Paul Lea 9/18/2024

ECON4570/6560 : Problem Set 1

Due by 11:59pm on Sep 13, 2024 [1]

1. Below is the joint probability table for the discrete random variables $X$ and $Y$. Suppose that the conditional probability $P(X = 1|Y = 10) = 0.5$.

| | $Y = 6$ | $Y = 8$ | $Y = 10$ | |
|---|---|---|---|---|
| $X = 1$ | $A$ | $B$ | $C$ | $G$ |
| $X = 2$ | 0 | $D$ | $E$ | 0.2 |
| $X = 3$ | 0.2 | $F$ | 0.2 | $I$ |
| | 0.4 | $H$ | 0.4 | 1 |

$A = 0.2$
$B = 0$
$C = 0.2$
$D = 0.2$
$E = 0$
$F = 0$
$G = 0.4$
$H = 0.2$
$I = 0.4$

• C is 0.5 of 0.4 conditional probability.
Given $C = 0.5 \cdot 0.4 = 0.2$
& $C + E + 0.2 = 0.4$
$\Rightarrow E = 0$

$0 + E + 0 = 0.2 \therefore D = 0.2$
$A + 0 + 0.2 = 0.4 \therefore A = 0.2$
$H + 0.4 + 0.4 = 1 \therefore H = 0.2$
$B + 0 + F = 0.2 \therefore B, F = 0$
$G = A + B + C = 0.4$
$G + I + 0.2 = 1 \therefore I = 0.4$

| | $Y = 6$ | $Y = 8$ | $Y = 10$ | |
|---|---|---|---|---|
| $X = 1$ | 0.2 | 0 | 0.2 | 0.4 |
| $X = 2$ | 0 | 0.2 | 0 | 0.2 |
| $X = 3$ | 0.2 | 0 | 0.2 | 0.4 |
| | 0.4 | 0.2 | 0.4 | 1 |

**(b)** Compute mean $\mu_X$ ($= E(X)$) and variance $\sigma_X^2$ of $X$?

$$\sum^{3} x_i \left(P(X_i)\right) = 1 \cdot 0.4 + 2 \cdot 0.2 + 3 \cdot 0.4$$

$$= 2 \qquad \mu_X = 2$$

$\sigma_x^2 = E(x^2) - \mu_x^2 = ?$

$$E(X^2) = \sum^{3} X_i^2 P(X_i) = 1^2 \cdot 0.4 + 2^2 \cdot 0.2 + 3^2 \cdot 0.4$$

$$\Rightarrow 0.4 + 0.8 + 3.6$$

$$= 4.8$$

$$\mu_X^2 = 2^2 = 4$$

$$4.8 - 4 = 0.8 \qquad \sigma_X^2 = 0.8$$

**(c)** Compute conditional mean of $X$ given $Y = 6, 8, 10$, that is, compute $E(X|Y = 6), E(X|Y = 8), E(X|Y = 10)$. Are $X$ and $Y$ mean independent?

$$E(X|Y=6) \equiv \sum_{i=1}^{3} P(X=X_i \mid Y=6)\, X_i$$

$$\frac{2}{10} \cdot \frac{10}{4} = \frac{20}{40} = 0.5$$

$$0.5(1) + 0(2) + 0.5(3) = 2$$

$$E(X|Y=6) = 2$$

$$E(X|Y=8) \equiv \sum_{i=1}^{3} P(X=X_i \mid Y=8)\, X_i$$

$$\frac{0.2}{0.2} = 1$$

$$0 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 = 2$$

$$E(X|Y=8) = 2$$

$$E(X \mid Y=10) = \sum_{i=1}^{3} P(X=x_i \mid Y=10)\, x_i$$

$$\frac{0.2}{0.4} = 0.5 \qquad 0.5 \cdot 1 + 0.2 + 0.5 \cdot 3 = 2$$

$$E(X \mid Y=10) = 2$$

all 3 are the same, $\therefore$ X & Y are mean independent.

**(d)** Compute the covariance between $X$ and $Y$? Are $X$ and $Y$ correlated?

The two variables are mean independent, therefore the covariance is 0, & they are uncorrelated

#2

2. You are having guests over for a mussel feast. In the morning you are at Joes Not-so-Fresh Fish Market trying to decide how many mussels to buy. From experience you know that about one out of every ten will not open (dead and hence not fresh) when cooked and must be thrown away. If you buy 60 mussels, what is the approximate probability that your feast will consist of less than 6 mussels?

$$X_i \begin{cases} 1 : \text{i'th mussel is good} \\ 0 : \text{else} \end{cases}$$

$$X_i \sim \text{Bernoulli}(p)$$

$$p = P(X=1) = 0.9$$

we want to find

$$P(\sum X_i < 6)$$

$$P\left(\frac{\sum^{60} X_i - 60 \cdot p}{\sqrt{60 \cdot p(1-p)}} < \frac{6 - 60 \cdot p}{\sqrt{60 \cdot p(1-p)}}\right)$$

$$\approx P\left(Z < \frac{6 - 60 \cdot p}{\sqrt{60 \cdot p(1-p)}}\right)$$

$$\frac{-48}{2.3238} = -20.6559$$

$$P(Z < -20.6559)$$

very small!! $\approx 0$

#3 a: $X_i \sim \text{Bernoulli}(P)$

where $P = P(X=1)$

$$n_1 = \sum_{i=0}^{n} (X_i)$$

Using Law of Large #'s:

$$\frac{n_1}{n} = \frac{1}{n} \sum X_i$$

$$E\left(\frac{n_1}{n}\right) = E\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) = P$$

↳ ∴ it is not a biased estimator, & is consistent.

b: $N = 200,\ n = 80,\ n_1 = 30$

$$P = \frac{150}{200} = 0.75$$

$$P\left(\sum X_i > 55\right)?$$

$$= P\left(\frac{\sum^{80} X_i - 80 \cdot P}{\sqrt{80\,P(1-P)}} > \frac{\overbrace{55 - 80P}^{-5}}{\underbrace{\sqrt{80P(1-P)}}_{15}}\right)$$

$\Leftrightarrow P\left(Z > \frac{-3}{\sqrt{13}}\right)$

$\Leftrightarrow 1 - P\left(Z \le \frac{-3}{\sqrt{13}}\right) = 1 - 0.09835$

$\Leftrightarrow 0.902 \rightarrow$ 90.2%

c. $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{n_1}{n} = \frac{50}{80} = 0.625$

Confidence Interval:

$$\left[\hat{p} - Z_{\frac{\alpha}{2}} \cdot s.e(\hat{p}),\ \hat{p} + Z_{\frac{\alpha}{2}}\ s.e(\hat{p})\right]$$

$$\Leftrightarrow \left[\hat{p} - Z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] - Z_{\frac{\alpha}{2}}$$

90% CI

$$\left[0.625 - Z_{0.05} \cdot \sqrt{\frac{0.625(1-0.625)}{80}},\ 0.625 + Z_{0.05} \cdot \sqrt{\frac{0.625(1-0.625)}{80}}\right.$$

$$\left[0.625 \pm 1.645 \cdot 0.05412659\right]$$

0.536 , 0.714

The proportion is between

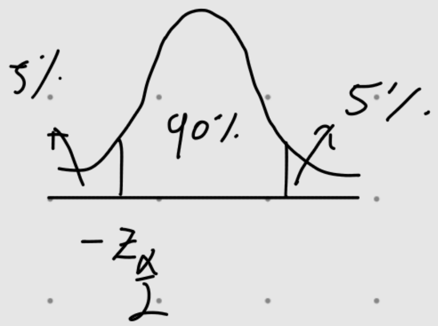
0.54686 & 0.65314

d: $H_0 : p = 0.75$ vs $H : p \neq 0.75$

Under $H_0$

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \sim N(0,1)$$ using CLT

$$Z_{out} = \frac{5/8 - 0.75}{\sqrt{\frac{5/8(1-5/8)}{80}}} = -2.31$$




$\alpha = 10\%$ CV $= Z_{\frac{\alpha}{2}} = Z_{0.05}$

$|Z_{out}| = 1.645$

p-Value $= 2 \cdot P(Z < -|Z_{out}|)$

$\hookrightarrow 2 \cdot P(Z < -1.645)$

$2 \cdot (0.049985) = 0.09997$

$0.1 > 0.09997$

P-Value test rejects
$H_0$ b/c P-value $< 0.1$