Class 5

Laplace’s Equation

(01/25/2024)

Outline

- Electrostatic’s in a nutshell
- My favorite example in electrostatics: The charged spherical shell
- Solving the 1-dimensional Laplace’s & Poisson’s equations using direct integration

Electrostatics in a nutshell

- Electrostatic fields are described by:

$$\vec{\nabla} \times \vec{E} = 0 \quad , \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

- Electrostatic fields can be derived from electric potentials as:

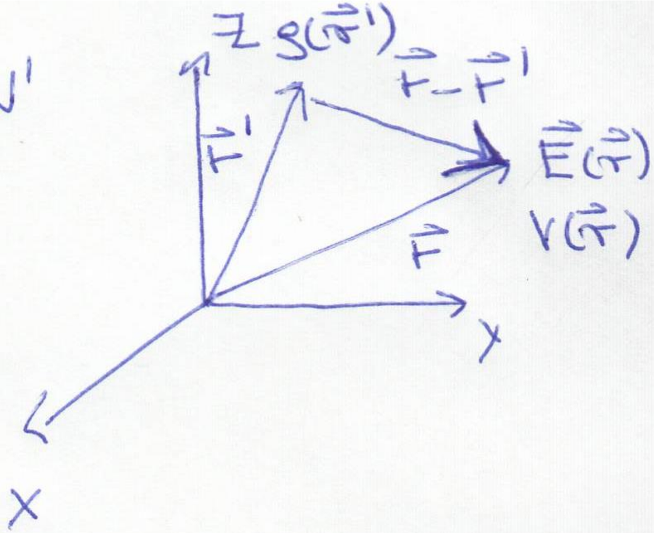
$$\vec{E} = -\vec{\nabla} V$$

- For electrostatic potentials the following differential $\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0} \,,\, \vec{\nabla}^2 V = 0 .$

General description of the electric field and the electric potential

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} (\vec{r}-\vec{r}\,')\, dV'$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dV'$$



The Physics of the
Boundary Conditions at a charged surface:

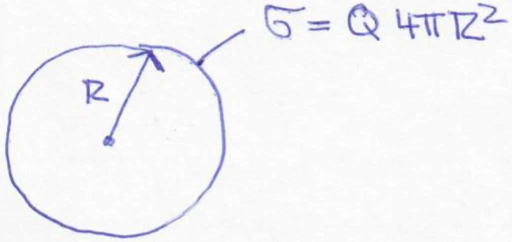
$$E_{\perp\,outside} - E_{\perp\,inside} = \frac{\sigma}{\varepsilon_0}$$ discontinuous

$$E_{\parallel\,outside} = E_{\parallel\,inside}$$ continuous

$$\vec{\nabla} V_{outside} - \vec{\nabla} V_{inside} \Rightarrow$$ discontinuous

$$V_{outside} = V_{inside}$$ continuous

Example of the charged hollow sphere



hollow sphere of radius R
with surface charge σ

$$E_{in}(r) = 0$$

$$\vec{E}_{out}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,\hat{r}$$

$$E_{out}(R) - E_{in}(R) = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 4\pi R^2}{R^2} = \frac{\sigma}{\varepsilon_0}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) \qquad \vec{\nabla}V_{out} - \vec{\nabla}V_{in} = -\vec{E}_{out} - (-\vec{E}_{in})$$

$$= -(\vec{E}_{out} - \vec{E}_{in}) = -\frac{\sigma}{\varepsilon_0}$$

$$V_{out}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

from observation, also agrees with $\vec{E}(\vec{r}) = -\vec{\nabla}V(r)$
i.e. $\vec{E}(\vec{r}) = -\hat{r}\frac{\partial}{\partial r}V(r)$

Methods for solving Laplace’s & Poisson’s equations

cylindrical coordinates
(Bessel function)

multipole
expansion

Green's
function

separation
of variables

spherical
coordinates
(Legendre Polynoms)

$$\vec{\nabla}^2 V = 0 \Leftarrow \vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$$

image
charges

cartesian coordinates
(Fourierseries)

direct integration of the 1-dimensional Laplace’s and Poisson’s equations

Let's try to figure out V in by solving

uniformly distributed surface charge σ

$\vec{\nabla}^2 V = 0$ inside hollow sphere

in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=0 \quad \text{because } \frac{\partial V}{\partial\theta}=0,\ \frac{\partial V}{\partial\phi}=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=0$$

$$\Leftrightarrow \quad r^2\frac{\partial V}{\partial r}=C_1$$

$$\frac{\partial V}{\partial r}=\frac{C_1}{r^2} \rightarrow V(r)=\int\frac{C_1}{r^2}dr=-\frac{C_1}{r}+C_2$$

from boundary condition

$$V_{in}(r=R)=V_{out}(r=R)$$

$$-\frac{C_1}{R}+C_2=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R}+C_2$$

$$\Leftrightarrow C_2=0,\quad -C_1=\frac{Q}{4\pi\varepsilon_0 R}$$

$$V_{in}(r)=\frac{Q}{4\pi\varepsilon_0 R}=\text{const}$$

Solution for $V_{in}(r)$ in agreement with $\vec{E}_{in}=-\vec{\nabla}V=0$

and in agreement with $\vec{\nabla}V_{in}-\vec{\nabla}V_{out}=-\frac{\sigma}{\varepsilon_0}$

Let's look at $V(\vec{r}) = \int_{V'} \rho(\vec{r}\,') \frac{1}{|\vec{r}-\vec{r}\,'|} \, dV'$

and ask, does $V(\vec{r})$ fulfill $\vec{\nabla}^2 V(\vec{r}) = -\frac{\rho}{\varepsilon_0}$?

$$\vec{\nabla}^2_{\vec{r}} V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,') \, \vec{\nabla}^2_{\vec{r}} \frac{1}{|\vec{x}-\vec{x}\,'|} \, dV'$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{V'} \rho(\vec{r}\,') \left(-4\pi \, \delta(\vec{r}-\vec{r}\,')\right) dV'$$

$$= -\frac{1}{\varepsilon_0} \int_{V'} \rho(\vec{r}\,') \, \delta(\vec{r}-\vec{r}\,') \, dV'$$

$$\vec{\nabla}^2_{\vec{r}} V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0} \quad \checkmark$$

useful pieces of math

$$-\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\vec{\nabla} \cdot \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = 4\pi \, \delta(\vec{r}-\vec{r}\,')$$

The proof that elliptic differential equations, e.g. Laplace's and Poisson's equations, do have unique solutions, is part of the mathematical theory of partial differential equations.
The following key points are reformulations of the proven mathematical Uniqueness Theorems for solutions to Laplace's and Poisson's equations in terms of electrostatics, i.e. electric field E, electric potential V and charge q.

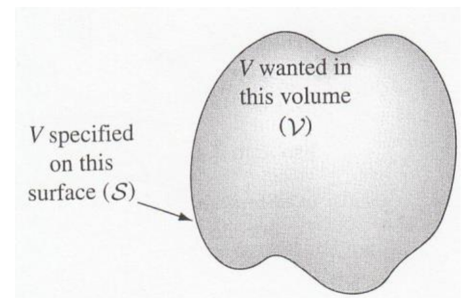
Solving the Laplace & Poisson equation:

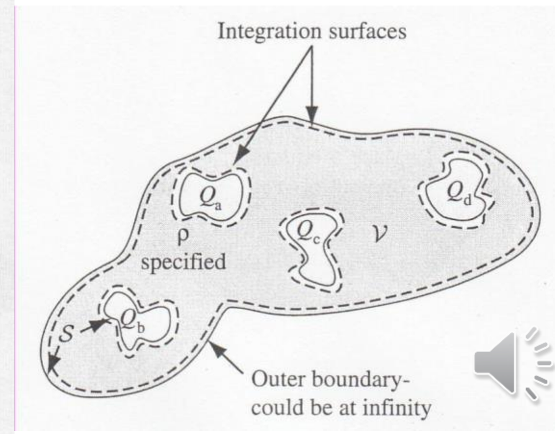
math problem: homogenous & inhomogenous linear, 2nd order, partial differential equation

partial differential equations

Some key points

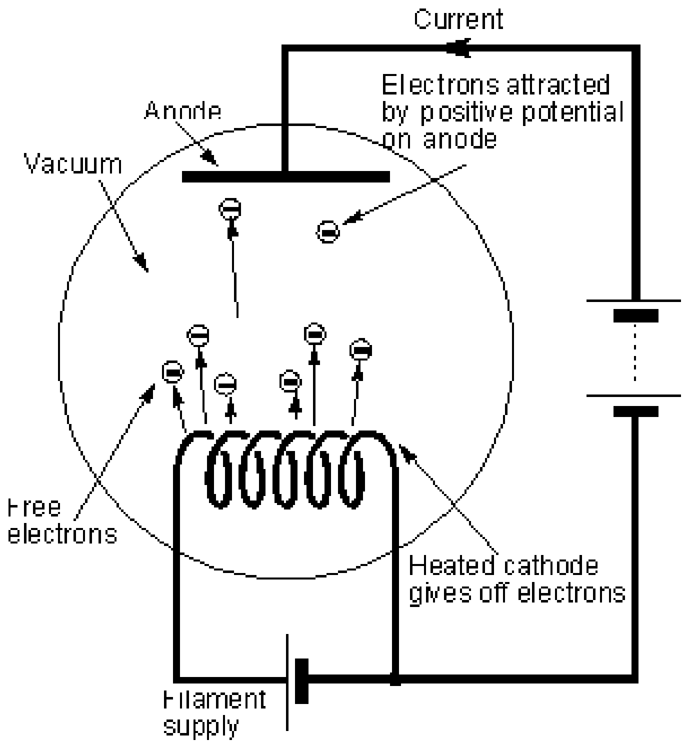
* $\vec{\nabla}^2 V = 0$ all extrema occur at boundaries
* $V(\vec{r})$ fulfills $\vec{\nabla}^2 V(\vec{r}) = 0$ and has the (physically) correct value at the boundary, then $V(\vec{r})$ is the proper solution.
* $\vec{E}(\vec{r})$ is uniquely known in a volume with charge q and bounded by conductors when the total charge Q on each conductor is known.
* $V(\vec{r})$ for $\vec{\nabla}^2 V(\vec{r}) = -\rho/\varepsilon_0$ can be uniquely determined if ρ in volume and $V(\vec{r})$ on boundary are known.

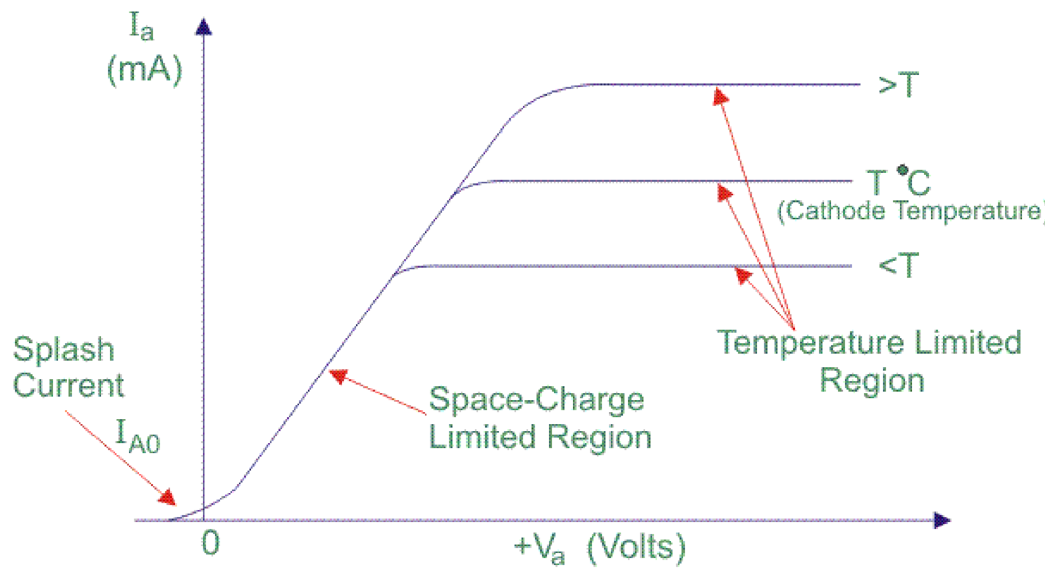




In-Class Problems

- Problem 3.3 : Solving the 1-dimensional Laplace equation in spherical & cylindrical coordinates
- Problem 2.53/2.54: Solving Poisson's equation for a real-world examples (some background physics provided on the next slide)





I-V Characteristics of Vacuum Diode under forward bias

Problem 2.53 In a vacuum diode, electrons are "boiled" off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

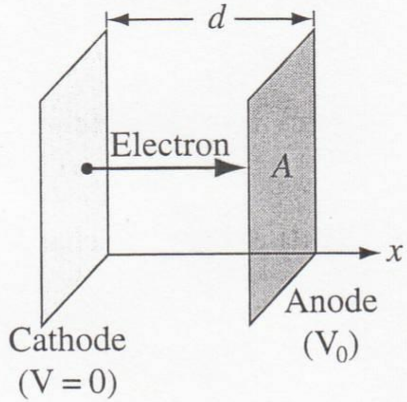


FIGURE 2.55

(a) Write Poisson's equation for the region between the plates.

(b) Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$?

(c) In the steady state, I is independent of x. What, then, is the relation between ρ and v?

(d) Use these three results to obtain a differential equation for V, by eliminating ρ and v.

(e) Solve this equation for V as a function of x, V_0, and d. Plot $V(x)$, and compare it to the potential *without* space-charge. Also, find ρ and v as functions of x.

(f) Show that

$$I = K V_0^{3/2}, \tag{2.56}$$

and find the constant K. (Equation 2.56 is called the **Child-Langmuir law**. It holds for other geometries as well, whenever space-charge limits the current. Notice that the space-charge limited diode is *nonlinear*—it does not obey Ohm's law.)