Quantum Physics 1
Class 22

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Spherical Harmonics

Last Time :

$$\left(\frac{P_r^2}{2M_0} + \frac{\hat{L}^2}{2M_0 r^2}\right)\Psi(r,\theta,\phi) = (E - V(r))\,\Psi(r,\theta,\phi)$$

where $\Psi(r,\theta,\phi) = R(r)\underbrace{\Theta(\theta)\,\Phi(\phi)}_{Y(\theta,\phi)\,\equiv\,\text{spherical harmonics}}$

Recall :

① $\begin{cases} \hat{L}^2\, Y_{\ell m}(\theta,\phi) = \ell(\ell+1)\hbar^2\, Y_{\ell m}(\theta,\phi) \\ \hat{L}_z\, Y_{\ell m}(\theta,\phi) = m\hbar\, Y_{\ell m}(\theta,\phi), \quad \text{for } \Phi(\phi): \end{cases}$

$$L_z^2\,\Phi(\phi) = m^2\hbar^2\,\Phi(\phi)$$

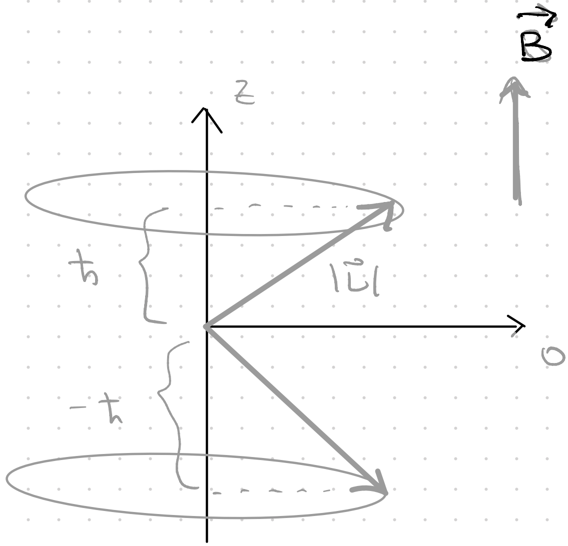
② $\hat{L}$: total angular momentum

NB : $[\hat{L}^2, \hat{L}_z] = 0$

w/ $\ell = 0, 1, 2, \ldots$

& $m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$

~~+~~ each ℓ has matching m value(s) eg) Graphically!



if $l = 1$

$|\vec{L}| = \sqrt{l(l+1)}\,\hbar = \sqrt{2}\,\hbar$

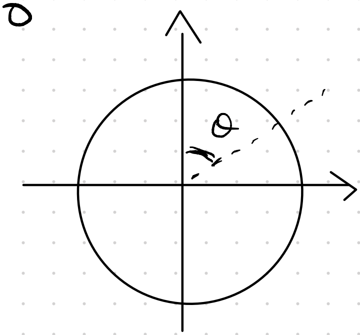
w/ $m = 0, \pm 1$

eg) if $l = 0$
then $m = 0$

then eigenstate : $Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}}$

Now, $Y^* Y$ for $l = 0$, $m = 0$

$$Y_{l,m}(\theta, \phi) = \Theta(\theta)\underbrace{\Phi(\phi)}_{e^{im\phi}}$$



Now if $l = 1$, $m = -1, 0, +1$

Spherical Harmonics :

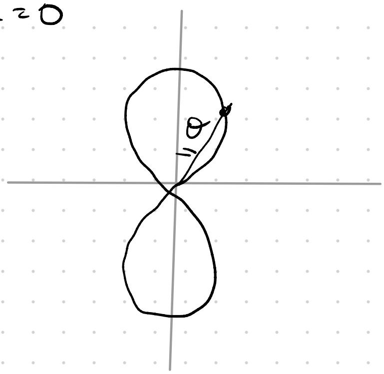
$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

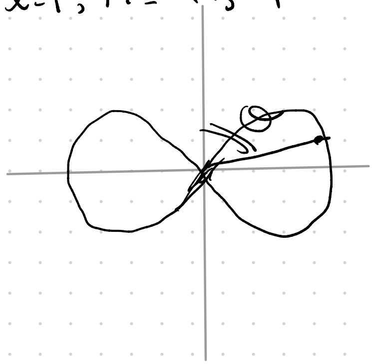
$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

** There is a three-fold degeneracy

$l=1, m=0$



$l=1, m=+1, -1$



NB $\exists$ orthonormality condition:

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{l,m}(\theta,\phi)\, Y_{l',m'}(\theta,\phi)\, d\Omega = \delta_{ll'}\delta_{mm'}$$

Recall: $d\Omega = \sin\theta \, d\theta \, d\phi$
SOLID ANGLE

$\boxed{\text{In-class 2201}}$

eg) $\hat{L}_z$ for $l=1$; $\phi_1 \equiv Y_{1,1} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\phi_2 \equiv Y_{1,0} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\phi_3 \equiv Y_{1,-1} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$(L_z) \Rightarrow \begin{pmatrix} \phi_1 \hat{L}_z \phi_1 & \phi_1 \hat{L}_z \phi_2 & \phi_1 \hat{L}_z \phi_3 \\ \phi_2 \hat{L}_z \phi_1 & \phi_2 \hat{L}_z \phi_2 & \phi_2 \hat{L}_z \phi_3 \\ \phi_3 \hat{L}_z \phi_1 & \phi_3 \hat{L}_z \phi_2 & \phi_3 \hat{L}_z \phi_3 \end{pmatrix}$$

$$\left\{ \text{eg)} \ \phi_3 \hat{L}_z \phi_3 = \int Y_{1,-1}^* \hat{L}_z Y_{1,1} \, d\Omega \right\}$$

$$(L_z) \Rightarrow \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boxed{\text{In-class } 22.2}$$

Recall: $\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$

$$L_z = x p_y - y p_x,$$

$$L_x = y p_z - z p_y,$$

$$L_y = z p_x - x p_z$$

In-class 22.3., 22.4