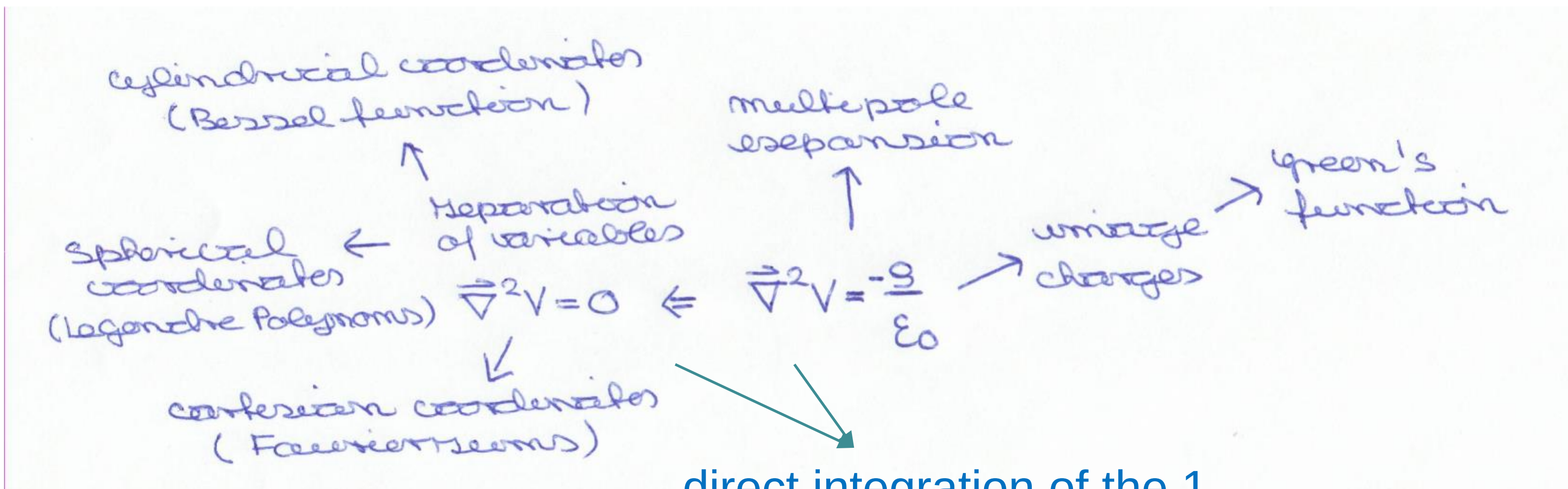
Class 7 (02/01/24)

Solving Laplace Equation by Separation of Variables in Cartesian Coordinates

Methods for solving Laplace’s & Poisson’s equations



direct integration of the 1-dimensional Laplace’s and Poisson’s equations

Derivation of the electrostatic Laplace's equation

$$\vec{\nabla} \times \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla} \cdot \vec{E} = (\vec{\nabla} \cdot (-\vec{\nabla} V)) = -\vec{\nabla}^2 V = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$$

Poisson's equation

$$\vec{\nabla}^2 V = 0$$

Solving the Laplace's equation: Finding V(**r**) in a volume empty of electric charge and bounded by conducting surfaces.

Some mathematical facts about the Laplace equation:

- It is a homogenous, 2^{nd} order, linear partial differential equation.
- In mathematical sciences, it is classified as an elliptic differential equation.
- Analytically, the Laplace equation can solved by the methods of separation of variables in 11 coordinate systems.
- So, any electrostatic problem where the geometry can be appropriately described by one of the 11 coordinate systems, the Laplace equation can be solved by separation of variables.
- In PHYS4210 we limit ourselves to separation of variables in Cartesian coordinates x, y, z & spherical coordinates r, θ, ϕ .

Laplace equation: Separation of variables in x, y, z

1. Derivation of the general solution
2. Example illustrated in Figure 2.9 of the textbook
3. Math tutorials
   - Orthogonality of harmonic functions
   - Simple Fourier Series
   - Double Fourier Series
3. Determination of the Fourier coefficients of the general solution by boundary conditions

Finding the general solution:

cartesian coordinates x, y, z $\quad \vec{\nabla}^2 V(x,y,z) = 0$

$$\vec{\nabla}^2 V = \frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0$$

assume $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0 \quad \Big| \frac{1}{XYZ}$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{Fct. of } x} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{fct. of } y} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{fct. of } z} = 0$$

$\Downarrow$ each term is constant

$$\alpha^2 + \beta^2 - \gamma^2 = 0$$

$$\frac{1}{X}X'' = -\alpha^2$$

$$\frac{1}{Y}Y'' = -\beta^2$$

$$\frac{1}{Z}Z'' = \gamma^2$$

$$\frac{1}{X} X'' = -\alpha^2$$

$$\frac{d^2 X}{d x^2} = -\alpha^2 X$$

Ansatz: $X = e^{\pm i\alpha x}$

$$\frac{d^2 X}{d x^2} = (\pm i\alpha)(\pm i\alpha) e^{\pm i\alpha x}$$

$$= -\alpha^2 e^{\pm i\alpha x}$$

solution for $Y = e^{\pm i\beta y}$, $Z = e^{\pm \gamma z} = e^{\pm \sqrt{\alpha^2+\beta^2}\, z}$

$$V(x, y, z) = e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z}$$

α, β determined from boundary conditions

Example illustrated in Figure 2.9

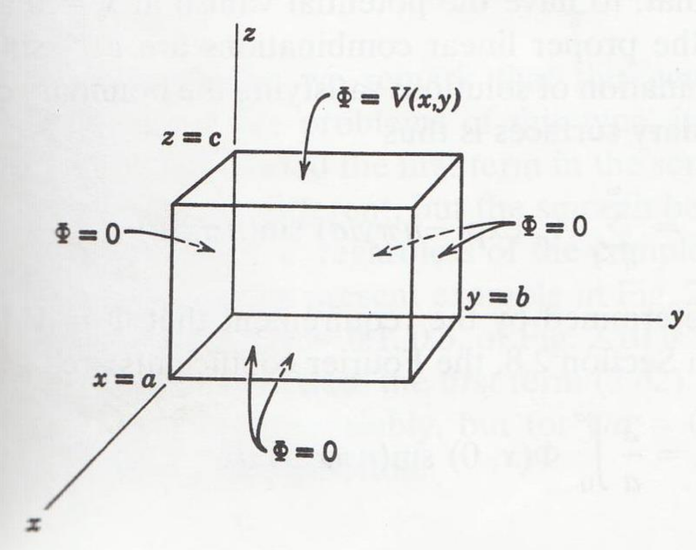


Figure 2.9 Hollow, rectangular box with five sides at zero potential, while the sixth ($z = c$) has the specified potential $\Phi = V(x, y)$.

example: rectangular box with dimensions a, b, c in x, y, z directions

Fig. 2.9

all surfaces are $V = 0$

except $V(x, y, c) = V(x, y)$

$\leadsto$ $X(0) = 0$, $Y(0) = 0$, $Z(0) = 0$

$X(a) = 0$, $Y(b) = 0$

$Z(c) = V(x, y)$

$$X(0) = e^{i\alpha x = 0} = \overset{1}{\cancel{\cos\alpha\emptyset}} \pm i \sin\alpha\emptyset = 0$$

$\leadsto$ $X = \sin\alpha x$

$X(a) = 0$ $\quad \sin\alpha a = 0$ $\leadsto$ $\alpha_n = \frac{n\pi}{a}$

$Y(0) = 0$ $\leadsto$ $Y = \sin\beta x$

$Y(b) = 0$ $\quad \beta_m = \frac{m\pi}{b}$

$Z(0) = 0$ $\leadsto$ $Z = \sinh\sqrt{\alpha^2 + \beta^2}\, z$

$$\left[\sinh x = \tfrac{1}{2}(e^{x} - e^{-x})\Big|_{x=0} = 0\right]$$

$$\gamma_{mn} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$V_{nm}(x,y,z) = A_{nm} \sin(\alpha_n x)\sin(\beta_m y) \sinh(\gamma_{nm} z)$$

expand $V(x,y,z)$ in terms of V_{nm} with arbitrary A_{nm}

$$V(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y) \sinh(\gamma_{nm} z)$$

A_{nm} determined by last boundary condition

$$V(x, y, z=c) = V(x,y)$$

$$V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m \cancel{x}) \sinh(\gamma_{mn} c)$$

$\rightsquigarrow$ double Fourier series

$$A_{nm} = \frac{4}{ab \sinh(\gamma_m c)} \int_0^a dx \int_0^b dy \, V(x,y) \sin(\alpha_n x) \sin(\beta_m y)$$

Solution for example

general solution for Laplace Equation:
in cartesian coordinates

$$V(x,y,z) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} e^{i\alpha_n x} e^{i\beta_m y} \left(A_{nm} e^{\sqrt{\alpha_n^2+\beta_m^2}\, z} + B_{nm} e^{-\sqrt{\alpha_n^2+\beta_m^2}\, z} \right)$$

Math tutorial: Orthogonality of harmonic function

(1)

$$\int \sin ax \sin bx\, dx = \begin{cases} \dfrac{\sin(a-b)x}{2(a-b)} - \dfrac{\sin(a+b)x}{2(a+b)} & a \neq b \\[2ex] \dfrac{1}{2}x - \dfrac{1}{4a}\sin 2ax & a = b \end{cases} \qquad a \geq 0,\ b \geq 0$$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \left.\frac{\sin\frac{(n-m)\pi x}{L}}{2\left(\frac{(n-m)\pi}{L}\right)} - \frac{\sin\frac{(n+m)\pi x}{L}}{2\frac{(n+m)\pi x}{L}}\right|_0^L$$

The dimension of the integral is length, I missed writing down L. It does not matter in the end because the solution to the integral is “0” for n≠m.

$$= \frac{\overset{=0}{\sin(n-m)\pi}}{2(n-m)\pi} - \frac{\overset{=0}{\sin(n+m)\pi}}{2(n+m)\pi} - \left[\frac{\overset{=0}{\sin 0}}{2(n-m)\pi} - \frac{\overset{=0}{\sin 0}}{2(n+m)\pi}\right] \qquad n \neq m,\ n, m \text{ integer} >$$

$$= 0$$

for $n = m$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \left.\frac{1}{2}x - \frac{1}{\frac{4n\pi}{L}}\sin\frac{2n\pi}{L}x\right|_0^L$$

$$= \frac{1}{2}L - \frac{1}{\frac{4n\pi}{L}}\underbrace{\sin n2\pi}_{0} - \left[0 - \frac{1}{\frac{4n\pi}{L}}\sin(\right]$$

$$= \frac{1}{2}L$$

generally

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{1}{2}L\,\delta_{nm}$$

Kronecker δ :
δ_{nm}=1 for n=m and
δ_{nm}=0 for n≠m.

Math tutorial: Expanding a function in a Fourier series

$$F(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

$$F(x) \sin \frac{m\pi x}{L} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}$$

$$\int_0^L F(x) \sin \frac{m\pi x}{L} \, dx = \int_0^L \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin \frac{n\pi x}{L} \cdot \sin \frac{m\pi x}{L} \, dx}_{\frac{1}{2} L \delta_{nm}}$$

$$\int_0^L F(x) \sin \frac{m\pi x}{L} \, dx = a_m \frac{1}{2} L$$

Kronecker δ :
δ_{nm}=1 for n=m and
δ_{nm}=0 for n≠m.

Math tutorial: Double Fourier Series

$$V(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y).$$

$$V(x,y) \sin(\alpha_{n'} x) \sin(\beta_{m'} y)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\alpha_{n'} x) \sin(\beta_{m'} y) \sin(\beta_m y)$$

$$\int_0^a \int_0^b V(x,y) \sin(\alpha_{n'} x) \sin(\beta_{m'} t)\, dx\, dy$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \underbrace{\int_0^a \sin \alpha_n x \sin \alpha_{n'} x\, dx}_{\frac{a}{2}\delta_{nn'}} \underbrace{\int_0^b \sin(\beta_{m'} y) \sin \beta_m y\, dy}_{\frac{b}{2}\delta_{mm'}}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \frac{ab}{4} \delta_{nn'} \delta_{mm'}$$

Kronecker δ :
$\delta_{nn'}=1$ for $n=n'$ and
$\delta_{nn'}=0$ for $n \neq n'$.
Same for $\delta_{mm'}$.

$$\int_0^L \int_0^L V(x,y) \sin(\alpha_{n'} x) \sin(\beta_{m'} y)\, dx\, dy = A_{n'm'} \frac{ab}{4}$$

Follow up on last slide

$$\int_0^L \int_0^L V(x,y)\,\sin(\alpha_{n'}x)\,\sin(\beta_{m'}y)\,dx\,dy = A_{n'm'}\frac{ab}{4}$$

e.g. $V(x,y) = V_0$

$$A_{nm} = \frac{4V_0}{ab}\int_0^a \sin(\alpha_{n'}x)\,dx \int_0^b \sin(\beta_{m'}y)\,dy$$

$$= \frac{4V_0}{ab}\left.\frac{-\cos\alpha_{n'}x}{\alpha_n}\right|_0^a \left.\frac{-\cos\beta_{m'}y}{\alpha_m}\right|_0^b$$

$$= \frac{4V_0}{ab}\left[\frac{\cos\frac{n'\pi a}{a}}{n'\pi/a} - \frac{\overset{1}{\cos 0}}{n'\pi/a}\right]\left[\frac{\cos\frac{m'\pi b}{b}}{n'\pi/b} - \frac{\overset{1}{\cos 0}}{m'\pi/b}\right]$$

$$= \frac{4V_0\,ab}{ab\,\pi^2}\left[\frac{\cos n'\pi - 1}{n'}\right]\left[\frac{\cos m'\pi - 1}{m'}\right]$$

$\cos n\pi = 1$ n even

$\cos n\pi = -1$ n odd

n, m even $A_{nm} = 0$

n, m odd $A_{nm} = 16\,V_0 / (\pi^2\, n\, m)$

Now we apply the previously discussed math to the general solution: x, y, z, with separation constants α, β, γ

$$\int V(x,y) \sin \alpha_{n'} x \sin \beta_{m'} y \, dx\, dy$$

$$= \sum_{n,m=1}^{\infty} A_{nm} \int_0 \sin \alpha_n x \sin \alpha_{n'} x \int \sin \beta_m y \sin \beta_{m'} y \, dx\, dy \; \sinh(\gamma_{mn} c)$$

$$\Rightarrow$$

$$\iint V(x,y) \sin \alpha_{n'} x \sin \beta_{m'} y \, dx\, dy$$

$$= \sum_{n,m 1}^{\infty} A_{nm} \sinh(\gamma_{mn} c) \frac{1}{2} a \delta_{nn'} \frac{1}{2} b \delta_{mm'}$$

$$\iint V(x,y) \sin \alpha_{n'} \sin \beta_{m'} y \, dx\, dy$$

$$= A_{n'm'} \frac{1}{4} ab \sinh(\gamma_{mn} c)$$

$$A_{nm} = \frac{4}{ab} \frac{1}{\sinh(\gamma_{mn} c)} \iint V(x,y) \sin \alpha_{n'} \sin \beta_{m'} y \, dx\, dy$$

- That's it for today's class.
- Next class, I will provide a summary of the essentials for problem solving with an overview of common rectangular geometries and boundary conditions.