Class 3 (01/18/24)

Electric Potential

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

Coulomb force

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} dV'$$

Electric Field

The Coulomb force has the wonderful property that it is a force acting over distance. The two electrical charges do not have to be in physical contact to experience the force. This property enables the introduction of electric fields. The advantage of working with electric fields is that an electric field is the property of a single charge. It describes the Coulomb force on electric charges when they are situated in the space around an electric charge.

Maxwell's equations for electrostatic fields

True electrostatic fields obey the following two fundamental relationships:

$$\vec{\nabla} \times \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

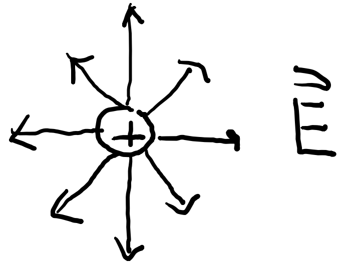
Any real electrostatic fields observed in our world obey Maxwell's equations for electrostatic fields !

Any electric fields which do not obey Maxwell's equations for electrostatic fields are not electrostatic fields !

Maxwell’s equation for electrostatic fields are the mathematical description of electrostatic (and magnetostatic) E and B-field lines as observed and measured in nature.

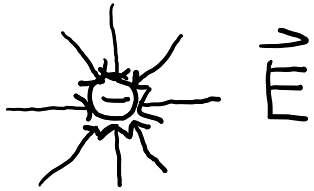
$$\vec{\nabla} \cdot \vec{f} = C$$

A positive constant (C > 0) as the result of calculating the divergence of a vector **f,** describes field lines where **f** spreads out from one point in space (the “source c”), like electric field lines point radially away from a positive charge q.



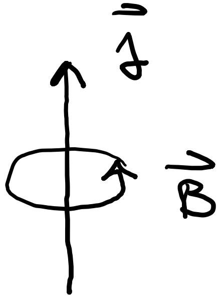
$$\vec{\nabla} \cdot \vec{f} = -C$$

A negative constant (-C < 0) as the result of the divergence of a vector f, describes field lines where f converges to one point in space (the “drain c”), like electric field lines radially converge onto a negative charge q.



$$\vec{\nabla} \times \vec{f} = \vec{a}$$

A vector **a** as the result of the curl of a vector **f** describes field lines where f forms closed loops around **a**. The plane defined by the closed loop field lines is perpendicular to **a**, like magnetic field lines **B** in the space around a conducting wire carrying a current density **j**.



Electric potential V is a mathematical concept introduced to electromagnetic theory with the objective of simplifying calculations of electric fields **E**.

Let's look at the curl of the electric field **E** and recall the following piece of very useful math:

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times (\vec{\nabla} F) = 0$$

Since the curl of **E** is zero, math offers the great opportunity to introduce a scalar function V(r) (named electric potential) and the electric field **E** as the negative gradient of V.

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

$$\vec{\nabla}_{\vec{r}} \times \vec{E}(\vec{r}) = \vec{\nabla}_{\vec{r}} \times \left(-\vec{\nabla}_{\vec{r}} V(\vec{r})\right) = 0$$

Useful math previously introduced:

$$-\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

Let’s rewrite **E**(**r**) as:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\vec{r}\,')\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dV' = \frac{1}{4\pi\varepsilon_0}\int_{V'} \rho(\vec{r}\,')\left[-\vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r}-\vec{r}\,'|}\right] dV'$$

When we pull the differential operator out of the integral (which is permitted because the differential operator acts on r while the integral is over r’, and find that the generalized description of V(r) is:

$$\vec{E}(\vec{r}) = -\vec{\nabla}_{\vec{r}}\left[\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV'\right]$$

$$= -\vec{\nabla}_{\vec{r}}\, V$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV'$$

Useful math: Divergence theorem

$$\int_{A(s)} (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = \oint \vec{E} \cdot d\vec{s}$$

$$\vec{\nabla} \times \vec{E} = 0 \sim \oint \vec{E} \cdot d\vec{s} = 0 \rightarrow \int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{s}$$ path independent line integral

A definition for electric potential V when the electric field E is given:

$$V(\vec{r}) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{s} = -\left[V(\vec{r}_b) - V(\vec{r}_a)\right]$$

Electrostatic potential energy:

$$U = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}_c \cdot d\vec{s} = -\int_{\vec{r}_a}^{\vec{r}_b} q\vec{E} \cdot d\vec{s} = q\left[V(\vec{r}_b) - V(\vec{r}_a)\right]$$

Boundary Conditions

An important aspect of in electromagnetic theory are boundary conditions for fields and potentials at the interface between different materials (media). In electrostatics boundary conditions define how the electric field and the electric potential changes when the electric field or the electric potential cross a boundary between two different materials.

Typical boundaries:
dielectric (electrically insulating) / metal (electrically conducting)
real world examples:
power cord connector: metallic cylindrical rode has a boundary with air

dielectric / dielectric:
real word examples:
Window: the glass (dielectric) has two boundaries with air
Swimming pool: water has one boundary with air

Let's explore boundary conditions in electrostatics by looking at the electric field inside and outside of a conducting shell of radius R with a uniform surface charge density σ when the shell sits in air:

$$E(r) = 0 \quad r < R$$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma \cdot 4\pi R^2}{r^2} = \frac{\sigma}{\varepsilon_0}\frac{R^2}{r^2} \quad r \geq R$$

$\vec{E} \perp$ at surface of shell

$$E(r=R) = \frac{\sigma}{\varepsilon_0}$$

In-class problem 2.11

The electric field E (r) inside the shell is zero, the magnitude of the electric field E(r=R) at the boundary to air is finite. Therefore, the electric field is discontinuous at the boundary. The magnitude of the discontinuity is described by the value of the electric field at the boundary. Another important observation is the fact that the electric field at the conducting boundary is perpendicular to the conducting surface this is a fundamental property of electrostatic fields at the surface of conductors.

Let's make an educated guess of the electric potent V(r) of a shell with surface charge density σ:

$$E(r) = 0 \quad r < R$$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma \cdot 4\pi R^2}{r^2} = \frac{\sigma}{\varepsilon_0}\frac{R^2}{r^2} \quad r \geq R$$

$$E(r=R) = \frac{\sigma}{\varepsilon_0}$$

$$V_{inside} = V_{outside} \quad r = R$$

$$const = \frac{Q}{4\pi\varepsilon_0 R}$$

$$V(\vec{r}) = const.$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

fulfills

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) = 0$$

for $r < R$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) = -\vec{\nabla}\left(\frac{Q}{4\pi\varepsilon_0 r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

for $r \geq R$

Boundary conditions at electrically charged surfaces

$E_{\perp \, outside} - E_{\perp \, inside} = \frac{\sigma}{\varepsilon_0}$ discontinuous

$E_{\parallel \, outside} = E_{\parallel \, inside}$ continuous

$\vec{\nabla} V_{outside} - \vec{\nabla} V_{inside} \rightarrow \sigma/\varepsilon_0$ discontinuous

$V_{outside} = V_{inside}$ continuous

Derivation of Poisson's & Laplace's equations for the electric potential V

$$\vec{\nabla} \times \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla} \cdot \vec{E} = \left(\vec{\nabla} \cdot \left(-\vec{\nabla} V\right)\right) = -\vec{\nabla}^2 V = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$$

Poisson's equation

$$\vec{\nabla}^2 V = 0$$

Laplace's equation