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Quantum Error Correction

Idea: Protect your quantum info. from local noise by encoding nonlocally in multiqubit states.

Why? - Need very low error rates for QC [~ 1 in 10^{17} classical CPU]
- QI very sensitive to noise - even a single photon.

Can think about in 2 ways:

Storage
↓
Quantum memory

1. Encode
2. Store / send over noisy channel
3. Decode

Processing
↓
Fault tolerant QC (FTQC)

1. Encode
2. Apply encoded gates
3. Do error correction
4. Decode

Repeat for circuit

only subset of codes allowed

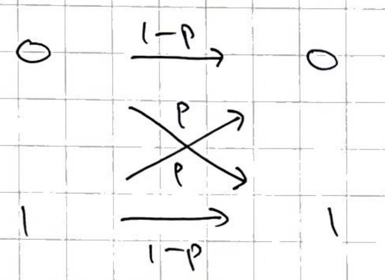
Shor '96: First q. codes & FTQC

Kitaev '97: Topological codes in 2D

Gottesman '97: Stabilizer codes

Threshold Thm: If physical error rate smaller than some threshold [a #, O(1)], then FTQC possible (can make logical error as small as want).

Classical Repetition Code



Binary symmetric channel

3-bit repetition code

encode $0 \rightarrow \bar{0} = 000$ logical 0

$1 \rightarrow \bar{1} = 111$ & 1

cloning?

Each physical bit independently noisy.

$$\bar{0} = 000 \xrightarrow[\text{noise}]{} 100 \xrightarrow[\text{majority vote}]{} 000 = \bar{0}$$

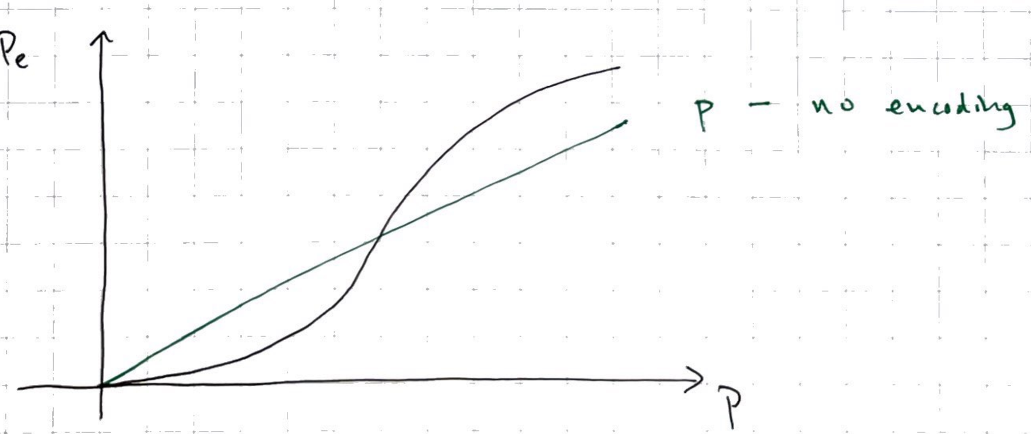
majority vote ← measure

Prob. of 2, 3 flips:

only discrete errors!

$$P_e = \binom{3}{2} p^2 (1-p) + p^3 = 3p^2 - 2p^3$$

$$P_e < p \quad \text{for} \quad p \leq 1/2$$



3-Qubit Bit Flip Code

Now quantum! Might worry about:

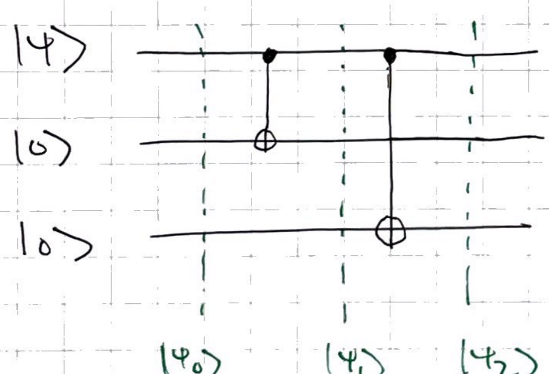
- No cloning theorem
- Errors are continuous
- Measurement destroys quantum info.

$\Big\}$ → not a problem!

3-qubit bit flip code corrects a single bit flip

$$\left.\begin{array}{l} |0\rangle \rightarrow |\bar{0}\rangle = |000\rangle \\ |1\rangle \rightarrow |\bar{1}\rangle = |111\rangle \end{array}\right\} \quad a|0\rangle + b|1\rangle \rightarrow a|000\rangle + b|111\rangle$$

Encoding circuit

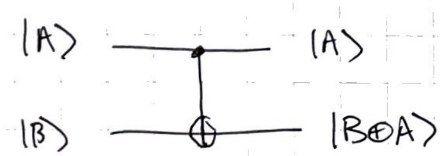


$$|\psi_0\rangle = |\psi\rangle|0\rangle|0\rangle = (a|0\rangle + b|1\rangle)|0\rangle|0\rangle$$

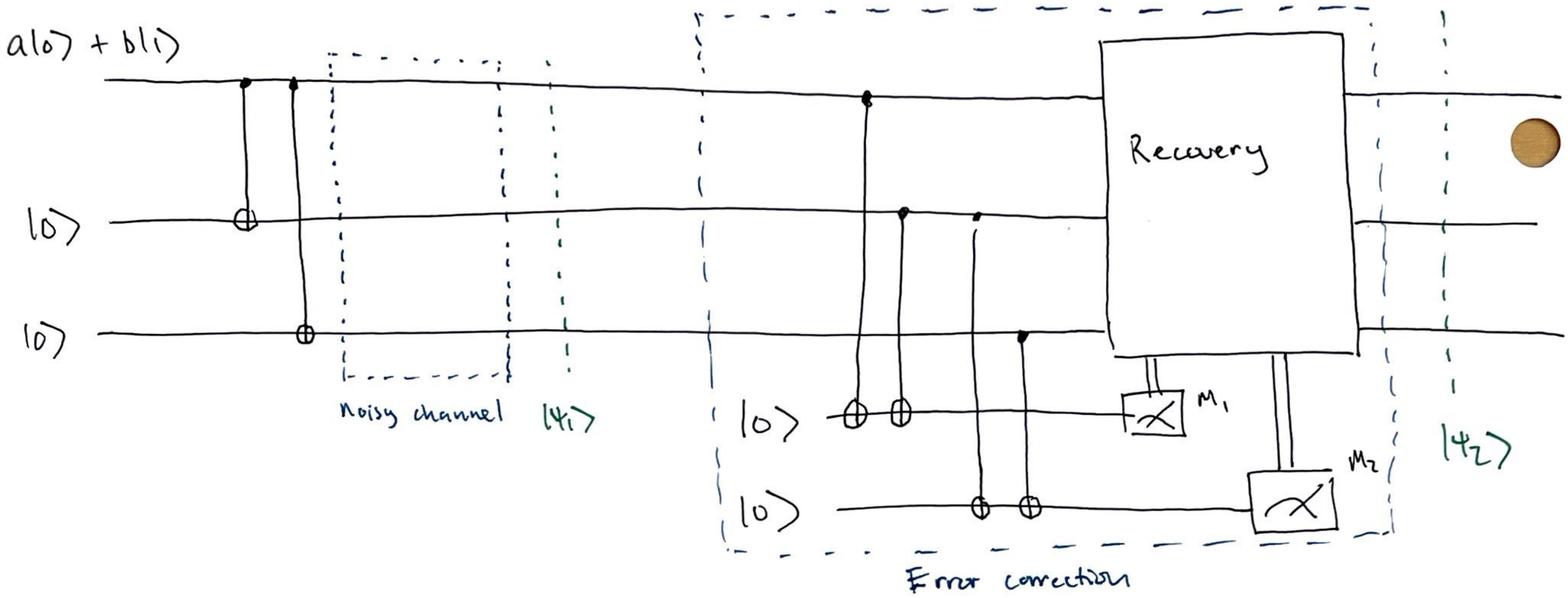
$$|\psi_1\rangle = (a|00\rangle + b|11\rangle)|0\rangle$$

$$|\psi_2\rangle = (a|000\rangle + b|111\rangle)$$

Recall CNOT



$$|A, B\rangle \rightarrow |A, A \oplus B\rangle$$



Bit flip	$\|\psi_1\rangle$	M_1	M_2	Recovery	$\|\psi_2\rangle$
—	$a\|000\rangle + b\|111\rangle$	0	0	$I \otimes I \otimes I$	$a\|000\rangle + b\|111\rangle$
1	$a\|100\rangle + b\|011\rangle$	1	0	$X \otimes I \otimes I$	$a\|000\rangle + b\|111\rangle$
2	$a\|010\rangle + b\|101\rangle$	1	1	$I \otimes X \otimes I$	$a\|000\rangle + b\|111\rangle$
3	$a\|001\rangle + b\|110\rangle$	0	1	$I \otimes I \otimes X$	$a\|000\rangle + b\|111\rangle$

Puts logical qubit in codespace, which is spanned by $|000\rangle, |111\rangle$

Code: space $C = \text{span}\{|\bar{0}\rangle, |\bar{1}\rangle\}$

↳ A 2d Hilbert space.

Ex: $a|0\rangle + b|1\rangle \longrightarrow a|000\rangle + b|111\rangle$

$$\downarrow \text{ bit flip } X_1$$

$$a|100\rangle + b|011\rangle$$

Naïve measurement:

$$\left.\begin{array}{l} |100\rangle \text{ w.p. } |a|^2 \\ |011\rangle \text{ w.p. } |b|^2 \end{array}\right\} \text{Collapse!}$$

a, b gone. Need something more subtle.

Two Step Error Correction

1. Error detection / syndrome diagnosis: Measure which error occurred (syndrome) w/o fully measuring state.
2. Recovery: Apply gates based on syndrome to go back

Observables — Parity checks detect mismatch

	$Z_1 Z_2$	$Z_2 Z_3$	Are simul. obs.	Recovery
~~[illegible]~~	+1	+1	$|000\rangle, |111\rangle$	do nothing
bit flip X_1	−1	+1	$|011\rangle, |100\rangle$	X_1
bit flip X_3	+1	−1	$|001\rangle, |110\rangle$	X_2
bit flip X_2	−1	−1	$|010\rangle, |101\rangle$	X_3

Ex: $|\psi'\rangle = a|100\rangle + b|011\rangle$

1. Measure $Z_1 Z_2 |\psi'\rangle = -a|100\rangle - b|011\rangle = -|\psi'\rangle$

$Z_2 Z_3 |\psi'\rangle = a|100\rangle + b|011\rangle = |\psi'\rangle$

Measure $(-, +)$ w/o disturbing $|\psi'\rangle$

2. Recovery: Apply X_1 : $X_1|\psi'\rangle = a|000\rangle + b|111\rangle$

Phase Errors

3-qubit bit flip code utilized:

1. Entanglement for repetition
2. Detect errors using parity-checks that do not destroy QI

But does <u>NOT</u> address that errors can be continuous. (Practically, 3-qubit bit flip fails against Z-errors.)

X flips bits

Z flips phase

↑ continuous

Ex: $|\overline{+}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} \overset{Z}{\longleftrightarrow} \frac{|000\rangle - |111\rangle}{\sqrt{2}} = |\overline{-}\rangle$

Not in code space

Encoded operations

What gates perform Paulis on code space?

$\overline{Z} = Z_1$ $(Z_2,\ Z_3,\ Z_1 Z_2 Z_3$ also work$)$

$\overline{X} = X_1 X_2 X_3$

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Generally, parity checks $Z_1 Z_2 = +1$ on codespace so

$$Z_1 = Z_1 Z_2 Z_3 = Z_2 \quad \text{restricted to } C$$

Note that for $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$ Z takes

$$Z|\pm\rangle = |\mp\rangle$$

So suggests using $|\bar{0}\rangle = |+++\rangle$ as logical states.
$|\bar{1}\rangle = |---\rangle$

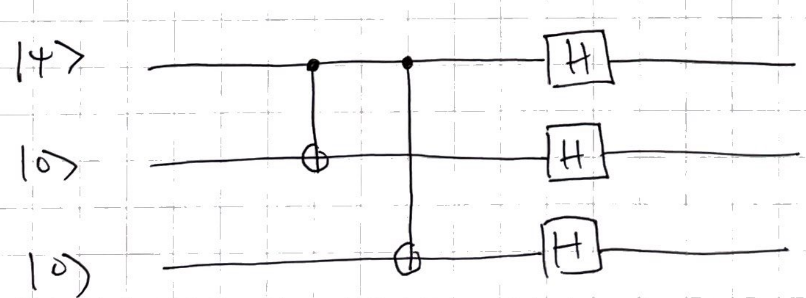
3-Qubit Phase Flip Code

: To protect alg 1 Z-error use bit flip code in X-basis.

$$|0\rangle \to |\bar{0}\rangle = |+++\rangle$$

$$|1\rangle \to |\bar{1}\rangle = |---\rangle$$

Encoding circuit

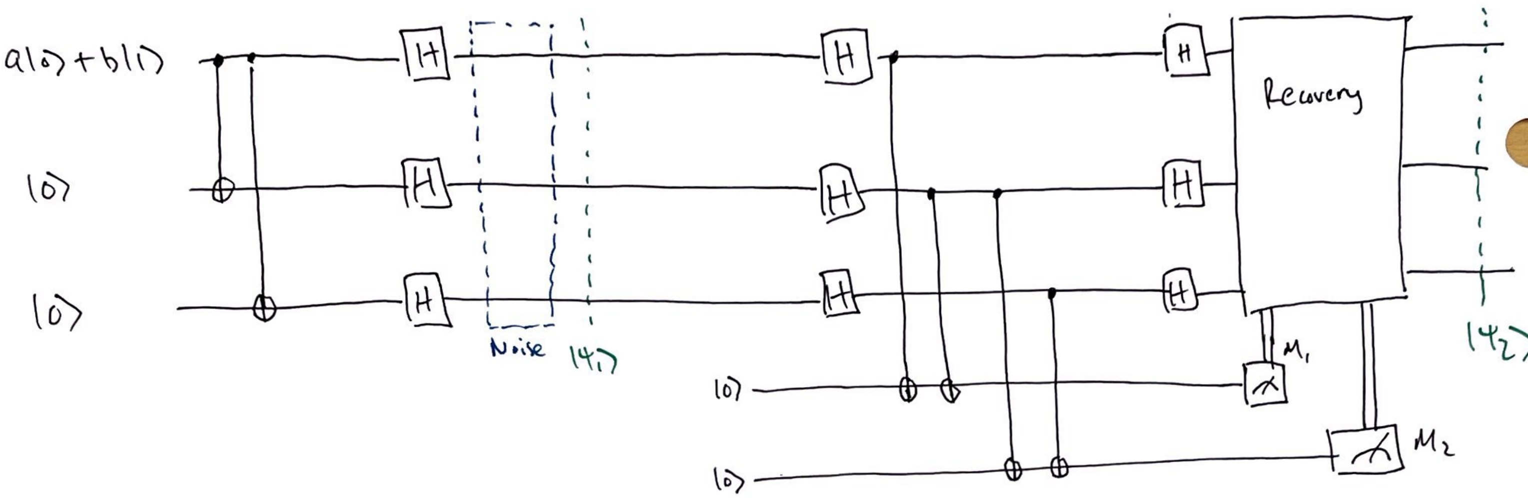


Recall Hadamard transforms from computational basis to qubit basis

Unitarily equivalent to bit flip channel $(Z \leftrightarrow X)$
[∃ a unitary op. U (Hadamard) that maps one to other.]

Syndrome detected by: $H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2$ "1 & 2 agree in X-basis"

$$H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$$



Recall phase flip sends

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle \mp |1\rangle\right) = |\mp\rangle$$

Phase flip	$\vert\psi_1\rangle$	M_1	M_2	Recovery	$\vert\psi_2\rangle$
—	$a\vert{+}{+}{+}\rangle + b\vert{-}{-}{-}\rangle$	0	0	$I \otimes I \otimes I$	$a\vert{+}{+}{+}\rangle + b\vert{-}{-}{-}\rangle$
1	$a\vert{-}{+}{+}\rangle + b\vert{+}{-}{-}\rangle$	1	0	$Z \otimes I \otimes I$	$a\vert{+}{+}{+}\rangle + b\vert{-}{-}{-}\rangle$
2	$a\vert{+}{-}{+}\rangle + b\vert{-}{+}{-}\rangle$	1	1	$I \otimes Z \otimes I$	$a\vert{+}{+}{+}\rangle + b\vert{-}{-}{-}\rangle$
3	$a\vert{+}{+}{-}\rangle + b\vert{-}{-}{+}\rangle$	0	1	$I \otimes I \otimes Z$	$a\vert{+}{+}{+}\rangle + b\vert{-}{-}{-}\rangle$

The Shor Code

arbitrary single qubit errors

X Z

A 9-qubit which protects against 1 bit flip, 1 phase flip

Idea: Concatenate 3-qubit bit flip and 3-qubit phase flip codes.

$$|0\rangle \xrightarrow{\text{phase flip}} |+++\rangle \xrightarrow{\text{bit flip}} \frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)$$

$$|1\rangle \longrightarrow |---\rangle \longrightarrow \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

How to check for errors?

Bit flips — Parity checks

$Z_1 Z_2$	$Z_2 Z_3$
$Z_4 Z_5$	$Z_5 Z_6$
$Z_7 Z_8$	$Z_8 Z_9$

compare neighbors and repair

Phase flips

Z_1 flips sign of first block: $|000\rangle + |111\rangle \rightarrow |000\rangle - |111\rangle$

So syndrome: $\bar{X}_1 \bar{X}_2 = X_1 X_2 X_3 X_4 X_5 X_6$

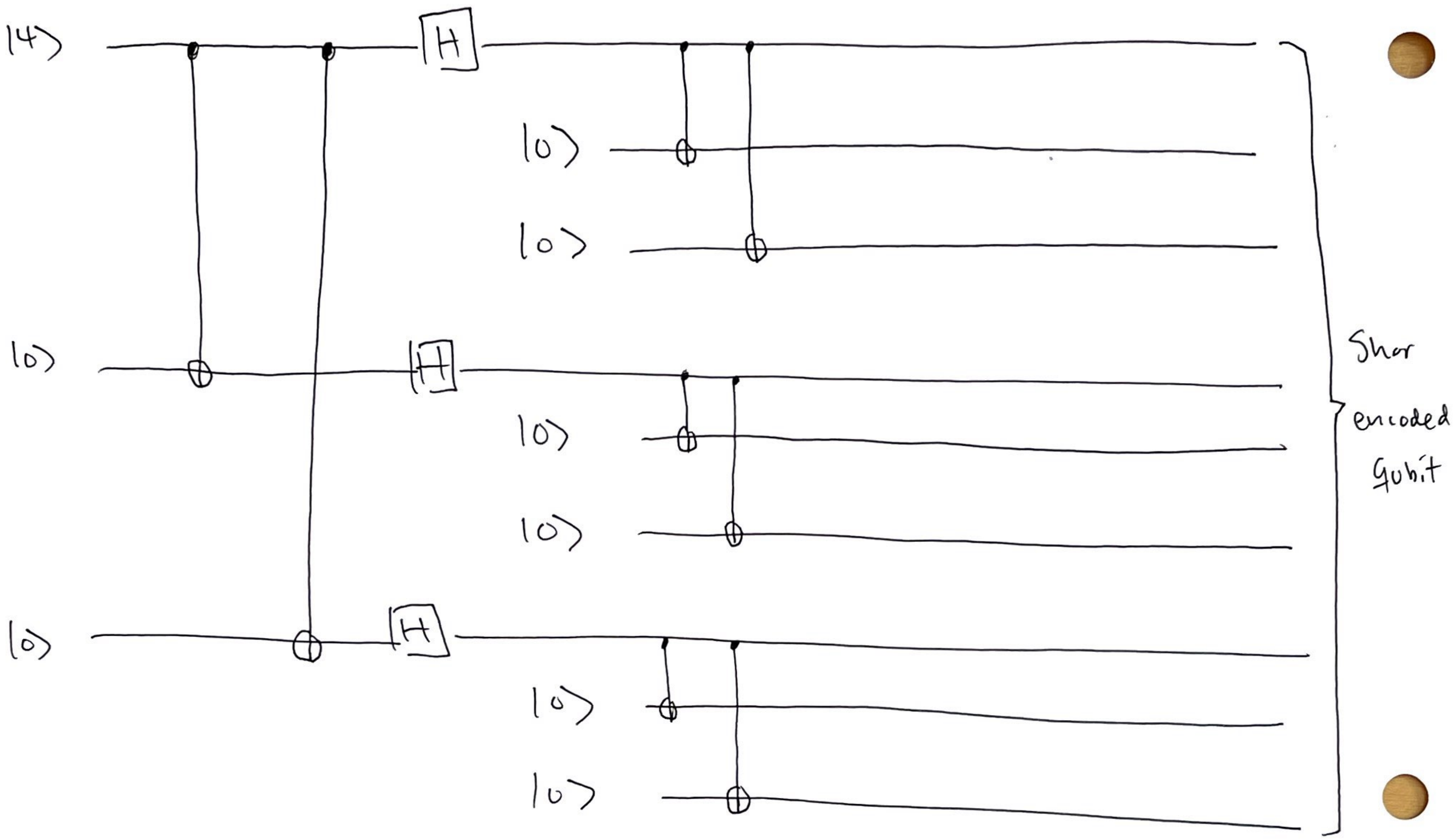
$\bar{X}_2 \bar{X}_3 = X_4 X_5 X_6 X_7 X_8 X_9$

Recovery: $\bar{Z}_1 = Z_1 Z_2 Z_3$ if phase flip on first block, etc.

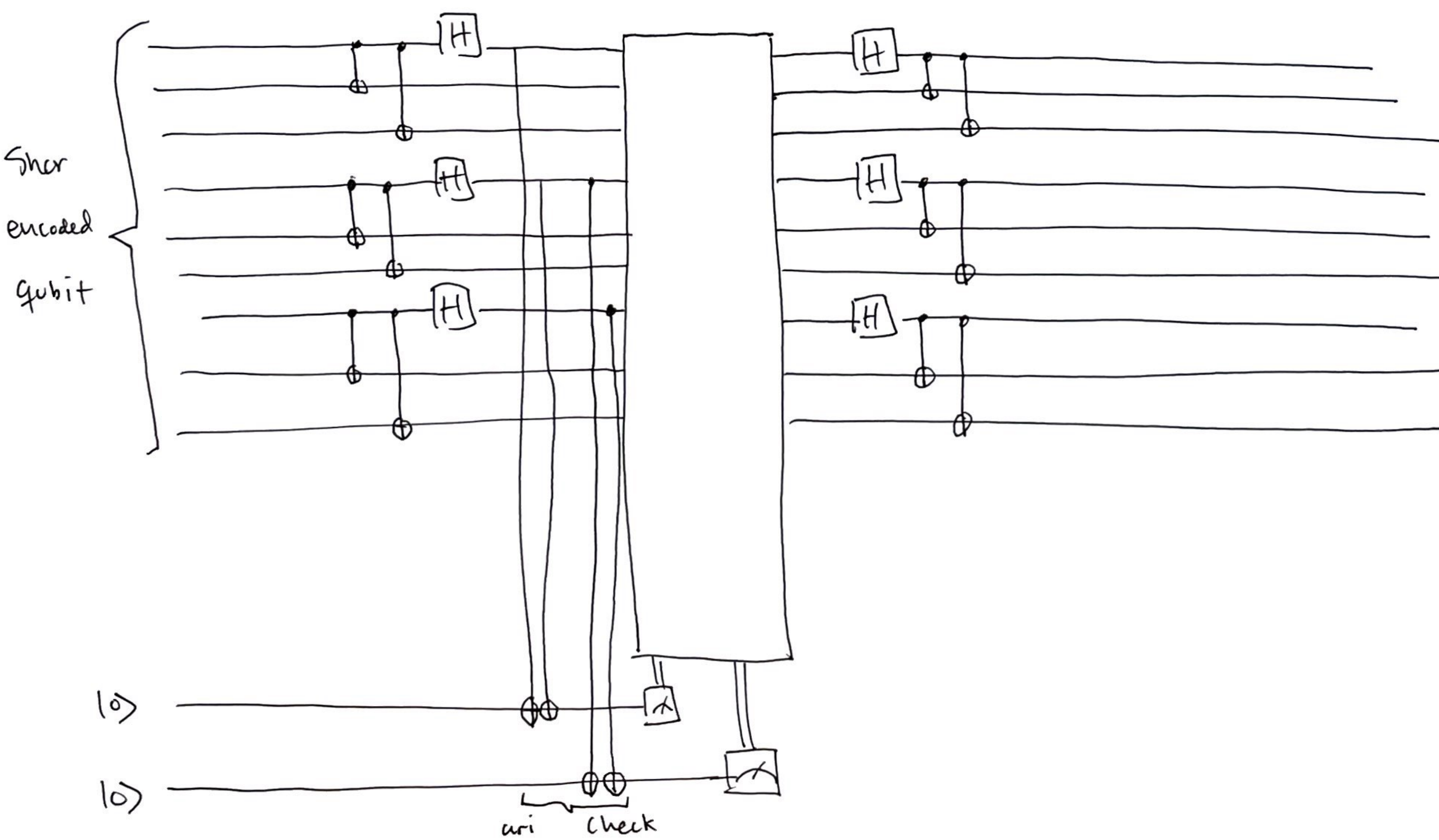
Bit and Phase flip

ZX on single qubit corrected by above procedure.

Encoding circuit



Correcting circuit:



Arbitrary Single Qubit Errors

Assume error on 1st qubit (for simplicity)

Single qubit channel $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$

arbitrary error — small, large, etc.

Kraus ops: $E_i = e_{i0} I + e_{i1} X_1 + e_{i2} \underset{XZ}{Y} + e_{i3} Z$

$|\psi\rangle = a|\bar{0}\rangle + b|\bar{1}\rangle \xrightarrow{\text{Noise}} \mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i E_i |\psi\rangle\langle\psi| E_i^\dagger = \rho'$

with probability $\langle\psi|E_i^\dagger E_i|\psi\rangle$ get state $E_i|\psi\rangle$ (unnormalized)

Measuring syndrome on $E_i|\psi\rangle$ collapses to:

$|\psi\rangle \qquad X|\psi\rangle \qquad Y|\psi\rangle = XZ|\psi\rangle \qquad Z|\psi\rangle$

All correctable!

Only to check the Shor code corrects X, Y, Z errors! Syndrome discretizes errors.

Summary

Shor code uses 9 qubits to encode 1 logical qubit, which it can protect from an arbitrary error on a single qubit.

(Smallest such code is 5-qubit error correcting code.)