Quantum Physics 1
Class 21

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Angular momentum

Review :

$$-\frac{\hbar^2}{2m}\nabla\psi(x,y,z) + V(x,y,z)\,\psi(x,y,z) = E\,\psi(x,y,z)$$

separation of variables in x, y, z :

$V(x,y,z) = 0$

$V(x,y,z) = \frac{1}{2}k_0 r^2$

Infinite square well

$V(r) \sim -\frac{1}{r}$; $r = \sqrt{x^2+y^2+z^2}$

* But separation of variables works in spherical coords.

NB : separation of variables NOT possible for x, y, z

$$\Rightarrow \left(\frac{\hat{P}_r^2}{2m_0} + \frac{\hat{L}^2}{2m_0 r}\right)\psi(r,\theta,\phi) = \left[E - V(r)\right]\psi(r,\theta,\phi)$$

where $\psi(r,\theta,\phi) = R(r)\,\underbrace{Y(\theta,\phi)}_{\text{spherical harmonics}}$

yields:

$$\frac{2mr^2}{\hbar^2}\frac{\hat{p}_r^2}{2m}R(r) + \lambda R(r) = \frac{2mr^2}{\hbar^2}(E - V(r))R(r) \quad \ldots (1)$$

$$\hat{L}^2 Y(\theta,\phi) = \lambda \hbar^2 Y(\theta,\phi) \; ; \; \lambda = l(l+1) \quad \ldots (2)$$

$$l = 0, 1, 2, \ldots$$

Now can separate variables for $Y(\theta,\phi)$ even further!

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

yields:

$$\left[-\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta}\right]\Theta(\theta) = \lambda\,\Theta(\theta) \quad \ldots (3)$$

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = -m^2\Phi(\phi) \quad \ldots (4)$$

with solution: $e^{im\phi}$; $m = 0, \pm 1, \pm 2, \ldots$

for $l=0$; $\Theta_0 \sim 1$

$l=1$; $\Theta_1 \sim \cos\theta$

$l=2$; $\Theta_2 \sim \frac{1}{2}(3\cos^2\theta - 1)$

$\vdots$

What is the angular momentum, $\hat{L}$?

Classical defn:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix}$$

$$= \underbrace{(y p_z - z p_y)}_{L_x} \hat{i} + \underbrace{(z p_x - x p_z)}_{L_y} \hat{j} + \underbrace{(x p_y - y p_x)}_{L_z} \hat{k}$$

$$\boxed{\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}}$$

In-class 12.1

Recall that $\hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$, what is $\hat{L}$ in spherical coord sys?

Consider: L_x, L_y, L_z in r, θ, ϕ coord.

use
$$\left. \begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \right\}$$

we will find that:

$$\boxed{\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}}$$

What are the corresponding eigenfunctions and eigenvalues for the operator?

Consider, $\hat{L}_z \Phi = c\Phi \Rightarrow \frac{\hbar}{i}\frac{\partial \Phi}{\partial \phi} = c\Phi$

what is solution?

let's consider,

$$\hat{L}_z^2 \Phi \Rightarrow \frac{\partial^2 \Phi}{\partial \phi^2} = -m^2 \Phi$$

yields solutions: $e^{im\phi}$

$$\therefore \quad \boxed{L_z^2 \Phi = m^2 \hbar^2 \Phi}$$

$\therefore \quad \hat{L}_z \Phi = m\hbar \Phi$ NB $[\hat{L}_z, \hat{L}_z^2] = 0$

w/ $m = 0, \pm 1, \pm 2, \ldots$

For spherical harmonics:

$$\boxed{\hat{L}^2 Y(\theta,\phi) = \ell(\ell+1)\hbar^2 Y(\theta,\phi)}$$

where, $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

$$\boxed{\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]}$$

NB : $\hat{L}^2$ is the square of the total angular momentum w/ eigenfunctions $Y(\theta, \phi)$ & eigenvalues $l(l+1)\hbar^2$

so that $\hat{L} \longrightarrow \sqrt{l(l+1)}\,\hbar$

Recap :

$$\hat{L}^2 Y(\theta, \phi) = l(l+1)\hbar^2 Y(\theta, \phi)$$

$$\hat{L}_z Y(\theta, \phi) = m\hbar Y(\theta, \phi)$$

$$\& \quad [\hat{L}^2, L_z] = 0$$

$$\& \quad Y(\theta, \phi) = \Theta(\theta)\, \Phi(\phi)$$

In-class 21.2

NB

$$\boxed{[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z}$$

compare : $[\hat{x}, \hat{p}] = i\hbar$ & uncertainty interpretation.

In-class 21-3

Compare w/ $[\hat{x}, \hat{p}_x] = i\hbar$

$[\hat{y}, \hat{p}_y] = i\hbar$

etc.

} do not commute!

but $[\hat{x}, \hat{y}] = (xy - yx) = xy - xy = 0$

& $[\hat{x}, \hat{p}_y] = \left[x \frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{\hbar}{i} \frac{\partial}{\partial y} x\right] = 0$

} commute!

Compare w/ angular momentum!

$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$ Does not commute

In-class: 21-4