Properties of RG (The Basic Ideas of RG)

$$\hat{\mathcal{H}} = -\beta\mathcal{H} = \sum_n K_n \Theta_n\{S_i\}$$

(Wilson, 1971)

local operator, depend on the degrees of freedom $\{S_i\}$

e.g.,

$$\hat{\mathcal{H}}[K,\{S_i\}] = NK_0 + h\sum_i S_i + K_1\sum_{ij} S_i S_j + K_2 \sum_{ijk} S_i S_j S_k$$

integrate out of degrees of freedom within blocks
simply integrating out degrees of freedom "thinning" out

set of coupling constants $[K] = (K_1, K_2, K_3, \ldots)$

- change of scale (by factor ℓ)
- reducing the degrees of freedom

$$[K'] = R_\ell [K] \qquad \ell > 1$$

recursion relation

- very complicated, non-linear transformation
- no inverse ($\ell > 1$)

a renormalization group transformation

semi group: ℓ_1: $[K'] = R_{\ell_1}[K]$
ℓ_2: $[K''] = R_{\ell_2}[K']$

$$[K''] = R_{\ell_2}[K'] = R_{\ell_2}R_{\ell_1}[K] = R_{\ell_2\cdot\ell_1}[K]$$

$$\boxed{R_{\ell_2}\cdot R_{\ell_1} = R_{\ell_2\cdot\ell_1}}$$

systematic coarse-graining to successively eliminate correlated degrees of freedom.

$\{S_i\}$ $\quad i = 1, 2, \ldots, N$

$\{S_I'\}$ $\quad I = 1, 2, \ldots, N'$ $\quad N' = N/\ell^d$

$\hat{\mathcal{H}}[K,\{S_i\}] \xrightarrow{R_\ell} \hat{\mathcal{H}}'[K',\{S_I'\}]$

$$-\beta \mathcal{H} = \hat{\mathcal{H}}[K, \{s_i\}] \xrightarrow{R_e} \hat{\mathcal{H}}'[K', \{s_I'\}]$$

"0" $$\boxed{\hat{\mathcal{H}}' = N' K_0' + h' \sum_I s_I' + K_1' \sum_{IJ} s_I' s_J' + K_2' \sum_{IJK} s_I' s_J' s_K'}$$

e.g.: $$\boxed{s_I' = \mathrm{sign}\left(\sum_{i \in I} s_i\right) = \pm 1}$$

$(2\ell+1)a$ "odd" number of sites in block

$$Z_N[K] = \sum_{\{s_i\}} e^{\hat{\mathcal{H}}[K, \{s_i\}]} = \sum_{\{s_I'\}} \sum_{\substack{\{s_i\} \\ \omega_I(s_i) = s_I'}}' e^{\hat{\mathcal{H}}[K, \{s_i\}]}$$

coarse-graining (e.g., majority rule, see below)

$$= \sum_{\{s_I'\}} e^{\hat{\mathcal{H}}'[K', \{s_I'\}]} = Z_{N'}[K']$$

(2) $$\boxed{e^{\hat{\mathcal{H}}'[K', \{s_I'\}]} = \sum_{\substack{\{s_i\} \\ \omega_I(s_i) = s_I'}}' e^{\hat{\mathcal{H}}[K, \{s_i\}]}}$$

"majority rule"

coarse-graining $s_I' = \omega_I(s_i)$ e.g. $\boxed{\omega_I(s_i) = \mathrm{sign}\left(\sum_{i \in I} s_i\right)} \Rightarrow s_I' = \pm 1$

(3) $$\boxed{e^{\hat{\mathcal{H}}'[K', \{s_I'\}]} = \sum_{\{s_i\}} \prod_I \delta(s_I' - \omega_I(s_i))\, e^{\hat{\mathcal{H}}[K, \{s_i\}]}}$$

ω_I should preserve the symmetries of the systems and produce the same range of variable for s_I's

(188) R_e in principle can generate <u>new</u> local operator

N

$$Z_{N'}[K'] = \sum_{\{S'_I\}} e^{\mathcal{H}'[K',\{S'_I\}]} = \sum_{\{S'_I\}} \sum_{\substack{\{s_i\} \\ w_I(s_i)=S'_I}}' e^{\mathcal{H}[K,\{s_i\}]}$$

$$f[K] = \frac{1}{N}\ln Z_N[K]$$

$$= \sum_{\{s_i\}} e^{\mathcal{H}[K,\{s_i\}]} = Z_N[K] =$$

$$N' f[K'] = N f[K] \qquad N' = N/\ell^d$$

$$\boxed{f[K] = \ell^{-d} f[K']}$$

(e.g. starting with n.n.n. $\xrightarrow{R_\ell}$ n.n. should appear)

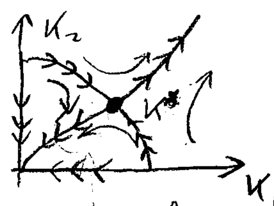
$\boxed{[K'] = R_\ell[K]}$ is expected to be analytic

singular behavior can emerge after infinite number of iterations

$$[K^{(n)}] = R_\ell^n[K] \qquad (\ell^n)$$

fixed points

$\boxed{R_\ell[K^*] = K^*}$



scaling behavior

while in dynamical systems governed by non-lin. eq. limit-cycles are possible, it is not typical for RG flow

nature of fixed point and global pattern of flow

$\Downarrow$

phase diagram

The correlation length

$$\boxed{\xi[K'] = \xi[K]/\ell}$$

critical manifold: basin of attraction of critical fixed point

$K \in$ crit. manif.:

$$\xi[K^*] = \xi[K^*]/\ell \Rightarrow \xi[K^*] = \begin{cases} 0 & \text{trivial fixed point} \\ \infty & \text{critical fixed point} \end{cases}$$

↳ basin of attraction of critical fixed point have $\xi = \infty$ as well

$$\xi[K^{(n)}] = \xi[K]/\ell^n \qquad \lim_{n\to\infty} \xi[K^*]\ell^n = \infty = \xi(K)$$

Local Behavior near fixed points $[K'] = R_\ell[K]$

$K \equiv (K_1, K_2, \ldots, K_n, \ldots)$

$$\boxed{K = K^* + \delta K}$$

in the vicinity of a fixed point $K_n = K_n^* + \delta K_n$

$$R_\ell: \quad K_n' = K_n'(K) = K_n'(K^* + \delta K) = K_n'(K^*) + \underbrace{\sum_m \frac{\partial K_n'}{\partial K_m}\Big|_{K_m = K_m^*} \delta K_m}_{\delta K_n'}$$

$\left(\hat{\mathcal{H}}^* \equiv \hat{\mathcal{H}}[K^*]; \quad \mathcal{H} = \mathcal{H}^* + \delta\mathcal{H}\right)$

$$+ o(\delta K)^2$$
$$\cong K_n^* + \sum_m \frac{\partial K_n'}{\partial K_m}\Big|_{K_m = K_m^*} \delta K_m \equiv K_n^* + \delta K_n'$$

$$\delta K_n' = \sum_m \frac{\partial K_n'}{\partial K_m} \delta K_m = \sum_m M_{nm} \delta K_m \qquad \boxed{M_{nm} = \frac{\partial K_n'}{\partial K_m}\Big|_{K_m = K_m^*}}$$

$R_\ell \rightsquigarrow \bar{\bar{M}}^{(\ell)} = (M_{nm}^{(\ell)})$ linearised RG transformation

eigenvectors and eigenvalues: (assume symmetric $\bar{\bar{M}}$)

$\bar{\bar{M}}^{(\ell)} \bar{e}_\sigma = \Lambda^{(\ell)} \bar{e}_\sigma$ $\sigma = 1, 2, \ldots$ $\{\bar{e}_\sigma\}_\sigma$ complete set

$R_\ell R_{\ell'} = R_{\ell\ell'} \implies \bar{\bar{M}}^{(\ell)} \cdot \bar{\bar{M}}^{(\ell')} = \bar{\bar{M}}^{(\ell\ell')} \implies \boxed{\Lambda_\sigma^{(\ell)} \cdot \Lambda_\sigma^{(\ell')} = \Lambda_\sigma^{(\ell \cdot \ell')}}$

(semi group)   e.g. use ($\ell = \ell'$)

$\boxed{f(\ell) f(\ell') = f(\ell\ell')}$ (*)

(1) $\ell' = 1 \implies f(1) = 1$

(2) $\frac{\partial}{\partial \ell'}: \quad f(\ell) \frac{df}{d\ell'}\Big|_{\ell'=1} = \ell \frac{df(\ell)}{d\ell} \implies \boxed{f(\ell) = \ell^y}$

only functional form which satisfies (*)

$$\implies \boxed{\Lambda_\sigma^{(\ell)} = \ell^{y_\sigma}} \implies y_\sigma = \frac{\ln(\Lambda_\sigma^{(\ell)})}{\ln(\ell)}$$

$$\delta \bar{K} = \sum_\sigma a_\sigma \bar{e}_\sigma \qquad \boxed{\Lambda_\sigma^{(\ell)} = \ell^{y_\sigma}} \qquad (\ell > 1)$$

$$\delta \bar{K}' = \bar{\bar{M}}^{(\ell)} \delta \bar{K} = \sum_\sigma a_\sigma \bar{\bar{M}}^{(\ell)} \bar{e}_\sigma = \sum_\sigma a_\sigma \Lambda_\sigma^{(\ell)} \bar{e}_\sigma$$

$$= \sum_\sigma a_\sigma \ell^{y_\sigma} \bar{e}_\sigma = \sum_\sigma a'_\sigma \bar{e}_\sigma$$

$$\boxed{a'_\sigma = a_\sigma \ell^{y_\sigma}}$$ projection along $\bar{e}_\sigma$

(i) $y_\sigma > 0$ $\quad (|\Lambda_\sigma^{(\ell)}| > 1)$ $\quad$ relevant variable

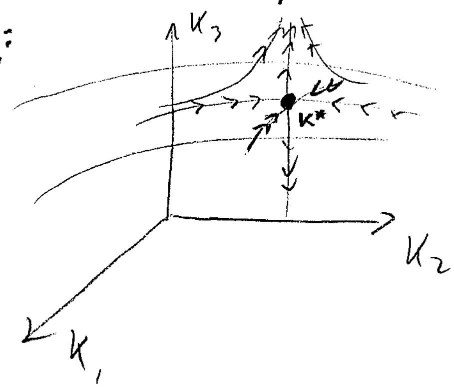
(ii) $y_\sigma < 0$ $\quad (|\Lambda_\sigma^{(\ell)}| < 1)$ $\quad$ irrelevant variable

(iii) $y_\sigma = 0$ $\quad (|\Lambda_\sigma^{\ell}| = 1)$ $\quad$ marginal variabl

after many iterations only (i) is important after many iterations

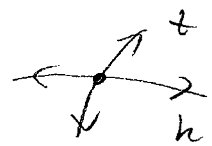
irrelevant eigenvectors span the critical manifold
(basin of attraction for critical point

example:



2 irrelevant variables (spanning the plane (around $\bar{K}_c$)
1 relevant variable

Ising:



Origin of Scaling

e.g. one coupling const.: $K = J/kT$ $\qquad K^*(T^*)$

$T' = R_\ell(T) \qquad T^* = R_\ell(T^*)$

$$\delta T' = T' - T^* = R_\ell(T) - T^* \simeq \underbrace{R_\ell(T^*)}_{T^*} + \left.\frac{\partial R_\ell}{\partial T}\right|_{T=T^*}(T-T^*) - T^*$$

$$\simeq \left.\frac{\partial R_\ell}{\partial T}\right|_{T=T^*} \underbrace{(T-T^*)}_{\delta T} \Rightarrow \Lambda_\ell = \left.\frac{\partial R_\ell}{\partial T}\right|_{T=T^*}$$

$$\boxed{\delta T' = \left.\frac{\partial R_\ell}{\partial T}\right|_{T=T^*} \delta T} \qquad \boxed{\Lambda_\ell = \ell^{y_t}}$$

$t = \dfrac{T-T^*}{T^*} \quad (>0)$

$$t' = t\,\ell^{y_t} \qquad t^{(n)} = (\ell^{y_t})^n\, t$$

correlation length (e.g. for $t>0$) $\qquad \xi(t^{(n)}) = \dfrac{\xi(t)}{\ell^n}$ $\qquad$ n-fold change of scale

$$\xi(t) = \ell^n\, \xi(\ell^{ny_t} t) \qquad \ell > 1 \text{ but arbitrary}$$

$$\ell^{ny_t} t = b \qquad \ell^n = (b/t)^{1/y_t} \Rightarrow \xi(t) = b^{1/y_t}\, t^{-1/y_t}\, \xi(b)$$

correlation length exponent $\qquad \boxed{\nu = 1/y_t}$

<u>Relevant & irrelevant variables</u> $\qquad f(K) = \ell^{-d} f(K') \overset{\text{use}}{\Rightarrow} f(\delta K) = \ell^{-d} f(\delta K')$

(and diagonal RG tr.)

$\delta K_1 = t, \quad \delta K_2 = h, \quad \delta \tilde{K}_3$

$$f(t, h, \delta\tilde{K}_3) = \ell^{-d} f(t', h', \delta\tilde{K}_3') = \ell^{-d} f(t\ell^{y_t}, h\ell^{y_h}, \delta K_3 \ell^{y_3})$$

$y_t > 0, \quad y_h > 0, \quad \boxed{y_3 < 0}$ $\qquad$ choose: $t\ell^{ny_t} = b \quad (b \gg 1)$

after n iterations: $\quad f(t, h, \delta\tilde{K}_3) = \ell^{-dn} f(\overbrace{t\ell^{ny_t}}^{b}, h\ell^{ny_h}, \delta K_3 \ell^{ny_3})$

$\ell^n = (b/t)^{1/y_t}$

$\ell^{ny_t} t = b \qquad f(t, h, \delta\tilde{K}_3) = b^{-d/y_t}\, t^{d/y_t} f\left(b, b^{y_h/y_t} \dfrac{h}{t^{y_h/y_t}}, \underbrace{\delta K_3\, b^{y_3/y_t}\, t^{-y_3/y_t}}_{\to 0 \; (t \to 0)}\right)$

$$\simeq b^{-d/y_t}\, t^{d/y_t} f\left(b, b^{y_h/y_t} h / t^{y_h/y_t}, 0\right) \qquad 2 - \alpha = d\nu, \quad \Delta = y_h/y_t$$