1. Consider a hypothetical Fermi system with N particles in volume V and with the density of states $g(\varepsilon)$ given by

$$g(\varepsilon) = \begin{cases} 0 & if \quad \varepsilon < 0 \\ \alpha V & if \quad \varepsilon > 0 \end{cases},$$

where α is a constant.

(a) Find the Fermi energy ε_F and the internal energy of the system at zero temperature.

(b) Using the Sommerfeld expansion, find the chemical potential, the internal energy, and the specific heat at low temperatures.

a Fermi Energy:

$$N = \int_0^{\varepsilon_F} a(\varepsilon)\, d\varepsilon = \int_0^{\varepsilon_F} \alpha V \, d\epsilon \rightarrow \alpha V \varepsilon_F = N$$

$$\therefore \boxed{\varepsilon_F = \frac{N}{\alpha V}}$$

Internal energy @ 0°

- Sum all particle energies from $0 \to \varepsilon_F$

$$U = \int_0^{\epsilon_F} \epsilon (\alpha V)\, d\epsilon = \alpha V \int_0^{\epsilon_F} \epsilon \, d\epsilon \Rightarrow \alpha V \left(\frac{\epsilon^2}{2}\right)\Bigg|_0^{\epsilon_F} = \boxed{\frac{\epsilon_F^2}{2} \alpha V}$$

or $\boxed{\frac{N^2}{2\alpha V}}$

$$b.\ N = \int_0^\infty g(\epsilon) n_F(\epsilon)\, d\epsilon \approx \int_0^{\epsilon_F} f(\epsilon)\, d\epsilon + \frac{\pi^2}{6}(kT)^2 f'(\epsilon_F) + \dots$$

$$n_F(\epsilon) = \frac{1}{e^{(\epsilon - \mu)/kT} + 1}$$

$$\mu(T) = \epsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right)$$

$$\mu(T) \approx \epsilon_F - \frac{\pi^2}{12}\frac{(kT)^2}{\epsilon_F}$$

$$\mu(T) \sim \epsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right)$$

Chemical potential

$$U(0) = \frac{\alpha V \varepsilon_F^2}{2}, \quad U(T) = \int_0^\infty \epsilon\, g(\epsilon)\, n_F(\epsilon)\, d\epsilon$$

$$g(\epsilon) = \alpha V \quad \text{for } \epsilon > 0$$

$$\Delta U(T) = \int_{\epsilon_F}^{\infty} \epsilon \alpha V n_F(\epsilon)\, d\epsilon$$

using Sommerfeld expansion

$$\sim \int_0^{\epsilon_F} \underbrace{\cancel{f(\epsilon)}}_{\epsilon \alpha V} d\epsilon + \frac{\pi^2}{6}(kT)^2 \underbrace{\cancel{f'(\epsilon_F)}}_{\alpha V} + \dots$$

$$\Delta U(T) = \frac{\pi^2}{6}(kT)^2 \alpha V$$

$$\boxed{U(T) = \frac{\alpha V \varepsilon_F^2}{2} + \frac{\pi^2}{6} \alpha V (kT)^2}$$

$$C_V = \frac{\partial U}{\partial T} = \boxed{\frac{\pi^2}{3} \alpha V k^2 T}$$

2. The dimensionality plays a fundamental role in phase transitions and critical phenomena. Its importance also shows in the peculiar phenomenon of Bose-Einstein condensation. Show that there is *no* Bose-Einstein condensation in the *one or two-dimensional* ideal Bose gas with the dispersion relation $\varepsilon(p) = p^2/2m$ at any nonzero temperature. (At $T = 0$, of course, all particles are in the ground state.)

$$\epsilon(p) = \frac{p^2}{2m}$$

$$g(\epsilon) \propto \epsilon^{\frac{d}{2}-1}$$

$$N = \int_0^\infty \frac{g(\epsilon)}{e^{\epsilon/k_B T} - 1}\, d\epsilon$$

1-D: $g(\epsilon) \propto \epsilon^{-\frac{1}{2}}$

2-D: $g(\epsilon) \propto \cancel{\epsilon}$ 1

* either one of these substituting into, diverges as $\epsilon \to 0$

Therefore 1-D & 2-D Bose-Einstein condensation is impossible

3. Find the behavior of the *isotherm compressibility*, κ_T, in as the critical temperature is approached *from above*, i.e., in the $T \to T_c + 0$ limit. This means that you should obtain an expression for κ_T as a function of $(T - T_c)$ in the vicinity of the transition temperature. You should review the notes posted on LMS regarding the behavior of the Bose gas just above T_c. In particular, use

$$\mu(T) \approx -kT\left(\frac{3\zeta(3/2)}{4\sqrt{\pi}}\right)^2\left(\frac{T-T_c}{T_c}\right)^2 \approx -kT_c\left(\frac{3\zeta(3/2)}{4\sqrt{\pi}}\right)^2\left(\frac{T-T_c}{T_c}\right)^2$$

and

$$f_{3/2}^-(e^{-\alpha}) \approx \zeta(3/2) - 2\sqrt{\pi}\alpha^{1/2}, \text{ where } \alpha = -\frac{\mu}{kT}.$$

Also, you will need to employ the basic thermodynamic relation

$$-\left(\frac{\partial V}{\partial P}\right)_{N,T} = \frac{V^2}{N^2}\left(\frac{\partial N}{\partial \mu}\right)_{V,T}.$$

given:

$$\mu(T) \approx -kT_c\left(\frac{3\xi(3/2)}{4\sqrt{\pi}}\right)^2 \cdot \left(\frac{T-T_c}{T_c}\right)^2$$

$$\alpha = -\mu/kT$$

$$\alpha \approx \cancel{\frac{T_c}{T}}^{\sim 1 \text{ for } T_c \sim T}\left(\frac{3\xi(3/2)}{4\sqrt{\pi}}\right)^2\left(\frac{T-T_c}{T_c}\right)^2$$

$$f_{3/2}^-(e^{-\alpha}) \sim \xi(3/2) - 2\sqrt{\pi}\,\alpha^{1/2} \quad \alpha = -\frac{\mu}{kT}$$

$$\Rightarrow f_{3/2}^-(e^{-\alpha}) \sim \xi(3/2) - \cancel{2\sqrt{\pi}}\left(\frac{3\xi(3/2)}{\cancel{4\sqrt{\pi}}}\right)\frac{T-T_c}{T_c}$$

$$\Rightarrow \xi(3/2) - \frac{3\xi(3/2)}{2}\,\frac{T-T_c}{T_c} \quad 2$$

$$N = V f_{3/2}^-(e^{-\alpha})$$

$$\frac{\partial N}{\partial \mu} = V\,\frac{\partial f_{3/2}^-(e^{-\alpha})}{\partial \alpha}\,\frac{\partial \alpha}{\partial \mu}$$

$$= -\frac{V}{kT}\,\frac{\partial f_{3/2}(e^{-\alpha})}{\partial \alpha}$$

$$\frac{\partial f_{3/2}^-(e^{-\alpha})}{\partial \alpha} = -\sqrt{\pi}\,\alpha^{-1/2}$$

$$\frac{\partial f_{3/2}(e^{-\alpha})}{\partial \alpha} = -\sqrt{\pi}\left(\frac{3\zeta(3/2)}{4\sqrt{\pi}}\frac{T-T_c}{T_c}\right)^{-1}$$

$$N \sim V\left(\xi(3/2) - 2\sqrt{\pi}\left(-\frac{\mu}{kT}\right)^{1/2}\right)$$

$$\mu(t) \approx -kT\left(\frac{3\xi(3/2)}{4\sqrt{\pi}}\right)^2\left(\frac{T-T_c}{T}\right)^2$$

$$-\mu\left(\frac{4\sqrt{\pi}}{3\xi(3/2)}\right)^2\left(\frac{T}{T-T_c}\right)^2 = kT$$

4. Consider the simple model for a one-dimensional solid consisting of N atoms of identical mass m along a chain. Atoms are connected only with nearest-neighbor atoms by "springs" with spring constant κ. The equilibrium separation between atoms (i.e., the lattice constant) is a. The springs are relaxed in equilibrium and we allow only for *longitudinal* oscillations. For simplicity, use periodic boundary conditions, $u_{j+N} = u_j$, where u_j is the displacement of the jth atom, measured from its equilibrium position.

(a) Show that the equation of motion for the displacements is

$$m\ddot{u}_j = \kappa(u_{j+1} + u_{j-1} - 2u_j), \quad j = 1,2,\ldots,N.$$

(b) Solve the above set of equation by means of complex Fourier series, yielding the normal modes and the spectrum. In particular, show that the frequency – wave-number dispersion relation is given by

$$\omega(k) = 2\sqrt{\frac{\kappa}{m}}\left|\sin\left(\frac{ka}{2}\right)\right|,$$

where $k = \frac{2\pi}{Na}n, \quad n = -\frac{N}{2},\ldots,\frac{N}{2}$ (assumed even N for simplicity).

(c) From the above dispersion relation it follows that for small k values we can use $\omega(k) = ck$, where c is the speed of longitudinal vibrations in this solid. Obtain the low-temperature behavior of the specific heat of this *one-dimensional* solid of size $L = Na$ in the Debye approximation.

a. Force on j from $j+1$, $j-1$

$$F_j = -k(u_j - u_{j-1}) - k(u_j - u_{j-1})$$

$$= -k(u_{j+1} + u_{j-1} - 2u_j$$

$$m\ddot{u}_j = k(u_{j+1} + u_{j-1} - 2u_j) \quad \text{Newton's 2nd law}$$

6. $u_j = Ae^{i(kja-\omega t)}$

$$\ddot{u}_j = -\omega^2 A e^{i(kja-\omega t)}$$

$$u_{j+1} = Ae^{i(kja-\omega t)} e^{ika}$$
$$u_{j-1} = Ae^{i(k(j-1)a-\omega t)}$$

$$-M\omega^2 A e^{i(kja-\omega t)} = k\left(Ae^{i(kja-\omega t)}\left(e^{ika}+e^{-ika}-2\right)\right.$$
$$-M\omega^2 = k\left(e^{ika}+e^{-ika}-2\right)$$
$$-M\omega^2 = k\left(2\cos(ka)-2\right.$$
$$\omega^2 = \frac{2k}{M}\left(1-\cos(ka)\right)$$
$$\omega^2 = \frac{4k}{M}\sin^2\left(\frac{ka}{2}\right)$$
$$\omega(k) = 2\sqrt{\frac{k}{M}}\left|\sin\left(\frac{ka}{2}\right)\right|$$

C small k

$$\omega(k) = C(k)$$

$$k = \frac{\omega}{c}, \quad k = \frac{2\pi n}{L} \quad n = 0,1,2\ldots$$

$$dk = \frac{d\omega}{c}, \quad g(\omega)\,d\omega = \frac{L}{2\pi}\frac{d\omega}{c} = \frac{L}{2\pi c}\,d\omega$$

$$\langle E \rangle = \frac{\hbar\omega}{e^{\hbar\omega/k_\beta T} - 1} \qquad U = \int_0^{\omega_D} g(\omega) \langle E \rangle \, d\omega$$

$$U = \int_0^{\omega_D} \frac{L}{2\pi c} \cdot \frac{\hbar\omega}{e^{\hbar\omega/k}} \, d\omega$$

$$C = \frac{\partial U}{\partial T} \qquad \boxed{C \propto T}$$

5. Obtain an *estimate* for the
(a) *Fermi energy and Fermi temperature in copper* (assume one conduction or "free" electron per atom). For the mass density of copper use $\rho = 9\frac{g}{cm^3}$, and the atomic mass is 63.5g/mol.
(b) critical temperature for the Bose-Einstein condensation in an ideal He^4 "gas" with density $\rho = 0.145\frac{g}{cm^3}$. The atomic mass is 4g/mol.
(c) *Debye temperature in copper* (use the parameters given in (a) and the effective sound velocity $c \approx 4000\frac{m}{s}$).

a.

$$\varepsilon_F = \left(\frac{3N}{8\pi V}\right)^{2/3} \frac{h^2}{2m}$$

$$T_F = \varepsilon_F / k$$

$$n = \frac{\rho}{M} \times N_A = \frac{9\,g/cm^3}{63.5\,g/mol} \times 6.022E23$$

$$= 8.535E22 \qquad E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

$$E_F = 1.13119e{-22}$$

$$T_F = \frac{E_F}{k_B} = 8.197$$

b.

$$n = \frac{\rho}{M} N_A = \frac{0.145}{4} \cdot 6.022E23$$

$$\approx 2.1829E22$$

$$T_c = \frac{2\pi\hbar^2}{k_B m}\left(\frac{n}{\zeta(3/2)}\right)^{2/3} = 7.834^\circ$$

C. $$\Theta_D = \frac{\hbar c}{K_B}\left(6\pi^2 n\right)^{1/3}$$

$$c = 4000$$

$$n = \frac{4}{63.5} \cdot 6.022e23$$

$$\underline{\Theta_D = 5.2429^\circ}$$