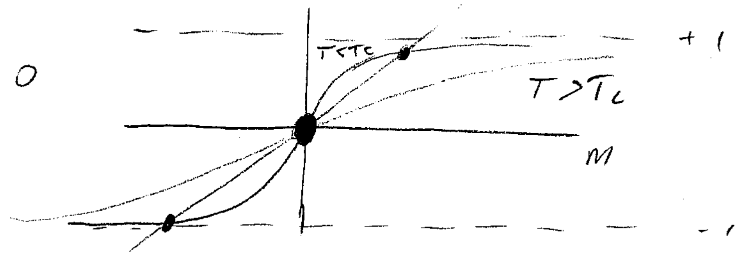
$$Z_N(T,H) = e^{-\frac{1}{2}\beta J N q m^2}\left[2\cosh\left(\beta(Jqm+H)\right)\right]^N$$

$$m = \frac{1}{N}\frac{1}{\beta}\frac{\partial \ln Z_N}{\partial H}$$

self. cons. eq. $\quad m = \tanh(\beta J q m + \beta H) \Rightarrow m(H)$ eq. of state

special case : $H = 0$

$$T_c = \frac{Jq}{k}$$



$T > T_c$ $\quad m = 0$ (one solution)

$T < T_c$ $\quad m = 0$ and $m = \pm m_{sp}$

How to choose the one which corresponds to the thermodynamically stable equilibrium?

variational free energy: $$\tilde{f}(T,m,H) = -kT\frac{1}{N}\ln Z_N(T,m,H) =$$

$$= \frac{1}{2}Jqm^2 - kT\ln 2 - kT\ln\cosh\left[\beta(Jqm+H)\right]$$

free energy: $$f(T,H) = \tilde{f}(T, m(H), H)$$

from eq. of state that we obtained from self-consistency condition

$$f(T,H) = \min_m\left\{\tilde{f}(T,m,H)\right\}$$

from the thermodyn. interpretation of the free energy to find m which minimizes $\tilde{f}(T, H, m)$:

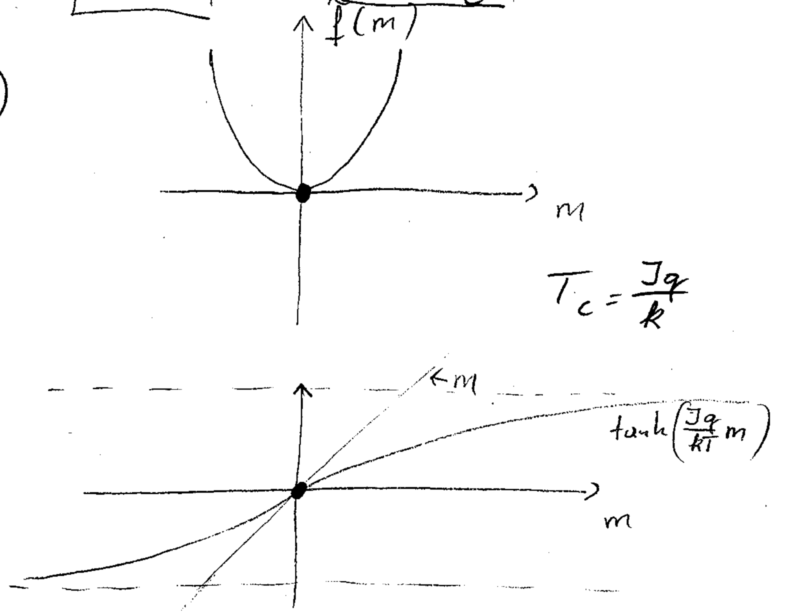
$$\frac{\partial \tilde{f}}{\partial m} = 0 \implies Jqm - Jq \tanh\left[\beta (Jqm + H)\right] = 0$$

$$(*) \quad m = \tanh\left(\frac{Jq}{kT} m + \frac{H}{kT}\right)$$ same as self-consistency condition!

a few sketches for $\tilde{f}(T, H, m)$ and the solution of $(*)$

$\boxed{H = 0}$ $\boxed{T > T_c}$

(A)

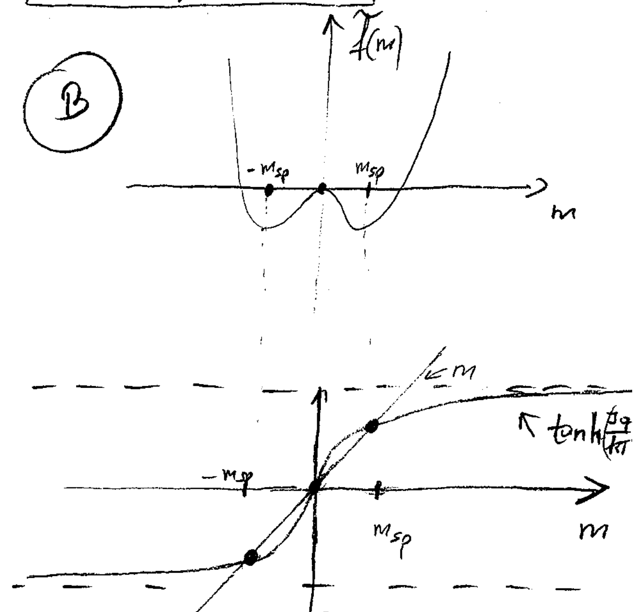


$T_c = \frac{Jq}{k}$

only one solution : $m = 0$

$\boxed{H = 0,\ T < T_c}$

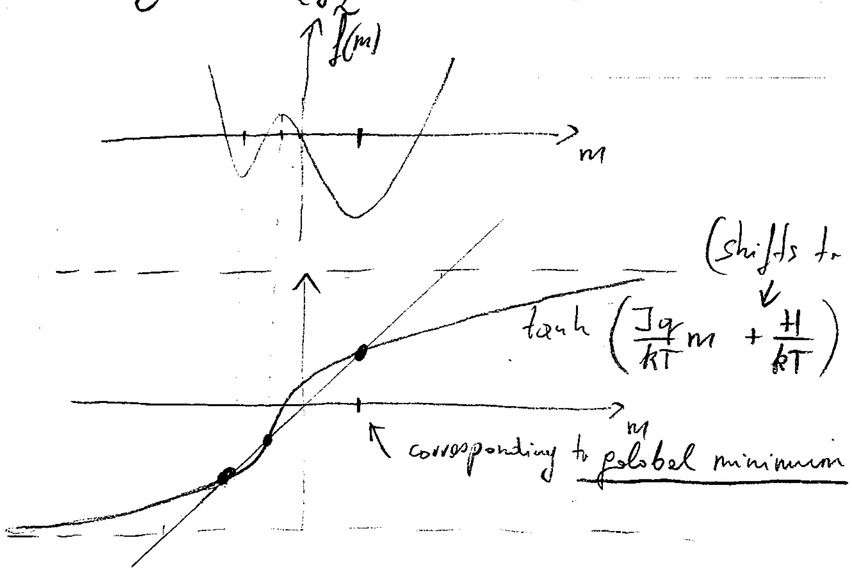
(B)



3 solutions: $m = 0, \pm m_{sp}$

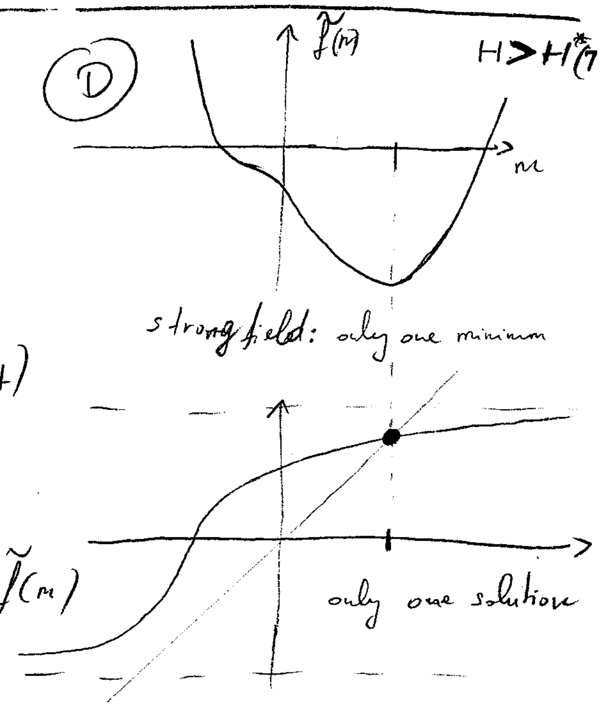
(C)

$T < T_c$ (e.g.) $0 < H < H^*(T)$ (spinodal field)



(D)

$H > H^*(T)$



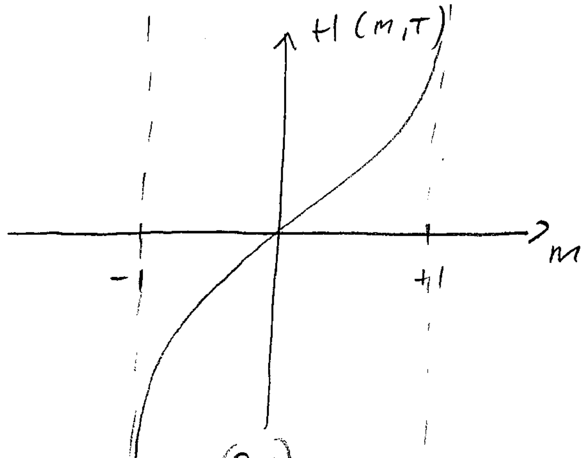
$$m = \tanh\left(\frac{Jq}{kT}m + \frac{H}{kT}\right)$$

using $\tanh^{-1}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$

$$\frac{Jq}{kT}m + \frac{H}{kT} = \frac{1}{2}\ln\frac{1+m}{1-m}$$

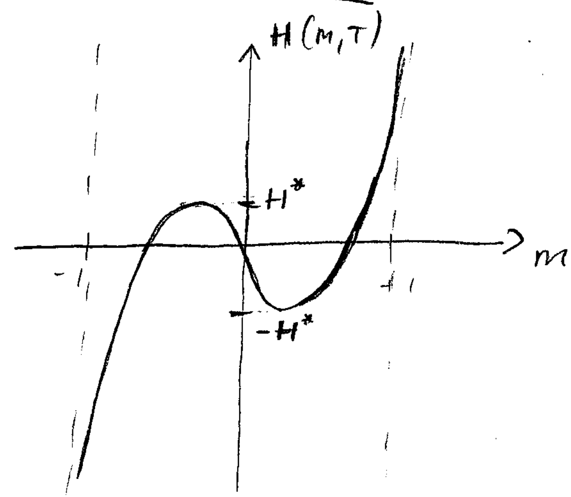
$$H(m,T) = \frac{kT}{2}\ln\frac{1+m}{1-m} - Jqm$$

$\boxed{T > T_c}$

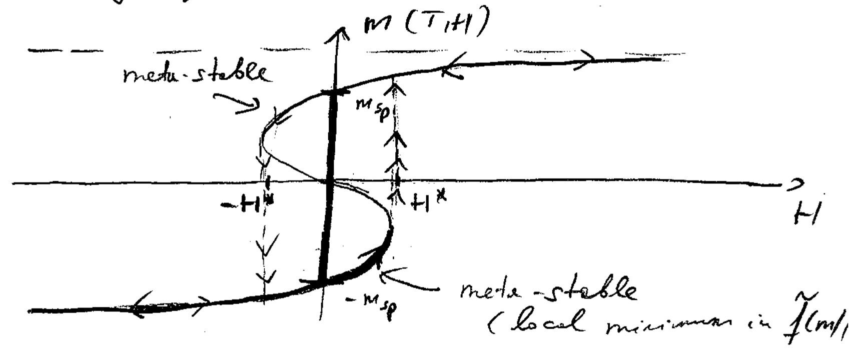


$$\left(\frac{\partial H}{\partial m}\right)_T > 0$$

$\boxed{T < T_c}$



for $T < T_c$ and $|H| < H^*$:
$m(T,H)$ is multi-valued, with a region of $\frac{\partial m}{\partial H} < 0$: instability :



Thus, for $T < T_c$: $|H| > H^*$: only one (local) minimum exist, solution is unique, $m(T,H)$ single-valued

$|H| < |H^*|$: two solutions ; $m(T,H)$ multi-valued (one local and global minima)

<u>metastable</u> branches

Critical Behavior in mean-field approximation

$$m = \tanh\left(\frac{Jq}{kT} m + \frac{H}{kT}\right) \qquad T_c = \frac{Jq}{k} \qquad \tanh(x) = x - \frac{x^3}{3} \pm \dots$$

spontaneous symmetry breaking: $H = 0$ $\quad m(T, H=0) \neq 0$
phase transition

for $H=0$ $\quad T < T_c$ (m is small, as spontaneous magnetization emerges)

a) $$m \simeq \frac{Jq}{kT} m - \frac{1}{3}\left(\frac{Jq}{kT}\right)^3 m^3 = \frac{T_c}{T} m - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 \qquad m \neq 0$$

$$1 = \frac{T_c}{T} - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^2$$

$$m^2 = 3\left(\frac{T}{T_c}\right)^3\left(\frac{T_c}{T} - 1\right) \overset{*}{=} 3\frac{T^3}{T_c^3}\,\frac{T_c - T}{T} = 3\frac{T^2}{T_c^2}\,\frac{T_c - T}{T_c} \simeq 3\,\frac{T_c - T}{T_c}$$

/up to linear order in $(T_c - T)$/

$|m| \sim \sqrt{3}\,|t|^{1/2}$ for $t < 0$ $\qquad$ where $t = \frac{T - T_c}{T_c}$

$|m| \sim |t|^{\beta} \Rightarrow \boxed{\beta = 1/2}$

b) at T_c: $\quad \frac{Jq}{kT_c} = 1$ (critical isotherm)

$$\cancel{m} \simeq \left(m + \frac{H}{kT_c}\right) - \frac{1}{3}\left(m + \frac{H}{kT_c}\right)^3 \pm \dots = \cancel{m} + \frac{H}{kT_c} - \frac{1}{3} m^3 - o(m^2 H)$$

$$H = kT_c \frac{1}{3} m^3 + o(m^2 H)$$

$\hookrightarrow o(m^5)$ after successive approx

$H \sim m^3$

$H \sim m^{\delta} \Rightarrow \boxed{\delta = 3}$ $\qquad |m| \sim |H|^{1/\delta}$

c) isotherm susceptibility: $\chi_T = \lim_{H \to 0}\left(\frac{\partial m}{\partial H}\right)_T$

since $H \to 0$ is taken, sufficient to keep linear terms in H from equation of state:

$$m \simeq \frac{T_c}{T} m + \frac{H}{kT} - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 \Leftarrow \left(\frac{\partial}{\partial H}\right)_T$$

$$\left(\frac{\partial m}{\partial H}\right)_T = \chi_T$$

$$\chi_T \simeq \frac{T_c}{T}\chi_T + \frac{1}{kT} - \left(\frac{T_c}{T}\right)^3 m^2 \chi_T$$

$$\chi_T\left(1 - \frac{T_c}{T} + \left(\frac{T_c}{T}\right)^3 m^2\right) = \frac{1}{kT}$$

$$m^2 = 3\left(\frac{T}{T_c}\right)^3 \frac{T_c - T}{T}$$

* from previous page

$$\chi_T\left(T - T_c + \left(\frac{T_c}{T}\right)^3 \cdot T m^2\right) = \frac{1}{k}$$

$$\chi_T \simeq \frac{1}{k\left(T - T_c + \left(\frac{T_c}{T}\right)^3 T m^2\right)}$$

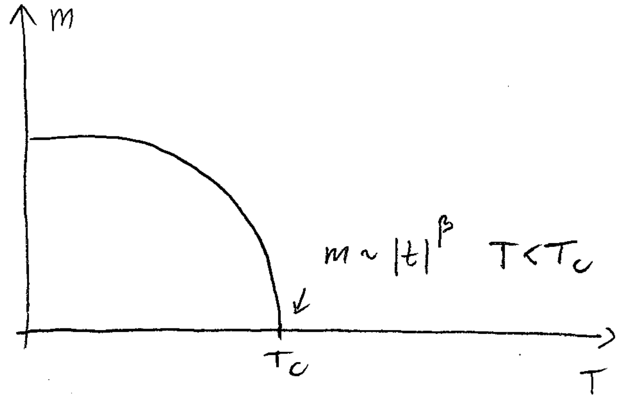
$\underline{T > T_c}$: $m = 0 \Rightarrow \chi_T = \dfrac{1}{k(T - T_c)} = A_> |t|^{-\gamma}$

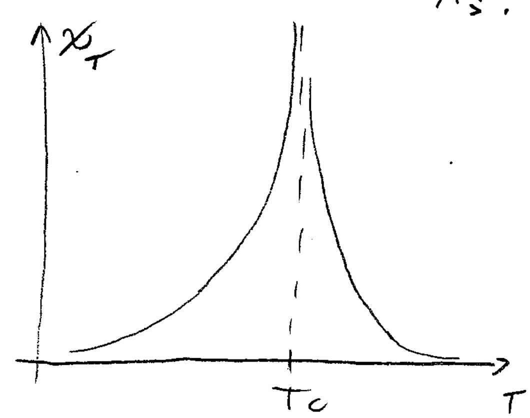
$A_> = \dfrac{1}{kT_c}$ $\quad \gamma = 1$

$\underline{T < T_c}$: $m = 3\left(\frac{T}{T_c}\right)^3 \frac{T_c - T}{T} \Rightarrow \chi_T = \dfrac{1}{k[T - T_c + 3(T_c - T)]} = \dfrac{1}{2k(T_c - T)}$

$$= A_< |t|^{-\gamma} \qquad A_< = \frac{1}{2kT_c} \qquad \gamma = 1$$

same γ exponent above and below T_c, but different amplitude $A_{\gtrless}$.





Landau Theory of Phase Transitions

we saw that $\tilde{f}(T,H,m)$ worked as a variational free energy to select global equilibrium states

$$\tilde{f}(T,H,m) = \frac{1}{2} Jq m^2 - kT \ln 2 - kT \ln \cosh [\beta (Jqm + H)]$$

can expand into power series of m & H

keeping terms up to $o(m^4)$ and $o(m \cdot H)$:

$$\tilde{f}(T,H,m) \simeq \frac{1}{2} Jqm^2 - kT \ln 2 - kT \left\{ \frac{\beta^2}{2} (Jqm + H)^2 - \frac{2}{24} \beta^4 (Jqm + H)^4 + \dots \right\}$$

$$= -kT \ln 2 + \frac{1}{2} Jqm^2 - \frac{1}{2} \frac{1}{kT} (Jqm)^2 - \frac{1}{kT} JqmH + \frac{1}{12} \frac{1}{(kT)^3} (Jqm)^4 + \dots$$

$$= -kT \ln 2 + \frac{1}{2} Jq \left(1 - \frac{Jq}{kT}\right) m^2 + \frac{1}{12} Jq \left(\frac{Jq}{kT}\right)^3 m^4 - \frac{Jq}{kT} m \cdot H$$

$$= -kT \ln 2 + \frac{1}{2} \frac{Jq}{kT} \underbrace{(kT - Jq)}_{k(T-T_c)} m^2 + \frac{1}{12} Jq \left(\frac{Jq}{kT}\right)^3 m^4 - \frac{Jq}{kT} mH$$

$T \approx T_c = \frac{Jq}{k}$

$$\boxed{\mathcal{L}(T,H,m) = a(T) + \frac{1}{2} b(T) m^2 + \frac{1}{4} c(T) m^4 - mH}$$

General Landau free energy

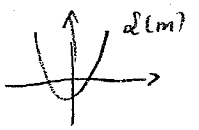
$a(T)$ and $c(T)$ approach a nonzero value as $T \to T_c$ (see in $\tilde{f}$)

$b(T) \propto T - T_c$ i.e., changes sign @ T_c!

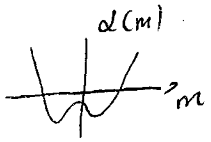
$$\boxed{b(T) = b_0 (T - T_c)}$$

$\mathcal{L}(T, H, m)$ fully captures the mean-field behavior: as a variational free energy. $C(T) \approx C(T_c) \equiv C \neq 0$

$H = 0$ E.g. $\frac{\partial \mathcal{L}}{\partial m} = b_0 (T - T_c) m + C m^3$



$T > T_c$ $m = 0$



$T < T_c$ $m = 0$

$$m = \pm \sqrt{\frac{b_0 (T_c - T)}{C}} \sim \pm |t|^{1/2} \qquad \beta = 1/2$$

(same mean-field exponent as for the original model) $\left(t = \frac{T - T_c}{T_c} \right)$

if for example $C(T)$ changes sign, then higher order terms in m should be retained in series to obtain "stablizing" term: $d(T) > 0$ for all T (does not change sign)

$$\mathcal{L}(T, H, m) = a(T) + \frac{1}{2} b(T) m^2 + \frac{1}{4} C(T) m^4 + \frac{1}{6} d(T) m^6 + \ldots - Hm$$