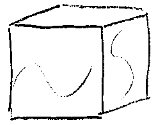
Black-Body Radiation



$V = L^3$ "photon gas" e.g. with periodic boundary condition

- second quantization of ElectroMagnetic field.

$$\text{energy} = \frac{1}{8\pi}\int_V \left(\bar{E}^2 + \bar{H}^2\right) dV = \ldots \text{ second quantization } =$$

$$= \sum_{s,\bar{k}} \hbar\omega_{\bar{k}} \left(a^{+}_{s,\bar{k}} a_{s,\bar{k}} + \cancel{\tfrac{1}{2}}\right) \qquad s = 1,2 \qquad k_x = \frac{2\pi}{L} n_x \quad (\text{p.b.c.}) \qquad n_x = 0, \pm 1, \pm 2, \ldots$$

$$\left(\bar{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{A} = 0 \qquad \boxed{\omega = kc} \qquad (\sim \text{from wave equation})$$

$$\operatorname{div}\bar{A} = 0 \qquad \bar{A}(\bar{x},t) \propto \sum_{k,s} q_{\bar{k}s}(t)\, \bar{A}_{\bar{k}s}(\bar{x}) + \text{c.c.} \qquad \ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = 0 \qquad \left(\bar{\nabla}^2 + \frac{\omega_\lambda^2}{c^2}\right)\bar{A}_\lambda(\bar{x}) = 0$$

since the number of modes is infinite: $\frac{1}{2} \to \infty$

"renormalized" Hamiltonian

$$\omega = kc \qquad \boxed{\mathcal{H} = \sum_{s,\bar{k}} \hbar\omega_{\bar{k}}\, n_{s,\bar{k}}} \qquad n_{s,\bar{k}} = 0, 1, 2, \ldots$$

$s = \pm 1$ polarization

for each $\bar{k}$ vector $s = -1, +1$ (no longitudinal photon exist)

for one oscillator we saw earlier: $\omega: Z_1 = \dfrac{1}{1 - e^{-\beta\hbar\omega}}$

$$\varepsilon_{\bar{k},s} = \hbar\omega_k n_{\bar{k},s} \quad , \quad n_{\bar{k},s} = 0, 1, 2, \ldots$$

$$Z = \prod_{\bar{k},s} Z(\bar{k},s) = \prod_{\bar{k},s}\sum_{n_{\bar{k},s}} e^{-\beta\hbar\omega_{\bar{k}} n_{\bar{k},s}} = \prod_{\bar{k},s}\frac{1}{1 - e^{-\beta\hbar\omega_k}}$$

$$\boxed{F = -kT\ln Z = kT\sum_{\bar{k},s}\ln\left(1 - e^{-\beta\hbar\omega_k}\right)} \qquad s = \pm 1 \qquad k_x = \frac{2\pi}{L} n_x$$

- infinite number of modes $(\bar{k}, s)$
- indefinite number of photons $\left(\sum_{\bar{k},s} n_{\bar{k},s}\right)$

$$F = kT \sum_{k,s} \ln\left(1 - e^{-\beta\hbar\omega_k}\right) \longrightarrow kT \frac{V}{(2\pi)^3} \cdot 2 \cdot \int_0^\infty 4\pi k^2 \ln\left(1 - e^{-\beta\hbar\omega(k)}\right) dk$$

$\omega = kc$ dispersion relation

$$= kT \, \frac{8\pi V}{8\pi^3} \int_0^\infty \frac{\omega^2}{c^2} \cdot \frac{d\omega}{c} \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$= kT \frac{V}{\pi^2 c^3} \int_0^\infty \omega^2 \ln\left(1 - e^{-\beta\hbar\omega}\right) d\omega = kT \int_0^\infty g(\omega) \ln\left(1 - e^{-\beta\hbar\omega}\right) d\omega$$

"density of states" $g(\omega)\,d\omega$: number of oscillators between energy ω, $\omega + d\omega$

$$g(\omega) = \frac{V\omega^2}{\pi^2 c^3}$$

$$\boxed{F(T,V) = kT \int_0^\infty g(\omega) \ln\left(1 - e^{-\beta\hbar\omega}\right) d\omega}$$

$$-S = +\left(\frac{\partial F}{\partial T}\right)_V = \frac{F}{T} + kT \int_0^\infty g(\omega) \frac{-e^{-\beta\hbar\omega} \frac{\hbar\omega}{kT^2}}{1 - e^{-\beta\hbar\omega}} \, d\omega$$

$$= \frac{F}{T} - \frac{1}{T} \int_0^\infty g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \, d\omega$$

$$F = E - TS$$

$$E = F + TS = \int_0^\infty g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \, d\omega$$

$$\boxed{E = \int_0^\infty g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \, d\omega} \qquad \boxed{g(\omega) = \frac{V\omega^2}{\pi^2 c^3}}$$

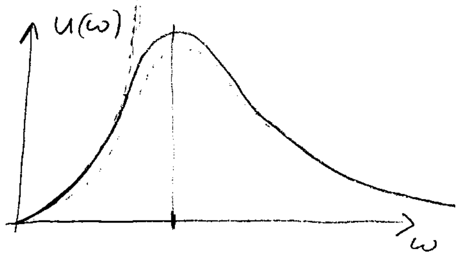
(Planck's result)

$$\left.\begin{aligned} \frac{E}{V} &= \int_0^\infty \frac{\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\, d\omega \\ \frac{E}{V} &\equiv \int_0^\infty d\omega\, u(\omega) \end{aligned}\right\} \qquad \boxed{u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\beta\hbar\omega}-1}}$$

energy density per unit frequency in $(\omega, \omega+d\omega)$

Spectral density

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\beta\hbar\omega}-1}$$



extreme limits:

1) $\frac{\hbar\omega}{kT} \ll 1$ (classical limit)

$$u(\omega) \simeq \frac{\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{\beta\hbar\omega} = \frac{\omega^2}{\pi^2 c^2}\, 2\left(\frac{1}{2} kT\right) \quad \text{(Rayleigh-Jeans)}$$

$\hookrightarrow$ equipartition for classical harmonic oscillator

$$\int_0^\infty u_{R\text{-}J}(\omega)\, d\omega = \infty \quad \text{(UV catastrophe)}$$

2) $\frac{\hbar\omega}{kT} \gg 1$

$$u(\omega) \simeq \frac{\hbar\omega^3}{\pi^2 c^3} e^{-\beta\hbar\omega} \qquad \text{(Wien's Law)}$$

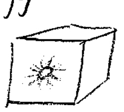
exact:

$\frac{\hbar\omega}{kT} \equiv x$

$$\frac{E}{V} = \int_0^\infty d\omega\, u(\omega) = \int_0^\infty \frac{\hbar \left(\frac{kT}{\hbar}\right)^3 x^3}{\pi^2 c^3} \frac{\left(\frac{kT}{\hbar}\right) dx}{e^x - 1} = \frac{(kT)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3\, dx}{e^x - 1}$$

$$= \frac{(kT)^4}{\pi^2 c^3 \hbar^3} \Gamma(4)\, \zeta(4) = \frac{(kT)^4}{\pi^2 c^3 \hbar^3}\, 6 \cdot \frac{\pi^4}{90} = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4$$

"effusion"



energy flux through small opening: $\boxed{\frac{1}{4}\frac{E}{V} \cdot c = \sigma T^4}$ ← Stefan - Boltzmann Law

$$\boxed{\sigma = \frac{\pi^2 k^4}{60 c^2 \hbar^3}} \quad \text{Stefan constant}$$

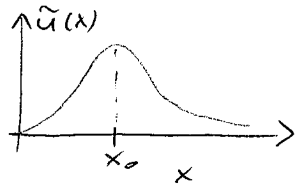
$$u(\omega)\,d\omega = \tilde{u}(x)\,dx \qquad x = \frac{\hbar\omega}{kT}$$

$$\frac{\hbar\omega^3}{\pi^2 c^3}\,\frac{d\omega}{e^{\frac{\hbar\omega}{kT}}-1} = \frac{\hbar\left(\frac{kT}{\hbar}\right)^3 x^3 \left(\frac{kT}{\hbar}\right)dx}{\pi^2 c^3\,(e^x-1)} = \frac{(kT)^4}{\pi^2 c^3 \hbar^3}\;\frac{x^3\,dx}{e^x-1}$$

$$\tilde{u}(x) = \frac{(kT)^4}{\pi^2 c^3 \hbar^3}\;\frac{x^3}{e^x-1} \qquad \text{dimensionless form}$$

$$\frac{d}{dx}\frac{x^3}{e^x-1} = \frac{3x^2(e^x-1) - x^3 e^x}{(e^x-1)^2} = \frac{3x^2e^x - x^3e^x - 3x^2}{e^x-1} = 0$$

$$x_0 \cong 2.82$$



$$\left(\frac{\hbar\omega}{kT}\right)_0 \cong 2.82 \quad \Rightarrow \quad \boxed{\omega_0 \simeq 2.82\,\frac{kT}{\hbar}} \qquad \text{determines color}$$

dominant color changes with temperature

$$\frac{E}{V} = \frac{4\sigma}{c}T^4 \qquad \sigma = \frac{\pi^2 k^4}{60 c^2 \hbar^3}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V \propto T^3$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = -kT\frac{1}{\pi^2 c^3}\int_0^\infty \omega^2 \ln\left(1-e^{-\beta\hbar\omega}\right)d\omega = \text{integrating by parts} =$$

$$= \dots =$$

$$= \frac{1}{3}\frac{E}{V} \qquad \boxed{PV = \frac{E}{3}} \qquad \text{trivial consequence of } \omega = kc$$

linear dispersion relation

$$P = \frac{1}{3}\frac{E}{V} = \frac{4\sigma}{3c}T^4 \qquad \underline{\text{pressure of radiation}}$$

$$F = -PV = -\frac{E}{3}$$ *specifically for the black-body*

and

$F = E - TS$ *in general*

$$\Rightarrow S = \frac{1}{T}(E - F) = \frac{1}{T}\frac{4}{3}E$$

$$= \frac{16}{3}\frac{\sigma}{c} V T^3 \ \underset{T\to 0}{\longrightarrow} 0$$

From basic thermo: $-PV = \phi = E - TS - \mu N = F - \mu N$

$$\Rightarrow \boxed{\mu = 0 \quad \text{implied}}$$

photons are ~~quasi particles~~. Their average number relates to thermal equilibrium

Photon picture: $\quad \varepsilon_{\bar{k},s} = \hbar\omega_{\bar{k}}\, n_{\bar{k},s} \qquad \omega_{\bar{k}} = kc$

$$Z_{\bar{k},s} = \sum_{n_{\bar{k},s}} e^{-\beta\hbar\omega_{\bar{k}} n_{\bar{k},s}} \qquad n_{\bar{k},s} = 0, 1, 2, \ldots$$

$$\langle n_{\bar{k},s}\rangle = -\frac{\partial}{\partial(\beta\hbar\omega_{\bar{k}})} \ln Z_{\bar{k},s} = -\frac{\partial}{\partial(\beta\hbar\omega_{\bar{k}})} \ln\left(1 - e^{-\beta\hbar\omega_{\bar{k}}}\right)^{-1}$$

$$= \frac{\partial}{\partial(\beta\hbar\omega_{\bar{k}})} \ln\left(1 - e^{-\beta\hbar\omega_k}\right) = \frac{e^{-\beta\hbar\omega_k}}{1 - e^{-\beta\hbar\omega_k}} = \frac{1}{e^{\beta\hbar\omega_{\bar{k}}} - 1}$$

<u>Bose-Einstein</u> distribution with $\mu = 0$

$$\langle N\rangle = \sum_{\bar{k},s} \langle n_{\bar{k},s}\rangle = \int_0^\infty g(\omega)\, n(\omega)\, d\omega = \int_0^\infty \frac{V\omega^2}{\pi^2 c^3} \frac{d\omega}{e^{\beta\hbar\omega} - 1}$$

$\frac{\hbar\omega}{kT} \equiv x$

$$= \frac{V}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^3 \int_0^\infty \frac{x^2\, dx}{e^x - 1} = \frac{V k^3}{\pi^2 c^3 \hbar^3} T^3\, \underbrace{\Gamma(3)}_{2!}\, \underbrace{\zeta(3)}_{\sim 1.202}$$

$$= \frac{2\zeta(3)\, k^3}{\pi^2 c^3 \hbar^3} V T^3$$

decreases as T decreases

but $\langle(\Delta N)^2\rangle$ ... ∂N ... $K = -\frac{1}{V}\partial V$

Lattice Vibrations - Phonons

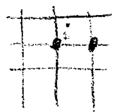
$\{x_i\}_{i=1}^{3N}$ potential energy: $\phi(x_i) = \phi_0 + \sum_i \underbrace{\frac{\partial \phi}{\partial x_i}}_{=0}(x_i - \bar{x}_i) + \frac{1}{2}\sum \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_i - \bar{x}_i)(x_j - \bar{x}_j) + \dots$

but $\bar{x}_j$ is the equilibrium position

$u_i \equiv x_i - \bar{x}_i$ displacements

$i = 1, 2, \dots, N$

$$\phi(u_i) = \phi_0 + \frac{1}{2}\sum_{i,j} \alpha_{ij} u_i u_j \quad , \text{where} \quad \alpha_{ij} = \left.\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right|_{x_i = \bar{x}_j}$$



$N = N_x N_y N_z$

$k_x = \frac{2\pi}{N_x a} n_x$

$$\mathcal{H}(u_i, \dot{u}_i) = \sum_{i=1}^{3N} \frac{1}{2} m \dot{u}_i^2 + \sum_{i,j} \frac{1}{2}\alpha_{ij} u_i u_j$$

normal coordinates: q_i (then diagonalization)

$n_x = 0, 1, 2, \dots, N_x - 1$

$u(x,t) \propto e^{i(kx - \omega t)}$

$$\mathcal{H}(q_i, \dot{q}_i) = \sum_{i=1}^{3N}\left(\frac{1}{2} m \dot{q}_i^2 + \frac{1}{2} m \omega_i^2 q_i^2\right)$$

$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

$$\Rightarrow \mathcal{H} = \sum_{\bar{k}, s} \hbar \omega_{\bar{k}, s}\left(n_{\bar{k}, s} + \frac{1}{2}\right) \qquad n_{\bar{k}, s} = 0, 1, 2, \dots$$

$k = \frac{2\pi}{L} n$

wave equation

$\omega_{\bar{k}, tr} = k c_{tr}$

$\omega_{\bar{k}, long} = k c_l$

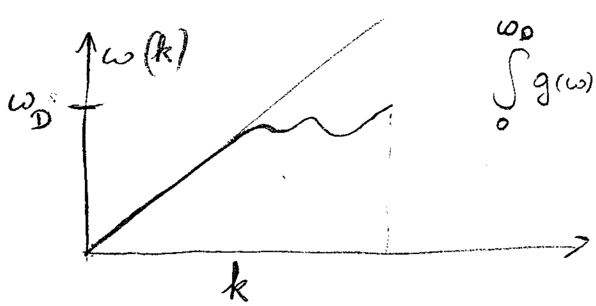
$$\sum_{\bar{k}, s} \frac{V}{(2\pi)^3} \Delta k_x \Delta k_y \Delta k_z = \sum_s \int \frac{V}{(2\pi)^3} 4\pi k^2 dk$$

$$= \int \frac{V}{(2\pi)^3} 4\pi \left[2\frac{\omega^2}{c_{tr}^3} + \frac{\omega^2}{c_l^3}\right] d\omega$$

$$g(\omega) = \frac{V\omega^2}{2\pi^2}\left(\frac{2}{c_{tr}^3} + \frac{1}{c_l^3}\right) = \frac{3}{2}\frac{V\omega^2}{\pi^2 c^3}$$

"effective" sound velocity

$$\frac{2}{c_{tr}^3} + \frac{1}{c_l^3} \equiv \frac{3}{c^3}$$



$$\int_0^{\omega_D} g(\omega) d\omega = 3N$$

Debye interpolation