Class 15

Maxwell’s Equations

March 14th, 2024

What is Physics ?

- Observation of nature
- Experiments in a laboratory (reduction of nature)

Laws of Nature – Mathematical Description – Mathematical Sciences

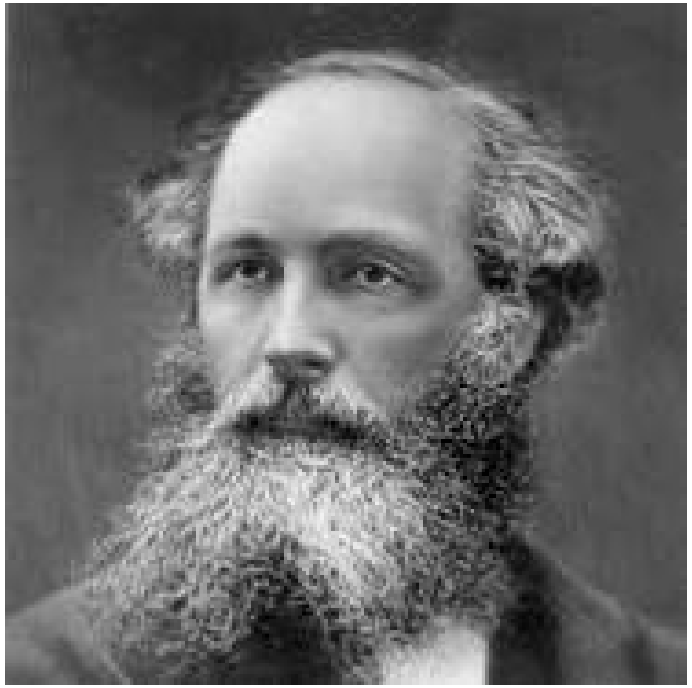
Maxwell’s Eqs. Classical Field Theory

The contents (electrical & magnetic phenomena) of the physical laws of nature does not change because they are a property of our world as it is.

What we designate as a “law of nature” and its mathematical description evolves over time.

Maxwell’s Equations

- James Clerk Maxwell (1831 – 1879) was a Scottish scientist in the field of mathematical physics. His most notable achievement was to formulate the classical theory of electromagnetic radiation, bringing together for the first time electricity, magnetism, and light as different manifestations of the same phenomenon.
- [Wikipedia]



Education: University of Edinburgh, University of Cambridge
Employment: University of Cambridge, Cavendish Professor of Physics

Class 14 Maxwell's Equations in Vacuum

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

painstakingly derived from experiments by e.g. Michael Faraday

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

this term in Ampère's law was initially guessed and later accepted because it helps to describe electromagnetic waves

displacement current density $\vec{J}_D = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$

Maxwell’s (= Heaviside) Equations

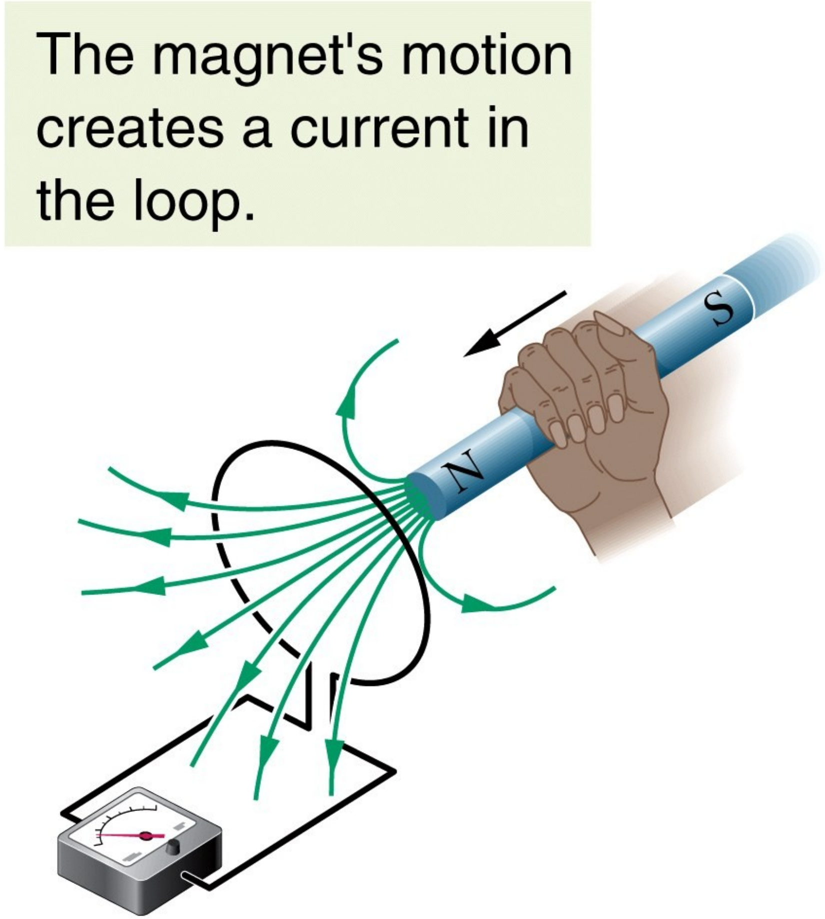
- **Oliver Heaviside** (1850 –1925) was an English self-taught electrical engineer, mathematician, and physicist who adapted complex numbers to the study of electrical circuits, invented mathematical techniques for the solution of differential equations (equivalent to Laplace transforms), reformulated Maxwell's field, and independently co-formulated vector analysis. Although at odds with the scientific establishment for most of his life, Heaviside changed the face of telecommunications, mathematics, and science. Wikipedia

Professor Wilke’s Hero in Electromagnetic Theory !

Faraday’s Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \iff \oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\phi_B \quad \text{and} \quad \phi_B = \int \vec{B} \cdot d\vec{A}$$

- Michael Faraday (1791 – 1867) was an English scientist who contributed to the study of electromagnetism and electrochemistry. His main discoveries include the principles underlying electromagnetic induction, diamagnetism and electrolysis. [Wikipedia]



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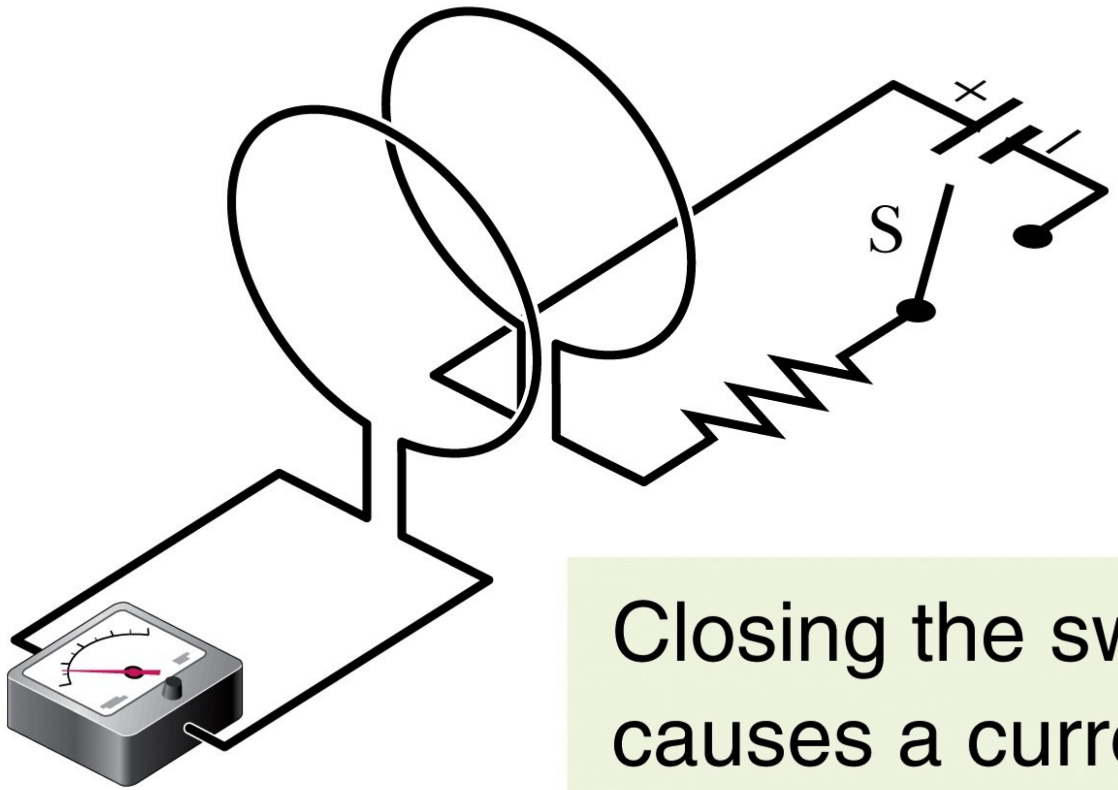
$\Phi_B(t) = A\,B(t)$ area of current loop $A = \text{const}$

B of bar magnet changes



Closing the switch causes a current in the left-hand loop.

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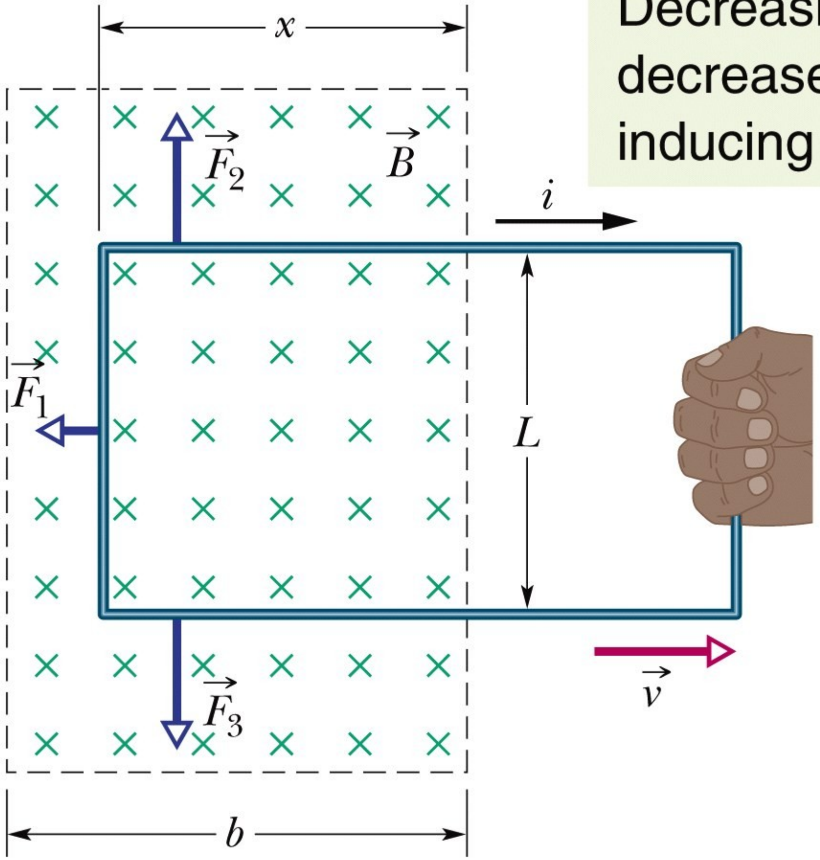
$\Phi_B(t) = A\,B(t)$ area of current loop $A = \text{const}$

$$B = B(t) = \frac{\mu_0 i(t)}{2R}$$

Decreasing the area decreases the flux, inducing a current.



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$\Phi_B(t) = A(t)\,B$ area of current loop exposed to $B = \text{const.}$ changes

In Electrodynamics:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$ new

$$\vec{B} = \vec{\nabla} \times \vec{A}$$ unchanged

everything now depends also on time

$\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$, $V(\vec{r},t)$, $\vec{A}(\vec{r},t)$

In electrodynamics $\vec{E} = -\vec{\nabla}V$ has to be modified as $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$ because according to Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{or} \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \quad (1)$$

Now, $\vec{B} = \vec{\nabla} \times \vec{A}$ or $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$ (2)

If we put (2) in (1): $\vec{\nabla} \times \vec{E} + \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} = 0$,

or $\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$. (3)

If we define $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V$ eq. (3) is fulfilled, (because $\vec{\nabla} \times \vec{\nabla} F = 0$, i.e. the rotation of the gradient of a scalar vanishes because $\vec{\nabla}$ and $\vec{\nabla} F$ are parallel vectors). So finally, $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$.

Ampère's Law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{Free} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$.

The displacement current density $\vec{J}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ is required to explain electromagnetic waves (it was "guessed" for this purpose).

Derivation of the wave equation for electric fields $\vec{E}(\vec{r},t)$ from Maxwell's eqs. for vacuum:
In vacuum, el. charge density $\rho = 0$ and electric current density $\vec{J} = 0$.

Starting point:

(i) $\vec{\nabla}\cdot\vec{E}=0$, (ii) $\vec{\nabla}\cdot\vec{B}=0$, (iii) $\vec{\nabla}\times\vec{E}=-\dot{\vec{B}}$, (iv) $\vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

Look at $\vec{\nabla}\times$ on (iii):

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}=-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0\text{ (see (i))}})-\vec{\nabla}^2\vec{E}=-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

So we get $\vec{\nabla}^2\vec{E}=\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$, wave eq. for $\vec{E}(\vec{r},t)$.

Do the derivation of the wave eq. for $\vec{B}(\vec{r},t)$ yourself, start with $\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \ldots$.

During WebEx class hour you can check in with me about this derivation.

Result $\quad \vec{\nabla}^2 \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{B}}{\partial t^2}$.

Maxwell's equations in materials which can be electrically polarized and/or magnetized

$$\vec{\nabla}\cdot\vec{D} = \rho_{free} \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t}$$

Materials equations

$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ and $\vec{P} = \varepsilon_0\chi_e\vec{E}$, $\varepsilon = \varepsilon_0(1+\chi_e)$

$\vec{B} = \mu_0(\vec{H} + \vec{M})$ and $\vec{M} = \chi_m\vec{H}$, $\mu = \mu_0(1+\chi_m)$

χ_e, χ_m are material properties (susceptibilities)

Boundary conditions at the boundary of 2 different media labeled ① and ② with free surface charge density σ_F and / or surface current $\vec{K}$ at the boundary.

$\perp$ refers to field component perpendicular to the boundary.

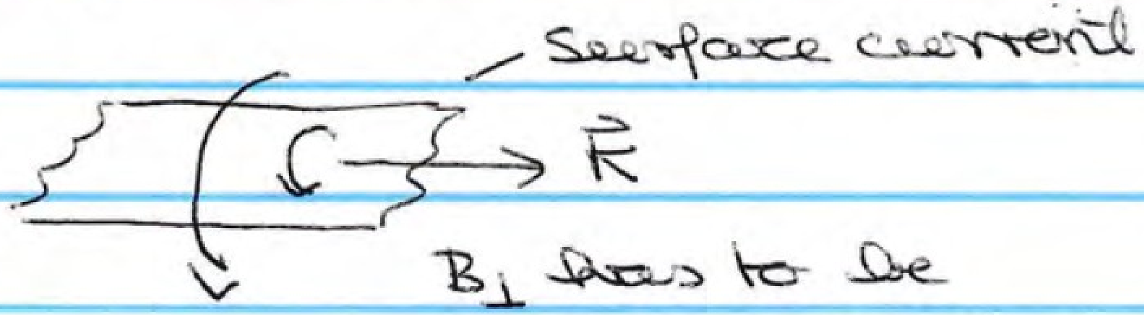
$\parallel$ refers to the field component parallel to the boundary.

$D_1^\perp - D_2^\perp = \sigma_F$ (not new, recall from electrostatics, spherical shell with surface charge density σ:

$$E_1^\perp - E_2^\perp = \sigma/\varepsilon_0)$$

$E_1^\parallel - E_2^\parallel = 0$ or $E_1^\parallel = E_2^\parallel$ (also not new)

$B_1^\perp - B_2^\perp = 0$ $\longrightarrow$



$B_\perp$ has to be continuous for closed $\vec{B}$-field loops around current $\vec{K}$

$$\frac{H_1^\parallel}{\mu_1} - \frac{H_2^\parallel}{\mu_2} = \vec{K} \times \vec{n}$$