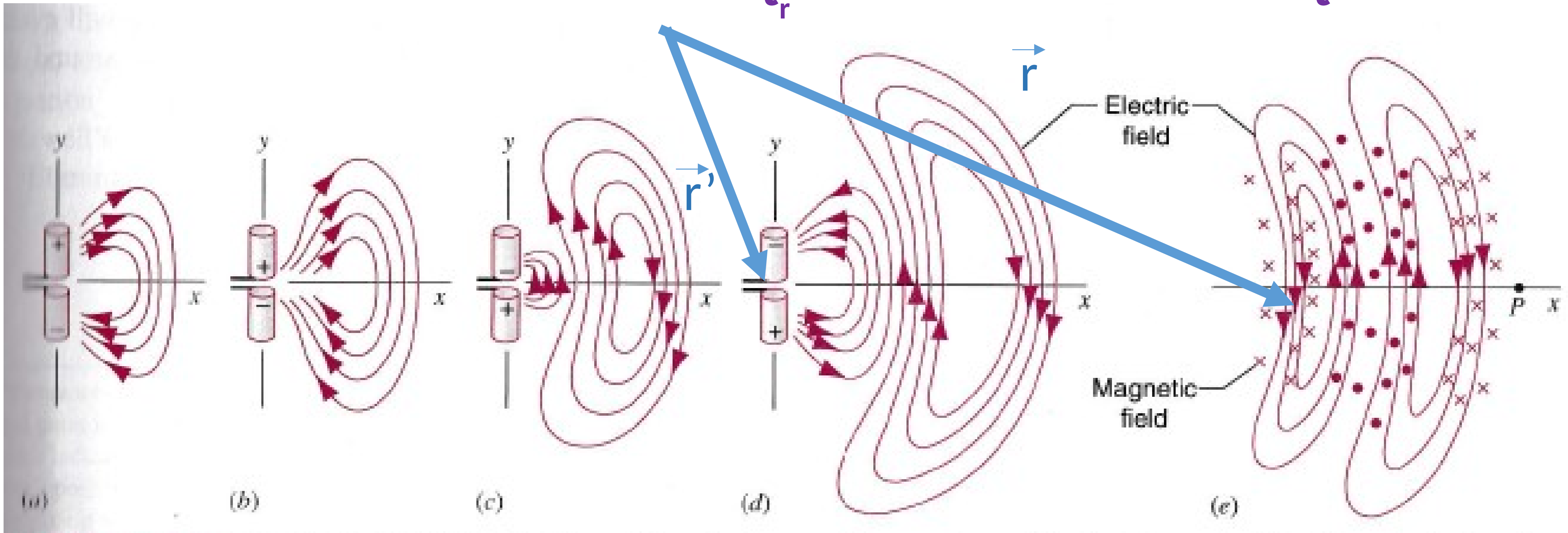
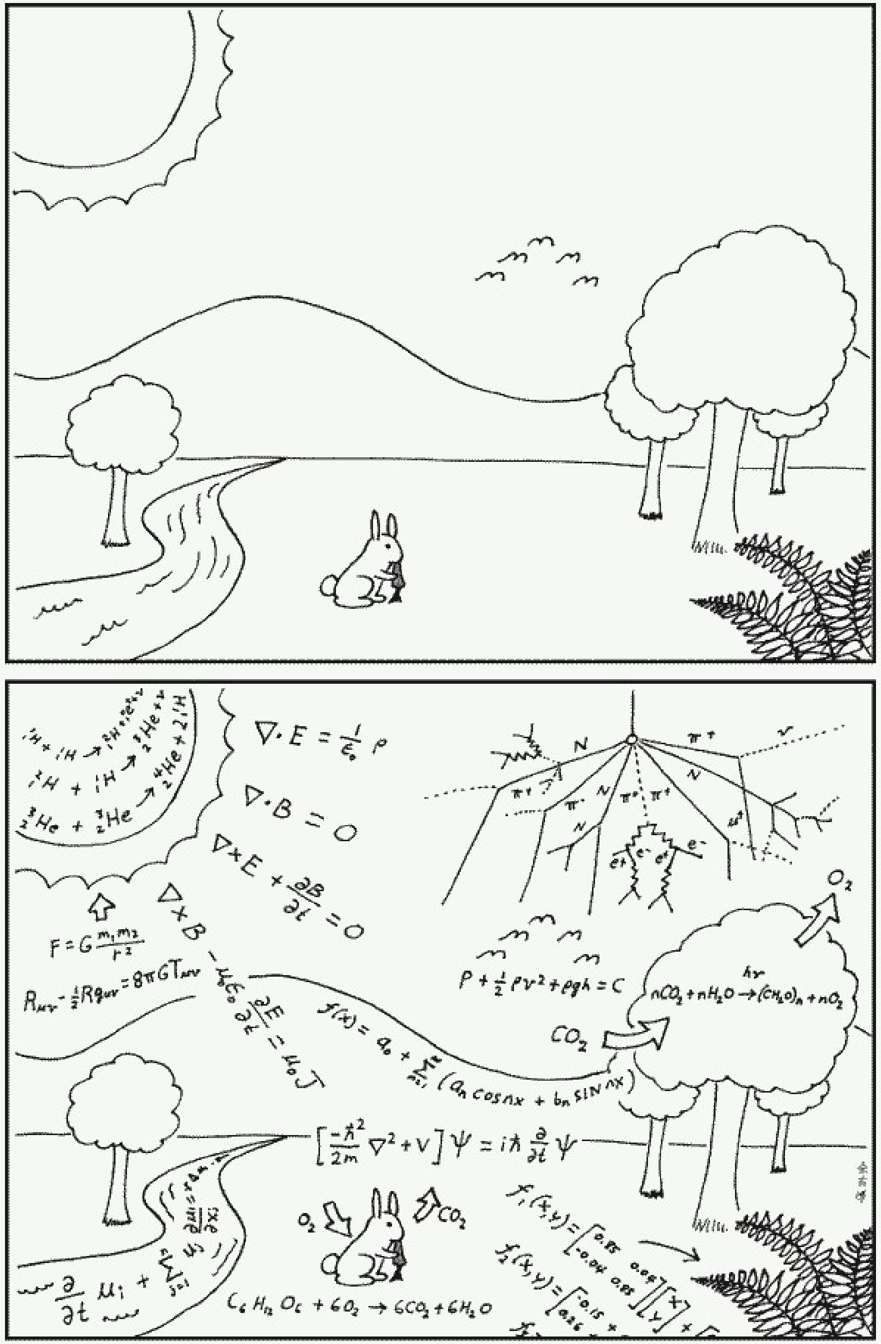


FIGURE 38-6. Successive stages in the emission of an electromagnetic wave from a dipole antenna. In (*a*)–(*d*), only the electric field patterns are shown. In (*e*), the magnetic field is shown as perpendicular to the plane of the page.



This is how scientists see the world.

Solving the inhomogeneous wave equation for $\varphi(\vec{x}, t)$ using Green's function and Fouriertransforms

$$\vec{\nabla}^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\vec{\nabla}^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

Solving the inhomogeneous wave equation for $\varphi(\vec{x},t)$ using Green's function and Fouriertransforms

① Green's function for

$$\vec{\nabla}^2 \varphi(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(\vec{x},t) = \varsigma(\vec{x},t)$$

solution:

$$\varphi(\vec{x},t) = \varphi_0(\vec{x},t) + \iint_{V'\,t'} G(\vec{x},t,\vec{x}',t')\,\varsigma(\vec{x}',t')\,dt'\,dV'$$

solution to homogeneous diff. eq.

Green's function

$$G(\underbrace{\vec{x}-\vec{x}'}_{\vec{r}},\ \underbrace{t-t'}_{\tau})$$

Property of Green's function:

$$\vec{\nabla}^2 G(\vec{x},t,\vec{x}',t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G(\vec{x},t,\vec{x}',t') = \delta(\vec{x}-\vec{x}')\,\delta(t-t')$$

exercise to slide B

$$\varphi(\vec{x},t) = \varphi_0 + \varphi_1$$

$$\vec{\nabla}^2 \varphi(\vec{x},t) - \frac{1}{c^2}\ddot{\varphi} = \underbrace{\vec{\nabla}^2 \varphi_0 - \frac{1}{c^2}\ddot{\varphi}_0}_{=0} + \vec{\nabla}^2 \varphi_1 - \frac{1}{c^2}\ddot{\varphi}_1 = \rho(\vec{x},t)$$

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi_1 = \iint\limits_{t'V'} \underbrace{\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x},t;\vec{x}',t')}_{\delta(\vec{x}-\vec{x}')\,\delta(t-t')} \rho(\vec{x}',t')\,dV'dt'$$

$$= \iint\limits_{t'V'} \delta(\vec{x}-\vec{x}')\,\delta(t-t')\,\rho(\vec{x}',t')\,dV'dt'$$

$$= \rho(\vec{x},t)$$

(2) Calculation of the Green's function

$$G(\vec{r}, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_\omega(\vec{r})\, e^{-i\omega T}\, d\omega$$

$G(\vec{r}, T)$ is expanded in a Fourier integral as a function of time.

$$\left(\vec{\nabla}^2 + \frac{\omega^2}{c^2}\right) G_\omega(\vec{r}) = \delta(\vec{r})$$

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, \tau) = \left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_\omega(\vec{r})\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\vec{\nabla}^2 + \frac{\omega^2}{c^2}\right) G_\omega(\vec{r})\, e^{i\omega\tau}\, d\omega$$

We can also expand $\delta(t-t')$ in a Fourier integral:

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(t-t')}\, d\omega$$

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, \tau) = \delta(\vec{r})\, \delta(\tau)$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\vec{\nabla}^2 + \frac{\omega^2}{c^2}\right) G_\omega(\vec{r})\, e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\int e^{i\omega\tau}\, d\tau\, \delta(\vec{r})$$

$$\left(\vec{\nabla}^2 + \frac{\omega^2}{c^2}\right) G_\omega(\vec{r}) = \delta(\vec{r})$$ Helmholtz eq.

solutions are $G_\omega^{(\pm)}(\vec{r}) = \dfrac{-1}{4\pi}\dfrac{e^{\pm i\frac{\omega}{c}r}}{r}$

obtain $G(\vec{r},\tau)$ through inverse Fourier transform

$$G(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(-\frac{1}{4\pi}\frac{e^{\pm i\frac{\omega}{c}r}}{r}\,e^{-i\omega\tau}\right)d\omega$$

$$= -\frac{1}{4\pi r}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm i\frac{\omega}{c}r}\,e^{-i\omega\tau}\,d\omega}_{\delta\left(\tau-\frac{r}{c}\right)}$$

Green's function is $G(\vec{r},t) = \delta\left(\tau-\frac{r}{c}\right)$

solution

$$\varphi(\vec{x},t) = \varphi_0(\vec{x},t) - \frac{1}{4\pi}\iint_{V'\,t'}\frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\,\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right)dt'\,dV'$$

integrate over t

$$= \varphi_0(\vec{x},t) - \frac{1}{4\pi}\int_V\frac{\rho\left(\vec{x}',\,t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\,dV'$$

<u>exercise</u> to slide D

$$\vec{\nabla}^2 G^+_\omega(\vec{r}) = \vec{\nabla}^2 \left(\frac{1}{4\pi} \frac{e^{i\frac{\omega}{c}r}}{r} \right)$$

$$= \frac{1}{4\pi} \left[e^{i\frac{\omega}{c}r} \, \vec{\nabla}^2 \frac{1}{r} + \frac{1}{r} \vec{\nabla}^2 e^{i\frac{\omega}{c}r} \right]$$

$$= \frac{1}{4\pi} \left[e^{i\frac{\omega}{c}r} \, \delta(r) + \frac{1}{r} \left(i\frac{\omega}{c}\right)^2 e^{i\frac{\omega}{c}r} \right]$$

$$= \frac{1}{4\pi} \left[-4\pi\delta(r) - \frac{\omega^2}{c^2} \frac{e^{i\frac{\omega}{c}r}}{r} \right]$$

$$\left(\vec{\nabla}^2 + \frac{\omega^2}{c^2}\right) G_\omega(\vec{r}) = \frac{1}{4\pi} \left[-4\pi\delta(r) - \frac{\omega^2}{c^2} + \frac{\omega^2}{c^2} \right] \frac{e^{i\frac{\omega}{c}r}}{r}$$

$$= \delta(r)$$

Comparison : General Solution of the Inhomogeneous Wave Equation with Retarded Potentials

Green's function is $G(\vec{r},t) = \delta\left(\tau - \frac{r}{c}\right)$

solution

$$f(\vec{x},t) = f_0(\vec{x},t) - \frac{1}{4\pi}\iint_{V'\,t'} \frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\,\delta\left(t-t' - \frac{|\vec{x}-\vec{x}'|}{c}\right)dt'\,dV'$$

integrate over t

$$= f_0(\vec{x},t) - \frac{1}{4\pi}\int_{V'} \frac{\rho\left(\vec{x}',\ t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\,dV'$$

Retarded Potentials

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}\,dV'$$

with

$$t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}\,dV'$$