Underlying Structure: <u>scaling for the free energy</u>

for the singular part of the free energy per unit volume

e.g. $t<0$ $\mathcal{F}_f^{\pm}(x)$ would be needed

$$\boxed{f_s(t,h) = |t|^{2-\alpha}\,\mathcal{F}_f\left(h/|t|^{\Delta}\right)} \qquad h = \frac{H}{kT}$$

$$m = -\frac{\partial}{\partial H} f_s = -\frac{1}{kT}\frac{\partial f_s}{\partial h} = -\frac{1}{kT}|t|^{2-\alpha}\frac{1}{|t|^{\Delta}}\mathcal{F}_f'\left(h/|t|^{\Delta}\right)$$

$$\sim |t|^{2-\alpha-\Delta}\,\mathcal{F}_f'\left(h/|t|^{\Delta}\right) \xrightarrow[h\to 0]{} \sim |t|^{\beta}$$

clearly

$\mathcal{F}_m(x) = -\frac{1}{kT}\mathcal{F}_f'(x)$

$$\beta = 2-\alpha-\Delta \qquad \text{using (2):}$$

$$\beta = 2-\alpha-\beta-\gamma \quad \Rightarrow \quad \boxed{\alpha + 2\beta + \gamma = 2}$$

Rushbrooke scaling law

$$\chi_T = \frac{\partial m}{\partial H} = \frac{1}{kT}\frac{\partial m}{\partial h} = \frac{-1}{(kT)^2}|t|^{\beta-\Delta}\,\mathcal{F}_f''\left(h/|t|^{\Delta}\right) \sim |t|^{-\gamma}$$

$$\left.\begin{aligned}\beta &= 2-\alpha-\Delta \\ \beta-\Delta &= -\gamma\end{aligned}\right\} \quad \boxed{\alpha + 2\beta + \gamma = 2} \qquad (4)$$

<u>Scaling for the correlation function</u>

recall: $\xi \sim |t|^{-\nu}$

$r/\xi \sim r|t|^{\nu}$

$$G(\bar{r},t,h) = \frac{1}{r^{d-2+\eta}}\,\mathcal{F}_G\left(r|t|^{\nu}, h/|t|^{\Delta}\right)$$

$h=0$

$$\chi_T \sim \int d^d r\, G(\bar{r},t,0) = \int d^d r \frac{1}{r^{d-2+\eta}}\mathcal{F}_G(r|t|^{\nu},0) \qquad \bar{r}|t|^{\nu} \equiv \bar{x}$$

$$\int \frac{d^d x}{t^{d\nu}}\, t^{\nu(d-2+\eta)}\frac{1}{x^{d-2+\eta}}\mathcal{F}_G(x,0) = t^{\nu\eta-2\nu}\underbrace{\int d^d x \frac{1}{x^{d-2+\eta}}\mathcal{F}_G(x,0)}_{\text{const}}$$

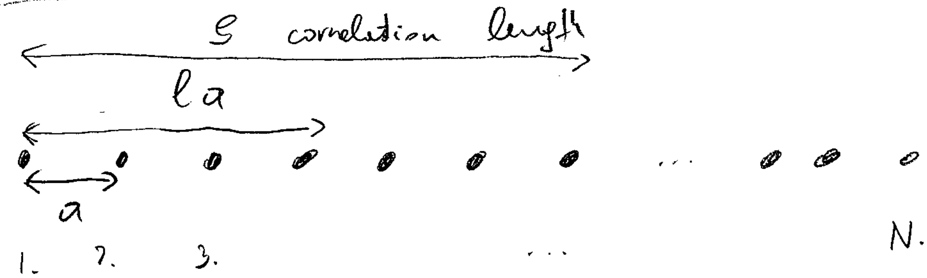
on the other hand: $\chi_T \sim |t|^{-\gamma}$ $\Rightarrow$ $\boxed{\gamma = 2\nu - \eta\nu}$

Kadanoff Block Spins (Kadanoff, 1966)

$$-\beta \mathcal{H}\{s_i\} = \underbrace{\beta J}_{K} \sum_{\langle ij \rangle} s_i s_j + \underbrace{\beta H}_{h} \sum_i s_i = K \sum_{\langle ij \rangle} s_i s_j + h \sum_i s_i$$

determines the partition function

$\boxed{K \equiv \beta J}$ $\boxed{h = \beta H}$ the usual scaled coupling constants



as approaching T_c: ξ is large, but finite

spins on length scale la act as a 'single unit'

$a \ll la \ll \xi(T)$ $\qquad \xi \sim |t|^{-\nu} \quad (t \ll 1)$

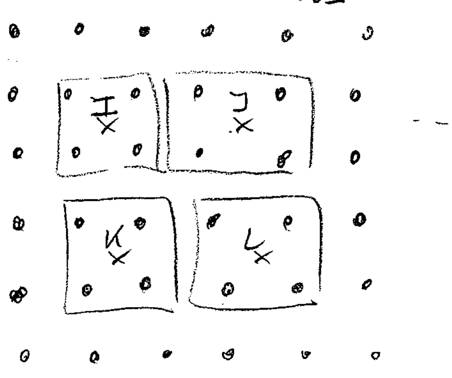
spins $\xrightarrow{\text{'coarse-graining'}}$ block spins

in d-dimension: one block contains l^d of the original spins

total number of blocks: $\frac{N}{l^d}$ $\quad$ N is the total # of original spins

block spin variable: $\quad S_I \equiv \frac{1}{|\overline{m}_l|} \frac{1}{l^d} \sum_{i \in I} s_i$

where $\overline{m}_l \equiv \frac{1}{l^d} \sum_{i \in I} \langle s_i \rangle \quad \Longrightarrow \quad \langle S_I \rangle = \pm 1$



"•" original spin

"x" block spin

N sites

$l = 2$

original Hamiltonian: $\ell = 1$ $K_1 = K$, $h_1 = h$

for the block-spin system: **Assumption II**: K n.n. $\to K_\ell$ n.n only as well

$$\boxed{-\beta \mathcal{H}_\ell = K_\ell \sum_{\langle IJ \rangle}^{N/\ell^d} s_I s_J + h_\ell \sum_{I=1}^{N/\ell^d} s_I}$$

observation: new lattice spacing: ℓa
number of blockspins N/ℓ^d (fewer degrees of freedom)

correlation length: ξ (physical distance)

$$([\mathring{A}] =) \xi = \xi_\ell (\ell a) = \xi_1 a$$

measured in the units of the underlying lattice (ℓa) and (a) respectively

$$\boxed{\xi_\ell = \xi_1 / \ell}$$

i.e. $\xi_\ell < \xi_1 \Rightarrow |t_\ell| > |t_1|$
getting further away from criticality

$$t = \left|\frac{T - T_c}{T_c}\right|$$

also,

$$h \sum_i s_i = h \sum_I \sum_{i \in I} s_i = h \sum_I |\bar{m}_\ell| \ell^d s_I = \underbrace{h |\bar{m}_\ell| \ell^d}_{h_\ell} \sum_I s_I$$

$$= h_\ell \sum_I s_I$$

$$\boxed{h_\ell = h |\bar{m}_\ell| \ell^d}$$

same form of hamiltonian with different coupling const. and degrees of freedom

$$N f_s(t, h) = N/\ell^d \, f_s(t_\ell, h_\ell)$$

$$\boxed{f_s(t, h) = \ell^{-d} f_s(t_\ell, h_\ell)}$$

singular part of the free energy per spin

Origin of Scaling

Assumption II

$$\boxed{\begin{aligned} t_\ell &= t\,\ell^{y_t} \\ h_\ell &= h\,\ell^{y_h} \end{aligned}}$$

this is how the coupling constant transform under the block-spin transformation

$$\boxed{f_s(t,h) = \ell^{-d} f_s(t\ell^{y_t}, h\ell^{y_h})}$$

$y_t > 0$, $y_h > 0$
$\ell > 1$

ℓ is arbitrary ($\ell > 1$ block spin grouping parameter)
specifically:

choose $\ell = |t|^{-1/y_t} = \frac{1}{|t|^{1/y_t}}$

$$\Rightarrow \quad f_s(t,h) = |t|^{d/y_t} f_s\left(\pm 1, \frac{h}{|t|^{\frac{y_h}{y_t}}}\right)$$

making the correspondence with scaling exponents introduced last time:

$$\boxed{y_h/y_t \equiv \Delta} \qquad \boxed{d/y_t \equiv 2-\alpha}$$

recall:
$\Delta = \beta + \gamma$
$\beta + \gamma = \beta\delta$
$\alpha + 2\beta + \gamma = 2$

$\Downarrow$ y_h, y_t
all others can be determined

$y_h/y_t = \beta + \gamma$
$d/y_t = 2 - \alpha$
$\alpha + 2\beta + \gamma = 2$

recalling: $f_s(t,h) = |t|^{2-\alpha} F_f^{\pm}\left(h/|t|^{\Delta}\right)$

$$\boxed{F_f^{\pm}(x) = f_s(\pm 1, x)}$$

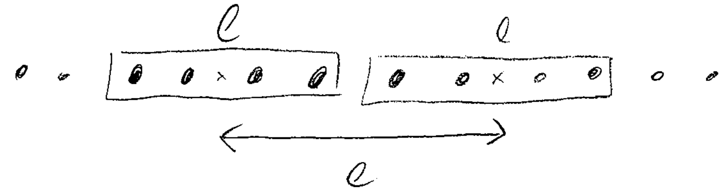
$\pm$ above or below T_c — $t > 0$, $t < 0$

for the free energy

Two-Point correlation function

$$G(\bar{r}_\ell, t_\ell, h_\ell) \equiv \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle$$

$\bar{r}_\ell$ is the displacement vector between the center of block I and J in units of the new lattice constant ℓa, i.e., $\bar{r}_\ell = \bar{r}/\ell$ after the block-spin tr.



$$\boxed{\bar{r}_\ell (\ell a) = \bar{r} a}$$

block spin distance in units of $\underline{a}$ — original distance in units of a

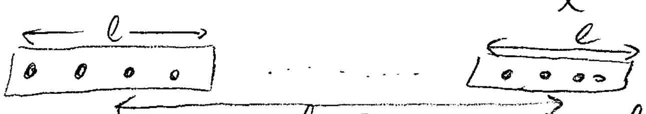
$$\boxed{\bar{r}_\ell = \bar{r}/\ell}$$

using $|\bar{m}_\ell| = \dfrac{h_\ell}{h \ell^d} = \dfrac{h \ell^{y_h}}{h \ell^d} = \ell^{y_h - d}$ $\qquad \left(h_\ell = |\bar{m}_\ell| \ell^d h \right)$ from earlier

and

$$S_I = \frac{1}{|\bar{m}_\ell| \ell^d} \sum_{i \in I} S_i$$

$$G(\bar{r}_\ell, t_\ell, h_\ell) = \frac{1}{\ell^{2(y_h - d)} \cdot \ell^{2d}} \sum_{\substack{i \in I \\ j \in J}} \underbrace{\left[\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \right]}_{\approx\, G(\bar{r}, t, h)\, \ell^d \cdot \ell^d}$$



long $\bar{r}$ wavelength behaviour $\qquad r \gg \ell > 1$

$$\approx \frac{1}{\ell^{2(y_h - d)}} G(\bar{r}, t, h)$$

$$G(\bar{r},t,h) = l^{2(y_h-d)} G(\bar{r}_e, t_e, h_e)$$

use assumption 2. $t_e = t\, l^{y_t}$, $h_e = h\, l^{y_h}$
and $\bar{r}_e = \bar{r}/l$

$$\boxed{G(\bar{r},t,h) = l^{2(y_h-d)} G(\bar{r}/l,\, t l^{y_t},\, h l^{y_h})}$$

specifically, choose: $l = |t|^{-1/y_t}$ $\boxed{l = 1/|t|^{1/y_t}}$

$$G(\bar{r},t,h) = |t|^{\frac{2(d-y_h)}{y_t}} G\left(\bar{r}|t|^{1/y_t}, \pm 1, h/|t|^{y_h/y_t}\right) \quad \left(\begin{matrix} + & t>0 \\ - & t<0 \end{matrix}\right)$$

defining:
$$G\left(r|t|^{1/y_t}, \pm 1, h/|t|^{y_h/y_t}\right) \equiv \left(r|t|^{1/y_t}\right)^{-2(d-y_h)} F_G^{\pm}\left(r|t|^{1/y_t}, h/|t|^{y_h/y_t}\right)$$

$$\boxed{G(\bar{r},t,h) = \frac{1}{r^{2(d-y_h)}} F_G^{\pm}\left(r|t|^{1/y_t}, h/|t|^{y_h/y_t}\right)}$$

yielding: $y_h/y_t = \Delta$, $\boxed{1/y_t = \nu}$

$$\boxed{2(d-y_h) = d-2+\eta}$$

Conclusion: if y_h and y_t are known,
all others $(\alpha, \beta, \gamma, \delta, \nu, \eta)$ can be obtained
y_t and y_h describe the change of the coupling constants when the lengthscale is changed
$h_e = h\, l^{y_h}$
$t_e = t\, l^{y_t}$
Goal is to find y_t, y_h