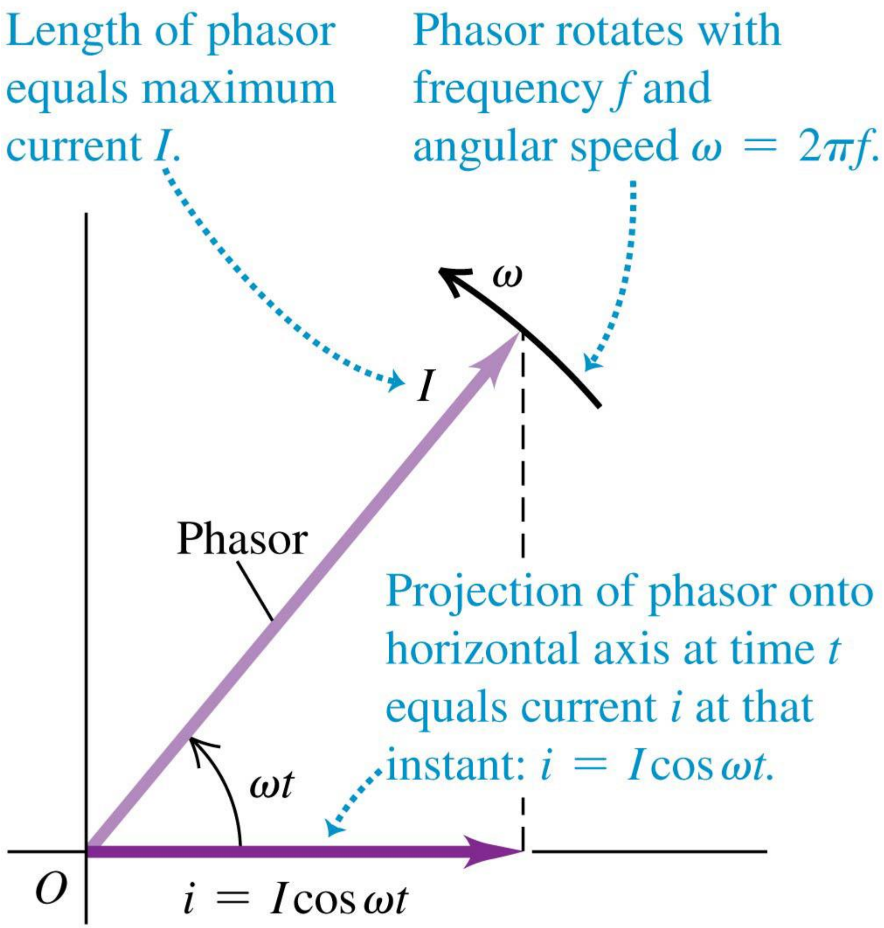
Quantum Physics I

Notes-2B

Phasors

Complex numbers

Phasor representation of a harmonic quantity



A harmonic quantity $v = V_0 \cos(\omega t+\varphi)$ can be represented by a rotating vector known as a phasor using the following conventions:

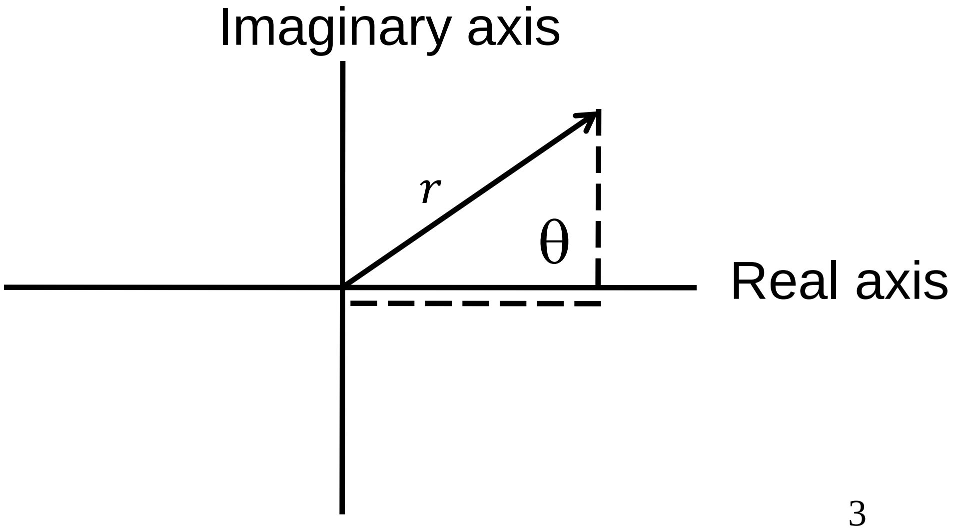
1. Phasors rotate in the counterclockwise direction with angular speed ω
2. The length of each phasor is proportional to the ac quantity amplitude
3. The projection of the phasor on the horizontal axis gives the instantaneous value of the ac quantity.

Phasors represented by complex numbers

A complex number includes both real and imaginary parts: $\tilde{z} = a + jb$

A complex number can also be written as an exponential using Euler's relation.

$$\tilde{z} = x + jy$$
$$= r(\cos\theta + j\sin\theta) = re^{j\theta}$$
$$\text{with } r = \sqrt{\vec{r} \bullet \vec{r}} = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$



$$j \equiv \sqrt{-1} = e^{\frac{j\pi}{2}}$$

The phase effect of multiplying a harmonic function by *j*$= \sqrt{-1}$

$$\tilde{A}(t) = jae^{j\omega t} = ae^{j\frac{\pi}{2}}e^{j\omega t} = ae^{j\left(\omega t+\frac{\pi}{2}\right)}$$

Remember:

$$\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right) \quad \sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

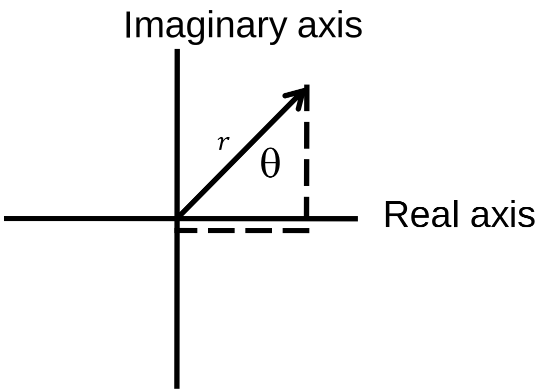
j increases the phase by $\pi/2$
(rotates the phasor 90º ccw)

Basic complex arithmetic

$$If\ z_1 = A_1 e^{i\varphi}; \quad z_2 = A_2 e^{i(\varphi+\delta)}$$
$$z_1 + z_2 = e^{i\varphi}\left(A_1 + A_2 e^{i\delta}\right)$$
$$= e^{i\varphi}[(A_1 + A_2\cos\delta) + iA_2\sin\delta]$$
$$z_1 + z_2 = e^{i\varphi}Me^{i\beta} = Me^{i(\beta+\varphi)}$$

where:

$$M = \sqrt{(A_1 + A_2\cos\delta)^2 + (A_2\sin\delta)^2};$$
$$\beta = \tan^{-1}\left(\frac{(A_2\sin\delta)}{(A_1 + A_2\cos\delta)}\right)$$



$$\text{If } \mathrm{z}_1 = r_1\mathrm{e}^{\mathrm{j}\theta_1} \text{ and } z_2 = r_2\mathrm{e}^{\mathrm{j}\theta_2}$$
$$\mathrm{z}_1\mathrm{z}_2 = r_1\mathrm{e}^{\mathrm{j}\theta_1}r_2\mathrm{e}^{\mathrm{j}\theta_2} = r_1 r_2\mathrm{e}^{\mathrm{j}(\theta_2+\theta_2)}$$
$$\frac{\mathrm{z}_1}{\mathrm{z}_2} = r_1\mathrm{e}^{\mathrm{j}\theta_1}\left(\frac{1}{r_2}\right)\mathrm{e}^{-\mathrm{j}\theta_2} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{j}(\theta_2-\theta_2)}$$

See the math appendix in Y&F or Complex Numbers in Schaum's

Complex Conjugate

- Complex Conjugate – wherever a *j* appears in a complex number, negate it.

  $(a + jb)^* = a - jb$ and $\left(Re^{j\theta}\right)^* = Re^{-j\theta}$
- Multiplying by the complex conjugate always results in a real positive number:

$$(a + jb)(a - jb) = a^2 + b^2$$

- Multiplying by the complex conjugate and taking the square root yields the magnitude of a complex number.

Manipulating complex ratios (1)

It is frequently useful to take a complex ratio, like $\frac{a+jb}{c+jd}$ and separate it into real and imaginary components.

This allows us to rapidly determine how much of the complex number is "in-phase" and "out-of-phase" with the driving signal.

To do this, multiply top and bottom by the complex conjugate of the bottom.

$$\frac{a+jb}{c+jd}\left(\frac{c-jd}{c-jd}\right)=\frac{(ac+bd)+j(-ad+bc)}{c^2+d^2}$$

$$=\left(\frac{ac+bd}{c^2+d^2}\right)+j\left(\frac{-ad+bc}{c^2+d^2}\right)$$

Manipulating complex ratios (2)

It is also useful to find the magnitude of a complex ratio.
To do this, multiply by the complex conjugate and take the square root.

$$A = \frac{a + jb}{c + jd}$$

$$|A| = \sqrt{AA^*} = \sqrt{\left(\frac{a + jb}{c + jd}\right)\left(\frac{a - jb}{c - jd}\right)} = \sqrt{\frac{a^2 + b^2}{c^2 + d^2}}$$