Lattice Data

Spatial autoregressive models
SAR and CAR models

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Inference for areal data

For areal units the inferential issues are:

- Is there a spatial pattern? How strong is it?
  Spatial pattern suggest that observations close to each other have more similar values than those far from each other.

- Do we want to smooth the data? How much?

Introduction

The most popular models in lattice data remind us of the autoregressive models to time series

For example AR(1) (autoregressive order 1)

$$Z_t = \rho Z_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N\left(0, \sigma^2\right), \quad t = 0, \pm 1, \ldots$$

Where $\rho \in (-1,1)$ is called the autocorrelation coefficient

$$\mathrm{corr}\left(Z_t, Z_{t-k}\right) = \rho^k$$

Simultaneous autoregressive model ordre 1 SAR(1)

We transfer these ideas in the space data, where ***Z*** will be depending on the neighboring regions

$$\boxed{Z_i = \sum_j b_{ij} Z_j + \varepsilon_i \qquad \varepsilon_i \sim N\left(0, \sigma_i^2\right)}$$

Matricial form: $$Z = BZ + \varepsilon \qquad \varepsilon \sim N\left(0, \Lambda\right)$$

Reduced form $$Z = \left(I - B\right)^{-1} \varepsilon$$

$$\boxed{\begin{aligned} &E(Z) = 0 \\ &Var\left(Z\right) = \sigma^2 \left(1 - B\right)^{-1} I \left(1 - B\right)^{-T} \end{aligned}}$$

Re-parametrization of the model

$(I - B)$ Must be non-singular

Re-paremetrization of this model can by writing $B = \rho W$

$$Z = \rho W Z + \varepsilon \qquad \varepsilon \sim N(0, \Lambda) \quad \Longrightarrow \quad Z = (I - \rho W)^{-1} \varepsilon$$

$$E(Z) = 0$$

$$Var(Z) = \sigma^2 (1 - \rho W)^{-1} I (1 - \rho W)^{-T}$$

$$Z \sim N\left(0, (I - \rho W)^{-} \Lambda (I - \rho W)^{-T}\right)$$

Conditions to variance and covariance matrix will be non-singular:

W: neighborhood matrix binary (B)

(I-ρB) is non-singular, we need to impose that ρ,

$\rho \in \left(1/\lambda_1 , 1/\lambda_n\right)$ where λ are eigenvalues of W

W : neighborhood standardized by rows

(I-ρW) is non-singular, we need impose that $\rho \in (-1,1)$

Explanatory variables into a spatial regression model

Spatial error **SAR**$_{\textbf{error}}$

$$\boxed{\begin{aligned} Z &= \beta X + u \\ u &= \rho W u + \varepsilon \\ \varepsilon &\sim N\left(0, \sigma^2 I\right) \end{aligned}}$$

Spatial autocorrelation in the error, is generally considered a nuisance model, because the primary interest is in the relationship between the explanatory variables X and response variable Z

Reduced form $$Z = X\beta + (I - \lambda W)^{-}\varepsilon$$

$$E(Z) = X\beta$$

$$Var(Z) = \sigma^2 (I - \lambda W)^{-} (I - \lambda W)^{-T}$$

$$Z \sim N(X\beta, \sigma^2 (I - \lambda W)^{-} (I - \lambda W)^{-T})$$

Interpretation of the SAR error model

SAR model for the random error generate a global spatial dependence

Global: covariance matrix involved all the locations of the system

$$Var(Z) = \sigma^2 (1 - \rho W)^{-1} I (1 - \rho W)^{-T}$$

$$u = [I - \rho W]^{-1} \varepsilon$$

$[I - \rho W]^{-1} = I + \rho W + \rho^2 W^2 + \ldots$ Leontieff expansion

$$Var(Z) = I + \rho W + \rho W' + \rho^2 (W^2 + WW' + W^{2\prime}) \ldots$$

Potency of W involves a contiguity of higher order.
Potency of ρ involves progressive loss of the value of the covariance

Conditionally autoregressive models (CAR)

Introduced by Besag 1974, is called **conditionally autoregressive models (CAR).**

It is used in Gibbs sampling, particularly Monte Carlo Markov Chain to construct a hierarchical model

Definition: Conditioned distributions Z_i

$$f\left(Z_i \mid Z_j \quad j \neq i\right)$$

Gaussian distribution (autonormal)

$$E[Z_i|Z_{-i}] = X\beta + \sum_{j\neq i} c_{ij}\left(Y_j - X\beta\right)$$

$$Var[Z_i|Z_{-i}] = \sigma^2 \qquad\qquad \mathrm{i} = 1....\mathrm{n}$$

Joint distribution

Multivariate distribution of Z

$$Z \sim NMV(X\beta, Q^{-}) \qquad Q = D^{-1}(I - C)$$

Where C=c_{ij} and D is diagonal matrix with elements $d_i = \tau_i^2$

Restrictions for Q:

Ensure symmetric matrix $Q = D^{-1}(I - C)$

$$\frac{c_{ij}}{\tau_i^2} = \frac{c_{ji}}{\tau_j^2}$$ For all i and j

1. Binary definition for C (non-standardized by row)

No problems of symmetry.

2. Row-standardized weights

If C is **standardized by rows** of the binary matrix

$$\begin{cases} w_{ij}=1 \text{ if } i \text{ and } j \text{ are neighbors} \\ w_{ij}=0 \text{ other case} \end{cases}$$

$$c_{ij} = w_{ij} / w_{i+} \qquad \tau_i^2 = \tau^2 / w_{i+}$$

Conditional distribution of Z_i is Normal with expectation and variance

$$E[Z_i|Z_{-i}] = X\beta + \sum_{j\neq i} \frac{w_{ij}}{w_{i+}} (Y_j - X\beta) \qquad Var[Z_i|Z_{-i}] = \frac{\sigma^2}{w_{i+}}$$

Joint distribution of Z

$$Z \sim NMV(X\beta, \sigma^2 Q^-)$$ where $$Q = (I_w - W)$$ $$I_w = \text{diag}(w_{i+})$$

Handicap

$$\left(I_w - W\right)1 = 0 \text{ then } \mathrm{Q} \text{ is singular}$$

$$\Downarrow$$

Joint distribution might be improper

The joint distribution is improper but the conditionals are proper Model called as **intrinsically autoregressive model (IAR)**

IAR it is used to define a random spatial effects

The impropriety can be remedied. Redefine:

$\mathrm{Q} = \left(\mathrm{I}_{\mathrm{w}} - \rho \mathrm{W}\right)$ Standardized by row

Choice ρ to make Q^{-1} nonsingular

It is guarantee if

$$\rho \in \left(1/\lambda_1, 1/\lambda_n\right)$$

where $\lambda_1 < \ldots \lambda_n$ *are the* ordered eigenvalues of

$$I_w^{-1/2} \rho W I_w^{-1/2}$$

$$\lambda_1 < 0, \; and \; \lambda_n < 1$$

Binary definition for W (non-standardized by row)

$\mathrm{Q} = \mathrm{D}^{-1}\left(\mathrm{I} - \rho \mathrm{W}\right)$ is nonsingular if $\rho \in \left(1/\lambda_1, 1/\lambda_n\right)$

values $\lambda_1 < \ldots \lambda_n$ *eigenvalues* of W

COMPARATION BETWEEN SAR AND CAR

Grid 3x3 (definition of neighbors tower criterion). MATRIX W

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1		1		1					
2	1		1		1				
3		1				1			
4	1				1		1		
5		1		1		1		1	
6			1		1				1
7				1				1	
8					1		1		1
9						1		1	

SAR

$$\Sigma = \sigma^2 \left(I - \rho W\right)^{-1} \left(I - \rho W\right)^{-T}$$

CAR

$$\Sigma = \sigma^2 \left(I - \rho W\right)^{-1}$$

ρ=0.25

	1	2	3	4	5	6	7	8	9
1	1,86	1,33	0,63	1,33	1,25	0,67	0,63	0,67	0,39
2	1,33	2,48	1,33	1,25	2,00	1,25	0,67	1,02	0,67
3	0,63	1,33	1,86	0,67	1,25	1,33	0,39	0,67	0,63
4	1,33	1,25	0,67	2,48	2,00	1,02	1,33	1,25	0,67
5	1,25	2,00	1,25	2,00	3,50	2,00	1,25	2,00	1,25
6	0,67	1,25	1,33	1,02	2,00	2,48	0,67	1,25	1,33
7	0,63	0,67	0,39	1,33	1,25	0,67	1,86	1,33	0,63
8	0,67	1,02	0,67	1,25	2,00	1,25	1,33	2,48	1,33
9	0,39	0,67	0,63	0,67	1,25	1,33	0,63	1,33	1,86

	1	2	3	4	5	6	7	8	9
1	1,20	0,39	0,13	0,39	0,25	0,11	0,13	0,11	0,05
2	0,39	1,32	0,39	0,25	0,50	0,25	0,11	0,18	0,11
3	0,13	0,39	1,20	0,11	0,25	0,39	0,05	0,11	0,13
4	0,39	0,25	0,11	1,32	0,50	0,18	0,39	0,25	0,11
5	0,25	0,50	0,25	0,50	1,50	0,50	0,25	0,50	0,25
6	0,11	0,25	0,39	0,18	0,50	1,32	0,11	0,25	0,39
7	0,13	0,11	0,05	0,39	0,25	0,11	1,20	0,39	0,13
8	0,11	0,18	0,11	0,25	0,50	0,25	0,39	1,32	0,39
9	0,05	0,11	0,13	0,11	0,25	0,39	0,13	0,39	1,20

1. Different values but the properties are the same.
2. The interior sites, and those with more neighbors have a larger variance. The variance are not stationary

SAR Binary weights

$$\Sigma = \sigma^2 (I - \rho W)^{-1} (I - \rho W)^{-T}$$

CAR Binary weights

$$\Sigma = \sigma^2 (I - \rho W)^{-1}$$

ρ=0.25

	1	2	3	4	5	6	7	8	9
1	1,86	1,33	0,63	1,33	1,25	0,67	0,63	0,67	0,39
2	1,33	2,48	1,33	1,25	2,00	1,25	0,67	1,02	0,67
3	0,63	1,33	1,86	0,67	1,25	1,33	0,39	0,67	0,63
4	1,33	1,25	0,67	2,48	2,00	1,02	1,33	1,25	0,67
5	1,25	2,00	1,25	2,00	3,50	2,00	1,25	2,00	1,25
6	0,67	1,25	1,33	1,02	2,00	2,48	0,67	1,25	1,33
7	0,63	0,67	0,39	1,33	1,25	0,67	1,86	1,33	0,63
8	0,67	1,02	0,67	1,25	2,00	1,25	1,33	2,48	1,33
9	0,39	0,67	0,63	0,67	1,25	1,33	0,63	1,33	1,86

	1	2	3	4	5	6	7	8	9
1	1,20	0,39	0,13	0,39	0,25	0,11	0,13	0,11	0,05
2	0,39	1,32	0,39	0,25	0,50	0,25	0,11	0,18	0,11
3	0,13	0,39	1,20	0,11	0,25	0,39	0,05	0,11	0,13
4	0,39	0,25	0,11	1,32	0,50	0,18	0,39	0,25	0,11
5	0,25	0,50	0,25	0,50	1,50	0,50	0,25	0,50	0,25
6	0,11	0,25	0,39	0,18	0,50	1,32	0,11	0,25	0,39
7	0,13	0,11	0,05	0,39	0,25	0,11	1,20	0,39	0,13
8	0,11	0,18	0,11	0,25	0,50	0,25	0,39	1,32	0,39
9	0,05	0,11	0,13	0,11	0,25	0,39	0,13	0,39	1,20

1. Different values but the properties are the same.
2. The interior sites, and those with more neighbors have a larger variance. The variance are not stationary

3. Covariance decreases with the distance

SAR Binary weights

	Z_2	Z_5	Z_3	Z_6	Z_9
Z_1	1.33	1.25	0.63	0.67	0.39

CAR Binary weights

	Z_2	Z_5	Z_3	Z_6	Z_9
Z_1	0.39	0.25	0.13	0.11	0.05

The correlation is not constant(effect of variance is not constant).

	Z_2	Z_5	Z_3	Z_6	Z_9
Z_1	0.62	0.49	0.34	0.31	0.21

	Z_2	Z_5	Z_3	Z_6	Z_9
Z_1	0,31	0,19	0,10	0,09	0,04

4. The ρ parameter controls the amount of autocorrelation.

If ρ =0.10 cor(Z_1,Z_2)= 0.21
If ρ =-0.25 cor(Z_1,Z_2)=-0.62

If ρ =0.10 cor(Z_1,Z_2)= 0.10
If ρ =-0.25 cor(Z_1,Z_2)=-0.31

5. The covariances are not invariant to translation:

$\mathrm{cov}(Z_1,Z_3)= 0.63$ $\mathrm{cov}(Z_1,Z_3)= 0.13$

$\mathrm{cov}(Z_4,Z_6)= 1.25$ $\mathrm{cov}(Z_4,Z_6)= 0.18$

6. The models are anisotropic:

$\mathrm{cov}(Z_2,Z_3)= 1.33$ but $\mathrm{cov}(Z_2,Z_5)= 2.00$

$\mathrm{cor}(Z_2,Z_3)= 0.62$ but $\mathrm{cor}(Z_2,Z_5)= 0.69$

$\mathrm{cov}(Z_2,Z_3)= 0.39$ but $\mathrm{cov}(Z_2,Z_5)= 0.5$

$\mathrm{cor}(Z_2,Z_3)= 0.31$ but $\mathrm{cov}(Z_2,Z_5)= 0.36$

Grid of 3x3 (definition of neighbors move tower). **MATRIX W* Row Standardized weights**

	1	2	3	4	5	6	7	8	9
1		1/2		½					
2	1/3		1/3		1/3				
3		1/2				1/2			
4	1/3				1/3		1/3		
5		1/4		1/4		1/4		1/4	
6			1/3		1/3				1/3
7				1/2				1/2	
8					1/3		1/3		1/3
9						1/2		1/2	

Example matrix Q^{-} with ρ=0.9 $$Q = (I_w - \rho W)$$

	1	2	3	4	5	6	7	8	9
1	0.95	0.50	0.36	0.50	0.36	0.29	0.36	0.29	0.26
2	0.50	0.75	0.50	0.36	0.39	0.36	0.29	0.29	0.29
3	0.36	0.5	0.95	0.29	0.36	0.50	0.26	0.29	0.36
4	0.50	0.36	0.29	0.75	0.39	0.29	0.50	0.36	0.29
5	0.36	0.39	0.36	0.39	0.61	0.39	0.36	0.39	0.36
6	0.29	0.36	0.50	0.29	0.39	0.75	0.29	0.36	0.50
7	0.36	0.29	0.26	0.50	0.36	0.29	0.95	0.50	0.36
8	0.29	0.29	0.29	0.36	0.39	0.36	0.50	0.75	0.5
9	0.26	0.29	0.36	0.29	0.36	0.50	0.36	0.5	0.95

1. The interior sites, those with more neighbors have smaller variance
2. The variances are not stationary
3. Covariance decreases with “distance”.

Example matrix Q^{-} with ρ=0.9 $$Q = \left(I_w - \rho W\right)$$

	1	2	3	4	5	6	7	8	9
1	0.95	0.50	0.36	0.50	0.36	0.29	0.36	0.29	0.26
2	0.50	0.75	0.50	0.36	0.39	0.36	0.29	0.29	0.29
3	0.36	0.5	0.95	0.29	0.36	0.50	0.26	0.29	0.36
4	0.50	0.36	0.29	0.75	0.39	0.29	0.50	0.36	0.29
5	0.36	0.39	0.36	0.39	0.61	0.39	0.36	0.39	0.36
6	0.29	0.36	0.50	0.29	0.39	0.75	0.29	0.36	0.50
7	0.36	0.29	0.26	0.50	0.36	0.29	0.95	0.50	0.36
8	0.29	0.29	0.29	0.36	0.39	0.36	0.50	0.75	0.5
9	0.26	0.29	0.36	0.29	0.36	0.50	0.36	0.5	0.95

1. The interior sites, those with more neighbors have smaller variance
2. The variances are not stationary
3. The covariance decreases with “distance”.

4. The covariance decrease with the distances

	Z_2	Z_5	Z_3	Z_6	Z_9
Z_1	0.5	0.36	0.36	0.29	0.26

And the correlations also decrease with the distances

	Z_2	Z_5	Z_3	Z_6	Z_9
Z_1	0.59	0.47	0.37	0.35	0.28

5. For $\rho=0.912$ $\quad$ $cor(Z_1,Z_2)=0.6171$

	Z_2	Z_5	Z_3	Z_6	Z_9
Z_1	0.62	0.50	0.41	0.38	0.31

6 . The covariances are not translation invariant

$cov(Z_1,Z_3)=0.5$ $cov(Z_4,Z_6)=0.29$

7. The model is not isotropic.

$cov(Z_2,Z_3)= 0.5$ but $cov(Z_2,Z_5)= 0.39$
$cor(Z_2,Z_3)= 0.42$ but $cov(Z_2,Z_5)= 0.58$

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