N I.3 1, 3, 5, 6
N II.1 4
N II.2 1

Homework 2
Paul Lea

1. Vectors $\vec{p}$ & $\vec{q}$

Prove $\begin{pmatrix} p^2 q^3 \\ p^3 q^1 \\ p^1 q^2 \end{pmatrix}$ is not a vector. I'll call it $\vec{V}$

$\vec{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}$, $\vec{q} = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}$ rotated around X

$q_2 \leftrightarrow q_3$, $p_2 \leftrightarrow p_3$

$\vec{p}\,' = \begin{pmatrix} p_1 \\ p_3' \\ p_2' \end{pmatrix}$, $\vec{q}\,' = \begin{pmatrix} q_1 \\ q_3' \\ q_2' \end{pmatrix}$ ← X-component is left unchanged

Applying this transformation to $\vec{V}$

$\vec{V}' = \begin{pmatrix} p_3' q_2' \\ p_2' q_1 \\ p_1 q_3' \end{pmatrix}$ ← X component is changed by a rotation around X. Does not behave like a vector under rotation, ∴ not a vector.

3. $j_x = \begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$, $j_y = \begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}$, $j_z = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$

Paul Len

$$R(\theta) = e^{\theta j_x}$$

$$j_x^2 = \begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix} = \underline{\begin{pmatrix}0&0&0\\0&-1&0\\0&0&-1\end{pmatrix}}$$

$$j_x^4 = \begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$R_x(\theta) = e^{\theta j_x}$$

$$j_x^3 = \begin{pmatrix}0&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix} = \underline{\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}}$$

$$e^x \sim 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{\theta j_x} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \theta\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix} + \frac{\theta}{2!}\begin{pmatrix}0&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

$$+ \frac{\theta}{3!}\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix} + \frac{\theta}{4!}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} + \dots = \begin{pmatrix}1&0&0\\0, & \left(1-\frac{\theta}{2!}+\frac{\theta}{4!}\dots\right), & \left(\theta-\frac{\theta}{3!}\dots\right)\\0, & \left(-\theta+\frac{\theta}{3!}\dots\right), & \left(1-\frac{\theta}{2!}+\frac{\theta}{4!}\right)\end{pmatrix}$$

$$1 - \frac{\theta}{2!} + \frac{\theta}{4!} \dots = \cos\theta$$

$$\theta - \frac{\theta}{3!} \dots = \sin\theta$$

$$\Longrightarrow \begin{pmatrix}1, & 0, & 0\\0, & \cos\theta_x, & \sin\theta_x\\0, & -\sin\theta_x, & \cos\theta_x\end{pmatrix} = R_x\theta_x$$

$$R_y(\theta_y) = \begin{pmatrix}\cos\theta_y, & 0, & \sin\theta_y\\0, & 1, & 0\\-\sin\theta_y, & 0, & \cos\theta_y\end{pmatrix}$$

$$\begin{pmatrix}\cos\theta_y, & 0, & \sin\theta_y\\0, & 1, & 0\\-\sin\theta_y, & 0, & \cos\theta_y\end{pmatrix} \qquad \begin{pmatrix}1, & 0, & 0\\0, & \cos\theta_x, & \sin\theta_x\\0, & -\sin\theta_x, & \cos\theta_x\end{pmatrix}$$

$$\begin{pmatrix}1, & 0, & 0\\0, & \cos\theta_x, & \sin\theta_x\\0, & -\sin\theta_x, & \cos\theta_x\end{pmatrix}\begin{pmatrix}\cos\theta_y, & 0, & \sin\theta_y\\-\sin\theta_x\sin\theta_y, & \cos\theta_x, & \sin\theta_x\cos\theta_y\\-\sin\theta_y\cos\theta_x, & -\sin\theta_x, & \cos\theta_x\cos\theta_y\end{pmatrix} \neq \begin{pmatrix}\cos\theta_y, & 0, & \sin\theta_y\\0, & 1, & 0\\-\sin\theta_y, & 0, & \cos\theta_y\end{pmatrix}\begin{pmatrix}\cos\theta_y, & -\sin\theta_x\sin\theta_y, & \cos\theta_x\sin\theta_y\\0, & \cos\theta_x, & \sin\theta_x\\-\sin\theta_y, & -\cos\theta_y\sin\theta_x, & \cos\theta_x\cos\theta_y\end{pmatrix}$$

$$5.\ \left(J_{(mn)}\right)^{ij} = -i\left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right)$$

Paul Lea

Commutator: $|A, B| = AB - BA$

$$|J_{(mn)}, J_{(pq)}|^{ij} = \left(J_{(mn)} J_{(pq)}\right)^{ij} - \left(J_{(pq)} J_{(mn)}\right)^{ij}$$

$$= \sum_{j=1}^{n} J^{ij}_{(m,n)} J^{jk}_{(pq)} - \sum_{ij}^{n} J^{ij}_{(pq)} J^{jk}_{(m,n)}$$

$$-\sum \cancel{(-i)^2}\left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right)\left(\delta^{pi}\delta^{qj} - \delta^{pi}\delta^{pj}\right)$$

$$+\sum \cancel{(-i)^2}\left(\delta^{pi}\delta^{qj} - \delta^{pi}\delta^{pj}\right)\left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right)$$

$$\sum_{k=1}^{N} \delta^{ik}\delta^{kj} = \delta^{ij}$$

$$= -\sum_k \left(\delta^{mi}\delta^{nk} - \delta^{mk}\delta^{ni}\right)\delta^{pk}\delta^{qj} + \sum_k \left(\delta^{mi}\delta^{nk} - \delta^{mk}\delta^{ni}\right)\delta^{pj}\delta^{qk}$$

$$+\sum_k \left(\delta^{pi}\delta^{qk} - \delta^{pk}\delta^{qi}\right)\delta^{mk}\delta^{nj} - \sum_k \left(\delta^{pi}\delta^{qk} - \delta^{pk}\delta^{qi}\right)\delta^{mj}\delta^{nk}$$

$$= -\delta^{mi}\delta^{np}\delta^{qj} + \delta^{mp}\delta^{ni}\delta^{qj} + \delta^{mi}\delta^{nq}\delta^{pj} - \delta^{mq}\delta^{ni}\delta^{pj}$$

$$+\delta^{pi}\delta^{mq}\delta^{nj} - \delta^{mp}\delta^{qi}\delta^{nj} - \delta^{pi}\delta^{nq}\delta^{mj} + \delta^{np}\delta^{qi}\delta^{mj}$$

$$\Rightarrow \delta^{mp}\left(\delta^{ni}\delta^{qj} - \delta^{qi}\delta^{nj}\right) + \delta^{nq}\left(\delta^{mi}\delta^{pj} - \delta^{pi}\delta^{mj}\right)$$

$$-\delta^{np}\left(\delta^{mi}\delta^{qj} - \delta^{qi}\delta^{mi}\right) - \delta^{mq}\left(\delta^{ni}\delta^{pj} - \delta^{pi}\delta^{nj}\right)$$

$$\Rightarrow i\delta^{mp} J^{ji}_{(nq)} + i\delta^{nq} J^{ji}_{(mp)} - i\delta^{np} J^{ij}_{(mq)} - i\delta^{mz} J^{ij}_{(np)}$$

$$= |J_{(mn)}, J_{(pq)}| = i\left(\delta^{mp} J_{(nq)} + \delta^{nw} J_{(mp)} - \delta^{(np)} J_{(mp)} - \delta J_{(np)}\right)$$

Paul Lee

6. 2, because to be simultanuously diagonalizable they must commute. Two matrices that share an indice do not commute. Therefore, out of the generators of SO(4)

$$J_{12},\ J_{23},\ J_{31},\ J_{14},\ J_{24},\ J_{34}$$

You can only pick 2 without repeating indicies, therefore, Only 2 matrices are Simultanuously diagonalizble

- credit Himani Menta

2N.11 #4 $\chi_c^{(r)} = \text{tr}\, D$

these are the classes & character is a function of class

$$(12)(34) \Rightarrow \text{tr}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = 0$$

$$I \Rightarrow \text{tr}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4$$

Paul
Len

(12)

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\Rightarrow \operatorname{tr} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b \\ a \\ c \\ d \end{bmatrix} = 2$$

(123)

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\Rightarrow \operatorname{tr} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ a \\ b \\ d \end{bmatrix} = 1$$

(1234)

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} d \\ a \\ b \\ c \end{bmatrix} = 0$$

Attribute Mayo, Paul

6 11.2 #1

Paul Le

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\ \omega\\ \omega^*\end{pmatrix} = \left(1(1)+1(\omega)+1(\omega^*)\right)$$

$$1+e^{i2\pi/3}+e^{-i2\pi/3}$$

$$1+\cos\tfrac{2\pi}{3}+i\sin\tfrac{2\pi}{3}+\cos-\tfrac{2\pi}{3}+i\sin\left(-\tfrac{2\pi}{3}\right)$$

$$1+(-0.5)+i\sin\tfrac{2\pi}{3}+(-0.5)-i\sin\left(\tfrac{2\pi}{3}\right)$$

$$1-1+i\sin\left(\tfrac{2\pi}{3}\right)-i\sin\left(\tfrac{2\pi}{3}\right)$$

$$=0$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\equiv\cdot\begin{pmatrix}1\\ \omega^*\\ \omega\end{pmatrix}$$

$$\begin{pmatrix}1\\ \omega\\ \omega^*\end{pmatrix}\cdot\begin{pmatrix}1\\ \omega^*\\ \omega\end{pmatrix} = 1(1)+\omega(\omega^*)+\omega^*(\omega)$$

$$=1+\omega(\omega^*)^*+\omega^*(\omega^*)$$

$$=1+\omega^2+\omega^{*2}$$

$$=1+e^{i4\pi/3}+e^{-i4\pi/3}$$

$$=1+\left(\cos\left(\tfrac{4\pi}{3}\right)+i\sin\left(\tfrac{4\pi}{3}\right)\right)+\left(\cos\left(\tfrac{-4\pi}{3}\right)+i\sin\left(\tfrac{-4\pi}{3}\right)\right)$$

$$=\cancel{1+(-0.5)+(-0.5)}+\cancel{i\sin\left(\tfrac{4\pi}{3}\right)-i\sin\left(\tfrac{4\pi}{3}\right)}$$

$$=0$$

These 3 vectors are orthogonal, dot product is 0

- attribute: Zach, Himnani, Tristan

Paul
Len

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}u\\v\\w\end{pmatrix} = u+v+w=0$$

$$\begin{pmatrix}1\\\omega\\\omega^*\end{pmatrix}\cdot\begin{pmatrix}u\\v\\w\end{pmatrix} = u+v\omega+w\omega^*=0$$

$$\begin{pmatrix}1\\\omega^*\\\omega\end{pmatrix}\cdot\begin{pmatrix}u\\v\\w\end{pmatrix} = u+v\omega^*+w\omega=0$$

$u+v+w=0 \rightarrow w=-u-v$

$u+v\omega+w\omega^*=0 \rightarrow u+v\omega-(u+v)\omega = u+\cancel{v\omega-v\omega}-u\omega=0$

$u+v\omega^*+w\omega=0$

$\downarrow$

$u-u\omega=0$

$u=u\omega$

$\omega\neq 0$

$\therefore u=0$

$\&\ w=\cancel{-0}-v$

$w=-v$

$2u + v(1+\omega) + w(1+\omega^*)=0$

$v(1+\omega) = -w(1+\omega^*)$

$v=-w$

$$\begin{pmatrix}u\\v\\w\end{pmatrix}=\vec{0} \quad \therefore\ \underline{v \,\&\, w=0} \text{ b/c } (1+\omega^*)\neq(1+\omega)$$

Paul Len

In an n-dimensional space there can only exist n orthogonal basis vectors. If there is any more than n orthogonal vectors, one of the vectors will have a non-zero dot product w/ another basis vector, making it non-orthogonal

$$\text{basis} \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}$$

$1, 2 \ldots n$