$\{s_i\}_{i=1}^{N}$ $\quad \langle s_i \rangle = m \quad \langle (s_i - m)^2 \rangle = \sigma_s^2 \quad \forall i$

and independent

$$\Delta s_i = s_i - \langle s_i \rangle$$
$$\langle (\Delta s_i)^2 \rangle = \sigma_s^2$$

$$M = \sum_{i=1}^{N} s_i$$

1) $$\langle M \rangle = \left\langle \sum_{i=1}^{N} s_i \right\rangle = \sum_{i=1}^{N} \langle s_i \rangle = \sum_{i=1}^{N} m = Nm = N\langle s_i \rangle$$

2) $$\langle (\Delta M)^2 \rangle = \langle (M - \langle M \rangle)^2 \rangle = \left\langle \left(\sum_{i=1}^{N} s_i - \sum_{i=1}^{N} \langle s_i \rangle \right)^2 \right\rangle = \left\langle \left(\sum_{i=1}^{N} \underbrace{(s_i - \langle s_i \rangle)}_{\Delta s_i} \right)^2 \right\rangle$$

$$= \left\langle \left(\sum_{i=1}^{N} \Delta s_i \right)^2 \right\rangle = \left\langle \sum_{i=1}^{N} \Delta s_i \sum_{j=1}^{N} \Delta s_j \right\rangle = \sum_{i,j} \langle \Delta s_i \Delta s_j \rangle =$$

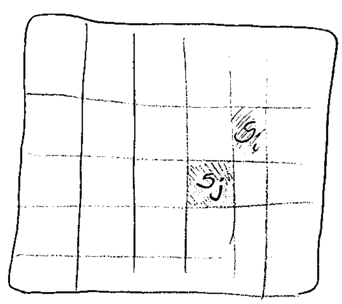
$$= \sum_{i \neq j} \underbrace{\langle \Delta s_i \Delta s_j \rangle}_{\langle \Delta s_i \rangle \langle \Delta s_j \rangle} + \sum_{i=1}^{N} \langle (\Delta s_i)^2 \rangle = N \langle (\Delta s_i)^2 \rangle = N \sigma_s^2$$

$\langle \Delta s_i \rangle \langle \Delta s_j \rangle$ (if s_i and s_j are independent)

$\quad \| \quad\quad \|$
$\quad 0 \quad\quad 0$

$$\boxed{\sigma_M = \sqrt{N}\, \sigma_s}$$

Stat. Mech implications: N independent subsystem



relative fluctuations: $$\frac{\sigma_M}{\langle M \rangle} = \frac{\sqrt{N}\sigma_s}{Nm} = \frac{1}{\sqrt{N}} \frac{\sigma_s}{m}$$

$$\propto \frac{1}{\sqrt{N}}$$

$i \neq j$: $\sim$ independent

Generating function

$p(x)$ (prob. density)

$$\phi(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{\infty} dx\, e^{ikx} p(x) \qquad \text{(Fourier tr.)}$$

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{-ikx} \phi(k) \qquad \text{(inverse Fourier tr.)}$$

Properties: $\phi(k)$: moment generating function

1) $$\phi(k)\Big|_{k=0} = \int_{-\infty}^{\infty} dx\, p(x) = 1 \qquad \text{by normalization}$$

2) $$\frac{d\phi}{dk}\Big|_{k=0} = \int_{-\infty}^{+\infty} dx\, (ix)\, p(x) = i\langle x \rangle$$

$$\frac{d^2\phi}{dk^2}\Big|_{k=0} = \int_{-\infty}^{+\infty} dx\, (ix)^2 p(x) = (i)^2 \langle x^2 \rangle = -\langle x^2 \rangle$$

$$\frac{d^n\phi}{dk^n}\Big|_{k=0} = \int_{-\infty}^{+\infty} dx\, (ix)^n p(x) = i^n \langle x^n \rangle$$

$$\boxed{\langle x^n \rangle = \frac{1}{i^n} \frac{d^n\phi}{dk^n}\Big|_{k=0}} \qquad \boxed{\phi(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle}$$

"free energy"

$$\boxed{\psi(k) = \ln \phi(k)}$$ partition function

$$\psi(k) = \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} C_m$$

$$\psi(k)\Big|_{k=0} = 0 \qquad C_m = \frac{1}{i^k} \frac{d^m \psi(k)}{dk^m}\Big|_{k=0}$$

$$\boxed{C_0 = 0}$$

$$\left.\frac{d\Psi}{dk}\right|_{k=0} = \left.\frac{1}{\phi(k)}\frac{d\phi}{dk}\right|_{k=0} = i\langle x\rangle$$

$$\left.\frac{d^2\Psi}{dk^2}\right|_{k=0} = \left.\frac{\frac{d^2\phi}{dk^2}\phi(k) - \left(\frac{d\phi}{dk}\right)^2}{\phi^2(k)}\right|_{k=0} = i^2\langle x^2\rangle - (i\langle x\rangle)^2 = -(\langle x^2\rangle - \langle x\rangle^2)$$

$$\vdots$$

$$\boxed{\Psi(k) = \ln\phi(k) = \sum_{n=1}^{\infty}\frac{(ik)^n}{n!}C_n} \qquad C_n = \frac{1}{i^n}\left.\frac{d^n\Psi}{dk^n}\right|_{k=0}$$

$$C_1 = \langle x\rangle$$
$$C_2 = \langle x^2\rangle - \langle x\rangle^2 = \sigma^2$$
$$C_3 = \langle x^3\rangle - 3\langle x^2\rangle\langle x\rangle + 2\langle x\rangle^3$$
$$\vdots$$
$$C_n = \dots$$

The Central Limit Theorem

assume $\{x_i\}_{i=1}^{N}$ identically distributed, independent variables and $\underline{\sigma = \text{finite}}$

$\langle x_i \rangle = \langle x \rangle \quad \forall_i$
$\sigma_i = \sigma \quad$ (iid)

Then, for $\displaystyle \mathbf{z} = \frac{\sum_{i=1}^{N} x_i - N\langle x \rangle}{\sqrt{N}\,\sigma}$:

$$\boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}}$$

__Proof:__ for simplicity: $\langle x_i \rangle = 0$ (w. l. g.)
(if not we could consider $x_i' = x_i - \langle x \rangle \qquad \langle x_i' \rangle = 0$)

$$\boxed{z = \frac{\sum x_i}{\sqrt{N}\,\sigma}} \qquad (\text{guarantees } \sigma_z^2 = 1)$$

$$\phi_z(k) = \langle e^{ikz} \rangle = \langle e^{ik\frac{\sum_i x_i}{\sqrt{N}\sigma}} \rangle = \langle \prod_{i=1}^{N} e^{i\frac{k}{\sqrt{N}\sigma}x_i} \rangle = \prod_{i=1}^{N} \langle e^{i\frac{k}{\sqrt{N}\sigma}x_i} \rangle$$

$$= \prod_{i=1}^{N} \phi\left(\frac{k}{\sqrt{N}\sigma}\right) = \phi^N\left(\frac{k}{\sqrt{N}\sigma}\right)$$

$$\psi_z(k) = \ln \phi^N\left(\frac{k}{\sqrt{N}\sigma}\right) = N \ln \phi\left(\frac{k}{\sqrt{N}\sigma}\right) = N\,\psi\left(\frac{k}{\sqrt{N}\sigma}\right)$$

$$= N \sum_{n=1}^{\infty} \frac{\left(i\, k/\sqrt{N}\sigma\right)^n}{n!} C_n = N \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} (N\sigma^2)^{-n/2} C_n$$

$$= \sum_{n=1} \frac{(ik)^n}{n!} \underbrace{\sigma^{-n} N^{1-n/2} C_n}_{C_n^z} \qquad \boxed{C_n^z = \frac{C_n}{\sigma^n N^{n/2-1}}}$$

for z: $\displaystyle C_1^z = \langle z \rangle = \frac{C_1}{\sigma/\sqrt{N}} = \frac{\langle x \rangle}{\sigma/\sqrt{N}} \equiv 0$ (all N)

$$C_2^z = \sigma_z^2 = \frac{1}{\sigma^2} C_2 = \frac{1}{\sigma^2}\sigma^2 = 1$$

$$C_n^z = \frac{C_n}{\sigma^n N^{n/2-1}} \xrightarrow[N\to\infty]{} 0 \qquad \text{for } n > 2$$

$n=3$: $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$

Thus, for $N \to \infty$

$$\Psi_z(k) = -\frac{k^2}{2}$$

$$\Psi_z(k) = \ln \phi_z(k) \Rightarrow \phi_z(k) = e^{-k^2/2}$$

$$p(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-ikz}\phi_z(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-k^2/2 - ikz} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Details

$$-\frac{k^2}{2} - ikz = -\frac{1}{2}(k+iz)^2 - \frac{z^2}{2}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-\frac{1}{2}(k+iz)^2 - \frac{z^2}{2}} = \frac{1}{2\pi} e^{-z^2/2}\int_{-\infty}^{+\infty} dk\, e^{-\frac{1}{2}(k+iz)^2} =$$

$$= \frac{1}{2\pi} e^{-z^2/2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$