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On the $M/M/1$ Queue with Catastrophes and Its Continuous Approximation

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Abstract. For the $M/M/1$ queue in the presence of catastrophes the transition probabilities, densities of the busy period and of the catastrophe waiting time are determined. A heavy-traffic approximation to this discrete model is then derived. This is seen to be equivalent to a Wiener process subject to randomly occurring jumps for which some analytical results are obtained. The goodness of the approximation is discussed by comparing the closed-form solutions obtained for the continuous process with those obtained for the $M/M/1$ catastrophized queue.

Keywords: busy period, catastrophe waiting time, Wiener process, first-passage time

1. Introduction

The $M/M/1$ queueing system has been the object of systematic and through investigations for a long time. In recent years the attention has been focused on certain extensions that include the effect of catastrophes. This consists of adding to the standard assumptions the hypothesis that the number of customers is instantly reset to zero at certain random times.

Well-known is the role played by the notion of catastrophes in various areas of science and technology. In particular, birth-and-death models including catastrophes have been discussed within the context of populations dynamics (see, for instance, [2,3,16]) and some related continuous processes have also been considered [2]. However, it does not appear to have been noticed that $M/M/1$ catastrophized processes may be suitable to approach a current hot topic of great biological relevance, concerning the interaction between myosin heads and actin filaments that is responsible for force generation during muscle contraction. Indeed, by means of a new and extremely sophisticated instrumentational approach (see [6]), it has been possible to prove that the sliding of a myosin head over the actin filament occurs by randomly distributed minuscule steps eventually followed by a sudden reset (a "catastrophe" in our paradigm) with an approximately exponentially distributed dwell time [11].

In the present paper we undertake anew the analysis of the $M/M/1$ queue in the presence of catastrophes in order to obtain some analytical results that do not appear to be present in the literature. In section 2 we shall determine the transition probabilities by a non-conventional straightforward method. We then show that the transition probabilities can be expressed as the sum of the steady-state probabilities and of a time-dependent term. This appears to be more elegant and computationally more manageable than that obtained via the general expressions given in [12]. In section 3 density, mean and variance of the busy period and of the catastrophe waiting time will be obtained. A heavy-traffic approximation for the discrete queueing model will then be obtained in section 4. In particular, the conditional density of the approximating continuous process will be determined. This turns out to be a Wiener-type process with jumps. In this section the approximating busy period density for the continuous process will be obtained as a first-passage-time probability density function (pdf) through the origin. The goodness of such an approximation is, finally, discussed in section 5.

2. The $M/M/1$ queue with catastrophes

We consider an $M/M/1$ queueing system with FIFO discipline that is subject to catastrophes at the service station. By α and β we denote the customers' arrival rate and service rate, respectively. When the system is not empty, catastrophes occur according to a Poisson process of rate ξ. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept new customers.

Let $N(t)$ denote the number of customers in the system at time t. The state probabilities

$$p_n(t) := P\{N(t) = n \mid N(0) = 0\}, \quad n = 0, 1, \ldots,$$

of $N(t)$ satisfy

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} p_0(t) &= -(\alpha + \xi) p_0(t) + \beta p_1(t) + \xi, \\ \frac{\mathrm{d}}{\mathrm{d}t} p_n(t) &= -(\alpha + \beta + \xi) p_n(t) + \alpha p_{n-1}(t) + \beta p_{n+1}(t), \quad n = 1, 2, \ldots. \end{aligned} \tag{1}$$

Assuming that the system is initially empty, we have:

$$p_n(0) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Denoting by N the number of customers in the system in the equilibrium regime, if $\xi > 0$ one has (see [12]):

$$q_n := \mathrm{P}(N = n) = \lim_{t \to +\infty} p_n(t) = \left(1 - \frac{1}{z_1}\right)\left(\frac{1}{z_1}\right)^n = \left(1 - \frac{\alpha}{\beta} z_2\right)\left(\frac{\alpha}{\beta} z_2\right)^n, \\ n = 0, 1, \ldots, \tag{3}$$

where

$$z_{1,2} = \frac{\alpha + \beta + \xi \pm \sqrt{(\alpha + \beta + \xi)^2 - 4\alpha\beta}}{2\alpha}, \quad 0 < z_2 < 1 < z_1, \tag{4}$$

mean and variance of N being

$$E(N) = \frac{1}{z_1 - 1}, \qquad \mathrm{Var}(N) = \frac{z_1}{(z_1 - 1)^2}, \tag{5}$$

respectively.

The following lemma shows that $N(t)$ is distributed as the minimum between N and the number of customers in an $M/M/1$ queue with particularly assigned rates.

Lemma 2.1. Let $\widetilde{N}(t)$ denote the number of customers in the $M/M/1$ queueing system having arrival rate αz_1 and service rate β/z_1, with z_i $(i = 1, 2)$ defined by (4) and such that $\mathrm{P}[\widetilde{N}(0) = 0] = 1$. Further, let N be a random variable with distribution (3) independent of $\widetilde{N}(t)$. Then,

$$M(t) := \min\{\widetilde{N}(t), N\}, \quad \forall t \geqslant 0, \tag{6}$$

has the same distribution as the catastrophized process $N(t)$.

Proof. Setting $p_n^*(t) = \mathrm{P}\{M(t) = n\}$ and $\tilde{p}_n(t) = \mathrm{P}\{\widetilde{N}(t) = n\}$ for $n = 0, 1, \ldots,$ from (3) and (6) we have

$$p_n^*(t) = q_n \sum_{k=n}^{+\infty} \tilde{p}_k(t) + \left(\frac{1}{z_1}\right)^{n+1} \tilde{p}_n(t), \quad n = 0, 1, \ldots. \tag{7}$$

Recalling the equations and initial conditions satisfied by $\tilde{p}_n(t)$, it is not hard to see that the right-hand side of (7) satisfies system (1) with initial condition (2). Hence, $p_n(t) \equiv p_n^*(t)$ for all $n = 0, 1, \ldots$ and for all $t \geqslant 0$. $\square$

Let us now come to the transient analysis of the $M/M/1$ queue with catastrophes. In the following theorem we obtain the probabilities $p_n(t)$ and their Laplace transforms $\pi_n(\lambda)$.

Theorem 2.1. For all $t > 0$ one has

$$p_n(t) = \frac{1}{\alpha(z_1 - z_2)} \frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{t} \Bigg\{ z_1^{-n-1}(z_1 - 1) \sum_{r=n+1}^{+\infty} r\left(z_1\sqrt{\frac{\alpha}{\beta}}\right)^r I_r\big(2\sqrt{\alpha\beta}t\big) + z_2^{-n-1}(1 - z_2) \sum_{r=n+1}^{+\infty} r\left(z_2\sqrt{\frac{\alpha}{\beta}}\right)^r I_r\big(2\sqrt{\alpha\beta}t\big)\Bigg\}, \quad n = 0, 1, \ldots, \tag{8}$$

with z_1 and z_2 defined in (4) and where

$$I_k(z) = \sum_{m=0}^{+\infty} \frac{(z/2)^{k+2m}}{m!(m+k)!}, \quad k = 0, 1, \dots, \tag{9}$$

is the modified Bessel function of first kind. Moreover, for $\lambda > 0$ it is

$$\begin{aligned}\pi_n(\lambda) &:= \int_0^{+\infty} \mathrm{e}^{-\lambda t} p_n(t)\,\mathrm{d}t = \frac{1}{\lambda}\big[\psi_1(\lambda) - 1\big]\big[\psi_1(\lambda)\big]^{-n-1} \\ &= \frac{1}{\lambda}\left[1 - \frac{\alpha}{\beta}\psi_2(\lambda)\right]\left[\frac{\alpha}{\beta}\psi_2(\lambda)\right]^n, \quad n = 0, 1, \dots, \end{aligned} \tag{10}$$

where

$$\psi_{1,2}(\lambda) = \frac{\lambda + \alpha + \beta + \xi \pm \sqrt{(\lambda + \alpha + \beta + \xi)^2 - 4\alpha\beta}}{2\alpha}, \tag{11}$$

with $0 < \psi_2(\lambda) < 1 < \psi_1(\lambda)$.

Proof. Noting that z_1 and z_2 are the roots of $\alpha z^2 - (\alpha + \beta + \xi)z + \beta = 0$, and recalling that the transient probabilities of queue $\widetilde{N}(t)$ are given by (cf. [4])

$$\tilde{p}_n(t) = \frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{\beta t} z_1 \left(\frac{\alpha}{\beta} z_1^2\right)^n \sum_{r=n+1}^{+\infty} r\left(\frac{\beta}{\alpha}\frac{1}{z_1^2}\right)^{r/2} I_r\big(2\sqrt{\alpha\beta}t\big), \quad n = 0, 1, \dots,$$

after some straightforward calculations from (7) we obtain (8). Moreover, since from lemma 2.1 $p_n(t) \equiv p_n^*(t)$, we easily have

$$\pi_n(\lambda) = q_n \sum_{k=n}^{+\infty} \tilde{\pi}_k(\lambda) + \left(\frac{1}{z_1}\right)^{n+1} \tilde{\pi}_n(\lambda), \quad n = 0, 1, \dots,$$

where $\tilde{\pi}_n(\lambda)$ are the Laplace transforms of the transient probabilities $\tilde{p}_n(t)$ of queue $\widetilde{N}(t)$. Equation (10) then finally follows after performing some calculations and recalling that (cf. [4])

$$\tilde{\pi}_n(\lambda) = \frac{1}{\lambda}\left[1 - \frac{\alpha}{\beta} z_1 \psi_2(\lambda)\right]\left[\frac{\alpha}{\beta} z_1 \psi_2(\lambda)\right]^n, \quad n = 0, 1, \dots. \qquad \square$$

As $\xi \downarrow 0$, (i.e. by removal of catastrophes) expression (4) yields $z_1 \to \max(1, \beta/\alpha)$ and $z_2 \to \min(1, \beta/\alpha)$. Hence, for all $t > 0$, in the limit as $\xi \downarrow 0$, (8) yields the well-known expression (cf. [4]):

$$\lim_{\substack{\xi \downarrow 0 \\ \alpha \neq \beta}} p_n(t) = \frac{\mathrm{e}^{-(\alpha+\beta)t}}{\beta t} \left(\frac{\alpha}{\beta}\right)^n \sum_{r=n+1}^{+\infty} r\left(\frac{\beta}{\alpha}\right)^{r/2} I_r\big(2\sqrt{\alpha\beta}t\big). \tag{12}$$

The case $\alpha = \beta$ can be obtained from (12) by a limiting procedure.

As Conolly and Langaris [5] have then shown, the right-hand side of (12) is equivalent to the following more manageable form due to Sharma [14]:

$$\tilde{p}_n(t) = \left(1-\frac{\alpha}{\beta}\right)\left(\frac{\alpha}{\beta}\right)^n + \mathrm{e}^{-(\alpha+\beta)t}\left(\frac{\alpha}{\beta}\right)^n \sum_{m=0}^{+\infty}\frac{(\alpha t)^m}{m!}\sum_{k=0}^{m+n}(m-k)\frac{(\beta t)^{k-1}}{k!}. \tag{13}$$

Analogously to Sharma's formula (13), for $n = 0, 1, \ldots$ the following theorem expresses $p_n(t)$ as the sum of a time-independent term, now coinciding with the steady-state probability (3), and of a time-dependent term whose contribution vanishes in the limit as $t \to +\infty$.

Theorem 2.2. For all $t > 0$, and for $n = 0, 1, \ldots$ one has

$$\begin{aligned} p_n(t) = q_n + \frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{\alpha t(z_1-z_2)}\Bigg\{&\frac{z_1-1}{z_1^{n+1}}\sum_{m=0}^{+\infty}\frac{1}{m!}\left(\frac{\beta t}{z_1}\right)^m\sum_{r=0}^{m+n}(m-r)\frac{(\alpha z_1 t)^r}{r!}\\ &+\frac{1-z_2}{z_2^{n+1}}\sum_{m=0}^{+\infty}\frac{1}{m!}\left(\frac{\beta t}{z_2}\right)^m\sum_{r=m+n+1}^{+\infty}(r-m)\frac{(\alpha z_2 t)^r}{r!}\Bigg\}. \end{aligned} \tag{14}$$

Proof. Recalling (9) we have

$$\begin{aligned} &\sum_{r=n+1}^{+\infty} r\left(z_1\sqrt{\frac{\alpha}{\beta}}\right)^r I_r\big(2\sqrt{\alpha\beta}t\big)\\ &= \sum_{m=0}^{+\infty}\frac{1}{m!}\left(\frac{\beta t}{z_1}\right)^m\sum_{k=m+n+1}^{+\infty}(k-m)\frac{(\alpha z_1 t)^k}{k!}\\ &= \sum_{m=0}^{+\infty}\frac{1}{m!}\left(\frac{\beta t}{z_1}\right)^m\left\{(\alpha z_1 t - m)\mathrm{e}^{\alpha z_1 t} + \sum_{k=0}^{m+n}(m-k)\frac{(\alpha z_1 t)^k}{k!}\right\}\\ &= \left(\alpha z_1 - \frac{\beta}{z_1}\right)t\exp\left\{\left(\alpha z_1 + \frac{\beta}{z_1}\right)t\right\} + \sum_{m=0}^{+\infty}\frac{1}{m!}\left(\frac{\beta t}{z_1}\right)^m\sum_{k=0}^{m+n}(m-k)\frac{(\alpha z_1 t)^k}{k!}\\ &= \alpha(z_1-z_2)t\mathrm{e}^{(\alpha+\beta+\xi)t} + \sum_{m=0}^{+\infty}\frac{1}{m!}\left(\frac{\beta t}{z_1}\right)^m\sum_{k=0}^{m+n}(m-k)\frac{(\alpha z_1 t)^k}{k!}, \end{aligned} \tag{15}$$

where in the last equality use of identities $\alpha z_1 - \beta/z_1 = \alpha(z_1 - z_2)$ and $\alpha z_1 + \beta/z_1 = \alpha + \beta + \xi$ has been made. We also have

$$\sum_{r=n+1}^{+\infty} r\left(z_2\sqrt{\frac{\alpha}{\beta}}\right)^r I_r\big(2\sqrt{\alpha\beta}t\big) = \sum_{m=0}^{+\infty}\frac{1}{m!}\left(\frac{\beta t}{z_2}\right)^m\sum_{r=m+n+1}^{+\infty}(r-m)\frac{(\alpha z_2 t)^r}{r!}. \tag{16}$$

Making use of (15) and (16) in (8), we, finally, obtain (14). □

It is noteworthy that when $\xi \downarrow 0$, probabilities (14) identify with Sharma's expression (13) if $\alpha < \beta$, whereas for $\alpha > \beta$ they yield the right-hand side of (12).

It should be underlined that the very same remark by Sharma and Bunday [15], concerning the evident valuable advantages of expression (13) of the state transition probabilities for the $M/M/1$ model as compared to (12), is certainly most appropriate for the expressions (14) for the model admitting catastrophes when compared with the corresponding ones given by (8).

3. Busy period and catastrophe distribution

A busy period starts with the arrival of a customer who finds the system empty and ends when the system becomes again idle. Let B denote the random variable describing the duration of the busy period and $b(t)$ its pdf. Notice that a busy period ends (i) when the server becomes idle because a customer leaves the system and there are no waiting customers, or (ii) when the effect of a catastrophe empties the system. Hence, under the assumptions in section 2, we have

$$b(t) = \xi \mathrm{e}^{-\xi t}\left[1 - \int_0^t k(s)\,\mathrm{d}s\right] + \mathrm{e}^{-\xi t}k(t), \quad t > 0, \tag{17}$$

where $k(t)$ denotes the pdf of a variable K representing the busy period of an $M/M/1$ queueing system without catastrophes characterized by rates α and β. Since (see, for instance, [4])

$$k(t) = \frac{\sqrt{\alpha\beta}}{\alpha t}\mathrm{e}^{-(\alpha+\beta)t} I_1\big(2\sqrt{\alpha\beta}t\big), \quad t > 0,$$

equation (17) yields

$$b(t) = \frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{\alpha t}\left\{\sqrt{\alpha\beta}I_1\big(2\sqrt{\alpha\beta}t\big) + \xi\sum_{k=1}^{+\infty} k\left(\frac{\alpha}{\beta}\right)^{k/2} I_k\big(2\sqrt{\alpha\beta}t\big)\right\}, \quad t > 0. \tag{18}$$

Notice that, if $\alpha < \beta$, then B can be viewed as the minimum of two independent random variables: K and an exponentially distributed variable with mean $1/\xi$.

Using (11) and (see, for instance, [8, equation (49), p. 237])

$$\int_0^{+\infty} \mathrm{e}^{-\lambda t}\left[\frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{t} n\left(\frac{\beta}{\alpha}\right)^{-n/2} I_n\big(2\sqrt{\alpha\beta}t\big)\right]\mathrm{d}t = \big[\psi_1(\lambda)\big]^{-n}, \quad n = 1, 2, \ldots, \tag{19}$$

after some calculations from (18) we obtain the Laplace transform of $b(t)$:

$$\mathcal{B}(\lambda) = \frac{\lambda\psi_2(\lambda) + \xi}{\lambda + \xi}, \quad \lambda > 0. \tag{20}$$

We note that the busy period ends with certainty, and that

$$\begin{aligned} E(B) &= \frac{1}{\alpha(z_1-1)} = \frac{1-z_2}{\xi}, \\ \mathrm{Var}(B) &= \frac{\alpha+\beta+\xi}{\alpha^3(z_1-1)^2(z_1-z_2)} = \frac{(z_1+z_2)(1-z_2)^2}{\xi^2(z_1-z_2)}, \end{aligned} \tag{21}$$

with z_1 and z_2 given in (4). The first equality in (21) is in agreement with the well-known result $E(B) = (1-q_0)/(\alpha q_0)$ (see, for instance, [13, section 1.8.1]).

It should be underlined that, differently from the case of the $M/M/1$ queue (in which the busy period ends with certainty if and only if $\alpha < \beta$), the presence of catastrophes always implies $\mathrm{P}(B < +\infty) = 1$.

Let us now analyze the occurrence time of catastrophes. To this end, let us denote by D the random variable that describes the first occurrence of a catastrophe when initially the system is empty. For the characterization of D, hereafter we consider a time-homogeneous Markov process $\{\overline{N}(t);\ t \geqslant 0\}$ with state-space $\{-1, 0, 1, \ldots\}$ and such that $\mathrm{P}[\overline{N}(0) = 0] = 1$. We conventionally assume that $\overline{N}(t) = -1$ means that a catastrophe occurs before t, while by $\overline{N}(t) = n \geqslant 0$ we mean that n customers are present in the system at time t and that no catastrophe has yet occurred. Thus, state -1 acts as an absorbing state for $\overline{N}(t)$, which is attained as soon as the first catastrophe occurs. The state probabilities

$$r_n(t) := \mathrm{P}\big\{\overline{N}(t) = n \mid \overline{N}(0) = 0\big\}, \quad n = -1, 0, 1, \ldots,$$

satisfy the differential system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} r_{-1}(t) &= \xi\big[1 - r_{-1}(t) - r_0(t)\big], \\ \frac{\mathrm{d}}{\mathrm{d}t} r_0(t) &= -\alpha r_0(t) + \beta r_1(t), \\ \frac{\mathrm{d}}{\mathrm{d}t} r_n(t) &= -(\alpha+\beta+\xi) r_n(t) + \alpha r_{n-1}(t) + \beta r_{n+1}(t), \quad n = 1, 2, \ldots, \end{aligned} \tag{22}$$

with initial conditions

$$r_n(0) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Theorem 3.1. For all $t > 0$ one has

$$\begin{aligned} r_{-1}(t) &= 1 - \frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{(\beta+\xi)t} \sum_{j=1}^{+\infty} j \left(\frac{\beta+\xi}{\sqrt{\alpha\beta}}\right)^j I_j\big(2\sqrt{\alpha\beta}t\big) \sum_{k=0}^{j-1} \left(\frac{\alpha}{\beta+\xi}\right)^k, \\ r_n(t) &= \frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{(\beta+\xi)t} \left(\frac{\alpha}{\beta+\xi}\right)^n \sum_{j=n+1}^{+\infty} j \left(\frac{\beta+\xi}{\sqrt{\alpha\beta}}\right)^j I_j\big(2\sqrt{\alpha\beta}t\big), \quad n = 0, 1, \ldots. \end{aligned} \tag{24}$$

Proof. Using a standard procedure, from (22) and (23) one obtains:

$$\begin{gathered}(\lambda+\xi)R_{-1}(\lambda)=\xi\left[\frac{1}{\lambda}-R_0(\lambda)\right],\qquad (\lambda+\alpha)R_0(\lambda)=1+\beta R_1(\lambda),\\ (\lambda+\alpha+\beta+\xi)R_n(\lambda)=\alpha R_{n-1}(\lambda)+\beta R_{n+1}(\lambda),\quad n=1,2,\ldots,\end{gathered}\tag{25}$$

where $R_n(\lambda)$ is the Laplace transform of $r_n(t)$. From the last of (25), it follows $R_n(\lambda)=A[\psi_1(\lambda)]^{-n}$, where $A=[\alpha\psi_1(\lambda)-(\beta+\xi)]^{-1}$ as a consequence of the second of (25). Hence,

$$\begin{aligned}R_n(\lambda)&=\frac{[\psi_1(\lambda)]^{-n}}{\alpha\psi_1(\lambda)-(\beta+\xi)}=\frac{[\psi_1(\lambda)]^{-n-1}}{\alpha}\left[1-\frac{\beta+\xi}{\alpha\psi_1(\lambda)}\right]^{-1}\\ &=\frac{1}{\alpha}\sum_{k=0}^{+\infty}\left(\frac{\beta+\xi}{\alpha}\right)^k\big[\psi_1(\lambda)\big]^{-n-k-1},\quad n=0,1,\ldots,\end{aligned}\tag{26}$$

the last equality following from $0<(\beta+\xi)/(\alpha\psi_1(\lambda))<1$. Making then use of (19), the second of (24) follows. Furthermore, due to $r_{-1}(t)=1-\sum_{n=0}^{+\infty}r_n(t)$, one, finally, obtains the first of (24). □

Note that the distribution function of the first occurrence of a catastrophe is given by $\mathrm{P}(D<t)\equiv r_{-1}(t)$. Denoting by $d(t)$ the pdf of D and making use of the first equation in (22) and of (24), we have:

$$\begin{aligned}d(t)&:=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}(D<t)=\frac{\mathrm{d}}{\mathrm{d}t}r_{-1}(t)=\xi\big[1-r_{-1}(t)-r_0(t)\big]\\ &=\xi\frac{\mathrm{e}^{-(\alpha+\beta+\xi)t}}{(\beta+\xi)t}\sum_{j=2}^{+\infty}j\left(\frac{\beta+\xi}{\sqrt{\alpha\beta}}\right)^j I_j\big(2\sqrt{\alpha\beta}\,t\big)\sum_{k=1}^{j-1}\left(\frac{\alpha}{\beta+\xi}\right)^k,\quad t>0.\end{aligned}\tag{27}$$

Mean and the variance of the first occurrence of a catastrophe can be obtained for instance via the Laplace transform $\mathcal{D}(\lambda)$ of $d(t)$:

$$\mathcal{D}(\lambda)=\frac{\xi}{\xi+\lambda}\left[1-\frac{\lambda}{\alpha\psi_1(\lambda)-(\beta+\xi)}\right].\tag{28}$$

It is not difficult to prove that catastrophe is a sure event and that

$$E(D)=\frac{z_1}{\xi},\qquad \mathrm{Var}(D)=\frac{z_1}{\xi^2}\left[2-z_1+\frac{2(z_1-1)^2}{z_1-z_2}\right],\tag{29}$$

with z_1 and z_2 given in (4).

It should be pointed out that the non-exponential form of the catastrophe pdf is a consequence of our having excluded that a catastrophe can occur while the system is empty. However, in the limit as α tends to $+\infty$, the distribution of D becomes exponential. Indeed, noting that $\lim_{\alpha\to+\infty}\psi_1(\lambda)=1$, equation (28) yields $\lim_{\alpha\to+\infty}\mathcal{D}(\lambda)=\xi/(\lambda+\xi)$, i.e. time to catastrophe becomes exponential with mean ξ^{-1}.

In the remaining part of this section we shall briefly discuss the role played by the parameters α, β and ξ appearing in the $M/M/1$ model with catastrophes. From (3) we note that the probability q_0 that the system is empty in the equilibrium regime decreases as α increases, whereas it increases as either β or ξ increase. As shown by (5), the mean $E(N)$ of number of customers in the system in the equilibrium regime increases with α, whereas it decreases as either β or ξ increase. Furthermore, the coefficient of variation $\mathrm{CV}(N) = \sqrt{z_1} > 1$ decreases as α increases, and $\lim_{\alpha\to+\infty} \mathrm{CV}(N) = 1$. Moreover $\mathrm{CV}(N)$ increases as either β or ξ increase. We note from (21) that the mean $E(B)$ of the busy period increases with α and decreases as either β or ξ increase. Furthermore, $\mathrm{CV}(B) = [(z_1+z_2)/(z_1-z_2)]^{1/2} > 1$: as a function of α, it increases for $\alpha < \beta+\xi$, decreases for $\alpha > \beta+\xi$ and it is $\lim_{\alpha\to+\infty} \mathrm{CV}(B) = 1$. Instead, as α function of β, $\mathrm{CV}(B)$ increases for $\beta < \alpha + \xi$, decreases for $\beta > \alpha + \xi$, and $\lim_{\beta\to+\infty} \mathrm{CV}(B) = 1$. Furthermore, $\mathrm{CV}(B)$ decreases as ξ increases, and it is $\lim_{\xi\to+\infty} \mathrm{CV}(B) = 1$. From (29) we see that the mean $E(D)$ of the catastrophe waiting time increases with α and decreases as either β or ξ increase. Moreover,

$$\mathrm{CV}(D) = \left\{1 - \frac{2(z_1-1)(1-z_2)}{z_1(z_1-z_2)}\right\}^{1/2} < 1:$$

as a function of α, it decreases for $\alpha < \beta + \xi$, increases for $\alpha > \beta + \xi$ and $\lim_{\alpha\to+\infty} \mathrm{CV}(B) = 1$.

Tables 1–3 provide the numerical counterpart of the analytical behaviors just outlined.

Table 1
For the $M/M/1$ queue with catastrophes q_0, $E(N)$, $\mathrm{CV}(N)$, $E(B)$, $\mathrm{CV}(B)$, $E(D)$ and $\mathrm{CV}(D)$ are listed for $\alpha = 0.1i$ $(i = 1, 2, \ldots, 20)$, $\beta = 0.5$ and $\xi = 0.1$.

α	q_0	$E(N)$	$\mathrm{CV}(N)$	$E(B)$	$\mathrm{CV}(B)$	$E(D)$	$\mathrm{CV}(D)$
0.1	0.838516	0.192582	2.48849	1.92582	1.14012	61.9258	0.96955
0.2	0.689898	0.44949	1.79576	2.24745	1.27789	32.2474	0.934559
0.3	0.558258	0.791288	1.50458	2.63763	1.40141	22.6376	0.898447
0.4	0.447214	1.23607	1.345	3.09017	1.49535	18.0902	0.867633
0.5	0.358258	1.79129	1.2483	3.58258	1.54932	15.5826	0.848481
0.6	0.289898	2.44949	1.1867	4.08248	1.56508	14.0825	0.842676
0.7	0.238516	3.19258	1.14596	4.56083	1.55372	13.1323	0.846872
0.8	0.2	4.0	1.11803	5.0	1.52753	12.5	0.856349
0.9	0.17082	4.8541	1.09819	5.39345	1.49535	12.0601	0.867633
1.0	0.148331	5.74166	1.08359	5.74166	1.46222	11.7417	0.878853
1.1	0.130662	6.65331	1.07252	6.04847	1.43058	11.503	0.889206
1.2	0.116515	7.58258	1.0639	6.31881	1.40141	11.3188	0.898447
1.3	0.104988	8.52494	1.05703	6.55764	1.37498	11.173	0.906579
1.4	0.0954451	9.47723	1.05144	6.76945	1.3512	11.0552	0.913702
1.5	0.0874342	10.4372	1.04681	6.95811	1.32986	10.9581	0.919944
1.6	0.0806248	11.4031	1.04293	7.12695	1.31069	10.877	0.925428
1.7	0.0747727	12.3739	1.03962	7.27874	1.29345	10.8082	0.930268
1.8	0.0696938	13.3485	1.03678	7.41582	1.27789	10.7491	0.934559
1.9	0.0652476	14.3262	1.03431	7.54013	1.2638	10.698	0.938383
2.0	0.0613248	15.3066	1.03215	7.65331	1.25101	10.6533	0.941805

Table 2
Same of table 1 for $\alpha = 1.5$, $\beta = 0.1i$ $(i = 1, 2, \ldots, 20)$ and $\xi = 0.1$.

β	q_0	$E(N)$	CV(N)	$E(B)$	CV(B)	$E(D)$	CV(D)
0.1	0.066373	14.0664	1.03494	9.37758	1.0599	10.7109	0.936274
0.2	0.0707142	13.1414	1.03735	8.76095	1.12261	10.761	0.93267
0.3	0.075604	12.2268	1.04009	8.15121	1.18838	10.8179	0.928752
0.4	0.0811388	11.3246	1.04322	7.5497	1.25743	10.883	0.924508
0.5	0.0874342	10.4372	1.04681	6.95811	1.32986	10.9581	0.919944
0.6	0.0946274	9.56776	1.05096	6.37851	1.40558	11.0452	0.91509
0.7	0.102879	8.72015	1.05578	5.81344	1.48425	11.1468	0.910024
0.8	0.112372	7.89898	1.06141	5.26599	1.56508	11.266	0.904883
0.9	0.123308	7.10977	1.06801	4.73985	1.6467	11.4065	0.899899
1.0	0.13589	6.3589	1.07576	4.23927	1.72696	11.5726	0.89541
1.1	0.150301	5.65331	1.08484	3.76887	1.80289	11.7689	0.891861
1.2	0.166667	5.0	1.09545	3.33333	1.87083	12.0	0.889757
1.3	0.18501	4.40512	1.1077	2.93675	1.92693	12.2701	0.889552
1.4	0.205213	3.87298	1.12169	2.58199	1.96799	12.582	0.891508
1.5	0.227008	3.40512	1.1374	2.27008	1.99227	12.9367	0.895576
1.6	0.25	3.0	1.1547	2.0	2.0	13.3333	0.901388
1.7	0.273724	2.65331	1.17341	1.76887	1.99317	13.7689	0.908369
1.8	0.297717	2.3589	1.19328	1.5726	1.97486	14.2393	0.915907
1.9	0.321567	2.10977	1.21408	1.40651	1.94841	14.7398	0.923486
2.0	0.344949	1.89898	1.23556	1.26599	1.91683	15.266	0.930746

Table 3
Same of table 1 for $\alpha = 1.5$, $\beta = 0.5$ and $\xi = 0.1i$ $(i = 1, 2, \ldots, 20)$.

ξ	q_0	$E(N)$	CV(N)	$E(B)$	CV(B)	$E(D)$	CV(D)
0.1	0.0874342	10.4372	1.04681	6.95811	1.32986	10.9581	0.919944
0.2	0.156466	5.39116	1.0888	3.59411	1.27352	5.92744	0.86675
0.3	0.213275	3.68879	1.12743	2.45919	1.23284	4.23697	0.8295
0.4	0.261325	2.82666	1.16352	1.88444	1.20193	3.38444	0.80268
0.5	0.302776	2.30278	1.19761	1.53518	1.1776	2.86852	0.783102
0.6	0.339072	1.94923	1.23005	1.29948	1.15795	2.52171	0.768754
0.7	0.371232	1.69374	1.26111	1.12916	1.14174	2.27202	0.758287
0.8	0.4	1.5	1.29099	1.0	1.12815	2.08333	0.750757
0.9	0.425941	1.34774	1.31984	0.898497	1.11661	1.93553	0.745484
1.0	0.44949	1.22474	1.34777	0.816497	1.10668	1.8165	0.741964
1.1	0.470992	1.12318	1.37489	0.748785	1.09807	1.71848	0.739816
1.2	0.490725	1.0378	1.40128	0.691868	1.09054	1.63631	0.738749
1.3	0.508914	0.964967	1.42699	0.643311	1.0839	1.56639	0.738538
1.4	0.525748	0.902053	1.4521	0.601369	1.07801	1.50613	0.739007
1.5	0.541381	0.847127	1.47664	0.564751	1.07275	1.45364	0.740014
1.6	0.555947	0.798733	1.50066	0.532489	1.06804	1.40749	0.74145
1.7	0.569557	0.755752	1.5242	0.503835	1.06379	1.36658	0.743225
1.8	0.582307	0.717307	1.54729	0.478205	1.05995	1.33006	0.745267
1.9	0.594281	0.682706	1.56996	0.455137	1.05646	1.29724	0.747519
2.0	0.605551	0.651388	1.59223	0.434259	1.05328	1.26759	0.749932

4. A heavy-traffic approximation

We shall now focus our attention on those situations in which arrival and service rates are much larger than the rate of occurrence of catastrophes. We shall then construct a continuous approximation to $N(t)$ that yields a particularly manageable description of the queueing system under heavy-traffic regime. To this purpose, in (1) we perform a substitution of parameters by setting

$$\alpha = \frac{\hat{\alpha}}{\varepsilon} + \frac{\sigma^2}{2\varepsilon^2}, \qquad \beta = \frac{\hat{\beta}}{\varepsilon} + \frac{\sigma^2}{2\varepsilon^2}, \tag{30}$$

where $\hat{\alpha}$, $\hat{\beta}$, σ and ε are positive parameters, and make use of a customary scaling (see [7,9,10]). In the limit as $\varepsilon \downarrow 0$, the scaled process $\{\varepsilon \cdot N(t)\}$ yields a continuous process $\{X(t);\ t \geqslant 0\}$ defined on the non-negative half-line and such that

$$f(x,t) = \frac{\partial}{\partial x} \mathrm{P}\big\{X(t) < x \mid X(0) = 0\big\}, \quad x \geqslant 0,\ t \geqslant 0$$

is solution of

$$\frac{\partial}{\partial t} f(x,t) = -\xi f(x,t) - (\hat{\alpha} - \hat{\beta}) \frac{\partial}{\partial x} f(x,t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} f(x,t), \quad x \in (0,+\infty), \tag{31}$$

with boundary and initial conditions

$$\begin{aligned} &\lim_{x \downarrow 0} \left\{ \xi - (\hat{\alpha} - \hat{\beta}) f(x,t) + \frac{\sigma^2}{2} \frac{\partial}{\partial x} f(x,t) \right\} = 0, \\ &\lim_{t \downarrow 0} f(x,t) = \delta(x). \end{aligned} \tag{32}$$

If $\xi = 0$ (no catastrophe allowed), (31), (32) define the transition pdf of a Wiener process with drift $\hat{\alpha} - \hat{\beta}$ and infinitesimal variance σ^2 restricted to $[0,+\infty)$ with 0 a reflecting boundary. The more interesting case $\xi > 0$ makes $X(t)$ a jump–diffusion process of the Wiener type with randomly occurring jumps at rate ξ, each jump making $X(t)$ instantly attain the state 0. Such a process does not appear to have been studied so far.

In the following, we shall denote by $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$ the roots of $\sigma^2 z^2 + 2(\hat{\alpha} - \hat{\beta}) z - 2(\lambda + \xi) = 0$:

$$\varphi_{1,2}(\lambda) = \frac{\hat{\beta} - \hat{\alpha} \pm \sqrt{(\hat{\beta} - \hat{\alpha})^2 + 2\sigma^2(\lambda + \xi)}}{\sigma^2}, \quad \varphi_1(\lambda) > \varphi_2(\lambda), \tag{33}$$

and set

$$y_{1,2} = \varphi_{1,2}(0) = \frac{\hat{\beta} - \hat{\alpha} \pm \sqrt{(\hat{\beta} - \hat{\alpha})^2 + 2\sigma^2 \xi}}{\sigma^2}, \quad y_1 > 0 > y_2. \tag{34}$$

Let X denote the random variable that describes the asymptotic regime of $X(t)$. By making use of (31), it is not hard to obtain the steady-state density:

$$W(x) := \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}(X < x) = \lim_{t\to+\infty} f(x,t) = y_1 \mathrm{e}^{-y_1 x}, \quad x \geqslant 0, \tag{35}$$

showing that the geometric steady-state distribution of $N(t)$ becomes exponential under the heavy-traffic approximation.

Let us now study the transient behavior of $X(t)$.

Lemma 4.1. Let $\widetilde{X}(t)$ be a Wiener process with drift

$$\mu = \hat{\alpha} - \hat{\beta} + \sigma^2 y_1 = \sqrt{(\hat{\alpha} - \hat{\beta})^2 + 2\sigma^2\xi}$$

and infinitesimal variance σ^2, restricted to $[0, +\infty)$ by a reflecting boundary at 0, such that $\mathrm{P}[\widetilde{X}(0) = 0] = 1$. Further, let X be an absolutely continuous random variable with pdf (35), independent of $\widetilde{X}(t)$. Then,

$$Z(t) = \min\{\widetilde{X}(t), X\}, \quad \forall t \geqslant 0, \tag{36}$$

has the same distribution as $X(t)$.

Proof. Setting

$$f_Z(x,t) = \frac{\partial}{\partial x}\mathrm{P}\{Z(t) < x\} \quad \text{and} \quad \tilde{f}(x,t) = \frac{\partial}{\partial x}\mathrm{P}\{\widetilde{X}(t) < x\},$$

from (35) and (36) we have:

$$f_Z(x,t) = y_1 \mathrm{e}^{-y_1 x}\int_x^{+\infty} \tilde{f}(u,t)\,\mathrm{d}u + \mathrm{e}^{-y_1 x}\tilde{f}(x,t), \quad x \geqslant 0. \tag{37}$$

It is not difficult to prove that $f_Z(x,t)$ is a solution of (31) and that it satisfies conditions (32) since $\tilde{f}(x,t)$ satisfies

$$\frac{\partial}{\partial t}\tilde{f}(x,t) = -\mu\frac{\partial}{\partial x}\tilde{f}(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\tilde{f}(x,t), \quad x \in (0, +\infty),$$

with conditions

$$\lim_{x\downarrow 0}\left\{-\mu\tilde{f}(x,t) + \frac{\sigma^2}{2}\frac{\partial}{\partial x}\tilde{f}(x,t)\right\} = 0$$
$$\lim_{t\downarrow 0}\tilde{f}(x,t) = \delta(x).$$

Hence, $f(x,t) \equiv f_Z(x,t)$ for all $x \geqslant 0$ and for all $t \geqslant 0$. □

Let us now discuss the transient phase of $X(t)$.

Theorem 4.1. For all $x \geqslant 0$ and $t > 0$ we have:

$$
\begin{aligned}
f(x,t) = {} & y_1 \mathrm{e}^{-y_1 x} + \frac{2}{\sigma\sqrt{2\pi t}} \mathrm{e}^{-\xi t} \exp\left\{-\frac{[x-(\hat{\alpha}-\hat{\beta})t]^2}{2\sigma^2 t}\right\} \\
& + \frac{y_2}{2}\mathrm{e}^{-y_2 x}\left[1-\mathrm{Erf}\left(\frac{x+t\sqrt{(\hat{\alpha}-\hat{\beta})^2+2\sigma^2\xi}}{\sigma\sqrt{2t}}\right)\right] \\
& - \frac{y_1}{2}\mathrm{e}^{-y_1 x}\left[1+\mathrm{Erf}\left(\frac{x-t\sqrt{(\hat{\alpha}-\hat{\beta})^2+2\sigma^2\xi}}{\sigma\sqrt{2t}}\right)\right],
\end{aligned}
\tag{38}
$$

where y_1 and y_2 have been defined in (34). Furthermore, for $\lambda > 0$ it is

$$
f_\lambda(x) := \int_0^{+\infty} \mathrm{e}^{-\lambda t} f(x,t)\,\mathrm{d}t = \frac{\varphi_1(\lambda)\mathrm{e}^{-\varphi_1(\lambda)x}}{\lambda}, \tag{39}
$$

where $\varphi_1(\lambda)$ has been defined in (33).

Proof. It is well known (cf., for instance, [1]) that

$$
\tilde{f}(x,t) = \frac{2}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(x-\mu t)^2}{2\sigma^2 t}\right\} - \frac{\mu}{\sigma^2}\exp\left\{\frac{2\mu x}{\sigma^2}\right\}\left[1-\mathrm{Erf}\left(\frac{x+\mu t}{\sigma\sqrt{2t}}\right)\right], \quad x \geqslant 0. \tag{40}
$$

Hence, (38) follows from (37) and (40). From lemma 4.1 we also have:

$$
f_\lambda(x) = y_1 \mathrm{e}^{-y_1 x} \int_x^{+\infty} \tilde{f}_\lambda(u)\,\mathrm{d}u + \mathrm{e}^{-y_1 x}\tilde{f}_\lambda(x), \tag{41}
$$

where $\tilde{f}_\lambda(x)$ is the Laplace transform of the transition density of Wiener process $\widetilde{X}(t)$:

$$
\tilde{f}_\lambda(x) = \frac{1}{\lambda}\frac{\sqrt{\mu^2+2\sigma^2\lambda}-\mu}{\sigma^2}\exp\left\{-\frac{\sqrt{\mu^2+2\sigma^2\lambda}-\mu}{\sigma^2}x\right\}. \tag{42}
$$

From (41) and (42) one obtains (39). □

Quite similarly to the transition probability (14) of the process $N(t)$ describing the $M/M/1$ queue, $f(x,t)$ is expressed as the sum of a time-independent term, given by the steady-state density (35), and of time-dependent terms vanishing as $t \to +\infty$.

As $\xi \downarrow 0$, (38) tends to the transition pdf of a Wiener process with drift $\hat{\alpha} - \hat{\beta}$ and infinitesimal variance σ^2 restricted to $[0, +\infty)$, with 0 a reflecting boundary.

In order to obtain a description of the busy period under the above heavy-traffic approximation, we denote by $g(0,t|x_0)$ the first-passage-time pdf of $X(t)$ from $x_0 > 0$ to 0. By analogy with the busy period of the queueing system, for all $x_0 > 0$ it is

$$
g(0,t|x_0) = \xi\mathrm{e}^{-\xi t}\left[1-\int_0^t \tilde{k}(0,s|x_0)\,\mathrm{d}s\right] + \mathrm{e}^{-\xi t}\tilde{k}(0,t|x_0), \quad t > 0, \tag{43}
$$

where $\tilde{k}(0,t|x_0)$ denotes the first-passage-time pdf of $\widetilde{X}(t)$ from x_0 to 0 (see [1]). Hence, for all $t > 0$ and $x_0 > 0$ we have

$$\begin{aligned} g(0,t|x_0) = {} & \xi \mathrm{e}^{-\xi t} + \frac{x_0 \mathrm{e}^{-\xi t}}{\sigma\sqrt{2\pi t^3}} \exp\left\{-\frac{[x_0 + (\hat{\alpha}-\hat{\beta})t]^2}{2\sigma^2 t}\right\} \\ & - \frac{\xi}{2}\mathrm{e}^{-\xi t}\left\{1 - \mathrm{Erf}\left(\frac{x_0 - (\hat{\beta}-\hat{\alpha})t}{\sigma\sqrt{2t}}\right) + \exp\left\{\frac{2(\hat{\beta}-\hat{\alpha})}{\sigma^2}x_0\right\} \right. \\ & \left. \times \left[1 - \mathrm{Erf}\left(\frac{x_0 + (\hat{\beta}-\hat{\alpha})t}{\sigma\sqrt{2t}}\right)\right]\right\}. \end{aligned} \tag{44}$$

From (43) we note that if $\hat{\alpha} < \hat{\beta}$ the first-passage time of $X(t)$ from $x_0 > 0$ to 0 can be expressed as the minimum between an exponentially distributed random variable with mean $1/\xi$ and the first-passage time through 0 of an independent Wiener process with initial state x_0, drift $\hat{\alpha} - \hat{\beta}$ and infinitesimal variance σ^2.

Since the Laplace transform of $\tilde{k}(0,t|x_0)$ is (see [1])

$$\tilde{k}_\lambda(0|x_0) = \exp\left\{\frac{\hat{\beta}-\hat{\alpha}-\sqrt{(\hat{\beta}-\hat{\alpha})^2 + 2\sigma^2\lambda}}{\sigma^2}x_0\right\}, \quad \lambda > 0,$$

from (43) the Laplace transform of $g(0,t|x_0)$ is obtained:

$$g_\lambda(0|x_0) = \frac{\lambda \mathrm{e}^{\varphi_2(\lambda)x_0} + \xi}{\lambda + \xi}, \quad \lambda > 0, \tag{45}$$

where $\varphi_2(\lambda)$ has been defined in (33).

Use of (45) yields the first two moments and the variance of the first-passage time:

$$\begin{aligned} t_1(0|x_0) &:= \int_0^{+\infty} t g(0,t|x_0)\,\mathrm{d}t = \frac{1}{\xi}\left(1 - \mathrm{e}^{y_2 x_0}\right), \\ t_2(0|x_0) &:= \int_0^{+\infty} t^2 g(0,t|x_0)\,\mathrm{d}t = \frac{2}{\xi^2}\left\{1 - \mathrm{e}^{y_2 x_0} - \frac{2\xi x_0 \mathrm{e}^{y_2 x_0}}{\sigma^2(y_1 - y_2)}\right\}, \\ \mathrm{Var}(0|x_0) &:= t_2(0|x_0) - \left[t_1(0|x_0)\right]^2 = \frac{1}{\xi^2}\left\{1 - \mathrm{e}^{2y_2 x_0} - \frac{4\xi x_0 \mathrm{e}^{y_2 x_0}}{\sigma^2(y_1 - y_2)}\right\}. \end{aligned} \tag{46}$$

Note that, under the continuous approximation, the catastrophe waiting time becomes exponential with mean $1/\xi$.

5. Some comparisons

In the previous section the number of customers in the $M/M/1$ system with catastrophes has been approximated under heavy-traffic condition by the continuous process $X(t)$, whose conditional pdf has been obtained in theorem 4.1.

A first confirmation of the goodness of the approximating procedure follows by comparing mean and variance of $N(t)$, when α and β are chosen as in (30) (namely, under the assumption of large arrival and service rates), with those of $X(t)/\varepsilon$ after the equilibrium regime has been attained. Recalling that $E(X) = \sqrt{\mathrm{Var}(X)} = 1/y_1$ and making use of (5), one has:

$$\lim_{\varepsilon\downarrow 0} \frac{E(N)}{E(X/\varepsilon)} = 1, \qquad \lim_{\varepsilon\downarrow 0} \frac{\mathrm{Var}(N)}{\mathrm{Var}(X/\varepsilon)} = 1. \tag{47}$$

Hence, $E(N) \sim E(X/\varepsilon)$ and $\mathrm{Var}(N) \sim \mathrm{Var}(X/\varepsilon)$ as far as ε is near 0.

In order to discuss the goodness of the continuous heavy-traffic approximation, we denote by $p_n^{(\varepsilon)}(t)$ the transition probabilities of the discrete process $N(t)$ when α and β are given by (30). The following theorem holds.

Theorem 5.1. For all $t > 0$ one has

$$\lim_{\varepsilon\downarrow 0,\ n\varepsilon=x} \frac{p_n^{(\varepsilon)}(t)}{\varepsilon} = f(x,t). \tag{48}$$

Proof. To prove (48), we show that

$$\lim_{\varepsilon\downarrow 0,\ n\varepsilon=x} \frac{\pi_n^{(\varepsilon)}(\lambda)}{\varepsilon} = f_\lambda(x),$$

where $\pi_n^{(\varepsilon)}(\lambda)$ is the Laplace transform of $p_n^{(\varepsilon)}(t)$. Indeed, from (10) we have:

$$\lim_{\varepsilon\downarrow 0,\ n\varepsilon=x} \frac{\pi_n^{(\varepsilon)}(\lambda)}{\varepsilon} = \frac{1}{\lambda} \lim_{\varepsilon\downarrow 0} \frac{\psi_1^{(\varepsilon)}(\lambda) - 1}{\varepsilon\psi_1^{(\varepsilon)}(\lambda)} \big[\psi_1^{(\varepsilon)}(\lambda)\big]^{-x/\varepsilon}, \tag{49}$$

where $\psi_1^{(\varepsilon)}(\lambda)$ is obtained from (11) after setting α and β as given by (30). Since

$$\lim_{\varepsilon\downarrow 0} \frac{\psi_1^{(\varepsilon)}(\lambda) - 1}{\varepsilon\psi_1^{(\varepsilon)}(\lambda)} = \varphi_1(\lambda), \qquad \lim_{\varepsilon\downarrow 0} \big[\psi_1^{(\varepsilon)}(\lambda)\big]^{-x/\varepsilon} = \mathrm{e}^{-\varphi_1(\lambda)x},$$

with $\varphi_1(\lambda)$ given in (33), from (49) we finally obtain:

$$\lim_{\varepsilon\downarrow 0,\ n\varepsilon=x} \frac{\pi_n^{(\varepsilon)}(\lambda)}{\varepsilon} = \frac{\varphi_1(\lambda)}{\lambda} \mathrm{e}^{-\varphi_1(\lambda)x} = f_\lambda(x),$$

where the last equality is due to (39). Equation (48) then immediately follows. □

A consequence of (48) is that $p_n^{(\varepsilon)}(t) \sim \varepsilon f(n\varepsilon, t)$ as far as ε is near to 0.

Let us now test the goodness of the continuous approximation for the busy period. Hereafter we show that, under the limit as $\varepsilon \downarrow 0$, the first-passage-time density $g(0, t|\varepsilon)$ of $X(t)$ is close to $b^{(\varepsilon)}(t)$, the latter being obtained from (18) when α and β are chosen as in (30). This will be accomplished by comparison of the Laplace transforms of such densities.

Let $\mathcal{B}^{(\varepsilon)}(\lambda)$ denote the Laplace transform of $b^{(\varepsilon)}(t)$.

Theorem 5.2. For all $\lambda > 0$

$$\lim_{\varepsilon\downarrow 0} \frac{1-\mathcal{B}^{(\varepsilon)}(\lambda)}{1-g_\lambda(0|\varepsilon)} = 1. \tag{50}$$

Furthermore, one has

$$\lim_{\varepsilon\downarrow 0} \frac{E(B^{(\varepsilon)})}{t_1(0|\varepsilon)} = 1, \qquad \lim_{\varepsilon\downarrow 0} \frac{\mathrm{Var}(B^{(\varepsilon)})}{\mathrm{Var}(0|\varepsilon)} = 1, \tag{51}$$

where $B^{(\varepsilon)}$ denotes the random variable with probability density $b^{(\varepsilon)}(t)$.

Proof. Making use of (20) and (45) we have

$$\lim_{\varepsilon\downarrow 0} \frac{1-\mathcal{B}^{(\varepsilon)}(\lambda)}{1-g_\lambda(0|\varepsilon)} = \lim_{\varepsilon\downarrow 0} \frac{1-\psi_2^{(\varepsilon)}(\lambda)}{\varepsilon} \frac{\varepsilon}{1-\mathrm{e}^{\varphi_2(\lambda)\varepsilon}}$$

where $\psi_2^{(\varepsilon)}(\lambda)$ is obtained from (11) by setting α and β as in (30). Since

$$\lim_{\varepsilon\downarrow 0} \frac{1-\psi_2^{(\varepsilon)}(\lambda)}{\varepsilon} = -\varphi_2(\lambda), \qquad \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{1-\mathrm{e}^{\varphi_2(\lambda)\varepsilon}} = -\frac{1}{\varphi_2(\lambda)},$$

relation (50) immediately follows. Furthermore, recalling (21) with α and β chosen as in (30), and making use of (46), after some calculations we obtain relations (51). □

Table 4

For the $M/M/1$ queue with catastrophes, with α and β given by (30), $E(N)$, CV(N), $E(B)$, CV(B), $E(D)$ and CV(D) are listed for $\hat{\alpha} = 1$, $\hat{\beta} = 0.5$, $\xi = 0.1i$ $(i = 1, 2, \ldots, 20)$, $\sigma^2 = 1$ and $\varepsilon = 0.01$. Furthermore, for the continuous approximation $E(X|\varepsilon)$, $t_1(0|\varepsilon)$ and CV$(0|\varepsilon)$ are also reported for the same choices of parameters.

ξ	$E(N)$	$E(X/\varepsilon)$	CV(N)	$E(B)$	$t_1(0\|\varepsilon)$	CV(B)	CV$(0\|\varepsilon)$	$E(D)$	CV(D)
0.1	586.027	585.41	1.00085	0.114907	0.116399	12.2803	12.2096	10.0171	0.998516
0.2	326.985	326.556	1.00153	0.0641147	0.0648866	11.1947	11.1372	5.01529	0.997535
0.3	237.305	236.992	1.0021	0.0465303	0.0470631	10.465	10.4148	3.34738	0.996772
0.4	190.818	190.587	1.00262	0.0374152	0.0378283	9.9245	9.87893	2.5131	0.996131
0.5	161.973	161.803	1.00308	0.0317595	0.0321003	9.4999	9.4576	2.01235	0.995572
0.6	142.135	142.013	1.00351	0.0278697	0.028162	9.15296	9.11313	1.67839	0.99507
0.7	127.549	127.466	1.00391	0.0250097	0.0252671	8.86137	8.82348	1.43977	0.994613
0.8	116.31	116.259	1.00429	0.0228059	0.0230369	8.61101	8.5747	1.26075	0.994189
0.9	107.345	107.321	1.00465	0.021048	0.0212583	8.39245	8.35747	1.12146	0.993794
1.0	100.0	100.0	1.00499	0.0196078	0.0198013	8.1991	8.16524	1.01	0.993421
1.1	93.8542	93.8749	1.00531	0.0184028	0.0185824	8.02615	7.99326	0.918777	0.993069
1.2	88.6228	88.6618	1.00563	0.017377	0.017545	7.87003	7.838	0.842736	0.992734
1.3	84.106	84.1613	1.00593	0.0164914	0.0166494	7.72801	7.69674	0.778377	0.992413
1.4	80.1594	80.2295	1.00622	0.0157175	0.015867	7.59795	7.56736	0.723197	0.992105
1.5	76.6758	76.7592	1.0065	0.0150345	0.0151764	7.47814	7.44817	0.675361	0.99181
1.6	73.5738	73.6693	1.00677	0.0144262	0.0145615	7.36723	7.33782	0.633495	0.991524
1.7	70.7904	70.897	1.00704	0.0138805	0.0140099	7.26409	7.23518	0.596545	0.991248
1.8	68.276	68.3928	1.0073	0.0133875	0.0135115	7.16779	7.13935	0.563692	0.990981
1.9	65.9911	66.1174	1.00755	0.0129394	0.0130587	7.07755	7.04955	0.534291	0.990722
2.0	63.9038	64.0388	1.00779	0.0125301	0.0126451	6.99272	6.96513	0.507824	0.99047

To conclude, we shall now make use of the results outlined in the foregoing to indicate by means of some examples the goodness of our continuous approximation to the $M/M/1$ queueing system with catastrophes. Hence, in tables 4–6, obtained by choosing $\varepsilon = 0.01$, $\varepsilon = 0.001$ and $\varepsilon = 0.0001$, respectively, we have listed the values of $E(N)$, $\mathrm{CV}(N)$, $E(B)$, $\mathrm{CV}(B)$, $E(D)$ and $\mathrm{CV}(D)$ for the $M/M/1$ queue with catastrophes when α and β are given as in (30), with $\hat{\alpha} = 1$, $\hat{\beta} = 0.5$, $\sigma^2 = 1$ and $\xi = 0.1i$ $(i = 1, 2, \ldots, 20)$. The values of $E(N)$ are listed in the second columns, whereas the values of $E(X/\varepsilon) = (y_1\varepsilon)^{-1}$ are given in the third columns; the goodness of the agreement between $E(N)$ and $E(X/\varepsilon)$ is evident. In the fourth columns the values of $\mathrm{CV}(N)$ are listed; we note that $\mathrm{CV}(N)$ approaches the value $\mathrm{CV}(X/\varepsilon) = \mathrm{CV}(X) = 1$ to a very high degree of precision, a reminder of the exponential nature of the steady-state density $W(x)$. In columns five and seven $E(B)$ and of $\mathrm{CV}(B)$ are shown, whereas in columns six and eight the values of $t_1(0|\varepsilon)$ and of $\mathrm{CV}(0|\varepsilon)$, both obtained from (46) by setting $x_0 = \varepsilon$, are listed. The agreement between $E(B)$ and $t_1(0|\varepsilon)$, and between $\mathrm{CV}(B)$ and $\mathrm{CV}(0|\varepsilon)$ is again evident. Finally, in columns nine and ten the values of $E(D)$ and $\mathrm{CV}(D)$ are reported. We underline that the values of $E(D)$ are near $1/\xi$ and that the values of $\mathrm{CV}(D)$ are near 1, a confirmation of the circumstance that under the continuous approximation the catastrophe waiting time density becomes exponential with mean $1/\xi$. As pointed out, the approximation of $N(t)$ by $X(t)/\varepsilon$ is expected to improve as ε decreases. This is confirmed by the data of tables 4–6 showing that the

Table 5
Same as in table 4 for $\hat{\alpha} = 1$, $\hat{\beta} = 0.5$, $\xi = 0.1i$ $(i = 1, 2, \ldots, 20)$, $\sigma^2 = 1$ and $\varepsilon = 0.001$.

ξ	$E(N)$	$E(X/\varepsilon)$	$\mathrm{CV}(N)$	$E(B)$	$t_1(0\|\varepsilon)$	$\mathrm{CV}(B)$	$\mathrm{CV}(0\|\varepsilon)$	$E(D)$	$\mathrm{CV}(D)$
0.1	5854.72	5854.1	1.00009	0.0116861	0.0117014	38.6323	38.6097	10.0017	0.999851
0.2	3265.99	3265.56	1.00015	0.00651895	0.00652687	35.2368	35.2186	5.00153	0.999752
0.3	2370.24	2369.92	1.00021	0.00473101	0.00473648	32.95	32.9341	3.33474	0.999675
0.4	1906.1	1905.87	1.00026	0.00380459	0.00380883	31.2539	31.2394	2.50131	0.99961
0.5	1618.2	1618.03	1.00031	0.00322995	0.00323345	29.9204	29.907	2.00124	0.999553
0.6	1420.26	1420.13	1.00035	0.00283484	0.00283785	28.8303	28.8176	1.66784	0.999502
0.7	1274.74	1274.66	1.00039	0.0025444	0.00254705	27.9136	27.9016	1.42969	0.999456
0.8	1162.64	1162.59	1.00043	0.00232065	0.00232302	27.1264	27.1148	1.25108	0.999413
0.9	1073.24	1073.21	1.00047	0.00214219	0.00214435	26.439	26.4278	1.11215	0.999372
1.0	1000.0	1000.0	1.0005	0.00199601	0.001998	25.8307	25.8199	1.001	0.999334
1.1	938.728	938.749	1.00053	0.00187371	0.00187556	25.2865	25.276	0.910059	0.999298
1.2	886.578	886.618	1.00056	0.00176962	0.00177135	24.7952	24.785	0.834273	0.999264
1.3	841.557	841.613	1.00059	0.00167976	0.00168139	24.3482	24.3382	0.770145	0.999231
1.4	802.224	802.295	1.00062	0.00160125	0.00160279	23.9387	23.929	0.715176	0.9992
1.5	767.508	767.592	1.00065	0.00153195	0.00153342	23.5616	23.5521	0.667535	0.999169
1.6	736.597	736.693	1.00068	0.00147025	0.00147165	23.2124	23.2031	0.625848	0.99914
1.7	708.862	708.97	1.00071	0.0014149	0.00141623	22.8876	22.8785	0.589065	0.999112
1.8	683.811	683.928	1.00073	0.00136489	0.00136617	22.5844	22.5754	0.556368	0.999084
1.9	661.047	661.174	1.00076	0.00131945	0.00132069	22.3002	22.2914	0.527112	0.999058
2.0	640.252	640.388	1.00078	0.00127795	0.00127914	22.0331	22.0243	0.500781	0.999032

Table 6
Same as in table 4 for $\hat{\alpha} = 1$, $\hat{\beta} = 0.5$, $\xi = 0.1i$ $(i = 1, 2, \ldots, 20)$, $\sigma^2 = 1$ and $\varepsilon = 0.0001$.

ξ	$E(N)$	$E(X/\varepsilon)$	CV(N)	$E(B)$	$t_1(0\|\varepsilon)$	CV(B)	CV$(0\|\varepsilon)$	$E(D)$	CV(D)
0.1	58541.6	58541.	1.00001	0.0011706	0.00117075	122.102	122.095	10.0002	0.999985
0.2	32656.1	32655.6	1.00002	0.000652991	0.00065307	111.377	111.371	5.00015	0.999975
0.3	23699.6	23699.2	1.00002	0.000473896	0.000473951	104.152	104.147	3.33347	0.999967
0.4	19058.9	19058.7	1.00003	0.000381102	0.000381145	98.7922	98.7877	2.50013	0.999961
0.5	16180.5	16180.3	1.00003	0.000323546	0.000323581	94.5784	94.5742	2.00012	0.999955
0.6	14201.5	14201.3	1.00004	0.000283972	0.000284002	91.1333	91.1293	1.66678	0.99995
0.7	12746.7	12746.6	1.00004	0.000254883	0.000254909	88.2364	88.2326	1.42868	0.999946
0.8	11626.0	11625.9	1.00004	0.000232473	0.000232497	85.7483	85.7447	1.25011	0.999941
0.9	10732.1	10732.1	1.00005	0.0002146	0.000214622	83.5757	83.5721	1.11121	0.999937
1.0	10000.0	10000.0	1.00005	0.00019996	0.00019998	81.6531	81.6497	1.0001	0.999933
1.1	9387.47	9387.49	1.00005	0.000187712	0.00018773	79.9331	79.9298	0.909188	0.99993
1.2	8866.14	8866.18	1.00006	0.000177287	0.000177305	78.3802	78.377	0.833427	0.999926
1.3	8416.08	8416.13	1.00006	0.000168288	0.000168304	76.9673	76.9642	0.769322	0.999923
1.4	8022.88	8022.95	1.00006	0.000160425	0.000160441	75.6733	75.6702	0.714375	0.99992
1.5	7675.83	7675.92	1.00007	0.000153486	0.000153501	74.4812	74.4782	0.666754	0.999917
1.6	7366.83	7366.93	1.00007	0.000147307	0.000147321	73.3775	73.3745	0.625085	0.999914
1.7	7089.59	7089.7	1.00007	0.000141763	0.000141777	72.3509	72.348	0.588318	0.999911
1.8	6839.16	6839.28	1.00007	0.000136756	0.000136769	71.3924	71.3896	0.555637	0.999908
1.9	6611.61	6611.74	1.00008	0.000132206	0.000132218	70.4942	70.4914	0.526395	0.999906
2.0	6403.75	6403.88	1.00008	0.000128049	0.000128061	69.6498	69.6471	0.500078	0.999903

goodness of the approximation of $N(t)$ by $X(t)/\varepsilon$ improves as ε is reduced. Indeed, let us define the relative errors

$$\varrho = \left|\frac{E(N) - E(X/\varepsilon)}{E(N)}\right|, \qquad \eta = \left|\frac{E(B) - t_1(0|\varepsilon)}{E(B)}\right|, \qquad \omega = \left|\frac{E(D) - 1/\xi}{E(D)}\right|.$$

From tables 4–6 we see that for $\varepsilon = 0.01$ one has $\varrho < 2\cdot 10^{-3}$, $\eta < 1\cdot 10^{-2}$, $\omega < 1\cdot 10^{-2}$; for $\varepsilon = 0.001$ it is $\varrho < 2 \cdot 10^{-4}$, $\eta < 1 \cdot 10^{-3}$, $\omega < 1 \cdot 10^{-3}$; finally, for $\varepsilon = 0.0001$ one obtains $\varrho < 2 \cdot 10^{-5}$, $\eta < 1 \cdot 10^{-4}$, $\omega < 1 \cdot 10^{-4}$. It is thus concluded that if the continuous approximation is used in place of the original queue, the order of magnitude of the absolute value of each relative error linearly decreases with ε.

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