RESNICK + HALLIDAY, BASIC CONCEPTS IN RELATIVITY AND EARLY QUANTUM THEORY, (WILEY, 1972)

SUPPLEMENTARY TOPIC A

the geometric representation of spacetime

Oh, that Einstein, always cutting lectures—I really would not have believed him capable of it.

Hermann Minkowski (ca. 1908)

A-1 SPACETIME DIAGRAMS

We have seen that in classical physics it is proper to treat the space and time coordinates separately. In relativity, however, it is natural to treat them together, their intimate interconnection being clearly displayed in the Lorentz transformation equations; see Tables 2-2 and 2-3. The common use of the single word "spacetime" (without a hyphen) to represent the coordinate description of events is symbolic of the general acceptance of this view.

As we have learned, it was Einstein [1] who first set forth, in his special theory of relativity, the physical basis for the proper description of events in space and time. Shortly afterwards the mathematician Hermann Minkowski (who, incidentally, had formerly been Einstein's mathematics professor in Zurich) [2] presented a simple and symmetrical geometric representation of these ideas, a representation that permits a ready understanding in geometric terms of such matters as the relativity of simultaneity, the length contraction, and the time dilation, including their reciprocal nature.

In what follows, we shall consider only one space axis, the x axis, and shall ignore the y and z axes. We lose no generality by this algebraic simplification, and this procedure will enable us to focus more clearly on the interdependence of space and time and its geometric representation. The coordinates of an event are given, then, by x and t. All possible spacetime coordinates can be represented on a spacetime diagram in which the space axis is horizontal and the time axis is vertical. It is convenient to keep the dimensions of the coordinates the same; this is easily done by multiplying the time t by the universal constant c, the velocity of light. Let ct be represented by the symbol w. Then, the Lorentz transformation equations (see Table 2-2 and Problem 11 of Chapter 2) can be written as follows:

$$\begin{aligned} &(a)\ \ x' = \gamma(x - \beta w) \qquad &(a')\ \ x = \gamma(x' + \beta w') \\ &(b)\ \ w' = \gamma(w - \beta x) \qquad &(b')\ \ w = \gamma(w' + \beta x') \end{aligned} \tag{A-1}$$

Notice the symmetry of this form of the equations.

To represent the situation geometrically, we begin by drawing the x and w axes of frame S at right angles to one another, as in Fig. A-1. If we want to represent a moving particle in this frame, we draw a curve, called the *world line* of the particle, which gives the loci of spacetime points corresponding to the motion.

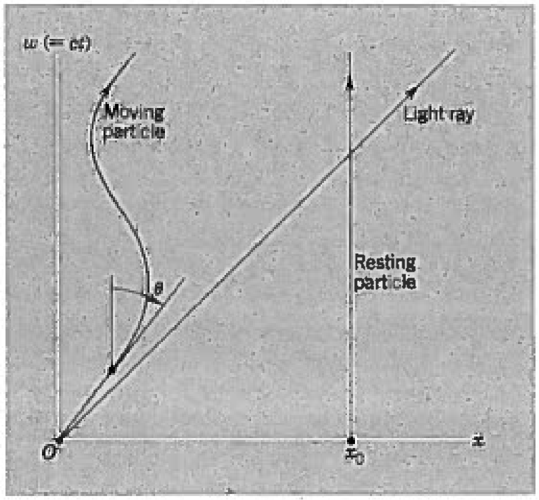

Figure A-1. The *world lines* of light and some particles.

The tangent to the world line at any point makes an angle θ with the direction of the time axis that is given by $\tan\theta = dx/dw = (dx/dt)(1/c) = u/c$. Because we must have $u < c$ for a material particle, the angle θ at any point on its world line must always be less than 45°. If the particle is at rest, say, at position x_0 on the x axis of Fig. A-1, its world line is parallel to the w axis, with $\theta\,(=\tan^{-1} u/c) = 0$ at all points. For a light ray traveling along the x axis we have $u = c$, so its world line is a straight line making an angle of 45° with the axes.

Consider now the primed frame (S'), which moves relative to S with a velocity $\mathbf{v}$ along the common x-x' axis. The equation of motion of the origin of S' relative to S can be obtained by setting $x' = 0$; from Eq. A-1a, we see that this corresponds to $x = \beta w$. We draw the line $x' = 0$ (that is, $x = \beta w$) on our diagram (Fig. A-2) and note that since $v < c$ and $\beta < 1$, the angle this line makes with the w axis, $\phi(=\tan^{-1}\beta)$, is less than 45°. Just as the w axis corresponds to $x = 0$ and is the time axis in frame S, so the line $x' = 0$ gives the time axis w' in S'. Now, if we draw the line $w' = 0$ (giving the location of clocks that read $t' = 0$ in S'), we shall have the space axis x'. That is, just as the x axis corresponds to $w = 0$, so the x' axis corresponds to $w' = 0$. But, from Eq. A-1b, $w' = 0$ gives us $w = \beta x$ as the equation of this axis on our w-x diagram (Fig. A-2). The angle between the space axes is the same as that between the time axes. Note that, for simplicity, we have shown in Fig. A-2 only the quadrant in which both x and w are positive.

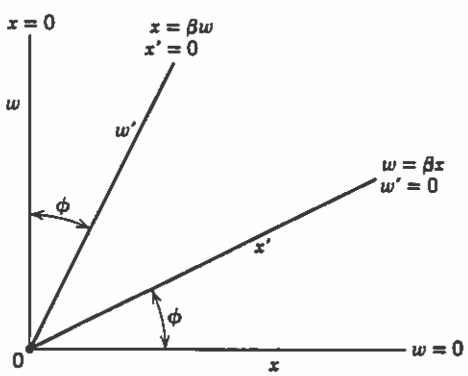

Figure A-2. The *Minkowski diagram* for frames S and S'.

You should compare Fig. A-2 carefully with the standard representation of Fig. 1-1, which we have used exclusively in the main body of the text. A point in the coordinate reference frames of Fig. 1-1 shows only the space coordinates of the event to which it corresponds; the time of occurrence of the event must be given separately. A point on the Minkowski diagram of Fig. A-2, however, shows both the space and the time coordinates of the event in a single geometric representation.

A-2 CALIBRATING THE SPACETIME AXES

Before we can make practical use of the spacetime diagram we must establish scales on its x, w and its x', w' axes. We can use the Lorentz transformation equations of Eq. A-1 for this purpose. Consider first point O, located at the common origin of the two pairs of axes in Fig. A-3. It has coordinates $x = w = 0$ and $x' = w' = 0$, and the event to which it corresponds is the coincidence in time of the origins of the S and S' reference frames.

Point P_1 on the x' axis of Fig. A-3 has been chosen as a point to which we wish to assign the value $x' = 1$, representing a unit of length on this axis. As for all points on the x' axis, the time coordinate w' of P_1 is zero. Putting $x' = 1$ and $w' = 0$ into Eq. A-1a' yields, by simple inspection, $x = \gamma$ for the x coordinate of P_1. With this information we can easily construct numerical scales for both the x and the x' axes, based on our initially assumed unit length.

Consider now point P_2 on the w' axis of Fig. A-3, to which we wish to assign the value $w' = 1$, representing a unit of time (measured in terms of ct', to be sure) on that axis. We wish the scales on both the x' and the w' axes to be based on the same unit length, so we choose to locate P_2 so that the line segment OP_2 is equal in length to the segment OP_1. As for all points on the w' axis, the space coordinate x' of P_2 is zero. Putting $w' = 1$ and $x' = 0$ into Eq. A-1b' yields, again by simple inspection, $w = \gamma$ for the w coordinate of P_2. We are now able to construct numerical scales for both the w and the w' axes, based on the same unit length as we assumed in calibrating the space axes.

To gain some physical familiarity with the Minkowski diagram, let us consider a clock at rest at the origin of the S' frame. For that clock we have $x' = 0$ (always), so events involving it must correspond to points along the w' axis of Fig.

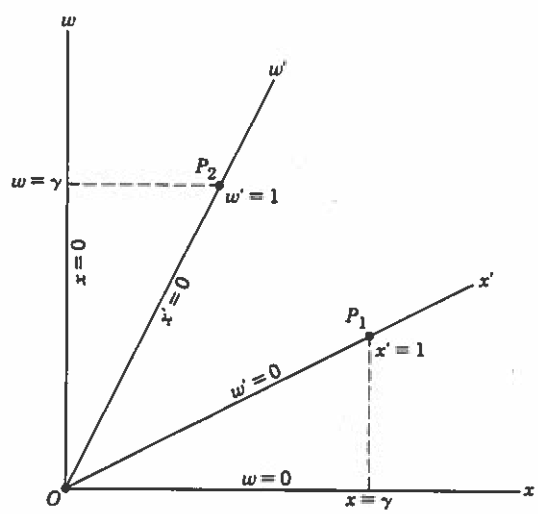

Figure A-3. Establishing the scales on the spacetime axes.

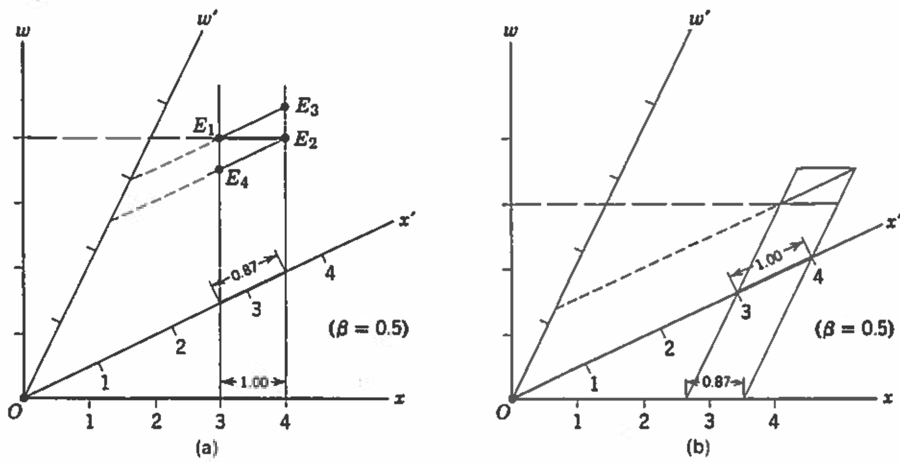

Figure A-7. Showing the space contraction, (*a*), and its reciprocal nature, (*b*).

The length of the stick is defined as the distance between the end points measured simultaneously. In S, the rest frame, the length is the distance in S between the intersections of the world lines with the x axis, or any line parallel to the x axis, for these intersecting points represent simultaneous events in S. The rest length is one meter. To get the length of the stick in S', where the stick moves, we must obtain the distance in S' between end points measured simultaneously. This will be the separation in S' of the intersections of the world lines with the x' axis, or any line parallel to the x' axis, for these intersecting points represent simultaneous events in S'. The length of the (moving) stick is clearly less than one meter in S' (see Fig. A-7*a*).

Notice how very clearly Fig. A-7*a* reveals that it is a disagreement about the simultaneity of events that leads to different measured lengths. Indeed, the two observers do not measure the same pair of events in determining the length of a body (for example, the S observer uses E_1 and E_2, say, whereas the S' observer would use E_1 and E_3, or E_2 and E_4) for events that are simultaneous to one inertial observer are *not* simultaneous to the other. We should also note that the x' coordinate of each end point decreases as time goes on (simply project from successive world-line points parallel to w' onto the x' axis), consistent with the fact that the stick that is at rest in S moves towards the left in S'.

The reciprocal nature of this result is shown in Fig. A-7*b*. Here, we have a meter stick at rest in S', and the world lines of its end points are parallel to w' (the end points are always at $x' = 3$ and $x' = 4$, say). The rest length is one meter. In S, where the stick moves to the right, the measured length is the distance in S between intersections of these world lines with the x axis, or any line parallel to the x axis. The length of the (moving) stick is clearly less than one meter in S (Fig. A-7*b*).

It remains now to demonstrate the time-dilation result geometrically. For this purpose consider Fig. A-8. Let a clock be at rest in frame S, ticking off units of time there. The solid vertical line in Fig. A-8, at $x = 2.3$, is the world line corresponding to such a single clock. T_1 and T_2 are the events of ticking at $w\,(= ct) = 2$ and $w\,(= ct) = 3$, the time interval in S between ticks being unity. In S', this clock is moving to the left so that it is at a different place there each time it ticks. To measure the time interval between events T_1 and T_2 in S', we use two different clocks, one at the location of event T_1 and the other at the location of event T_2.

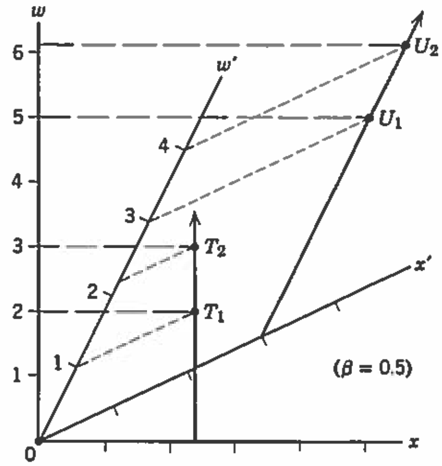

Figure A-8. Showing time dilation.

The difference in reading of these clocks in S' is the difference in times between T_1 and T_2 as measured in S'. From the graph, we see that this interval is greater than unity. Hence, from the point of view of S', the moving S clock appears slowed down. During the interval that the S clock registered unit time, the S' clock registered a time greater than one unit.

The reciprocal nature of the time-dilation result is also shown in Fig. A-8. You should construct the detailed argument. Here a clock at rest in S' emits ticks U_1 and U_2 separated by unit proper time. As measured in S, the corresponding time interval exceeds one unit.

A-4 THE TIME ORDER AND SPACE SEPARATION OF EVENTS

We can also use the geometric representation of spacetime to gain further insight into the concepts of simultaneity and the time order of events that we discussed in Chapter 2. Consider the shaded area in Fig. A-9, for example. Through any point P in this shaded area, bounded by the world lines of light waves, we can draw a w' axis from the origin; that is, we can find an inertial frame S' in which the events O and P occur at the same place ($x' = 0$) and are separated only in time.* As shown in Fig. A-9, event P follows event O in time (it comes later on S' clocks), as is true wherever event P is in the upper half of the shaded area. Hence, events in the upper half (region 1 on Fig. A-10) are absolutely in the future relative to O, and this region is called the Absolute Future. If event P is at a spacetime point in the lower half of the shaded area (region 2 on Fig. A-10), then P will precede event O in time. Events in the lower half are absolutely in the past relative to O, and this region is called the Absolute Past. In the shaded regions, therefore, there is a definite time order of events relative to O, for we can always find a frame in which O and P occur at the same place; a single clock will determine absolutely the time order of the event at this place.

* We cannot draw an x' axis through points such as P in Fig. A-9 because the angle ϕ in Fig. A-2 would then exceed 45°, which requires that $\beta > 1$ (or, equivalently, that $v > c$). For the same reason, we cannot draw a w' axis through points such as Q in Fig. A-9.

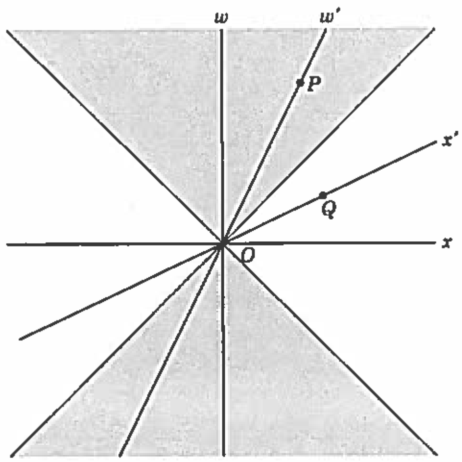

Figure A-9. The time order and space separation of events.

Consider now the unshaded regions of Fig. A-9. Through any point Q we can draw an x' axis from the origin; that is, we can find an inertial frame S' in which the events O and Q occur at the same time ($w' = ct' = 0$) and are separated only in space. We can always find an inertial frame in which events O and Q appear to be simultaneous for spacetime points Q that are in the unshaded regions (region 3 of Fig. A-10), so that this region is called the Present. In other inertial frames, of course, O and Q are not simultaneous, and there is no absolute time order of these events but a relative time order, instead.

If we ask about the space separation of events, rather than their time order, we see that events in the present are absolutely separated from O, whereas those in the absolute future or absolute past have no definite space order relative to O. Indeed, region 3 (present) is said to be "spacelike" whereas regions 1 and 2 (absolute past or future) are said to be "timelike." That is, a world interval such as OQ is spacelike and a world interval such as OP is timelike.

The geometric considerations that we have presented are connected with the invariant nature of the spacetime *interval*, described in Section 2.3. As presented there, the interval involves a pair of events. For our purposes we can choose as one universal member of this pair the standard reference event represented by point O in Fig. A-9. It corresponds to the coincidence in time of the

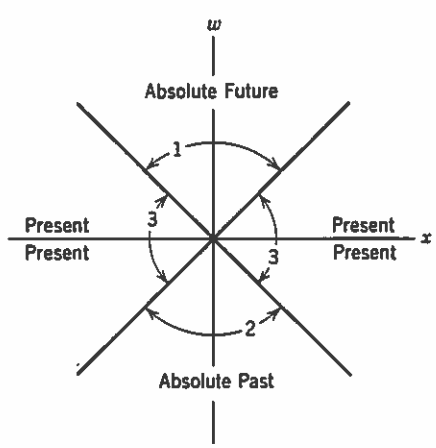

Figure A-10. Location in time of events relative to the origin.

origins of the two reference frames, S and S', and has the spacetime coordinates $x = w = 0$ and $x' = w' = 0$. The other member of the event pair can then be a generalized event represented by points such as P or Q in Fig. A-9. In this way we can associate the spacetime interval with P and Q alone, and can write (from Eq. 2-16, recalling that $w = ct$),

$$s^2 = w^2 - x^2 = w'^2 - x'^2. \tag{A-2}$$

We have seen that s^2, which has the same numerical value in all reference frames, can be either positive, negative, or zero, depending on the relative magnitudes of w and x (or of w' and x'). If $w > x$, as it is for points such as P in Fig. A-9, then s^2 is positive and s is a real quantity; we write it as $c\tau$, where τ is the *proper time interval* associated with the event pairs such as OP; see Eq. 2-17. If $w < x$, as it is for points such as Q in Fig. A-9, then $-s^2$ is a positive quantity; we call its square root σ, the *proper distance interval* for the event pairs such as OQ. We have then two relations,

$$c^2\tau^2 = w^2 - x^2 \tag{A-3a}$$

and

$$\sigma^2 = x^2 - w^2. \tag{A-3b}$$

Now consider Fig. A-10. In regions 1 and 2 we have spacetime points for which $w > x$, so the proper time is a real quantity, $c^2\tau^2$ being positive; see Eq. A-3a. In regions 3 we have spacetime points for which $x > w$, so the proper distance σ is a real quantity; see Eq. A-3b. Hence either τ or σ is real for any two events (that is, the event at the origin and the event elsewhere in spacetime) and either τ or σ may be called the spacetime interval between the two events. When τ is real the interval is called "timelike"; when σ is real the interval is called "spacelike." Because σ and τ are invariant properties of two events, it does not depend at all on what inertial frame is used to specify the events whether the interval between them is spacelike or timelike.

In the spacelike region we can always find a frame S' in which the two events are simultaneous, so that σ can be thought of as the spatial interval between the events in that frame. (That is, $\sigma^2 = x^2 - w^2 = x'^2 - w'^2$. But $w' = 0$ in S', so $\sigma = x'$.) In the timelike region we can always find a frame S' in which the two events occur at the same place, so that τ can be thought of as the time interval between the events in that frame. [That is, $\tau^2 = t^2 - (x^2/c^2) = t'^2 - (x'^2/c^2)$. But $x' = 0$ in S', so $\tau = t'$.]

What can we say about points on the 45° lines? For such points, $x = w$. Therefore, the proper time interval between two events on these lines vanishes, for $c^2\tau^2 = w^2 - x^2 = 0$ if $x = w$. We have seen that such lines represent the world lines of light rays and give the limiting velocity ($v = c$) of relativity. On one side of these 45° lines (shaded regions in Fig. A-9), the proper time interval is real; on the other side (unshaded regions), it is imaginary. An imaginary value of τ would correspond to a velocity in excess of c. But no signals can travel faster than c. All this is relevant to an interesting question that can be posed about the unshaded regions.

In this region, which we have called the Present, there is no absolute time order of events; event O may precede event Q in one frame but follow event Q in another frame. What does this do to our deep-seated notions of cause and effect? Does relativity theory negate the causality principle? To test cause and effect, we would have to examine the events at the same place so that we could say absolutely that Q followed O, or that O followed Q, in each instance. But in the Present, or spacelike, region these two events occur in such rapid succession that the time difference is less than the time needed by a light ray to traverse the spatial distance between two events. We cannot fix the time order of such events absolutely, for no signal can travel from one event to the other faster than c. In other words, no frame of reference exists with respect to which the two events occur at the same place; thus, we simply cannot test causality for such events even in principle. Therefore, there is no violation of the law of causality implied by the relative time order of O and events in the spacelike region. We can arrive at this same result by an argument other than this operational one. If the two events, O and Q, are related causally, then they must be capable of interacting physically. But no physical signal can travel faster than c, so events O and Q cannot interact physically. Hence, their time order is immaterial, for they cannot be related causally. Events that can interact physically with O are in regions other than the Present. For such events, O and P, relativity gives an unambiguous time order. Therefore, relativity is completely consistent with the causality principle.

questions and problems

1. Interpreting events on a spacetime diagram (I). Draw a spacetime diagram and on it locate an event P whose coordinates are $x = 450$ m and $t = 1.00\ \mu s$ ($w = ct = 300$ m). With respect to the standard reference event O at the origin, (*a*) does P represent an event in the future? The present? The past? (*b*) Is the interval OP spacelike? Timelike? Lightlike? (*c*) What proper time interval is associated with OP? (*d*) What proper space interval? (*e*) Can you find another frame S' for which the events OP would occur at the same time? If so, draw the spacetime axes of that frame on your diagram and give the speed parameter β of the frame. (*f*) Can you find another frame S'' in which the events OP would occur at the same place? If so, draw the spacetime axes of *that* frame on your diagram and give *its* speed parameter β.

2. Interpreting events on a spacetime diagram (II). Solve Problem 1 for an event whose coordinates are $x = 450$ m and $t = 1.50\ \mu s$, and for an event whose coordinates are $x = 450$ m and $t = 1.00\ \mu s$. Plot both events on the spacetime diagram of Fig. A-10 and compare.

3. Present or Absolute Future? Consider two events, both of which have $x = 1$ in, say, Fig. A-4. Event A has $w = 0.9$ and event B has $w = 1.1$. Comparison with Fig. A-10 shows that, with respect to the reference event O at the origin, the first of these events would be classified as Absolute Future and the second as Present. However, they both seem to occur in the future from the point of view of observer S. If Present means "right now," then neither of these events seems to qualify. It also seems clear that these two events could differ as slightly as you please; one, for example, could have $w = 0.99999$ and the other $w = 1.00001$, and a fundamental difference would still remain between them. Can you identify and clarify this difference?

★4. Calibrating the axes. (*a*) Let d represent the length (measured with a ruler) of a unit interval on the x (or the w) axis of Fig. A-4 and let d' represent the corresponding quantity for the x' (or the w') axis of that figure. Show that

$$\frac{d'}{d} = \frac{\gamma}{\cos\phi} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}.$$

(*b*) Evaluate this ratio for $\beta = 0.50$ (for which Fig. A-4 is drawn) and verify (using a ruler) that the axes in that figure are calibrated correctly.

5. Changing the scales of a spacetime diagram. (*a*) Redraw the spacetime diagram of Fig. A-4 but, in place of the dimensionless scale factor of unity, take 200 m as the unit scale distance. On this diagram, locate an event P whose spacetime coordinates, as determined by observer S, are $x = 800$ m and $t = 1.00\ \mu s$. Determine the spacetime coordinates of this event as determined by observer S' (*b*) directly from your diagram and (*c*) using the Lorentz transformation equations.

6. Learning about the time axes. In the spacetime (or Minkowski) diagram of Fig. A-2, the w axis, from the point of view of observer S, represents the world line of a particle resting at the origin of the S frame. Identify on the diagram the world line (*a*) of the S origin from the point of view of S'; (*b*) of the S' origin from the point of view of S; (*c*) of the S' origin from the point of view of S'. (*d*) Write down the equations of all four of these world lines, in coordinates appropriate to the observer, using the Lorentz transformation equations (see Eqs. A-1) as needed.

7. Learning about the position axes. In the spacetime diagram of Fig. A-2, the x axis, from the point of view of observer S, is made up of points at each of which there is a clock, fixed in the S frame, that reads $w = 0$. (*a*) What times would observer S' read on these same S clocks? (*b*) Identify the locus of points each of which contains a clock fixed in the S' frame that reads $w' = 0$. (*c*) What times would the S observer read on those S' clocks?

8. S and S' watch a clock. In Fig. A-4, consider a clock at rest at the origin of the S' frame and consider an event corresponding to a reading of "3" (as seen by observer S') on that clock. What reading will observer S (who uses his own clocks) record for this event? Solve by direct measurement from the spacetime diagram and also by use of the Lorentz transformation equations. Recall that $\beta = 0.50$ for the conditions of Fig. A-4.

9. S and S' measure a rod. In Fig. A-4, consider a rod 2.00 units long at rest along the x' axis of the S' reference frame, one end of the rod being at the origin of that frame. Both observers, S and S', measure the length of the rod. What values do they find? Solve by direct measurement from the spacetime diagram and also by use of the Lorentz transformation equations. Show on the spacetime diagram the two events that are used by each observer in the measuring process.

10. You can't get there that fast. Let the departure of a plane from Boston be an event whose coordinates are $x = 0$ and $w = t = 0$. Let a second event be the arrival of that plane in Seattle. Plot these two events qualitatively on the spacetime diagram of Fig. A-10. (*a*) Can you find a second frame S' in which these two events are simultaneous? If so, describe that frame. (*b*) Can you find a frame in which these two events occur at the same place? If so, describe *that* frame. (*c*) Is the interval associated with these two events spacelike or timelike?

11. Three reference frames. A system S' moves to the right relative to S at a speed of $0.60c$ and another system S'' moves to the right relative to S at a speed $0.35c$. (*a*) Using the Minkowski diagram, find the velocity of S'' relative to frame S'. (*b*) Repeat, with S'' moving to the right at a speed of $0.50c$. (*Hint:* Construct lines of constant x' and t' on the diagram. Using this gridwork of lines, find the slope of the world line for $\beta = 0.35$ and for $\beta = 0.50$).

12. An event viewed from three reference frames. (*a*) Redraw the spacetime diagram of Fig. A-4 and superimpose on it a set of axes corresponding to frame S'', which is moving in the positive x direction with a speed $0.80c$. Calibrate the x''-w'' axes described in Problem 4. (*b*) Consider an event, represented by P in Fig. A-4, whose coordinates in S are $x = 3.0$ and $w = 2.5$. Find the coordinates of this event in frame S' and in frame S'', both directly from the diagram and by use of the Lorentz transformation equations.

13. A collision on a spacetime diagram. A particle of mass m is at rest at $x = 3$m along the x axis of a coordinate system. A second particle, of mass $2m$, passes through the origin at $t = 0$ and moves toward the first particle with a speed of 1 m/s. The two particles then undergo a head-on collision, both particles moving forward along the x axis after the collision. (*a*) Show that, according to classical physics, after the collision the initially moving particle moves forward with speed of $\frac{1}{3}$m/s and the initially resting particle does so with a speed of $\frac{4}{3}$m/s. (*b*) Draw the world lines for these particles on a spacetime diagram, for an interval encompassing the collision. (*c*) Draw on the same diagram the world line representing the motion of the center of mass of the colliding particles.

14. Using spacetime diagrams (I). Read again Problems 31 and 32 of Chapter 2. (*a*) Draw a world diagram for the problem, including on it the world lines for observers A, B, C, D, and E. Label the points AD, BD, AC, BC, and EC. (*b*) Show, by means of the diagram, that the clock at A records a shorter time interval for the events AD, AC than do clocks at D and C. (*c*) Show that, if observers on the cart try to measure the length DC by making simultaneous markings on a measuring stick in their frame, they will measure a length shorter than the rest length DC. Explain this result in terms of simultaneity, using the diagram. For convenience, take $v = \frac{1}{2}c$.

15. Using spacetime diagrams (II). Do Example 4, Chapter 2, by means of a Minkowski diagram. Check your results by calculating the invariants $c^2\tau^2$ or σ^2.

16. Using spacetime diagrams (III). Do Example 5, Chapter 2, by means of a Minkowski diagram. Check your results by calculating the invariants $c^2\tau^2$ or σ^2.

17. Calibration by hyperbolas. Equations A-3 represent the equations of hyperbolas, each of which has two branches. If we choose both σ and $c\tau$ equal to one scale unit on the x and w axes of a spacetime diagram, we can write, for the equations of these hyperbolas,

$$w^2 - x^2 = 1 \qquad \text{(timelike region)}$$

and

$$x^2 - w^2 = 1 \qquad \text{(spacelike region)}.$$

Figure A-11 shows these hyperbolas; they represent but one typical family of an infinite set of hyperbolas, corresponding to different choices for σ and for $c\tau$ in Eqs. A-3. Figure A-12 shows the upper right quadrant of Fig. A-11, with the spacetime axes of two reference frames S' ($\beta = 0.50$) and S'' ($\beta = 0.80$) drawn in.

(*a*) Prove that the hyperbola branches in Fig. A-11 approach the 45° lightlike lines asymptotically. (*b*) Sketch in roughly the curves that would correspond to a choice of $\sigma = 2$ in Eqs. A-3 and to a choice of $c\tau = 2$. (*c*) With respect to the standard reference event O, what proper time interval (in terms of w) can you assign to events such as I,

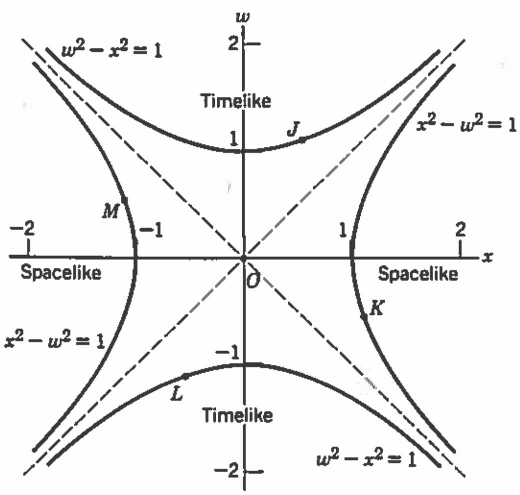

Figure A-11. Problem 17; calibration by hyperbolas.

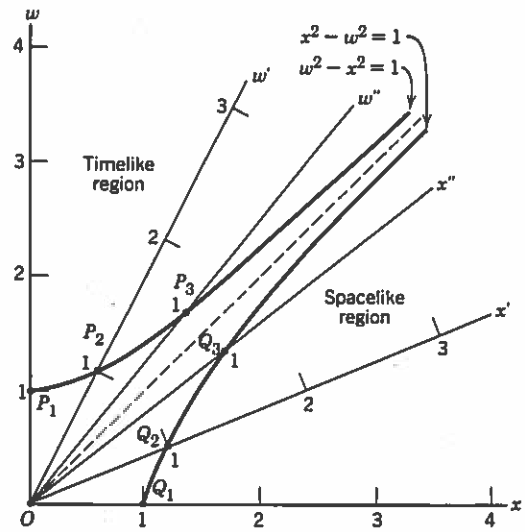

Figure A-12. Problem 17; the first quadrant in detail.

K, L, and M in Fig. A-11? What proper space interval? (*d*) In Fig. A-11, convince yourself that the point representing the intersection of the w axis and the upper branch of the timelike hyperbola corresponds to a clock at rest at the origin of the S frame and reading "1." Other points on this hyperbola branch read times greater than "1"; where are these clocks located? Are they fixed in the S frame, or are they moving? What does it mean when we say that, though one of these clocks may read, say, "1.5," its *proper* time is still "1"? (*e*) In Fig. A-12 we have seen that P_1 represents a clock at the origin of the S frame reading "1." Can you see that P_2 represents a clock resting at the origin of the S' frame and also reading "1"? What does P_3 represent? Can you see how the hyperbolas of Figs. A-11 and A-12 can be used to establish scales for the axes of a given reference frame? (The hyperbolas are often called "calibration curves," for this reason.) (*f*) Analyze the spacelike hyperbola branch shown in Fig. A-12 in physical terms, following the pattern outlined above for the timelike branch. In particular, to what do events Q_1, Q_2, and Q_3 correspond physically?

references

1. A. Einstein, H. Minkowski, H. A. Lorentz, and H. Weyl, *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity*, notes by A. Sommerfeld, Dover, New York, 1952.

2. Hermann Minkowski, "Space and Time" (a translation of an address given September 21, 1908), in *The Principle of Relativity* (Dover, New York).

SUPPLEMENTARY TOPIC B
the twin paradox

If we placed a living organism in a box . . . one could arrange that the organism, after any arbitrary lengthy flight, could be returned to its original spot in a scarcely altered condition, while corresponding organisms which had remained in their original positions had already long since given way to new generations. For the moving organism the lengthy time of the journey was a mere instant, provided the motion took place with approximately the speed of light.

Albert Einstein (1911)

In the above statement Einstein describes what has come to be called the twin paradox or the clock paradox [1]. If the stationary organism is a man and the traveling one is his twin, then the traveler returns home to find his twin brother much aged compared to himself. The paradox centers around the contention that, in relativity, either twin could regard the other as the traveler, in which case each should find the other younger—a logical contradiction. This contention assumes that the twins' situations are symmetrical and interchangeable, an assumption that is not correct. Furthermore, the accessible experiments have been done and support Einstein's prediction. In succeeding sections, we look with some care into the many aspects of this problem.

B-1 THE ELAPSED PROPER TIME DEPENDS ON THE ROUTE

Figure B-1*a* shows the world lines of three particles, their motions being confined to the x axis of an inertial reference frame. Particle 1 is at rest on the x axis, its world line being a simple vertical line. Particle 2 is moving along this axis in the direction of increasing x, its (constant) speed v being $dx/dt = c\,dx/(c\,dt) = c \tan \theta$, where θ is the angle made by the world line of this particle with the vertical. A reference frame moving with particle 2 would thus be an inertial frame. Particle 3 is in accelerated motion along the x axis, its speed v being, in general, different for every position of the particle. An inertial frame moving with *this* particle would *not* be an inertial frame.

Let us assume that particle 3 in Fig. B-1*a* is, in fact, a clock, and let us consider the problem of calculating the elapsed proper time on this clock as it travels from one point to another. We can assume that this traveling clock is in an inertial frame only for differential elements of its path. We can compute the elapsed proper time $d\tau$ for such an element from Eq. 2-17, which we write in differential form as

$$(c\,d\tau)^2 = (c\,dt)^2 - (dx)^2$$

or

$$d\tau = \sqrt{(dt)^2 - \left(\frac{dx}{c}\right)^2}. \qquad \text{(B-1)}$$

The total elapsed proper time between any two points would then be the integral of this quantity between those points.