Exact Solution for the one dimensional Ising Chain

$d=1$ periodic boundary conditions , general H

Transfer Matrix method

$$\boxed{s_{N+1} = s_1}$$

$$\mathcal{H}[\{s_i\}] = -J \sum_{\langle ij \rangle} s_i s_j - H \sum_{i=1}^{N} s_i \qquad s_i = \pm 1 \qquad \forall i = 1, 2, \dots, N$$

$$Z_N(T,H) = \sum_{s_1, s_2, \dots s_N} e^{-\beta \mathcal{H}[\{s_i\}]} = \sum_{s_1, s_2, \dots s_N} e^{K \sum_{i=1}^{N} s_i s_{i+1} + h \sum_{i=1}^{N} s_i}$$

where $K = \beta J$ $\qquad h = \beta H \qquad s_{N+1} \equiv s_1$ (p.b.c)

$$K \sum_{i=1}^{N} s_i s_{i+1} + h \sum_{i=1}^{N} s_i = K \sum_{i=1}^{N} s_i s_{i+1} + \frac{h}{2} \sum_{i=1}^{N} (s_i + s_{i+1})$$

$$Z_N(T,H) = \sum_{s_1, s_2, \dots, s_N} e^{K \sum_{i=1}^{N} s_i s_{i+1} + \frac{h}{2} \sum_{i=1}^{N} (s_i + s_{i+1})} = \sum_{s_1, s_2, \dots, s_N} \prod_{i=1}^{N} e^{K s_i s_{i+1} + \frac{h}{2}(s_i + s_{i+1})}$$

$s, s' = \pm 1 \qquad T_{ss'} \equiv e^{K s s' + \frac{h}{2}(s + s')} \qquad \hat{T} = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$

transfer matrix

$$Z_N = \sum_{s_1, s_2, \dots s_N} \prod_{i=1}^{N} T_{s_i s_{i+1}} = \sum_{s_1, s_2, \dots, s_N} T_{s_1 s_2} T_{s_2 s_3} \dots T_{s_{N-1} s_N} T_{s_N s_1} =$$

$$= \sum_{s_1} (\hat{T}^N)_{s_1 s_1} = \mathrm{Tr}(\hat{T}^N)$$

$$\boxed{Z_N(T,H) = \mathrm{Tr}(\hat{T}^N)}$$

Trace is independent of the representation: diagonal representation

$$\mathrm{Tr}\left(\hat{T}^N\right) = \mathrm{Tr}\left[\underbrace{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\cdots\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}}_{N \text{ times}}\right] = \mathrm{Tr}\left\{\begin{pmatrix}\lambda_1^N & 0\\ 0 & \lambda_2^N\end{pmatrix}\right\} = \lambda_1^N + \lambda_2^N$$

$$Z_N(T,H) = \lambda_1^N + \lambda_2^N$$

eigenvalues of $\hat{T}$: λ_1, λ_2
eigenvalues of $\hat{T}^N$: λ_1^N, λ_2^N
$\Rightarrow \boxed{\mathrm{Tr}\left(\hat{T}^N\right) = \lambda_1^N + \lambda_2^N}$

Thermodynamic limit: $N \to \infty$ $\quad |\lambda_1| > |\lambda_2|$

$$Z_N(T,H) = \lambda_1^N\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right) \simeq \lambda_1^N \qquad \left(\frac{\lambda_2}{\lambda_1}\right)^N \to 0$$

$$F(T,H,N) = -kT \ln Z_N(T,H) = -kT \ln\left[\lambda_1^N\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right)\right]$$

$$f(T,H) = \frac{F(T,H,N)}{N} = -kT\ln\lambda_1 - \frac{1}{N}kT\ln\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right)$$

$$\xrightarrow[N\to\infty]{} -kT\ln\lambda_1$$

$$\boxed{\begin{aligned} Z_N(T,H) &\simeq \lambda_1^N \\ f(T,H) &= -kT\ln(\lambda_1)\end{aligned}} \qquad N \to \infty$$

determine λ_1, λ_2 eigenvalues:

$$\begin{vmatrix} e^{K+h} - \lambda & e^{-K} \\ e^{-K} & e^{K-h} - \lambda \end{vmatrix} = \left(e^{K+h} - \lambda\right)\left(e^{K-h} - \lambda\right) - e^{-2K} = 0$$

$$\lambda^2 - \lambda\, 2e^{K}\cosh(h) + 2\sinh(2K) = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left\{ 2e^{K}\cosh(h) \pm \sqrt{4e^{2K}\cosh^2(h) - 8\sinh(2K)} \right\} =$$

$$(\cosh^2(h) - \sinh^2(h) = 1) \quad = e^{K}\cosh(h) \pm \sqrt{e^{2K}\sinh^2(h) + e^{-2K}}$$

λ_1 is the $\oplus$ eigenvalue (the largest)

$\boxed{h = \beta H \\ K = \beta J}$ coupling co

$\boxed{H=0}$: $\Rightarrow h = 0$ (no external field)

$$\lambda_1 = e^{K} + e^{-K} = 2\cosh(K)$$
$$\lambda_2 = e^{K} - e^{-K} = 2\sinh(K)$$

$$Z_N \simeq 2^N \cosh^N(K) = \lambda_1^N = 2^N \cosh^N(K)$$
$$f(T,0) = -kT\ln(\lambda_1) = -kT\ln(2\cosh(K)) = -kT\ln 2 - kT\ln\cos$$

$$\boxed{f(T,0) = -kT\ln 2 - kT\ln\cosh(K)}$$

$\boxed{H \neq 0 \quad \text{general case}}$

$$m = -\frac{\partial f}{\partial H} = -\frac{\partial f}{\partial h}\cdot\beta = -\beta\frac{\partial}{\partial h}\left[-kT\ln\lambda_1\right] = \frac{\partial}{\partial h}\ln\lambda_1$$

$$\lambda_1 = e^{K}\left[\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}\right]$$

$m(T,H)$

$$\frac{\partial}{\partial h}\ln\lambda_1 = \frac{\partial}{\partial h}\ln\left[\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}\right] =$$

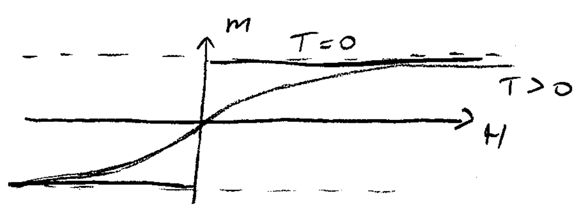
$$= \frac{1}{\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}}\left[\sinh(h) + \frac{\sinh(h)\cosh(h)}{\sqrt{\sinh^2(h) + e^{-4K}}}\right] =$$

$$= \boxed{\frac{\sinh(h)}{\sqrt{\sinh^2(h) + e^{-4K}}}} \qquad h = \beta H \qquad K = \beta J = \frac{J}{kT}$$

$$\lim_{H\to 0} m(T,H) = 0 \qquad \text{for all } T > 0$$

no phase transition in the one-dimensional model!!!

$$\lim_{T\to 0} m(T,H) = \mathrm{sgn}(H)$$



no spontaneous magnetization at any non-zero temperature

$$\chi = \frac{\partial m}{\partial H} = \frac{e^{-4K}}{\left(\sinh^2(h) + e^{-4K}\right)^{3/2}}\cosh(h)\cdot\frac{1}{kT} = \frac{1}{kT}\frac{\cosh(h)\,e^{-4K}}{\left(\sinh^2(h) + e^{-4K}\right)^{3/2}}$$

$H = 0$: $\quad T \to \infty$: $\quad \chi \simeq \frac{1}{kT}$ $\qquad$ Curie's Law

$T \to 0 \qquad \chi = \frac{1}{kT}\, e^{2J/kT}$ $\qquad$ (essential singularity)

1d Ising

The Correlation length $\xi(T)$ /open boundary conditions/

we saw that there is no finite temperature phase tr.
$m(T, H=0) \equiv 0$ for all $T>0$
$\langle S_i \rangle \simeq m = 0$

$\bullet \; \bullet \; \bullet \; \bullet \cdots \bullet$
$1 \; 2 \; 3 \; i \quad N$

$$G(i,j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle = \langle (S_i - \langle S_i \rangle)(S_j - \langle S_j \rangle) \rangle$$

two-point correlation function

Interpretation: $P_{ij} = \langle \delta_{S_i S_j} \rangle$ = probability that spin at site i and site j has the same value

$$P_{ij} = \langle \delta_{ij} \rangle = \left\langle \tfrac{1}{2}(1 + S_i S_j) \right\rangle = \frac{1}{2} + \frac{1}{2}\langle S_i S_j \rangle = \frac{1}{2} + \frac{1}{2}\left[G(i,j) + \langle S_i \rangle \langle S_j \rangle \right]$$

For the 1d open end Ising Chain:

$$K = \beta J \qquad -\beta \mathcal{H} = K \sum_{l=1}^{N-1} S_l S_{l+1} \longrightarrow \sum_{l=1}^{N-1} K_l S_l S_{l+1} \qquad S_l = \pm 1$$

$$\langle S_i S_{i+j} \rangle = \frac{1}{Z_N} \sum_{S_1, S_2, \dots, S_N} S_i S_{i+j} \, e^{\sum_{l=1}^{N-1} K_l S_l S_{l+1}} \qquad Z_N = \sum_{S_1, S_2, \dots, S_N} e^{\sum_{l=1}^{N-1} K_l S_l S_{l+1}}$$

$$= \frac{1}{Z_N} \sum_{\{S_l\}} (S_i S_{i+1})(S_{i+1} S_{i+2}) \dots (S_{i+j-1} S_{i+j}) \, e^{\sum_{l=1}^{N-1} K_l S_l S_{l+1}}$$

$$= \frac{1}{Z_N} \frac{\partial^j}{\partial K_i \partial K_{i+1} \dots \partial K_{i+j-1}} Z_N \Bigg|_{K_l \equiv K}$$

$$Z_N = \sum_{\{S_l\}} e^{\sum_{l=1}^{N-1} K_l S_l S_{l+1}} = \sum_{S_1 S_2 \dots S_{N-1}} e^{\sum_{l=1}^{N-2} K_l S_l S_{l+1}} \underbrace{\sum_{S_N} e^{K_{N-1} S_{N-1} S_N}}_{2\cosh(K_{N-1} S_{N-1})} = \qquad S_{N-1} = \pm 1$$

$$= Z_{N-1} \, 2\cosh(K_{N-1}) \qquad \text{recursion relation}$$

$N=2 \qquad Z_2 = \sum_{S_1, S_2} e^{K_1 S_1 S_2} = 2\cdot 2\cosh(K_1)$

$$\Rightarrow \boxed{Z_N(K_\ell) = 2^N \prod_{\ell=1}^{N-1} \cosh(K_\ell)}$$

$$\frac{1}{Z_N} \cdot \frac{\partial^j}{\partial K_i \partial K_{i+1} \dots \partial K_{i+j-1}} Z_N(K_\ell) = \prod_{\ell=i}^{i+j-1} \tanh(K_\ell)$$

$K_\ell = K \quad \Rightarrow \quad \langle S_i S_{i+j} \rangle = [\tanh(K)]^j = e^{-j \ln\left(\frac{1}{\tanh(K)}\right)}$

$\tanh(K) < 1$

$= e^{-j/\xi}$

where $\quad \xi \equiv \frac{1}{\ln\coth(K)} = \frac{1}{\ln\coth(J/kT)}$ correlation length

$$\boxed{G(i, i+j) = G(j) = e^{-j/\xi}}$$

exponential decay for $T>0$

$T \to 0$: $\quad \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \underset{x\to\infty}{\simeq} \frac{1+e^{-2x}}{1-e^{-2x}} \simeq (1+e^{-2x})(1+e^{-2x}) \simeq 1+2e^{-2x}$

$\ln\coth(J/kT) \simeq \ln(1+2e^{-2J/kT}) \simeq 2e^{-2J/kT}$

$$\boxed{\xi(T) \simeq \frac{1}{2} e^{2J/kT}}$$

for low temperatures

$\lim_{T\to 0} \xi(T) = \infty \qquad$ spins become infinitely correlated as $T\to 0$ approached

Scaling Hypothesis & RG

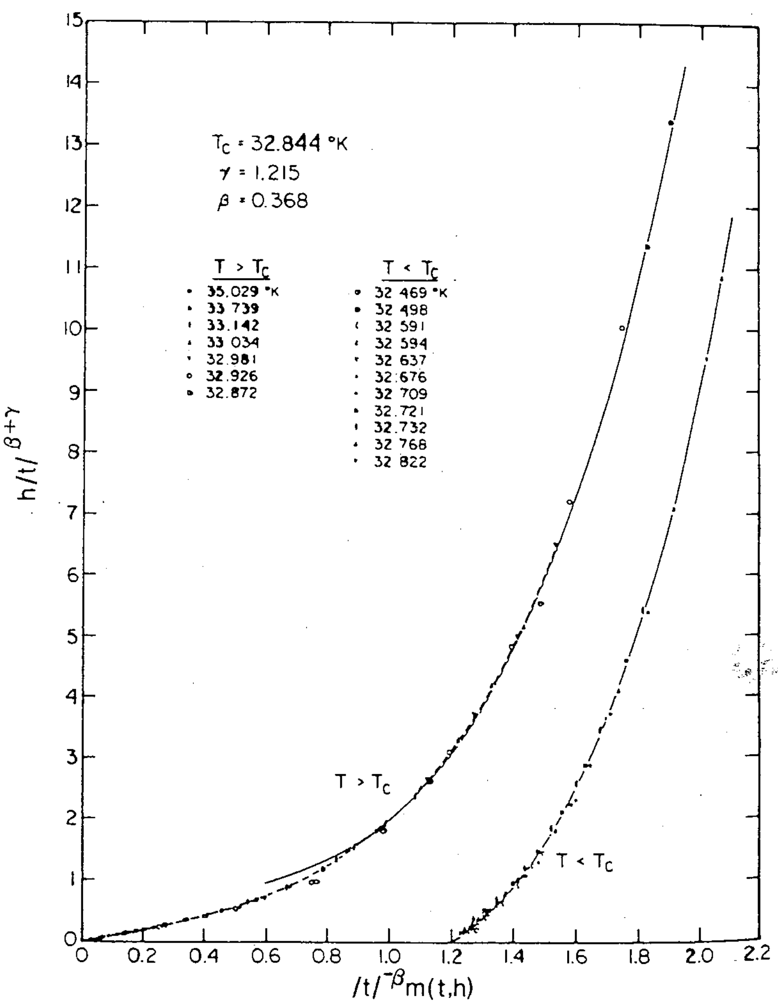


Figure 5.6: Magnetization of $CrBr_3$ in the critical region plotted in a scaled form (see the text). (From Ho and Litster [112].)

Scaling Hypothesis

magnetization:

motivation: common description of two experimental results:

$m(t,h) = ?$

$$t = \frac{T - T_c}{T_c}$$

$$h = \beta H = \frac{H}{kT} \quad \text{(for ferromagn.)}$$

$$m(t,0) = \begin{cases} 0 & t > 0 \\ \pm A|t|^\beta & t < 0 \end{cases} \qquad \pm : \ h \to \pm 0$$

$$m(0,h) = B|h|^{1/\delta} \operatorname{sign}(h)$$

the above two behavior is captured by

$$m(t,h) = |t|^\beta \mathcal{F}_\pm \left(h/|t|^\Delta \right) \qquad \Delta: \text{"gap" exponent}$$

valid for $h, t \ll 1$ but for arbitrary ratio $h/|t|^\Delta$

$h \to \pm 0$ $\quad t > 0 \quad m(t,0) \equiv 0$, i.e. $\mathcal{F}_+(0) \equiv 0$

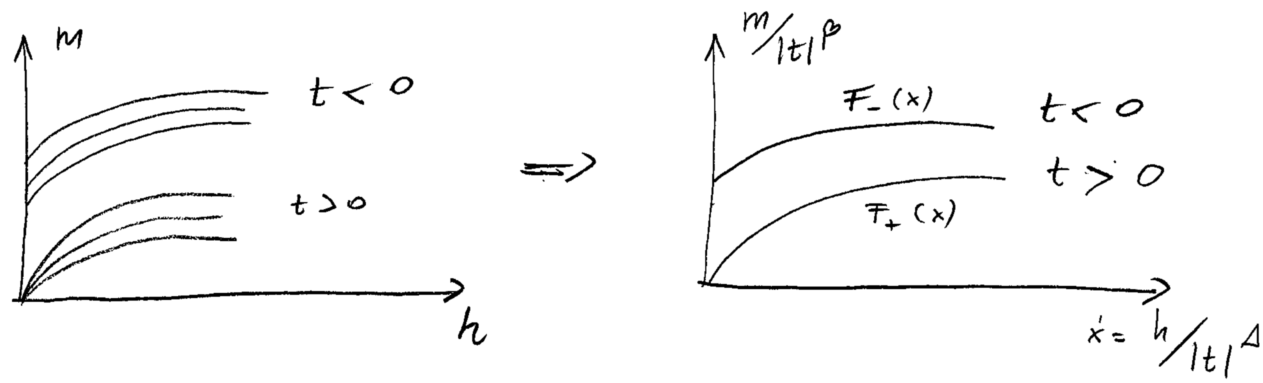
$\quad t < 0 \quad m(t,\pm 0) = \pm A|t|^\beta$ i.e., $\mathcal{F}_-(\pm 0) = \pm A$

$$m(0,h) \sim |t|^{\beta} \lim_{x\to\infty} \mathcal{F}_{\pm}(x) \qquad x = \frac{h}{|t|^{\Delta}} \qquad \mathcal{F}_{\pm}(x) \underset{x\to\infty}{\simeq} x^{\lambda}$$

$$\sim |t|^{\beta} \left(\frac{h}{|t|^{\Delta}}\right)^{\lambda} \sim |t|^{\beta} \frac{h^{\lambda}}{|t|^{\Delta\lambda}} \sim h^{1/\delta}$$

$\Rightarrow \quad \lambda = 1/\delta$ and $\beta = \Delta\lambda$

(1) $\boxed{\delta = \frac{\Delta}{\beta}}$ e.g. $h > 0$



$$m(t,-h) = -m(t,h) \quad \Rightarrow \quad \boxed{\mathcal{F}_{\pm}(-x) = -\mathcal{F}_{\pm}(x)}$$

<u>Susceptibility:</u> (zero field)

$$\chi_T(t, h=0) = \left.\frac{\partial m}{\partial H}\right|_{h=0} = \frac{1}{kT}\left(\frac{\partial m}{\partial h}\right)_{h=0} =$$

$$= \frac{1}{kT} |t|^{\beta} \frac{\partial}{\partial h} \mathcal{F}_{\pm}\left(h/|t|^{\Delta}\right)\Big|_{h=0} = \frac{1}{kT} |t|^{\beta-\Delta} \mathcal{F}_{\pm}'(0) \sim |t|^{-\gamma}$$

assuming that $\mathcal{F}_{\pm}'(0) \neq 0$, or ∞ $\qquad \mathcal{F}_{\pm}'(0) = \mathcal{B}_{\pm}$

$\Rightarrow \quad \beta - \Delta = -\gamma \qquad \boxed{\Delta = \beta + \gamma} \qquad (2)$

combining (1) & (2): $\boxed{\delta\beta = \beta + \gamma} \qquad (3)$

Underlying Structure: <u>scaling for the free energy</u>

for the singular part of the free energy per unit volume

e.g. $t<0$
($\mathcal{F}_f^{\pm}(x)$ would be needed)

$$\boxed{f_s(t,h) = |t|^{2-\alpha}\,\mathcal{F}_f\left(h/|t|^{\Delta}\right)}$$

$h = \frac{H}{kT}$

$$m = -\frac{\partial}{\partial H} f_s = -\frac{1}{kT}\frac{\partial f_s}{\partial h} = -\frac{1}{kT}|t|^{2-\alpha}\frac{1}{|t|^{\Delta}}\mathcal{F}_f'\left(h/|t|^{\Delta}\right)$$

$$\sim |t|^{2-\alpha-\Delta}\,\mathcal{F}_f'\left(h/|t|^{\Delta}\right) \xrightarrow[h\to+0]{} \sim |t|^{\beta}$$

clearly
$\mathcal{F}_m(x) = -\frac{1}{kT}\mathcal{F}_f'(x)$

$$\beta = 2-\alpha-\Delta$$

using (2):

$$\beta = 2-\alpha-\beta-\gamma \quad \Rightarrow \quad \boxed{\alpha+2\beta+\gamma = 2}$$

Rushbrooke scaling law

$$\chi_T = \frac{\partial m}{\partial H} = \frac{1}{kT}\frac{\partial m}{\partial h} = \frac{-1}{(kT)^2}|t|^{\beta-\Delta}\,\mathcal{F}_f''\left(h/|t|^{\Delta}\right) \sim |t|^{-\gamma}$$

$$\left.\begin{array}{l}\beta = 2-\alpha-\Delta\\ \beta-\Delta = -\gamma\end{array}\right\} \quad \boxed{\alpha+2\beta+\gamma = 2} \qquad (4)$$

<u>Scaling for the correlation function</u>

$$G(\bar{r},t,h) = \frac{1}{r^{d-2+\eta}}\,\mathcal{F}_G\left(r|t|^{\nu}, \frac{h}{|t|^{\Delta}}\right)$$

$h=0$

$$\chi_T \sim \int d^d r\, G(\bar{r},t,h) = \int d^d r \frac{1}{r^{d-2+\eta}}\mathcal{F}_G(r t^{\nu},0) \qquad \bar{r}t^{\nu} \equiv \bar{x}$$

$$\int \frac{d^d x}{t^{d\nu}}\, t^{\nu(d-2+\eta)}\frac{1}{x^{d-2+\eta}}\mathcal{F}_G(x,0) = t^{\nu\eta-2\nu}\underbrace{\int d^d x \frac{1}{x^{d-2+\eta}}\mathcal{F}_G(x,0)}_{\text{const}}$$

on the other hand: $\chi_T \sim |t|^{-\gamma}$ $\Rightarrow$ $\boxed{\gamma = 2\nu-\eta\nu}$