

HW 10

Problem 1 N9.3

The contribution to the anomaly by a given irreducible representation is determined by the trace of the product of a generator and the anti-commutator of two generators, namely, $\text{Tr} T^A \{T^B T^C\}$. Here T^A denotes a generator of the gauge group G . Show that for $G = \text{SU}(5)$, the anomaly cancels between the 5^* and the 10. Note that for $\text{SU}(N)$, we can, with no loss of generality, take A, B , and C to be the same, so that the anomaly is determined by the trace of a generator cubed (namely, $\text{Tr} T^3$). It is rare that we get something cubed in physics, and so any cancellation between irreducible representations can hardly be accidental.

Easiest to use diagonal generator.

$$T = \text{diag} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{2} \right)$$

Taking this diagonal generator, we evaluate the trace of the generator cubed in the 5^* dimensional representation.

All entries are real: (multiply by 6)

$$T^* = \text{diag}(2, 2, 2, -3, -3)$$

Cubing this matrix:

$$T^{*3} = \text{diag}(8, 8, 8, -27, -27)$$

Taking the trace:

$$\text{Tr}(T^{*3}) = -30$$

Now for the 10-dimensional representation:

Basis for 10-d representation:

$$(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)$$

Converting the 5-d diagonal matrix to a 10-d representation

$$T_{10\text{d}} = \text{diag}(d_1 + d_2, d_1 + d_3, d_1 + d_4, d_1 + d_5, d_2 + d_3, d_2 + d_4, d_2 + d_5, d_3 + d_4, d_3 + d_5, d_4 + d_5)$$

$$T_{10\text{d}} = \text{diag}(4, 4, -1, -1, 4, -1, -1, -1, -1, -6)$$

Cubing this diagonal matrix:

$$T_{10\text{d}}^3 = \text{diag}(64, 64, -1, -1, 64, -1, -1, -1, -1, -216)$$

Taking the trace:

$$\text{Tr} T_{10\text{d}}^3 = 3(64) - 222 = 192 - 222 = -30$$

$$30 - (-30) = 0$$

Problem 3 N9.3

Work out how the 3-indexed antisymmetric 10 dimensional tensor in $SO(10)$ decomposes on restriction to $SO(4) \otimes SO(6)$.

- 3 indexed antisymmetric 10-d tensor has 120 elements.
- Symmetric tensor rep of $SO(6)$ has 20 elements $\frac{1}{2}(6)(7) - 1$
- Adjoint rep of $SO(6)$ has 15 elements $\frac{1}{2}(6)(5)$
- The vector representation of $SO(6)$ has 6 (obviously)
- Trivial representation has 1
- The vector representation of $SO(4)$ has 4 elements
- The trivial representation has 1 element
- Tensor product representations (3,1) and (1,3)

Decomposing

Using the following equation (credit Nate Laposky and Hannah Turner)

$$\Lambda^3(V \oplus W) = \Lambda^3(V) \oplus (\Lambda^2(V) \otimes W) \oplus (V \otimes \Lambda^2(W)) \oplus \Lambda^3(W)$$

We can plug in our representations:

$$\Lambda^3(10) = \Lambda^3(4, 1) \oplus (\Lambda^2(4, 1) \otimes (1, 6)) \oplus ((4, 1) \otimes \Lambda^2(1, 6)) \oplus \Lambda^3(1, 6)$$

And from this we can start pulling out our representations:

We have a (4,1) representation from $\Lambda^3(4, 1)$

$$\text{From } (\Lambda^2(4, 1) \otimes (1, 6)) \text{ we get}$$

(6,6)

$$\text{From } ((4, 1) \otimes \Lambda^2(1, 6)) \text{ we get}$$

$$\Lambda^2(6) \cong 15 \rightarrow (4, 15)$$

$$\text{From } \Lambda^3(1, 6) \text{ we get the 20 dimensional representation of } SO(6)$$

(1,20)

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$$\Lambda^2(6) \cong 15 \rightarrow (4,15)$$

From $\Lambda^3(1,6)$ we get the 20 dimensional representation of $SO(6)$

$$(1,20)$$

Taking this collection of representations, we can

$$\begin{aligned} 120 &\rightarrow (20,1) \oplus (15,4) \oplus (6,6) \oplus (1,4) \\ &\rightarrow (20,1) \oplus (15,4) \oplus (6,3) \oplus (6,3) \oplus (1,4) \end{aligned}$$