**Analog Computer Analysis & Theory**

# Section I

- Provide labelled scope traces of each of the circuit configurations described in the Analog Computer Instructions document. The labels should briefly describe the particular configuration used for the measurement.

# Section II

- Show a circuit diagram of your simple harmonic oscillator with numerical values
- Use the formulas of integrating and adding circuits to numerically describe your circuit: what is the value of the factor *k/m*? (*Hint: The Appendix section in this document has a discussion of how one could perform this analysis.*)
- Fit a sine curve to your simple harmonic oscillator. Show that there is some natural damping by including an exponential term in your fit. Determine the frequency of the oscillation and the damping constant.
- Compare the frequency of the circuit with the mathematically expected frequency according to your computed value of *k/m*.

# Section III

- Determine the damping constants of both oscillators and their ratio from the waveforms. Apply a correction to this, taking into consideration the natural damping of the circuit
- Compare the ratio of the constants to what you would expect based on the gain of the circuit, that is determined by the choice of components, that drives the damping term in the circuit.

# Section IV

- Show the scope trace of the beat pattern
- Provide the general solution of the mathematical problem. You do not need to determine any numerical solutions.
- Based on the general solution, explain what beat frequency you would expect and why. Base this expectation on the frequency of the sine wave from the function generator and on the frequency from part II.
- Compare the calculated beat frequency to your scope trace.

# Analog Computer Theory

When we think of computers we think of digital computers, which store and manipulate discrete data in binary form. Analog computers use any set of electrical and mechanical properties to model solutions to problems. By using the electrical properties of circuits, we can model differential equations as a rate of change in voltage.

## Op-Amps (Operational Amplifier)

An op-amp has two voltage inputs: one positive (V+) non-inverting input, one negative (V-) inverting input; two power inputs: one positive ($V_{S+}$), one negative ($V_{S-}$); and a voltage output ($V_{out}$), shown in figure 1.

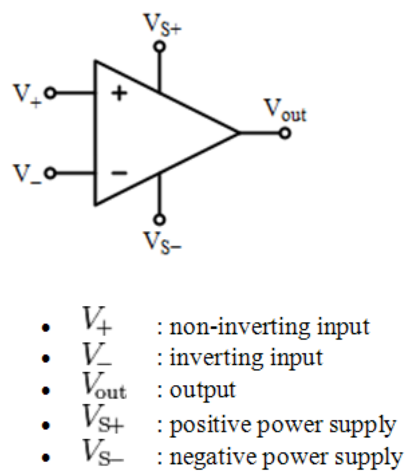


Figure 1: Basic op-amp symbol

Recall “**The Golden Rules”** for understanding Op-Amp operation with external feedback (from Electronic Lab documentation):

Here are some simple rules for working out op-amp behavior with external feedback:

**First**, the gain of the op-amp (open loop i.e. without feedback - ref. Figure 1) is so high that a fraction of a millivolt between its input terminals will swing the output over its full range. If the non-inverting (+) input is grounded, and the inverting (-) input has a small positive input voltage, the op-amp will output -9V (*if +/-9V are applied to the power terminals*), because the voltage difference is positive and applied to the non-inverting input. Open-loop circuits are valuable in electronics when measuring high and low signals, where high and low effectively correspond to ON and OFF states, respectively, but for our application, closed-loop circuits are much more valuable.

**GOLDEN RULE 1**: The output attempts to do whatever is necessary to make the voltage difference between the inputs zero

Second, op-amps draw very little input current.

**GOLDEN RULE 2**: The inputs (both the inverting and non-inverting) draw no current

Op-amps have two powering terminals - for our lab, these are +9V and -9V using the rechargeable batteries. (*Note: There is a voltage range for powering the terminals. In general, you will need to look up the datasheet for the specific op-amp to identify the range but for this lab, the +/-9V is valid for all the Op-Amps that are found in the lab*).

## Inverting Amplifier

Closed loop circuits include a feedback loop from the output, typically to the negative input.The behavior of the inverting op-amp was covered in our earlier Electronic Lab.  A negative feedback, closed-loop circuit is shown in figure 2, where $R\_f$ is the mechanism of the feedback.

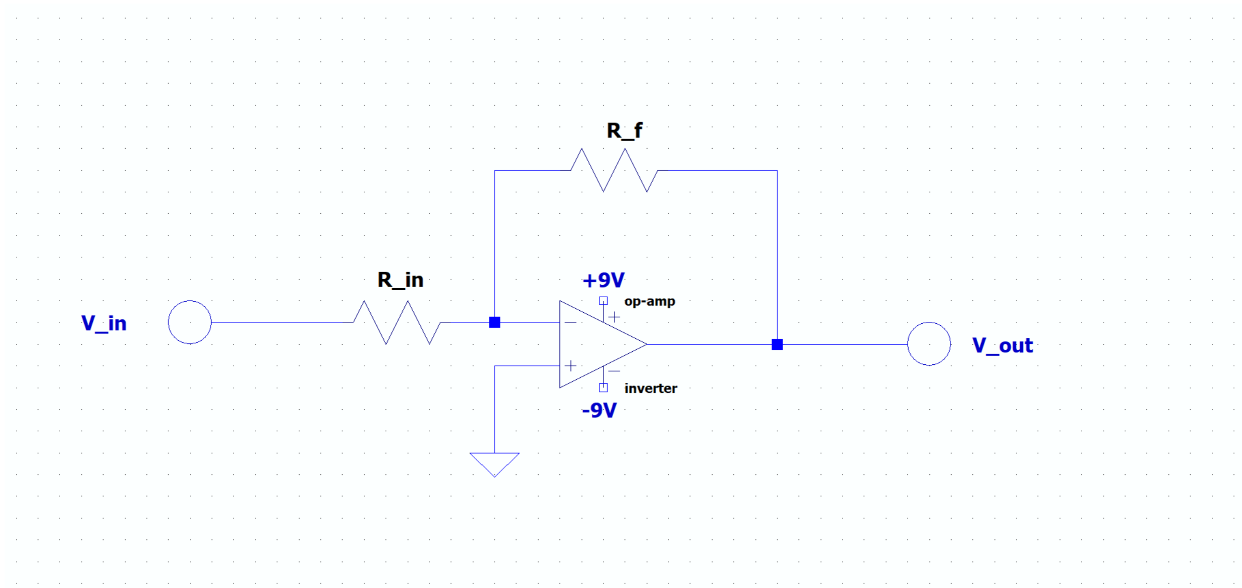


Figure 2: closed-loop circuit

In a closed-loop circuit, the op-amp adjusts output voltage such that the input voltage in both inputs are close to the same, meaning when $V_+$ is grounded, and therefore is equal to zero, the op-amp will decrease the potential difference of $V_-$ such that it too will go to zero. This satisfies the rule that current cannot flow into the input terminals. The current, $I_{in}$ , into the input resistor $R\_in$ therefore must be the same as the current, $I_f$ , in the feedback resistor $R\_f$ according to Kirchhoff's current law. Notice the use of Ohm's Law.

$$I_{in} + I_f = 0$$

$$I_{in} = -I_f$$

$$\frac{V\_in}{R\_in} = -\frac{V\_out}{R\_f}$$

$$-R\_f \cdot \frac{V\_in}{R_{in}} = V\_out$$

Changing the values of $R\_in$ and $R\_f$ allows us to change the output voltage accordingly. Notice that if $R\_in = R\_f$, then $V\_out = -V\_in$ , which is why this circuit is known as an **inverting circuit**.

## Integrating Amplifier

By using a reactive element in the feedback loop of the op-amp, one can develop an integrator (figure 3). In this case, the feedback is performed with a capacitor.

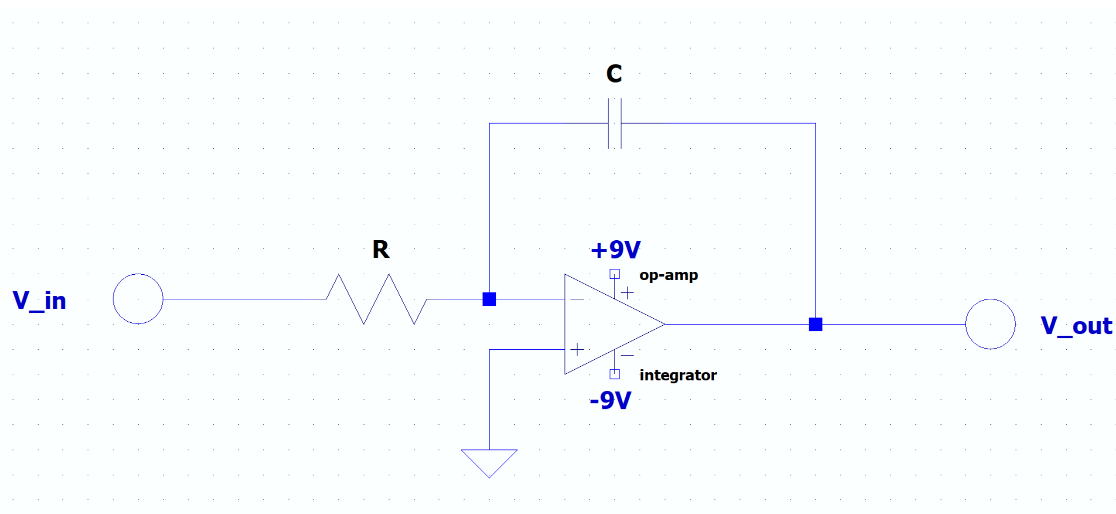


Figure 3: Integrating Amplifier

The current flowing through the capacitor follows a similar analysis as for the inverting amplifier (figure 2). However, unlike the resistor, this feedback current, $I_f$, generates a change in the voltage across the capacitor which is essentially *C.dV_out/dt*.

$$I_{in} = \frac{V\_in}{R\_in} = -C.\frac{dV\_out}{dt} = -I_f$$

$$\therefore V_{out} = V_o(t=0) - \frac{1}{R\_in \cdot C}\int V\_out \cdot dt$$

## Differentiating Amplifier

For differentiating amplifiers, the capacitor is now at the input of the op-amp and the resistor is used as the feedback element. Figure 4 shows the circuit of an op-amp performing differentiation of the input signal, *V_in*.

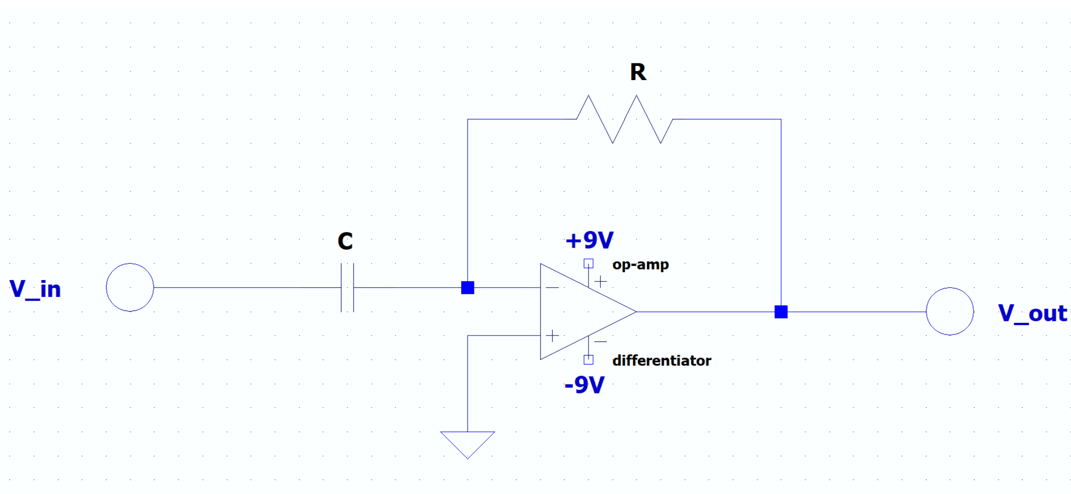


Figure 4: Differentiating Amplifier

Now the input current, $I_{in}$, is driven by the rate of change of input voltage *V_in* across the capacitor, *C*.

$$I_{in} = C \cdot \frac{dV\_in}{dt} = -\frac{V\_out}{R} = -I_f$$

$$\therefore V\_out = -RC\frac{dV\_in}{dt}$$

## Summing Amplifier

Op-amps can also be used for voltage addition following the same application of Kirchhoff's current law in a closed-loop, negative feedback circuit. The current flowing in the feedback loop, $I_f$, is the sum of the individual currents that make up $I_{in}$. Notice below, the voltage is summed and inverted. Inversion is a small price to pay for the simplicity of a negative feedback loop circuit, as opposed to a positive feedback loop circuit, which will not be discussed in this lab. Inversion and addition are shown in figure 5:

$$I_{in} = \frac{V\_1}{R\_1} + \frac{V\_2}{R\_2} + \cdots + \frac{V_N}{R\_N} = -\frac{V_{out}}{R_f} = -I_f$$

$$-\frac{R\_f}{R\_1} \cdot V\_1 - \frac{R\_f}{R\_2} \cdot V\_2 \cdots - \frac{R\_f}{R\_N} \cdot V\_N = V\_out$$

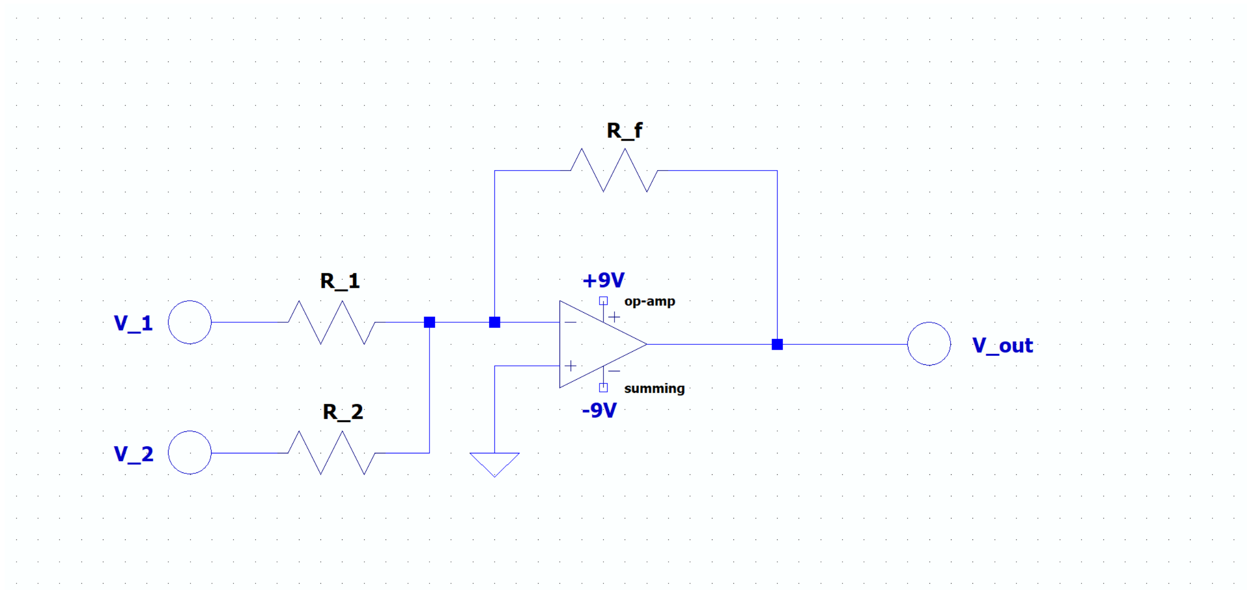


Figure 5: Example of a summing amplifier with two inputs.

Input and output voltages are variables; for the harmonic oscillator, the input voltage is the initial displacement of the mass. The computer solution is simply a voltage waveform whose time dependency is the same as that of the desired variable.

## Curve fitting for exponentially decaying sine curves

All of the fits in this lab can be modelled using the product of a sine function and an exponential term. This kind of curve fit therefore requires five fit parameters: the amplitude, the voltage offset, the frequency and phase of the sign, and the constant in the exponential term.
The ***scipy.odr*** curve fit is able to determine all parameters within one fit very efficiently. Remember to include the number of parameters as degrees of freedom when computing ***chi squared***. If you get a fit that is obviously very wrong, include an argument with an array of initial guesses to the argument of the fit function, e.g. p0=[1, 2, 3, 4, 5] in the order of the parameters of your initial function. You can use the "measure" function on the scope to get initial guesses for your parameters e.g. amplitude, frequency and decay time.

## Differential Equations and their solutions

Since this is a model and not a physical system with a mass and a spring, we use the usual nomenclature of assigning *m* to the mass attached to the spring and *k* to the spring constant. The familiar general solutions below should help you set up the curve fit functions. It might be expected to also see an offset from having the square pulse as a constant added to the differential equation.

Equations that define motion of a simple harmonic oscillator:

$$F = -kx$$

$$\ddot{x} = -\frac{k}{m} \cdot x$$

With the familiar general solution:

$$x(t) = A \cdot cos(\omega t - \varphi) = A \cdot \cos(\sqrt{\frac{k}{m}} \cdot t - \varphi)$$

Differential equation for the damped or driven harmonic oscillator are :

$F = -kx + \gamma\dot{x}$

$\ddot{x} = -\frac{k}{m} \cdot x + \frac{\gamma}{m} \cdot \dot{x}$

With the general solution:

$$x(t) = A \cdot e^{-\frac{\gamma}{2m}t} \cdot cos(\omega t - \varphi) = A \cdot e^{-\frac{\gamma}{2m}t} \cdot \cos(\sqrt{\frac{k}{m}} \cdot t - \varphi)$$

The damped and driven harmonic oscillator differ only in the sign of γ - when this term is positive the system is damped and when negative it is driven. In this experiment, we will focus only on dampening the simple harmonic oscillator. Either way, this is an undercritically damped/driven harmonic oscillator because γ<<k. The general solution stated above is the real part only, since the quadratic formula returns complex roots for the under critical case when obtaining the general solution.

# Appendix - How to relate Op-Amp Integrators to Differential Equations

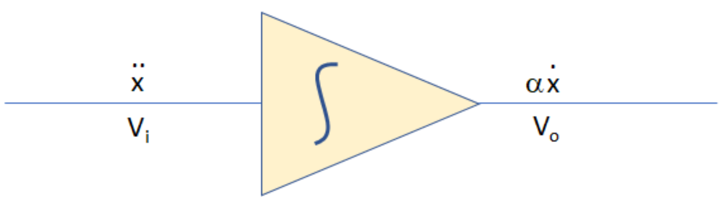


Figure 5: Symbolic representation of the Op-Amp Integrator (first integrator)

Figure 5 sets up the essential question of how does one relate the proportionality factor αto the fundamental values of the passive components that create an integrator from an op-amp. The equation that drives the integration process for an op-amp configured for continuous integration is:

$$V_o = -\frac{1}{RC}\int V_i \, dt \qquad \text{(A1)}$$

When compared with the integration process described in Figure 5, we can create the following relationships:

$$V_o = \alpha\dot{x}$$

and

$$V_i = \ddot{x} \tag{A2}$$

Substituting (A2) into (A1) yields the following equations:

$$\alpha \dot{x} = -\frac{1}{RC}\int V_i \, dt = -\frac{1}{RC}\int \ddot{x} \, dt \tag{A3}$$

Recognizing that:

$$\dot{x} = \int \ddot{x} \, dt$$

implies that

$$\alpha = -\frac{1}{RC} \tag{A4}$$

Equations A1-A4 can be applied for the first integrator that we will set up for the simple harmonic oscillator.

The second integrator which generates a term proportional to *x* has the following symbolic diagram:

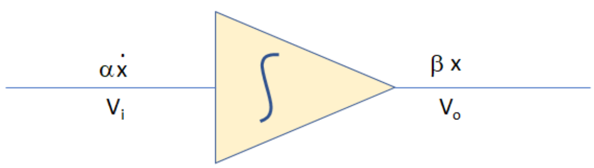


Figure 6: Symbolic representation of the Op-Amp Integrator (second integrator)

Using the same type of relations shown in equations A1-A4, what is the value of $\beta$ in terms of the passive components of both the first and second integrator? Note that in this experiment we want to keep both integrators identical, although the actual values of the passive components may differ slightly when measured with the DMM.

The final step in completing the circuit is to include the addition or summing amplifier. This inverts the input signals with an associated gain that depends on the ratio of the feedback resistor and input resistor (figure 5). Compute the final output signal (i.e. the proportionality constant for $x$ ) that will be tied back to the input of the first integrator. This would be equivalent to $k/m$ which is related to the oscillator frequency, $\omega_0$.