1. Consider a hypothetical Fermi system with N particles in volume V and with the density of states $g(\varepsilon)$ given by

$$g(\varepsilon) = \begin{cases} 0 & if \quad \varepsilon < 0 \\ \alpha V & if \quad \varepsilon > 0 \end{cases},$$

where α is a constant.

(a) Find the Fermi energy ε_F and the internal energy of the system at zero temperature.

(b) Using the Sommerfeld expansion, find the chemical potential, the internal energy, and the specific heat at low temperatures.

2. The dimensionality plays a fundamental role in phase transitions and critical phenomena. Its importance also shows in the peculiar phenomenon of Bose-Einstein condensation. Show that there is *no* Bose-Einstein condensation in the *one or two-dimensional* ideal Bose gas with the dispersion relation $\varepsilon(p) = p^2/2m$ at any nonzero temperature. (At $T = 0$, of course, all particles are in the ground state.)

3. Find the behavior of the *isotherm compressibility*, κ_T, in as the critical temperature is approached *from above*, i.e., in the $T \to T_c + 0$ limit. This means that you should obtain an expression for κ_T as a function of $(T - T_c)$ in the vicinity of the transition temperature. You should review the notes posted on LMS regarding the behavior of the Bose gas just above T_c. In particular, use

$$\mu(T) \approx -kT\left(\frac{3\zeta(3/2)}{4\sqrt{\pi}}\right)^2\left(\frac{T-T_c}{T_c}\right)^2 \approx -kT_c\left(\frac{3\zeta(3/2)}{4\sqrt{\pi}}\right)^2\left(\frac{T-T_c}{T_c}\right)^2$$

and

$$f_{3/2}^{-}(e^{-\alpha}) \approx \zeta(3/2) - 2\sqrt{\pi}\alpha^{1/2}, \text{ where } \alpha = -\frac{\mu}{kT}.$$

Also, you will need to employ the basic thermodynamic relation

$$-\left(\frac{\partial V}{\partial P}\right)_{N,T} = \frac{V^2}{N^2}\left(\frac{\partial N}{\partial \mu}\right)_{V,T}.$$

4. Consider the simple model for a one-dimensional solid consisting of N atoms of identical mass m along a chain. Atoms are connected only with nearest-neighbor atoms by "springs" with spring constant κ. The equilibrium separation between atoms (i.e., the lattice constant) is a. The springs are relaxed in equilibrium and we allow only for

longitudinal oscillations. For simplicity, use periodic boundary conditions, $u_{j+N} = u_j$, where u_j is the displacement of the j th atom, measured from its equilibrium position.

(a) Show that the equation of motion for the displacements is

$$m\ddot{u}_j = \kappa(u_{j+1} + u_{j-1} - 2u_j),\ j = 1,2,\ldots,N.$$

(b) Solve the above set of equation by means of complex Fourier series, yielding the normal modes and the spectrum. In particular, show that the frequency – wave-number dispersion relation is given by

$$\omega(k) = 2\sqrt{\frac{\kappa}{m}}\left|\sin\left(\frac{ka}{2}\right)\right|,$$

where $k = \frac{2\pi}{Na}n, \quad n = -\frac{N}{2},\ldots,\frac{N}{2}$ (assumed even N for simplicity).

(c) From the above dispersion relation it follows that for small k values we can use $\omega(k) = ck$, where c is the speed of longitudinal vibrations in this solid. Obtain the low-temperature behavior of the specific heat of this *one-dimensional* solid of size $L = Na$ in the Debye approximation.

5. Obtain an *estimate* for the

(a) *Fermi energy and Fermi temperature in copper* (assume one conduction or "free" electron per atom). For the mass density of copper use $\rho = 9\frac{g}{cm^3}$, and the atomic mass is 63.5g/mol.

(b) critical temperature for the Bose-Einstein condensation in an ideal He^4 "gas" with density $\rho = 0.145\frac{g}{cm^3}$. The atomic mass is 4g/mol.

(c) *Debye temperature in copper* (use the parameters given in (a) and the effective sound velocity $c \approx 4000\frac{m}{s}$).