HW 8 Paul Lea N.VII.2 #'s 3,4

N.VII.3 #'s 4,5

VII.2

#3 Commutation relations of SO(2,2)

Verify $[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho})$

SO(4) have similar generators to SO(2,2)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

SO(4)

SO(2,2) has 2 rotations & 4 boosts

$J_{12}, J_{34}, K_{13}, K_{14}, K_{23}, K_{24}$

$$\underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{J_{12}}, \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{K_{13}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{K_{14}}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{K_{24}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}_{J_{34}}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{K_{23}}$$

See Mathematica Doc for comutator calc

$[J_{12}, K_{13}] = K_{23}$ $[J_{12}, K_{24}] = -K_{14}$ $[J_{12}, K_{23}] = -K_{13}$

$[J_{12}, K_{14}] = K_{24}$ $[J_{12} J_{34}] = 0$ $[K_{13}\ K_{14}] = J_{34}$

$[K_{13}, K_{24}] = 0$ $[K_{13}\ J_{34}] = K_{14}$ $[K_{13}\ K_{23}] = -J_{12}$

$[K_{14}, K_{24}] = -J_{12}$ $[K_{14}, J_{34}] = -K_{13}$ $[K_{14}, K_{23}] = 0$

$[K_{24}, J_{34}] = -K_{23}$ $[K_{24}, K_{23}] = -J_{34}$ $[J_{34}, K_{23}] = -K_{24}$

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho})$$

$$J_{\mu\nu} J_{\rho\sigma} - J_{\rho\sigma} J_{\mu\nu}$$

$$-(X_\mu d_\nu - X_\nu d_\mu)(X_\rho d_\sigma - X_\sigma d_\rho) + (X_\rho d_\sigma - X_\sigma d_\rho)(X_\mu d_\nu - X_\mu d_\mu)$$

$$-[X_\mu d_\nu, X_\rho d_\sigma] + [X_\mu d_\nu, X_\sigma d_\rho] - [X_\nu d_\mu, X_\rho d_\sigma] - [X_\nu d_\mu, X_\sigma d_\rho]$$

$$X_\mu [d_\mu, X_\rho] d_\sigma + [X_\mu, X_\rho] d_\nu d_\sigma + X_\mu [d_\nu, X_\sigma] d_\rho +$$
$$[X_\mu, X_\sigma] \partial_\mu d_\rho + X_\nu [d_\mu, X_\rho] d_\sigma + [X_\nu, X_\rho] d_\mu d_\sigma$$
$$+ X_\nu [\partial_\mu, X_\sigma] d_\rho + [X_\nu, X_\sigma] \partial_\mu d_\rho - X_\mu d^\mu_\rho d_\sigma - X_\mu d^\mu_\sigma d_\rho$$
$$-X_\nu d^\mu_\rho d_\sigma - X_\nu d^\mu_\sigma d_\rho$$
$$= X_\mu (d^\mu_\rho d_\sigma - d^\nu_\sigma d_\rho + X_\nu (d^\mu_\rho d_\sigma - d^\mu_\sigma d_\rho)$$

$$\Rightarrow -i(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho})$$

#4 Show that $\eta_{\mu\nu} L^{\mu}_{\ \sigma} L^{\nu}_{\ \rho} = \eta_{\sigma\rho}$

implies $L^{0}_{\ 0} \geq 1$ or ≤ -1

$\eta_{\mu\nu} L^{\mu}_{\ \sigma} L^{\nu}_{\ \rho} = \eta_{\sigma\rho}$ for $SO[3,1]$

$L^{0}_{\ 0}$

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\sigma = \rho = 0$

always ± 1 for Lorentz group

VII.3 #4 Decompose $\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right)$

$$j \otimes j' = (j + j') \oplus (j + j' - 1) \oplus \ldots \oplus |j - j'|$$

$$\left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2} + \frac{1}{2}, 0 + \frac{1}{2}\right) \oplus \left(\left|\frac{1}{2} - \frac{1}{2}\right|, \left|\frac{1}{2} - \frac{1}{2}\right|\right)$$

$$= \left(1, \frac{1}{2}\right) \oplus \left(0, \frac{1}{2}\right)$$

#5 $$(E - \underbrace{\vec{\sigma} \cdot \vec{p}}) u = 0$$

$$\vec{\sigma} \cdot \vec{p} = \sigma_1 p^x + \sigma_2 p^y \sigma_3 p^z$$

Variation

$$(\Delta E - \vec{\sigma} \cdot \delta \vec{p}) + (E - \vec{\sigma} \cdot \vec{p}) \Delta$$

$$= \left(p^z - \sigma_3 E\right) + \left(E - \vec{\sigma}\cdot\vec{p}\right)\tfrac{1}{2}\sigma_3$$

$$2\sigma^3\left[\left(p^z - \sigma_3 E\right) + \left(E - \vec{\sigma}\cdot\vec{p}\right)\tfrac{1}{2}\sigma_3\right]$$

$$\Leftrightarrow -\left(E - \sigma_3 p^z\right) + \left(\sigma_1 p^x + \sigma_2 p^y\right)$$

$$\Leftrightarrow \underbrace{-\left(E - \vec{\sigma}\cdot\vec{p}\right) = 0}$$