1. N. II.3 Exercise 4
2. N. II.4 Exercise 1, 4, 5, 7, 8

Paul Lea HW 3

N II.3 #4

Complete D_4 table $\xrightarrow{r}$

D_4	n_c		1	1'	1''	1'''	2
	1	I	1	1	1	1	2
Z_2	1	$-I$	1	1	1	1	-2
Z_4	2	R, R^3	1	a	d	g	0
Z_2	2	r_x, r_y	1	b	e	h	0
Z_2	2	d_1, d_2	1	c	f	i	0

$c \downarrow$

Using column orthogonality: $\sum_c n_c (\chi^{(r)}(c))^* \chi^{(s)}(c) = N(G)\delta^{rs}$

$1^2 + 1^2 + 2a^2 + 2b^2 + 2c^2 = 8$

$2(a^2+b^2+c^2) = 6$

$a^2+b^2+c^2 = 3$

$a, b \,\&\, c = \pm 1$

$1+1+2a+2b+2c = 0$

$\frac{2(a+b+c)}{2} = \frac{-2}{2}$

$a+b+c = -1$

one $+1$, two -1

possibilities: $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Answer:

D_4	n_c		1	$1'$	$1''$	$1'''$	2
	1	I	1	1	1	1	2
Z_2	1	$-I$	1	1	1	1	-2
Z_4	2	R, R^3	1	1	1	-1	0
Z_2	2	r_x, r_y	1	1	-1	1	0
Z_2	2	d_1, d_2	1	-1	1	1	0

~ Credit: Himani M

N 11.4 #1

We know that $D(g)^*$ is a representation

$$D(g_1)^* D(g_2)^* = D(g_1 g_2)^*$$

but if applying the transpose operation, the order is reversed in distribution

$$D(g_1)^T D(g_2)^T = D(g_2 g_1)^T \neq D(g_1 g_2)^T$$

Thus the transpose of a representation cannot form a representation.

#4 3 of A_4

Reality Checker

$$\sum_{g \in G} \chi^{(r)}(g^2) = \eta^{(r)} N(G)$$

3 of A_4

A_4	n_c		...	3
	1	I	...	3
Z_2	3	$(12)(34)$		-1
Z_3	4	(123)		0
Z_3	4	(132)		0

$\chi(I) \Rightarrow 3 \cdot n_c = 3$

$((12)(34))^2 = I, \ \chi(I) = 3 \cdot n_c = 9$

$\chi([(12)(34)]^2) = 3$

$\chi([123]^2) =$

$$\downarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = [3\,2\,1]$$

$\& \ \chi[321] = 0$

$$\chi((132)^2) = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$9 + 3 + 0 + 0 = 12$$

$$N(g) = 12$$

$\therefore \eta^{(r)} = 1$ & it is real

#5 Check the 3 of S_4

S_4	n_c			3
	1	I	...	3
Z_2	3	$(12)(34)$		-1
Z_3	8	(123)		0
Z_2	6	(12)		1
Z_4	6	(1234)		-1

$\chi^3(I^2) = \chi^3(I) = \underline{3}$

$\chi^3([(12)(34)]^2) = \chi^3(I) = \underline{3}$

$\chi^3((123)^2) = \underline{0}$ from #4

$\chi^3((12)^2) = \chi^3(I) = \underline{3}$

$\chi^3((1234)^2) = \chi^3((1234)) = \underline{-1}$

$$\sum_{g \in G} \chi^{(r)}(g^2) = \eta^{(r)} N(G)$$

$= 3\cdot 1 + 3\cdot 3 + 0\cdot 8 + 6\cdot 3 + 6\cdot(-1)$

$3 + 9 + 18 - 6 = 24$

$24 = \eta\, 24$

$\underline{\eta = 1}$ This is a real representation.

#7 $\sigma_f = \sum_r \eta^r \chi^r(f)$

A_4	n_C		1	1'	1''	3
	1	I	1	1	1	3
Z_2	3	(12)(34)	1	1	1	-1
Z_3	4	(123)	1	ω	ω^*	0
Z_3	4	(132)	1	ω^*	ω	0

$$\sum_{g\in G} \chi^{(r)}(g^2) = \eta^{(r)} N(G)$$

$$\eta^r = \frac{1}{N(G)} \sum_{g\in G} \chi^{(r)}(g^2)$$

A_4 of 3 : $\eta^r = 1$

A_4 of 1'' :

A_4 of $1''$:

$$4\chi((123)^2) = 4\chi(132) = 4\omega$$
$$4\chi((132)^2) = 4\chi(123) = 4\omega^*$$
$$3\chi([(12)(34)]^2) = 3\chi(I) = 3$$
$$\chi(I^2) = 1$$

$$4(\omega + \omega^*) + 3 + 1$$
$$-4 + 3 + 1 = 0$$
$$\eta^{1''} = \frac{0}{12} = 0$$

A_4 of $1'$:

$$4\chi((123)^2) = 4\chi(132) = 4\omega^*$$
$$4\chi((132)^2) = 4\chi(123) = 4\omega$$
$$3\chi([(12)(34)]^2) = 3\chi(I) = 3$$
$$\chi(I^2) = 1$$

$$4(\omega^* + \omega) + 3 + 1$$
$$-4 + 3 + 1 = 0$$
$$\eta^{1'} = \frac{0}{12} = 0$$

A_4 of 1

$$4\chi((123)^2) = 4\chi(132) = 4$$
$$4\chi((132)^2) = 4\chi(123) = 4$$
$$3\chi([(12)(34)]^2) = 3\chi(I) = 3$$
$$\chi(I^2) = 1$$

$$12 = \eta^1 \, 12$$
$$\eta = 1$$

$$\chi^r(123)$$
$$= 1, \omega, \omega^*, 0$$

$$\sum_r \eta^r \chi^r(123)$$
$$= 1\cdot 1 + \cancel{\omega\cdot 0} + \cancel{\omega^*\cdot 0} + \cancel{0\cdot 1}$$
$$= 1$$

(123) has $\underline{1}$ square root

8 S_4 $\quad \sigma_f = \sum_r \eta^r \chi^r(f)$

S_4	n_c		1	$\bar{1}$	2	3	$\bar{3}$
	1	E	1	1	2	3	3
Z_2	3	$(12)(34)$	1	1	2	-1	-1
Z_3	8	(123)	1	1	-1	0	0
Z_2	6	(12)	1	-1	0	1	-1
Z_4	6	(1234)	1	-1	0	-1	1
$\eta =$			1	1	1	1	1

$\sigma_{(12)(34)}$

$1+1+4-3-3=0$

$\sigma_{(123)}$

$1+1-1=1$

$\sigma_{(12)}$

$1-1+0+1-1=0$

$\sigma_{(1234)}$

$1-1+0-1+1=0$

We can see $\eta_3 \equiv \eta_{\bar{3}}$, n_c is same & last 2 rows are switched.

for η_2 $\quad \chi(g^2)$

E	2	1	2
$(12)(34)$	2	3	6
(123)	-1	8	-8
(12)	2	6	12
(1234)	2	6	12

$= 24 \to \eta = 1$

for η_3

	$\chi(g^2)$	
I	1	1
$(12)(34)$	1	3
(123)	1	8
(12)	1	6
(1234)	1	6

$\to = 24 \to \eta = 1$