Lecture 17 - Transient Circuits

How does the current or potential in a circuit respond when a source is switched in or out of the circuit?

Let's start with the relationships between electromotive force, current, and charge from the past few classes:

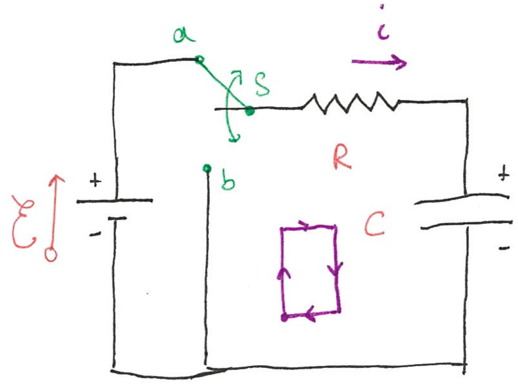
$$EMF_L = -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$$

$$EMF_R = iR = R\frac{dq}{dt}$$

$$EMF_C = \frac{1}{C}\int i\,dt = \frac{q_c}{C}$$

Then let's put these elements into circuits and switch switches to see how they behave ...

RC circuit analysis



1) Start with the switch in position a
2) Write the potential for each element.
3) Write the loop law paying attention to signs.

$$\Rightarrow \begin{cases} EMF_R = IR = \dfrac{dq}{dt} R \\[2ex] EMF_c = \dfrac{q}{C} \end{cases}$$

Loop law starting in lower left corner and going clockwise:

$$\mathcal{E} - iR - \frac{q}{C} = 0$$

$$\mathcal{E}C - RC\frac{dq}{dt} - q = 0$$

$$\frac{dq}{\mathcal{E}C - q} = \frac{dt}{RC}$$

$$\int_{Q_i}^{Q_f} \frac{dq}{\mathcal{E}C - q} = \int_{t_1}^{t_2} \frac{dt}{RC} = \frac{t_2 - t_1}{RC}$$

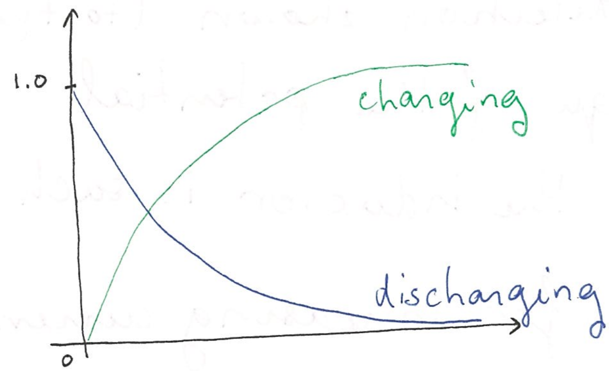
Letting $u = \mathcal{E}C - q$ so $dq = -du$

$$-\int_{\mathcal{E}C - Q_i}^{\mathcal{E}C - Q_f} \frac{du}{u} = -\ln\left(\frac{\mathcal{E}C - Q_f}{\mathcal{E}C - Q_i}\right) = \frac{t_2 - t_1}{RC}$$

Now we can introduce initial conditions:
For instance, uncharged capacitor at $t = 0$ $(Q_i = 0)$

$$\Rightarrow -\ln\left(\frac{\mathcal{E}C - Q(t)}{\mathcal{E}C}\right) = \frac{t}{RC} \Rightarrow \mathcal{E}C - Q(t) = \mathcal{E}C e^{-t/RC}$$

$$\Rightarrow Q(t) = \mathcal{E}C\left(1 - e^{-t/RC}\right)$$



$$Q(t) = \mathcal{E}C\left(1 - e^{-t/RC}\right)$$

Does this make sense ?

- At infinite time, $Q(\infty) = Q_{max} = \mathcal{E}$
- At zero time : $Q(0) = 0$

Now, the switch is moved to position b after the capacitor is charged. Using the same loop, the new loop equation is : $- IR - \frac{q}{C} = 0$ or $\frac{dq}{dt} + \frac{q}{RC} = 0$

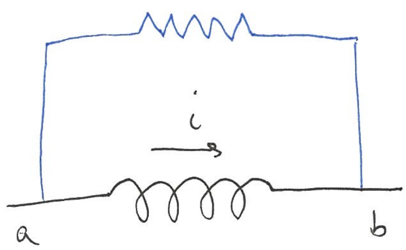
Rearranging: $\frac{dq}{dt} = -\frac{1}{RC}\,dt \Rightarrow q(t) - q(0) = -\frac{t}{RC}$

So finally, $Q(t) = Q(0)\,e^{-t/RC}$

On the sign of the potential difference across an inductor

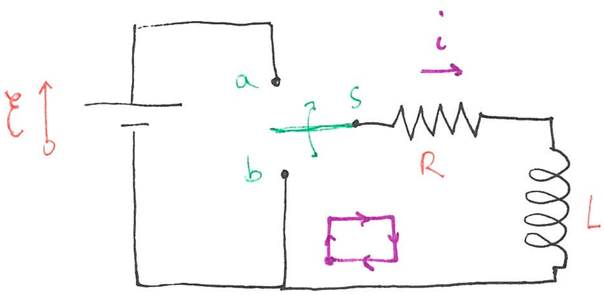
There is a potential difference across an inductor only when the current is changing.

The sign of the potential drop is such that the inductor can be thought of as maintaining the current at its previous value in the external circuit. (As though it is attempting to drive current through an external resistor.)



For i in the direction shown (to the right) the sign of the potential change across the inductor is such that: $V_a - V_b = +\frac{di}{dt}$ (positive for increasing current and negative for decreasing current).

RL circuit analysis



With the switch in the a position so that current flows clockwise around the loop:

$$\mathcal{E} + V_R + V_L = 0$$

Starting in the lower left corner and going clockwise:

$$+\mathcal{E} - IR - L\frac{dI}{dt} = 0 \iff \frac{dI}{-I + \frac{\mathcal{E}}{R}} = \frac{R}{L}\,dt$$

$$\int_{I_i}^{I_f} \frac{dI}{I - \mathcal{E}/R} = \int_{t_i}^{t_f} \frac{R}{L}\,dt$$

Letting $u = I - \mathcal{E}/R$,

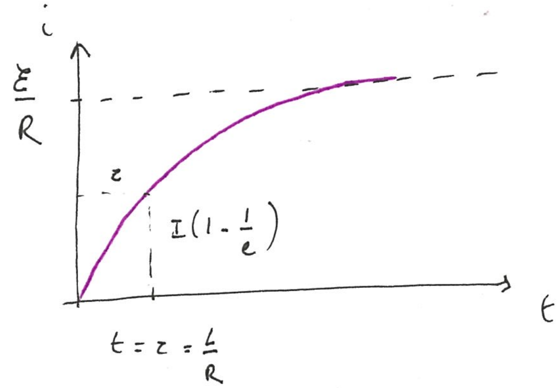
$$\int_{I_i - \frac{\mathcal{E}}{R}}^{I_f - \frac{\mathcal{E}}{R}} \frac{du}{u} = -\frac{(t_f - t_i)}{L/R} = \ln\left(\frac{I_f - \mathcal{E}/R}{I_i - \mathcal{E}/R}\right)$$

Integrating:

$$\frac{I_f - \mathcal{E}/R}{I_i - \mathcal{E}/R} = e^{-(t_f - t_i)/L/R}$$

Assuming that the current is zero just before the switch is closed: $I_i = 0$ at $t = 0$, so

$$\frac{I(t) - \mathcal{E}/R}{-\mathcal{E}/R} = e^{-t/L/R}$$

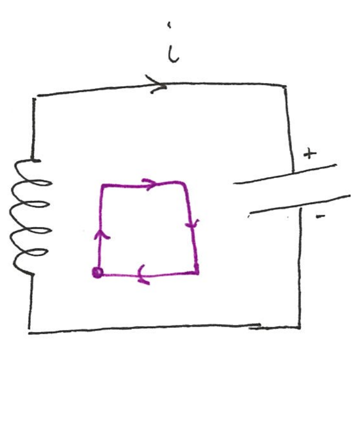


$$\Rightarrow \quad I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/L/R}\right)$$

$$EMF_L = -L\frac{dI}{dt} = -\mathcal{E}\, e^{-t/L/R}$$

This says that initially, all the battery voltage is dropped across the inductor (no current) and finally, all the voltage is dropped across the resistor.

LC circuit analysis



Assuming:

- current is flowing clockwise,
- starting our loop in lower left corner,
- taking the upper plate of the capacitor as positive, and
- following a clockwise path:

$$-L\frac{di}{dt} - \frac{q}{C} = 0 \quad . \quad \text{Substituting } i = \frac{dq}{dt} :$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

The solution to this equation is:

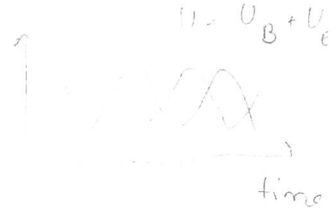
$$q(t) = Q_{max}\cos(\omega t + \phi)$$

where $\omega = \frac{1}{\sqrt{LC}}$

In PHYS 1150, you saw how a mass on a spring oscillated in a similar manner, and how one form of energy (kinetic) was exchanged for another (potential).

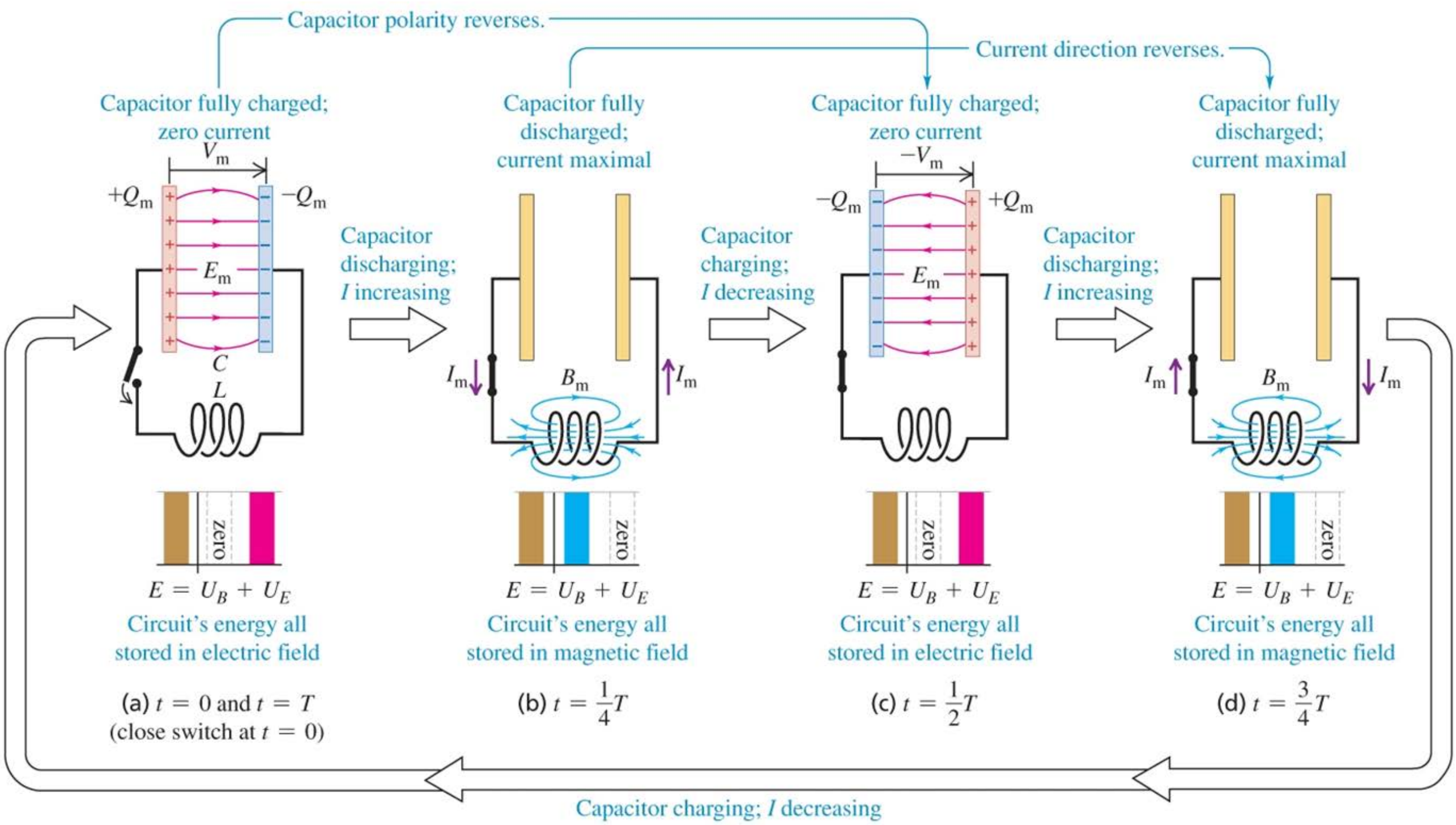
In an inductor/capacitor circuit, energy stored in the electric field of the capacitor can be alternated with energy stored in the magnetic field of the inductor.

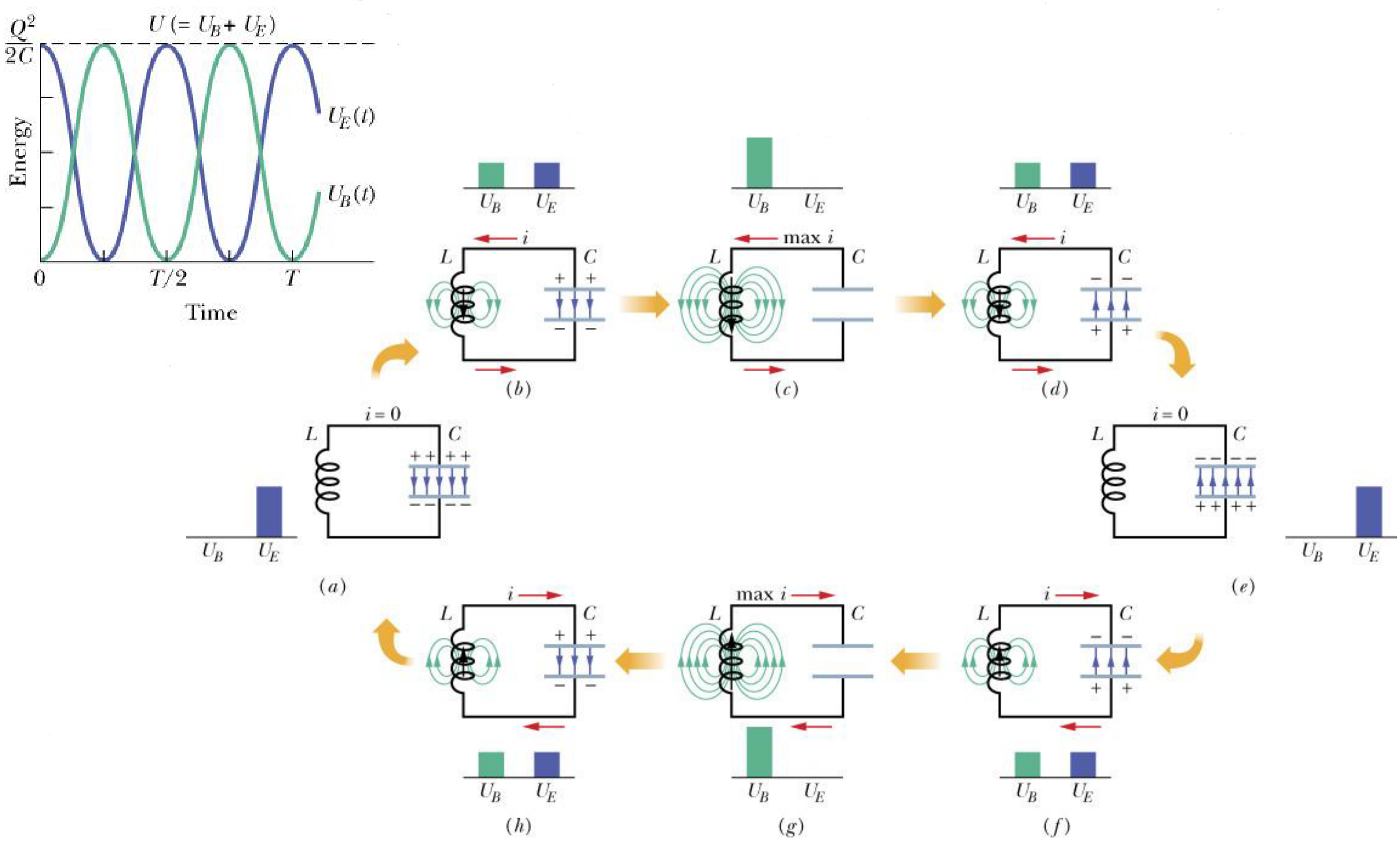
$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi)$$



$$U_B = \frac{1}{2}Li^2 = \frac{1}{2}LI^2\sin^2(\omega t + \phi)$$

so $U_E + U_B = \text{constant}$


Capacitor polarity reverses.
Current direction reverses.
Capacitor fully charged; zero current
V_m
$+Q_m$
$-Q_m$
E_m
C
L
Capacitor discharging; I increasing
Capacitor fully discharged; current maximal
I_m
B_m
Capacitor charging; I decreasing
Capacitor fully charged; zero current
$-V_m$
Capacitor discharging; I increasing
Capacitor fully discharged; current maximal
zero
$E = U_B + U_E$
Circuit's energy all stored in electric field
Circuit's energy all stored in magnetic field
Circuit's energy all stored in electric field
Circuit's energy all stored in magnetic field
(a) $t = 0$ and $t = T$
(close switch at $t = 0$)
(b) $t = \frac{1}{4}T$
(c) $t = \frac{1}{2}T$
(d) $t = \frac{3}{4}T$
Capacitor charging; I decreasing


$\frac{Q^2}{2C}$
$U\ (= U_B + U_E)$
$U_E(t)$
$U_B(t)$
Energy
0
$T/2$
T
Time
U_B
U_E
i
max i
$i = 0$
L
C
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

RC, RL, LC transients summary

- $EMF_L = -L \frac{di}{dt} = -L \frac{d^2q}{dt^2}$

- $EMF_R = iR = R \frac{dq}{dt}$

- $EMF_C = \frac{1}{C} \int i\, dt = \frac{q_c}{C}$

- $V_c(t) = V_0 e^{-t/\tau_{RC}}$ (discharging) with $\tau_{RC} = RC$

- $I_L(t) = I_0 e^{-t/\tau_{LR}}$ (decay) with $\tau_{LR} = \frac{L}{R}$; $I_0 = \frac{V_0}{R}$

- $V_{LC}(t) = V_0 \cos(\omega t + \phi)$ with $\omega = \frac{1}{\sqrt{LC}}$

- Energy in capacitor : $U = \frac{1}{2} \frac{q^2}{C}$

- Energy in inductor : $U = \frac{1}{2} L i^2$

↳ The maximum energy stored in the inductor is equal to the maximum energy stored in the capacitor.

The maxima are 90° out of phase with one another.