Mathematical Detour

Gamma Function

$$\boxed{\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt}$$

except $x = 0, -1, -2, \ldots$ simple poles

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1$$

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t} \, dt = \left[-t^x e^{-t}\right]_0^\infty + \int_0^\infty x\, t^{x-1} e^{-t} \, dt =$$

$$= x \int_0^\infty t^{x-1} e^{-t} \, dt = x\, \Gamma(x)$$

$$\boxed{\Gamma(x) = (x-1)\Gamma(x-1)}$$

integers

$$\Gamma(2) = 1\,\Gamma(1) = 1$$

$$\Gamma(3) = 2\,\Gamma(2) = 2$$

$$\Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2$$

$$\vdots$$

$\Gamma(n) = (n-1)!$ or $\boxed{\Gamma(n+1) = n!}$ for integer n

half-integers

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} \, dt \overset{\substack{dt = 2u\,du \\ t = u^2}}{=} \int_0^\infty u^{-1} e^{-u^2} 2u\,du = 2\int_0^\infty e^{-u^2} du = 2 \cdot \frac{1}{2}\sqrt{\pi} = \sqrt{\pi}$$

$$\Gamma(3/2) = \frac{1}{2} \cdot \Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma(5/2) = 3/2\, \Gamma(3/2) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma(7/2) = 5/2\, \Gamma(5/2) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}$$

$$\vdots$$

$$\boxed{\Gamma\left(\frac{2m+1}{2}\right) = \frac{(2m-1)!!}{2^m}\sqrt{\pi}}$$

Gaussian integrals

$$I = \int_{-\infty}^{+\infty} e^{-ax^2}\, dx$$

$x = r\cos\varphi \qquad dx\,dy \to$

$y = r\sin\varphi \qquad dF = r\,dr\,d\varphi$

$$I^2 = I \cdot I = \int_{-\infty}^{+\infty} e^{-ax^2} dx \int_{-\infty}^{+\infty} e^{-ay^2} dy = \iint_{-\infty\,-\infty}^{+\infty\,+\infty} e^{-a(x^2+y^2)}\, dx\,dy =$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-ar^2} r\,dr\,d\varphi = 2\pi \int_0^{\infty} dr\, r\, e^{-ar^2} = \frac{\pi}{a}\int_0^{\infty} dr\, 2ar\, e^{-ar^2} =$$

$$= \frac{\pi}{a}\left[-e^{-ar^2}\right]_0^{\infty} = \frac{\pi}{a} \qquad I = \sqrt{\frac{\pi}{a}}$$

$$\boxed{\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}}$$

Exercise: $\boxed{\int_{-\infty}^{+\infty} x^{2m} e^{-ax^2} dx = \frac{(2m-1)!!}{(2a)^m}\sqrt{\frac{\pi}{a}}}$

moments of a Gaussian variable: (zero mean, σ^2 variance)

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x^{2m} e^{-x^2/2\sigma^2} dx = (2m-1)!!\, \sigma^{2m} = \langle x^{2m} \rangle$$

$$\boxed{\int_0^{\infty} x^{2m+1} e^{-ax^2} dx = \frac{m!}{2a^{m+1}}}$$

Relationship to Gamma Function

$ay^2 = x \qquad y = \left(\frac{x}{a}\right)^{1/2} \qquad dy = \frac{1}{2}\frac{dx}{\sqrt{ax}}$

$$I_m = \int_0^{\infty} e^{-ay^2} y^m dy = \int_0^{\infty} e^{-x} \frac{x^{m/2}}{a^{m/2}} \frac{1}{2}\frac{1}{(ax)^{1/2}} dx = \frac{1}{2a^{\frac{m+1}{2}}} \int_0^{\infty} e^{-x} x^{\frac{m-1}{2}} dx$$

$$= \frac{1}{2a^{\frac{m+1}{2}}} \int_0^{\infty} e^{-x} x^{\frac{m+1}{2}-1} dx = \boxed{\frac{1}{2a^{\frac{m+1}{2}}}\Gamma\left(\frac{m+1}{2}\right)} \qquad m \geq -1$$

Thus: $I_{2m} = \frac{1}{2}\frac{(2m-1)!!}{2^m a^m}\sqrt{\frac{\pi}{a}}$

$$I_{2m+1} = \frac{1}{2}\frac{1}{a^{m+1}} m!$$

The factorial $(n!)$, its integral representation $\Gamma(n+1)$, and its asymptotic expansion for large n

$$n! = \Gamma(n+1)$$

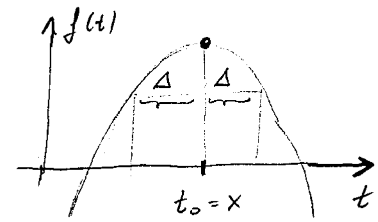
$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x\, dt = \int_0^\infty e^{-t}\, e^{x \ln t}\, dt = \int_0^\infty e^{-t + x\ln t} = \int_0^\infty e^{f(t)}\, dt$$

$f(t) = -t + x \ln t$ Taylor expand: (about max) $f(t) = f(t_0) + f'(t_0)(t-t_0) + \frac{1}{2} f''(t_0)(t-t_0)^2$

$f'(t) = -1 + \frac{x}{t} \implies f'(t_0) = 0$ $t_0 = x$ (at max.) $f(t_0) = -x + x\ln x$

$f''(t) = -\frac{x}{t^2}$ $f''(t_0) = -\frac{1}{x}$

$m \geq 3$: $f^{(m)}(t) \propto \frac{x}{t^m}$ $f^{(m)}(t_0) \propto \frac{1}{x^{m-1}}$



$\Delta = \sqrt{x}$

$$f(t) = \underbrace{-x + x\ln x}_{f(t_0)} - \frac{1}{2x}(t-x)^2 + \sum_{m\geq 3} o\left(\frac{x^{m/2}}{x^{m-1}}\right)$$

$$\searrow o\left(\frac{1}{x^{\frac{m}{2}-1}}\right) \xrightarrow[x \gg 1]{} 0$$

$|t - x| \lesssim \sqrt{x} = x^{1/2}$ (region of dominant contribution)

$$\Gamma(x+1) \approx \int_{x-\Delta}^{x+\Delta} e^{-x + x\ln x - \frac{1}{2x}(t-x)^2}\, dt \approx e^{-x + x\ln x} \int_{-\infty}^{+\infty} e^{-\frac{1}{2x}(t-x)^2}\, dt$$

$$= e^{-x + x\ln x}\sqrt{2\pi x} = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$$

$$\ln n! = \Gamma(n+1) \approx n\ln n - n + \ln\sqrt{2\pi n}$$

$\longrightarrow$ negligible (if written for logarithm)

$$\boxed{\ln n! \approx n \ln n - n + \ln\sqrt{2\pi n}}$$

(Stirling approximation)

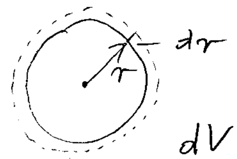
The "area" and "volume" of the d-dimensional sphere $\quad dV = A(r)\,dr$

$d=2 \qquad V_2 = \pi r^2 \qquad A_2 = 2\pi r$

$d=3 \qquad V_3 = \frac{4\pi}{3} r^3 \qquad A_3 = 4\pi r^2$

$\vdots$

$V(r) = \int_0^r A(r')\,dr' \qquad \frac{dV}{dr} = A(r)$



$dV = A\,dr$

$A = \frac{dV}{dr}$

$V_d = C_d\, r^d \implies \boxed{A_d = d\cdot C_d\, r^{d-1}}$

$I \equiv \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$ (*)

$$I^d = \int_{-\infty}^{+\infty} e^{-x_1^2}dx_1 \int_{-\infty}^{+\infty} e^{-x_2^2}dx_2 \cdots \int_{-\infty}^{+\infty} e^{-x_d^2}dx_d = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{-(x_1^2+x_2^2+\dots+x_d^2)}\,dx_1 dx_2 \dots dx_d$$

$$= \int e^{-r^2}\,dV \qquad r^2 = x_1^2 + x_2^2 + \dots + x_d^2$$

$$= \int_0^\infty e^{-r^2} A_d\,dr = \int_0^\infty e^{-r^2} d\cdot C_d\, r^{d-1}\,dr = d\cdot C_d \int_0^\infty e^{-r^2} r^{d-1}\,dr$$

$r^2 = u \qquad r = u^{1/2} \qquad dr = \frac{1}{2}u^{-1/2}\,du$

$$I^d = d\cdot C_d \int_0^\infty e^{-u}\, u^{\frac{d-1}{2}}\, \frac{1}{2} u^{-1/2}\,du = \frac{1}{2} d\cdot C_d \int_0^\infty e^{-u}\, u^{\frac{d}{2}-1}\,du$$

$$= \frac{d}{2}\cdot C_d\, \Gamma\left(\frac{d}{2}\right) = C_d\, \Gamma\left(\frac{d}{2}+1\right) = \pi^{d/2} \text{ (from (*))}$$

$$\boxed{C_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}}$$

$$\boxed{\begin{aligned} V_d &= \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\, r^d \\ A_d &= d\cdot C_d\, r^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\, r^{d-1} \end{aligned}}$$