Fourier Series and Transforms

Math for Physicists

Orthogonality and Completeness

- The solutions to certain types of differential equations with boundary conditions form sets of functions that are orthogonal to one another (explanation later) and when added to one another can make any physical function in the solution region.
- A good analogy is that Cartesian unit vectors are orthogonal to one another and can be added to make any vector in the space.

Orthogonality

$$\int_{solution region} \rho(x)\psi_n\psi_m dv = \delta_{nm}$$

where ψ_n and ψ_m are independent solutions

to a Sturm-Liouville equation.

S-L equations have the form: $Ly(x) = \lambda\rho(x)y(x)$

with $L = -\frac{\partial}{\partial x}p(x)\frac{\partial}{\partial x} + q(x)$

Examples of S-L equations are:

Wave equation, Legendre's equation, Laguerre's equation,

Hermite's equation, Bessel's equation.

Orthogonality of Harmonic Functions

Solutions to: $$\frac{\partial \psi}{\partial x^2} + k^2 \psi = 0$$

For n,m = integers

$$\frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} \sin \frac{2\pi n}{\lambda} \sin \frac{2\pi m}{\lambda} dx = \delta_{nm} \text{ (zero unless } n = m\text{)}$$

$$\frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} \cos \frac{2\pi n}{\lambda} \cos \frac{2\pi m}{\lambda} dx = \delta_{nm} \text{ (zero unless } n = m\text{)}$$

$$\frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} \sin \frac{2\pi n}{\lambda} \cos \frac{2\pi m}{\lambda} dx = 0$$

Orthogonality and Completeness allow us to do this

Express any physical function $f(x)$ as the sum of harmonic functions (Fourier series)

$$f(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos k_n x + B_n \sin k_n x)$$

$$k_n = n\frac{2\pi}{\lambda}$$

Finding the coefficients A_n and B_n

- Fourier's trick – multiply both sides of the series by one of the harmonic functions and integrate over one period

$$\frac{2}{\lambda} f(x)\sin k_m x = \frac{2}{\lambda} A_0 \sin k_m x + \frac{2}{\lambda}\sum_{n=1}^{\infty}(A_n \cos k_n x \sin k_m x + B_n \sin k_n x \sin k_m x)$$

$$\frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\sin k_m x dx = \frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2}\sum_{n=1}^{\infty}(A_n \cos k_n x \sin k_m x + B_n \sin k_n x \sin k_m x)dx$$

$$\frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\sin k_m x dx = B_m$$

$$\frac{2}{\lambda} f(x)\cos k_m x = \frac{2}{\lambda} A_0 + \frac{2}{\lambda}\sum_{n=1}^{\infty}(A_n \cos k_n \cos k_m x + B_n \sin k_n x \cos k_m x)$$

$$\text{for } k = 0 \quad A_0 = \frac{1}{\lambda}\int f(x)dx$$

$$\frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\sin k_m x dx = \frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2}\sum_{n=1}^{\infty}(A_n \cos k_n x \cos k_m x + B_n \sin k_n x \cos k_m x)dx$$

$$\frac{2}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\cos k_m x dx = A_m$$