# HW 10 Paul Lea

Problem 1 N9.3

The contribution to the anomaly by a given irreducible representation is determined by the trace of the product of a generator and the anti-commutator of two generators, namely, $\mathrm{Tr} T^A\{T^B T^C\}$ Here T A denotes a generator of the gauge group G. Show that for G = SU (5), the anomaly cancels between the $5^*$ and the 10. Note that for SU (N), we can, with no loss of generality, take A, B, and C to be the same, so that the anomaly is determined by the trace of a generator cubed (namely, $\mathrm{Tr} T^3$ ). It is rare that we get something cubed in physics, and so any cancellation between irreducible representations can hardly be accidental.

Easiest to use diagonal generator.

$$T = \mathrm{diag}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}\right)$$

Taking this diagonal generator, we evaluate the trace of the generator cubed in the 5* dimensional representation.

All entries are real: (multiply by 6)

$$T^* = \mathrm{diag}\,(2, 2, 2, -3, -3)$$

Cubing this matrix:

$$T^{*3} = \mathrm{diag}\,(8, 8, 8, -27, -27)$$

Taking the trace:

$$\mathrm{Tr}(T^{*3}) = -30$$

Now for the 10-dimensional representation:

Basis for 10-d representation:

$$(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)$$

Converting the 5-d diagonal matrix to a 10-d representation

$$T_{10\text{ d}} = \mathrm{diag}(d_1 + d_2, d_1 + d_3, d_1 + d_4, d_1 + d_5, d_2 + d_3, d_2 + d_4, d_2 + d_5, d_3 + d_4, d_3 + d_5, d_4 + d_5)$$
$$T_{10\text{ d}} = \mathrm{diag}(4, 4, -1, -1, 4, -1, -1, -1, -1, -6)$$

Cubing this diagonal matrix:

$$T^3_{10\text{ d}} = \mathrm{diag}(64, 64, -1, -1, 64, -1, -1, -1, -1, -216)$$

Taking the trace:

$$\mathrm{Tr} T^3_{10\text{ d}} = 3(64) - 222 = 192 - 222 = -30$$

$$30 - (-30) = 0$$

Problem 3 N9.3
Work out how the 3-indexed antisymmetric 10 dimensional tensor in SO(10) decomposes on restriction to $SO(4) \otimes SO(6)$.

- 3 indexed antisymmetric 10-d tensor has 120 elements.
- Symmetric tensor rep of SO(6) has 20 elements $\frac{1}{2}(6)(7) - 1$
- Adjoint rep of SO(6) has 15 elements $\frac{1}{2}(6)(5)$
- The vector representation of SO(6) has 6 (obviously)
- Trivial representation has 1
- The vector representation of SO(4) has 4 elements
- The trivial representation has 1 element
- Tensor product representations (3,1) and (1,3)

Decomposing
Using the following equation (credit Nate Laposky and Hannah Turner)

$$\Lambda^3(V \oplus W) = \Lambda^3(V) \oplus \left(\Lambda^2(V) \otimes W\right) \oplus \left(V \otimes \Lambda^2(W)\right) \oplus \Lambda^3(W)$$

We can plug in our representations:

$$\Lambda^3(10) = \Lambda^3(4,1) \oplus \left(\Lambda^2(4,1) \otimes (1,6)\right) \oplus \left((4,1) \otimes \Lambda^2(1,6)\right) \oplus \Lambda^3(1,6)$$

And from this we can start pulling out our representations:
We have a (4,1) representation from $\Lambda^3(4,1)$

From $\left(\Lambda^2(4,1) \otimes (1,6)\right)$ we get

$$(6,6)$$

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$$\mathit{\Lambda}^2(6) \cong 15 \to (4,15)$$

From $\Lambda^3(1,6)$ we get the 20 dimensional representation of $SO(6)$

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Taking this collection of representations, we can

$$120 \to (20,1) \oplus (15,4) \oplus (6,6) \oplus (1,4)$$

$$\to (20,1) \oplus (15,4) \oplus (6,3) \oplus (6,3) \oplus (1,4)$$