APPENDIX

BASICS OF PROBABILITY, RANDOM VARIABLES, RANDOM PROCESSES, AND QUEUEING SYSTEMS

A.1 INTRODUCTION

In this appendix we review the basics of probability, random variables, exponential random process, birth–death processes, and queueing systems.

A.2 PROBABILITY

A.2.1 Set Operations

The following are basic set operations:

1. $A \cap B = B \cap A$
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$
4. $A \cup B = B \cup A$
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
6. $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$
7. $(\overline{A \cup B}) = \overline{A} \cap \overline{B}$
8. $(\overline{A \cap B}) = \overline{A} \cup \overline{B}$

Wireless Broadband Networks, By David Tung Chong Wong, Peng-Yong Kong, Ying-Chang Liang, Kee Chaing Chua, and Jon W. Mark
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A.2.2 Elements of Probability

Let us first define a trial, a sample space, and an event before introducing probability through set operations.

- A *trial* is a single performance of an experiment for which there is an outcome.
- A *sample space*, S, is the set of all possible outcomes in any given experiment.
- An *event*, A, is a subset of the sample space. If two events have no common outcomes, they are *mutually exclusive*.

Axioms

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n)$ if$(A_m \cap A_n) = \emptyset$, where $\emptyset$ is the null set.

Joint Probability

1. $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

For $A \cap B = \emptyset,\ P(A \cap B) = P(\emptyset) = 0$ (mutually exclusive).

Conditional Probability The conditional probability of an event A, given B, with $P(B) > 0$, is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{A.1}$$

If A and B are mutually exclusive ($A \cap B = \emptyset$), we have

$$P(A|B) = 0. \tag{A.2}$$

If $A \cap C = \emptyset$, we have

$$P[(A \cup C)|B] = P(A|B) + P(C|B). \tag{A.3}$$

Total Probability If $B_m \cap B_n = \emptyset,\ m \neq n = 1, 2, \ldots, N, \bigcup_{n=1}^{N} B_n = S,$ we have

$$P(A) = \sum_{n=1}^{N} P(A|B_n)P(B_n). \tag{A.4}$$

Bayes' Theorem

$$P(B_n|A) = \frac{P(B_n \cap A)}{P(A)}, \qquad P(A) \neq 0, \tag{A.5}$$

$$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)}, \qquad P(B_n) \neq 0, \tag{A.6}$$

$$P(B_n|A) = \frac{P(A|B_n)P(B_n)}{P(A)} = \frac{P(A|B_n)P(B_n)}{\sum_{n=1}^{N} P(A|B_n)P(B_n)}. \tag{A.7}$$

Independent Events Given that events A, B, and C are independent, we have

$$P(A|B) = P(A), \tag{A.8}$$

$$P(B|A) = P(B), \tag{A.9}$$

$$P(A \cap B) = P(A) \cdot P(B), \tag{A.10}$$

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C). \tag{A.11}$$

A.3 RANDOM VARIABLES

A random variable is a real function of the elements of a sample space, S.

A.3.1 Conditions

1. Set $\{X \leq x\}$ is an event for any real number x.
2. $P(X = -\infty) = 0$.
3. $P(X = \infty) = 0$.

A.3.2 Discrete Random Variables

A discrete random variable has only discrete values. The sample space can be discrete, continuous, or a mixture.

Probability Mass Function

$$f_X(x) = P(X = x), \qquad x = 0, 1, 2, \ldots \quad \text{(discrete sample space)} \tag{A.12}$$

Total Probability

$$\sum_{x=0}^{\infty} f_X(x) = 1 \tag{A.13}$$

Cumulative Distribution Function

$$F_X(x) = \sum_{i=0}^{x} f_X(i) \tag{A.14}$$

Expected or Mean Value

$$E[X] = \sum_{x=0}^{\infty} x f_X(x) \tag{A.15}$$

or

$$E[X] = \sum_{x=0}^{\infty} (1 - F_X(x)) \qquad \text{if } x > 0 \tag{A.16}$$

Variance

$$\mathrm{Var}[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2, \qquad \text{where } E[X^2] = \sum_{x=0}^{\infty} x^2 f_X(x). \tag{A.17}$$

A few common discrete random variables, with their mean, second moments, and variance, are shown in Table A.1.

TABLE A.1 Common Discrete Random Variables

Discrete Random Variable	$f_X(x)$	$E[X]$ or $\overline{X}$	$E[X^2]$ or $\overline{X^2}$	$\mathrm{Var}[X]$
Bernoulli	$\begin{cases} 1-p, x=0 \\ p, x=1 \end{cases}$	p	p	$p(1-p)$
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2,\ldots,n$ $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$	np	$n^2p^2 + np(1-p)$	$np(1-p)$
Geometric	$p(1-p)^{x-1}, \quad x = 1,2,\ldots$	$\dfrac{1}{p}$	$\dfrac{2-p}{p^2}$	$\dfrac{1-p}{p^2}$
Poisson	$\dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\ldots$	λ	$\lambda^2 + \lambda$	λ

A.3.3 Continuous Random Variables

A continuous random variable has a continuous range of values. The sample space is continuous.

Probability Density Function

$$f_X(x) = P(X = x), x \in (-\infty, \infty) \tag{A.18}$$

Total Probability

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1. \tag{A.19}$$

Cumulative Distribution Function

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx \tag{A.20}$$

Expected or Mean Value

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \tag{A.21}$$

or

$$E[X] = \int_{0}^{\infty} [1 - F_X(x)]\, dx, \qquad \text{if } x \geq 0 \tag{A.22}$$

Variance

$$\begin{aligned}\mathrm{Var}[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2, \qquad \text{where} \quad E[X^2]\\ = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx\end{aligned} \tag{A.23}$$

A few common continuous random variables, with their mean, second moments, and variance, are shown in Table A.2.

TABLE A.2 Common Continuous Random Variables

Continuous Random Variables	$f_X(x)$	$E[X]$ or $\overline{X}$	$E[X^2]$ or $\overline{X^2}$	Var[X]
Uniform	$\dfrac{1}{b-a}, x \in [a, b]$	$\dfrac{a+b}{2}$	$\dfrac{a^2+b^2+ab}{3}$	$\dfrac{(b-a)^2}{12}$
Exponential	$\lambda e^{-\lambda x}, \quad x \in [0, \infty)$	$\dfrac{1}{\lambda}$	$\dfrac{2}{\lambda^2}$	$\dfrac{1}{\lambda^2}$
Gaussian (normal)	$\dfrac{1}{\sqrt{2\pi}\sigma} \exp\left[-\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right]$	μ	$\mu^2 + \sigma^2$	σ^2

Memoryless Property of Exponential Distribution

Exponential pdf

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0, \quad \lambda > 0 \tag{A.24}$$

Exponential CDF

$$F_X(x) = 1 - e^{-\lambda x} \tag{A.25}$$

Memoryless Property

$$\begin{aligned} P[X > s + t | X > t] &= \frac{P[X > s + t, X > t]}{P[X > t]} \\ &= \frac{P[X > s + t]}{P[X > t]} \\ &= \frac{P[X > s]P[X > t]}{P[X > t]} \\ &= P[X > s] \\ &= e^{-\lambda s} \quad \text{(independent of } t\text{)} \end{aligned} \tag{A.26}$$

A.4 POISSON RANDOM PROCESS

Let $X(t)$ be the number of Poisson points in $[0, t)$ for $t \geq 0$. Given t, $X(t)$ is a Poisson random variable with parameter λt.

$$P[X(t) = x] = \frac{(\lambda t)^x}{x!} e^{-\lambda t}, \tag{A.27}$$

$$E[X(t)] = \lambda t, \tag{A.28}$$

$$E[X^2(t)] = \lambda t + \lambda^2 t^2, \tag{A.29}$$

$$\text{Var}[X(t)] = \lambda t. \tag{A.30}$$

A.4.1 Interarrival Times of a Poisson Process

The interarrival times of a Poisson process are independent, identically distributed exponential random variables with mean $1/\lambda$.

A.4.2 Decomposition of a Poisson Process

$X(t)$, $t \geq 0$ is a Poisson random process with rate λ. Event i is class i with probability p_i such that

$$\sum_{i=1}^{n} p_i = 1. \tag{A.31}$$

Let $X_i(t)$ = number of class i arrivals in $[0, t)$ such that

$$X(t) = \sum_{i=1}^{n} X_i(t). \tag{A.32}$$

$X_i(t)$ is a Poisson random process with rate $p_i\lambda$, and the $X_i(t)$'s are independent. That is, the decomposition of a Poisson random process will result in n other Poisson random processes.

A.4.3 Superposition of Poisson Processes

If the $X_i(t)$'s are Poisson random processes with rate λ_i such that

$$X(t) = \sum_{i=1}^{n} X_i(t), \tag{A.33}$$

then $X(t),\ t \geq 0$ is a Poisson random process with rate λ such that

$$\lambda = \sum_{i=1}^{n} \lambda_i. \tag{A.34}$$

That is, the superposition of Poisson random processes results in a Poisson random process with a rate equal to the sum of the individual rates.

A.5 BIRTH–DEATH PROCESSES

A birth–death process is a continuous-time Markov chain with discrete state space in which only transitions to neighboring states are permitted. The state transition diagram is shown in Figure A.1. $K(t) = k$ is the number in the system (customers, packets, calls, channels, etc.). λ_k is the arrival rate when $K(t) = k$. μ_k is the service rate when $K(t) = k$. $\{K(t) : t \geq 0\}$ is a continuous-time Markov chain. The steady-state (equilibrium) distribution of the probability of the number in the system

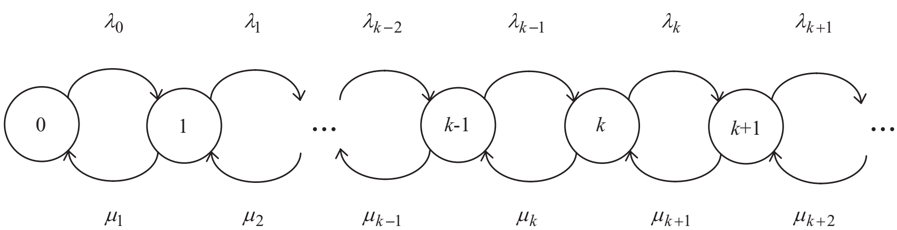

FIGURE A.1 State transition diagram of a birth–death process.

(state k), p_k, is given by

$$p_k = \lim_{t\to\infty} P[K(t) = k]. \tag{A.35}$$

By global balance equations:

$$\begin{aligned} \lambda_0 p_0 &= \mu_1 p_1, \qquad k = 0, \\ p_1 &= \frac{\lambda_0}{\mu_1} p_0, \end{aligned} \tag{A.36}$$

$$(\lambda_k + \mu_k)p_k = \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}, \qquad k = 1, 2, \ldots. \tag{A.37}$$

By local balance equations (across a boundary), we have

$$\begin{aligned} \lambda_{k-1}p_{k-1} &= \mu_k p_k, \qquad k = 1, 2, \ldots, \\ p_k &= \frac{\lambda_{k-1}}{\mu_k} p_{k-1} \\ &= p_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}. \end{aligned} \tag{A.38}$$

By the total probability property, we have

$$\begin{aligned} &\sum_{k=0}^{\infty} p_k = 1, \\ &p_0 + p_1 + p_2 + \cdots = 1, \\ &p_0 \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \right] = 1, \\ &p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}]}. \end{aligned} \tag{A.39}$$

The steady-state probability of being in state k, p_k, is given by

$$p_k = \frac{\prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}]}. \tag{A.40}$$

The mean number of customers in the system, $\overline{N}$, is given by

$$\overline{N} = \sum_{k=0}^{\infty} k p_k. \tag{A.41}$$

A.6 BASIC QUEUEING SYSTEMS

A.6.1 Kendall's Notation

An $A/B/C/K/m/Z$ queueing system means:

A	arrival process
B	service process
C	number of servers
K	maximum capacity (buffer size)
m	population of users
Z	service discipline

For arrival or service processes:

M	exponential distribution
D	deterministic (constant) distribution
G	general distribution

For principal service disciplines:

FIFO	first in, first out
LCFS	last come, first served
FIRO	first in, random out

When the last three elements of Kendall's notation are not specified, it is understood that Z = FIFO, $m = +\infty$, and $K = +\infty$.

A.6.2 *M/M/*1

The following are the assumptions for an $M/M/1$ queue:

- Poisson arrival process (exponential interarrival times)
- Exponential service time
- One server
- Infinite storage spaces

The state transition diagram for an $M/M/1$ queue is shown in Figure A.2. From this diagram, we have

$$\lambda_k = \lambda \qquad \forall k, \tag{A.42}$$

$$\mu_k = \mu \qquad \forall k, \tag{A.43}$$

$$p_0 = 1 - \rho, \qquad \rho = \frac{\lambda}{\mu} < 1. \tag{A.44}$$

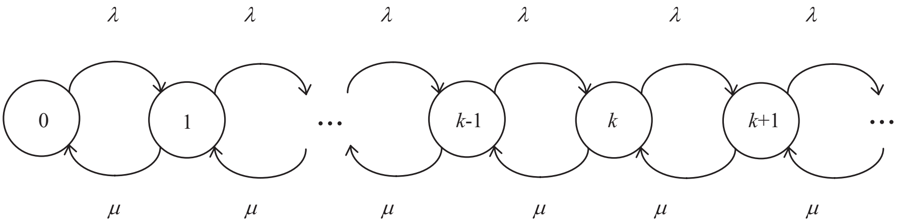

FIGURE A.2 State transition diagram of an $M/M/1$ queue.

The steady-state probability of being in state k, p_k, is given by

$$p_k = (1-\rho)\rho^k \quad \text{(geometric).} \tag{A.45}$$

The mean number of customers in the system, $\overline{N}$, is given by

$$\overline{N} = \sum_{k=0}^{\infty} k p_k = \frac{\rho}{1-\rho}. \tag{A.46}$$

Figure A.3 shows an $M/M/1$ queue. From Little's law,

$$\overline{N} = \lambda \overline{T}, \tag{A.47}$$

where λ is the mean arrival rate for the system and $\overline{T}$ is the mean time spent in the system. The mean time spent in the system, $\overline{T}$, is given by

$$\overline{T} = \frac{\overline{N}}{\lambda} = \frac{\frac{1}{\mu}}{1-\rho} = \frac{1}{\mu-\lambda}, \qquad \rho = \frac{\lambda}{\mu}. \tag{A.48}$$

The mean queueing delay, $\overline{W}$, is given by

$$\begin{aligned} \overline{W} &= \overline{T} - \frac{1}{\mu}, \qquad \frac{1}{\mu} \text{ is the mean service time} \\ &= \frac{\mu - (\mu - \lambda)}{\mu(\mu - \lambda)} \\ &= \frac{1}{\mu}\frac{\rho}{1-\rho}. \end{aligned} \tag{A.49}$$

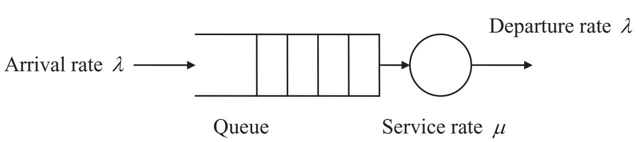

FIGURE A.3 $M/M/1$ queue.

Using Little's law, the mean number of customers in the queue, $\overline{N}_q$, is given by

$$\overline{N}_q = \lambda \overline{W} = \frac{\rho^2}{1-\rho}. \tag{A.50}$$

Using Little's law, the mean number of customers in service, $\overline{N}_s$, is given by

$$\overline{N}_s = \lambda \frac{1}{\mu} = \rho. \tag{A.51}$$

A.6.3 *M/M/*1*/K* (Finite Storage)

The following are the assumptions for an $M/M/1/K$ queue:

- Poisson arrival process (exponential interarrival times)
- Exponential service time
- One server
- K storage spaces

The state transition diagram for an *M/M/1/K* queue is shown in Figure A.4. From this diagram we have

$$\lambda_k = \begin{cases} \lambda, & k < K \\ 0, & k = K, \end{cases} \tag{A.52}$$

$$\mu_k = \mu \qquad \forall k, \tag{A.53}$$

$$p_k = \rho p_{k-1}, \qquad \rho = \frac{\lambda}{\mu}, \quad k \le K \tag{A.54}$$

$$= \rho^k p_0$$

$$= 0, \qquad k > K. \tag{A.55}$$

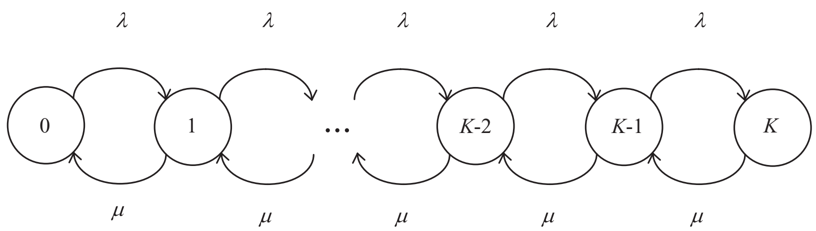

FIGURE A.4 State transition diagram of an $M/M/1/K$ queue.

By the total probability property,

$$\begin{aligned}
&\sum_{k=0}^{K} p_k = 1, \\
&p_0[1 + \rho + \rho^2 + \cdots + \rho^K] = 1, \\
&p_0 = \frac{1-\rho}{1-\rho^{K+1}}.
\end{aligned} \tag{A.56}$$

The steady-state probability of being in state k, p_k, is given by

$$p_k = \begin{cases} \dfrac{(1-\rho)\rho^k}{1-\rho^{K+1}}, & k \le K \\ 0, & k > K. \end{cases} \tag{A.57}$$

The blocking probability, P_B, is given by

$$\begin{aligned}
P_B &= P[\text{an arrival sees the system full}] \\
&= p_K \\
&= \frac{(1-\rho)\rho^K}{1-\rho^{K+1}}.
\end{aligned} \tag{A.58}$$

The mean number in the system, $\overline{N}$, is given by

$$\overline{N} = \sum_{k=0}^{\infty} k p_k = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}. \tag{A.59}$$

Figure A.5 shows an $M/M/1/K$ queue. The actual arrival rate to the system, λ_a, is given by

$$\lambda_a = (1 - P_B)\lambda. \tag{A.60}$$

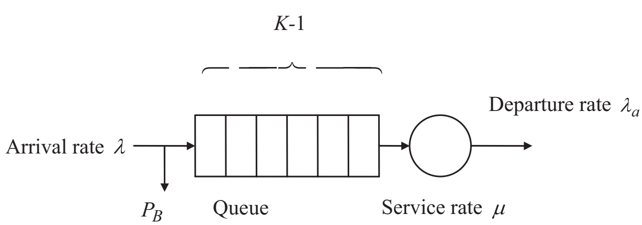

FIGURE A.5 $M/M/1/K$ queue.

The mean time spent in the system, $\overline{T}$, is given by

$$\overline{T} = \frac{\overline{N}}{\lambda_a}. \tag{A.61}$$

The mean number of customers in service, $\overline{N}_s$, is given by

$$\overline{N}_s = 1 - p_0. \tag{A.62}$$

The mean number of customers in the queue, $\overline{N}_q$, is given by

$$\overline{N}_q = \overline{N} - \overline{N}_s. \tag{A.63}$$

The mean queueing delay, $\overline{W}$, is given by

$$\overline{W} = \frac{\overline{N}_q}{\lambda_a}. \tag{A.64}$$

A.6.4 *M/M/m* (*m* Servers System)

The following are the assumptions for an *M/M/m* queue:

- Poisson arrival process (exponential interarrival times)
- Exponential service time
- m servers
- Infinite storage spaces

The state transition diagram for an *M/M/m* queue is shown in Figure A.6. From this diagram we have

$$\lambda_k = \lambda \qquad \forall k, \tag{A.65}$$

$$\mu_k = \begin{cases} k\mu, & k \le m \\ m\mu, & k > m, \end{cases} \tag{A.66}$$

$$\begin{aligned} p_k &= p_0 \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu}, \qquad k \le m \\ &= p_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}, \end{aligned} \tag{A.67}$$

$$\begin{aligned} p_k &= p_0 \prod_{i=0}^{m-1} \frac{\lambda}{(i+1)\mu} \prod_{j=m}^{k-1} \frac{\lambda}{m\mu}, \qquad k > m \\ &= p_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{m!\, m^{k-m}}. \end{aligned} \tag{A.68}$$

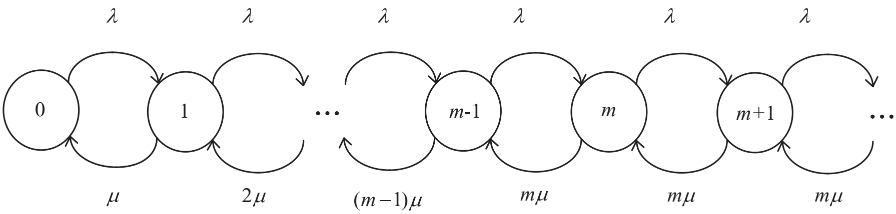

FIGURE A.6 State transition diagram of an *M/M/m* queue.

The steady-state probability of being in state k, p_k, is given by

$$p_k = \begin{cases} p_0 \dfrac{(m\rho)^k}{k!}, & k \le m, \quad \rho = \dfrac{\lambda}{m\mu} < 1 \\ p_0 \dfrac{\rho^k m^m}{m!}, & k > m. \end{cases} \tag{A.69}$$

By the total probability property,

$$\begin{aligned} &\sum_{k=0}^{\infty} p_k = 1, \\ &p_0 \left[1 + \sum_{k=1}^{m-1} \frac{(m\rho)^k}{k!} + \frac{m^m}{m!} \sum_{k=m}^{\infty} \rho^k \right] = 1, \\ &p_0 \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \sum_{k=0}^{\infty} \rho^k \right] = 1, \\ &p_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho} \right]^{-1}. \end{aligned} \tag{A.70}$$

Figure A.7 shows an *M/M/m* queue. The probability that an arrival will find all servers busy and will be forced to wait in queue is an important performance measure of the *M/M/m* system. Since an arriving customer finds the system in a "typical" state, we have

$$\begin{aligned} P[\text{queueing}] &= \sum_{k=m}^{\infty} p_k \\ &= p_0 \frac{(m\rho)^m}{m!\,(1-\rho)}, \end{aligned} \tag{A.71}$$

known as the *Erlang C formula*. This formula is often used in telephony (and more generally, in circuit-switching systems) to estimate the probability of a call request finding all the m circuits of a transmission line busy. In an *M/M/m* model it is assumed

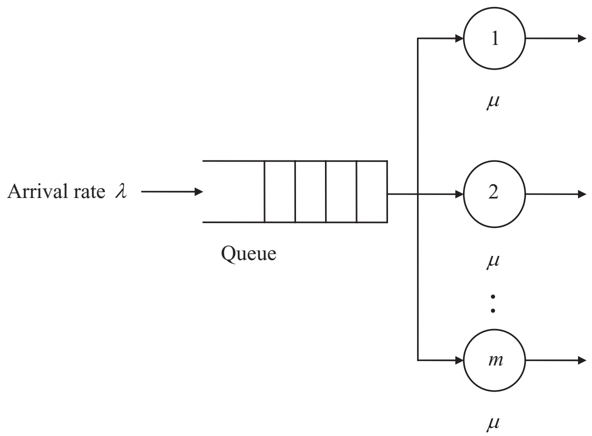

FIGURE A.7 *M/M/m* queue.

that such a call request "remains in queue," that is, attempts continuously to find a free circuit. The mean number of customers in the queue, $\overline{N}_q$, is given by

$$\overline{N}_q = \sum_{k=m}^{\infty} (k - m)p_k = \frac{(m\rho)^m \rho}{m!\,(1-\rho)^2} p_0. \tag{A.72}$$

The mean queueing delay, $\overline{W}$, is given by

$$\overline{W} = \frac{\overline{N}_q}{\lambda}. \tag{A.73}$$

The mean time spent in the system, $\overline{T}$, is given by

$$\overline{T} = \overline{W} + \frac{1}{\mu}. \tag{A.74}$$

The number of customers in the system, $\overline{N}$, is given by

$$\overline{N} = \lambda \overline{T}. \tag{A.75}$$

A.6.5 *M/M/m/m* (*m* Servers Loss System)

The following are the assumptions for an *M/M/m/m* queue:

- Poisson arrival process (exponential interarrival times)
- Exponential service time
- m servers
- m storage spaces

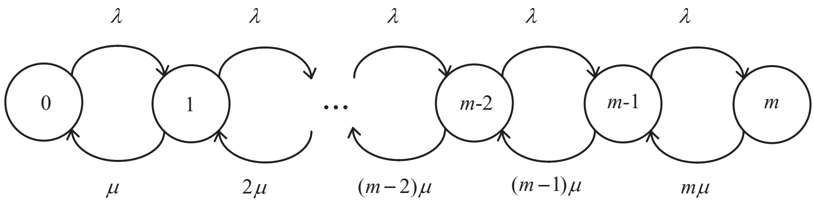

FIGURE A.8 State transition diagram of an *M/M/m/m* queue.

The state transition diagram for an *M/M/m/m* queue is shown in Figure A.8. From this diagram we have

$$\lambda_k = \begin{cases} \lambda, & 0 \le k < m \\ 0, & k = m, \end{cases} \tag{A.76}$$

$$\mu_k = k\mu, \qquad 0 \le k \le m, \tag{A.77}$$

$$\begin{aligned} p_k &= p_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}, \qquad 0 \le k \le m \\ &= p_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}. \end{aligned} \tag{A.78}$$

By the total probability property,

$$\begin{aligned} \sum_{k=0}^{m} p_k &= 1, \\ p_0 \left[\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}\right] &= 1, \\ p_0 &= \left[\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}\right]^{-1}. \end{aligned} \tag{A.79}$$

The steady-state probability of being in state k, p_k, is given by

$$p_k = \frac{\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}{\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}. \tag{A.80}$$

Figure A.9 shows an *M/M/m/m* queue. The probability that an arrival will find all servers busy and will therefore be lost is an important performance measure of the *M/M/m/m* system. The blocking probability, P_B, is given by

$$P_B = p_m = \frac{\left(\frac{\lambda}{\mu}\right)^m \frac{1}{m!}}{\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}, \tag{A.81}$$

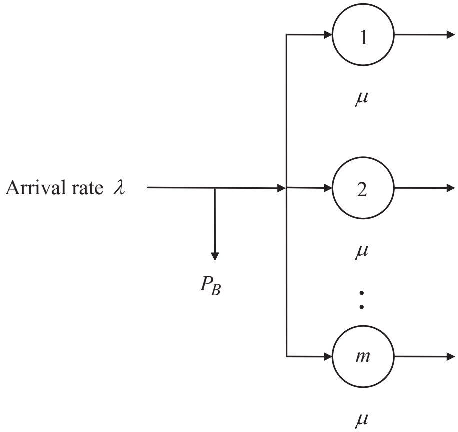

FIGURE A.9 *M/M/m/m* queue.

known as the *Erlang B formula*. This formula is widely used to evaluate the blocking probability of telephone systems. The actual arrival rate at the system, λ_a, is given by

$$\lambda_a = (1 - P_B)\lambda. \tag{A.82}$$

The mean time spent in the system, $\overline{T}$, is given by

$$\overline{T} = \frac{1}{\mu}. \tag{A.83}$$

The mean number of customers in the system, $\overline{N}$, is given by

$$\overline{N} = \lambda_a \overline{T} = \frac{\lambda}{\mu}(1 - P_B). \tag{A.84}$$

Note that there is no waiting room or queue. Therefore, there is no queueing delay and each customer is either served or lost.

A.6.6 *M*/*M*/∞ (∞ Servers)

The following are the assumptions for an $M/M/\infty$ queue:

- Poisson arrival process (exponential interarrival times)
- Exponential service time
- Infinite servers
- Infinite storage spaces

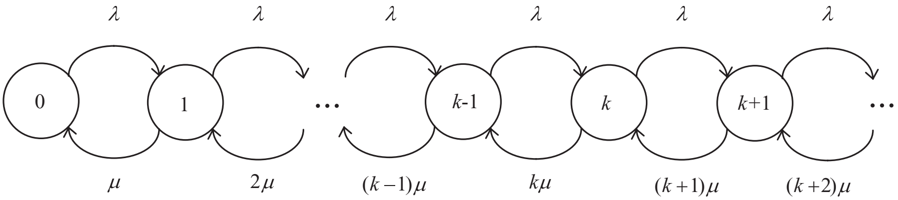

FIGURE A.10 State transition diagram of an $M/M/\infty$ queue.

The state transition diagram for an $M/M/\infty$ queue is shown in Figure A.10. From this diagram we have

$$\lambda_k = \lambda, \tag{A.85}$$

$$\mu_k = k\mu, \tag{A.86}$$

$$\begin{aligned} p_k &= \frac{1}{k!}\frac{\lambda}{\mu}p_{k-1}, \qquad k = 1, 2, 3, \ldots \\ &= \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k p_0. \end{aligned} \tag{A.87}$$

By the total probability property,

$$\begin{aligned} \sum_{k=0}^{\infty} p_k &= 1, \\ p_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k &= 1, \\ p_0 e^{\lambda/\mu} &= 1, \\ p_0 &= e^{-\lambda/\mu}. \end{aligned} \tag{A.88}$$

The steady-state probability of being in state k, p_k, is given by

$$p_k = \frac{\left(\frac{\lambda}{\mu}\right)^k e^{-\lambda/\mu}}{k!} \quad \text{(Poisson)} \tag{A.89}$$

It can be shown that the number in the system is Poisson distributed even if the service time is not exponential (i.e., $M/G/\infty$). The mean time spent in the system, $\overline{T}$, is given by

$$\overline{T} = \frac{1}{\mu}. \tag{A.90}$$

The mean number in the system, $\overline{N}$, is given by

$$\overline{N} = \frac{\lambda}{\mu}. \tag{A.91}$$

There is no waiting in the queue.

A.6.7 *M/G/*1 Queueing System

The following are the assumptions for an $M/G/1$ queue:

- Poisson arrival process (exponential interarrival times)
- General service time, Y (packet service time which depends on the distribution of the packet length)
- One server
- Infinite storage spaces (infinite buffer)

The mean service time, $\overline{Y}$, is given by

$$\overline{Y} = \frac{1}{\mu}. \tag{A.92}$$

Let $\overline{Y^2}$ denote the second moment of service time. From queueing theory, the mean queueing delay, $\overline{W}$, is given by

$$\overline{W} = \frac{\lambda \overline{Y^2}}{2(1-\rho)} = \frac{\rho}{2(1-\rho)} \frac{\overline{Y^2}}{\overline{Y}}, \qquad \text{where } \rho = \frac{\lambda}{\mu} = \lambda \overline{Y}. \tag{A.93}$$

The total time spent in the system, in queue and in service, is

$$\overline{T} = \overline{Y} + \frac{\lambda \overline{Y^2}}{2(1-\rho)}. \tag{A.94}$$

Applying Little's law, the mean number of customers in the queue, $\overline{N}_q$, and the mean number of customers in the system, $\overline{N}$, are given by

$$\overline{N}_q = \frac{\lambda^2 \overline{Y^2}}{2(1-\rho)} = \lambda \overline{W}, \tag{A.95}$$

$$\overline{N} = \rho + \frac{\lambda^2 \overline{Y^2}}{2(1-\rho)}. \tag{A.96}$$

When the service times are *exponentially distributed* ($M/M/1$), we have $\overline{Y^2} = 2/\mu^2 = 2(\overline{Y})^2$, and the mean queueing delay reduces to

$$\overline{W} = \frac{1}{\mu}\frac{\rho}{(1-\rho)} = \frac{\rho}{1-\rho}\overline{Y} \qquad (M/M/1). \tag{A.97}$$

When the service times are *deterministic or identical for all customers* ($M/D/1$), we have $\overline{Y^2} = 1/\mu^2 = (\overline{Y})^2$, and the mean queueing delay reduces to

$$\overline{W} = \frac{1}{\mu}\frac{\rho}{2(1-\rho)} = \frac{\rho}{2(1-\rho)}\overline{Y} \qquad (M/D/1) \tag{A.98}$$

A.6.8 *M/G/*1 with Vacation

The following are the assumptions for an $M/G/1$ queue with vacation:

- Poisson arrival process (exponential interarrival times)
- General service time, Y (packet service time which depends on the distribution of the packet length)
- One server
- Infinite storage spaces (infinite buffer)
- Server goes on vacation with duration V

From queueing theory, the mean queueing delay, $\overline{W}$, is

$$\begin{aligned}\overline{W} &= \frac{\lambda\overline{Y^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}}, \qquad \text{where } \rho = \frac{\lambda}{\mu} = \lambda\overline{Y}\\ &= \frac{\rho}{2(1-\rho)}\left(\frac{\overline{Y^2}}{\overline{Y}}\right) + \frac{\overline{V^2}}{2\overline{V}}.\end{aligned} \tag{A.99}$$

The total time spent in the system, in queue and in service, is

$$\overline{T} = \overline{Y} + \overline{W}. \tag{A.100}$$

A.6.9 *M/G/*1 with Vacations and with *M* Users

The following are the assumptions for an $M/G/1$ queue with vacation and M users:

- Poisson arrival process (exponential interarrival times)
- General service time, Y (packet service time which depends on the distribution of the packet length)
- One server

- Infinite storage spaces (infinite buffer)
- Server goes on vacation with duration V
- M users

From queueing theory, the mean queueing delay, $\overline{W}$, is

$$\begin{aligned}\overline{W} &= \frac{\lambda\overline{Y^2}}{2(1-\rho)} + \frac{(M-\rho)\overline{V}}{2(1-\rho)} + \frac{\mathrm{Var}[V]}{2\overline{V}}, \qquad \text{where } \rho = \lambda\overline{Y} \\ &= \frac{\rho}{2(1-\rho)}\frac{\overline{Y^2}}{\overline{Y}} + \frac{(M-\rho)\overline{V}}{2(1-\rho)} + \frac{\mathrm{Var}[V]}{2\overline{V}}. \end{aligned} \tag{A.101}$$

The total time spent in the system, in queue and in service, is

$$\overline{T} = \overline{Y} + \overline{W}. \tag{A.102}$$

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