ECON4570/6560 : Problem Set 4

Due by 11:59pm on Nov 30, 2024.

1. Consider observations (Y_{it}, X_{it}) from the linear panel data model

$$Y_{it} = X_{it}\beta_1 + \alpha_i + \lambda_i t + u_{it},$$

where $t = 1, \cdots, T$; $i = 1, \cdots, n$; and $\alpha_i + \lambda_i t$ is an unobserved individual specific time trend. How would you estimate β_1(two approaches)?

Answer: We can first remove the time effect and then remove the time-invariant individual effect. Let $\Delta Y_{it} = Y_{it} - Y_{i(t-1)}$, $\Delta X_{it} = X_{it} - X_{i(t-1)}$, $\Delta u_{it} = u_{it} - u_{i(t-1)}$. Then we have

$$\Delta Y_{it} = \Delta X_{it}\beta_1 + \lambda_i + \Delta u_{it}$$

Next let

$$\Delta^2 Y_{it} = \Delta Y_{it} - \Delta Y_{i(t-1)} = Y_{it} - 2Y_{i(t-1)} + Y_{i(t-2)}$$

$$\Delta^2 X_{it} = \Delta X_{it} - \Delta X_{i(t-1)} = X_{it} - 2X_{i(t-1)} + X_{i(t-2)}$$

$$\Delta^2 u_{it} = \Delta u_{it} - \Delta u X_{i(t-1)} = u_{it} - 2u_{i(t-1)} + u_{i(t-2)}$$

As a result,

$$\Delta^2 Y_{it} = \Delta^2 X_{it}\beta_1 + \Delta^2 u_{it}$$

Applying pooled OLS to the above equation, we have

$$\hat{\beta}_{1,SD} = \frac{\sum_{i=1}^{n}\sum_{t=3}^{T}\Delta^2 Y_{it}\Delta^2 X_{it}}{\sum_{i=1}^{n}\sum_{t=3}^{T}(\Delta^2 X_{it})^2}$$

Alternatively, we can define

$$\Delta X_{it}^* = \Delta X_{it} - \frac{1}{T-1}\sum_{t=2}^{T}\Delta X_{it}$$

$$\Delta Y_{it}^* = \Delta Y_{it} - \frac{1}{T-1}\sum_{t=2}^{T}\Delta Y_{it}$$

$$\Delta u_{it}^* = \Delta u_{it} - \frac{1}{T-1}\sum_{t=2}^{T}\Delta u_{it}$$

It follows that

$$\Delta Y_{it}^* = \Delta X_{it}^*\beta_1 + \Delta u_{it}^*$$

Applying pooled OLS to the above equation, we have

$$\hat{\beta}_{1,FDWT} = \frac{\sum_{i=1}^{n}\sum_{t=2}^{T}\Delta Y_{it}^{*}\Delta X_{it}^{*}}{\sum_{i=1}^{n}\sum_{t=2}^{T}(\Delta X_{it}^{*})^2}$$

The third method: After first differencing the data, we can then apply least square dummy variable method to the differenced data to obtain the estimator for β_1, which is numerically identical to $\hat{\beta}_{1,FDWT}$.

2. Suppose we have the following AR(2) process

$$y_t = 0.3y_{t-1} + 0.1y_{t-2} + \varepsilon_t,$$

where $\varepsilon_t \backsim WN(0,1)$

(a) Determine whether y_t is weakly stationary? Please justify your answer.

Answer: Since

$$(1 - 0.3L - 0.1L^2)y_t = \varepsilon_t,$$

to see whether y_t is weakly stationary, it suffices to check whether the the solutions to the following characteristic function

$$1 - 0.3z - 0.1z^2 = 0$$

lie outside the unit circle. Now

$$z_1 = \frac{-(-0.3) + \sqrt{(-0.3)^2 - 4 \times (-0.1) \times 1}}{2 \times (-0.1)} = -5,$$

$$z_2 = \frac{-(-0.3) - \sqrt{(-0.3)^2 - 4 \times (-0.1) \times 1}}{2 \times (-0.1)} = 2$$

So $|z_1| = 5 > 1$ and $|z_2| = 2 > 1$. Hence y_t is weakly stationary.

(b) Compute $E(y_t)$, $Var(y_t)$ and $Cov(y_t, y_{t-k})$ for $k = 1, 2$.

Answer: Since y_t is weakly stationary, we have

$$E(y_t) = E(y_{t-1}) = E(y_{t-2})$$

Since

$$E(y_t) = 0.3E(y_{t-1}) + 0.1E(y_{t-2}) + E\varepsilon_t$$

So

$$0.6E(y_t) = E\varepsilon_t = 0$$

Hence

$$E(y_t) = 0$$

Now since

$$y_t = 0.3y_{t-1} + 0.1y_{t-2} + \varepsilon_t$$

we have

$$E(y_t^2) = 0.3E(y_ty_{t-1}) + 0.1E(y_ty_{t-2}) + E(y_t\varepsilon_t)$$

$$E(y_ty_{t-1}) = 0.3E(y_{t-1}^2) + 0.1E(y_{t-1}y_{t-2}) + E(y_{t-1}\varepsilon_t)$$

$$E(y_ty_{t-2}) = 0.3E(y_{t-1}y_{t-2}) + 0.1E(y_{t-2}^2) + E(y_{t-2}\varepsilon_t)$$

So it follows that

$$\gamma_0 = 0.3\gamma_1 + 0.1\gamma_2 + 1$$

$$\gamma_1 = 0.3\gamma_0 + 0.1\gamma_1$$

$$\gamma_2 = 0.3\gamma_1 + 0.1\gamma_0$$

$\gamma_1 = \frac{1}{2.64} = 0.379$, $\gamma_0 = 3\gamma_1 = 3 \times \frac{1}{2.64} = 1.136$, $\gamma_2 = 0.6\gamma_1 = 0.227$.

(c) Compute the ACF ρ_k for $k = 1, 2$.
Answer:

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{1}{3} = 0.333,$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0.6\gamma_1}{3\gamma_1} = 0.2,$$

(d) Compute the PACF a_{kk} for $k = 1, 2$.
Answer:

$$a_{11} = \rho_1 = 0.333$$

$$a_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = 0.100$$

3. Suppose we have the following AR(1) process

$$y_t = 2 + y_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \backsim WN(0, 1)$ for all t, $y_0 \backsim WN(0, 1)$ and is independent of ε_t for all t,

(a) Compute $E(y_t)$, $Var(y_t)$ and $Cov(y_t, y_{t-k})$ for $k = 1, 2$.
Answer: Note that

$$y_t - y_{t-1} = 2 + \varepsilon_t$$

so

$$(y_t - y_{t-1}) + (y_{t-1} - y_{t-2}) + \cdots + (y_{t-k+1} - y_{t-k}) = 2k + \sum_{i=t-k+1}^{t} \varepsilon_i$$

That is,

$$y_t - y_{t-k} = 2k + \sum_{i=t-k+1}^{t} \varepsilon_i \quad \text{or} \quad y_t = 2k + y_{t-k} + \sum_{i=t-k+1}^{t} \varepsilon_i$$

So let $k = t$, we have

$$y_t = 2t + y_0 + \sum_{i=1}^{t} \varepsilon_i$$

and hence

$$E(y_t) = 2t + E(y_0) + \sum_{i=1}^{t} E(\varepsilon_i) = 2t + 0 + 0 = 2t$$

by virtue of the fact $\varepsilon_t \backsim WN(0,1)$ for all t, $y_0 \backsim WN(0,1)$.

$$\begin{aligned}
Var(y_t) &= E[y_t - E(y_t)]^2 = E[(2t + y_0 + \sum_{i=1}^{t} \varepsilon_i) - 2t]^2 \\
&= E[y_0 + \sum_{i=1}^{t} \varepsilon_t]^2 \\
&= E(y_0^2) + 2E[y_0(\sum_{i=1}^{t} \varepsilon_i)] + E(\sum_{i=1}^{t} \varepsilon_i)^2 \\
&= E(y_0^2) + 2[E(y_0)]E(\sum_{i=1}^{t} \varepsilon_i) + \sum_{i=1}^{t} E(\varepsilon_i^2) + 2\sum_{i \neq j} E(\varepsilon_i \varepsilon_j) \\
&= 1 + 2 \cdot 0 + t \cdot 1 + 2 \cdot 0 = 1 + t.
\end{aligned}$$

$$\begin{aligned}
Cov(y_t, y_{t-k}) &= Cov(2k + y_{t-k} + \sum_{i=t-k+1}^{t} \varepsilon_i, y_{t-k}) \\
&= Cov(2k, y_{t-k}) + Cov(y_{t-k}, y_{t-k}) + Cov(\sum_{i=t-k+1}^{t} \varepsilon_i, y_{t-k}) \\
&= 0 + Var(y_{t-k}) + Cov(\sum_{i=t-k+1}^{t} \varepsilon_i, 2(t-k) + y_0 + \sum_{i=1}^{t-k} \varepsilon_i) \\
&= Var(y_{t-k}) + Cov(\sum_{i=t-k+1}^{t} \varepsilon_i, 2(t-k)) + Cov(\sum_{i=t-k+1}^{t} \varepsilon_i, y_0) \\
&\quad + Cov(\sum_{i=t-k+1}^{t} \varepsilon_i, \sum_{i=1}^{t-k} \varepsilon_i) \\
&= Var(y_{t-k}) + 0 + 0 + 0 \\
&= 1 + t - k
\end{aligned}$$

Thus $Cov(y_t, y_{t-1}) = t$, $Cov(y_t, y_{t-2}) = t - 1$

(b) Compute the ACF ρ_k for $k = 1, 2$.
Answer

$$\rho_1 = \frac{Cov(y_t, y_{t-1})}{Var(y_t)} = \frac{t}{1+t}$$

$$\rho_2 = \frac{Cov(y_t, y_{t-2})}{Var(y_t)} = \frac{t-1}{1+t}$$

(c) Compute the PACF a_{kk} for $k = 1, 2$.
Answer:

$$a_{11} = \rho_1 = \frac{t}{1+t}$$

$$a_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\frac{t-1}{1+t} - \left(\frac{t}{1+t}\right)^2}{1 - \left(\frac{t}{1+t}\right)^2} = \frac{-1}{1+2t}$$