Class 22 (04/11/24)

Electric Dipole Radiation

Electromagnetic Radiation

- Transports electromagnetic energy :

Energy flux density (Pointing Vector) $\mathbf{S}=(1/\mu_0)\ \mathbf{E} \times \mathbf{B}$

- Transports information

Engineered Forms of Electromagnetic Radiation

- Electric dipole radiation (source oscillating electric dipole)
- Magnetic dipole radiation (wire loop carrying oscillating current)
- Radiation from an arbitrary source

Retarded Potentials
&
Radiation Fields
of an Oscillating Electric
Dipole

Retarded Potentials

$$V(\vec{r},t)=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\,dV'$$

with

$$t_r = t-\frac{|\vec{r}-\vec{r}\,'|}{c}$$

$$\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\,dV'$$

j (**r**, t_r = t - (**r**-**r**’)/c)

E (**r**, t), **B** (**r**,t)

Origin of Coordinate System O

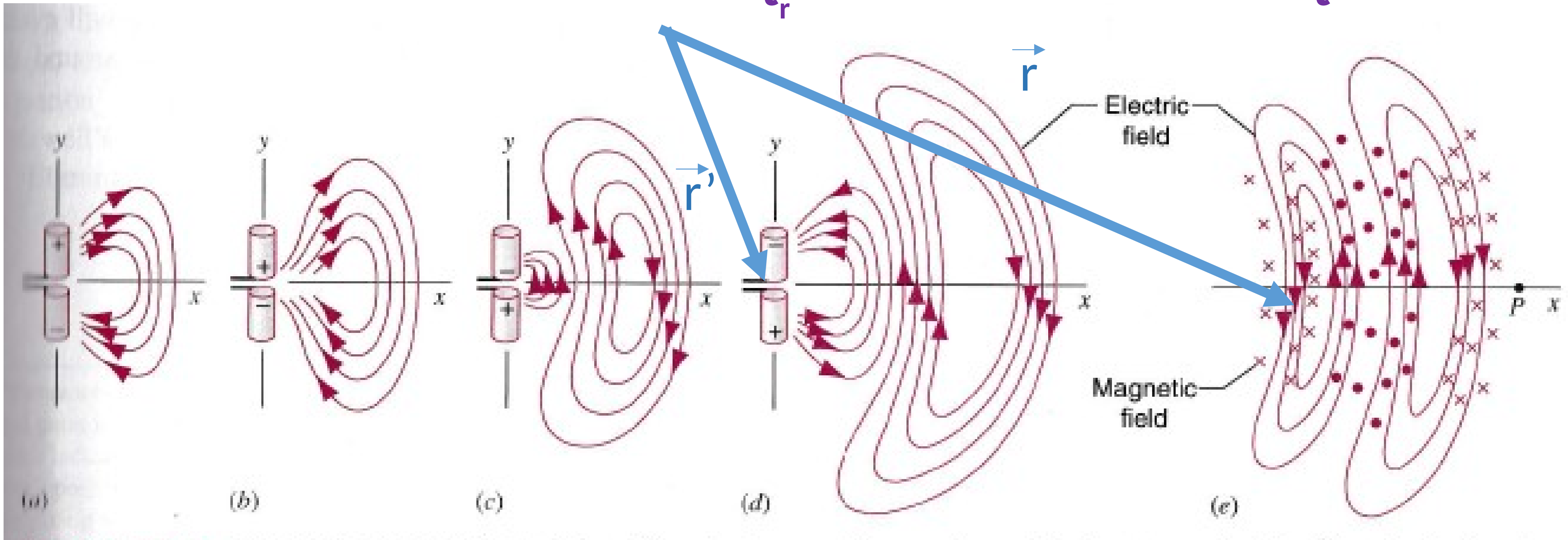


FIGURE 38-6. Successive stages in the emission of an electromagnetic wave from a dipole antenna. In (a)–(d), only the electric field patterns are shown. In (e), the magnetic field is shown as perpendicular to the plane of the page.

Electric Dipole Radiation

Characteristic length scales

- Size of oscillating electric dipole d
- Wavelengths of the electromagnetic radiation λ
- Distance $|r-r'|$ between field point **r** and location of the electric dipole **r'**

Example: 5G cellular data networks have an operating frequency of $f \approx 28$ GHz.

$\lambda(@\ 28\ \text{GHz}) \approx 1.1\text{cm}$,

antenna size ($\lambda/2$, $\lambda/4$) $d \approx 0.25$ to 0.55cm,

cell tower range $|r-r'| \approx 10$ Miles (10^6 to 10^7 cm)

Electrical dipole radiation

characteristic lengths : dipole lengths d,

wavelengths λ, field point distance r

generally $d < \lambda \ll r$

for radio, cellphone etc. $d \approx \frac{\lambda}{4}, \frac{\lambda}{2}$ Hertz dipole

oscillating charge $q = q_0 \cos\omega t$ over distance d

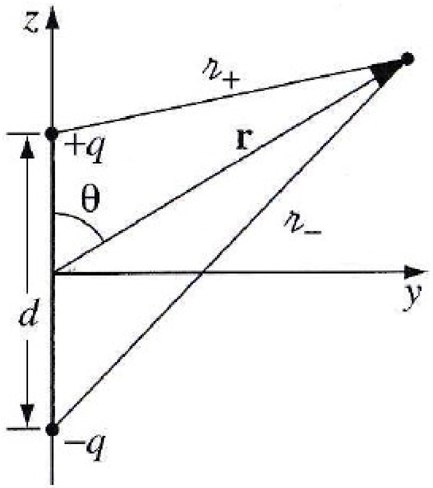
oscillating electrical dipole : $\vec{p}(t) = p_0 \cos\omega t \,\hat{z}$

$$p_0 = q_0 d$$

oscillating current $i = \frac{dq}{dt} = -\omega q_0 \sin\omega t \qquad \vec{i} = i\,\hat{z}$

retarded electric & magnetic potentials :

$$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} dV' , \quad A(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)\, dV'}{|\vec{r} - \vec{r}'|}$$



for retarded potential $V(\vec{r},t) = V_{+q} + V_{-q}$

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_0\cos[\omega(t-r_+/c)]}{r_+} - \frac{q_0\cos[\omega(t-r_-/c)]}{r_-}\right]$$

point charges at $+d/2$, $-d/2$ where point charge magnitude changes periodically with time

$r_\pm$ $\quad r_+^2 = \frac{d^2}{2^2} + r^2 - \frac{2dr}{2}\cos\theta$

$r_-^2 = \frac{d^2}{2^2} + r^2 - \frac{2dr}{2}\cos(180^\circ-\theta)$, $\cos(180^\circ-\theta) = -\cos\theta$

$$r_\pm^2 = r^2 \mp dr\cos\theta + \frac{d^2}{2^2}$$

$r_\pm^2 = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4}\right)$ → expand in $\frac{d}{r}$

$F \approx F(0) + F'(0)\cdot x$

Characteristic length scale d/r <<< 1 !!!!

result of expanding $r_\pm = r\left(1 \mp \frac{d}{2r}\cos\theta\right)$

expand again $\frac{1}{r_\pm}$ in Taylor series $F = F(0) + F'(0)x$

result of expansion $\frac{1}{r_\pm} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\cos[\omega(t - r_\pm/c)] = \cos\left[\omega\left(t - \frac{r}{c}\right) \pm \frac{\omega d}{2c}\cos\theta\right]$$

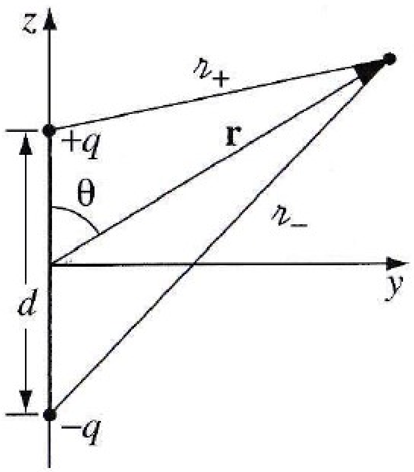
$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right)$$

$$\mp \sin[\omega(t - r/c)]\sin\frac{\omega d}{2c}\cos\theta$$

$d \ll \lambda \Rightarrow d \ll \frac{c}{f} \Rightarrow d \ll \frac{c}{2\pi f} = \frac{c}{\omega} \Rightarrow \frac{\omega d}{c} \ll 1$

small angle approximation $\cos x \approx 1$, $\sin x \approx x$

$$= \cos[\omega(t - r/c)] \mp \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\frac{\omega d}{2c}\cos\theta$$



$$\frac{\cos\left[\omega\left(t-\frac{r_+}{c}\right)\right]}{r_+} = \left\{\cos[\omega(t-r/c)] - \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\frac{\omega d}{2c}\cos\theta\right\} \times \frac{1}{r}\left(1+\frac{d}{2r}\cos\theta\right)$$

$$\frac{\cos\left[\omega\left(t-\frac{r_-}{c}\right)\right]}{r_-} = \left\{\cos[\omega(t-r/c)] + \sin\left[\omega\left(t-\frac{r}{c}\right)\right]\frac{\omega d}{2c}\cos\theta\right\} \times \frac{1}{r}\left(1-\frac{d}{2r}\cos\theta\right)$$

$$V(\vec{r},t) = \frac{q_0}{4\pi\varepsilon_0}\cdot\frac{1}{r}\,d\cos\theta\left[-\frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \frac{1}{r}\cos[\omega(t-r/c)]\right]$$

$$\omega \Rightarrow 0 \qquad V(\vec{r},t) = \frac{q_0 d\cos\theta}{4\pi\varepsilon_0 r^2} \propto \frac{1}{r^2}$$

$$\text{radiation term } V(\vec{r},t) = \frac{q_0 d\cos\theta}{4\pi\varepsilon_0 r}\left(-\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

Radiation term V (**r**,t)≈ (1/r) matters in the far-field because it is larger than the ($1/r^2$) term V(**r**,t).

vector potential

$$\vec{I} = -q_0 \omega \sin \omega t \ \hat{z}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{-q_0 \omega \sin[\omega(t - r_+/c)]\ \hat{z}}{r_+} \, dz$$

d very short

replace integrand by its value at the center d = 0

$$\Rightarrow \quad r_+ (d=0) = r \quad , \quad \frac{1}{r_+}(d=0) = \frac{1}{r}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 d}{4\pi} \frac{q_0 \omega \sin[\omega(t - r/c)]}{r}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 q_0 d \omega}{4\pi r} \sin\left[\omega\left(t - \frac{r}{c}\right)\right] \hat{z}$$

in spherical coordinates

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}$$

$\vec{A} = A\,\hat{z}$ $\qquad$ $\hat{z}$ in $\hat{r}, \hat{\theta}, \hat{\phi}$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}$$

1.37 $\qquad \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ $\qquad\qquad \vec{a} = X\vec{d} \Rightarrow X^{-1}\vec{a} = \underbrace{X^{-1}X}_{=1}\vec{d}$

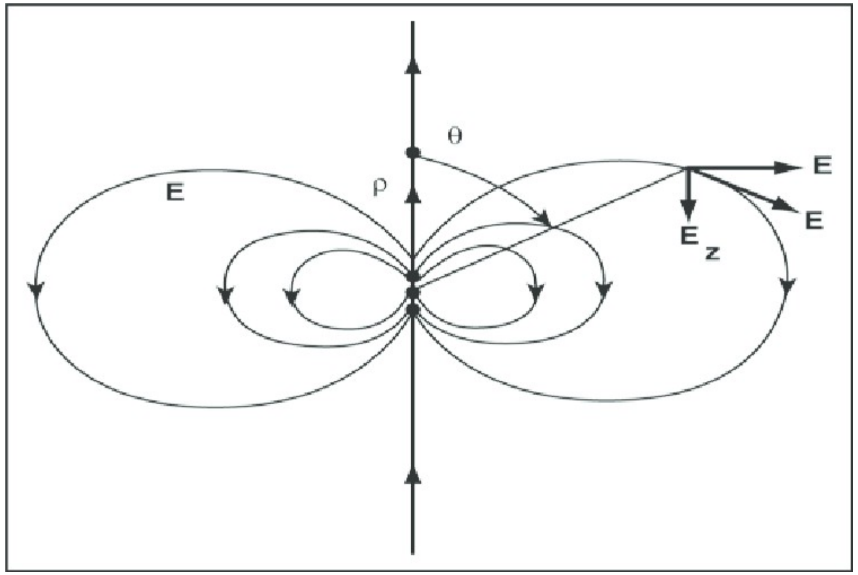
$$\vec{A}(r,\theta,\phi) = \frac{\mu_0 q_0 d\,\omega}{4\pi r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

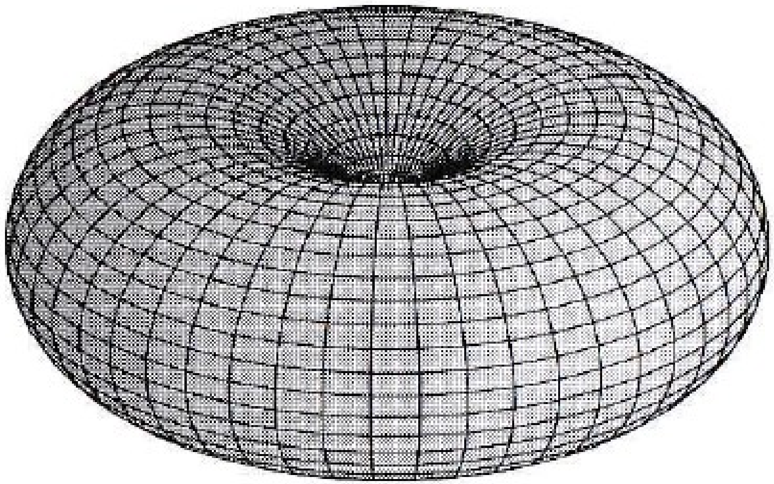
$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\phi}$$

$$\langle S \rangle = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r}$$

$$\langle P \rangle = \int_{\text{sphere}} \vec{S}\cdot d\vec{a} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

Electric field radiation pattern<S>





<S> $= \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \frac{\sin^2\theta}{r^2} \hat{r}$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi} \left(\frac{\sin\theta}{r}\right) \cos\left[\omega\left(t - \frac{r}{c}\right)\right] \hat{\theta}$$