single-particle wave functions: **Quantum Statistics**

$$\hat{H} = \hat{\mathcal{H}}(x_1) + \hat{\mathcal{H}}(x_2)$$ (same mass, etc.)

$$\hat{H}\,\psi(x_1, x_2) = E\,\psi(x_1, x_2)$$

$$= \psi_r(x_1)\,\psi_s(x_2) \quad : \quad \mathcal{H}\,\psi_r(x) = \varepsilon_r\,\psi_r(x)$$

single-part. wavefunc

$$E = \varepsilon_r + \varepsilon_s$$

particle permutation oper.

$$P\,\psi(x_1, x_2) = \psi(x_2, x_1)$$

permutation of particles

$$\boxed{P\,\psi(x_1, x_2) = \pm\psi(x_1, x_2)}$$

indistinguishability

($P^2 = P \Rightarrow \pm 1$ eigenvalue)

Symm: $$\psi_E(x_1, x_2) = \frac{1}{\sqrt{2}}\left\{\psi_r(x_1)\,\psi_s(x_2) + \psi_r(x_2)\,\psi_s(x_1)\right\}$$

$$\mathcal{H}\,\psi_E(x_1, x_2) = E\,\psi_E(x_1, x_2) \quad , \quad E = \varepsilon_r + \varepsilon_s$$

Antisymm $$\psi_E(x_1, x_2) = \frac{1}{\sqrt{2}}\left\{\psi_r(x_1)\,\psi_s(x_2) - \psi_r(x_2)\,\psi_s(x_1)\right\}$$

General: $$\hat{H} = \sum_{i=1}^{N} \hat{\mathcal{H}}(x_i) \qquad \hat{\mathcal{H}}(x)\,\psi_r(x) = \varepsilon_r\,\psi_r(x)$$

$r = 1, 2, \ldots,$

$$P\,\psi_E(x_1, x_2, \ldots, x_N) = \pm\,\psi_E(x_1, x_2, \ldots, x_N)$$ required

bosons: $$\psi_{Symm}(x_1, \ldots x_N) = C \sum_P P\,\psi_q(x_1)\,\psi_r(x_2) \cdots \psi_s(x_N) \qquad E = \sum_r \varepsilon_r$$

fermions: $$\psi_{Antisym}(x_1, \ldots, x_N) = C \sum_P \delta_P\, P\,\psi_q(x_1)\,\psi_r(x_2) \ldots \psi_s(x_N)$$

$\delta_P = \pm 1$ for even/odd permutations

C: normalization constant

Quantum Statistical Physics : the density operator

$|\psi\rangle$ (pure state) $\langle\psi|\psi\rangle = 1$

$\hat{A}$ observable: $\langle A \rangle = \langle\psi|\hat{A}|\psi\rangle$

$|e_i\rangle$ basis $\mathbb{1} = \sum_i |e_i\rangle\langle e_i|$

$$\langle A \rangle = \langle\psi|\hat{A}\sum_i |e_i\rangle\langle e_i|\psi\rangle = \sum_i \langle\psi|\hat{A}|e_i\rangle\langle e_i|\psi\rangle =$$

$$\sum_i \langle e_i|\underbrace{\psi\rangle\langle\psi|}_{\hat{\rho} = |\psi\rangle\langle\psi|}\hat{A}|e_i\rangle = \sum_i \langle e_i|\hat{\rho}\hat{A}|e_i\rangle = \mathrm{Tr}(\hat{\rho}\hat{A})$$

density matrix/operator

Specifically: $\hat{A} = \mathbb{1}$ (unity)

$\langle A \rangle = Tr(\hat{\rho}\hat{A}) \quad \Rightarrow \quad Tr(\hat{\rho}) = 1$

$$\hat{\rho}^2 = \underbrace{|\psi\rangle\langle\psi|}_{\hat{\rho}} \underbrace{|\psi\rangle\langle\psi|}_{\hat{\rho}} = |\psi\rangle\langle\psi| = \hat{\rho}$$

Ensemble: p_r, $|\psi_r\rangle$ $\quad \Rightarrow \quad \hat{\rho} = \sum_r p_r \underbrace{|\psi_r\rangle\langle\psi_r|}_{\hat{\rho}_r}$

$\sum_r p_r = 1$

mixed state

$$\langle A \rangle \equiv \sum_r p_r \langle \psi_r | \hat{A} | \psi_r \rangle = \sum_r p_r \, Tr(\hat{\rho}_r \hat{A}) = Tr\left[\left(\sum_r p_r \hat{\rho}_r\right)\hat{A}\right]$$

$$= Tr(\hat{\rho}\hat{A})$$

energy representation $|n\rangle$: $\hat{H}|n\rangle = E_n|n\rangle$ $\qquad p_n = \frac{e^{-\beta E_n}}{Z}$

$$\langle A \rangle = \sum_n p_n \langle n | \hat{A} | n \rangle = Tr(\hat{\rho}\hat{A})$$

$$\hat{\rho} = \sum_n \frac{e^{-\beta E_n}}{Z} |n\rangle\langle n| = \frac{1}{Z} e^{-\beta \hat{H}}$$

$Tr\,\hat{\rho} = 1 \quad \Rightarrow \quad \boxed{Z = Tr\left(e^{-\beta\hat{H}}\right)}$

$$\langle A \rangle = Tr\left(\frac{e^{-\beta\hat{H}}}{Z}\hat{A}\right)$$

$\hat{H} \to \hat{H} - \mu\hat{N} \qquad \hat{\rho} = \frac{1}{Z_G} e^{-\beta(\hat{H}-\mu\hat{N})} \qquad \boxed{Z_G = Tr\left(e^{-\beta(\hat{H}-\mu\hat{N})}\right)}$

density op: $\hat{\rho} = \dfrac{e^{-\beta \hat{H}}}{Z}$ $\qquad Z = Tr\,(e^{-\beta \hat{H}})$

$$e^{-\beta \hat{H}} = e^{-\beta \hat{H}}\,\mathbb{1} = e^{-\beta \hat{H}} \sum_k |E_k\rangle\langle E_k| = \sum_k e^{-\beta E_k} |E_k\rangle\langle E_k|$$

density matrix in coord. repr.: N particles, canonical ensemble

$$\langle x | \hat{\rho} | x' \rangle \qquad Tr(\hat{\rho}) = \int dx \langle x | \hat{\rho} | x \rangle = 1$$

$$Z = Tr(e^{-\beta \hat{H}}) = \int dx \langle x | e^{-\beta \hat{H}} | x \rangle$$

$$\langle x | e^{-\beta \hat{H}} | x' \rangle = \sum_k e^{-\beta E_k} \langle x | E_k \rangle \langle E_k | x' \rangle =$$

$$\boxed{= \sum_k e^{-\beta E_k} \psi_k(x) \psi_k^*(x')}$$

$x = \bar{x}_1, \bar{x}_2, \dots, \bar{x}_N$
$k = \bar{k}_1, \bar{k}_2, \dots, \bar{k}_N$

N free and <u>indistinguishable</u> particles, in a box, $V = L^3$

single particle wave function $\Rightarrow$ $\boxed{\varphi_{\bar{k}_i}(\bar{x}_i) = \dfrac{1}{\sqrt{V}} e^{i \bar{k}_i \cdot \bar{x}_i}}$ $\qquad \bar{k}_i = \dfrac{2\pi}{L} \bar{n}_i \qquad n_i^{x,y,z} = (0,1,2,\dots$

$\boxed{\text{Fermions}}$

$$\psi_{\bar{k}_1 \bar{k}_2 \dots \bar{k}_N}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N) = \frac{1}{\sqrt{N!}} \sum_P \delta_P\, \varphi_{\bar{k}_1}(P\bar{x}_1) \varphi_{\bar{k}_2}(P\bar{x}_2) \dots \varphi_{\bar{k}_N}(P\bar{x}_N)$$

$$E_{k_1, k_2, \dots k_N} = \frac{\hbar^2}{2m}(k_1^2 + k_2^2 + \dots + k_N^2)$$

$\delta_P = (\pm 1)^{[P]}$

for Fermions

$$\langle x_1, x_2, \ldots x_N | e^{-\beta \hat{H}} | x_1', x_2', \ldots, x_N' \rangle =$$

$$\sum_{k_1,k_2,\ldots k_N}{}' e^{-\frac{\hbar^2}{2m}(k_1^2+k_2^2+\ldots k_N^2)} \times$$

$$\times \frac{1}{N!}\left(\sum_P \delta_P \varphi_{k_1}(Px_1)\varphi_{k_2}(Px_2)\ldots\varphi_{k_N}(Px_N)\right)\times\left(\sum_{\tilde{P}} \delta_{\tilde{P}} \varphi^*_{k_1}(\tilde{P}x_1')\varphi^*_{k_2}(\tilde{P}x_2')\ldots\varphi^*_{k_N}(\tilde{P}x_N')\right)$$

$$= \sum_{k_1,k_2,\ldots,k_N}{}' e^{-\frac{\beta\hbar^2}{2m}(k_1^2+k_2^2+\ldots+k_N^2)} \times \frac{1}{N!}$$

$$\times\left(\sum_P \delta_P \varphi_{k_1}(P1)\varphi_{k_2}(P2)\ldots\varphi_{k_N}(PN)\right)\times\sum_{\tilde{P}} \delta_{\tilde{P}} \varphi^*_{\tilde{P}k_1}(1')\varphi^*_{\tilde{P}k_2}(2')\ldots\varphi^*_{\tilde{P}k_N}(N')$$

(extend $\sum'_{k_1 k_2 \ldots k_N}$ to over all k_i permutations : $\sum_{k_1,k_2,\ldots k_N}$ and keep only the first term in the $\sum_{\tilde{P}}$ summation)

$$= \frac{1}{N!}\sum_{k_1,k_2,\ldots,k_N} e^{-\frac{\beta\hbar^2}{2m}(k_1^2+k_2^2+\ldots+k_N^2)} \sum_P \delta_P \varphi_{k_1}(P1)\varphi^*_{k_1}(1')\varphi_{k_2}(P2)\varphi^*_{k_2}(2')\ldots\varphi_{k_N}(PN)\varphi^*_{k_N}(N')$$

$$\left[\sum_{k_i} = \left(\frac{L}{2\pi}\right)^3\int d^3k_i = \frac{V}{(2\pi)^3}\int d^3k_i\right] \qquad \left(\Delta n_i^x = \frac{L}{2\pi}\Delta k_i^x\right)$$

$$= \frac{1}{N!}\frac{V^N}{(2\pi)^{3N}}\int d^3k_1 d^3k_2\ldots d^3k_N\; e^{-\frac{\beta\hbar^2}{2m}(k_1^2+k_2^2+\ldots k_N^2)}\sum_P \delta_P \varphi_{k_1}(P1)\varphi^*_{k_1}(1')\ldots\varphi_{k_N}(PN)\varphi^*_{k_N}(N')$$

$$= \frac{1}{N!}\frac{1}{(2\pi)^{3N}} \sum_P \delta_P \int d^3k_1 \dots d^3k_N \, e^{-\frac{\beta\hbar^2}{2m}(k_1^2+\dots+k_N^2)} e^{ik_1(P1-1')} \dots e^{ik_N(PN-N')}$$

$$= \frac{1}{N!}\frac{1}{(2\pi)^{3N}} \sum_P \delta_P \underbrace{\int d^3k_1 \, e^{-\frac{\beta\hbar^2}{2m}k_1^2 + ik_1(P1-1')}}_{\left(\frac{2\pi mkT}{\hbar^2}\right)^{3/2} e^{-\frac{m}{2\beta\hbar^2}(P1-1')^2}} \dots \int d^3k_N \, e^{-\frac{\beta\hbar^2}{2m}k_N^2 + ik_N(PN-N')}$$

$$\hbar = \frac{h}{2\pi}$$

$$= \frac{1}{N!}\left(\frac{2\pi mkT}{h^2}\right)^{3N/2} \sum_P \delta_P \, f(P1-1') f(P2-2') \dots f(PN-N')$$

$$f(u) \equiv e^{-\frac{m}{2\beta\hbar^2}u^2} \qquad \lambda \equiv \left(\frac{h^2}{2\pi mkT}\right)^{1/2}$$

True for Bosons as well: with $\delta_P = 1$

diagonal elements:

$$\langle x_1, x_2, \dots, x_N | e^{-\beta \hat{H}} | x_1, x_2, \dots, x_N \rangle = \frac{1}{N!}\frac{1}{\lambda^{3N}} \sum_P \delta_P \, f(Px_1 - x_1) \dots f(Px_N - x_N)$$

$$\boxed{f(u) = e^{-\pi u^2/\lambda^2}}$$

$$Z_N = \int d^3x_1 \dots d^3x_N \, \langle x_1, x_2, \dots, x_N | e^{-\beta \hat{H}} | x_1, x_2 \dots x_N \rangle$$

$$\sum_P \delta_P \, f(Px_1 - x_1) f(Px_2 - x_2) \dots f(Px_N - x_N) = \qquad r_{ij} = |\bar{x}_i - \bar{x}_j|$$

$$= 1 \pm \sum_{i<j} f(r_{ij}) f(r_{ji}) + \dots$$

$$Z_N = \frac{1}{N!}\frac{1}{\lambda^{3N}} \int d^3x_1 d^3x_2 \dots d^3x_N \left\{ 1 \pm \sum_{i<j} f^2(r_{ij}) \right\} =$$

$$= \frac{1}{N!}\frac{1}{\lambda^{3N}} \left\{ V^N \pm \frac{N(N-1)}{2} V^{N-1} \int d^3x \, f^2(\bar{x}) \right\} = \frac{1}{N!}\frac{V^N}{\lambda^{3N}} \left\{ 1 \pm \frac{N(N-1)}{2V} \int_0^\infty 4\pi r^2 dr f(\dots$$

$f(r_{ij})$ vanishes rapidly for $\left(\frac{V}{N}\right)^{1/3} \gg \lambda$

classical limit $\quad \lambda^3\left(\frac{N}{V}\right) \ll 1$

$$Z_N \simeq \frac{1}{N!}\frac{1}{\lambda^{3N}} V^N = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N$$

Illustration: 2 particles

$$e^{-\frac{2\pi|\bar{x}_1-\bar{x}_2|^2}{\lambda^2}}$$

$$Z_2 = \frac{1}{2}\frac{1}{\lambda^6}\int d^3x_1\, d^3x_2 \left(1 \pm f^2(\bar{x}_1-\bar{x}_2)\right)$$

$$= \frac{1}{2}\frac{1}{\lambda^6}\left\{V^2 \pm V\int d^3x\, f^2(\bar{x})\right\} = \frac{1}{2}\left(\frac{V}{\lambda^3}\right)^2\left\{1 \pm \frac{1}{V}\int_0^\infty 4\pi r^2 dr\, f^2(r)\right\}$$

$$= \frac{1}{2}\left(\frac{V}{\lambda^3}\right)^2\left\{1 \pm \frac{1}{2^{3/2}}\left(\frac{\lambda^3}{V}\right)\right\} \approx \frac{1}{2}\left(\frac{V}{\lambda^3}\right)^2$$

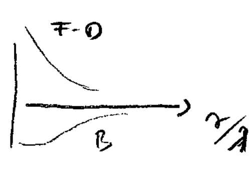
$$\langle x_1, x_2 | \hat{\rho} | x_1, x_2 \rangle = \frac{1}{V^2}\left[1 \pm e^{-\frac{2\pi r_{12}^2}{\lambda^2}}\right]$$

$$\langle x_1, x_2 | e^{-\beta\hat{H}} | x_1, x_2 \rangle = \frac{1}{2\lambda^6}\left\{1 \pm e^{-\frac{2\pi r_{12}^2}{\lambda^2}}\right\}$$

$r_{12} = |\bar{x}_2 - \bar{x}_1|$

probability density that a pair of particles are separated by a distance r: $\frac{1}{V^2}\left[1 \pm e^{-\frac{2\pi r^2}{\lambda^2}}\right]$

$$v_s = -kT \ln\left(1 \pm e^{-\frac{2\pi r_{12}^2}{\lambda^2}}\right)$$



Summary:

The density operator in the canonical ensemble:

$$\hat{\rho} = \frac{e^{-\beta \hat{H}}}{Z} \qquad Z = Tr(e^{-\beta H})$$

$$e^{-\beta \hat{H}} = e^{-\beta \hat{H}} \mathbb{1} = e^{-\beta \hat{H}} \sum_k |k\rangle\langle k|$$

where $\hat{H}|k\rangle = E_k|k\rangle$

$\{|k\rangle\}$ is a complete set of eigenstates of $\hat{H}$

$$e^{-\beta \hat{H}} = e^{-\beta \hat{H}} \sum_k |k\rangle\langle k| = \sum_k e^{-\beta \hat{H}} |k\rangle\langle k| = \sum_k e^{-\beta E_k} |k\rangle\langle k|$$

The matrix elements of the $e^{-\beta \hat{H}}$ operator

$$\langle x| e^{-\beta \hat{H}} |x'\rangle = \langle x| \sum_k e^{-\beta E_k} |k\rangle\langle k|x'\rangle =$$

$$= \sum_k e^{-\beta E_k} \underbrace{\langle x|k\rangle}_{\psi_k(x)} \underbrace{\langle k|x'\rangle}_{\psi_k^*(x')} = \sum_k e^{-\beta E_k} \psi_k(x)\, \psi_k^*(x')$$

for an N-particle system:

$x = x_1, x_2, \dots, x_N$

k; appropriate set of "good" quantum numbers

for free particles: $k = k_1, k_2, \dots, k_N$

where k_i's are the single-particle quantum numbers.

Density Matrix in Coordinate Representation for $\boxed{\text{Bosons}}$

$$\langle x_1 \dots x_N | \hat{\rho} | x_1' \dots x_N' \rangle = \frac{1}{Z} \langle x_1 \dots x_N | e^{-\beta \hat{H}} | x_1' \dots x_N' \rangle$$

in the canonical ensemble, where $Z = \mathrm{Tr}\,(e^{-\beta \hat{H}})$

The normalized fully symmetric boson wave function for N particle

$$\psi_{k_1, k_2 \dots k_N}(x_1, x_2, \dots x_N) = \sqrt{\frac{n_1! n_2! \dots}{N!}} \sum_P{}' \underbrace{\varphi_{k_1}(Px_1)\, \varphi_{k_2}(Px_2)}_{n_1} \dots \underbrace{\varphi_{k_N}(Px_N)}_{n_2} \dots$$

where $n_1, n_2, \dots$ are the number of k_i wave vectors which have the same value, and $\sum'$ runs over the permutations in which particles do not remain in the same state.

Thus, different N-particle wave functions differ in their partition $(n_\alpha, n_\beta, \dots)$

$$\langle x_1 \dots x_N | e^{-\frac{\beta \hbar^2}{2m}(k_1^2 + k_2^2 + \dots + k_N^2)} | x_1' \dots x_N' \rangle =$$

$$= \sum_{\{n_1, n_2, \dots\}}{}' e^{-\frac{\beta \hbar^2}{2m}(k_1^2 + \dots + k_N^2)} \psi_{k_1 \dots k_N}(x_1, \dots, x_N)\, \psi^*_{k_1, \dots, k_N}(x_1', \dots, x_N') =$$

$$= \sum_{\{n_1, n_2, \dots\}}{}' e^{-\frac{\beta \hbar^2}{2m}(k_1^2 + \dots k_N^2)} \left(\frac{n_1! n_2! \dots}{N!}\right) \sum_P{}' \varphi_{k_1}(Px_1) \dots \varphi_{k_N}(Px_N) \times$$

$$\times \sum_{\tilde{P}}{}' \varphi^*_{k_1}(\tilde{P}x_1') \dots \varphi^*_{k_N}(\tilde{P}x_N') =$$

$= \langle x | e^{-\beta \hat{H}} | x' \rangle$ for short $(x \equiv x_1, x_2, \dots, x_N)$

where $\sum'_{\{n_1, n_2 \ldots\}}$ is a sum over the k_i wave vectors with distinct $(n_1, n_2, \ldots,)$ partitioning

since the exponent $(k_1^2 + k_2^2 + \ldots + k_N^2)$ and the fully symmetric wave functions are insensitive to interchanging k_i's within the group of n_α, we can change this sum to one over <u>all</u> k_i's <u>independently</u>, and compensating by a factor $\dfrac{1}{\left(\dfrac{N!}{n_1! n_2! \ldots}\right)}$

$$\langle x | e^{-\beta \hat{H}} | x' \rangle = \sum_{k_1, k_2 \ldots k_N} e^{-\frac{\beta \hbar^2}{2m}} \left(\frac{n_1! n_2! \ldots}{N!} \right)^2 \sum_P{}' \varphi_{k_1}(Px_1) \ldots \varphi_{k_N}(Px_N) \times \sum_{\tilde{P}}{}' \varphi^*_{k_1}(\tilde{P}x_1') \ldots \varphi^*_{k_N}(\tilde{P}x_N')$$

also, $(n_1! n_2! \ldots) \sum_P{}' \underbrace{\varphi_{k_1}(Px_1) \varphi_{k_2}(Px_2) \ldots}_{n_1} \underbrace{\varphi_{k_N}(Px_N)}_{n_2 \ldots} =$

$$= \sum_P \varphi_{k_1}(Px_1) \varphi_{k_2}(Px_2) \ldots \varphi_{k_N}(Px_N)$$

where the sum now runs over <u>all</u> permutations P

$$\langle x | e^{-\beta \hat{H}} | x' \rangle = \sum_{k_1, k_2, \ldots k_N} e^{-\frac{\beta \hbar^2}{2m}} \frac{1}{(N!)^2} \left(\sum_P \varphi_{k_1}(Px_1) \ldots \varphi_{k_N}(Px_N) \right) \times \left(\sum_{\tilde{P}} \varphi^*_{\tilde{P}k_1}(x_1') \ldots \varphi_{\tilde{P}k_N}(x_N') \right)$$

(permutations over x_i and k_i's yield the same)

$$= \frac{1}{N!} \sum_{k_1, k_2, \ldots k_N} e^{-\frac{\beta \hbar^2}{2m}} \sum_P \varphi_{k_1}(Px_1) \varphi^*_{k_1}(x_1') \ldots \varphi_{k_N}(Px_N) \varphi^*_{k_N}(x_N')$$

Single-particle wave functions in a 3-dim box with linear size L:

$$\psi_{\bar{k}}(\bar{x}) = \frac{1}{\sqrt{V}} e^{i\bar{k}\cdot\bar{x}} \qquad k^{x,y,z} = \frac{2\pi}{L} n^{x,y,z} \qquad n^{x,y,z} = 0,1,2,\ldots$$

$$\sum_{\bar{k}} = \sum \left(\frac{L}{2\pi}\right)^3 (\Delta k)^3 \to \frac{V}{(2\pi)^3} \int d^3k$$

$$\langle x_1 x_2 \ldots x_N | e^{-\beta \hat{H}} | x_1' x_2' \ldots x_N' \rangle = \frac{1}{N!} \frac{V^N}{(2\pi)^{3N}} \int d^3k_1 d^3k_2 \ldots d^3k_N \; e^{-\frac{\beta\hbar^2}{2m}(k_1^2 + k_2^2 + \ldots + k_N^2)} \times$$

$$\times \sum_P \frac{1}{V} e^{i\bar{k}_1 (P\bar{x}_1 - \bar{x}_1')} \frac{1}{V} e^{i\bar{k}_2\cdot(P\bar{x}_2 - \bar{x}_2')} \ldots \frac{1}{V} e^{i\bar{k}_N\cdot(P\bar{x}_N - \bar{x}_N')} =$$

$$= \frac{1}{N!} \frac{1}{(2\pi)^{3N}} \sum_P \underbrace{\int d^3k_1 \, e^{-\frac{\beta\hbar^2}{2m}k_1^2 + i\bar{k}_1(P\bar{x}_1 - \bar{x}_1')}}_{\left(\frac{2\pi m kT}{\hbar^2}\right)^{3/2} e^{-\frac{m}{2\beta\hbar^2}(P\bar{x}_1 - \bar{x}_1')^2}} \ldots \int d^3k_N \, e^{-\frac{\beta\hbar^2}{2m}k_N^2 + i\bar{k}_N(P\bar{x}_N - \bar{x}_N')}$$

$$= \frac{1}{N!} \left(\frac{2\pi m kT}{h^2}\right)^{\frac{3N}{2}} \sum_P f(P\bar{x}_1 - \bar{x}_1') f(P\bar{x}_2 - \bar{x}_2') \ldots f(P\bar{x}_N - \bar{x}_N')$$

$$\text{where } f(u) \equiv e^{-\frac{m}{2\beta\hbar^2}u^2} = e^{-\frac{\pi u^2}{\lambda^2}}$$

$$\text{thermal wavelength: } \lambda \equiv \left(\frac{h^2}{2\pi m kT}\right)^{1/2}$$

$$\boxed{\langle x_1 \ldots x_N | e^{-\beta \hat{H}} | x_1' \ldots x_N' \rangle = \frac{1}{N!} \frac{1}{\lambda^{3N}} \sum_P f(P\bar{x}_1 - \bar{x}_1') f(P\bar{x}_2 - \bar{x}_2') \ldots f(P\bar{x}_N - \bar{x}_N')}$$

This is identical to what we obtained for fermions, except for the $\delta_P = (-1)^{[P]}$ in the $\sum_P$ summation

Thus, <u>the general result is</u> :

$$\langle x_1 x_2 \ldots x_N | e^{-\beta \hat{H}} | x_1' x_2' \ldots x_N' \rangle =$$

$$= \frac{1}{N!} \frac{1}{\lambda^{3N}} \sum_P \delta_P \, f(P\bar{x}_1 - \bar{x}_1') \, f(P\bar{x}_2 - \bar{x}_2') \ldots f(P\bar{x}_N - \bar{x}_N')$$

where $\delta_P = 1$ for bosons
and $\delta_P = (-1)^{[P]}$ for fermions ($[P]$ is the order of the permutation)

$$Z_N = Tr(e^{-\beta \hat{H}}) = \int d^3x_1 d^3x_2 \ldots d^3x_N \langle x_1 x_2 \ldots x_N | e^{-\beta \hat{H}} | x_1 x_2 \ldots x_N \rangle$$

(trace is a sum (integral when continuous variables) over the "diagonal" elements)