BREHM + MULLIN

It was noted long ago that a measurement of the threshold wavelength in a bremsstrahlung spectrum could be used in conjunction with Equation (2-50) to furnish another accurate determination of Planck's constant. Studies of the short-wavelength limit were conducted for this purpose in experiments performed by W. Duane and F. L. Hunt in 1915.

The discrete lines in the x-ray spectrum are peculiar to the specific metal in the target of the x-ray tube. These *characteristic x rays* are significant because they offer direct evidence for the quantum properties of matter.

X rays have enjoyed a wide variety of practical applications as a result of their remarkable penetrating power. X-ray sources have been put to use as diagnostic and therapeutic devices in medicine and in dentistry. The radiation has also been employed in industry to examine structures for possible hidden defects.

X rays have recently been observed from sources in the universe beyond the solar system. Galactic sources of the radiation are believed to exist in binary stars, where bremsstrahlung x rays are supposedly produced as ionized matter streams from one star to the other. The exciting new field of x-ray astronomy has developed out of these discoveries.

Example

The combination of constants hc appears often enough in atomic physics to warrant a separate calculation:

$$hc = \frac{(6.626 \times 10^{-34}\ \mathrm{J \cdot s})(2.998 \times 10^{8}\ \mathrm{m/s})(10^{9}\ \mathrm{nm/m})}{1.602 \times 10^{-19}\ \mathrm{J/eV}}$$

$$= 1240\ \mathrm{eV \cdot nm} = 1.240\ \mathrm{keV \cdot nm}.$$

We can apply the result immediately to Equation (2-50) and express the short-wavelength cutoff for the continuous x-ray spectrum as

$$\lambda_{\min} = \frac{1.240\ \mathrm{kV \cdot nm}}{\phi}.$$

Thus, a 40 kV x-ray tube has a cutoff at

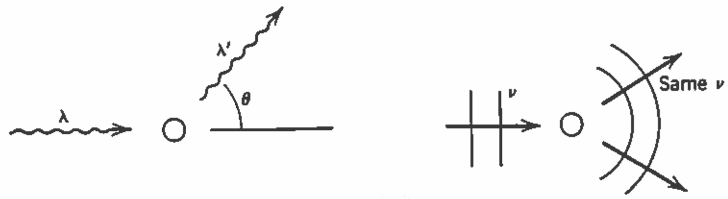
$$\lambda_{\min} = \frac{1.24}{40}\ \mathrm{nm} = 0.031\ \mathrm{nm}$$

and produces a continuous spectrum of x rays containing all longer wavelengths.

2-7 The Compton Effect

Einstein's original quantum theory of light proposed that the photon be regarded only as a quantum of *energy*. His conception of the photon continued to evolve until 1917, as he came to believe that a notion of *directedness* should also be incorporated in his idea. He made provision for this by letting the light quantum have characteristics of *momentum* as well as energy. It might seem odd that the father of special relativity should hesitate for more than a decade to bring together these complementary

Figure 2-18
Compton scattering of radiation. The scattered wavelength λ' is longer than the incident wavelength λ. In the classical picture, the incident plane wave and the scattered spherical wave have the same frequency and wavelength.



relativistic properties, but such was Einstein's provisional attitude toward the developing quantum theory. He looked for support from experiment to establish momentum and energy as joint aspects of the same hypothesis and was not able to find immediate evidence. A test of his revised photon concept was finally proposed by Debye and, independently, by A. H. Compton. The decisive experiment was then performed by Compton in 1923.

The Compton effect pertains to the scattering of monochromatic x rays by atomic targets and refers to the observation that the wavelength of the scattered x rays is *greater* than that of the incident radiation. Figure 2-18 illustrates the process and identifies the Compton wavelength shift in terms of the wavelength difference $\lambda' - \lambda$. This quantity is observed to vary as a function of the scattering angle θ shown in the figure. The experiment is performed with x rays because the short wavelengths are needed to have an observable effect. A pronounced x-ray wavelength shift is associated with a scattering of the x ray by an *electron* in an atom rather than by the atom as a whole. We demonstrate this experimentally by finding that the shift does not depend on the identity of the atomic scatterer, and so we attribute the effect to the electron as the common constituent of all target atoms. Our discussion of the Compton effect is given in terms of the scattering of an x ray by a *free* electron, since an x-ray quantum carries enough energy to make the distinction between a bound electron and a free electron essentially irrelevant.

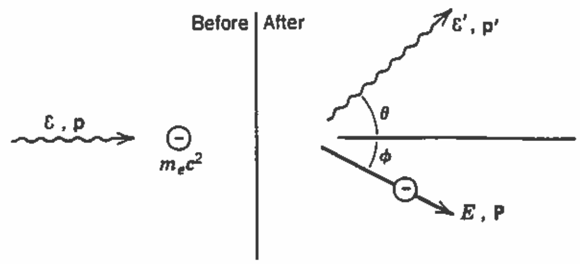
The figure goes on to show how the phenomenon presents another unexplainable problem for classical physics. The classical mechanism for the scattering of an electromagnetic wave is explained in a radiation theory developed by (and named after) Thomson. His model assumes that the incident wave comes in at frequency ν and causes the target electron to oscillate and radiate the outgoing wave. The oscillation frequency is also given by ν since the electron is driven at this frequency by the incoming wave. The outgoing classical radiation oscillates at the frequency of the source and therefore has to have the same frequency ν. Thus, the classical picture cannot account for the observed wavelength shift.

The quantum theory of radiation treats the x-ray beam as a stream of photons. For x rays of wavelength λ the photon energy is given by

$$\varepsilon = h\nu = \frac{hc}{\lambda}. \tag{2-51}$$

The photon is also assigned a momentum according to Einstein's revision of the

Figure 2-19
Kinematics of Compton scattering.



photon concept:

$$p = \frac{\varepsilon}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}. \tag{2-52}$$

Note that this assignment of energy and momentum reproduces the relativistic relation given in Equation (1-39). Recall that the equation holds for a particle of zero mass whose speed is equal to c in all Lorentz frames. Thus, the revised light-quantum prescription begins to attach some of the properties of a *massless particle* to the behavior of a photon.

The Compton process is then described in terms of a relativistic collision involving the elastic scattering of a photon by an electron,

$$\gamma + e \rightarrow \gamma + e.$$

We proceed by imposing the familiar conservation laws for the total relativistic momentum and energy, where the various kinematic quantities are identified in Figure 2-19. We take the electron to be at rest initially and to have energy and momentum E and $\mathbf{P}$ after the collision. These variables are related by the relativistic formula given in Equation (1-35):

$$E^2 = P^2c^2 + m_e^2c^4. \tag{2-53}$$

We may write conservation of momentum as

$$\mathbf{p} - \mathbf{p}' = \mathbf{P}$$

and square the equality to get

$$c^2p^2 - 2c^2\mathbf{p}\cdot\mathbf{p}' + c^2p'^2 = c^2P^2.$$

Since $cp = \varepsilon$ and $cp' = \varepsilon'$, the result can be rewritten as

$$\varepsilon^2 - 2\varepsilon\varepsilon'\cos\theta + \varepsilon'^2 = E^2 - m_e^2c^4,$$

where Equation (2-53) is used to eliminate P^2. We also employ conservation of relativistic energy by writing

$$\varepsilon + m_e c^2 = \varepsilon' + E.$$

Rearranging and squaring produce the following equality:

$$\varepsilon^2 - 2\varepsilon\varepsilon' + \varepsilon'^2 = E^2 - 2Em_e c^2 + m_e^2 c^4.$$

The two quadratic equations represent information obtained from independent conservation laws. We subtract equalities to get

$$\begin{aligned} 2\varepsilon\varepsilon'(1 - \cos\theta) &= 2m_e c^2(E - m_e c^2) \\ &= 2m_e c^2(\varepsilon - \varepsilon'), \end{aligned}$$

and we divide by $2\varepsilon\varepsilon'$ to obtain

$$\begin{aligned} 1 - \cos\theta &= m_e c^2\left(\frac{1}{\varepsilon'} - \frac{1}{\varepsilon}\right) \\ &= m_e c^2\left(\frac{\lambda'}{hc} - \frac{\lambda}{hc}\right), \end{aligned}$$

using Equation (2-51) at the last step. Finally, we solve for the wavelength difference and find

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) \tag{2-54}$$

as the desired result for the Compton wavelength shift.

It is interesting that the shift in wavelength depends only on θ and does not vary with the wavelength of the incident radiation. This feature of the result is seen clearly if we rewrite Equation (2-54) in the form

$$\Delta\lambda = \lambda_C(1 - \cos\theta).$$

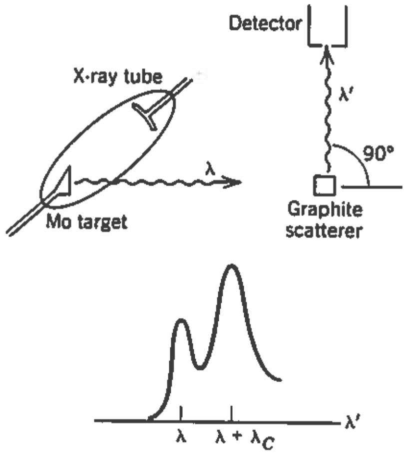
The parameter λ_C is known as the *Compton wavelength* of the electron, a constant defined in terms of fundamental constants as

$$\lambda_C = \frac{h}{m_e c}. \tag{2-55}$$

This quantity is numerically equal to 0.00243 nm, a length that sets the scale for the wavelength shift in the Compton effect. It is clear that the incident wavelength must be comparable to λ_C if λ' is to be noticeably different from λ. For this reason the effect is not detectable for visible light and only begins to be measurable for x rays.

Compton's experiment was performed with incident radiation at one of the characteristic wavelengths of molybdenum, taken from a Mo target in an x-ray tube. These x rays were scattered from graphite, and the scattered radiation was observed in a detector set at 90° with respect to the direction of incidence, as shown in Figure 2-20. Compton's data confirmed his prediction of a wavelength shift and thereby verified the formula in Equation (2-54) as a valid consequence of the premises

Figure 2-20
X-ray scattering at 90°. The observed distribution of scattered wavelengths shows Compton scattering by electrons at $\lambda' = \lambda + \lambda_C$ and Thomson scattering by atoms at $\lambda' = \lambda$.



underlying Equations (2-51) and (2-52). Further studies at several values of λ supported the conclusion that the shift was independent of wavelength. These results firmly established the quantum nature of electromagnetic radiation with respect to both energy and momentum aspects of Einstein's proposition.

Let us draw a final connection between these conclusions and the classical treatment of the scattering of radiation. We have alluded to the latter viewpoint, and to Thomson's classical theory, in our discussion of Figure 2-18. The classical picture requires $\lambda' = \lambda$, while the quantum picture predicts $\Delta\lambda \ll \lambda$ and $\lambda' \approx \lambda$ for $\lambda \gg \lambda_C$. This observation tells us that Compton scattering becomes Thomson scattering in the limit where the wavelength becomes much larger than the Compton wavelength. We can see the Thomson limit in another guise by referring to the lower part of Figure 2-20. The observed wavelength distribution for x rays scattered at 90° exhibits a peak at $\lambda' = \lambda$, in addition to the result expected at $\lambda' = \lambda + \lambda_C$. The second peak agrees with the Compton prediction in Equation (2-54) for $\theta = 90°$. The first peak represents no shift and evidently corresponds to photon scattering from the atom as a whole. In this circumstance, the factor $h/m_e c$ in Equation (2-54) is replaced by h/Mc, where M is the mass of the atom. The Compton wavelength of the atom is much smaller than λ_C since M is much larger than m_e. Hence, even x-ray wavelengths experience no observable shift in scattering from the whole atom, and the corresponding feature of the data exhibits the classical Thomson prediction.

The photon is often called a *particle* because it occurs in radiation discretely and because it has energy and momentum properties appropriate for a relativistic particle of zero mass. We refrain from adopting this usage in all its implications, however. We especially avoid contemplating any sort of localization of the photon, as we might imagine in the case of ordinary types of particles. None of our applications of the light-quantum concept includes any notion of electromagnetic energy and momentum at some localized position in space. We learn that the photon is absorbed by an electron in the photoelectric effect and is scattered by an electron in the Compton effect. The electron can be expected to have certain localization properties, but the interaction of the electron with the photon does not require the photon to have any localization of its own. Thus, when we speak of radiation as a system of photons with particle attributes, we realize that we are beginning to entertain a new *quantum–particle* idea in which discreteness and localization are entirely separate considerations.

Example

Let us begin with Equation (2-55) and compute the value for the Compton wavelength of the electron. We use the electron rest energy and the convenient constant hc to get

$$\lambda_C = \frac{hc}{m_e c^2} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.5110 \times 10^6 \text{ eV}} = 0.002427 \text{ nm},$$

as quoted above. If we then let the incident x ray have wavelength $\lambda = 0.0711$ nm as in Compton's 1923 experiment, we find from Equation (2-54) that the wavelength of the x ray scattered at $\theta = 90°$ is

$$\lambda' = \lambda + \lambda_C = (0.0711 + 0.0024) \text{ nm} = 0.0735 \text{ nm}.$$

Figure 2-20 shows how this feature would appear alongside the classical Thomson component at $\lambda' = 0.0711$ nm. (The broadening of the observed wavelengths may be attributed to the fact that the target particle is not necessarily at rest, as assumed in the analysis.) For 90° x-ray scattering, the recoil of the electron has momentum components, parallel to the incident photon and antiparallel to the scattered photon, given by

$$P_x = p = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{0.711 \times 10^{-10} \text{ m}} = 9.32 \times 10^{-24} \text{ kg} \cdot \text{m/s}$$

and

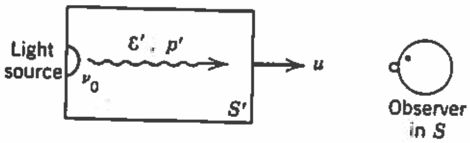
$$P_y = p' = \frac{h}{\lambda'} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{0.735 \times 10^{-10} \text{ m}} = 9.02 \times 10^{-24} \text{ kg} \cdot \text{m/s}.$$

We can compute the electron recoil angle ϕ from these quantities:

$$\tan\phi = \frac{P_y}{P_x} = \frac{9.02}{9.32} = 0.968 \quad \Rightarrow \quad \phi = 44°.$$

This calculation provides further illustration of the Compton effect as an exercise in relativistic elastic collisions.

Figure 2-21
Doppler effect for a photon from a moving source.



Example

Let us take the photon's energy and momentum from Equations (2-51) and (2-52) and introduce a momentum four-vector for the photon as

$$\not{p} = \begin{bmatrix} \mathbf{p} \\ i\varepsilon/c \end{bmatrix}, \quad \text{where} \quad p = \frac{\varepsilon}{c} = \frac{h\nu}{c}.$$

We know how the four-vector $\not{p}$ transforms under the Lorentz transformation to produce the photon's energy and momentum in another Lorentz frame. We can use this approach as an elegant alternative to our earlier derivation of the Doppler shift for the frequency of light, since both ε and p are directly related to the frequency ν. Figure 2-21 shows the notation for the case of a light source of frequency ν_0 at rest in S', where S' moves with speed u toward an observer at rest in S. We let the photon be directed at the observer so that we need only treat two components of the photon's momentum four-vector in the two Lorentz frames. Since S' is the rest frame of the source, the primed four-vector is

$$\not{p}' = \begin{bmatrix} p' \\ i\varepsilon'/c \end{bmatrix} = \frac{h\nu_0}{c}\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The corresponding components in S are found by applying the inverse of the Lorentz transformation defined in Equation (1-66):

$$\not{p} = \mathcal{L}^{-1}\not{p}' = \begin{bmatrix} \gamma & -i\gamma\beta \\ i\gamma\beta & \gamma \end{bmatrix}\begin{bmatrix} p' \\ i\varepsilon'/c \end{bmatrix},$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{and} \quad \beta = \frac{u}{c}.$$

We let the observed photon frequency be called $\bar{\nu}$ in S and perform the transformation as follows:

$$\frac{h\bar{\nu}}{c}\begin{bmatrix} 1 \\ i \end{bmatrix} = \gamma\frac{h\nu_0}{c}\begin{bmatrix} 1 & -i\beta \\ i\beta & 1 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix}$$
$$= \frac{h\nu_0}{c}\gamma\begin{bmatrix} 1+\beta \\ i(1+\beta) \end{bmatrix} = \frac{h\nu_0}{c}\gamma(1+\beta)\begin{bmatrix} 1 \\ i \end{bmatrix}.$$