Harmonic Functions, Fourier Series, and Fourier Transforms

In this section we will consider:

- periodic phenomena and their representation using harmonic functions (Fourier series).
- Representation of arbitrary non-periodic functions using harmonic functions (Fourier transforms).

- Reading on harmonic fns and Fourier series is in Ch 7 of Boas.
- Reading on Fourier transforms is in Boas, Ch. 15.4, 15.5, 15.6

Most physically interesting functions can be represented by the sum of harmonic (sin and cos) functions with various frequencies and amplitudes.

Many basic physical processes are periodic.

Many periodic processes are harmonic or can be approximated as periodic.

- Fourier series and transforms are important.
- Fourier series and transforms allow us to determine how much of what frequency is required to fabricate a certain waveform. This is important for signal processing. (Do we have a detector or amplifier with a broad enough frequency response?) It also turns up in quantum theory.

Simple Harmonic Functions

The simplest periodic processes are described by the sine or cosine function (examples: pendulum for small oscillations, tuning fork, string, sound waves, light waves, quantum wave functions)

For a process that repeats itself *f* times per second:

$$u = A\cos 2\pi f t = A\cos\omega t = \mathrm{Re}(Ae^{i\omega t})$$

For a process that repeats itself in space *1/λ* times per meter along the x direction:

$$u = A\cos\frac{2\pi}{\lambda}x = A\cos kx = \mathrm{Re}(Ae^{ikx})$$

And for a traveling wave (e.g. - sound, light):

$$u = A\cos(kx-\omega t) = \mathrm{Re}(Ae^{i(kx-\omega t)})$$

k=wavenumber=$2\pi/\lambda$; ω=angular frequency=$2\pi f$

It is frequently easier to do mathematics using the complex notation and then take the real part of the result after the calculation is complete.

Anharmonic periodic waves: the Fourier series

In general any periodic function can be represented as the sum of sine and cosine functions with wavelengths equal to the wavelength of the periodic function divided by an integer (λ/n) for a spatial wave.

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + B_m \sin mkx$$

All we need to do is find the coefficients.

To do this we will make use of some mathematical identities from calculus:

$$\int_0^{\lambda} \sin akx \cos bkx dx = 0; \qquad \int_0^{\lambda} \cos akx \cos bkx dx = \frac{\lambda}{2}\delta_{ab}$$

$$\int_0^{\lambda} \sin akx \sin bkx dx = \frac{\lambda}{2}\delta_{ab}; \qquad \text{for a, b = positive integers}$$

where $\delta_{ab}=1$ when a=b and $\delta_{ab}=0$ when $a \neq b$ (Kronicker delta fn)

To find the cosine coefficients:

Multiply both sides by cos(*nkx*) and integrate over one period or wavelength,

$$\int_0^{\lambda}\cos(nkx)\,f(x)dx=\int_0^{\lambda}\left(\cos(nkx)\frac{A_0}{2}+\sum_{m=1}^{\infty}A_m\cos(nkx)\cos(mkx)+B_m\cos nkx\sin mkx\right)$$

and using the identities above,

$$\int_0^{\lambda}\cos(nkx)\,f(x)dx=\int_0^{\lambda}\left(\cos(nkx)\frac{A_0}{2}dx\right)++\sum_{m=1}^{\infty}\left(A_m\delta_{nm}+B_m0\right)$$

so that $A_0=\frac{2}{\lambda}\int_0^{\lambda}f(x)\,dx;$ and $A_m=\frac{2}{\lambda}\int_0^{\lambda}f(x)\cos mkx\,dx$

$B_m=\frac{2}{\lambda}\int_0^{\lambda}f(x)\sin mkxdx;$ when the same is done for the sine coefficients

Example the positive square wave centered at zero:

$$f(x) = \begin{cases} 0, -L/2 < x < L/4 \\ 1, -L/4 < x < L/4 \\ 0, L/4 < x < L/2 \end{cases}$$

and repeating every L from $-\infty$ to $+\infty$

From even and odd functions, $B_m = 0$ for all m

$$A_0 = 1$$

$$A_m = \left(\frac{2}{m\pi}\right)\sin\left(\frac{m\pi}{2}\right)$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty}\left[\left(\frac{1}{m}\right)\sin\left(\frac{m\pi}{2}\right)\right]\cos\left(m\frac{2\pi}{L}x\right)$$

so that the coefficient in the [brackets] is just 1, 0, -1, 0, 1...
for m=1, 2, 3, 4, 5 ...

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[-\left(\frac{1}{2n-1}\right)(-1)^n\right]\cos\left((2n-1)\frac{2\pi}{L}x\right)$$

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\left\{\cos\left(\frac{2\pi}{L}x\right)-\frac{1}{3}\cos\left(3\frac{2\pi}{L}x\right)+\frac{1}{5}\cos\left(5\frac{2\pi}{L}x\right)-\ldots\right\}$$

see the Maple worksheet "fourier_series..." for an example of the sums of these functions to make a symmetric square wave about y=0.

Example: Periodic Square Pulse (arbitrary width)

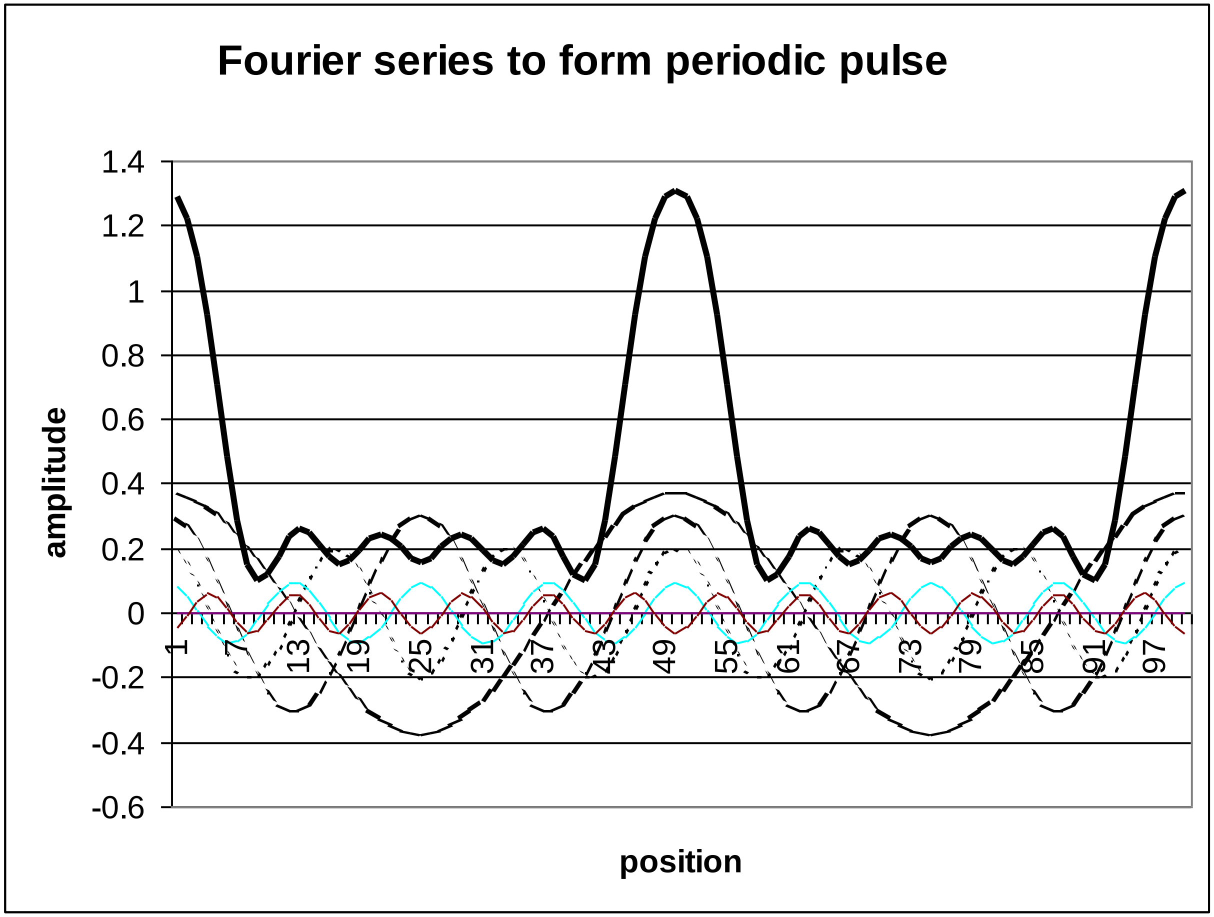
$$f(x) = \begin{cases} 0, -L/2 < x < L/a \\ 1, -L/a < x < L/a \\ 0, L/a < x < L/2 \end{cases}$$

$$B_m = 0 \text{ for all m}$$

$$A_0 = \frac{4}{a}$$

$$A_m = \left(\frac{4}{a}\right)\left(\frac{\sin\left(\frac{m2\pi}{a}\right)}{m2\pi/a}\right)$$

$$f(x) = \frac{2}{a} + \sum_{m=1}^{\infty}\left(\frac{2}{a}\right)\left(\frac{\sin\left(\frac{m2\pi}{a}\right)}{m2\pi/a}\right)\cos\left(m\frac{2\pi}{L}x\right)$$



(from an Excel worksheet)

Complex form of the Fourier Series

We discussed earlier that $\sin nx = \dfrac{e^{inx} - e^{-inx}}{2i}$ and $\cos nx = \dfrac{e^{inx} + e^{-inx}}{2}$

This suggests that we might be able to make a Fourier series representation like this:

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + c_3 e^{3ix} + c_{-3} e^{-3ix} + \ldots$$

To find c_n we multiply both sides by e^{-inx} and integrate over all x.

We find that $c_n = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

Example: The periodic square wave

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < +\pi \end{cases}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx} \cdot 0dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx} \cdot 1dx$$

$$= \frac{1}{2\pi}\frac{e^{-inx}}{-in}\bigg|_0^{\pi} = \frac{1}{-2\pi in}\left(e^{-in\pi} - 1\right) = \begin{cases} \dfrac{1}{\pi in}, & \text{n=odd} \\ 0, & \text{n=even} \end{cases}$$

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{1}{2} + \frac{1}{i\pi}\left(\frac{e^{ix}}{1} + \frac{e^{i3x}}{3} + \frac{e^{i5x}}{5} + \ldots\right) + \frac{1}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-i3x}}{-3} + \ldots\right)$$

$$= \frac{1}{2} + \frac{2}{\pi}(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots)$$

Some useful rules for convergence of Fourier series (so you can decide whether you want to use a Fourier series to represent a function)

1) If f(x) has a finite number of discontinuities in a period, then A_n and/or B_n converge like the members of the series 1/n.
2) If f(x) is continuous while its first derivative has a finite number of discontinuities, then its terms converge like $1/n^2$.
3) In general if f(x) and its derivatives are continuous up to order (m-1) while the mth derivative has a finite number of discontinuities in a period, then its terms converge like $1/n^{m+1}$

Parseval's Completeness Theorem

We can test whether we have got the most important terms in a series by comparing the sum of the coefficients with the square average of the function:

Let $f(x) = \dfrac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

then the average square of f(x) is $\langle f(x)^2 \rangle = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} [\mathrm{f(x)}]^2 \, dx$

and $\langle f(x)^2 \rangle = \left(\dfrac{a_o}{2}\right)^2 + \dfrac{1}{2}\left(\sum_{n=1}^{\infty} a_n{}^2 + \sum_{n=1}^{\infty} b_n{}^2\right)$

Nonperiodic (or finite) waves: the Fourier transform

Let us return to the short square pulse,

$$f(x)=\begin{cases}0, & -L/2<x<a\\ 1, & -a<x<a\\ 0, & a<x<L/2\end{cases}$$

$B_m = 0$ for all m

$$A_0=\frac{a}{L}$$

$$A_m=\left(\frac{a}{L}\right)\left(\frac{\sin\left(\frac{m2\pi a}{L}\right)}{m2\pi a/L}\right)$$

$$f(x)=\frac{a}{L}+\sum_{m=1}^{\infty}\left(\frac{a}{L}\right)\left(\frac{\sin\left(\frac{m2\pi a}{L}\right)}{m2\pi a/L}\right)\cos\left(m\frac{2\pi}{L}x\right)=\frac{a}{L}\left(1+\sum_{m=1}^{\infty}\left(\frac{\sin(k_m a)}{k_m a}\right)\cos k_m x\right)$$

If we allow the spacing between terms in a Fourier series to become small (for example by making L very large)), the periodic peaks move apart. The limit of zero spacing (infinite L) leads to the Fourier integral:

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(k)\cos kx\,dk + \int_0^\infty B(k)\sin kx\,dk\right]$$

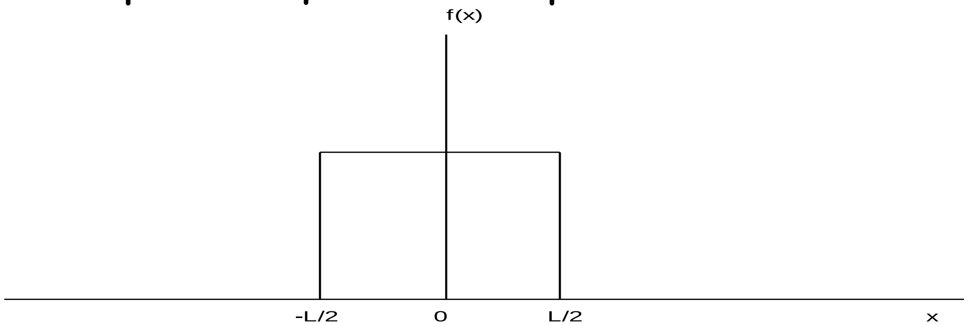
where :

$$A(k) = \int_{-\infty}^{\infty} f(x)\cos kx\,dx$$

$$B(k) = \int_{-\infty}^{\infty} f(x)\sin kx\,dx$$

A(k) and *B(k)* are the amplitudes of the sine and cosine contributions in the range from *k* to *k+dk*

Example: A square field pulse of width L and height E_0, centered at x=0.



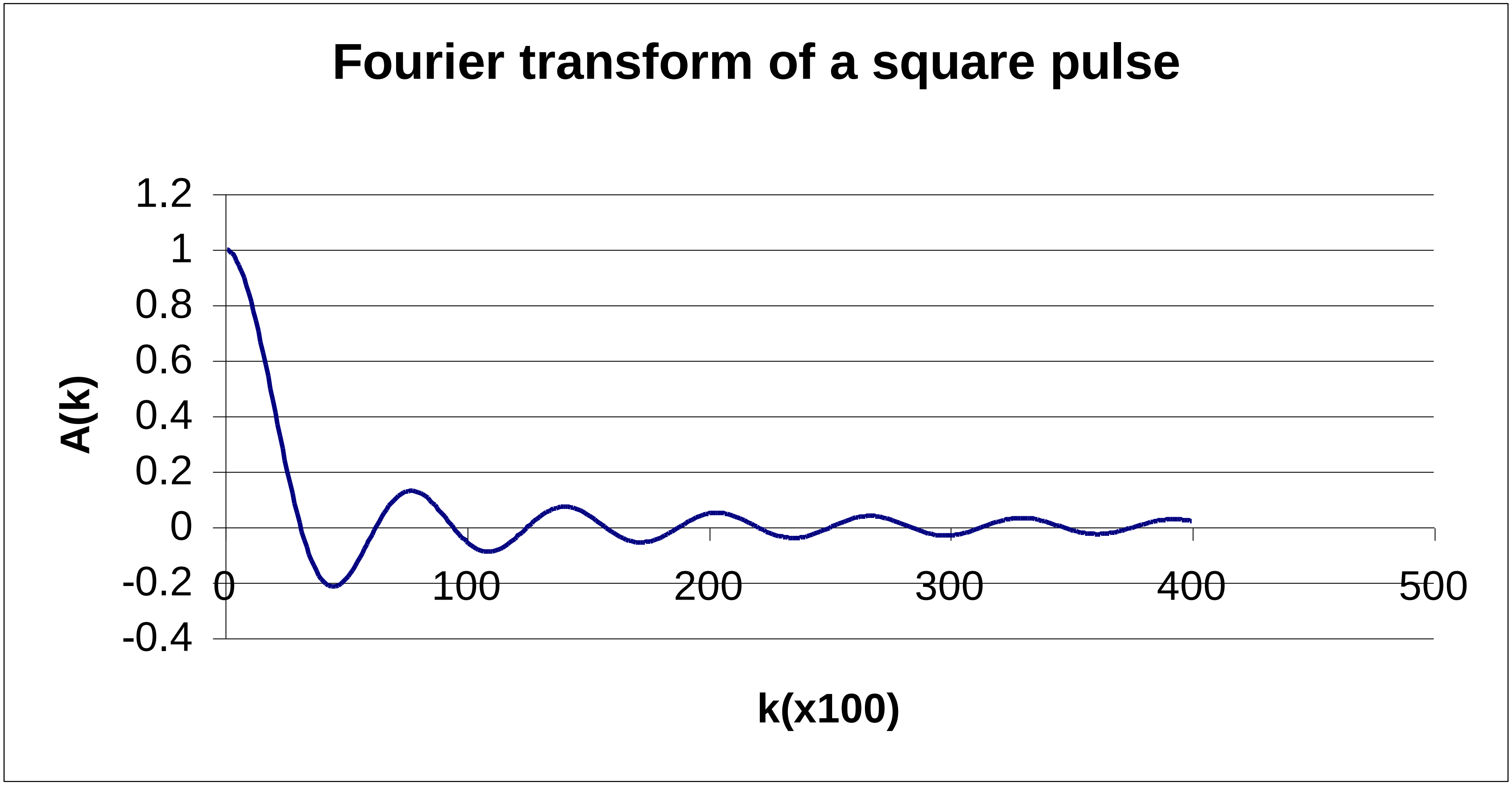
$$f(x) = \begin{cases} E_0, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$$

$$A(k) = \int_{-L/2}^{L/2} E_0 \cos kx\,dx$$

$$= \frac{E_0}{k} \sin kx \Big|_{-L/2}^{+L/2} = \frac{2E_0}{k} \sin \frac{kL}{2}$$

$$= E_0 L \frac{\sin kL/2}{kL/2}$$

(This solution is identical in form to the solution to the diffraction pattern of a monochromatic plane wave of light incident on an open slit of width L.)



Fourier transform of a square pulse
A(k)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
100
200
300
400
500
k(x100)

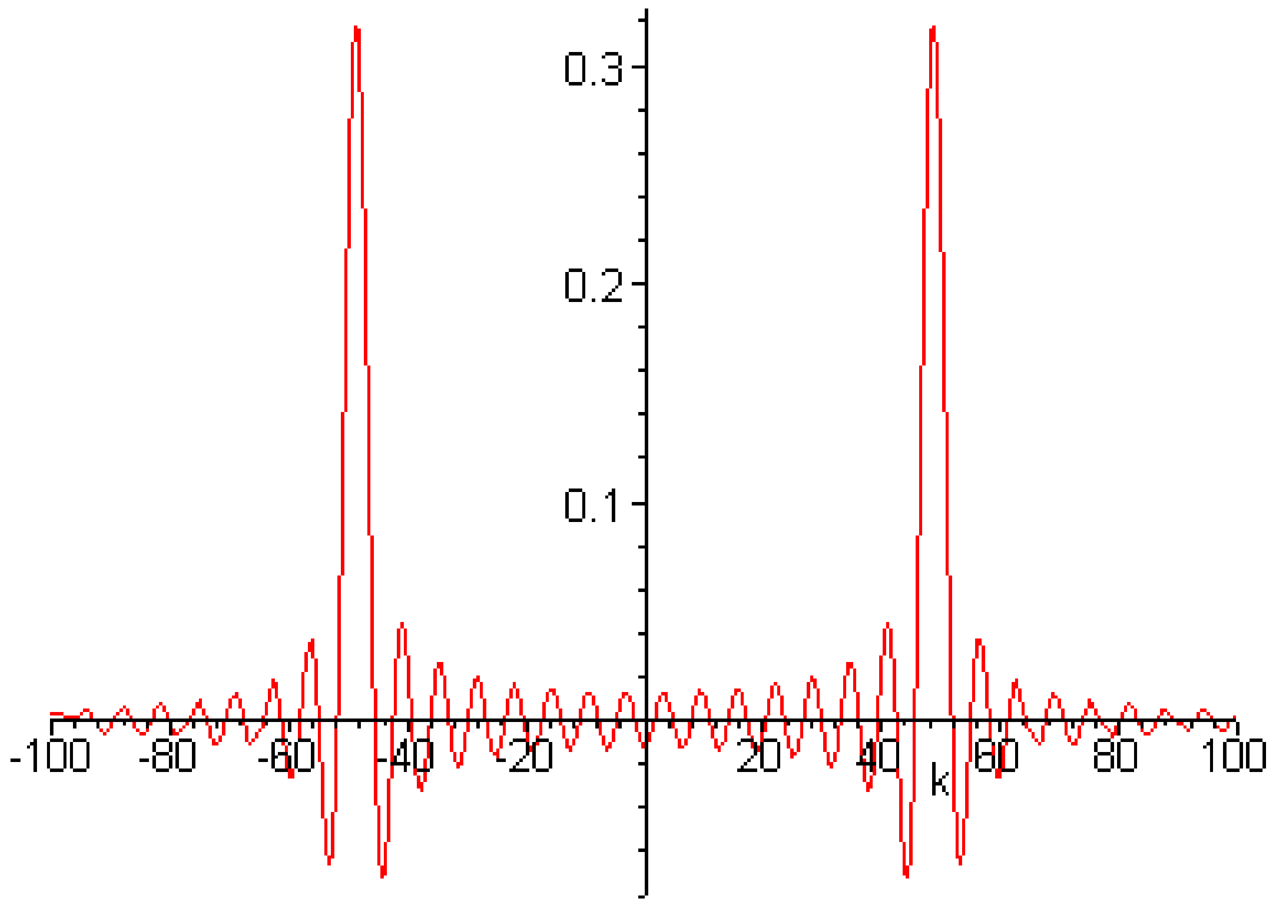
As an aside, this is what is happening in Prof. Zhang's laboratory when he uses short optical pulses to create "Terahertz" radiation.

If we transform the "wavelength-space" spectrum above back to "real-space", we recover the original waveform.

Example 2: The cosine wavetrain

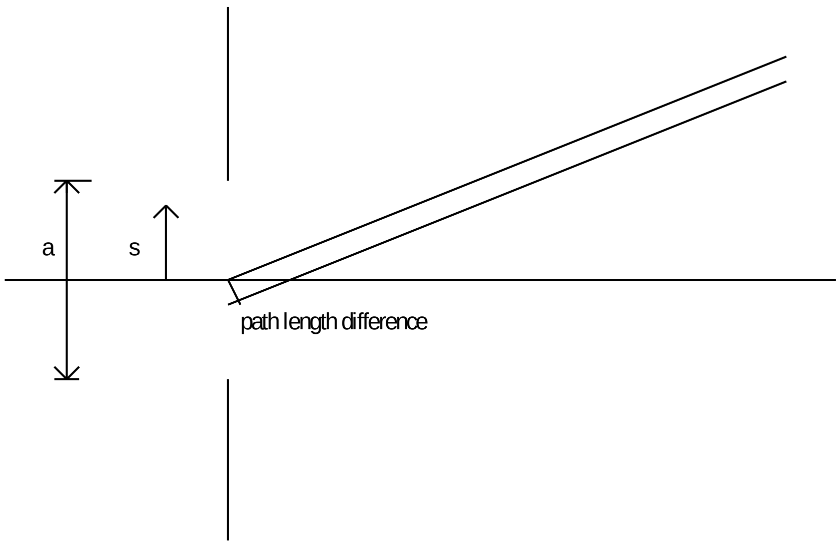
$$E(x) = \begin{array}{l} E_0 \cos k_p x \text{ for } \; -a < x < a \\ 0 \;\; \text{otherwise} \end{array}$$

$$\begin{aligned} A(k) &= \int_{-a}^{a} E_0 \cos k_p x \;\; \cos kx \;\; dx \\ &= \int_{-a}^{a} \frac{1}{2} E_0 \left[\cos(k_p + k)x + \cos(k_p - k)x \right] dx \\ &= E_0 a \left[\frac{\sin(k_p + k)a}{(k_p + k)a} + \frac{\sin(k_p - k)a}{(k_p - k)a} \right] \end{aligned}$$



An example with L=15.5 λ. Note that amplitude grows linearly with L.

Example 3: The Diffraction pattern from a slit



The path difference is $y\sin\theta$. The field at a distant point is just the sum of all the waves, which we will write as

$$dE = \mathrm{Re}(Edye^{i(kx-\omega t)})$$

$$x = x_0(1 + y\sin\theta)$$

$$E_T(k\sin\theta) = e^{-i\omega t}\int_{-\infty}^{\infty} E(y)e^{ikx_0(1+y\sin\theta)}dy = e^{i(kx_0-\omega t)}\int_{-\infty}^{\infty} E(y)e^{i(k\sin\theta)y}dy$$

$$= E_s e^{i(kz_0 - \omega t)} a \frac{\sin\beta}{\beta}$$

where

$$\beta = \frac{ak}{2}\sin\theta = \frac{n\pi a}{\lambda_o}\sin\theta$$

$$I = \frac{\varepsilon_0 c}{2}(E_s a)^2 \frac{\sin^2\beta}{\beta^2}$$

This is just the Fourier transform of the field at the plane of the aperture. Since the wavelength of the incident field is fixed, the transform variable is actually $\sin\theta$.

An application of the Fourier Transform and Distribution Functions to Quantum Theory: The Heisenberg Uncertainty Principle

Assume that we can describe the probability distribution of a particle in space by a Gaussian function.

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x/2\sigma_x^2}$$ (a "normal" distribution)

We can find the standard deviation (an effective width of the distribution) by solving:

$$SD = \sqrt{\int P(x) x^2 dx} = \sigma_x$$

What is the distribution of wavevectors ($\propto 1/\lambda$) necessary to produce this spatial distribution? (We don't care in this exercise what the absolute magnitude is, so I'm dropping extraneous terms whenever possible. I will use this example to show you the complex transform.)

$$g(k) \propto \int_{-\infty}^{\infty} e^{-x^2/2\sigma_x^2} e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx$$

$$where \qquad a = 1/2\sigma_x^2$$

We can solve this integral by completing the square.

$$g(k) \propto \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-\left(x\sqrt{a} - \frac{ik}{2\sqrt{a}}\right)^2 - k^2/4a} dx$$

$$\text{letting } \beta = x\sqrt{a} - \frac{ik}{2\sqrt{a}}$$

$$g(k) \propto \frac{1}{\sqrt{a}} e^{-k^2/4a} \int_{-\infty}^{\infty} e^{-\beta^2} d\beta = \sqrt{\frac{\pi}{a}} e^{-k^2/4a} = \sqrt{\frac{\pi}{a}} e^{-k^2\sigma_x^2/2}$$

We can rewrite this in the standard form of a Gaussian in k:

$$g(k) \propto e^{-k^2/2\sigma_k^2} \quad \text{where} \quad \sigma_k^2 = 1/\sigma_x^2$$

The result then is that $\sigma_x\sigma_k$=1 for a Gaussian pulse. You will find that the product of spatial and wavenumber widths is always equal to or greater than one. Since the deBroglie hypothesis relates wavelength to momentum, $p=h/\lambda$ we thus conclude that $\sigma_x\sigma_p >= h/2\pi$. This is a statement of the Heisenberg uncertainty principle.

We can do the same arithmetic for time and frequency, which can be translated into an uncertainty relation for energy and lifetime.

Useful trigonometric identities:

$$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A-B))$$

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\sin^2 A + \cos^2 A = 1$$

Standing waves

$$E = E_{01}(\sin(kx - \omega t + \varepsilon_1) + \sin(kx + \omega t + \varepsilon_2))$$

and using the $\sin A + \sin B$ identity:

$$\mathrm{E} = 2\mathrm{E}_{01} \sin kx \cos \omega t$$

$$I = 4I_1 \sin^2 kx$$

One way to get standing waves: reflection

Two waves of different frequencies: beats

$$E_1 = E_{01}\cos(k_1 x - \omega_1 t)$$

$$E_2 = E_{01}\cos(k_2 x - \omega_2 t)$$

$$E = 2E_{01}\cos\frac{1}{2}\left[(k_1 + k_2)x - (\omega_1 + \omega_2)t\right]$$

$$\times\cos\frac{1}{2}\left[(k_1 - k_2)x - (\omega_1 - \omega_2)t\right]$$

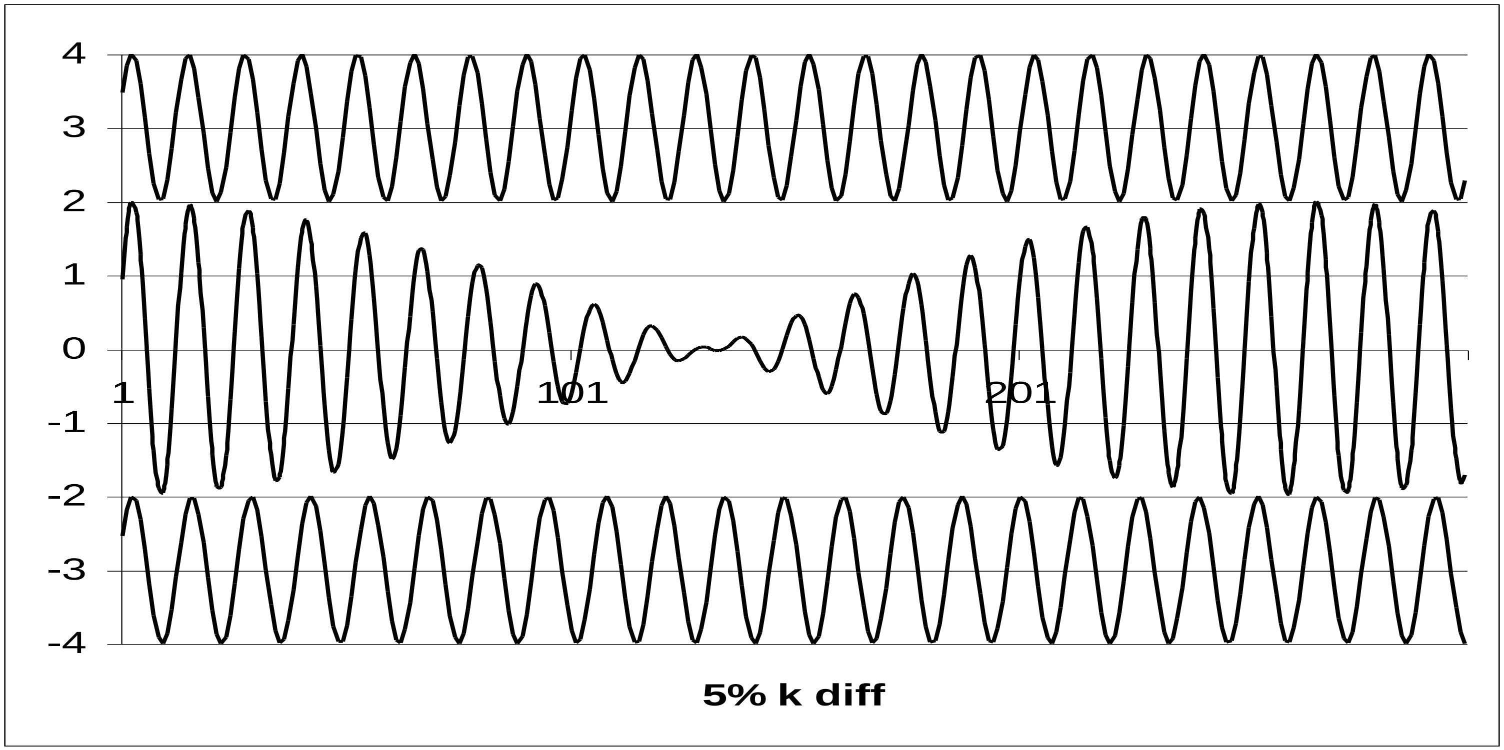
$$= E_{01}\cos\left[\bar{k}x - \bar{\omega}t\right]\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]$$

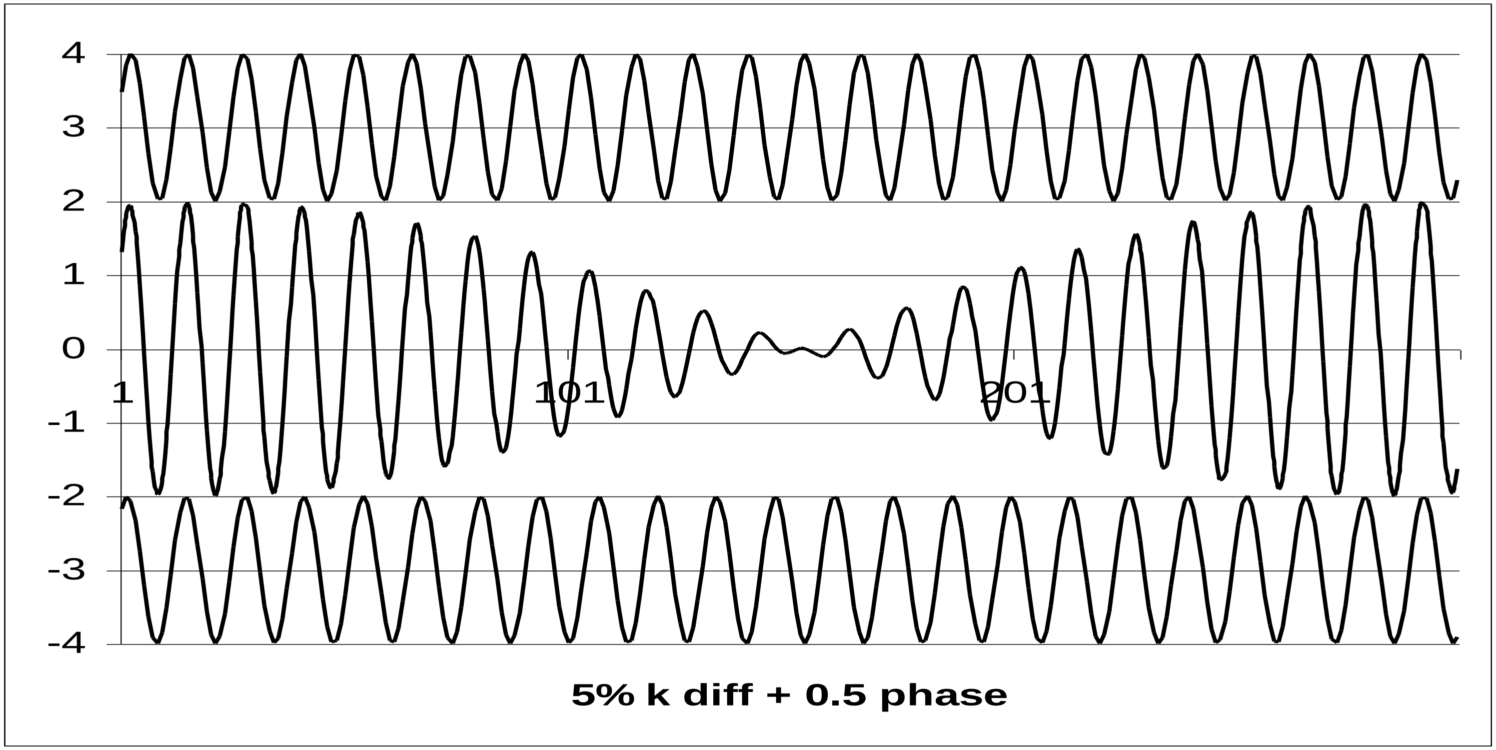
$$I = 4I_1\cos^2\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]$$

$$= 2I_1\left[1 + \cos(\Delta k - \Delta\omega)\right]$$

Group velocity

The beat envelope need not travel at the same speed as the individual waves!




4
3
2
1
0
-1
-2
-3
-4
1
101
201
5% k diff + 0.5 phase

The phase velocity of a single monochromatic wave is given by:

$v_p = \omega / k$

The beat envelope moves with velocity:

$v_e = \Delta\omega / \Delta k$

and since $\omega = k v_p$

$$v_e = v_p + k\frac{dv_p}{dk}$$

for light $v_p = c / n$ so

$$v_g \equiv v_e = \frac{c}{n}\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

group index of refraction

$n_g \equiv c / v_g$

The Delta Function

When we considered the cosine wavetrain of length *a*, we found:

$$A(k) = \int_{-a}^{a} E_0 \cos k_p x \ \ \cos kx \ \ dx$$

$$= \int_{-a}^{a} \frac{1}{2} E_0 \left[\cos(k_p + k)x + \cos(k_p - k)x \right] dx$$

$$= E_0 a \left[\frac{\sin(k_p + k)a}{(k_p + k)a} + \frac{\sin(k_p - k)a}{(k_p - k)a} \right]$$

and we noted that the amplitude at a peak of this function grows linearly with *a* as the wavetrain length grows. At the same time, the width shrinks linearly with a, so that the area under the peak remains constant. The Dirac delta δ function has the characteristic that is in infinitesimally narrow, but has area of one. For many calculations, it can be represented by the sink/k function or by a normal Gaussian.

It has the useful property that $\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = f(a)$.

If we think about Fourier transforms then we have,

$$f(x) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-ikx}\delta(k-k_0)\,dk = \frac{1}{2\pi}e^{-ik_0x}$$

$$g(k) = \delta(k-k_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik_0x}e^{ikx}\,dx$$

The delta fn selects out one frequency component.

The delta function is the transform of an infinite wavetrain of a single frequency.