Lecture 25: Interference

It is quite common for two or more waves to arrive at the same point in space or for them to co-exist together along the same propagation direction. Here, we will consider several important effects arising from the combination of two or more waves.

Such effects are governed by the superposition principle: when two or more waves overlap, the resultant displacement at any point and at any instant is found by adding the instantaneous displacements that would be produced at the point by the individual waves if each were present alone.

Consider the harmonic travelling (plane) waves:

$$y(t) = y_m \sin(kx - \omega t) = y_m \sin(k[x - vt])$$

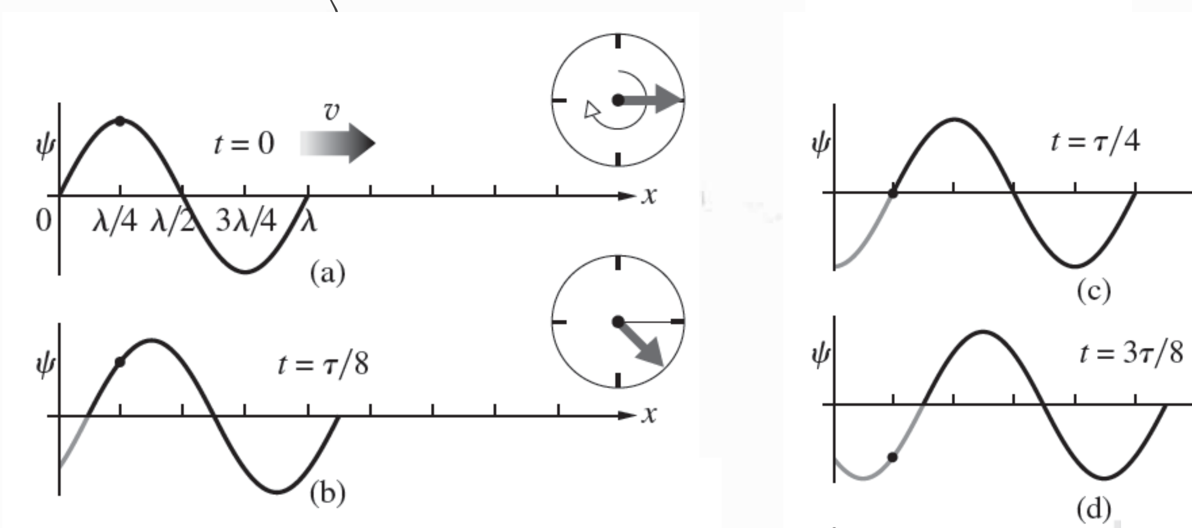
This represents a wave of amplitude y_m travelling in the $+x$-direction at wave speed: $v_{phase} = \frac{\lambda}{T} = \lambda f$

The argument of the sine function is called the phase of the wave:

$$\boxed{\varphi(t) = kx - \omega t + \varepsilon}$$

<u>initial</u> phase

A point with a specific phase moves to the right at velocity v



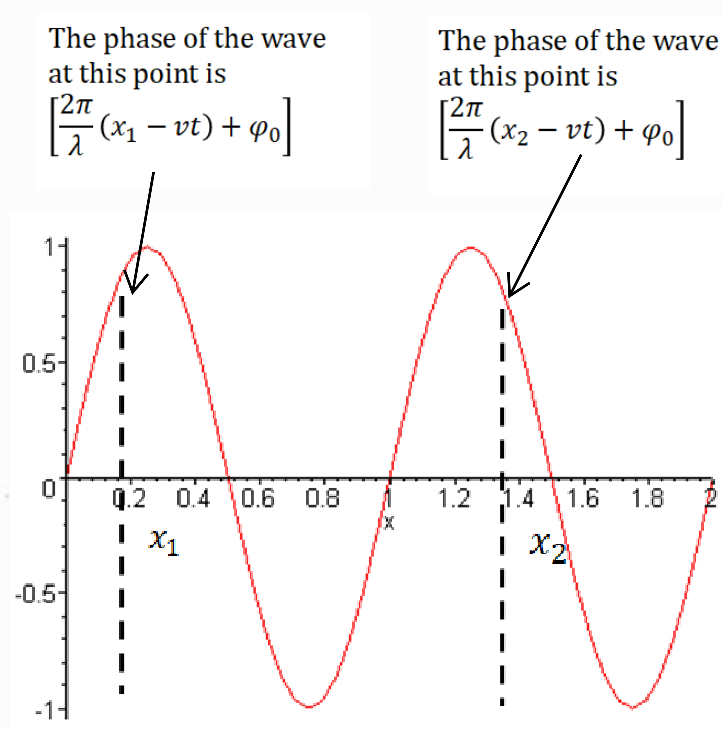
When we talk about phase difference $\Delta\varphi$, we are referring to how the argument differs for different times or places in the wave:

- 2 times at same point:

$$\Delta\varphi = \left[\frac{2\pi}{\lambda} v\Delta t\right] = \omega \Delta t$$

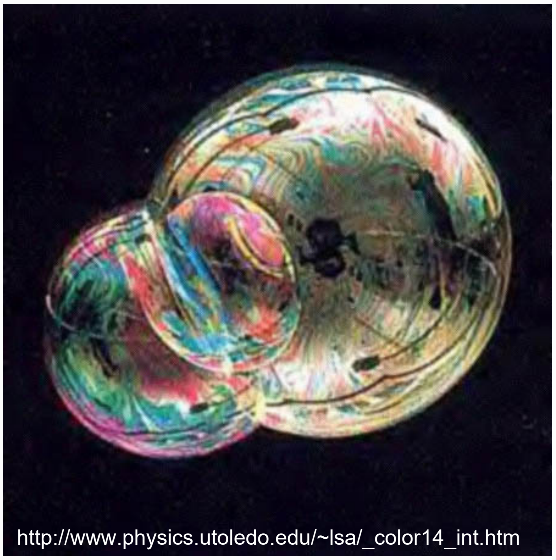
- 2 points at same time:

$$\Delta\varphi = \left[\frac{2\pi}{\lambda}\Delta x\right] = k\Delta x$$



The phase difference is: $\left[\frac{2\pi}{\lambda}(x_2 - x_1)\right]$

Applications and observations of thin film interference:



- Pretty reflections from soap bubbles and oil slicks
- Antireflection coatings on glass
- Wavelength selectors (for imaging, coloration, spectroscopy, ...)
- Determining the thickness of technically important layers (e.g. insulating oxides in computer chips).

Once again, consider a plane wave:

$$E(x,t) = E_0 \sin[\omega t - (kx + \varepsilon)]$$

To separate the space and time part of the phase, let $\alpha(x, \varepsilon) = -kx - \varepsilon$

In the case of two waves coexisting in space, we have:

$$E_1 = E_{01} \sin(\omega t + \alpha_1) = E_{01}[\sin(\omega t)\cos\alpha_1 + \cos(\omega t)\sin\alpha_1]$$

$$E_2 = E_{02} \sin(\omega t + \alpha_2) = E_{02}[\sin(\omega t)\cos\alpha_2 + \cos(\omega t)\sin\alpha_2]$$

And the resulting wave is

$$E_T = E_1 + E_2$$

$$= \left[E_{01} \cos\alpha_1 + E_{02} \cos\alpha_2 \right] \sin(\omega t) + \left[E_{01} \sin\alpha_1 + E_{02} \sin\alpha_2 \right] \cos(\omega t)$$

Now let $E_{01} \cos\alpha_1 + E_{02} \cos\alpha_2 = E_0 \cos\alpha$

and $E_{02} \sin\alpha_1 + E_{02} \sin\alpha_2 = E_0 \sin\alpha$

Taking the square and adding the two terms:

$$\boxed{E_0^2 = E_{01}^2 + E_{02}^2 + \overbrace{2 E_{01} E_{02} \cos(\alpha_2 - \alpha_1)}^{\text{interference term}}}$$

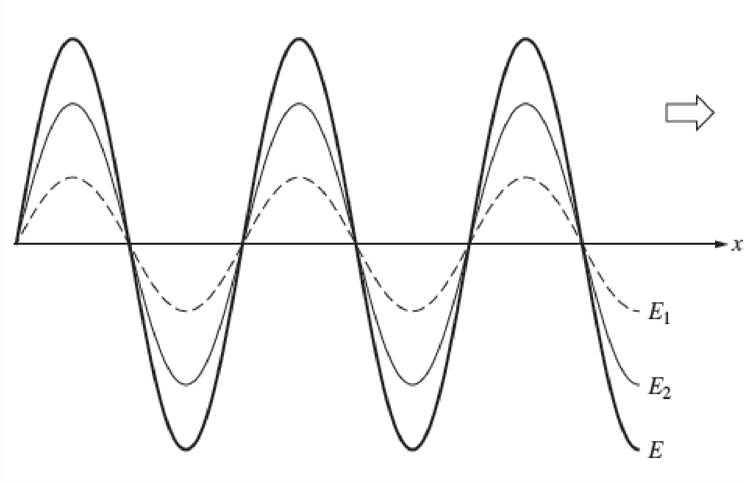
Once we have the field, we can find the intensity:

$$I = \frac{E^2}{2\mu_0 c}, \text{ so } \boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\delta}$$

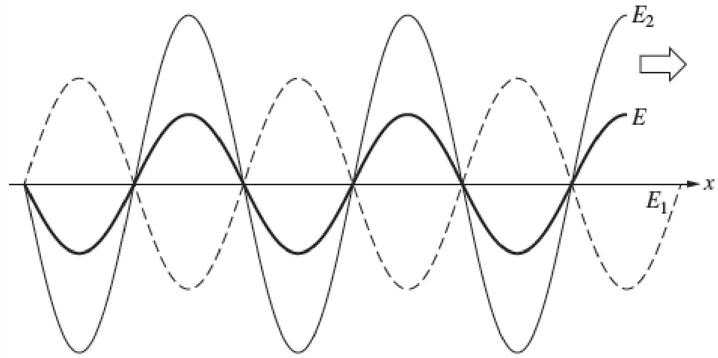
where $\delta \equiv \alpha_2 - \alpha_1$ is the phase difference

Thus, we see that the intensity of the resulting wave is not just the sum of the intensity of the original waves, but that there is an additional cross term: the interference term.

Constructive Interference: superposition of in-phase waves



Destructive Interference: superposition of out-of-phase waves



For the special case where $I_1 = I_2$,

$$I = 2I_1(1 + \cos\delta)$$

Using $\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$, $\boxed{I = 4I_1\cos^2\left(\frac{\delta}{2}\right)}$

Of course, the waves can also be expressed in complex notation: $E_1 = E_{01}e^{j(\omega t + \alpha_1)}$ & $E_2 = E_{02}e^{j(\omega t + \alpha_2)}$

Then $E_T = e^{j\omega t}\left[E_{01}e^{j\alpha_1} + E_{02}e^{j\alpha_2}\right] = E_0 e^{j\alpha} e^{j\omega t} = E_0 e^{j(\omega t + \alpha}$

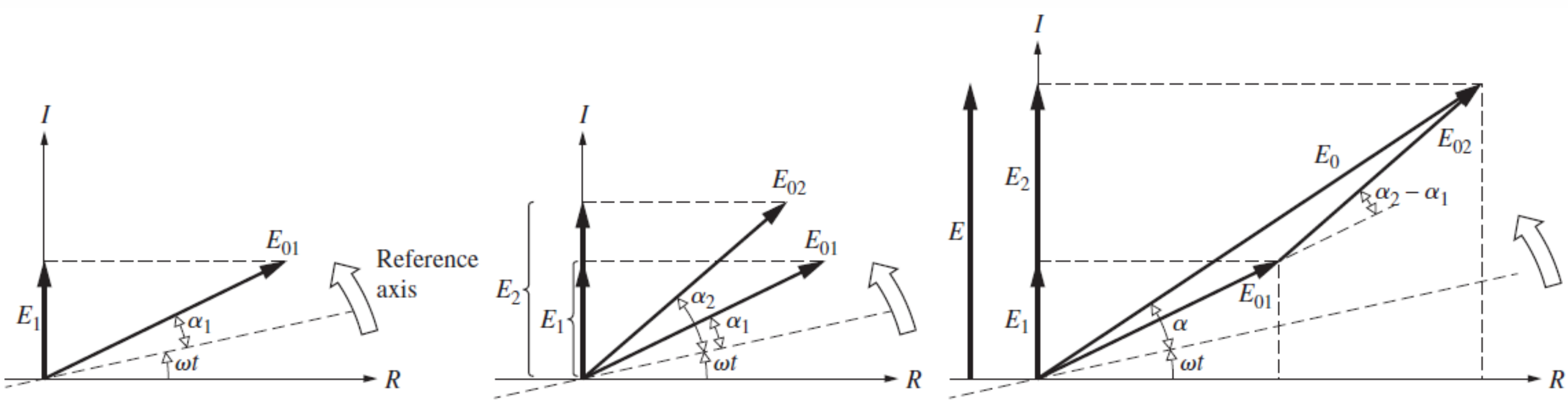
and $I = \left(E_0 e^{j\alpha}\right)\left(E_0 e^{j\alpha}\right)^*$

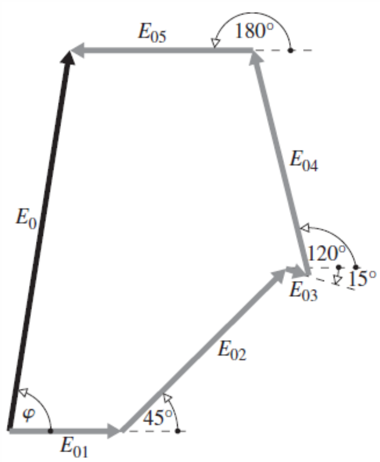
where $\tan\alpha = \dfrac{E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2}{E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2}$

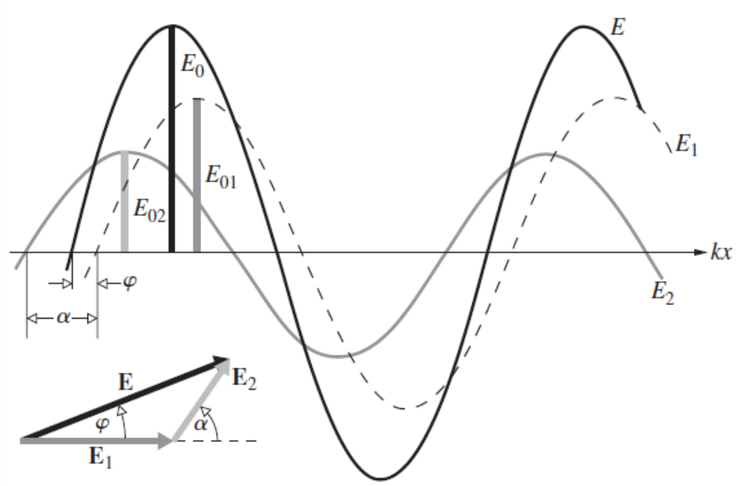
$E_T = E_0 \sin(\omega t + \alpha)$ is a harmonic wave of the same frequency as its constituents, but with a different amplitude and phase.

Since the complex amplitude can be represented by a phasor, we can represent superposition graphically as an addition of vectors in the complex plane.

If we're looking at cosine waves, E_0 will be the projection onto the horizontal axis, and sine waves will be the projection onto the vertical axis.







Since $\alpha = -kx - \varepsilon$, where $k = \frac{2\pi}{\lambda}$, we can write:

$$\delta = \alpha_2 - \alpha_1 = (k_1 x_1 + \varepsilon_1) - (k_2 x_2 + \varepsilon_2)$$

$$= \left[\frac{2\pi}{\lambda_1} x - \frac{2\pi}{\lambda_2} x_2\right] + (\varepsilon_1 - \varepsilon_2)$$

Thus, phase differences arise from:

- differences at the generation of the waves
- path length differences
- reflections

$$\Rightarrow \quad \delta = \delta_{\text{initial phase}} + \delta_{\text{path length}} + \delta_{\text{reflection}}$$

$$= \varepsilon + \delta_{OPL} + \delta_R$$

Supposing the two waves are initially in-phase $(\varepsilon_1 = \varepsilon_2)$, we have:

$$\delta = \frac{2\pi}{\lambda_1} k_1 - \frac{2\pi}{\lambda_2} k_2 = \frac{2\pi}{\lambda_0} n_1 L_1 - \frac{2\pi}{\lambda_0} n_2 L_2$$

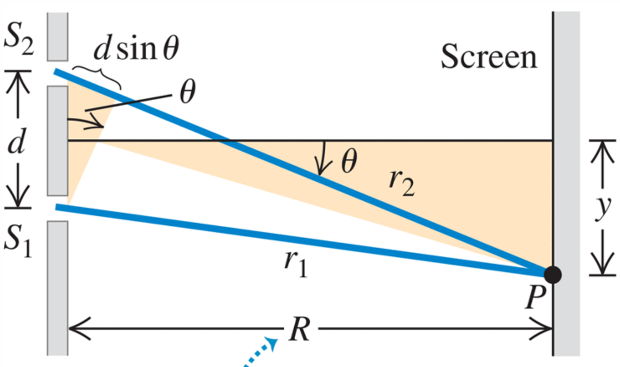
$$\boxed{\delta = \frac{2\pi}{\lambda_0} [\underbrace{n_1 L_1 - n_2 L_2}_{\text{optical path difference}}]}$$

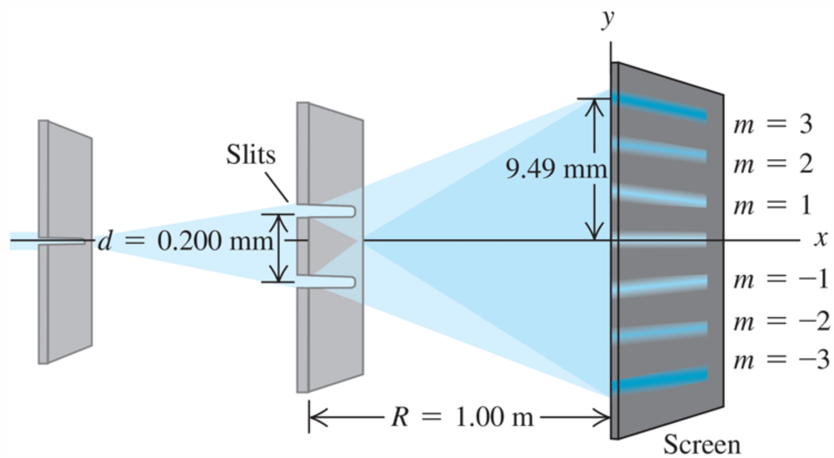
wavelength in vacuum

Now let's consider that E_1 and E_2 are waves from two sources, S_1 at (x_1, y_1) and S_2 at (x_2, y_2), emitting in phase ($\varepsilon_1 = \varepsilon_2$) in a homogeneous medium. 8

If the observation point is at (x_0, y_0), we have:

- $I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$
- $\delta = \dfrac{2\pi}{\lambda_0} n (L_1 - L_2)$; $L_i = \sqrt{(x_0 - x_i)^2 + (y_0 - y_i)^2}$





If the two sources are separated by a distance d and the view screen is a distance D from the source, the optical path difference is: (with $x_1 = x_2$; $x_0 = D$)

$$L_2 - L_1 = D\left[\sqrt{1 + \frac{(y_0 - y_1)^2}{D^2}} - \sqrt{1 + \frac{(y_0 - y_2)^2}{D^2}}\right]$$

$y_1, y_2 \ll y_0$

$$\approx D\left(1 + \frac{1}{2}\frac{(y_0 - y_1)^2}{D^2} - 1 - \frac{1}{2}\frac{(y_0 - y_2)^2}{D^2} + \dots\right)$$

$$L_2 - L_1 = \frac{D}{2}\left[\frac{y_0^2 - 2y_1y_0 + y_1^2}{D^2} - \frac{y_0^2 - 2y_2y_0 + y_2^2}{D^2}\right]$$

y_1^2, y_2^2 sma

$y_2 - y_1 = d$

$$\approx \frac{y_0}{D} d$$

[Including higher order terms, $L_2 - L_1 = d\sin\theta$]

This corresponds to Young's double slit experiment.

- Constructive interference occurs when

$$\Delta L = d\sin\theta = m\lambda$$

- Destructive interference occurs when

$$\Delta L = d\sin\theta = \left(m + \frac{1}{2}\right)\lambda$$

where $m = 0, \pm 1, \pm 2, \ldots$ is the order of the max or min.

If $D \gg d$, then the angle θ is small so that

$$\sin\theta \approx \tan\theta \approx \frac{y}{s}$$

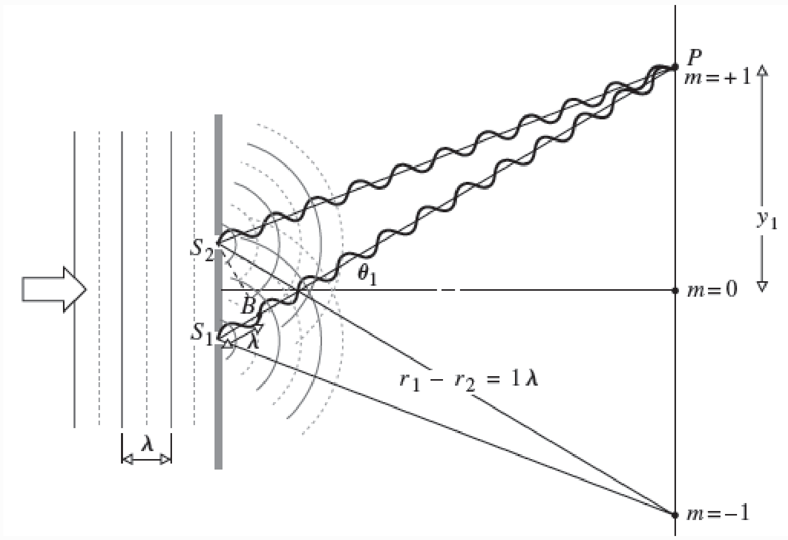
In this limit, the position of the m^{th} bright fringe is

$$\boxed{y_m = m\lambda\frac{D}{d}} \quad \text{or} \quad \boxed{\theta_m = \frac{m\lambda}{d}}$$

Then the distance between adjacent maxima is

$$\Delta y = y_{m+1} - y_m = \frac{(m+1)\lambda D}{d} - \frac{m\lambda D}{d}$$

$$\Rightarrow \quad \boxed{\Delta y = \frac{\lambda D}{d}}$$



The phase difference is

$$\delta = \frac{2\pi}{\lambda} d\sin\theta \approx \frac{2\pi d y}{\lambda D}$$

and the intensity is given by: $I = 4I_0 \cos^2\left(\frac{\delta}{2}\right)$

so that for the two-slit interference we have:

$$\boxed{I = 4I_0 \cos^2\left(\frac{\pi d}{\lambda D} y\right)}$$