Rohlf

CHAPTER 4

SPECIAL RELATIVITY

It is not good to introduce the concept of the mass $M = m/(1-v^2/c^2)^{1/2}$ of a moving body for which no clear definition can be given. It is better to introduce no other mass concept than the "rest mass" m. Instead of introducing M it is better to mention the expression for the momentum and energy of a body in motion.

Albert Einstein

NOTE: KEY MATERIAL IS EITHER BRACKETED [OR HAS A LARGE ASTERISK *

Electromagnetic waves are observed to propagate in a vacuum at the speed

$$c \approx 3.00 \times 10^8 \text{ m/s}. \tag{4.1}$$

The speed c is a universal constant, that is, c does not depend on the wavelength. The constant c is called the *speed of light*. We have seen from the photoelectric effect that electromagnetic radiation is quantized. Electromagnetic waves have a particle behavior. The particles of radiation are photons. The energy of an individual photon is directly proportional to the frequency of the wave, but all photons have the same speed (c) when moving in a vacuum, independent of their energies.

The speed (v) of any macroscopic object of mass m found on the earth is much smaller than the speed of light, $v \ll c$. The classical definitions of momentum ($\mathbf{p}$),

$$\mathbf{p} \equiv m\mathbf{v}, \tag{4.2}$$

and kinetic energy (E_k),

$$E_k \equiv \frac{1}{2}mv^2 = \frac{p^2}{2m}, \tag{4.3}$$

were established empirically as the quantities that are conserved in all collisions. When a particle has a speed approaching c, the momentum (4.2) and kinetic energy (4.3) are no longer conserved quantities in particle collisions. We shall need a revised description for particles that have very large speeds and large kinetic energies. This description is contained in the theory of special relativity.

4-1 FOUNDATIONS OF SPECIAL RELATIVITY

The Postulates

The theory of special relativity was first formulated by Einstein in 1905, the same year he published his famous papers on Brownian motion and the photoelectric effect. The two postulates of special relativity are as follows:

1. The laws of physics are identical in all inertial frames of reference.
2. The speed of electromagnetic radiation in vacuum is constant, independent of any motion of the source.

The first postulate is identical to that used by Newton in his formulation of the laws of classical physics. The definition of an inertial frame is a reference frame where a particle has no acceleration unless there is a force acting on the particle. The second postulate may at first sight seem counterintuitive; it is our common experience at low speeds, $v \ll c$, that speeds add linearly.

Consider a particle moving with speed (dx/dt) in the x direction as indicated in Figure 4-1. In a frame that is moving with a speed v in the x direction, the coordinates are

$$t' = t, \tag{4.4}$$

$$x' = x - vt, \tag{4.5}$$

$$y' = y, \tag{4.6}$$

and

$$z' = z. \tag{4.7}$$

This coordinate transformation is called the *Galilean transformation*. Since $dt = dt'$, differentiation of the Galilean transformation of the x coordinate (4.5) with respect to t leads to the velocity addition rule:

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v. \tag{4.8}$$

A headwind of speed v decreases the speed of an airplane with respect to the earth's surface by an amount v. The Galilean transformation works provided that the speeds of the objects are small compared to the speed of light.

The Galilean transformation does not work for very large speeds. You will need to develop a relativistic

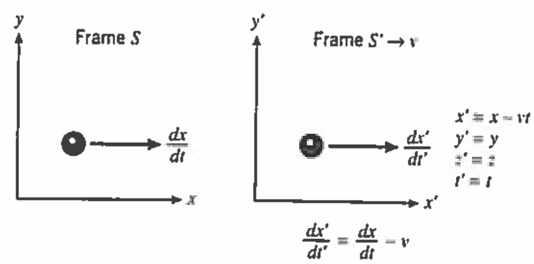


FIGURE 4-1 The Galilean transformation.
Consider the motion of a particle as observed in two frames of reference. In frame S, the particle has a speed dx/dt in the x direction. Frame S' is defined to be a frame that is moving with a speed v in the x direction. The speed of the particle in S' is $dx'/dt' = dx/dt - v$, provided that the coordinates are related by the transformation: $x' = x - vt$, $y' = y$, $z' = z$, and $t' = t$. This transformation is called the Galilean transformation. It holds true for speeds much less than the speed of light.

physics intuition! Consider a light source at rest. Light emitted by the source propagates at the speed c. Now consider a frame that is moving with a speed v in the x direction. In this frame, the light source has a speed v. The Galilean transformation predicts that the speed of light waves moving in the minus x direction in the moving frame is equal to $v+c$. The Galilean transformation is not correct. The speed of the light wave in the moving frame is unchanged (c). This is true even if the speed of the reference frame (v) is very large. The relative speeds of stars in distant galaxies with respect to the earth is an appreciable fraction of the speed of light, but the light from all stars propagates to the earth at a speed c. The speed of electromagnetic radiation is a universal constant.

The Michelson-Morley Experiment

Since sound waves need a medium for propagation, it was natural to suspect that electromagnetic waves might also need a medium for propagation. The hypothetical medium was called the *ether*. Albert Michelson designed an experiment in 1881 that was sensitive to the possible presence of an ether. This experiment was repeated by Michelson and Edward Morley with greater sensitivity in 1887. If the propagation of light waves depended on the ether, then motion with respect to the ether would necessarily have an effect on the speed of light. Search for the existence of an ether is equivalent to testing the second postulate of special relativity. The objective of the Michelson-Morley experiment was to see if the motion of the earth could affect the speed of light. The speed of the earth in its orbit about the sun is 3×10^4 m/s, or

$$\frac{v}{c} \approx \frac{3\times10^4 \text{ m/s}}{3\times10^8 \text{ m/s}} = 10^{-4}. \tag{4.9}$$

If the Galilean transformation was valid, then the speed of light from a source that was directed parallel to the motion of the earth would be increased by one ten-thousandth, while speed of light from a source that was directed antiparallel to the motion of the earth would be decreased by one ten-thousandth.

The Michelson-Morley experiment is shown in Figure 4-2. The waves from a light source are directed onto a partially transmitting mirror. The mirror divides the light into two beams that travel at right angles to each other. Each light beam travels a distance of about 1 m and is reflected by a mirror. After multiple reflections, the light beams merge. Suppose for a moment that the total path lengths for the two light beams were *exactly* equal. According to the theory of special relativity, the light beams arrive at precisely the same time. On the other hand, if the Galilean transformation were to hold true, the speed of light would not be constant along the two paths because of the motion of the earth, and the two light beams would not arrive at the same time. In practice, the two path lengths can never be made *exactly* equal. This means that an interference pattern is observed arising from the unequal path lengths. The light from the two paths arrives out of phase. The experimenters looked for a *change* in the interference pattern when the apparatus was rotated because the rotation causes a change in the direction of travel of the light beams with respect to the direction of motion of the earth. The rotation was accomplished by mounting the entire apparatus on a massive stone that was floated on mercury. The apparatus invented by Michelson is called an *interferometer*. Michelson used interferometers to measure fine structure of spectral lines in 1891 and to compare wavelengths with the standard meter in 1895.

According to the theory of special relativity, the speed of the light beams remains constant and the interference pattern does not change when the apparatus is rotated through 90°.

We now analyze the experiment using the Galilean transformation velocity addition rule (4.8). This analysis will tell us the sensitivity for an experiment to distinguish between the Galilean transformation and the second postulate of special relativity. Consider the case where the light beam is traveling parallel and antiparallel to the motion of the earth. The time (t_1) taken to reach the interferometer is

$$t_1 = \frac{L_1}{c+v} + \frac{L_1}{c-v}, \tag{4.10}$$

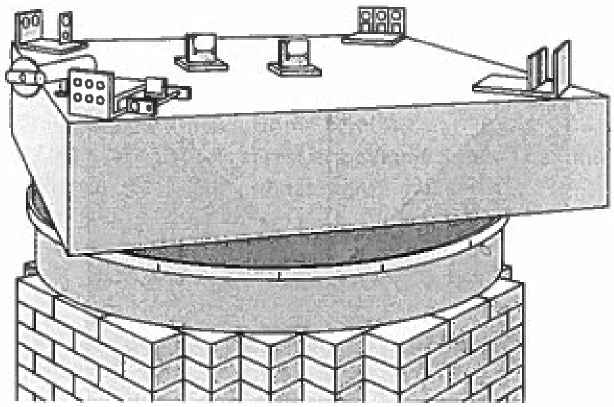
where $2L_1$ is the path length. For the case where the light beam travels perpendicular to the motion of the earth, the time (t_2) taken to reach the interferometer is

$$t_2 = \frac{L_2}{\sqrt{c^2 - v^2}} + \frac{L_2}{\sqrt{c^2 - v^2}}. \tag{4.11}$$

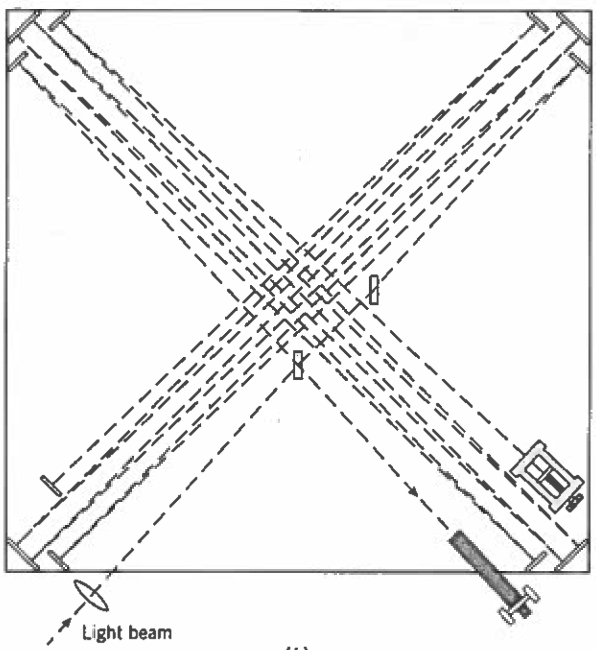
where $2L_2$ is the path length. The light beams are out of phase (ϕ) at the interferometer by

$$\phi = \frac{\left|c(t_2 - t_1)\right|}{\lambda}. \tag{4.12}$$

FIGURE 4-2 The Michelson-Morley experiment. (*a*) Sketch of the apparatus. (*b*) Path of the light. (*c*) Support system designed for rotation of the apparatus. After A. A. Michelson and E. M. Morley, *Phil. Mag.* **190**, 449 (1887).



(a)



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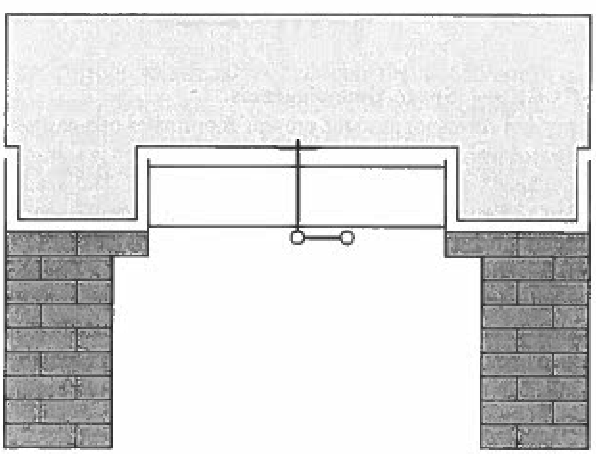


(c)

When the apparatus is rotated by 90°, the change in the phase or *fringe shift* is given by

$$\Delta\phi = \frac{2Lv^2}{\lambda c^2}, \tag{4.13}$$

where L is the approximate value of the path length, $L = L_1 \approx L_2$ (see problem 4). In the Michelson-Morley experiment $L \approx 10$ m. The wavelength of light is about 500 nm. The expected fringe shift is

$$\Delta\phi = \frac{(2)(10\,\mathrm{m})(10^{-4})^2}{5\times10^{-7}\,\mathrm{m}} \approx 0.4 \tag{4.14}$$

if the light beams have traveled at different speeds because of the earth's motion. No fringe shift was observed. The speed of light does not depend on the motion of the earth. The speed of light is an absolute constant, and there is no ether. Electromagnetic waves can propagate in a vacuum!

The Michelson-Morley experiment has been refined and repeated many times. Several of these results from the period 1881–1930 are summarized in Figure 4-3. On the vertical axis we plot the observed fringe shift and on the horizontal axis we plot the expected fringe shift as calculated from the Galilean transformation. If the speed of light depended on the motion of the earth, then the data would be expected to follow the 45° line. If the speed of light is constant, then zero fringe shift is expected. In practice a small fringe shift (much smaller than that predicted by the Galilean transformation) is observed due to the finite precision of the experimental apparatus. The

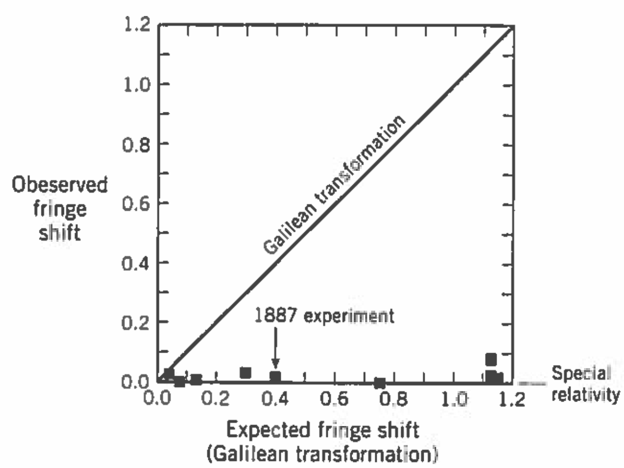


FIGURE 4-3 Results of the Michelson-Morley experiment.
The data are summarized in M. A. Handschy, *Am. J. Phys.* 50, 987 (1982).

main source of such a tiny fringe shift is a small change in the path length due to temperature variation.

The speed of light in vacuum (c) has been measured to great precision by measurement of the wavelength and frequency of light waves. (This will be discussed further in Chapter 13). In 1983, a new definition of the meter was adopted so as not to limit the accuracy of c. The new definition of the meter is the distance that light travels in vacuum in 1/299792458 second. By definition of the meter, the speed of light is *exactly*

$$c = 2.99792458 \times 10^8 \text{ m/s}. \tag{4.15}$$

Although the speed of light is very large, it is not infinite. It takes a photon a whole nanosecond to travel a distance of 0.3 m, roughly the distance from the page of this book to your eye. It takes 500 seconds for light to travel from the sun to the earth and about 15 billion years for light to travel from the furthest galaxy to the earth!

So far we have been discussing the speed of light in empty space. A medium that is transparent to light, such as air or glass, is characterized by an index of refraction (n). The speed of light is smaller in the medium than in a vacuum by a factor of n. Usually we may neglect this effect in air, where the index of refraction is 1.0003.

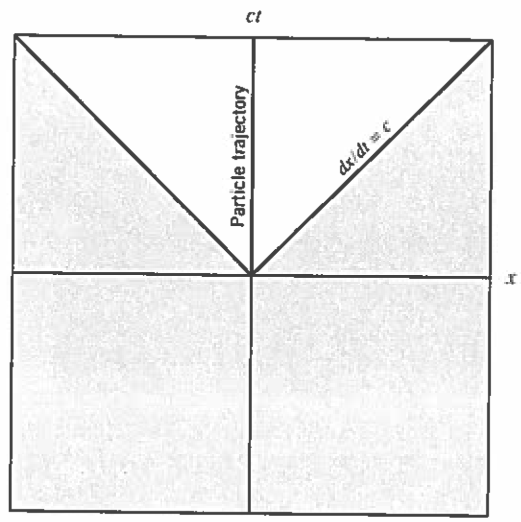
> The speed of light (in vacuum) is the same in all frames of reference:
>
> $$c = 2.99792458 \times 10^8 \text{ m/s}.$$

4-2 RELATIONSHIP BETWEEN SPACE AND TIME

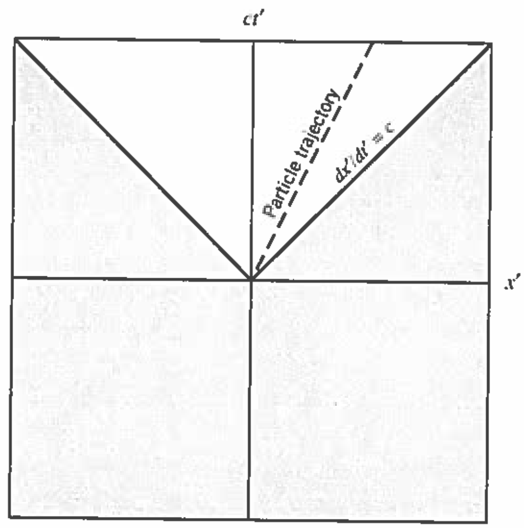
Since the definition of speed is the derivative of the space coordinate with respect to time, we see that the second postulate of relativity must have a profound implication on the relationship between space and time. Figure 4-4a shows a plot of the time coordinate of a particle times the speed of light (ct) versus the x coordinate of the particle. This type of plot is called a *space–time diagram*. At $ct = 0$, the present time, the position of the particle is at $x = 0$. The space–time coordinates of the particle may be written as

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.16}$$

Uniform motion corresponds to a straight line in Figure 4-4a with a slope of (c/v). Since it is impossible for the



(a) Frame S



(b) Frame S' $v \leftarrow$

FIGURE 4-4 Space–time diagrams.
The origin corresponds to the present coordinates of a particle. The trajectory of a particle moving at constant speed is a straight line. The 45 degree line represents the speed c. The shaded region is forbidden; the particle can never have those space–time coordinates. (a) In frame S the particle is at rest. The particle trajectory is along the vertical axis. (b) In a frame S' moving with relative speed v in the negative x direction, the particle has a speed v. The particle trajectory is indicated by the dashed line. Both the time and space coordinates are necessarily changed in this frame; otherwise a large speed would put the particle into the forbidden region.

particle to travel faster than c, the minimum slope is unity. The region below the 45° line corresponds to $v > c$. The particle can never have these space–time coordinates. (You can't get "there" from "here!")

Consider a particle at rest in the frame S. At a later time (t_1), its space–time coordinates are

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} ct_1 \\ 0 \end{pmatrix}. \tag{4.17}$$

The trajectory of the particle in the space–time diagram is along the vertical axis. The time interval t_1 has passed and the particle has not moved from the position $x = 0$. Now consider the analysis of the particle in the frame S', which is moving with a speed v in the negative x direction. The space–time diagram is shown in Figure 4-4*b*. The space–time coordinates in this frame are denoted by

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} ct_1' \\ x_1' \end{pmatrix}. \tag{4.18}$$

Both time and the space coordinates of the particle have increased ($t_1' > t_1$ and $x_1' > 0$). In the frame S' the particle has a speed v and follows the trajectory indicated by the dashed line. The particle speed is

$$v = \frac{x_1'}{t_1'}. \tag{4.19}$$

The time coordinate must increase because if it did not, a large relative speed would put the particle coordinates into the forbidden region. The Galilean transformation is incorrect at large speeds, in part, because it is assumed that the time coordinates in the two frames are identical. Of course, for small speeds, they are very nearly identical.

The Lorentz Transformation

The correct relationship between the coordinates of a stationary frame (t, x, y, and z) and the coordinates in a frame that is moving at a constant speed (t', x', y', and z') was discovered by Hendrik Lorentz in 1890. Lorentz found that a certain transformation of the space and time coordinates left the form of Maxwell's equations unchanged. This transformation is called the *Lorentz transformation*. Maxwell's equations are *invariant* under the Lorentz transformation, that is, Maxwell's equations have the same form in all inertial frames of reference. This is true even for very large speeds, as long as $v < c$. *How could it be that Maxwell's equations in their original form were relativistically correct, before the discovery of relativity?* The reason is that the magnetic force on a charge moving with $v << c$ is so small compared to the electric force, that one needs to include effects of size (v/c) to account for it. This is why the speed of light appears in Ampère's law (see Appendix B).

The Lorentz transformation relating the coordinates t, x, y, and z in a stationary frame to the coordinates t', x', y', and z' in a frame moving with a speed v in the x direction has the following properties:

1. The transformation is a linear function of x and t.
2. The transformation does not change the y and z coordinates.
3. The transformation does not affect the speed of a light wave.
4. Making a second transformation with a speed v in the minus x direction gives back the original space–time coordinates.
5. The transformation reduces to the Galilean transformation at small speeds ($v << c$).

The space–time coordinate transformation that has these properties is

$$t' = \gamma\left(t - \frac{xv}{c^2}\right), \tag{4.20}$$

$$x' = \gamma(x - vt), \tag{4.21}$$

$$y' = y, \tag{4.22}$$

$$z' = z, \tag{4.23}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4.24}$$

This is the famous Lorentz transformation. For small speeds ($v << c$), we have $\gamma = 1$ and the Lorentz transformation reduces to the Galilean transformation. The form of the Lorentz transformation is a direct consequence of the fact that the speed of light is the same in all frames of reference.

We may solve the above equations for t, x, y, and z to get the inverse transformation. We note, however, that the inverse transformation may be obtained by merely exchanging the primed and unprimed variables and also changing the sign of v. The inverse transformation is

$$t = \gamma\left(t' + \frac{x'v}{c^2}\right), \tag{4.25}$$

$$x = \gamma(x' + vt'), \tag{4.26}$$

$$y = y', \tag{4.27}$$

and

$$z = z'. \tag{4.28}$$

EXAMPLE 4-1

Show that if we transform first in the x direction and then in the minus x direction, with the same speed (v), we end up with the original space–time coordinates.

SOLUTION:

The first transformation gives

$$t' = \gamma\left(t - \frac{xv}{c^2}\right),$$

and

$$x' = \gamma(x - vt)$$

with y and z unchanged. Now let the coordinates ct'', x'', y'', and z'' be the result of the second transformation. The time part is

$$\begin{aligned} t'' &= \gamma\left(t' + \frac{x'v}{c^2}\right) \\ &= \gamma^2 t - \frac{\gamma^2 xv}{c^2} + \frac{\gamma^2 xv}{c^2} - \frac{\gamma^2 tv^2}{c^2} \\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\right)t = t. \end{aligned}$$

The space part is

$$\begin{aligned} x'' &= \gamma v t' + \gamma x' \\ &= \gamma^2 vt - \frac{\gamma^2 xv^2}{c^2} + \gamma^2 x - \gamma^2 vt \\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\right)x = x. \end{aligned}$$

The space–time coordinates are unchanged if we transform first in the x direction and then in the minus x direction with the same speed. ■

The Wave Equation

Lorentz discovered the transformation (4.20–4.23) by analyzing the behavior of the electric and magnetic fields of a charge moving with constant speed. The Lorentz transformation was identified as the transformation of space–time coordinates that did not change the form of Maxwell's equations. Since the electromagnetic wave equation is contained in Maxwell's equations, the Lorentz transformation leaves the form of the wave equation unchanged. The wave equation is

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}. \tag{4.29}$$

The Galilean transformation does not preserve the form of the wave equation. Under the Galilean transformation the space derivatives are

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial F}{\partial t'}\frac{\partial t'}{\partial x} = \frac{\partial F}{\partial x'}, \tag{4.30}$$

or

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial x'^2}, \tag{4.31}$$

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial y'^2}, \tag{4.32}$$

and

$$\frac{\partial^2 F}{\partial z^2} = \frac{\partial^2 F}{\partial z'^2}. \tag{4.33}$$

The time derivatives are

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial F}{\partial x'}\frac{\partial x'}{\partial t} = \frac{\partial F}{\partial t'} - v\frac{\partial F}{\partial x'}, \tag{4.34}$$

and

$$\begin{aligned} \frac{\partial^2 F}{\partial t^2} &= \left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)\left(\frac{\partial F}{\partial t'} - v\frac{\partial F}{\partial x'}\right) \\ &= \frac{\partial^2 F}{\partial t'^2} + v^2\frac{\partial^2 F}{\partial x'^2} - 2v\frac{\partial^2 F}{\partial x'\partial t'}. \end{aligned} \tag{4.35}$$

Using the expressions for the second derivatives (4.31–4.33 and 4.35), the wave equation in the moving frame (primed coordinates) becomes

$$\frac{\partial^2 F}{\partial x'^2}+\frac{\partial^2 F}{\partial y'^2}+\frac{\partial^2 F}{\partial z'^2} = \frac{1}{c^2}\left(\frac{\partial^2 F}{\partial t'^2}+v^2\frac{\partial^2 F}{\partial x'^2}-2v\frac{\partial^2 F}{\partial t'\partial x'}\right). \quad (4.36)$$

The form of the wave equation has changed because of the appearance of the last two terms. Note, however, that we get the wave equation back in the limit $v \to 0$.

The Lorentz transformation, however, preserves the form of the wave equation. Under the Lorentz transformation,

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial F}{\partial t'}\frac{\partial t'}{\partial x}=\gamma\frac{\partial F}{\partial x'}-\frac{\gamma v}{c^2}\frac{\partial F}{\partial t'}, \quad (4.37)$$

and

$$\frac{\partial F}{\partial t}=\frac{\partial F}{\partial t'}\frac{\partial t'}{\partial t}+\frac{\partial F}{\partial x'}\frac{\partial x'}{\partial t}=\gamma\frac{\partial F}{\partial t'}-\gamma v\frac{\partial F}{\partial x'}. \quad (4.38)$$

Applying the derivatives twice, the components of the wave equation are

$$\frac{\partial^2 F}{\partial x^2}=\gamma^2\frac{\partial^2 F}{\partial x'^2}+\frac{\gamma^2 v^2}{c^4}\frac{\partial^2 F}{\partial t'^2}-\frac{2\gamma^2 v}{c^2}\frac{\partial^2 F}{\partial x'\partial t'}, \quad (4.39)$$

and

$$\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}=\frac{\gamma^2}{c^2}\frac{\partial^2 F}{\partial t'^2}+\frac{\gamma^2 v^2}{c^2}\frac{\partial^2 F}{\partial x'^2}-\frac{2\gamma^2 v}{c^2}\frac{\partial^2 F}{\partial x'\partial t'}. \quad (4.40)$$

The y and z derivatives are the same as before. Using our calculated second derivatives (4.32, 4.33, 4.39, and 4.40), we have

$$\left(\gamma^2-\frac{\gamma^2 v^2}{c^2}\right)\frac{\partial^2 F}{\partial x'^2}+\frac{\partial^2 F}{\partial y'^2}+\frac{\partial^2 F}{\partial z'^2} = \left(\gamma^2-\frac{\gamma^2 v^2}{c^2}\right)\frac{1}{c^2}\frac{\partial^2 F}{\partial t'^2}, \quad (4.41)$$

or

$$\frac{\partial^2 F}{\partial x'^2}+\frac{\partial^2 F}{\partial y'^2}+\frac{\partial^2 F}{\partial z'^2}=\frac{1}{c^2}\frac{\partial^2 F}{\partial t'^2}, \quad (4.42)$$

because

$$\left(\gamma^2-\frac{\gamma^2 v^2}{c^2}\right)=\gamma^2\left(1-\frac{v^2}{c^2}\right)=1 \quad (4.43)$$

from the definition of γ (4.24). The Lorentz transformation preserves the form of the wave equation.

Transformation of Velocities

One important property of the Lorentz transformation is that it contains the experimental fact that the speed of light does not depend on the motion of the source. We now investigate this important property. Consider a particle in the frame S moving with a velocity dx/dt in the x direction. In a frame S', defined to be moving with a velocity v in the x direction relative to the frame S, the same particle has a speed dx'/dt'. The determination of dx'/dt' in terms of dx/dt is obtained from the Lorentz transformation by differentiation of the coordinates. From the Lorentz transformation (4.20 and 4.21), we have

$$\frac{dx'}{dt'}=\frac{dx-v\,dt}{\left(dt-\dfrac{v\,dx}{c^2}\right)}=\frac{\left(\dfrac{dx}{dt}-v\right)}{\left(1-\dfrac{v}{c^2}\dfrac{dx}{dt}\right)}. \quad (4.44)$$

The velocity transformation (4.44) is illustrated in Figure 4-5 for the case $v=-dx/dt$.

Let us examine our result in some limiting cases. When the relative speed of the two reference frames is small compared to c, then $v/c \approx 0$ and the particle speed (4.44) is

$$\frac{dx'}{dt'}=\frac{dx}{dt}-v, \quad (4.45)$$

which is the Galilean velocity addition rule (4.8).

If the particle is a photon, then $dx/dt = c$ and the transformed photon speed (4.44) is

$$\frac{dx'}{dt'}=\frac{(c-v)}{\left(1-\dfrac{v}{c^2}c\right)}=c. \quad (4.46)$$

The speed of the photon is unchanged. This is the second postulate of special relativity. The Lorentz transformation contains this important physics.

A corollary of the second postulate of relativity is that no particle may move faster than the speed of light (c). No particle has ever been observed to move at a speed faster than c.

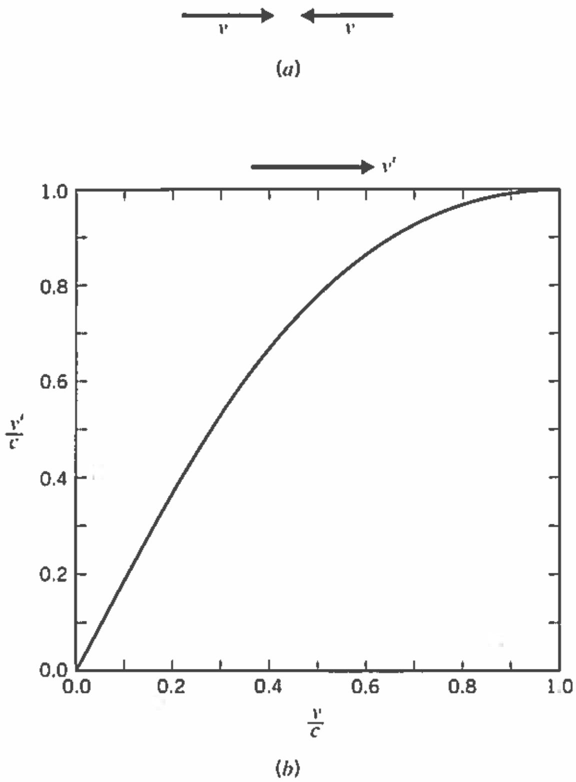


FIGURE 4-5 The velocity transformation.
(*a*) Two particles each have a speed v traveling in opposite directions. (*b*) In the frame where one of the particles is at rest, the other particle has a speed v' given by the velocity transformation (4.44).

> No particle has ever been observed to move at a speed faster than c.

In the limit where v approaches c, we have

$$\frac{dx'}{dt'} = \frac{\left(\dfrac{dx}{dt} - c\right)}{\left(1 - \dfrac{c}{c^2}\dfrac{dx}{dt}\right)} = c\,. \tag{4.47}$$

Thus, in a frame that is moving with a speed approaching the speed of light, the particle speed approaches c. In no case is it possible for dx/dt, dx'/dt', or v to be larger than c. This important result is verified directly by many experiments.

We now examine the more general case where the particle has velocity components dx/dt, dy/dt, and dz/dt in the x, y, and z directions in the frame S. In the frame S' the x component of velocity, dx'/dt', is given by the transformation (4.44) because the coordinates x' and t' do not depend on the coordinates y and z. Although the coordinates y' and z' are unchanged, the velocity components dy'/dt' and dz'/dt' are changed because the time coordinate t' is not equal to the time coordinate t. The Lorentz transformation (4.22) gives $dy' = dy$ and the y component of velocity is

$$\frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \dfrac{v\,dx}{c^2}\right)} = \frac{\dfrac{dy}{dt}}{\gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}. \tag{4.48}$$

The y component of velocity in the moving frame dy'/dt' depends on both the x and y components of velocity in the stationary frame (dx/dt and dy/dt) because the *total* speed of the particle cannot exceed c. Similarly, the z component of velocity is

$$\frac{dz'}{dt'} = \frac{dz}{\gamma\left(dt - \dfrac{v\,dx}{c^2}\right)} = \frac{\dfrac{dz}{dt}}{\gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}. \tag{4.49}$$

> If a particle has a velocity (dx/dt, dy/dt, dz/dt) in the frame S, then in the frame S', which is moving in the x direction with a speed v, the velocity components of the particle are
>
> $$\frac{dx'}{dt'} = \frac{\left(\dfrac{dx}{dt} - v\right)}{\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)},$$
>
> $$\frac{dy'}{dt'} = \frac{\dfrac{dy}{dt}}{\gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)},$$
>
> and
>
> $$\frac{dz'}{dt'} = \frac{\dfrac{dz}{dt}}{\gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}.$$

You may have the feeling that the velocity equations (4.44, 4.48 and 4.49) are not physically intuitive. We shall find that in most practical applications of special relativity, the velocity is not a useful variable. The reason for this is that in the extreme relativistic regime, particles move at speeds close to c. It is not easy to see the physical difference between a particle moving at the speed $0.99c$ and $0.999c$, but there is a big difference! The velocity transformation is an important result in that it illustrates the second postulate of special relativity. It is not too useful as a calculational tool. We shall see that the variables of choice for the description of particles are energy and momentum.

EXAMPLE 4-2

In the frame S, two electrons approach each other, each having a speed $v=c/2$. What is the relative speed of the two electrons?

SOLUTION:

The relative speed of the two electrons is the speed of one of the electrons in the frame where the other electron is at rest. Let the frame S' move with a speed $c/2$ in the minus x direction. In the frame S', one of the electrons is at rest and the other electron is moving in the x direction. The speed of the moving electron in the frame S' is

$$\frac{dx'}{dt'}=\frac{\left(\frac{dx}{dt}-v\right)}{\left(1-\frac{v}{c^2}\frac{dx}{dt}\right)}=\frac{\left(\frac{c}{2}-\frac{-c}{2}\right)}{\left(1-\frac{-c}{2c^2}\frac{c}{2}\right)}=\frac{4c}{5}. \quad \blacksquare$$

EXAMPLE 4-3

In the frame S, an electron has velocity $c/2$ in the x direction and a photon has a velocity c in the y direction. What is the relative speed of the electron and the photon?

SOLUTION:

We immediately know the answer from the second postulate of special relativity; the relative speed of the electron and photon is c. Let us see that this is so from the velocity transformation. Let the frame S' move with a speed $c/2$ in the x direction. In the frame S', the electron is at rest. The velocity components of the photon in the frame S' (4.44 and 4.48)

$$\frac{dx'}{dt'}=\frac{\left(\frac{dx}{dt}-v\right)}{\left(1-\frac{v}{c^2}\frac{dx}{dt}\right)}=\frac{\left(0-\frac{c}{2}\right)}{\left[1-\frac{c}{2c^2}(0)\right]}=\frac{-c}{2},$$

and

$$\frac{dy'}{dt'}=\frac{\frac{dy}{dt}}{\gamma\left(1-\frac{v}{c^2}\frac{dx}{dt}\right)}.$$

The gamma factor is

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\sqrt{\frac{4}{3}}.$$

The y component of velocity of the photon is

$$\frac{dy'}{dt'}=\frac{c}{\sqrt{\frac{4}{3}}\left[1-\frac{c}{2c^2}(0)\right]}=\frac{\sqrt{3}}{2}c.$$

The z component of velocity of the photon is zero. The speed of the photon is

$$\sqrt{\left(\frac{dx'}{dt'}\right)^2+\left(\frac{dy'}{dt'}\right)^2+\left(\frac{dz'}{dt'}\right)^2}$$

$$=\sqrt{\frac{c^2}{4}+\frac{3c^2}{4}+0}=c. \quad \blacksquare$$

Time Dilation

Consider a time interval (Δt) measured at a fixed position (x_0) in a stationary frame,

$$\Delta t=t_2-t_1. \tag{4.50}$$

If we use the Lorentz transformation (4.20) to calculate the time interval ($\Delta t'$) in a frame moving with a speed v, we arrive at the result

$$\begin{aligned}\Delta t'&=t_2'-t_1'\\&=\gamma\left(t_2-\frac{x_0v}{c^2}\right)-\gamma\left(t_1-\frac{x_0v}{c^2}\right)\\&=\gamma(t_2-t_1)=\gamma\Delta t.\end{aligned} \tag{4.51}$$

The time interval is longer in the moving frame. This result is called *time dilation.*

Time dilation is a direct consequence of the second postulate of special relativity. The time interval measured in the frame where a clock is at rest is called the *proper time*. Time intervals measured in other frames are longer. The physics of time dilation can be illustrated by the following example. Consider the length of time (Δt) that it takes light to travel the distance d from point A to point B in frame S as shown in Figure 4-6a. The "clock" in this case is apparatus that detects the light. Since the speed of light is c, we have

$$\Delta t = \frac{d}{c}. \tag{4.52}$$

Now consider the time interval for the same event, light traveling from point A to point B, in a moving frame (S') as shown in Figure 4-6b. The second postulate of special relativity states that the speed of light is constant. Examination of the velocity vector diagram shows that since the horizontal component of velocity is v and the net speed is c, that the vertical component of velocity is $(c^2 - v^2)^{1/2}$. Therefore,

$$\Delta t' = \frac{d}{\sqrt{c^2 - v^2}} = \frac{d}{c\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\gamma d}{c}. \tag{4.53}$$

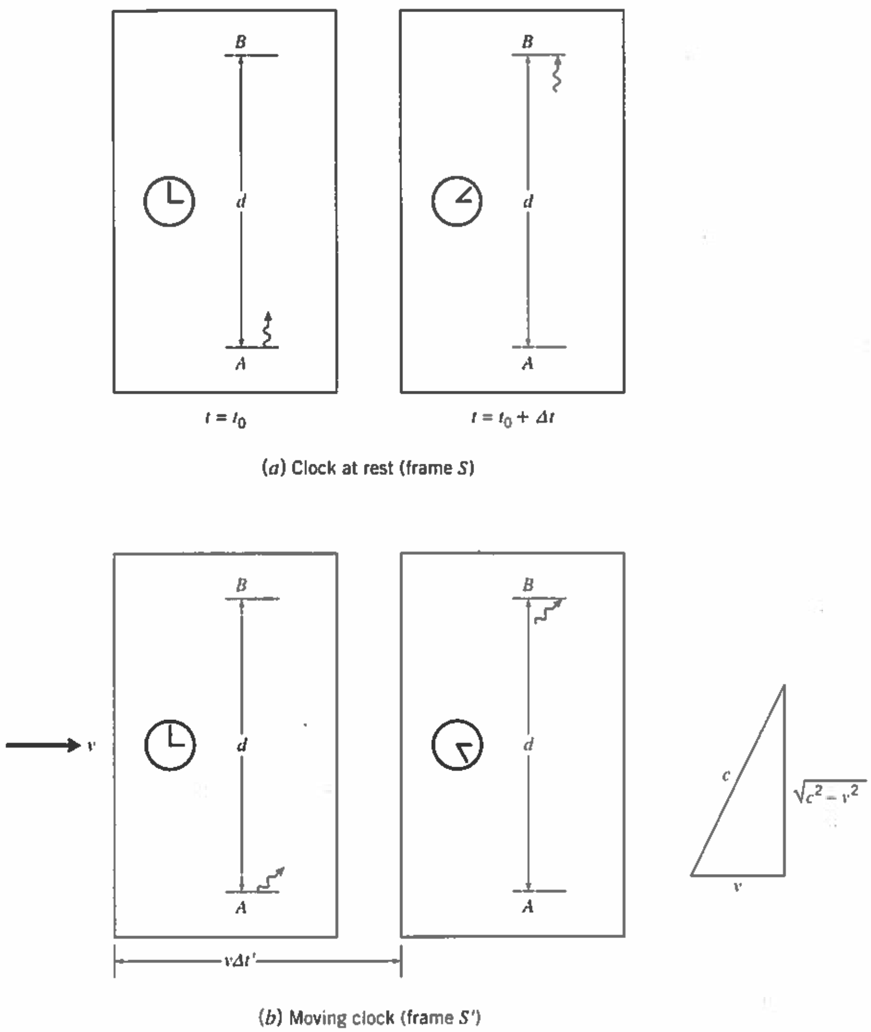


FIGURE 4-6. Time dilation.
(*a*) The clock is at rest. The time for the light to travel from point A to point B is $\Delta t = d/c$. The time measured in the frame where the clock is at rest is called the proper time. (*b*) The clock is moving with a speed v. Because the speed of light is constant, the time for the light to travel from point A to point B is $\Delta t' = d/(c^2 - v^2)^{1/2}$.

The time interval measured in frame S' is longer than in frame S by a factor of γ,

$$\Delta t' = \gamma \Delta t. \qquad (4.54)$$

The difference between the two frames is that the clock is stationary in frame S and the clock is moving in frame S'. For any time interval measurement, there is one special frame, the frame in which the clock is at rest. In the case of a particle that undergoes spontaneous radioactive decay, the clock is the particle itself and the particle lifetime is shortest in the frame in which the particle is at rest. In all other frames, the particle lifetime is observed to be longer.

EXAMPLE 4-4

A pion is created in a particle collision with a large speed, such that $\gamma = 100$, and it is observed to travel a distance of 300 m before it spontaneously decays. How long does the pion live in its rest frame?

SOLUTION:

Since $\gamma = 100$, the speed of the pion is very nearly equal to the speed of light,

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{10^4}} \approx 1.$$

The lifetime in the frame in which the pion travels 300 m is

$$\Delta t' = \frac{d}{v} = \frac{300\ \text{m}}{3 \times 10^8\ \text{m/s}} = 10^{-6}\ \text{s}.$$

The lifetime of the pion in its rest frame, the proper lifetime, is

$$\Delta t = \frac{\Delta t'}{\gamma} = \frac{10^{-6}\ \text{s}}{100} = 10^{-8}\ \text{s}.$$ ■

*

Length Contraction

The second postulate of special relativity also leads to a phenomenon called *length contraction*. The length measured in the frame in which the object is at rest is called the *proper length*. Consider a stick moving with speed v in frame S as shown in Figure 4-7*a*. The length of the stick (L) may be determined by measuring the time for the stick to pass a stationary clock,

$$L = v\Delta t. \qquad (4.55)$$

We could also make a length measurement (L') in a frame in which the stick is at rest, as shown in Figure 4-7*b*. In this frame; the clock is moving with a speed v,

$$L' = v\Delta t'. \qquad (4.56)$$

We know how the two time intervals are related by the time dilation rule: The time interval is longer by a factor of γ in the frame where the clock is moving,

$$\Delta t' = \gamma \Delta t. \qquad (4.57)$$

Therefore,

$$L = L' \frac{\Delta t}{\Delta t'} = \frac{L'}{\gamma}. \qquad (4.58)$$

The length of the stick is longest in the frame where the stick is at rest ($L' > L$). In a frame where the stick is moving in a direction parallel to its length, the stick is measured to be shorter by a factor of γ. Length contraction applies to any two points in space that may be considered connected by an imaginary stick.

EXAMPLE 4-5

A pion is created in a particle collision with $\gamma = 100$, and it is observed to travel a distance of 300 m before it spontaneously decays. What is the decay distance in the pion rest frame?

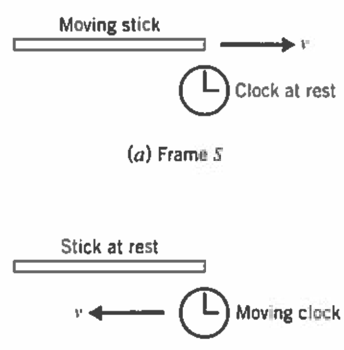


FIGURE 4-7 Length contraction.
(*a*) The length of a stick is measured in a frame S, where it is moving. The measurement is made by measuring the time that it takes to pass from end to end past a stationary clock. (*b*) The stick is measured in a frame S', where it is stationary. The measurement is made by measuring the time that it takes a moving clock to pass from end to end. The length measured in this frame where the stick is at rest is called the proper length. The length measured in frame S is shorter.

SOLUTION:

Let the pion be created at point A and decay at point B. In the frame where the pion is moving, the distance from A to B is not moving and is 300 m. In the rest frame of the pion, both points A and B are moving, and the distance from A to B (L') is contracted:

$$L' = \frac{L}{\gamma} = \frac{300\,\text{m}}{100} = 3\,\text{m}. \quad \blacksquare$$

Figure 4-8 illustrates time dilation and length contraction. A particle is created with speed v at position x_1' and spontaneously decays at position x_2'. The particle lives for a time $\Delta t'$ and travels a distance $\Delta x' = (x_1' - x_2')$. This decay-length distance is the proper length in this frame. The speed of the particle is

$$v = \frac{\Delta x'}{\Delta t'}. \tag{4.59}$$

In the rest frame of the particle, the lifetime is shorter:

$$\Delta t = \frac{\Delta t'}{\gamma}. \tag{4.60}$$

This decay time is the proper time. The decay length is also shorter:

$$\Delta x = x_1 - x_2 = \frac{x_1' - x_2'}{\gamma} = \frac{\Delta x'}{\gamma}. \tag{4.61}$$

The speed of the frame is

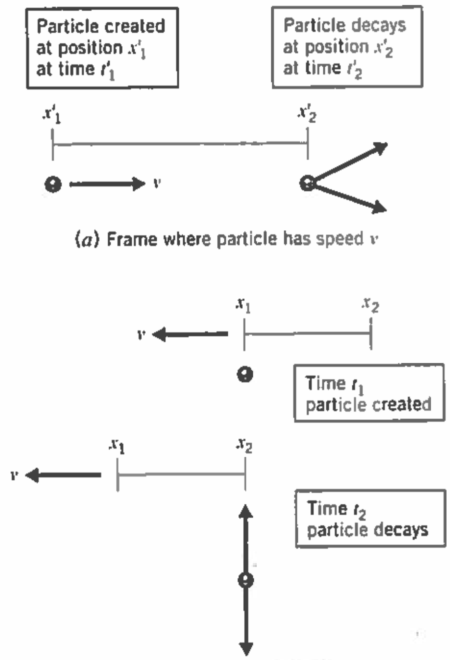
$$v = \frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\gamma \Delta t} = \frac{\Delta x'}{\Delta t'}. \tag{4.62}$$

The speed of the frame in Figure 4-8b must be equal to the speed of the particle in Figure 4-8a by definition of the rest frame. We see that the time dilation and length contraction gamma factors cancel each other.

There is an excellent example of time dilation and length contraction occurring constantly in nature all around us. Cosmic ray muons are generated in the upper atmosphere, a few kilometers above the surface of the earth. The muons are created with a large speed ($v \approx c$), corresponding to a $\gamma \approx 20$ on the average. A large flux of these cosmic ray muons are observed at sea level (about 180 $\text{m}^{-2}\cdot\text{s}^{-1}$). The mean proper lifetime of the muon (the average time that a muon lives in its rest frame) is

$$\tau_0 = 2.2\,\mu\text{s}. \tag{4.63}$$

Without time dilation, the muon could travel a mean distance of only



(a) Frame where particle has speed v

(b) Frame where particle is at rest

FIGURE 4-8 Time dilation and length contraction in particle decay.
(a) A particle is created with a speed v at the space-time coordinates (ct_1', x_1'). The particle decays at the coordinates (ct_2', x_2'). The particle speed is given by $v = (x_2' - x_1')/(t_2' - t_1')$. ($b$) In the frame where the particle is at rest, the proper lifetime is $\Delta t = (t_2 - t_1) = (t_2' - t_1')/\gamma$. The distance $(x_2' - x_1')$ is length contracted to $(x_2 - x_1) = (x_2' - x_1')/\gamma$. The speed of the frame is $v = (x_2 - x_1)/(t_2 - t_1)$, which is the same as the particle speed in the previous frame.

$$\begin{aligned} L &= c\tau_0 \\ &= \left(3.0\times10^8\ \text{m/s}\right)\left(2.2\times10^{-6}\ \text{s}\right) \\ &= 660\,\text{m}. \end{aligned} \tag{4.64}$$

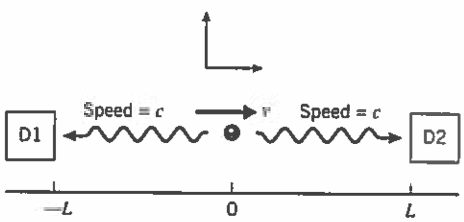
Including the time dilation factor, the muon travels a mean distance of

$$L_0 = \gamma c \tau_0 = (20)(660\ \text{m}) \approx 1.3\times10^4\ \text{m}. \tag{4.65}$$

The cosmic ray muons are observed to travel distances of several thousand meters to reach the surface of the earth, and the Lorentz γ factor is needed to explain this. In the muon rest frame the lifetime is shortest and the earth is moving toward it so the distance to the earth's surface is length contracted by a factor of γ. In the rest frame of the earth, the muon is moving and lives longer than the proper lifetime by a factor of γ. You may find this example of the cosmic ray muon to be a useful tool in sorting out from first principles what times are dilated and what distances are contracted in many problems. It is much more practical than trying to remember an equation!

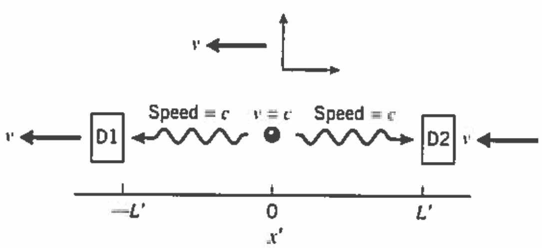
Simultaneity

The space–time relationship of special relativity forces us to give up the notion of universal simultaneity. Events that occur at the same time in one frame of reference do not necessarily occur at the same time in another frame of reference. At first, this may seem counterintuitive. Consider the following example, as illustrated in Figure 4-9. In the frame S, a pion (π^0) is produced with a speed v in the



Detectors D1 and D2 are hit at the same time.

(*a*) Frame S



Detector D2 is hit before D1.

(*b*) Frame S'

FIGURE 4-9 Simultaneity.
(*a*) In frame S, a particle is produced with a speed v, and when it is at $x = 0$, the particle decays into two photons. Detectors D1 and D2 are located at $x = -L$ and $x = L$. Since the speed of each photon is c, the photons arrive at the two detectors at the same time. (*b*) The same event is analyzed in the frame where the particle is at rest (frame S'). Detector D2 is moving toward the photon that hits it, and detector D1 is moving away from the photon that hits it. The speed of each photon is equal to c. Therefore, detector D2 gets hit before D1. Events that are simultaneous in frame S are not simultaneous in frame S'.

x direction. When the pion is at the space–time coordinates $(ct, x) = (0,0)$, it spontaneously decays into two photons. One of these photons travels in the negative x direction and strikes a detector (D1) that is placed at the location $x_1 = -L$. The other photon travels in the positive x direction and strikes a detector (D2) that is placed at $x_2 = L$. The speed of each photon is equal to c. The photon traveling in the negative x direction strikes the detector at a time $t_1 = L/c$, and the other photon strikes the other detector at a time $t_2 = L/c$:

$$\Delta t = t_2 - t_1 = \frac{L}{c} - \frac{L}{c} = 0. \tag{4.66}$$

In the frame S, the two photons strike the detector at the same time.

Now analyze the pion decay in the rest frame of the pion (frame S'). To evaluate the space–time coordinates in the moving frame, we do a Lorentz transformation with a speed v in the negative x direction. We want to evaluate the times t_1' and t_2'. The Lorentz transformation gives

$$ct_1' = \gamma ct_1 + \frac{\gamma v x_1}{c}, \tag{4.67}$$

and

$$ct_2' = \gamma ct_2 + \frac{\gamma v x_2}{c}. \tag{4.68}$$

The time difference is

$$\begin{aligned} c\Delta t' &= ct_2' - ct_1' \\ &= \left(\gamma ct_2 + \frac{\gamma v x_2}{c}\right) - \left(\gamma ct_1 + \frac{\gamma v x_1}{c}\right) \\ &= \frac{\gamma v (x_2 - x_1)}{c} = \frac{2Lv\gamma}{c}, \end{aligned} \tag{4.69}$$

where we have used the fact that $(t_2 - t_1) = 0$. Therefore,

$$\Delta t' = \frac{2Lv\gamma}{c^2}. \tag{4.70}$$

The photons do not strike the two detectors at the same time when the situation is analyzed in the frame S'. The time t_2' is larger than t_1', so that the detector D2 is struck before the detector D1. The reason for this is that the pion is traveling toward the detector D2 so that in the rest frame of the pion, detector D2 is moving toward the pion and detector D1 is moving away from the pion.

4-3 RELATIONSHIP BETWEEN ENERGY AND MOMENTUM

Classical Approximations

Mass is not a conserved quantity. Particles are created and destroyed in high-energy collisions. In experiments involving the collisions of energetic particles, we observe that kinetic energy may be converted into matter and matter may be converted into kinetic energy. In this sense, matter and energy are interchangeable. Neither mass nor kinetic energy are absolutely conserved.

For analysis of particles with large speeds, we shall need to modify the classical expression for kinetic energy, $mv^2/2$, which is inconsistent with the second postulate of special relativity. A photon has a constant speed (c) and zero mass energy ($mc^2 = 0$). Therefore, $mv^2/2$ for a photon is equal to zero, but a photon does have kinetic energy. We observe that a photon can knock an electron out of an atom.

The classical expression for kinetic energy, $mv^2/2$, does not work for massive particles moving at high speeds, either. The classical expression for kinetic energy predicts a violation of an important experimental observation: No particle has ever been observed to move faster than the speed of light (c).

EXAMPLE 4-6

Using the classical expression for kinetic energy, $E_k = mv^2/2$, calculate the speed of a proton that has a kinetic energy of 300 GeV.

SOLUTION:

The speed of the proton would be given by

$$v = \sqrt{\frac{2E_k}{m}}.$$

Dividing both sides by the speed of light (c), we get

$$\frac{v}{c} = \sqrt{\frac{2E}{mc^2}} \approx \sqrt{\frac{600 \text{ GeV}}{0.94 \text{ GeV}}} \approx 25.$$

The truth is that a proton with a kinetic energy of 300 GeV has a speed of approximately (but not exceeding) c. ■

The speed of a 300 GeV proton would be 25 times the speed of light if $mv^2/2$ was the correct expression for kinetic energy! The classical expression for kinetic energy, $mv^2/2$, is approximately correct as long as $v/c \ll 1$. It is grossly incorrect for $v/c \approx 1$.

Momentum (p) defined as mass times velocity (mv) is a useful quantity in classical mechanics because it is conserved in collisions provided that $v \ll c$. For large speeds, the quantity mv is not conserved. This can be seen by considering the acceleration of an electron in an electric field. The electrical force causes an increase in momentum given by

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{4.71}$$

If the force acts for a long enough time, the momentum would increase without bound. There is no limit to the electron momentum, however, the electron speed can never exceed c. The classical expression ($p = mv$) has a maximum value of mc. The classical expression does not work at large speeds.

Relativistic Momentum

Consider the elastic collision of two particles, A and B, of equal mass (m) as indicated in Figure 4-10. In the frame S' the particles have equal magnitudes of momentum in opposite directions both before and after the collision. Let v represent the speed of each particle in frame S', and let $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. Before the collision the velocities of both particles are along the x axis. After the collision the velocities of both particles are along the y axis.

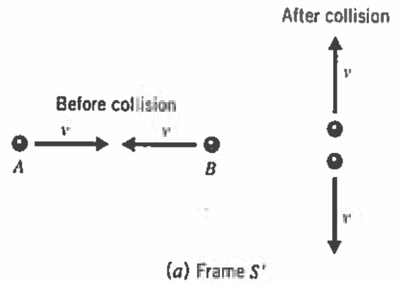
We now analyze the collision in the frame S where particle B is at rest. The speed of particle A before the collision (v_1) is given by the velocity transformation (4.44):

$$v_1 = \frac{v+v}{1+\frac{v^2}{c^2}} = \frac{2v}{1+\frac{v^2}{c^2}} = \frac{2v}{1+\beta^2}. \tag{4.72}$$

After the collision, the x component of velocity of each particle in the frame S must be equal to the relative speed of the two reference frames (v) according to the velocity transformation (4.44). The quantity mass times velocity is *not conserved* in frame S:

$$mv_1 \neq mv + mv. \tag{4.73}$$

We may see why mass times velocity is not conserved by examining the classical definition of momentum (4.2). The expression dr/dt is frame dependent; however, the expression dr/dt', where t' is the proper time, is invariant. The relationship between the differential time intervals is



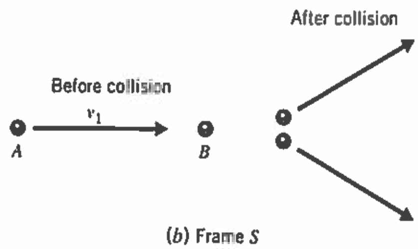


FIGURE 4-10 Particle collision analyzed in two frames.
(*a*) In frame *S'* the particles have equal speeds (v) directed in opposite directions. The particles scatter elastically at 90 degrees. (*b*) In frame *S* particle *B* is at rest and the speed of particle *A* is $v_1 = 2v/(1 + v^2/c^2)$.

$$dt' = \sqrt{1-\frac{v^2}{c^2}}\,dt = \frac{dt}{\gamma}, \tag{4.74}$$

We define the *relativistic momentum* to be

$$\mathbf{p} \equiv m\frac{d\mathbf{r}}{dt'} = m\frac{d\mathbf{r}}{dt}\frac{dt}{dt'}. \tag{4.75}$$

The relativistic momentum (4.75) is observed to be conserved in all processes and is commonly written as

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{4.76}$$

or

$$\mathbf{p} = \gamma m\mathbf{v}, \tag{4.77}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ is defined in terms of the particle speed in the frame where we are evaluating the momentum.

We shall now verify that the relativistic momentum (4.77) is conserved in the collision of Figure 4-10. We may calculate the gamma factor of particle *A* before the collision (γ_1) from the speed v_1 (4.72),

$$\gamma_1 = \frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{4\beta^2}{(1+\beta^2)^2}}}$$
$$= \frac{1}{\sqrt{\frac{1+\beta^4+2\beta^2-4\beta^2}{(1+\beta^2)^2}}} = \frac{1+\beta^2}{1-\beta^2}. \tag{4.78}$$

The total momentum before the collision is

$$p_1^{\text{tot}} = \gamma_1 m v_1$$
$$= \left(\frac{1+\beta^2}{1-\beta^2}\right)(m)\left(\frac{2v}{1+\beta^2}\right) = \frac{2vm}{1-\beta^2}. \tag{4.79}$$

The magnitude of the y component of velocity of each particle after the collision is $v(1 - v^2/c^2)^{1/2}$. The net speed (v_2) of each particle is

$$v_2 = \sqrt{v^2 + v^2\left(1-\frac{v^2}{c^2}\right)} = c\beta\sqrt{2-\beta^2}. \tag{4.80}$$

We may calculate the gamma factor for each particle after the collision from the speed v_2 (4.80)

$$\gamma_2 = \frac{1}{\sqrt{1-\frac{v_2^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2(2-\beta^2)}{c^2}}}$$
$$= \frac{1}{\sqrt{1-2\beta^2+\beta^4}} = \frac{1}{1-\beta^2}. \tag{4.81}$$

The total momentum after the collision is

$$p_2^{\text{tot}} = \gamma_2 m v + \gamma_2 m v = \frac{2vm}{1-\beta^2}. \tag{4.82}$$

Therefore,

$$p_1^{\text{tot}} = p_2^{\text{tot}}. \tag{4.83}$$

The momentum of a particle with nonzero mass (m) is

$$\mathrm{p} = \frac{m\mathrm{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m\mathrm{v}.$$

Relativistic Energy

Consider a force acting on a particle, for example, an electron in an electric field. The kinetic energy of the particle is given by the expression

$$E_k = \int_0^x dx'\,F' = \int_0^x dx'\,\frac{dp'}{dt'} = \int_0^p dp'\,\frac{dx'}{dt'} = \int_0^p dp'\,v' \qquad (4.84)$$

where the integration variables p' and v' are related by

$$p' = \frac{mv'}{\sqrt{1-\frac{v'^2}{c^2}}}. \qquad (4.85)$$

This expression may be integrated by parts. Using

$$d(p'v') = p'\,dv' + v'\,dp', \qquad (4.86)$$

we have

$$E_k = [p'v']_{v'=v} - \int_0^v dv'\,p' = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} - \int_0^v dv'\,\frac{mv'}{\sqrt{1-\frac{v'^2}{c^2}}}. \qquad (4.87)$$

Integrating, we get

$$E_k = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} - \left[-mc^2\sqrt{1-\frac{v'^2}{c^2}}\right]_{v'=0}^{v'=v} = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}} - mc^2, \qquad (4.88)$$

or

$$E_k = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{v^2}{c^2}+1-\frac{v^2}{c^2}\right) - mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2. \qquad (4.89)$$

For small speeds ($v \ll c$) the kinetic energy reduces to the classical form:

$$E_k = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2 \approx mc^2\left[1+\frac{v^2}{2c^2}\right] - mc^2 = \frac{1}{2}mv^2. \qquad (4.90)$$

The momentum and kinetic energy of an electron as a function of its speed are shown in Figure 4-11. At small speeds the momentum is proportional to v and the kinetic energy is proportional to v^2. At large speeds the kinetic energy is equal to pc.

Energy, Mass, and Momentum

The total energy (E) of a particle is defined to be the sum of the mass and kinetic energies,

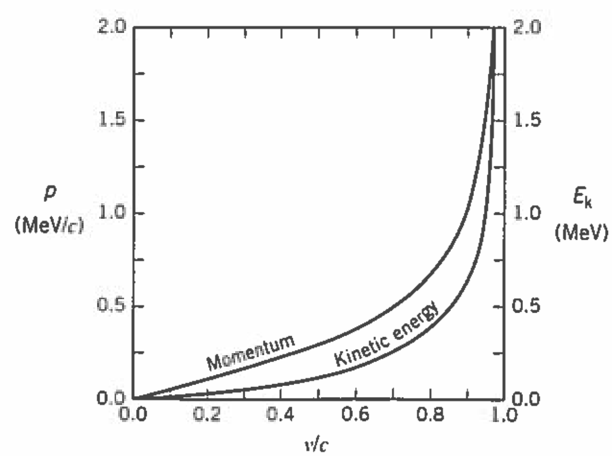


FIGURE 4-11 Momentum and kinetic energy of an electron as a function of speed.

$$E = E_k + mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma mc^2. \quad (4.91)$$

The total energy is observed to be conserved in all particle interactions,

$$\sum_{\substack{\text{before}\\\text{collision}}} E = \sum_{\substack{\text{after}\\\text{collision}}} E. \quad (4.92)$$

Energy and momentum are closely related because they both contain the factor γm:

$$E = \gamma mc^2 = \frac{pc^2}{v}. \quad (4.93)$$

The particle speed in terms of energy and momentum is

$$\frac{v}{c} = \frac{pc}{E}. \quad (4.94)$$

Thus, the energy may be written

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{(pc)^2}{E^2}}}. \quad (4.95)$$

Solving for E, we arrive at the expression relating the total energy (E), momentum (p) and mass (m) of a particle:

$$E = \sqrt{(pc)^2 + (mc^2)^2}. \quad (4.96)$$

This is the "master equation" of special relativity. When two out of three of the quantities E, p, and m are known, the energy equation determines the third quantity. The expression for energy (4.96) is universally valid for all particles; there are no exceptions.

> The total energy (E), momentum (p) and mass (m) of a particle are related by
>
> $$E = \sqrt{(pc)^2 + (mc^2)^2}.$$

> The speed of a particle divided by the speed of light is given by
>
> $$\frac{v}{c} = \frac{pc}{E}.$$

For massless particles ($m = 0$) like the photon, the energy is related to the momentum by

$$E = pc. \quad (4.97)$$

The momentum of a photon is equal to its energy divided by the speed of the photon. This expression for photon momentum is experimentally verified by scattering photons and electrons, a process called *Compton scattering*, and measuring how much momentum is transferred from the photons to the electrons. Compton scattering is discussed later in this chapter.

The energy equation (4.96) invites a convenient unit for momentum of a particle, (eV/c). One eV/c is defined to be one electronvolt divided by the speed of light. By defining this unit, we can save the trouble of dividing by c when calculating momentum from the energy equation. Our use of eV/c as a momentum unit is analogous to our use of eV/c^2 as a mass unit.

EXAMPLE 4-7

A particle has a momentum of 1.0 MeV/c. Express this momentum in SI units.

SOLUTION:

The particle momentum is

$$p = 1.0 \text{ MeV/c}$$
$$= (1.0 \text{ MeV/c})\left(\frac{1.6\times10^{-13}\text{ J}}{1.0\,\text{MeV}}\right)\left(\frac{c}{3\times10^8\text{ m/s}}\right)$$
$$\approx 5.3\times10^{-22}\text{ kg}\cdot\text{m/s}.$$ ■

EXAMPLE 4-8

Calculate the momentum of a photon that has an energy of 1 eV.

SOLUTION:

The photon momentum is

$$p = E/c = 1 \text{ eV}/c.$$

Apart from a factor of c, the energy and momentum of a photon are identical. ■

EXAMPLE 4-9

An electron is accelerated through a potential difference of 1.00 megavolts. Calculate the momentum of the electron.

SOLUTION:
The kinetic energy of the electron is

$$E_k = (e)\left(10^6 \text{ V}\right) = 1.00 \text{ MeV}.$$

The total energy of the electron is the kinetic energy plus the mass energy:

$$E = E_k + E_0.$$

The momentum of the electron times the speed of light is

$$pc = \sqrt{E^2 - E_0^2} = \sqrt{(E_k + E_0)^2 - E_0^2}$$
$$= \sqrt{E_0^2 + 2E_k E_0},$$

or

$$pc = \sqrt{(1.00 \text{ MeV})^2 + (2)(1.00 \text{ MeV})(0.511 \text{ MeV})}$$
$$= 1.42 \text{ MeV}.$$

The electron momentum is

$$p = 1.42 \text{ MeV}/c.$$ ■

EXAMPLE 4-10
Calculate the momentum of an electron that has a speed of $c/2$.

SOLUTION:
The gamma factor of the electron is

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{1}{4}}} = \sqrt{\frac{4}{3}}.$$

The electron momentum is

$$p = \gamma m v = \frac{\gamma m c}{2} = \frac{\gamma m c^2}{2c}$$
$$= \sqrt{\frac{4}{3}}(0.511 \text{MeV})\left(\frac{1}{2c}\right)$$
$$= 0.295 \text{MeV}/c.$$ ■

EXAMPLE 4-11
Calculate the energy of an electron that has a speed of 80% of the speed of light.

SOLUTION:
The gamma factor of the electron is

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-(0.8)^2}} = \frac{1}{0.6} = \frac{5}{3}.$$

The electron energy is

$$E = \gamma m c^2 = \left(\frac{5}{3}\right)(0.511 \text{MeV}) = 0.852 \text{ MeV}.$$ ■

EXAMPLE 4-12
A massless particle has an energy E. Calculate the speed of the particle.

SOLUTION:
The mass energy of the particle is zero

$$mc^2 = 0.$$

The particle energy is

$$E = pc,$$

$$\frac{v}{c} = \frac{pc}{pc} = 1,$$

and

$$v = c.$$

A particle with zero mass always travels at the speed of light. ■

EXAMPLE 4-13
The speed of a particle is c. Calculate the mass of the particle.

SOLUTION:

$$\frac{v}{c} = 1 = \frac{pc}{E}.$$

The particle energy is

$$E = pc.$$

The particle mass energy is

$$mc^2 = \sqrt{E^2 - (pc)^2} = 0.$$

The particle mass is zero. A particle that has a speed equal to c must be massless. Thus, the second postulate of special relativity guarantees that a particle has a speed c if and only if it is massless. ■

EXAMPLE 4-14

Calculate the total energy, kinetic energy, speed, and gamma factor of an electron that has a momentum of 1.00 MeV/c.

SOLUTION:

The total energy is

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$
$$= \sqrt{(1.00 \text{ MeV})^2 + (0.511 \text{ MeV})^2} = 1.12 \text{ MeV}.$$

The kinetic energy is

$$E_k = E - mc^2 = 1.12 \text{ MeV} - 0.511 \text{ MeV} = 0.61 \text{ MeV}.$$

The speed is given by

$$\frac{v}{c} = \frac{pc}{E} = \frac{1.00 \text{ MeV}}{1.12 \text{ MeV}} = 0.89,$$

or

$$v = 0.89c.$$

The gamma factor is

$$\gamma = \frac{E}{mc^2} = \frac{1.12 \text{ MeV}}{0.511 \text{ MeV}} = 2.19.$$ ■

EXAMPLE 4-15

A particle at rest with mass M decays into two particles of equal mass m. Calculate the speed of the two decay particles. Give a numerical answer for the decay of a rho particle ($M = 770$ MeV/c^2) into two charged pions ($m = 140$ MeV/c^2).

SOLUTION:

Let γ be the Lorentz gamma factor for the pions. By conservation of energy,

$$Mc^2 = \gamma mc^2 + \gamma mc^2 = 2\gamma mc^2.$$

The gamma factor is

$$\gamma = \frac{M}{2m}.$$

The speed of the pion is given by

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{4m^2}{M^2}}.$$

For the decay of the rho particle into two pions,

$$\frac{v}{c} = \sqrt{1 - \frac{(4)(0.14 \text{ MeV}/c^2)^2}{(0.77 \text{ MeV}/c^2)^2}} = 0.93.$$ ■

EXAMPLE 4-16

In the LEP accelerator at CERN, electrons are accelerated to energies of about 50 GeV. By how much do the electron speeds deviate from c?

SOLUTION:

The gamma of the electrons is the electron energy (E) divided by the electron mass energy (mc^2):

$$\gamma = \frac{E}{mc^2} = \frac{50 \text{ GeV}}{0.511 \text{ MeV}} \approx 10^5.$$

The electron speed is

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}.$$

The deviation from the speed of light is

$$\frac{c - v}{c} = \frac{1}{2\gamma^2} \approx 5 \times 10^{-11}.$$ ■

Force and Momentum

The classical relationship between force on a particle and the resulting acceleration ($\mathbf{a} = d\mathbf{v}/dt$), Newton's second law of mechanics, is given by

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt}. \quad (4.98)$$

The force is parallel to the acceleration. Let us examine the relationship between force and acceleration in light of our new definition of momentum. We have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v}\gamma)}{dt} = m\frac{d(\mathbf{v}\gamma)}{dt}$$
$$= m\gamma\frac{d\mathbf{v}}{dt} + m\mathbf{v}\frac{d\gamma}{dt}. \quad (4.99)$$

This expression contains the factor

$$\frac{d\gamma}{dt}=\left(1-\frac{v^2}{c^2}\right)^{-3/2}\frac{v}{c^2}\frac{dv}{dt}=\frac{\gamma^3 v}{c^2}\frac{dv}{dt}. \quad (4.100)$$

Writing the dot product of the force and velocity as

$$\begin{aligned}\mathbf{F}\cdot\mathbf{v} &= m\gamma v\frac{dv}{dt}+\frac{mv^3\gamma^3}{c^2}\frac{dv}{dt}\\ &= mv\gamma\frac{dv}{dt}\left(1+\frac{\gamma^2 v^2}{c^2}\right)=mv\gamma^3\frac{dv}{dt}, \quad (4.101)\end{aligned}$$

we see that

$$\frac{d\gamma}{dt}=\frac{\mathbf{F}\cdot\mathbf{v}}{mc^2}, \quad (4.102)$$

and the expression for the force (4.99) becomes

$$\begin{aligned}\mathbf{F} &= m\gamma\frac{d\mathbf{v}}{dt}+\frac{\mathbf{v}(\mathbf{F}\cdot\mathbf{v})}{c^2}\\ &= m\gamma\frac{d\mathbf{v}}{dt}+\boldsymbol{\beta}(\mathbf{F}\cdot\boldsymbol{\beta}). \quad (4.103)\end{aligned}$$

Therefore, the acceleration is

$$\mathbf{a}\equiv\frac{d\mathbf{v}}{dt}=\frac{\mathbf{F}-\boldsymbol{\beta}(\mathbf{F}\cdot\boldsymbol{\beta})}{m\gamma}. \quad (4.104)$$

This is the relativistically correct expression that relates force and acceleration (replaces $\mathbf{a} = \mathbf{F}/m$). The acceleration is not parallel to the force at large speeds because the total speed cannot exceed c, and the speed depends on all spatial components, not just on the component in the direction of the force.

Testing the Formula for Momentum

The relativistic expression for momentum is readily verified by experiment. Consider the motion of a charged particle of mass m and charge q in a magnetic field B. The electromagnetic force on the particle is

$$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\frac{d\mathbf{p}}{dt}=\frac{d(\gamma m\mathbf{v})}{dt}. \quad (4.105)$$

If the force is perpendicular to the velocity, the magnitude of the velocity does not change, only its direction. We have

$$\frac{d(\gamma m\mathbf{v})}{dt}=\gamma m\frac{d\mathbf{v}}{dt}=q\mathbf{v}\times\mathbf{B}. \quad (4.106)$$

The acceleration is

$$\frac{d\mathbf{v}}{dt}=\frac{q\mathbf{v}\times\mathbf{B}}{\gamma m}. \quad (4.107)$$

The orbit is circular and the acceleration may be written as

$$\frac{v^2}{r}=\frac{qvB}{\gamma m}. \quad (4.108)$$

The momentum is

$$p=\gamma mv=qrB. \quad (4.109)$$

This expression is the same as the nonrelativistic result.

EXAMPLE 4-17

An electron has a radius of curvature of 1 m in a uniform magnetic field of 1 T. What is the momentum of the electron?

SOLUTION:

The momentum of the electron is

$$p=erB=e(1\,\mathrm{m})(1\,\mathrm{T}).$$

One T·m is equal to 1 V·s/m. Therefore,

$$p=erB=e(1\,\mathrm{V\cdot s/m})=1\,\mathrm{eV\cdot s/m}.$$

The momentum times the speed of light is

$$pc=(1\,\mathrm{eV\cdot s/m})(3\times10^8\,\mathrm{m/s})=300\ \mathrm{MeV}.$$

The electron momentum is

$$p=300\ \mathrm{MeV}/c.$$ ■

In 1901 an experiment was performed by Walter Kaufmann that tested the relationship between momentum and speed. Kaufmann built a spectrometer similar to that used by J. J. Thomson (see Chapter 1). The source of electrons for this experiment was the beta decay of radioactive nuclei, as discovered earlier by Becquerel. Kaufmann simultaneously measured the speed of electrons and their radius of curvature in a magnetic field. From the relationship between momentum and radius of curvature (4.109), we expect that

$$r=\frac{p}{eB}=\frac{\gamma mv}{eB}=\frac{\gamma\beta mc}{eB}. \quad (4.110)$$

The radius of curvature should be proportional to $(\gamma\beta)$. Figure 4-12 shows the data of Kaufmann. These data were taken four years before the theory of special relativity was published by Einstein. From the work of Lorentz, Kaufmann knew about the gamma factor. Kaufmann used these data to deduce "an apparent mass of the electron," but what his data actually determined directly and conclusively was the *momentum* of the electron. These data were the first to show that the momentum of an electron depends on the speed as

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad (4.111)$$

Transformation of Energy and Momentum

The energy and momentum of a particle depend on the reference frame in which they are evaluated. The Lorentz transformation relates the energy and momentum (E, p_x, p_y, p_z) in one frame to the energy and momentum (E', p_x', p_y', p_z') in a frame moving with a relative speed v in the x direction. The transformation of energy and momentum may be obtained by combining the velocity transformation (4.44, 4.48, and 4.49) with the definition of energy (4.95) and momentum (4.77). The result (see problem 47) is

$$E' = \gamma E - \beta\gamma p_x c, \qquad (4.112)$$

$$p_x' c = -\beta\gamma E + \gamma p_x c, \qquad (4.113)$$

$$p_y' c = p_y c, \qquad (4.114)$$

$$p_z' c = p_z c, \qquad (4.115)$$

where, as usual, $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\beta = v/c$.

Consider a particle of mass m at rest in the frame S. In the frame S' that moves with a speed v in the x direction, the energy and momentum are given by

$$E' = \gamma mc^2, \qquad (4.116)$$

$$p_x' c = -\beta\gamma mc^2, \qquad (4.117)$$

$$p_y' c = 0, \qquad (4.118)$$

$$p_z' c = 0 \qquad (4.119)$$

This is often referred to as *boosting* a particle. The boosted energy is γmc^2 and the boosted momentum is γmv. These boosted quantities are just the energy and momentum of a particle with a speed v.

The inverse transformation is obtained by changing beta to minus beta:

$$E = \gamma E' + \beta\gamma p_x' c, \qquad (4.120)$$

$$p_x c = \beta\gamma E' + \gamma p_x' c, \qquad (4.121)$$

$$p_y c = p_y' c \qquad (4.122)$$

$$p_z c = p_z' c. \qquad (4.123)$$

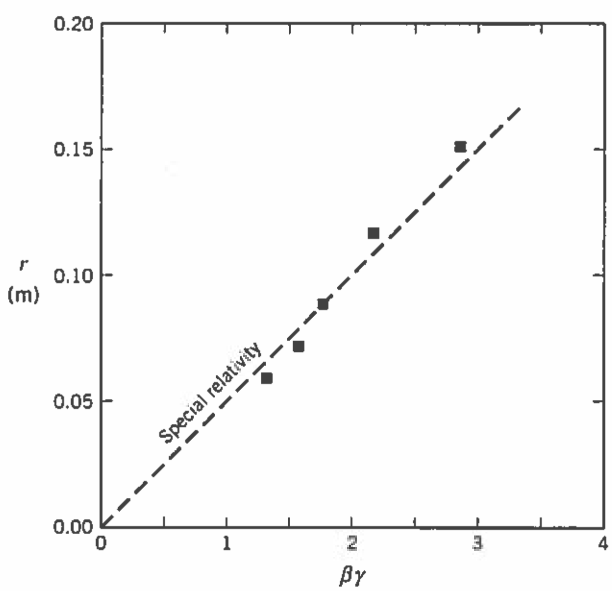


FIGURE 4-12 Measured speed dependence of the momentum of an electron.
These data were taken by W. Kaufmann in 1901, four years before Einstein published his famous paper on special relativity. Kaufmann simultaneously measured the electron speed and radius of curvature in a fixed magnetic field. From W. Kaufmann, *Göttingen Nachrichten,* 143 (1901).

EXAMPLE 4-18

Show that if we transform first in the x direction and then in the minus x direction we end up with the original energy and momentum.

SOLUTION:

The first transformation gives

$$E' = \gamma E - \beta\gamma p_x c,$$

and

$$p_x' c = -\beta\gamma E + \gamma p_x c,$$

with $p_y c$ and $p_z c$ unchanged. The inverse transformation gives for the energy

$$\begin{aligned} E'' &= \gamma E' + \beta\gamma p_x' c \\ &= \gamma(\gamma E - \beta\gamma p_x c) + \beta\gamma(-\beta\gamma E + \gamma p_x c) \\ &= \gamma^2 E - \beta^2\gamma^2 E = E, \end{aligned}$$

and for the x component of momentum

$$\begin{aligned} p_x'' c &= \beta\gamma E' + \gamma p_x' c \\ &= \beta\gamma(\gamma E - \beta\gamma p_x c) + \gamma(-\beta\gamma E + \gamma p_x c) \\ &= \gamma^2 p_x c - \beta^2\gamma^2 p_x c = p_x c\ . \end{aligned}$$

The energy and momentum are unchanged if we transform first in the x direction and then in the minus x direction with the same speed. ■

EXAMPLE 4-19

Show that the mass of the particle is unchanged by the Lorentz transformation.

SOLUTION:

The mass energy squared of the particle in the moving frame $(m'c^2)^2$ is

$$\begin{aligned} (m'c^2)^2 &= E'^2 - \left(p_x'^2 c^2 + p_y'^2 c^2 + p_z'^2 c^2\right) \\ &= (\gamma E - \beta\gamma p_x c)^2 - (-\beta\gamma E + \gamma p_x c)^2 \\ &\quad - p_y^2 c^2 - p_z^2 c^2 \\ &= (\gamma^2 - \beta^2\gamma^2)E^2 - (\gamma^2 - \beta^2\gamma^2)(p_x c)^2 \\ &\quad - p_y^2 c^2 - p_z^2 c^2 \\ &= E^2 - \left(p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2\right) = \left(mc^2\right)^2 . \end{aligned}$$

The transformation has left the mass unchanged. ■

The Addition of Velocities

We have already seen the velocity addition rule in the context of the space–time coordinates. The Lorentz transformation contains the important result that no particle can travel faster than the speed of light. Let us look at this in terms of energy and momentum. Consider two electrons in the frame S that have speeds $v_1 = \beta_1 c$ and $v_2 = \beta_2 c$ in opposite directions. Our question is: *What is the speed ($v = \beta c$) of one electron in the rest frame of the other?* This speed is the relative speed of the two electrons.

EXAMPLE 4-20

Two electrons are directed toward each other. As measured in the laboratory frame, electron 1 has speed $v_1 = \beta_1 c$ and electron 2 has speed $v_2 = \beta_2 c$. What is the speed of one electron in the rest frame of the other? Give numerical answers for $\beta_1 = \beta_2 = 0.01$, 0.5, and 0.9.

SOLUTION:

Consider electron 1, which has a speed $v_1 = \beta_1 c$. In the laboratory frame, the energy and momentum of this electron are given by

$$E = \gamma_1 mc^2,$$

and

$$pc = \gamma_1 \beta_1 mc^2,$$

where m is the electron mass and

$$\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}.$$

To get the energy and momentum of electron 1 in the frame where electron 2 is at rest, we must make a Lorentz transformation with the speed $v_2 = \beta_2 c$ in the direction that "stops" electron 2 and speeds up electron 1. The energy and momentum of electron 1 are

$$E' = \gamma_2 E + \gamma_2\beta_2 pc = \gamma_2\gamma_1 mc^2 + \gamma_2\beta_2\gamma_1\beta_1 mc^2,$$

and

$$p'c = \gamma_2\beta_2 E + \gamma_2 pc = \gamma_2\beta_2\gamma_1 mc^2 + \gamma_2\gamma_1\beta_1 mc^2,$$

where

$$\gamma_2 = \frac{1}{\sqrt{1-\beta_2^2}}.$$

The new speed of the electron (v') is given by

$$\frac{v'}{c} = \frac{p'c}{E'} = \frac{\gamma_2\beta_2\gamma_1 mc^2 + \gamma_2\gamma_1\beta_1 mc^2}{\gamma_2\gamma_1 mc^2 + \gamma_2\beta_2\gamma_1\beta_1 mc^2} = \frac{\beta_2 + \beta_1}{1 + \beta_1\beta_2}.$$

Thus, the transformation contains the correction factor $(1 + \beta_1\beta_2)^{-1}$. For $\beta_1 = \beta_2 = 0.01$, we have

$$\frac{v'}{c} = \frac{0.01 + 0.01}{1 + (0.01)(0.01)} = 0.02.$$

For $\beta_1 = \beta_2 = 0.5$,

$$\frac{v'}{c} = \frac{0.5+0.5}{1+(0.5)(0.5)} = 0.8.$$

For $\beta_1 = \beta_2 = 0.9$,

$$\frac{v'}{c} = \frac{0.9+0.9}{1+(0.9)(0.9)} = 0.9945.$$

■

Doppler Shift

The speed of light emitted from a moving source is c; however, the energy of the individual quanta (photons) depends on the velocity of the source. The shift in energy of a moving photon source compared to the source at rest is called the *Doppler shift*. A shift in energy also means a shift in wavelength and frequency. When the motion of the light source is toward the observer, the photon energy is higher, its frequency is higher, and its wavelength is shorter. This is referred to as a *blue shift*. If the motion of the source is away from the observer, the photon energy is lower, its frequency is lower, and its wavelength is longer. This is referred to as a *red shift*. The change in energy of a photon due to motion of the source (or observer) is easily calculated from the Lorentz transformation. Consider a source of photons with energy E, in the rest frame of the source. Since the photons are massless, their momentum is E/c. When these photons are measured in a frame that is moving toward the photon source with a speed $v = \beta c$, the observed energy is

$$E' = \gamma E + \beta\gamma E = E\gamma(1+\beta) = E\sqrt{\frac{1+\beta}{1-\beta}}. \quad (4.124)$$

The ratio of photon energies is

$$\frac{E'}{E} = \sqrt{\frac{1+\beta}{1-\beta}}. \quad (4.125)$$

In terms of the photon wavelength ($E = hc/\lambda$ and $E' = hc/\lambda'$), we have

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}}. \quad (4.126)$$

This is the blue shift result.

If the photons are measured in a frame that is moving away from the photon source with a speed $v = \beta c$, the observed energy is

$$E' = \gamma E - \beta\gamma E = E\gamma(1-\beta) = E\sqrt{\frac{1-\beta}{1+\beta}}. \quad (4.127)$$

The ratio of photon energies is

$$\frac{E'}{E} = \sqrt{\frac{1-\beta}{1+\beta}}. \quad (4.128)$$

In terms of the photon wavelength ($E = hc/\lambda$ and $E' = hc/\lambda'$), we have

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}}. \quad (4.129)$$

This is the red shift result. Figure 4-13 summarizes the Doppler shift. ■

EXAMPLE 4-21

A neighboring galaxy moves away from us with a relative speed, $v = 0.1\,c$. In the rest frame of the galaxy, photons from the L_α transition in hydrogen have a wavelength $\lambda = 122$ nm. Calculate the wavelength of the photons that are detected on earth.

SOLUTION:

In the rest frame of the galaxy the energy of the photons is

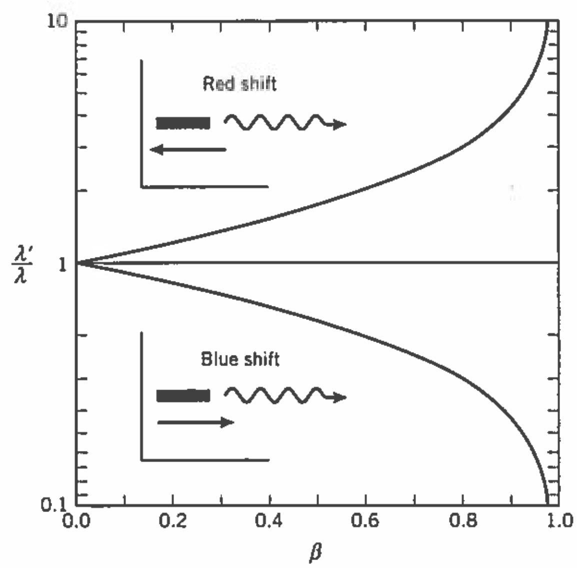


FIGURE 4-13 The Doppler shift.
The photons from a source at rest have a wavelength λ. When there is relative motion of the source toward the observer with speed v, the wavelength of the photons is observed to be blue-shifted to $\lambda' = \lambda[(1 - v/c)/(1 + v/c)]^{1/2}$. When there is relative motion of the source away from the observer with speed v, the wavelength of the photons is observed to be red shifted to $\lambda' = \lambda[(1 + v/c)/(1 - v/c)]^{1/2}$.

$$E = \frac{hc}{\lambda}.$$

The photons are Doppler red shifted. The observed energy of these photons is

$$E' = E\sqrt{\frac{1-\beta}{1+\beta}}.$$

The observed wavelength of the photons is

$$\lambda' = \frac{hc}{E'} = \frac{hc}{E}\sqrt{\frac{1+\beta}{1-\beta}} = (122\,\text{nm})\sqrt{\frac{1.1}{0.9}} = 135\,\text{nm}. \quad \blacksquare$$

EXAMPLE 4-22

Calculate the speed of a galaxy relative to the earth if the Doppler red shift causes the photon wavelengths to be doubled.

SOLUTION:

In the rest frame of the galaxy the energy of the photons is

$$E = \frac{hc}{\lambda}.$$

The observed energy of these photons (E') is one-half as large because the wavelength is doubled.

$$E' = \frac{E}{2} = E\sqrt{\frac{1-\beta}{1+\beta}}.$$

Therefore

$$\sqrt{\frac{1-\beta}{1+\beta}} = \frac{1}{2}.$$

Solving for β, we get

$$\beta = \frac{3}{5} = 0.6.$$

The speed of the galaxy is

$$v = \beta c = (0.6)\left(3\times10^8\ \text{m/s}\right) = 1.8\times10^8\ \text{m/s}. \quad \blacksquare$$

EXAMPLE 4-23

A π^0 particle (m=135 MeV/c^2) with an energy equal to 10 times its mass energy decays into two photons. One of the photons goes in the same direction as the π^0 particle and the other photon goes in the opposite direction. What are the resulting energies of the two photons?

SOLUTION:

In the rest frame of the π^0 particle, each photon has an energy

$$E = \frac{mc^2}{2} = \frac{135\,\text{MeV}}{2} = 67.5\,\text{MeV},$$

where m is the mass of the π^0 particle. In the frame where the π^0 particle is moving, both photons are Doppler shifted. The relativistic gamma factor of the pion is

$$\gamma = \frac{E}{mc^2} = 10.$$

The beta factor of the pion is

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} = \sqrt{0.99} = 0.995.$$

The energy of the blue shifted photon is

$$E' = \gamma E + \beta\gamma E = E\gamma(1+\beta)$$
$$= \left(\frac{135\ \text{MeV}}{2}\right)(10)(0.995) = 672\,\text{MeV}.$$

The energy of the red shifted photon is

$$E' = \gamma E - \beta\gamma E = E\gamma(1-\beta) = \frac{E}{\gamma(1+\beta)}$$
$$= \frac{135\,\text{MeV}}{(2)(10)(1.995)} = 3.38\,\text{MeV}.$$

Note that when beta is close to one, we want to avoid using factors of $(1-\beta)$ in getting a numerical answer. We do this by making use of the relationship: $\gamma^2 = [(1-\beta)(1+\beta)]^{-1}$. $\blacksquare$

`* Challenging`

4-4 FOUR-VECTORS

The second postulate of special relativity relates the space and time coordinates. The energy and momentum of a particle are related in the same fashion. We can make use of this relationship by defining space–time and energy–momentum vectors that each have four components. The main concept of a four-vector is that it is a quantity whose length is an invariant. The length of the space–time four-vector represents the physics that the speed of light is the same in all frames of reference. The length of the energy–momentum four-vector represents the physics that the mass of a particle is the same in all frames of reference. We shall have two benefits from the use of the four-vectors: (1) The Lorentz transformation will appear in a simple form that is very easy to remember, (2) many calculations are made easier with this simple notation.

The Space–Time Four-Vector

The zeroth component of the space–time four-vector (R_0) is defined to be the time coordinate times the speed of light:

$$R_0 \equiv ct. \tag{4.130}$$

The first, second, and third components of the four-vector (R_1, R_2, and R_3) are defined to be the space coordinates

$$R_1 \equiv x, \tag{4.131}$$

$$R_2 \equiv y, \tag{4.132}$$

and

$$R_3 \equiv z. \tag{4.133}$$

We write the four-vector as a column matrix,

$$\mathbf{R} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}. \tag{4.134}$$

The dot product between two four-vectors,

$$\mathbf{R}_a = \begin{pmatrix} ct_a \\ \mathbf{x}_a \end{pmatrix}, \tag{4.135}$$

and

$$\mathbf{R}_b = \begin{pmatrix} ct_b \\ \mathbf{x}_b \end{pmatrix}, \tag{4.136}$$

is defined to be

$$\mathbf{R}_a \cdot \mathbf{R}_b \equiv ct_a ct_b - \mathbf{x}_a \cdot \mathbf{x}_b. \tag{4.137}$$

The length of the four-vector is

$$\sqrt{\mathbf{R}\cdot\mathbf{R}} = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)^2}. \tag{4.138}$$

The length of the four-vector is an invariant; one obtains the same result in every inertial frame. This is a consequence of the fact that the speed of light is constant.

EXAMPLE 4-24

Show that the quantity $r_0{}^2 = \left[(ct)^2 - (x^2 + y^2 + z^2)\right]$ is unchanged by the Lorentz transformation.

SOLUTION:

Let

$$r_0{}'^2 = (ct')^2 - x'^2 - y'^2 - z'^2.$$

From the Lorentz transformation,

$$\begin{aligned} r_0'^2 &= \left(\gamma ct - \frac{\gamma x v}{c}\right)^2 - (\gamma x - \gamma v t)^2 - y^2 - z^2 \\ &= \gamma^2 c^2 t^2 + \frac{\gamma^2 x^2 v^2}{c^2} - (\gamma^2 v^2 t^2 + \gamma^2 x^2) - y^2 - z^2 \\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\right)c^2 t^2 - \gamma^2\left(1 - \frac{v^2}{c^2}\right)x^2 - y^2 - z^2. \end{aligned}$$

Notice that

$$\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

by definition of γ. Therefore,

$$r_0{}'^2 = (ct)^2 - (x^2 + y^2 + z^2) = r_0^2.$$

The transformation has left the quantity r_0^2 unchanged. ■

The Energy and Momentum Four-Vector

The energy and momentum of a particle form a four-vector, analogous to the space–time four-vector. The zeroth component of the energy–momentum four-vector (P_0) is defined to be the total energy:

$$P_0 \equiv E. \tag{4.139}$$

The first, second, and third components of the four-vector (P_1, P_2, and P_3) are defined to be the components of momentum times the speed of light:

$$P_1 \equiv p_x c, \tag{4.140}$$

$$P_2 \equiv p_y c, \tag{4.141}$$

and

$$P_3 \equiv p_z c. \tag{4.142}$$

The complete four-vector is

$$\mathbf{P} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} E \\ \mathbf{p}c \end{pmatrix}. \tag{4.143}$$

The dot-product between two four-vectors,

$$\mathbf{P}_a = \begin{pmatrix} E_a \\ \mathbf{p}_a c \end{pmatrix} \tag{4.144}$$

and

$$\mathbf{P}_b = \begin{pmatrix} E_b \\ \mathbf{p}_b c \end{pmatrix}, \tag{4.145}$$

is

$$\mathbf{P}_a \cdot \mathbf{P}_b = E_a E_b - \mathbf{p}_a \cdot \mathbf{p}_b c^2. \tag{4.146}$$

The length of the four-vector is

$$\sqrt{\mathbf{P} \cdot \mathbf{P}} = \sqrt{E^2 - (pc)^2}. \tag{4.147}$$

The length of the energy–momentum four-vector is the particle mass energy. It is an invariant; one obtains the same result no matter what frame it is evaluated in.

The sum of two four-vectors $(\mathbf{P}_a + \mathbf{P}_b)$ is also a four-vector.

$$\mathbf{P}_a + \mathbf{P}_b = \begin{pmatrix} E_a + E_b \\ \mathbf{p}_a c + \mathbf{p}_b c \end{pmatrix}. \tag{4.148}$$

Its length is an invariant,

$$\begin{aligned}(\mathbf{P}_a + \mathbf{P}_b)^2 &= (E_a + E_b)^2 - (\mathbf{p}_a c + \mathbf{p}_b c)^2 \\ &= (m_a c^2)^2 + (m_b c^2)^2 \\ &\quad + 2E_a E_b - 2\mathbf{p}_a \cdot \mathbf{p}_b c^2. \end{aligned} \tag{4.149}$$

It may be evaluated in any frame with identical results.

The Lorentz Transformation as Matrix Multiplication

The four equations of the Lorentz transformation may be written as matrix multiplication. If we represent a four-vector as a (4 × 1) column vector, then the Lorentz transformation may be written as a (4 × 4) matrix times the column vector. The result of the matrix multiplication is a new (4 × 1) column vector whose components are the transformed four-vector. The space–time four-vector transforms as

$$\begin{aligned}\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{pmatrix}. \end{aligned} \tag{4.150}$$

The energy–momentum four-vector transforms as

$$\begin{aligned}\begin{pmatrix} E' \\ p_x' c \\ p_y' c \\ p_z' c \end{pmatrix} &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \\ &= \begin{pmatrix} \gamma E - \beta\gamma p_x c \\ -\beta\gamma E + \gamma p_x c \\ p_y c \\ p_z c \end{pmatrix}. \end{aligned} \tag{4.151}$$

The inverse transformation is obtained by changing the sign of beta.

Center-of-Mass Energy

Consider the case where a particle with mass m_a, momentum $\mathbf{p}_a$, and energy E_a collides with a particle of mass m_b at rest. If we take the projectile particle to be moving in the x direction, then the sum of the initial four vectors is

$$\begin{pmatrix} E_a \\ p_a c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} m_b c^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_a + m_b c^2 \\ p_a c \\ 0 \\ 0 \end{pmatrix}. \tag{4.152}$$

Now consider the collision in the *center-of-mass* frame, the frame where the sum of the momenta of the colliding particles is zero. We denote the momentum of each particle in the center-of-mass frame by (p^*), and the energies by (E_a^* and E_b^*). The sum of the four vectors in the center-of-mass frame is

$$\begin{pmatrix} E_a^* \\ p^*c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} E_b^* \\ -p^*c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_a^*+E_b^* \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.153)$$

The total center-of-mass energy ($\sqrt{s}$) is

$$\begin{aligned}\sqrt{s} &= E_a^* + E_b^* \\ &= \sqrt{(p^*c)^2+(m_a c^2)^2} \\ &\quad + \sqrt{(p^*c)^2+(m_b c^2)^2}\end{aligned}. \quad (4.154)$$

Since the length of a four-vector is invariant, the total energy squared in the center-of-mass frame is

$$\begin{aligned}s &= (E_a^*+E_b^*)^2 = (E_a+m_b c^2)^2 - (p_a c)^2 \\ &= E_a^2 + (m_b c^2)^2 + 2E_a m_b c^2 - (p_a c)^2 .\end{aligned} \quad (4.155)$$

Using

$$E_a^2 = (p_a c)^2 + (m_a c^2)^2, \quad (4.156)$$

we have

$$s = (m_a c^2)^2 + (m_b c^2)^2 + 2E_a m_b c^2. \quad (4.157)$$

The total energy in the center-of-mass frame is

$$\sqrt{s} = \sqrt{(m_a c^2)^2 + (m_b c^2)^2 + 2E_a m_b c^2}\ . \quad (4.158)$$

We may use the expression for center-of-mass energy (4.154) to solve for the center-of-mass momentum:

$$\begin{aligned}p^*c = \frac{1}{2\sqrt{s}}\big[&s+(m_a c^2)^2+(m_b c^2)^2-2sm_a c^2 \\ &-2sm_b c^2 - 2m_a c^2 m_b c^2\big]^{1/2}.\end{aligned} \quad (4.159)$$

For the center-of-mass energy large compared to the particle masses, we have

$$p^*c \approx \frac{\sqrt{s}}{2} \approx \frac{\sqrt{2E_a m_b c^2}}{2}. \quad (4.160)$$

EXAMPLE 4-25

Two protons each with energy of 500 GeV travel in the opposite direction and collide. Calculate the energy of one of the protons in the frame where the other proton is at rest.

SOLUTION:

The energy in the center-of-mass is

$$\sqrt{s} = 500\,\text{GeV} + 500\,\text{GeV} = 1000\,\text{GeV}.$$

We have

$$\sqrt{s} = \sqrt{m_a c^2 + m_b c^2 + 2E_a m_b c^2} \approx \sqrt{2E_a m_b c^2}\ .$$

The proton energy is

$$E_a \approx \frac{s}{2m_b c^2} = \frac{(1000\,\text{GeV})^2}{(2)(0.94\,\text{GeV})} = 5.3\times 10^5\ \text{GeV}. \blacksquare\ *$$

4-5 COMPTON SCATTERING

The photon interacts with all particles that have electric charge. The process of a photon scattering from a charged particle is called Compton scattering after Arthur Compton, who made the first measurements of photon–electron scattering in 1922. Compton studied the process

$$\gamma + e \rightarrow \gamma + e. \quad (4.161)$$

We shall consider the scattering of a photon from an electron at rest, the "laboratory frame." In the initial state (before the scattering) we let the photon have a momentum of p_1. In the final state (after the scattering) the photon has a momentum p_2 and scattering angle θ, and the electron has a momentum p_e and scattering angle ϕ, where both angles are measured with respect to the incident photon direction as shown in Figure 4-14.

Conservation of energy gives

$$p_1 c + mc^2 = p_2 c + \sqrt{p_e^2 c^2 + m^2 c^4}\ . \quad (4.162)$$

Conservation of momentum gives

$$p_1 c = p_2 c\cos\theta + p_e c\cos\phi, \quad (4.163)$$

and

$$0 = -p_2 c\sin\theta + p_e c\sin\phi, \quad (4.164)$$

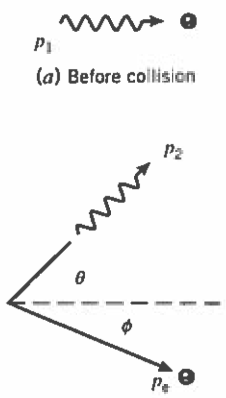


FIGURE 4-14 Definition of the variables in Compton scattering.
(*a*) A photon with momentum p_1 scatters from an electron (mass m) at rest. (*b*) After the scatter, the photon momentum is equal to p_2 and the electron momentum is equal to p_e. The scattering angle of the photon is θ and the scattering angle of the electron is ϕ.

where m is the electron mass. The electron energy and momentum are given by

$$\sqrt{p_e{}^2c^2 + m^2c^4} = p_1c + mc^2 - p_2c, \quad (4.165)$$

$$p_ec\cos\phi = p_1c - p_2c\cos\theta, \quad (4.166)$$

and

$$p_ec\sin\phi = p_2c\sin\theta. \quad (4.167)$$

We may solve for the electron mass energy squared by squaring the electron energy and subtracting the square of the electron momentum:

$$\begin{aligned} m^2c^4 &= (p_1c + mc^2 - p_2c)^2 \\ &\quad -[(p_1c - p_2c\cos\theta)^2 + (p_2c\sin\theta)^2] \\ &= p_1{}^2c^2 + m^2c^4 + p_2{}^2c^2 + 2p_1cmc^2 - 2p_2cmc^2 \\ &\quad -2p_1cp_2c - p_1{}^2c^2 - p_2{}^2c^2\cos^2\theta \\ &\quad +2p_1cp_2c\cos\theta - p_2{}^2c^2\sin^2\theta, \end{aligned} \quad (4.168)$$

which reduces to

$$mc^2(p_1c - p_2c) = p_1cp_2c(1-\cos\theta). \quad (4.169)$$

Solving for p_2c,

$$p_2c = \frac{mc^2p_1c}{mc^2 + p_1c(1-\cos\theta)}. \quad (4.170)$$

This is the *Compton formula*, which gives the momentum of the scattered photon (p_2) in terms of the incident photon momentum (p_1), the scattering angle (θ), and the electron mass (m).

We now look at some limiting cases. When the incident photon energy is much smaller than the electron mass energy ($p_1c \ll mc^2$), we have

$$p_2c \approx p_1c, \quad (4.171)$$

and the photon energy is unchanged by the scattering. This corresponds to a photon scattering from a "brick wall." Momentum is transferred but essentially no energy is transferred.

When the scattering angle is zero (forward scattering), we have

$$p_2c = p_1c, \quad (4.172)$$

and the photon energy is unchanged for any incident photon energy.

For $\theta = \pi$ (backward scattering), we have

$$p_2c = \frac{mc^2p_1c}{mc^2 + 2p_1c}. \quad (4.173)$$

For backward scattering in the case of a large incident photon energy ($p_1c \gg mc^2$), we have

$$p_2c = \frac{mc^2}{2}. \quad (4.174)$$

The Compton scattering formula may be written in terms of the photon wavelengths before (λ_1) and after (λ_2) the scattering:

$$p_1 = \frac{h}{\lambda_1}, \quad (4.175)$$

and

$$p_2 = \frac{h}{\lambda_2}. \quad (4.176)$$

We have

$$mc^2\left(\frac{hc}{\lambda_1} - \frac{hc}{\lambda_2}\right) = \frac{h^2c^2(1-\cos\theta)}{\lambda_1\lambda_2}, \quad (4.177)$$

or

$$(\lambda_2 - \lambda_1) = \frac{hc}{mc^2}(1 - \cos\theta). \qquad (4.178)$$

The change in wavelength of the photon depends only on the scattering angle θ. The quantity hc/mc^2 is called the *Compton wavelength* of the electron. The numerical value of the Compton wavelength (λ_C) is

$$\lambda_C = \frac{hc}{mc^2} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.511\,\text{MeV}} = 2.43\times10^{-6}\ \text{m}. \qquad (4.179)$$

The first data obtained by Compton are shown in Figure 4-15. Compton scattered energetic photons that he obtained from nuclear decays, and measured the wavelength at scattering angles of 0°, 45°, 90°, and 135°. The data show that the wavelength of the scattered photons depends linearly on $(1 - \cos\theta)$ and that the constant of proportionality is hc/mc^2.

4-6 DISCOVERY OF THE POSITRON

The *positron* was discovered in 1932 by Carl Anderson. Anderson observed 15 "positive electrons" in a sample of 1300 cosmic ray tracks photographed in a Wilson cloud chamber. The cloud chamber was placed in a magnetic field of 1.5 T. To establish the direction of motion of the particle, a 6-mm-thick lead plate was placed inside the chamber oriented such that the normal to the surface of the plate was perpendicular to the magnetic field. When a charged particle passes through the lead plate, it loses energy by electromagnetic interactions with the electrons in the lead atoms. The radius of curvature of the particle is therefore smaller after it passes through the plate. A cloud chamber photograph of the positron recorded by Anderson is shown in Figure 4-16. The radius of curvature is clearly smaller on the top portion of the photograph, indicating that the particle has traveled from bottom to top. The direction of the magnetic field, which is into the page in Figure 4-16, gives the sign of the electric charge to be positive. The momentum of the particle is determined by measurement of its radius of curvature and direction of travel. Before it enters the lead plate, the momentum of the particle is about 63 MeV/c. The momentum of the particle after it leaves the lead plate is about 23 MeV/c. The key to the identification of this and other similar tracks as being due to positrons is to establish that the particles have a positive charge and a small mass (much smaller than the proton mass).

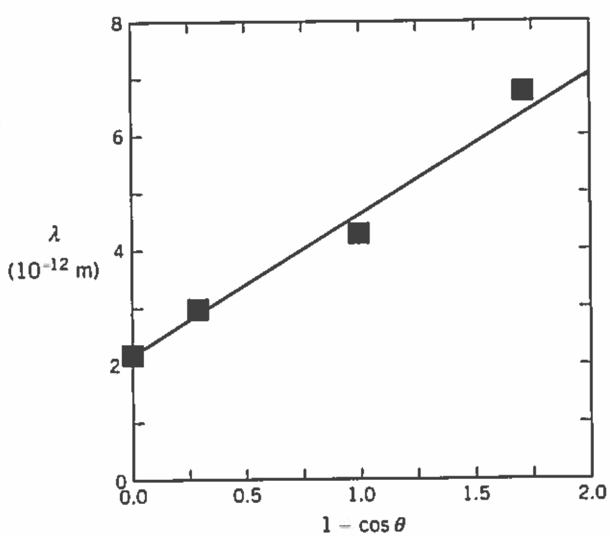


FIGURE 4-15 Data of Compton.
In his original experiment, Compton scattered photons of wavelength $\lambda = 2.2 \times 10^{-12}$ m from electrons and measured the wavelengths of radiation scattered at angles of 0°, 45°, 90°, and 135°. We plot his data in the form λ versus $(1-\cos\theta)$. The data fit a straight line with slope hc/mc^2. From A.H. Compton, *Phys. Rev.* **21**, 483 (1923).

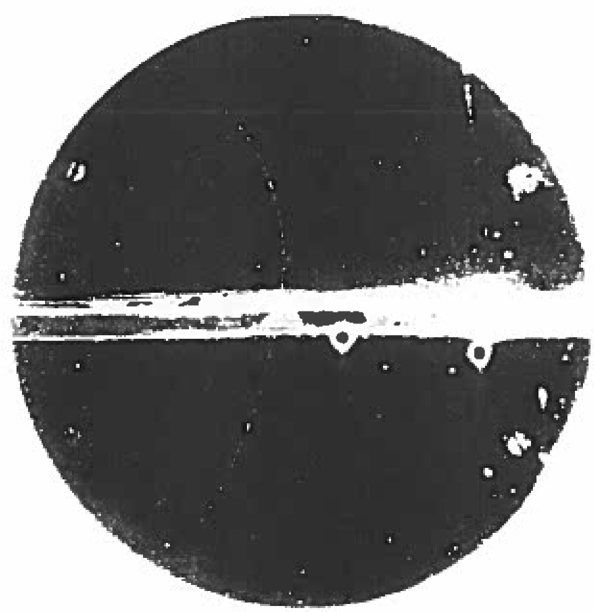


FIGURE 4-16 Discovery of the positron.
From C. Anderson, "The Positive Electron," *Phys. Rev.* **43**, 491 (1933).

EXAMPLE 4-26

Show that the change in momentum of the particle of Figure 4-16 is inconsistent with that of a proton. Find the maximum mass that is consistent with a change in momentum from 63 MeV/c to 23 MeV/c.

SOLUTION:

The momentum loss from passing through the lead plate is

$$\Delta p = 40 \text{ MeV}/c.$$

The minimum energy loss occurs when the particle is relativistic. Nonrelativistic particles lose more energy when passing through the plate because they spend more time experiencing the force of the atomic electrons. Since a relativistic particle has $E \approx pc$, the minimum energy loss (ΔE_{min}) is

$$\Delta E_{\text{min}} \approx 40 \text{ MeV}.$$

The kinetic energy of the particle, if it were a proton, would be

$$E_k = \frac{p^2}{2m} = \frac{(63 \text{ MeV}/c)^2}{(2)(940 \text{ MeV}/c)} \approx 2 \text{ MeV}.$$

The particle cannot be a proton because it does not have 40 MeV to lose! To be consistent with a change in momentum of $\Delta p = 40$ MeV/c, the particle must have a kinetic energy of at least 40 MeV. The kinetic energy of a particle of mass M is

$$E_k = \sqrt{(pc)^2 + (Mc^2)^2} - Mc^2.$$

The minimum kinetic energy is given by the expression

$$\sqrt{(pc)^2 + (Mc^2)^2} - Mc^2 = \Delta pc.$$

We solve this expression for the mass,

$$M = \frac{(pc)^2 - (\Delta pc)^2}{2\Delta pc}.$$

Numerically, we have

$$Mc^2 = \frac{(63 \text{ MeV})^2 - (40 \text{ MeV})^2}{(2)(40 \text{ MeV})} = 30 \text{ MeV}.$$

The maximum mass that is consistent with this change in momentum is 30 MeV/c^2. ■

The precise value of the positron mass was measured in subsequent experiments. The mass of a particle mass is determined by measuring both its energy and its momentum. The positron mass is found to be *identical* to the electron mass:

$$m_{\text{positron}} = m_{\text{electron}}. \tag{4.180}$$

The positron (e^+) is the *antiparticle* of the electron (e^-), and vice versa. An electron–positron pair can annihilate by

$$e^+ + e^- + N \rightarrow \gamma + N, \tag{4.181}$$

where N is a nucleus. A pair can be created by

$$\gamma + N \rightarrow e^+ + e^- + N. \tag{4.182}$$

The processes of pair-production is shown in Figure 4-17.

CHAPTER 4: PHYSICS SUMMARY

- The speed of light in vacuum is

  $$c = 3.00 \times 10^8 \text{ m/s},$$

  independent of the motion of the source.

- No particle can have a speed faster than the speed of light in vacuum (c).
- Space and time make a four–vector that transforms as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{pmatrix},$$

when going from one frame to another frame with relative speed (v) in the x direction, where

$$\beta = \frac{v}{c},$$

and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

The inverse transformation is obtained by changing β to $-\beta$.

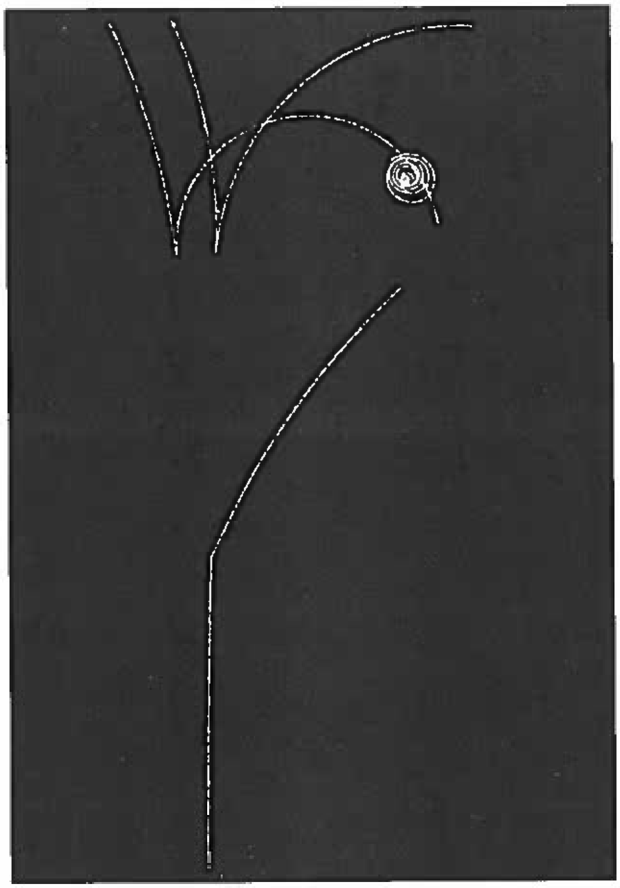


FIGURE 4-17 Electron-positron pair production.
The photograph was taken in a bubble chamber, a device similar to the cloud chamber except the chamber is filled with liquid hydrogen. The bubble chamber records the trajectories of charged particles in a magnetic field. A particle called a K^- meson (bottom of photo) enters the chamber and decays into a π^0 meson and a π^- meson. The π^0 decays into two photons each of which converts into an electron-positron pair on passing through a lead sheet. Lawrence Berkeley Laboratory/Science Photo Library/ Photo Researchers.

- A time interval measured in a frame where the clock is at rest is called the proper time. In all other frames of reference, the observed time interval is longer by a factor of γ, where the speed appearing in γ is the relative speed of the frame. Applied to the decay of a particle, the proper lifetime is the lifetime measured in the rest frame of the particle.
- A distance measured in a frame where the object is at rest is called the proper length. In all other frames of reference, the observed length is shorter by a factor of $1/\gamma$, where the speed appearing in γ is the relative speed of the frame.
- The total energy (E), momentum (p), and mass (m) of a particle are related by

$$E = \sqrt{(pc)^2 + (mc^2)^2}\,.$$

This quantity is observed to be conserved in all particle collisions.

- For the case where mass of a particle is not equal to zero, the momentum is given by

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = mv\gamma\,.$$

This quantity is observed to be conserved in particle collisions. An alternate expression for the energy in this case is

$$E = \gamma mc^2.$$

- The speed of a particle divided by the speed of light is

$$\frac{v}{c} = \frac{pc}{E}\,.$$

- The mass of a particle is the same in all frames of reference.
- Energy and momentum make a four-vector that transforms as

$$\begin{pmatrix} E' \\ p_x'c \\ p_y'c \\ p_z'c \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_xc \\ p_yc \\ p_zc \end{pmatrix}$$
$$= \begin{pmatrix} \gamma E - \beta\gamma p_x c \\ -\beta\gamma E + \gamma p_x c \\ p_y c \\ p_z c \end{pmatrix}.$$

- For the circular motion of a charged particle in a magnetic field, the momentum, charge, radius of curvature, and magnetic field are related by

$$pc = erB.$$

- For the case of a photon scattering off a free electron at rest, the change in wavelength of the photon depends only on the scattering angle,

$$(\lambda_2 - \lambda_1) = \frac{hc}{mc^2}(1 - \cos\theta).$$

- The positron is the antiparticle of the electron. The mass of the positron is identical to the electron mass. Electron–positron pairs are created when energetic photons interact with a nucleus,

$$\gamma + N \rightarrow e^+ + e^- + N.$$

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QUESTIONS AND PROBLEMS

The foundations of special relativity

1. Consider two observers each in an inertial frame of reference. What two speed measurements do the two observers agree on?
2. In Newton's theory the gravitational force between two masses depends inversely on the square of their distance of separation. Show that this is inconsistent with the postulates of special relativity. (*Hint:* To calculate the forces between the two masses in Newton's theory, we need to know their relative positions at the same time.)
3. An airplane travels at a constant speed v for a distance of 3000 km as measured by a stationary observer. The pilot measures the flight time to be Δt and the stationary observer measures the flight time to be $\Delta t'$. (a) Which time interval is longer? (b) If $|\Delta t - \Delta t'| = 4$ ns, determine the speed of the airplane. (Time dilation has been directly measured using atomic clocks flown on commercial airplanes.)
4. Show that the phase shift in the Michelson-Morley experiment is given by

$$\Delta\phi = \frac{2Lv^2}{\lambda c^2}.$$

The relationship between space and time

5. An electron is shot with a relativistic speed v in the x direction and another is shot at a time Δt later in the y direction with the same speed. Does the relative speed of the two electrons depend on the time interval Δt?
6. In the laboratory frame a particle with a speed $v = 0.99c$ travels a distance of 1 mm before spontaneously decaying. What is the proper lifetime of the particle?
7. The tau lepton has a proper lifetime of 0.3 ps. What speed must tau particles have in order to travel an average distance of 1 mm?
8. (a) The muon has a proper lifetime of 2.2 μs. If a muon has a speed, $v = 0.99c$, what is the average distance that it travels before decaying? (b) The charged pion has a proper lifetime of 26 ns. If a pion has a speed, $v = 0.99c$, what is the average distance that it travels before decaying?

*9. *The twin paradox.* An astronaut is accelerated in his spaceship to a cruising speed v and travels from the earth to a faraway destination and back, a total distance d. The astronaut's twin stays at home on the earth. Assume that the time needed to reach cruising speed and the time needed to turn the spaceship around are negligible compared to the time taken to complete the trip. (a) Analyze the trip in the frame of the twin that stays home. Which twin has aged more and by how much? (b) Analyze the trip in the frame of the traveling twin. Which twin has aged more and by how much? (c) Which twin has actually aged more and by how much? (Hint: Make use of the first postulate of special relativity.)

10. Relativistic protons that have a certain speed v in the laboratory frame are selected by measuring the time that it takes the proton to travel between two detectors separated by a distance L. Each detector produces an electronic pulse of very short duration ($\Delta t \ll L/v$) when a proton passes through it. A coincidence circuit is made by delaying the pulse from the first detector by an amount L/v. The signals from the two detectors are fed into a logic circuit that produces an output pulse if the input pulses arrive at the same time. For input pulses that arrive at the same time as measured in the laboratory frame, calculate the time difference between arrival of the input pulses as measured in the rest frame of the proton.

• **11.** An observer measures the velocity of two electrons and finds that one has a speed $c/2$ in the x direction and the other has a speed $c/2$ in the y direction. What is the relative speed of the two electrons?

The relationship between energy and momentum

• **12.** What is the speed of a particle that has a momentum equal to its mass times the speed of light?

• **13.** Determine the momentum and speed of a proton that has a kinetic energy of 1.00 GeV.

14. An electron has a kinetic energy of 1 MeV. What is the radius of curvature in a uniform magnetic field of 1 T, if the electron velocity is perpendicular to the field?

15. An electron and a proton have the same radius of curvature in a uniform magnetic field, and the electron speed is twice that of the proton. What is the momentum of each particle?

16. Calculate the radius of curvature of a 10-GeV electron in a magnetic field of 1 T.

17. Two protons and two neutrons bind together to form the nucleus of the helium atom (the alpha particle). The binding energy is 28.4 MeV. Calculate the mass of the alpha particle.

• **18.** The carbon-14 nucleus decays into a nitrogen-14 nucleus plus an electron and a massless particle called the electron-antineutrino. Calculate the amount of energy released in the decay.

19. Lambda particles ($m = 1116$ MeV/c^2) are produced with momenta of 100 GeV/c as the result of energetic proton–proton collisions. The lambda decays into a proton and a pion with a lifetime of 10^{-10} s. In the frame in which the lambda has $p = 100$ GeV/c, which particle has a larger momentum on the average, the proton or the pion? Why?

***20.** A Z^0 particle, which has a mass of about 91 GeV/c^2, is produced in proton–antiproton collisions, where each of the protons has an energy of 270 GeV. The proton and antiproton are composite objects, and the fundamental interaction for Z^0 production is the annihilation of a quark and an antiquark. If the quark has an energy of 30 GeV, calculate the energy of the antiquark. (Assume that the quark and antiquark are massless.)

21. An electron has a kinetic energy equal to its mass energy. Find the energy of a photon that has the same momentum as the electron.

22. A relativistic subatomic particle of mass m is moving away from a detector when it spontaneously decays, sending a photon toward the detector. The photon is observed to be red shifted by a factor of 100, that is, its energy is 1% of the energy of the photon measured in the frame where the decaying particle is at rest. What is the speed of the decaying particle?

• **23.** Two electrons have a relative speed of $0.9c$. Calculate the momentum of each electron in the center-of-mass frame, the frame where they have equal and opposite momenta.

24. A 10-MeV alpha particle ($mc^2 \approx 3700$ MeV) collides head on with an electron at rest. Find the speed of the electron after the collision.

• **25.** A very energetic cosmic ray proton has a speed that differs from the speed of light by one part in 10^{24}. What is the energy of the proton?

26. A π^0 meson moving with speed v in the z direction decays into two photons. One of the photons travels in the z direction and the other travels in the minus z direction. (a) If one photon has an energy that is nine times that of the other photon, calculate the speed of the π^0 meson. (b) If the speed of the π^0 meson is $c/2$, determine the energies of the two photons.

27. In the frame S, a beam of photons has an energy E. In the frame S', which moves opposite the direction of the photon beam with speed v', the photons have an energy of $E' = 10\,E$. Calculate the speed of the frame v'.

28. A π^0 meson is moving in the x direction when it spontaneously decays into two massless photons. One of the photons travels in the x direction and the other travels in the minus x direction. The difference in energy between the two photons is equal to the mass energy of the pion. Calculate the speed of the pion.

29. A Z^0 particle at rest decays into an electron and positron. The mass of the Z^0 particle is 91.2 GeV/c^2. Calculate the energy and momentum of the electron. By how much does the speed of the electron differ from c?

***30.** Quarks were discovered inside the proton by scattering high energy (E_i) electrons from a hydrogen target and measuring the energy (E_f) and angle (θ) of the scattered electrons. The process is electron–quark elastic scattering,

$$e + q \rightarrow e + q\,.$$

In this process we may neglect the masses of both the quark and the electron. Let the variable x represent the proton momentum fraction of the quark that scatters, that is, if p^*_p is the momentum of the proton in the electron–quark center-of-mass system, then the quark momentum is $p^*_q = xp^*_p$. Derive an expression for x in terms E_i, E_f, θ, and the proton mass (M).

Four-vectors

31. A proton with a momentum of 200 GeV/c collides with a proton at rest. Calculate the total energy in the center-of-mass system.

***32.** An experiment is designed to study pions produced by the interactions of relativistic protons with protons at rest. The experimenters wish to study relativistic pions produced at 90° with respect to the axis of the colliding protons in the proton–proton center-of-mass system. If the incident protons have momentum p ($pc >> mc^2$ where m is the proton mass), at what laboratory angle should the experimenters place their pion detector?

Compton scattering

33. A photon with kinetic energy equal to the mass energy of an electron collides with an electron at rest and scatters at an angle of $\pi/2$. Calculate the energy of the electron after the collision. (Give an exact answer in terms of the electron mass energy, mc^2).

34. A photon of energy E collides with an electron at rest. Calculate the maximum amount of energy E_k that may be transferred to the electron. Make a graph of E_k versus E, labeling the scale in electronvolts.

35. Consider Compton scattering of photons with energy E by electrons with mass m and momentum p moving in a direction opposite the photon direction. (a) Give an exact expression relating the energy E' of backward scattered photons with the incident photon energy E, the incident electron momentum p and the electron mass m. (b) What is the value of E' for the special case of $p = 0$ and $E = 1$ MeV? (c) What is the value of E' for the special case of $p = E/c$?

36. (a) A photon with an energy of 10 GeV scatters with an electron at rest and scatters backwards. What is the energy of the electron after the collision? What is the energy of the photon after the collision? (b) A photon with an energy of 10 eV collides with an electron that has an energy of 10 GeV. The photon scatters backwards. What is the energy of the photon after the collision? What is the electron energy after the collision?

37. A 1-MeV photon collides with a proton at rest and is scattered at an angle of 45°. (a) Calculate the energy of the scattered photon. (b) Calculate the kinetic energy of the proton.

38. Prove that in Compton scattering of a photon with an electron at rest, the electron may not be scattered at an angle greater than $\pi/2$ in the laboratory frame.

***39.** A beam of energetic photons is made by Compton scattering of a laser beam containing photons of energy 1 eV with electrons of energy 20 GeV. Calculate the maximum possible energy of the scattered photons.

***40.** A photon of momentum p_1 scatters from a charged particle of mass m at rest and the particle gains an energy E_k from the collision. Calculate the angle (ϕ) that the momentum vector of the struck particle makes with respect to the photon direction.

***41.** A photon with energy E collides with an electron at rest in the laboratory frame. (a) Calculate the speed of the electron in the center-of-mass frame. (b) Calculate the total energy in the center-of-mass frame. (c) The photon is scattered at 90° in the center-of-mass frame. Calculate the photon scattering angle in the laboratory frame.

Discovery of the positron

42. When a positron passes through a lead plate (see Figure 4-16), there is an electromagnetic force be-

an energy of $E' = 10\ E$. Calculate the speed of the frame v'.

28. A π^0 meson is moving in the x direction when it spontaneously decays into two massless photons. One of the photons travels in the x direction and the other travels in the minus x direction. The difference in energy between the two photons is equal to the mass energy of the pion. Calculate the speed of the pion.

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Discovery of the positron

42. When a positron passes through a lead plate (see Figure 4-16), there is an electromagnetic force be-

tween the positron and the atomic electrons. Why can't a positron *gain* energy by passing through a lead plate?

43. A photon collides with an electron at rest creating an electron–positron pair,

$$\gamma + e^- \rightarrow e^+ + e^- + e^-.$$

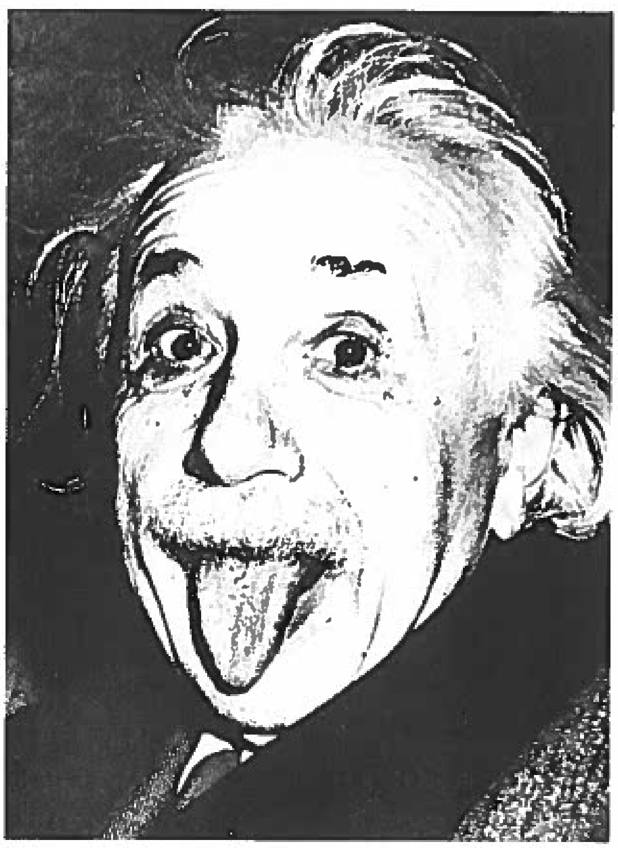
Calculate the minimum photon energy for this process.

Additional problems

44. The relativistic Doppler shift causes the light from a distant galaxy to be blue shifted by 5%. What is the speed of the galaxy relative to the earth?

45. The B meson has a mass energy equal to about 5.3 GeV and a mean lifetime of about 1 ps. With what energy do B mesons need to be produced if they are to travel an average distance of 1 mm?

46. A well-collimated beam of π^+ mesons of energy E is directed through two counters separated by a distance of 30 m. Some of the π^+ mesons decay in flight between the two counters so that the particle flux measured at the second counter is $1/e$ times that measured at the first counter. Determine the energy of the π^+ mesons. (The proper mean lifetime of the π^+ meson is 2.6×10^{-8} s; the mass of the π^+ meson is 140 MeV/c^2.)



UPI/Bettmann Archive.

*47. Verify the validity of the Lorentz transformation for energy and momentum (4.112 to 4.115). (*Hint:* Consider a particle with energy E and momentum $\mathbf{p}$ in the frame S. Write the three velocity components in terms of E and $\mathbf{p}$. Use the velocity transformation to determine the velocity components in a frame S' moving with relative speed v in the x direction. Calculate the velocity components in the frame S' in terms of E' and $\mathbf{p}'$ and show that they are identical to that obtained with the velocity transformation.)

48. A hydrogen atom in the ground state at rest absorbs a photon and makes a transition to the first excited state. Calculate the resulting speed of the atom.

49. An airplane with its running lights on is traveling at a speed of 300 m/s toward an observer. Calculate the Doppler shift of the light.

50. A galaxy is moving away from the earth with a speed of $c/2$. At what energy are the L_α photons in hydrogen (the $n = 2$ to $n = 1$ transition) expected to be observed, based on the relativistic motion of the source?

*51. If a particle has a very large momentum, its trajectory in a magnetic field will be nearly a straight line. The maximum deviation of the particle trajectory from a straight line of length L that connects the beginning and end points of the particle path is called the *sagitta* (s). (a) Show that the momentum of a particle with charge e is given by

$$p = \frac{eBL^2}{8s}.$$

(b) Calculate the sagitta of 40-GeV electron that has a track length of 2 m in a magnetic field of 0.7 T. (c) For the electron of (b) what is the maximum allowed error in determination of the sagitta if the momentum is to be determined to a precision of 25%?

*52. In 1955, a group led by Owen Chamberlain and Emilio Segrè working at the Berkeley Bevatron discovered the antiproton ($\bar{p}$) in the following reaction:

$$p + p \rightarrow p + p + p + \bar{p}.$$

If the initial state consists of a proton of energy E colliding with a proton at rest, calculate the minimum value of E for which the reaction may occur. (*Note:* The mass of the antiproton is identical to the mass of the proton).

53. Prove that the photoelectric effect cannot occur for a free electron, that is, that the process

$$\gamma + e^- \rightarrow e^-$$

is forbidden by conservation of energy and momentum.

*54. *The Bohr model.* Consider a scenario where the electron mass is nonzero but much smaller than 0.511 MeV/c^2 and where there is the possibility that an electron moves at relativistic speeds inside the hydrogen atom. Using the relativistic expression for momentum ($p = \gamma m v$) and the Bohr model quantization condition ($pr = n\hbar$), show that the speed of the electron in the ground state is equal to αc, independent of the electron mass.

55. *Positronium.* An electron and positron can form a bound state analogous to the hydrogen atom. (a) Use the Bohr model to show that the binding energy of positronium is equal to 6.8 eV. (b) Determine the Bohr radius.