Power laws and universality

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SIMPLE interactions sometimes lead to complex results. For example, dynamical systems with as few as three independent variables show chaotic behaviour that is, in any practical sense, unpredictable (at least for long times). Conversely, complex interactions can lead to simple results. This is particularly true for systems formed of many interacting subunits[1], such as liquids, solids or gases, which are poised at a 'critical point' where two or more macroscopic phases become indistinguishable. At a critical point, many of the precise details of the interactions between constituent subunits play virtually no role whatsoever in determining the bulk properties of the system.

To understand these surprising phenomena (known as 'critical' phenomena), 25 years ago scientists developed the twin concepts of 'scaling' and 'universality'. Systems near critical points exhibit self-similar properties; thus, to some degree, they are invariant under transformations of scale. This is the property of scaling. The word universality connotes the fact that quite disparate systems behave in a remarkably similar fashion near their respective critical points — because what matters most is not the details of the microscopic interactions but rather the nature of the 'paths along which order is propagated' from one subunit to another distant subunit.

The development of these ideas was made possible by a remarkable combination of experiment and phenomenological theory, combined with detailed study of a simple mathematical model — the Ising model. Most of the principal ideas that emerged were tested on this model system. Despite the tremendous wealth of experimental work on systems near their critical points, however, the equation of state predicted by the theory had never been tested with sufficient care. Such a test is the subject of the paper by C. H. Back and collaborators appearing on page 597 of this issue[2].

In the Ising model, classical spins localized on the sites of a lattice can point either 'up' or 'down'. The ferromagnetic Ising interaction is particularly simple: if two neighbouring spins are parallel (both up or both down), then there is a negative contribution to the energy. Hence the system has minimum energy when all the spins in the system are parallel. When the lattice is in equilibrium at a temperature T, it does not reside in this lowest energy state, but explores higher energy states. The Ising model can be regarded as a crude model of a ferromagnet if we think of the classical spins as representing the constituent microscopic moments that make up the ferromagnet.

Studies of the Ising model reveal a remarkable feature. If one tunes a 'control parameter' — the temperature T — then one finds that at a certain critical value T_c spins remarkably far apart have orientations that are strongly correlated. Such correlations do not lend themselves to a ready explanation. Normally in physics modelling you get out what you put in, yet in the case of the Ising model, we 'put in' an interaction that extends a finite distance, and magically, we 'get out' a correlation that spreads an infinite distance. How does this happen?

Our intuition tells us that the correlation $C(r)$ between subunits separated by a distance r should decay exponentially with r, for the same reason the value of money stored in one's mattress decays exponentially with time (each year it loses a constant fraction of its worth). Thus $C(r) = \exp\{-r/\xi\}$, where ξ is termed the correlation length — the characteristic length scale above which the correlation function is negligibly small.

Experiments and also calculations on mathematical models confirm that correlations do indeed decay exponentially so long as the system is not exactly *at* its critical point. At T_c, the rapid exponential decay turns into a long-range power-law decay of the form $C(r) = r^{-\eta}$ where η is called a critical exponent[1]. If correlations decay with a power-law form, we say the system is 'scale free', because there is no characteristic scale associated with a simple power law.

Critical exponents, such as η, are found empirically to depend most strongly upon the system dimension and on the general symmetry properties of the constituent subunits, and not on other details of the system under investigation. We understand power-law decay as arising primarily from the multiplicity of interaction paths that connect two spins in dimensions larger than one[3]. Exact enumeration methods take into account exactly the contributions of such paths, up to a maximum length that depends on the strength of the computer used and the patience of the investigator. To obtain quantitative results, the hierarchy of exact results is extrapolated to infinite order. Roughly stated, although the correlation along each path decreases exponentially with the length of the path, the number of such paths increases exponentially. The 'gently decaying' power-law correlation emerges as the victor in this competition between the two warring exponential effects.

Cyril Domb, Michael Fisher and their colleagues pioneered such exact enumeration approaches in the 1970s. Among the results emerging from their efforts was a complete calculation of the scaling equation of state of the Ising model in a magnetic field[4]. It is this equation of state that is tested by the experiments of Back and co-workers.

What Back *et al.* do is to grow, on a tungsten substrate, single-domain films of iron that consist of one complete atomic layer of iron followed by a two-dimensional network of irregular patches. They then measure the critical exponents characterizing the approximately two-dimensional iron, and use these exponents to form the scaling equation of state. A remarkable feature of their work is that the system they study does not perfectly mirror the conditions of the Ising model, yet the measured scaling equation of state conforms almost perfectly to the calculated result[4]. This is a consequence of universality. The experiments in question are essentially two-dimensional, and the component 'magnetic moments' have sufficient anisotropy in their interactions that when acting collectively they behave as one-dimensional classical spins. Hence the experimental system mirrors a two-dimensional Ising model.

At one time, it was imagined that the 'scale-free' case was relevant to only a fairly narrow slice of physical phenomena[1]. However, the range of systems that apparently display power-law correlations has increased dramatically in recent years, ranging from base-pair correlations in DNA[5], lung inflation[6] and interbeat intervals of the human heart[7], to complex systems involving large numbers of interacting subunits that display 'free will', such as city growth[8] and even economics[9]. The principle of universality seems to be reflected in the empirical fact that these quite different systems can have remarkably similar critical exponents — perhaps because the 'interaction paths' between the constituent subunits in such extremely complex systems dominate the observed cooperative behaviour more than the detailed properties of the subunits themselves, just as Back *et al.* find that the simple two-dimensional Ising model describes well the interaction paths in their more complex real-life experiment. □

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