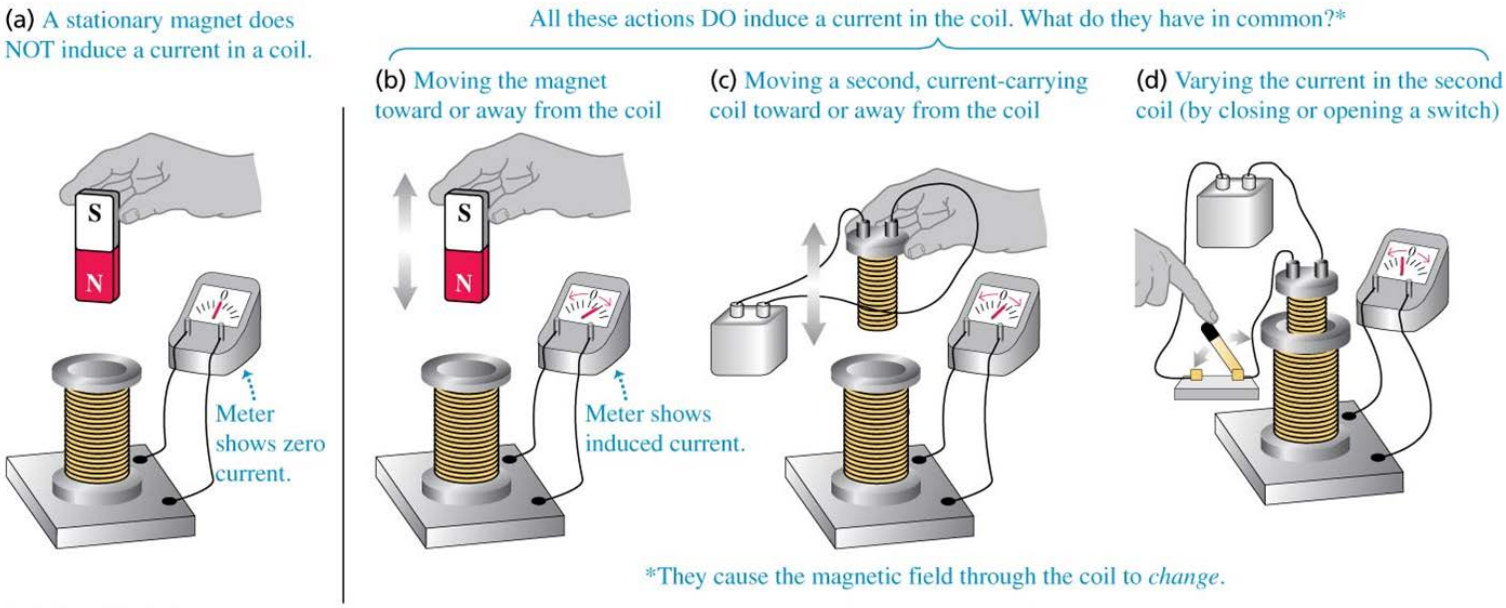
Lecture 15 : Maxwell's equations



Faraday's law : an emf is produced around a loop when the magnetic flux that passes through the loop changes.

$$\boxed{EMF_{loop} = -\frac{d\Phi_B}{dt}}$$

Since $EMF = \oint \vec{E} \cdot d\vec{l}$, we have :

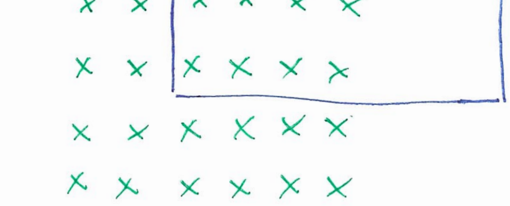
$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}}$$ Faraday's law

Faraday's law tells us that a non-static electric field can be induced by a non-static magnetic field. Thus, a current can be induced in a conducting loop by changing the flux of $\vec{B}$ thought it, no matter how the flux changes.

The minus sign in Faraday's law indicates that a changing magnetic flux will induce an electric field and current such that the $\vec{B}$ produced by the current leads to a flux change in the opposite direction. This is called Lenz's Law.

$\Rightarrow$ the induced current creates an induced field to oppose the change in magnetic flux.

Think 15.1: Which way does the current flow while the loop is moved to the right?

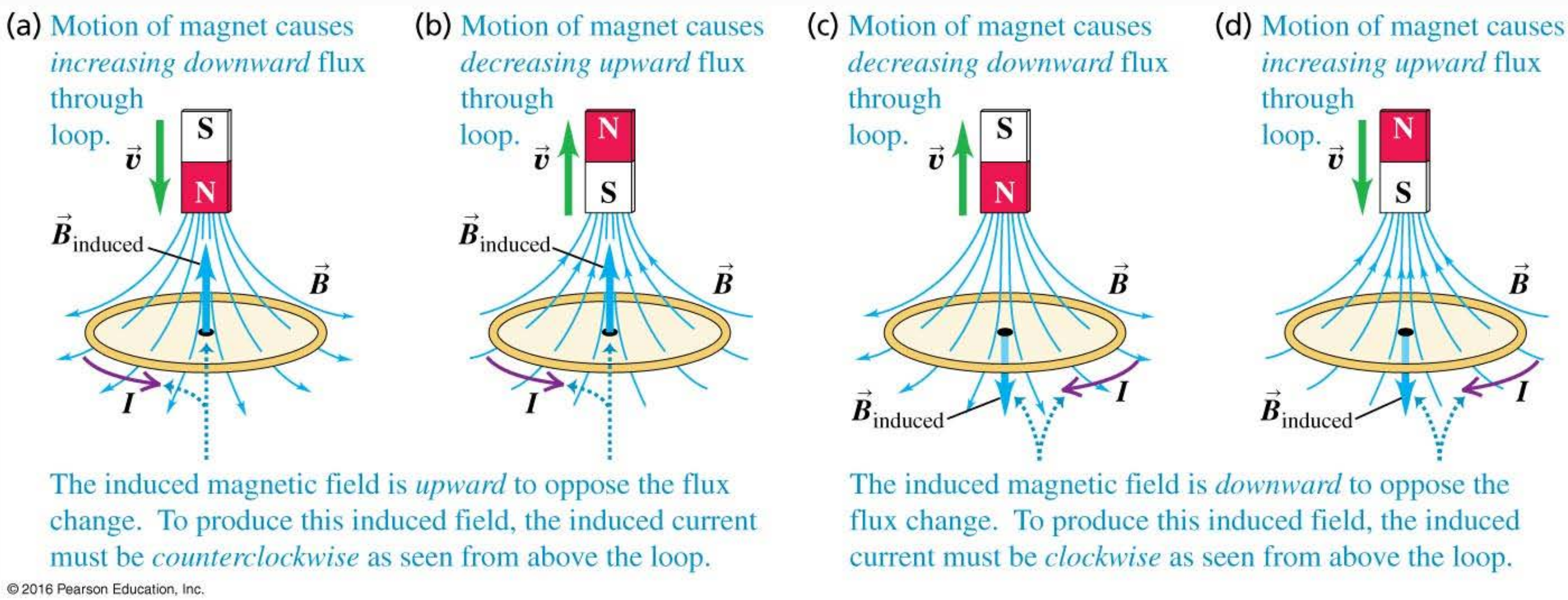


A) Clockwise

B) Counterclockwise

c) Out of the page

d) Into the page



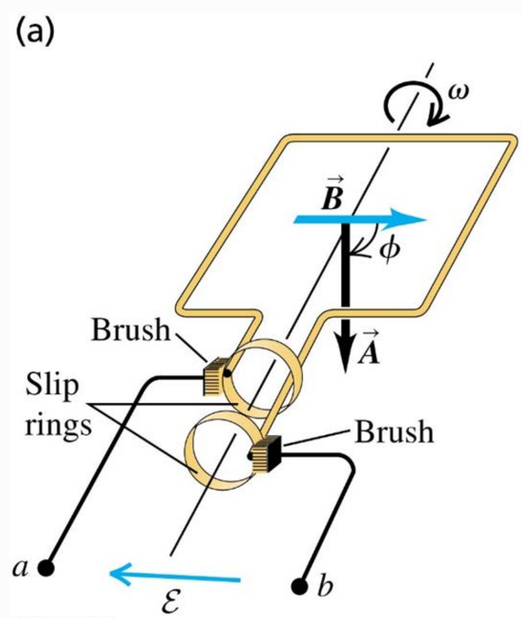
Example: A square loop (side length b) is mounted on a shaft and rotated at angular velocity ω. A uniform magnetic field is perpendicular to the axis. Find the emf for this alternating-current generator.

magnetic flux through loop.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(BA\cos\phi)$$

$$= -BA\frac{d}{dt}(\cos\phi) = -Bb^2\frac{d}{dt}(\cos\phi)$$

with $\phi = \omega t$



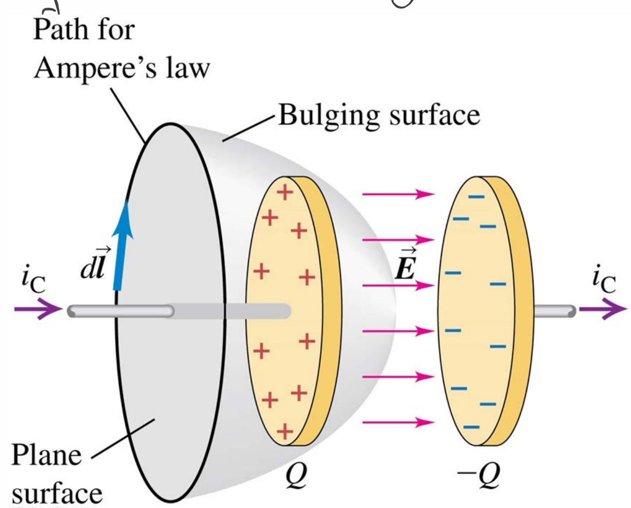
$$\Rightarrow \underline{\mathcal{E} = \omega B b^2 \sin(\omega t)}$$

Maxwell's equations so far:

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0} \qquad \oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

These equations represent the state of electromagnetic theory in the mid-nineteenth century, when Maxwell began his work. There is one big inconsistency in these formulas ...



Consider Ampere's law:

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \iint_{\substack{\text{area} \\ \text{bounded by loop}}} \mu_0 \vec{J} \cdot d\vec{a}$$

It should hold true for <u>any</u> loop, but doesn't in this form.

To illustrate this, let's apply Ampere's law to the parallel plate capacitor shown above. The magnetic field should be the same around the amperian loop no matter what surface we use. But in the above case, we get $I_{enc} = i_c$ for one and $I_{enc} = 0$ for the other.

Maxwell solved this by adding a term to Ampere's equation: the displacement current.

For a parallel plate capacitor:

$$Q = CV = \frac{\epsilon_0 A}{d} V = \epsilon_0 A E = \epsilon_0 \Phi_E$$

$$\boxed{i_{displacement} = \frac{dQ}{dt} = \epsilon_0 \frac{d\Phi_E}{dt}}$$

Then Ampere's law with Maxwell's correction becomes:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (i + i_{displacement}) = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

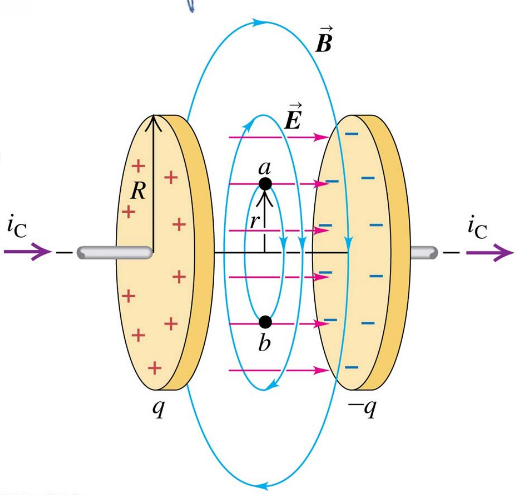
Complete Maxwell's equations:

$$\boxed{\begin{aligned} \oint \vec{E} \cdot d\vec{a} &= \frac{q_{enc}}{\epsilon_0} \\ \oint \vec{E} \cdot d\vec{l} &= -\frac{d\Phi_B}{dt} \\ \oint \vec{B} \cdot d\vec{a} &= 0 \\ \oint \vec{B} \cdot d\vec{l} &= \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \end{aligned}}$$

Electric fields can be produced by charges or changing magnetic fields; and magnetic fields can be produced by currents or changing electric fields.

Maxwell's equations tell you how charges produce fields and the force law tells you how fields affect charges. Together, these equations describe all of E&M except for some properties of matter.

Example: Circular capacitor



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

For a circular loop with $r < R$:

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B$$

$$\Phi_E = EA = \frac{\sigma}{\epsilon_0} A = \frac{Q_{tot}}{\epsilon_0}\left(\frac{\pi r^2}{\pi R^2}\right)$$

Then $$\frac{d\Phi_E}{dt} = \frac{r^2}{\epsilon_0 R^2} \frac{dQ_{tot}}{dt} = \frac{i_c}{\epsilon_0}\left(\frac{r^2}{R^2}\right)$$

And $$B_{r<R} = \frac{\mu_0 \epsilon_0}{2\pi r} \frac{i_c}{\epsilon_0} \frac{r^2}{R^2} = \frac{\mu_0 i_c}{2\pi} \frac{r}{R^2}$$

For $r > R$: $$B_{r>R} = \frac{\mu_0 i_c}{2\pi r}$$