HW 10

Problem 1 N9.3

The contribution to the anomaly by a given irreducible representation is determined by the trace of the product of a generator and the anti-commutator of two generators, namely, $\mathrm{Tr} T^A \{T^B T^C\}$ Here T A denotes a generator of the gauge group G. Show that for G = SU (5), the anomaly cancels between the 5^* and the 10. Note that for SU (N), we can, with no loss of generality, take A, B, and C to be the same, so that the anomaly is determined by the trace of a generator cubed (namely, $\mathrm{Tr} T^3$). It is rare that we get something cubed in physics, and so any cancellation between irreducible representations can hardly be accidental.

Easiest to use diagonal generator.

$$T = \mathrm{diag}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}\right)$$

Taking this diagonal generator, we evaluate the trace of the generator cubed in the 5* dimensional representation.

All entries are real: (multiply by 6)

$$T^* = \mathrm{diag}\,(2, 2, 2, -3, -3)$$

Cubing this matrix:

$$T^{*3} = \mathrm{diag}\,(8, 8, 8, -27, -27)$$

Taking the trace:

$$\mathrm{Tr}(T^{*3}) = -30$$

Now for the 10-dimensional representation:

Basis for 10-d representation:

$$(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)$$

Converting the 5-d diagonal matrix to a 10-d representation

$$T_{10\ \mathrm{d}} = \mathrm{diag}(d_1 + d_2, d_1 + d_3, d_1 + d_4, d_1 + d_5, d_2 + d_3, d_2 + d_4, d_2 + d_5, d_3 + d_4, d_3 + d_5, d_4 + d_5)$$

$$T_{10\ \mathrm{d}} = \mathrm{diag}(4, 4, -1, -1, 4, -1, -1, -1, -1, -6)$$

Cubing this diagonal matrix:

$$T_{10\ \mathrm{d}}^3 = \mathrm{diag}(64, 64, -1, -1, 64, -1, -1, -1, -1, -216)$$

Taking the trace:

$$\mathrm{Tr}T^3_{10\ \mathrm{d}} = 3(64) - 222 = 192 - 222 = -30$$

$$30 - (-30) = 0$$