# Monday, April 8th, 2024

**Solar Eclipse**

No in-person class in JROWL 2C22 between 2pm and 3:50 pm.

Lecture slides and video are available for self-study on the course website in LMS.

Key points of the topic will be reviewed in class on April 11th.

**Solar Eclipse Watch Party sponsored by the Astrophysical Society with the Schools of Science and Engineering**

When: Monday, April 8, 2024 2:00 PM - 4:00 PM
Where: '86 Field
Cost: free

Description: Join your friends, colleagues, staff, faculty at the Campus Watch party for the Solar Eclipse. Free Glasses, displays and fun science experiments for all to enjoy. The party will be on the '86 field. Come early to get a good seat. For more information, contact the Rensselaer Union- ext 6505.

# Class 20 (04/04/24)

Potentials & Fields I

# Remaining Course Topics

- Potentials & Fields → Radiation Fields

  ( → Lagrangian / Hamiltonian of the Electromagnetic Field, Quantum Electrodynamics )

- Relativistic Electrodynamics

# The topics discussed previously:

So far, we looked at Maxwell's equations for the scenarios that we do not have free charge $\rho$ and current $\vec{J}$, i.e. $\rho=0$, $\vec{J}=0$, in the space where we want to find $\vec{E}$ and $\vec{B}$.

We saw that for the scenarios, the solutions for the electric & magnetic fields are transverse electromagnetic waves which travel in vacuum at the speed of light $c$, in dielectrics at reduced speed $v=\frac{c}{n}$, and exponentially attenuated waves in conductors.

# The new topic today:

Now, we consider Maxwell's equations for scenarios where $\rho \neq 0$ and $\vec{J} \neq 0$ in the space where we want to find $\vec{E}$ and $\vec{B}$.

Maxwell's equations

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ , $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$ , $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dot{\vec{E}}$

Electric field $\vec{E}$ and magnetic field $\vec{B}$ are derived from potentials $\vec{A}$ and $V$ as: $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$.

# Derivation of the inhomogeneous wave equation for potentials **A** (**r**,t) and V (**r**,t) in the Lorentz Gauge

Let's put $\vec{E}$ & $\vec{B}$ in Maxwell's eqs. and derive the differential equations for $\vec{A}$ and $V$:

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right) = -\vec{\nabla}^2 V - \vec{\nabla}\cdot\left(\frac{\partial\vec{A}}{\partial t}\right) = -\vec{\nabla}^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \frac{\rho}{\varepsilon_0}$$

$$\text{or} \quad \vec{\nabla}^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right)$$

$$-\vec{\nabla}^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \mu_0\vec{J} - \mu_0\varepsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$-\vec{\nabla}^2\vec{A} + \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \mu_0\varepsilon_0\vec{\nabla}\frac{\partial V}{\partial t}$$

$$-\vec{\nabla}^2\vec{A} + \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J} + \vec{\nabla}\left[-\vec{\nabla}\cdot\vec{A} - \frac{1}{c^2}\frac{\partial V}{\partial t}\right]$$

# Inhomogeneous Wave Equations for **A** (**r**,t) and V (**r**,t)

with the Lorentz gauge, i.e. $-\vec{\nabla}\cdot\vec{A} = \frac{1}{c^2}\frac{\partial V}{\partial t}$,

and $\frac{1}{c^2} = \mu_0 \varepsilon_0$

the differential equations for $\vec{A}$ and V are

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\cdot\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

inhomogeneous wave equation

$$\vec{\nabla}^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

# What is a “Gauge”?

- 4: a function introduced into a field equation to produce a convenient form of equation but having no observable physical consequences. [Merriam-Webster Dictionary].
- Applying a Gauge transformation is the act of introducing this function in the field equation.
- The introduction of the function in the definition of the potentials does not change the observable electric field E and magnetic field B.
- The observation **E** = **E’** and **B** = **B’** although potentials **A**, V are different from **A’**, V’ is also described as “the fields are Gauge invariant”.

Adding $\vec{\nabla}\lambda$ to $\vec{A}$ does not change $\vec{B}$ because $\vec{\nabla} \times (\vec{\nabla}\lambda) = 0$ from mathematical point of view.

$\vec{B} = \vec{\nabla} \times \vec{A}$ , now if $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ , $\vec{B}' = \vec{\nabla} \times \vec{A}' = \underbrace{\vec{\nabla} \times \vec{A}}_{=\vec{B}} + \underbrace{\vec{\nabla} \times \vec{\nabla}\lambda}_{=0}$

Also, adding $-\frac{\partial \lambda}{\partial t}$ to $V$ does not change $\vec{E}$ because

$$\vec{E}' = -\vec{\nabla}V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\left(V - \frac{\partial \lambda}{\partial t}\right) - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\lambda)$$

$$= -\vec{\nabla}V + \vec{\nabla}\left(\frac{\partial \lambda}{\partial t}\right) - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\lambda \text{, since}$$

$\vec{\nabla}\left(\frac{\partial \lambda}{\partial t}\right) = \frac{\partial}{\partial t}\vec{\nabla}\lambda$ $\Downarrow$ $\vec{E}' = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = \vec{E}$

The (Lorentz) gauge, i.e. $\vec{\nabla}\cdot\vec{A} = \frac{-1}{c^2}\frac{\partial V}{\partial t}$, now states requirements for $\lambda(\vec{r},t)$:

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}(\vec{A}) + (\vec{\nabla}\cdot(\vec{\nabla}\lambda)) = -\frac{1}{c^2}\frac{\partial V'}{\partial t} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(V - \frac{\partial \lambda}{\partial t}\right)$$

$$\vec{\nabla}\cdot\vec{A} + \vec{\nabla}^2\lambda = -\frac{1}{c^2}\frac{\partial V}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2}$$

or
$$\vec{\nabla}^2\lambda - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = -\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right)}_{=\,0 \text{ in Lorentzgauge}}$$

So, for the Lorentz-gauge to be meaningful (correct assumption), $\lambda(\vec{r},t)$ (which connects $\vec{A}'$ with $V'$) has to fulfill $\vec{\nabla}^2\lambda - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = 0$.

Solutions to the differential equation for $\lambda$ are, e.g. functions like f (**r**±ct).

- Here, in Electromagnetic Theory, we apply previously developed, rock solid, proven Gauges: the Coulomb-Gauge in Electrostatics and the Lorentz-Gauge in Electrodynamics. The purpose of the Gauges is simplifying the differential equations for the magnetic vector potential **A** and the scalar electric potential V.
- In this course, we are not in the business of developing Gauge transformations.
- Still,  today's professional physicists make a living and are having careers by researching Gauge transformation and Gauge theories. See some current and exciting examples next !

PHYSICAL REVIEW D 100, 065002 (2019)

# Gauge transformation in macroscopic quantum electrodynamics near polarizable surfaces

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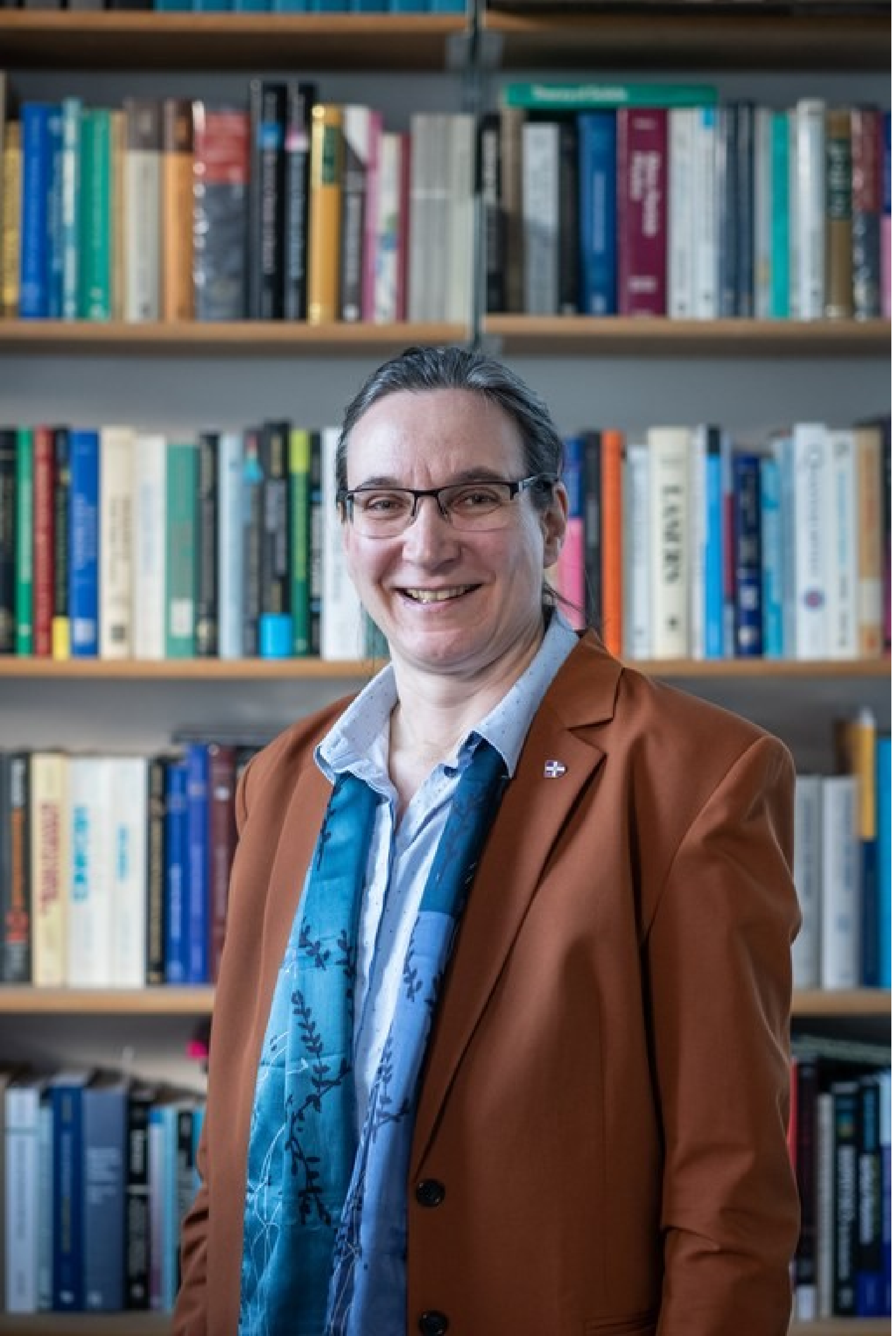
To describe charged particles interacting with the quantized electromagnetic field, we show the differences of working in the so-called generalized and the true Coulomb gauges. We find an explicit gauge transformation between them for the case of the electromagnetic field operators quantized near a macroscopic boundary described by a piecewise constant dielectric function. Starting from the generalized Coulomb gauge we transform operators into the true Coulomb gauge where the vector potential operator is truly transverse everywhere. We find the generating function of the gauge transformation to carry out the corresponding unitary transformation of the Hamiltonian and show that in the true Coulomb gauge the Hamiltonian of a particle near a polarizable surface contains extra terms due to the fluctuating surface charge density induced by the vacuum field. This extra term is represented by a second-quantized operator on equal footing with the vector field operators. We demonstrate that this term contains part of the electrostatic energy of the charged particle interacting with the surface and that the gauge invariance of the theory guarantees that the total interaction energy in all cases equals the well known result obtainable by the method of images when working in generalized Coulomb gauge. The mathematical tools we have developed allow us to work out explicitly the equal-time commutation relations and shed some light on typical misconceptions regarding issues of whether the presence of the boundaries should affect the field commutators or not, especially when the boundaries are modeled as perfect reflectors.

DOI: 10.1103/PhysRevD.100.065002

Take pleasure in reading the article after having completed the course PHYS 4210 in May. It starts out with Maxwell's equation and you are prepared to understand it up to the point in the manuscript where the discussion of quantum electrodynamics is picked up.



Claudia is a **Professor of Theoretical Physics** and Dean of the **School of Science** at Loughborough University (UK). Her research involves the application of quantum field theory to nanotechnology. Charged or polarizable quantum systems interact with the quantized electromagnetic field which leads to quantum corrections, e.g. the Lamb shift in atoms or the anomalous magnetic moment of the electron. This interaction is affected by the presence of material boundaries that reflect, refract, or absorb light, and this causes spatial variations in these quantum corrections that can be exploited for nanotechnology. However, after three decades of research in this field, Claudia now spends most of her time leading the School of Science, which comprises 5 academic disciplines and includes 240 staff and 2200 students, and contributing to the leadership and management of **Loughborough University**.

Source: The Weekly Women Physicist

PHYSICAL REVIEW D **99,** 064001 (2019)

# Ghost and tachyon free Poincaré gauge theories: A systematic approach

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A systematic method is presented for determining the conditions on the parameters in the action of a parity-preserving gauge theory of gravity for it to contain no ghost or tachyon particles. The technique naturally accommodates critical cases in which the parameter values lead to additional gauge invariances. The method is implemented as a computer program, and is used here to investigate the particle content of parity-conserving Poincaré gauge theory, which we compare with previous results in the literature. We find 450 critical cases that are free of ghosts and tachyons, and we further identify 10 of these that are also power-counting renormalizable, of which four have only massless tordion propagating particles and the remaining six have only a massive tordion propagating mode.

DOI: 10.1103/PhysRevD.99.064001

# Lorentz force acting on a moving charge q

Let's look at the classical Lorentz force of a charge q moving at speed $\vec{v}$ in electric & magnetic fields $\vec{E}$ & $\vec{B}$:

$$\vec{F} = \frac{d\vec{P}}{dt} = q\,(\vec{E} + \vec{v} \times \vec{B}).$$

Now, let's express $\vec{E}$ & $\vec{B}$ as $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$:

$$\vec{F} = \frac{d\vec{P}}{dt} = q\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A})\right)$$

using $\vec{\nabla}(\vec{v} \cdot \vec{A}) = \vec{v} \times (\vec{\nabla} \times \vec{A}) + (\vec{v} \cdot \vec{\nabla}) \cdot \vec{A}$

$$\vec{F} = \frac{d\vec{P}}{dt} = q\left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} + \vec{\nabla}(\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \vec{\nabla}) \cdot \vec{A}\right)$$

$$= -q\left(\frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \cdot \vec{A} + \vec{\nabla}(V - \vec{v} \cdot \vec{A})\right),$$

here $\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \cdot \vec{A}$

this term originates from the fact that charges are moving in electrodynamics

$$\vec{F} = \frac{d\vec{P}}{dt} = -q\left(\frac{d\vec{A}}{dt}\right) - q\,\vec{\nabla}(V - \vec{v} \cdot \vec{A})$$

$$\vec{F} = \frac{d\vec{p}}{dt} = -q\left(\frac{d\vec{A}}{dt}\right) - q\vec{\nabla}(V - \vec{v}\cdot\vec{A})$$

$$\frac{d}{dt}(\vec{p} + q\vec{A}) = -q\vec{\nabla}(V - \vec{v}\cdot\vec{A})$$

$\vec{P}_{can} = \vec{p} + q\vec{A}$ is the canonical momentum as in the independent variable $p$ in Hamilton mechanics $\dot{p} = -\frac{\partial H}{\partial q}$ and $\dot{q} = \frac{\partial H}{\partial p}$ with $H = H(q, p)$ (1-dim. example).

$U = V - \vec{v}\cdot\vec{A}$ is an example of a velocity dependent potential energy (this originates from the fact that charges are moving in electrodynamics).