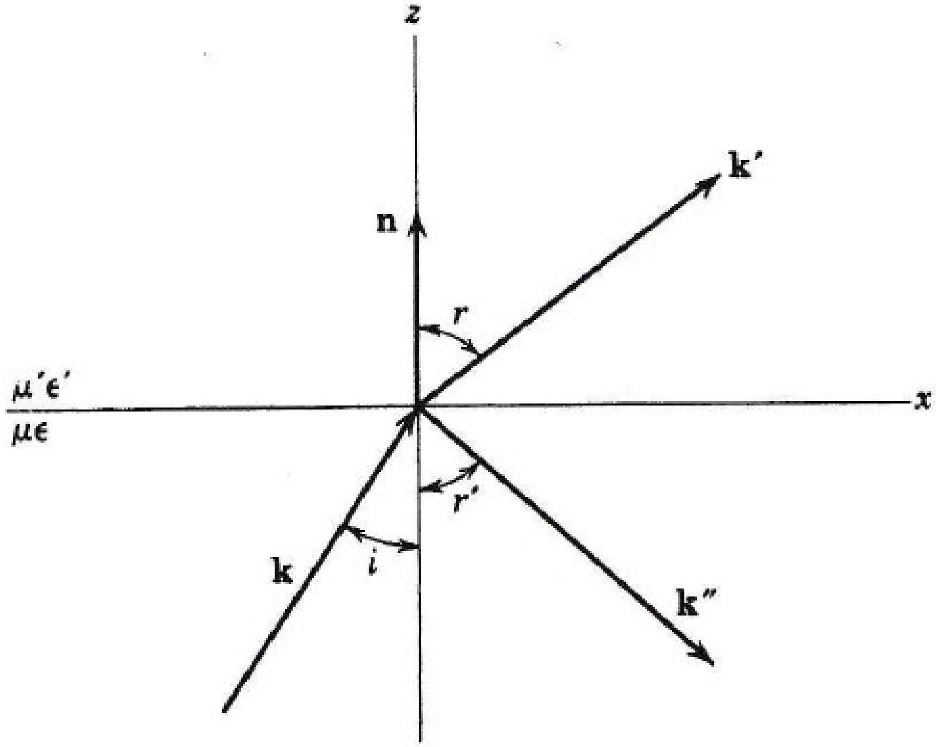
Class 19 (3/28/24)

- Fresnel's equation for normal & oblique incidence of electromagnetic waves at a boundary between non-conducting media
- Propagation of electromagnetic waves in conducting media
- Absorption & Dispersion

Reflection & Transmission of an Electromagnetic Wave



Reflection and Transmission of an electro-magnetic wave at the boundary (interface) between 2 different dielectrics

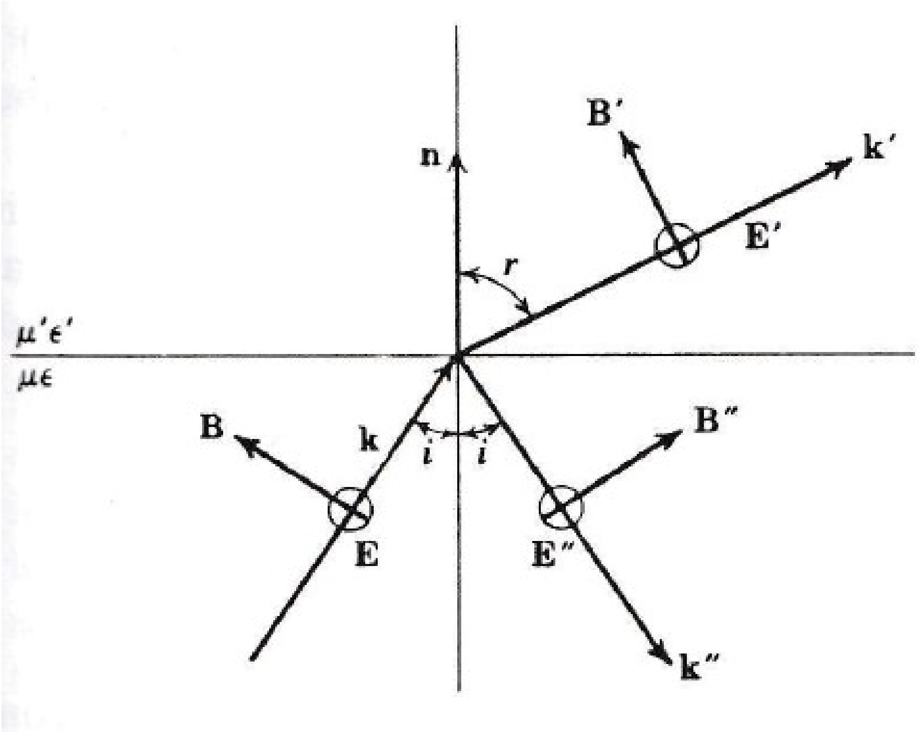
Wave: $\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

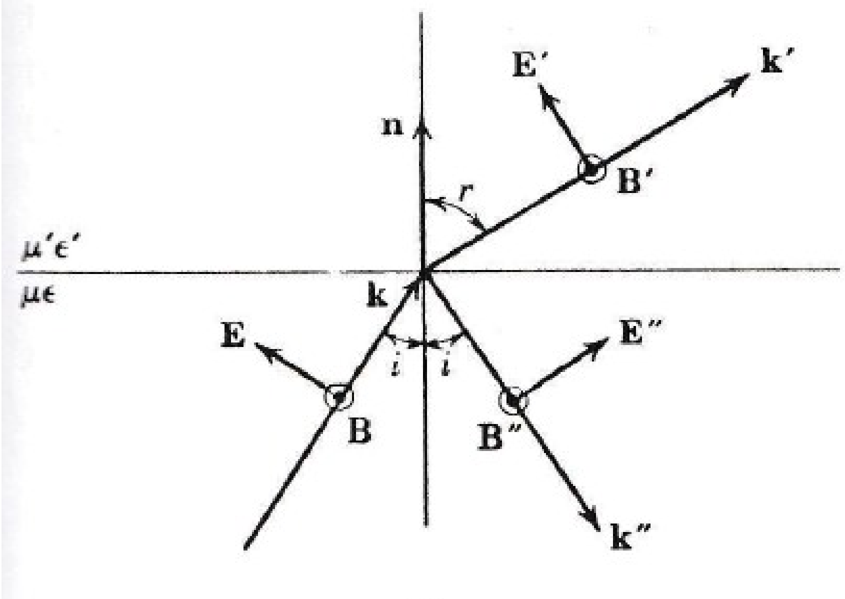
What happens to the amplitude of the wave $\vec{E}_0$ at the boundary is described by Fresnel's equations.

incident, reflected, and transmitted wave are in phase at boundary

$\rightsquigarrow$ $\theta_i = \theta_{r'}$, $n \sin\theta_i = n' \sin\theta_r$

Plane of Incidence and Electric Field Polarization



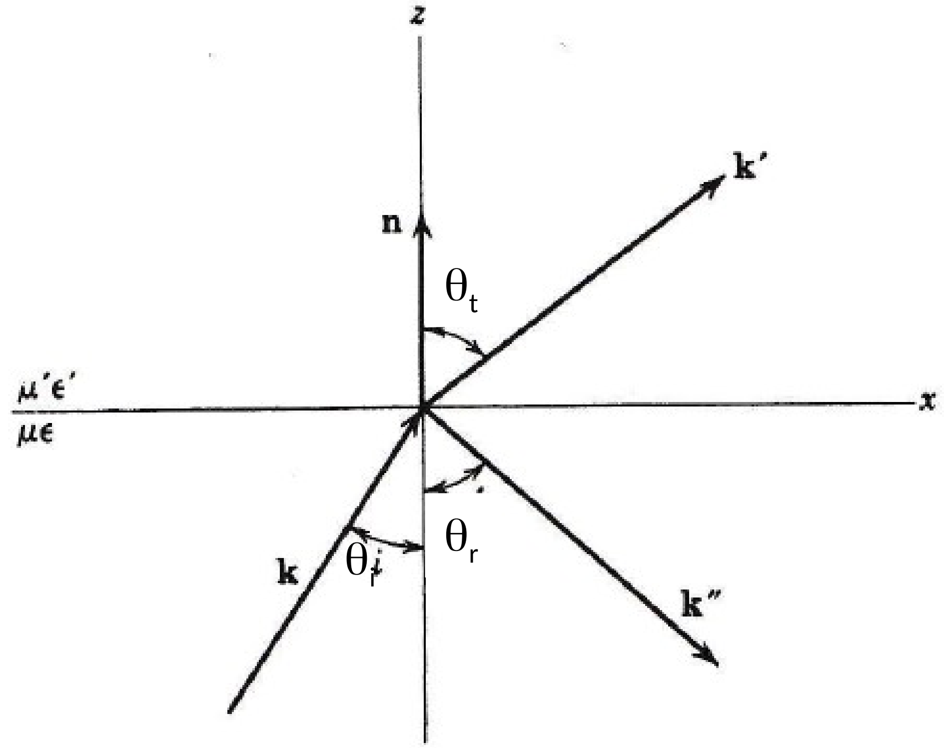


(b)

Electric field **E** polarized perpendicular to the plane of incidence formed by **k**, **n**, **k’** and **k”**.

Electric field **E** polarized parallel to the plane of incidence formed by **k**, **n**, **k’** and **k”**.

Fresnel's Equations



incident wave $\vec{E}_i = \vec{E}_{i0}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ with $k=\omega\sqrt{\mu\varepsilon}$

refracted wave $\vec{E}_t = \vec{E}_{t0}\, e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$ with $k'=\omega\sqrt{\mu'\varepsilon'}$

reflected wave $\vec{E}_r = \vec{E}_{r0}\, e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$ with $k''=k=\omega\sqrt{\mu\varepsilon}$

boundary conditions

$$\left(\varepsilon\left(\vec{E}_{i0}+\vec{E}_{r0}\right)-\varepsilon'\vec{E}_{t0}\right)\cdot\vec{n}=0$$

$$\left(\vec{k}\times\vec{E}_{i0}+\vec{k}''\times\vec{E}_{r0}-\vec{k}'\times\vec{E}_{t0}\right)\cdot\vec{n}=0$$

$$\left(\vec{E}_{i0}+\vec{E}_{r0}-\vec{E}_{t0}\right)\times\vec{n}=0$$

$$\left(\frac{1}{\mu}\left(\vec{k}\times\vec{E}_{i0}+\vec{k}''\times\vec{E}_{r0}\right)-\frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_{t0}\right)\right)\times\vec{n}=0$$

visualize $\vec{E}\cdot\vec{n}$ is a vector perpendicular to boundary

$\vec{B}\propto\vec{k}\times\vec{E}$ and $(\vec{k}\times\vec{E})\cdot\vec{n}$ vector $\perp$ to boundary

$\vec{E}\times\vec{n}$ vector parallel to boundary

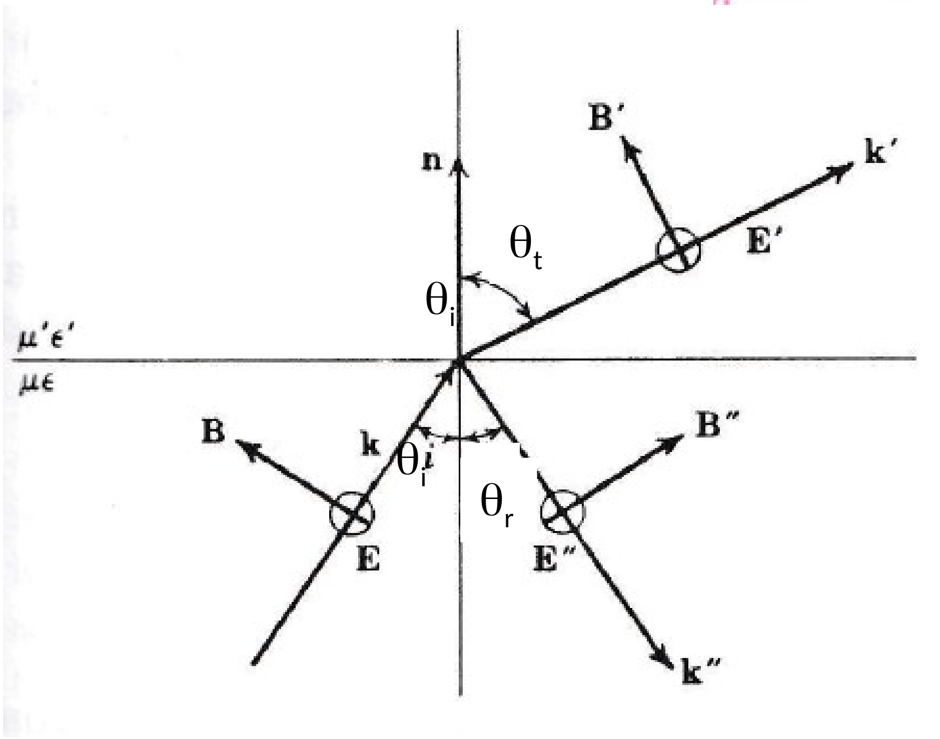
$\vec{B}=\mu\vec{H}$ or $\vec{H}=\frac{\vec{B}}{\mu}\propto\frac{1}{\mu}(\vec{k}\times\vec{E})$, $\frac{1}{\mu}(\vec{k}\times\vec{E})\times\vec{n}$ vector $\parallel$ to boundary

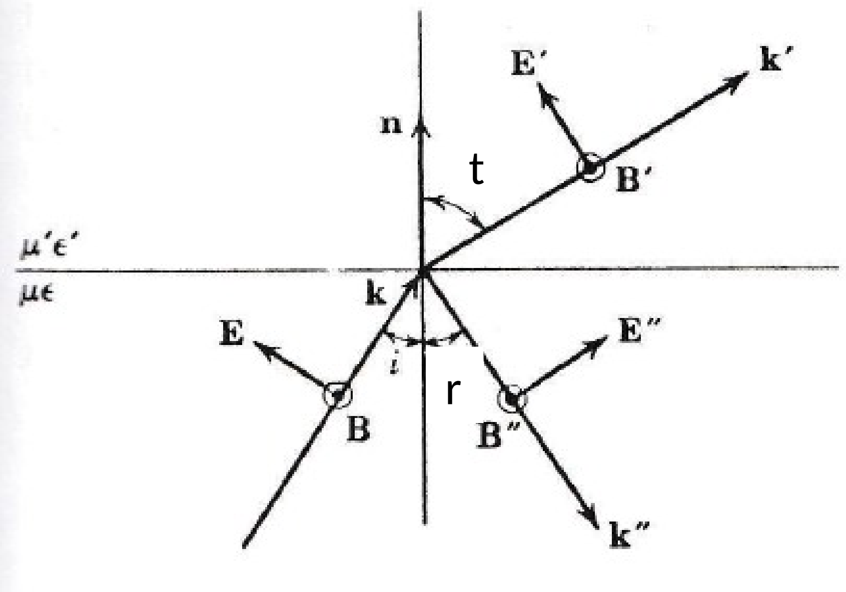
Fresnel's equation

$\vec{E}$ is perpendicular to plane of incidence

electric field transmission $t = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2n\cos\theta_i}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}}$

electric field reflection $r = \dfrac{E_{0r}}{E_{i0}} = \dfrac{n\cos\theta_i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}}{n\cos\theta_i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}}$





(b)

$\vec{E}$ is parallel to plane of incidence

electric field transmission $t = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2nn'\cos\theta_i}{\frac{\mu}{\mu'}n'^2\cos\theta_i + n\sqrt{n'^2 - n^2\sin^2\theta_i}}$

electric field reflection $r = \dfrac{E_{t0}}{E_{i0}} = \dfrac{\frac{\mu}{\mu'}n'^2\cos\theta_i - n\sqrt{n'^2 - n^2\sin^2\theta_i}}{\frac{\mu}{\mu'}n'^2\cos\theta_i + n\sqrt{n'^2 - n^2\sin^2\theta_i}}$

for normal incidence $t = \frac{2n}{n' + n}$ and $r = \frac{n' - n}{n' + n}$

intensity transmission $T = t^2 = \left(\frac{E_{t0}}{E_{i0}}\right)^2$ and $R = r^2 = \left(\frac{E_{r0}}{E_{i0}}\right)^2$

with $T + R = 1$

Augustin-Jean Fresnel (1788 – 1827) was a French civil engineer and physicist whose research in optics led to the almost unanimous acceptance of the wave theory of light. He is perhaps better known for inventing the Fresnel lens and for pioneering the use of "stepped" lenses to extend the visibility of lighthouses, saving countless lives at sea. (Wikipedia)

First-order lighthouse Fresnel lens, on display at the Point Arena Lighthouse Museum, Mendocino County, California.

Well, you can purchase Fresnel lenses, e.g. here X or there Y. Or design your own Fresnel lens and print it with a 3D-printer.

Electromagnetic Waves Traveling in Conducting Media

What is different for this situation compared to non-conducting media and vacuum?

* The electromagnetic wave gets attenuated!
* The electric field and magnetic field are out of phase.

The descriptions of electromagnetic waves travelling in a conductor (e.g. semiconductor, metal) are:

$$\vec{E}(z,t) = E_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E)\,\hat{x}$$

$$\vec{B}(z,t) = B_0 e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi)\,\hat{y}$$

The attenuation constant κ depends on the electric conductivity σ as $\kappa = \omega\sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}$.

The skin depth $d = \frac{1}{\kappa}$.

The phase difference ϕ between $\vec{E}$ and $\vec{B}$ is

$\phi = \tan^{-1}\left(\frac{\kappa}{k}\right)$ with κ attenuation constant and $k = \frac{n\omega}{c}$.

The previously described $\vec{E}$ and $\vec{B}$ of waves traveling in conducting media are solutions to the modified wave equations:

$$\vec{\nabla}^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} \quad \text{and} \quad \vec{\nabla}^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

which are derived from Maxwell's eqs. for conducting media: $\vec{\nabla}\cdot\vec{E}=0$, $\vec{\nabla}\cdot\vec{B}=0$, $\vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$, $\vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$ with $\vec{J} = \sigma\vec{E}$.

Generally, the electric properties of materials are frequency dependent.

Example: Attenuation constant k and skin depth d.

$$k = \omega \sqrt{\varepsilon\mu \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}} \qquad d = 1/k.$$

Table 14.1. *The 'penetration depth' d for copper for radiation in three familiar regions of the spectrum, calculated with the static conductivity* $\sigma \sim 5.14 \times 10^{17}\ \mathrm{s}^{-1}$ *and* $\mu = 1$.

Radiation	Infra-red	Microwaves	Long radio waves
λ_0	10^{-3} cm	10 cm	1000 m = 10^5 cm
d	6.1×10^{-7} cm	6.1×10^{-5} cm	6.1×10^{-3} cm

In non-conducting materials, the dielectric constant $\varepsilon(\omega)$ and the index of refraction $n(\omega)$ are also frequency dependent.

Electron is bound to the nucleus but can move around it's equilibrium position: modeled by considering the electron as a mass attached at the spring $F=-kx=-m\omega_0^2 x$.

The electron is exposed to other charges in the dielectric: modeled by a damping force $F=-m\gamma v$.

Electron is exposed to electric field of the wave: modeled by a harmonic external driving force $F=qE_0\cos\omega t$.

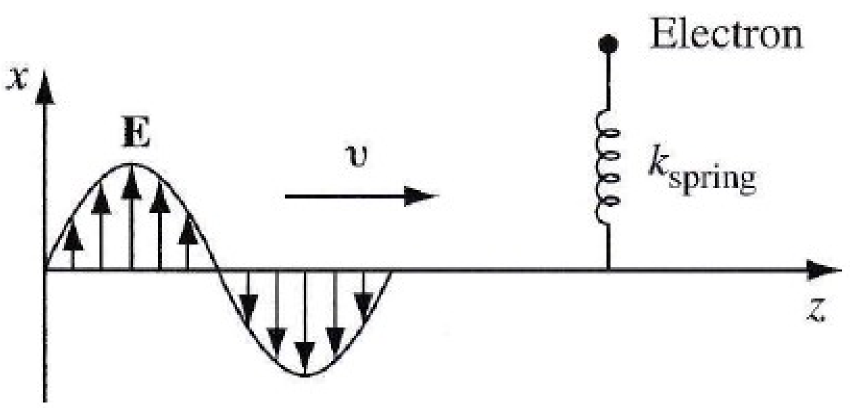


FIGURE 9.21

Differential equation for a damped harmonic oscillator driven by external force:

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = qE_0 \cos\omega t$$

Solution $\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$ with $\tilde{x}_0 = \dfrac{q/m\, E_0}{\omega_0^2 - \omega^2 - i\gamma\omega}$.

By considering the dipole moment $\tilde{p}(t) = q\tilde{x}(t)$, calculating the macroscopic polarization $\tilde{P} = \sum_{j=1}^{N} \dfrac{\langle \tilde{p}_j \rangle}{V}$, using $\tilde{P} = \varepsilon_0 \tilde{\chi}_e \tilde{E}$ in the end is obtained:

$$\varepsilon = 1 + \frac{Nq^2}{m\varepsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

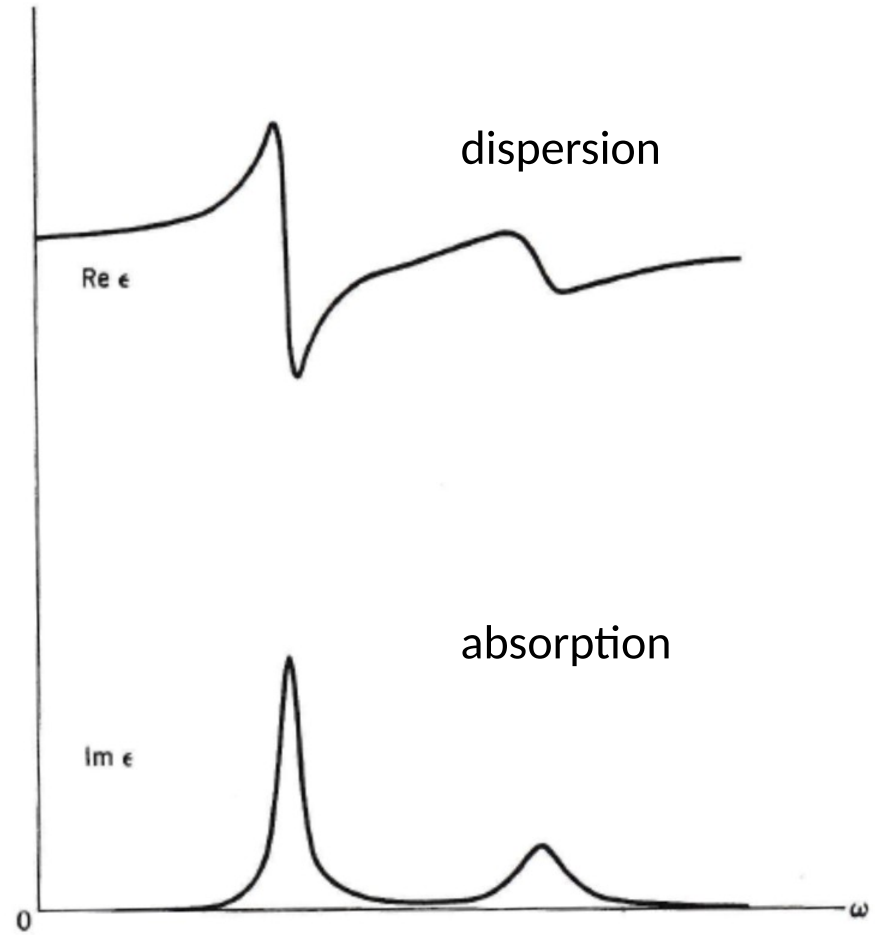


Figure 7.8 Real and imaginary parts of the dielectric constant $\epsilon(\omega)/\epsilon_0$ in the neighborhood of two resonances. The region of anomalous dispersion is also the frequency interval where absorption occurs.

$$\varepsilon = 1 + \frac{Nq^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

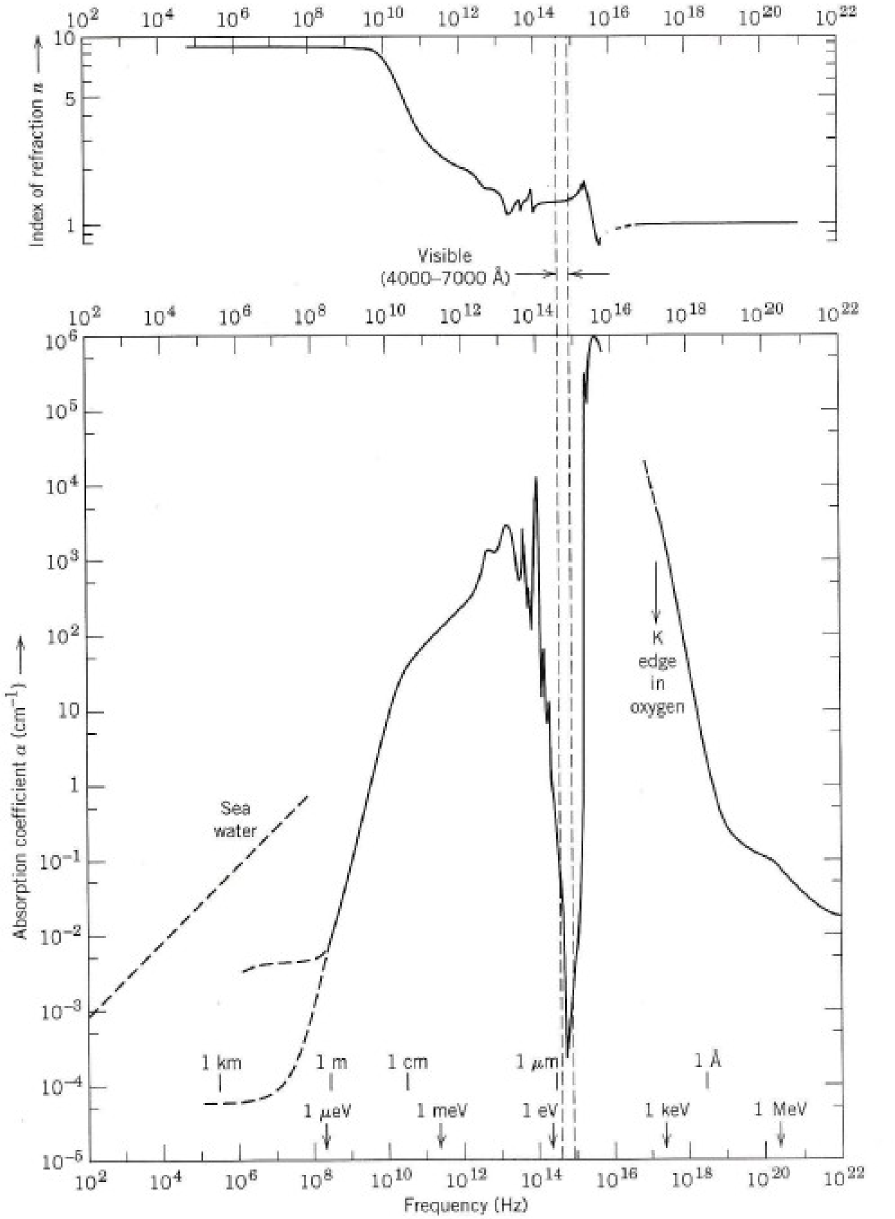


Figure 7.9 The index of refraction (top) and absorption coefficient (bottom) for liquid water as a function of linear frequency. Also shown as abscissas are an energy scale (arrows) and a wavelength scale (vertical lines). The visible region of the frequency spectrum is indicated by the vertical dashed lines. The absorption coefficient for seawater is indicated by the dashed diagonal line at the left. Note that the scales are logarithmic in both directions.