1. Does the central limit theorem (CLT) really work? Consider the example of N exponentially distributed random variables. Let $y = \sum_{i=1}^{N} x_i$, where x_i are *independent* exponentially distributed random variables, i.e., all x_i has the same exponential probability density:

$$p(x) = \begin{cases} ae^{-ax} & \textit{if } x \geq 0 \\ 0 & \textit{if } x < 0 \end{cases}.$$

(a) Find the generating function for a single exponential variable, $\Phi(k) = \langle e^{ikx} \rangle$.

(b) Find the generating function $\Phi_N(k) = \langle e^{iky} \rangle$, for the sum of N independent exponentially distributed random variables.

(c) Use your mathematical skills to inverse Fourier transform $\Phi_N(k)$ and obtain $p_N(y)$, the *exact* probability density function for y.

(d) What you found in (c) is obviously not a Gaussian. How can we to reconcile it with the CLT? Write $p_N(y)$ as $p_N(y) = ae^{f(y)}$ and expand $f(y)$ about its maximum y_o (show that $y_o = (N-1)/a$). You will also encounter $(N-1)!$ on the way so use Stirling's approximation to deal with it. Show that in the asymptotic large N limit, $p_N(y)$ converges to a Gaussian with mean N/a and variance N/a^2. Note that $1/a$ and $1/a^2$ are the mean $\langle x \rangle$ and the variance σ_x^2 of the individual exponential random variables, respectively. This example illustrates how the CLT works: the variable $\dfrac{y - N\langle x \rangle}{\sqrt{N}\sigma_x}$ will converge to a Gaussian variable with zero mean and unit variance in the $N \to \infty$ limit.

2. Consider the *microcanonical* ensemble of N independent *classical* (one-dimensional) oscillators with the same frequency ω. Note that the oscillators are localized, hence *distinguishable* by construction.

(a) Find the number of microstates between E and $E + \delta E$, $\Omega(E, \delta E)$.

(b) Find the entropy $S(E, N)$ for large values of N.

(c) Show that $E = NkT$.

(d) What would be the energy for N three-dimensional classical harmonic oscillators?

3. The number of states in classical N-particle systems with energy less than E can be typically written as $\Omega_<(E) = A_N E^{cN}$, where A_N is an N-dependent prefactor, while c is an N-independent constant. Using the notations from class, show that in the asymptotic large-N limit, the following three expressions become approximately equal (to leading order in N):

$$\ln(\Omega_<(E)) \approx \ln(\Omega(E,\delta E)) \approx \ln(g(E)).$$

(Thus, in the microcanonical ensemble, we can use any of the above measures to obtain the entropy.)

4. The dependence of the Hamiltonian of a classical system on a particular generalized coordinate q is given by $\mathcal{H} = \mathcal{H}' + \alpha q^2$, were $\mathcal{H}'$ may depend on all other coordinates and momenta, but not q. Working in the canonical ensemble, show that $\langle \alpha q^2 \rangle = \frac{1}{2}kT$.

5. Consider a system of N independent and localized (distinguishable) magnetic dipoles in the presence of an external magnetic field $\vec{H}$ which points to the z direction. The energy of an individual dipole is given by $\varepsilon = -\mu_z H$, where the possible values of μ_z are given by

$$\mu_z = g\mu_B m, \quad m = -J, -J+1, \ldots, J-1, J.$$

J is a *fixed* integer or half-integer, related to the magnitude of the angular momentum of the atom, g is the Lande factor, μ_B is the Bohr magneton, and H is amplitude of the external field. Working in the canonical ensemble:

(a) Find the average magnetization per dipole $\langle \mu_z \rangle = \dfrac{M_z}{N}$.

(b) Find the magnetic susceptibility $\chi = \lim_{H\to 0}\left(\dfrac{\partial M_z}{\partial H}\right)$

(c) Discuss your findings in the high- and low-temperature limits.