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Leaderless Consensus Control of Dynamical Agents Under Directed Interaction Topology

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Abstract— **This paper investigates the leaderless consensus control problem for a group of agents under fixed or switching directed interaction topology, where each agent is modeled as a generic linear system rather than the single- or double-integrator dynamics. For the case with fixed topology, it is shown that consensus can be reached by assigning an appropriate feedback matrix if the interaction topology has a directed spanning tree; while for the switching case, by imposing the balanced condition on the interaction topology, sufficient conditions are provided for the agents to reach consensus under arbitrary switching signal. Furthermore, the consensus equilibria are specified for both cases.**

I. Introduction

During the recent years, information consensus of autonomous agents has spurred increasing attention and study due to its wide application in various disciplines. Much work has been done, see e.g. [4], [5], [6], [9], [11], [12]. For more details and developments, see the survey paper [13] and references therein.

Most of the existing literature focuses on consensus algorithms for agents modeled by single- or double-integrator dynamics. Recently, the consensus problems for agents modeled as generic linear systems is investigated in [15], which aims at designing an appropriate feedback gain matrix guaranteeing the final consensus. The model is further investigated in [8] under fixed and switching interaction topology in a leader-following framework, in which different methods are used for the convergence analysis. Furthermore, a Riccati-inequality-based method is proposed in [8] to find the feedback gain matrix which thus reduces the computation load largely compared to that in [15]. Note here that the widely employed product properties of row-stochastic as well as the related nonnegative matrix theory in performing the convergence analysis for agents modeled by integrators under dynamic topology (see, e.g. [12], [11]) does not work in the context of generic linear system dynamics. In [8], the convergence analysis for the case with fixed and switching topology are performed by employing Lyapunov method together with graph theory, in which the interaction topology is much relaxed as opposed to that in [15]. The relaxation on the interaction in [8] relies heavily on the leader-following architecture, and the convergence analysis therein cannot be extended directly to deal with the leaderless consensus control. Moreover, it is worth pointing out that both the work in [8] and [15] are based on undirected interaction topology, and the convergence analysis employed therein does not work for the directed case, which, in general, is more challenging than that of the undirected case.

With the above motivation, we investigate in this paper the leaderless consensus control for multiple agents under directed interaction topology. For the case with fixed topology, it is shown that consensus control can be realized by assigning appropriate feedback matrix if the interaction topology has a spanning tree, in which the idea employed in performing the convergence analysis is motivated by the treatment of synchronization in complex dynamical networks [10], [14], [16]. It is further shown that the leader-following framework considered in [8] is a special case of the model considered in this paper. For the case with directed interaction topology, the consensus problem is investigated under arbitrarily switching balanced interaction topology. The main contribution of the part concerning the switching case lies in the extension of the existing literature to the case with dynamically changing weighting factors as opposed to the finite case in [8], [9], and [15].

II. Preliminary

The following notations will be used throughout the paper. Denote by $M > 0$ ($M < 0$) that M is symmetric positive (negative) definite. If all the eigenvalues of M are real, then denote by $\lambda_{max}(M)$ and $\lambda_{min}(M)$ the maximum and minimum eigenvalues of M, respectively. Denote by $diag\{A_1, A_2, \ldots, A_n\}$ the block diagonal matrix with its ith main diagonal matrix being a square matrix A_i, $i = 1, \ldots, n$. By abuse of notation, for any $m \times 1$ vector α, denote by $\mathrm{diag}(\alpha) \in \mathbb{R}^{m\times m}$ the diagonal matrix with the ith ($i = 1, 2, \ldots, m$) diagonal element being the ith element of α. A matrix is called nonnegative (positive) whenever all its elements are nonnegative (positive).

Let G be a weighted digraph of order N, and $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ be the associated adjacency matrix in which $a_{ij} > 0$ whenever there is a directed edge from node j to node i.

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Moreover, we assume $a_{ii}=0$, $i=1,\ldots,N$. Denote by $L=[\ell_{ij}]$ the Laplacian matrix associated with G, where $\ell_{ij}=-a_{ij}$, $i\neq j$, and $l_{ii}=\sum_{k=1,k\neq i}^{N}a_{ik}$. Apparently, any square matrix with non-positive off-diagonal elements and zero row sum, shall be called graph Laplacian for convenience, can be deemed as a Laplacian matrix of a weighted digraph. Digraph G is called balanced if and only if $\mathbf{1}^{\mathrm{T}}L=0$ [9].

A digraph is called strongly connected if any two distinct nodes of the graph can be connected by a directed path; while it is called weakly connected if replacing all of its directed edges with undirected edges produces a connected graph. A directed graph has a spanning tree if there exists at least one node, called the root node, having a directed path to all other nodes. Note that if G is strongly connected, then there exists a positive column $\beta=[\beta_1,\beta_2,\ldots,\beta_N]^{\mathrm{T}}\in\mathbb{R}^N$ satisfying $\beta^{\mathrm{T}}L=0$ and $\beta^{\mathrm{T}}\mathbf{1}_N=1$ [7]. With this notation, the following result regarding the algebraic connectivity of strongly connected digraph is proposed in [16].

Lemma 1: (Lemma 7, [16]) If L is the Laplacian matrix of a strongly connected graph, then the algebraic connectivity

$$a(L)=\min_{x^{\mathrm{T}}\beta=0,x\neq 0}\frac{x^{\mathrm{T}}\hat{L}x}{x^{\mathrm{T}}diag(\beta)x}>0, \tag{1}$$

where $\hat{L}=(diag(\beta)L+L^{\mathrm{T}}diag(\beta))/2$ and β is as defined above.

Lemma 2: Let $x\in\mathbb{R}^{nN\times nN}$ be any column vector satisfying

$$(\beta^{\mathrm{T}}\otimes e_i^{\mathrm{T}})x=0, \tag{2}$$

where $e_i\in\mathbb{R}^n$ stands for the column vector in which only the ith entry is 1 and all the other entries are 0, and $\otimes$ denotes the Kronecker product. Then for any symmetric positive-semidefinite matrix $B\in\mathbb{R}^{n\times n}$, we have

$$x^{\mathrm{T}}(\hat{L}\otimes B)x\geq a(L)x^{\mathrm{T}}(\Xi\otimes B)x, \tag{3}$$

where $\Xi=diag(\beta)$.

Proof: See the appendix. ■

When L is the Laplacian matrix of an undirected and connected graph, it is easy to get that $a(L)=\lambda_2(L)$, where $\lambda_2(L)$ is called the algebraic connectivity of G [3] and

$$\lambda_2(L)=\min_{x\neq 0,\mathbf{1}^{\mathrm{T}}x=0}\frac{x^{\mathrm{T}}Lx}{x^{\mathrm{T}}x}.$$

With this notation, as a corollary of Lemma 2, we have:

Corollary 1: Assume L is a Laplacian matrix of a connected undirected graph. Let $x=[x_1^{\mathrm{T}},x_2^{\mathrm{T}},\ldots,x_N^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{nN}$, where $x_i\in\mathbb{R}^n$, $i=1,\ldots,N$, be any column vector satisfying $\sum_{i=1}^{N}x_i=0$. Then for any symmetric positive-semidefinite matrix $B\in\mathbb{R}^{n\times n}$, we have

$$x^{\mathrm{T}}(L\otimes B)x\geq\lambda_2(L)x^{\mathrm{T}}(I_N\otimes B)x.$$

By using similar proof techniques as that for Lemma 2, one can easily derive the following result:

Lemma 3: Let $P\in R^{N\times N}$ be any symmetric matrix and $B\in\mathbb{R}^{n\times n}$ be any symmetric positive-semidefinite matrix. Then, for any column vector $x\in\mathbb{R}^{nN\times nN}$, we have $x^{\mathrm{T}}(P\otimes B)x\geq\lambda_{min}(P)x^{\mathrm{T}}(I_N\otimes B)x$.

Consider a multi-agent system consisting of N agents. Each agent is regarded as a node in a digraph G, and the dynamics of which is modeled by

$$\dot{x}_i=Ax_i+Bu_i, \tag{4}$$

where $x_i\in\mathbb{R}^n$ is the ith agent's state, and $u_i\in\mathbb{R}^m$ is the ith agent's input which can only use local information from its neighboring agents. The matrix B is of full column rank.

Throughout the paper, the matrix pair (A,B) satisfies the following assumption:

Assumption 1: The pair (A,B) is stabilizable.

III. Consensus under fixed interaction topology

The following control law will be used for agent i:

$$u_i=K\sum_{j\in N_i}a_{ij}(x_j-x_i),\ \ i=1,2,\ldots,N, \tag{5}$$

where $a_{ij}\geq 0$, $K\in\mathbb{R}^{m\times n}$ is a feedback matrix to be designed.

Without loss of generality, we shall assume that the interaction topology G has q $(1\leq q\leq N)$ strongly connected components, say $G_1,\ldots,G_q$, with, respectively, the node sets $\mathscr{V}(G_\ell)=\{\sum_{j=0}^{\ell-1}n_j+1,\ldots,\sum_{j=0}^{\ell}n_j\}$, $1\leq\ell\leq q$, where $n_0=0$; and the Laplacian matrix L associated with G takes in the following Frobenius normal form [2]:

$$\begin{bmatrix} L_{11} & 0 & \cdots & 0\\ L_{21} & L_{22} & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots\\ L_{q1} & L_{q2} & \cdots & L_{qq}\end{bmatrix},$$

where $L_{ii}\in\mathbb{R}^{n_i\times n_i}$, $i=1,\ldots,q$.

Note that if G has a spanning tree, for each $p=2,\ldots,q$, there must exist $1\leq k<p$ such that $L_{pk}\neq 0$ [14]. Then, following exactly the same proof as that in Lemma 4 in [10], we can get the following result:

Lemma 4: If G has a spanning tree, then for any $i=2,\ldots,q$, there exists a positive column vector β_i such that the matrix $\mathrm{diag}\{\beta_i\}L_{ii}+L_{ii}^{\mathrm{T}}\mathrm{diag}\{\beta_i\}>0$. In fact, $\beta_i\in\mathbb{R}^{n_i}$ can be chosen as the positive left eigenvector of $L_{ii}+\sum_{\ell=1}^{i-1}\mathscr{R}(L_{\ell i})$ associated with eigenvalue 0 satisfying $\beta_i^{\mathrm{T}}\mathbf{1}_{n_i}=1$, $i=2,\ldots,q$, where $\mathscr{R}(L_{\ell i})$ denotes the diagonal matrix with the kth $(k=1,\ldots,n_i)$ diagonal element being the ith row sum of $L_{\ell i}$.

In the sequel, denote $\Xi_i=diag\{\beta_i\}$, $i=2,\ldots,q$.

Theorem 1: Consider a group of agents (4) under fixed interaction topology G. If G has a spanning tree, then there exist a feedback matrix K such that all the agents reach consensus exponentially fast by using control law (5).

Proof: We first prove that the consensus of the group of agents with Laplacian matrix L_{11} can be realized.

Let $\beta_1=[\beta_1^1,\ldots,\beta_1^{n_1}]^{\mathrm{T}}\in\mathbb{R}^{n_1}$ be the positive vector satisfying $\beta_1^{\mathrm{T}}L=0$ and $\beta_1^{\mathrm{T}}\mathbf{1}_{n_1}=1$ and $\Xi_1=diag\{\beta_1\}$. Denote the state error between agent i and $\sum_{k=1}^{n_1}\beta_1^k x_k(t)$ as $e_i(t)=x_i(t)-\sum_{k=1}^{n_1}\beta_1^k x_k(t), i=1,\ldots,n_1$. In the sequel, t might be dropped for notational simplicity. Then, one obtain the following error dynamical system:

$$
\begin{aligned}
\dot{e}_i &= Ax_i + Bu_i - \sum_{k=1}^{n_1} \beta_1^k (Ax_k + Bu_k) \\
&= Ae_i + BK \sum_{j \in N_i} a_{ij}(e_j - e_i) - BK \sum_{k=1}^{n_1} \beta_1^k \sum_{j \in N_k} a_{kj}(x_j - x_k) \\
&= Ae_i + BK \sum_{j \in N_i} a_{ij}(e_j - e_i) + (\beta_1^{\mathrm{T}} L_{11} \otimes BK)x, \; i = 1, \ldots, n_1.
\end{aligned} \tag{6}
$$

Let $e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_{n_1}^{\mathrm{T}}]^{\mathrm{T}}$. Note that $\beta_1^{\mathrm{T}} L_{11} = 0$, then system (6) can be written in the following compact form:

$$
\dot{e} \;=\; [I_{n_1} \otimes A - L_{11} \otimes BK]e. \tag{7}
$$

Since $G(L_{11})$ is strongly connected, it follows from Lemma 1 that $a(L_{11}) > 0$. If (A, B) is stabilizable, then there exists a solution $P > 0$ to the following Riccati inequality

$$
PA + A^{\mathrm{T}}P - 2a(L)PBB^{\mathrm{T}}P + 2a(L)I_n < 0. \tag{8}
$$

Let P be a solution of Riccati inequality (8) and the feedback matrix K be $K = B^{\mathrm{T}}P$. Consider the following Lyapunov function candidate

$$
V_1(t) = e^{\mathrm{T}}(t)[\Xi_1 \otimes P]e(t).
$$

Differentiating $V_1(t)$ along the trajectories of (7) yields

$$
\begin{aligned}
&\dot{V}_1(t) \\
&= e^{\mathrm{T}}(t)[\Xi_1 \otimes (A^{\mathrm{T}}P + PA) - 2\hat{L}_{11} \otimes PBB^{\mathrm{T}}P]e(t) \qquad (9) \\
&\le e^{\mathrm{T}}(t)[\Xi_1 \otimes (A^{\mathrm{T}}P + PA) - 2a(L_{11})\Xi_1 \otimes PBB^{\mathrm{T}}P]e(t) \qquad (10) \\
&= \sum_{i=1}^{n_1} \beta_i e_i^{\mathrm{T}}(t)[A^{\mathrm{T}}P + PA - 2a(L_{11})PBB^{\mathrm{T}}P]e_i(t) \\
&\le -2a(L) \sum_{i=1}^{n_1} \beta_1^i e_i^{\mathrm{T}}(t) e_i(t) < 0, \qquad (11)
\end{aligned}
$$

for any $e(t) \neq 0$, where $\hat{L}_{11} = (\Xi_1 L_{11} + L_{11}^{\mathrm{T}} \Xi_1)/2$, (10) is obtained by observing that $\sum_{i=1}^{n_1} \beta_1^i e_i(t) = 0$ and applying Lemma 2 to the second term in (9).

Note that $V_1(t) = \sum_{i=1}^{n_1} \beta_1^i e_i^{\mathrm{T}} P e_i \le \lambda_{max}(P) \sum_{i=1}^{n_1} \beta_1^i e_i^{\mathrm{T}} e_i$, which, together with the inequality in (11), implies that $\dot{V}_1(t) \le -\frac{2a(L_{11})}{\lambda_{max}(P)} V(t)$ and thus $V_1(e(t)) \le V_1(e(0)) exp(-\frac{2a(L_{11})}{\lambda_{max}(P)} t)$. This means that $e(t)$ approaches 0 exponentially fast with a least speed of $\varepsilon = -\frac{a(L_{11})}{\lambda_{max}(P)}$. That is, the group of agents with Laplacian matrix L_{11} reach consensus exponentially fast. Let $x^*(t) = \sum_{k=1}^{n_1} \beta_1^k x_k(t)$. Clearly, $x^*(t)$ satisfies the following equation [14]:

$$
\dot{x}^*(t) = Ax^*(t) + \mathbf{O}(e^{-\varepsilon t}), \text{ for some } \varepsilon > 0,
$$

and $[x_1^{\mathrm{T}}, \ldots, x_{n_1}^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{1}_{n_1} \otimes x^* + \mathbf{O}(e^{-\varepsilon t})$.

Now we proceed to prove that consensus can be reached by the other groups of agents. Let $e_i = x_i - x^*$, $i = 1, 2, \ldots, N$, and $S_i = \sum_{j=1}^{i} n_j$, $i = 1, \ldots, q$. Obviously, $S_1 + \cdots + S_q = N$. We further let $y_k = [e_{S_{k-1}+1}^{\mathrm{T}}, \ldots, e_{S_k}^{\mathrm{T}}]^{\mathrm{T}}$, $k = 2, \ldots, q$ and $y = [y_2^{\mathrm{T}}, \ldots, y_q^{\mathrm{T}}]^{\mathrm{T}}$. Then, the network systems (4) with control law (5) can be written in the following compact form:

$$
\dot{y} = [I_{N-n_1} \otimes A - \bar{L} \otimes BK]y + \mathbf{O}(e^{-\varepsilon t}),
$$

where $\bar{L} = \begin{bmatrix} L_{22} & 0 & \cdots & 0 \\ L_{32} & L_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{q2} & L_{q3} & \cdots & L_{qq} \end{bmatrix}$

Consider the following Lyapunov function candidate

$$
V(t) = \sum_{i=2}^{q} \Delta_i y_i^{\mathrm{T}} (\Xi_i \otimes P) y_i,
$$

where P is a positive definite matrix to be chosen in order to satisfying: 1) $V(t)$ is a valid Lyapunov function; 2) the P chosen for $V_1(t)$ and $V(t)$ are the same and thus guarantee $K = B^{\mathrm{T}}P$ to be designed are consistent throughout the proof.

Note that

$$
\dot{y}_i = (I_{n_i} \otimes A)y_i - \sum_{j=2}^{i} (L_{ij} \otimes BK)y_j + \mathbf{O}(e^{-\varepsilon t}), \; i = 2, \ldots, q. \tag{12}
$$

Differentiating $V(t)$ along the trajectories of (12) gives

$$
\begin{aligned}
&\dot{V}(t) \\
&= \sum_{i=2}^{q} \Bigg\{ \Delta_i \Bigg[y_i^{\mathrm{T}}(I \otimes A^{\mathrm{T}}) - \sum_{j=2}^{i} y_j^{\mathrm{T}} (L_{ij}^{\mathrm{T}} \otimes K^{\mathrm{T}} B^{\mathrm{T}}) + \mathbf{O}(e^{-\varepsilon t}) \Bigg] (\Xi_i \otimes P) \\
&\times y_i + \Delta_i y_i^{\mathrm{T}} (\Xi_i \otimes P) \Bigg[(I \otimes A)y_i - \sum_{j=2}^{i} (L_{ij} \otimes BK)y_j + \mathbf{O}(e^{-\varepsilon t}) \Bigg] \Bigg\} \\
&= \sum_{i=2}^{q} \Delta_i y_i^{\mathrm{T}} [\Xi_i \otimes (A^{\mathrm{T}}P + PA)] y_i - 2 \sum_{i=2}^{q} \sum_{j=2}^{i} \Delta_i y_i^{\mathrm{T}} (\Xi_i L_{ij} \otimes PBB^{\mathrm{T}}P) y_j \\
&+ 2 \sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P) \mathbf{O}(e^{-\varepsilon t}) \\
&= y^{\mathrm{T}} \left[diag\{\Delta_2 \Xi_2, \ldots, \Delta_q \Xi_q\} \otimes (A^{\mathrm{T}}P + PA) - \Phi_q \otimes PBB^{\mathrm{T}}P \right] y \\
&+ 2 \sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P) \mathbf{O}(e^{-\varepsilon t}),
\end{aligned}
$$

where Φ_i, $i = 2, \ldots, q$, is defined as

$$
\Phi_i = \begin{bmatrix} \Delta_2(\Xi_2 L_{22} + L_{22}^{\mathrm{T}} \Xi_2) & \Delta_3 L_{32}^{\mathrm{T}} \Xi_3 & \cdots & \Delta_i L_{i2}^{\mathrm{T}} \Xi_i \\ \Delta_3 \Xi_3 L_{32} & \Delta_3(\Xi_3 L_{33} + L_{33}^{\mathrm{T}} \Xi_3) & \cdots & \Delta_i L_{i3}^{\mathrm{T}} \Xi_i \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_i \Xi_i L_{i2} & \Delta_i \Xi_i L_{i3} & \cdots & \Delta_i(\Xi_i L_{ii} + L_{ii}^{\mathrm{T}} \Xi_i) \end{bmatrix}. \tag{13}
$$

In order to find an appropriate matrix P such that $V(t)$ is a valid Lyapunov function, we first prove that matrix Φ_q is positive definite if the constant positives $\Delta_i, i = 2, \ldots, q$ are appropriately chosen. It follows directly from Lemma 4 that all the matrices $\Xi_i L_{ii} + L_{ii}^{\mathrm{T}} \Xi_i$, $i = 2, \ldots, q$, are positive definite. Thus, $\Phi_2 = \delta_2(\Xi_2 L_{22} + L_{22}^{\mathrm{T}} \Xi_2)$. Suppose that $\Phi_i > 0, 2 \le i < q - 1$, to complete the proof for the argument, it suffices to prove by induction that $\Phi_{i+1} > 0$. To this end, according to Schur complement lemma [1] and by noting the fact that $\Xi_{i+1} L_{i+1,i+1} + L_{i+1,i+1}^{\mathrm{T}} \Xi_{i+1} > 0$, it suffices to prove that

$$
\Phi_i - \Delta_{i+1} \Pi_{i+1}^{\mathrm{T}} (\Xi_{i+1} L_{i+1,i+1} + L_{i+1,i+1}^{\mathrm{T}} \Xi_{i+1})^{-1} \Pi_{i+1} > 0,
$$

where $\Pi_{i+1} = \left[\begin{array}{cccc} \Xi_{i+1}L_{i+1,2} & \Xi_{i+1}L_{i+1,3} & \cdots & \Xi_{i+1}L_{i+1,i} \end{array}\right]$, which can be guaranteed by choosing appropriate Δ_{i+1} so that Δ_{i+1} is sufficiently smaller than Δ_j for any $j \le i$.

Assume now that $\Delta_i, i = 2,\dots,q$ are properly chosen constants such that $\Phi_q > 0$. Recall that $\Xi_i = diag\{\beta_i\}$, where $\beta_i = \left[\beta_i^1, \beta_i^2, \dots, \beta_i^{n_i}\right]^{\mathrm{T}} \in \mathbb{R}^{n_i}$, $i = 2,\dots,q$. Let $\delta_{min} = \lambda_{min}(\Phi_q) \times \min\{(\Delta_i\beta_i^j)^{-1} : i = 2,\dots,q, 1 \le j \le n_i\}$ and $\delta = \min\{2a(L_{11}), \delta_{min}\}$. Since (A,B) is stabilizable, we can choose P as a solution to the following Riccati inequality

$$PA + A^{\mathrm{T}}P - \delta PBB^{\mathrm{T}}P + \delta I_n < 0. \tag{14}$$

It is clear from the discussion above that such P as chosen satisfying (14) can guarantee the final consensus of the group of agents with Laplacian matrix L_{11}. Let $\Delta = diag\{\Delta_2\Xi_2, \dots, \Delta_q\Xi_q\}$. With the above notations, we can get that

$$\begin{aligned}
&\dot{V}(t) \\
&\le y^{\mathrm{T}}\left[\Delta \otimes (A^{\mathrm{T}}P + PA) - \lambda_{min}(\Phi_q) \otimes PBB^{\mathrm{T}}P\right] y \\
&+ 2\sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P)\mathbf{O}(e^{-\varepsilon t}) \\
&= [(\Delta^{1/2} \otimes I_n)y]^{\mathrm{T}}\left[I \otimes (A^{\mathrm{T}}P + PA) - \lambda_{min}(\Phi_q)\Delta^{-1} \otimes PBB^{\mathrm{T}}P\right] \\
&\times [(\Delta^{1/2} \otimes I_n)y] + 2\sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P)\mathbf{O}(e^{-\varepsilon t}) \\
&\le [(\Delta^{1/2} \otimes I_n)y]^{\mathrm{T}}\left[I \otimes (A^{\mathrm{T}}P + PA) - \delta I \otimes PBB^{\mathrm{T}}P\right][(\Delta^{1/2} \otimes I_n)y] \\
&+ 2\sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P)\mathbf{O}(e^{-\varepsilon t}) \\
&= \sum_{i=1}^{N-n_1} \tilde{y}_i^{\mathrm{T}}(A^{\mathrm{T}}P + PA - \delta PBB^{\mathrm{T}}P)\tilde{y}_i + 2\sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P)\mathbf{O}(e^{-\varepsilon t}) \\
&\le \sum_{i=1}^{N-n_1} -\delta \tilde{y}_i^{\mathrm{T}}\tilde{y}_i + 2\sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P)\mathbf{O}(e^{-\varepsilon t}) \\
&= -\delta y^{\mathrm{T}}(\Delta \otimes I_n)y + 2\sum_{i=2}^{q} \Delta_i y_i^T (\Xi_i \otimes P)\mathbf{O}(e^{-\varepsilon t}),
\end{aligned}$$

where $\tilde{y} = (\Delta^{1/2} \otimes I_n)y = [\tilde{y}_1^{\mathrm{T}}, \tilde{y}_2^{\mathrm{T}}, \dots, \tilde{y}_{N-n_1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N-n_1)n}$, $\tilde{y}_i \in \mathbb{R}^n, i = 1,\dots,N-n_1$. This implies $V(t)$ converges to zero exponentially and therefore completing the proof. ■

Remark 1: Theorem 1 specifies that $x^*(t) = \sum_{k=1}^{n_1} \beta_1^k x_k(t)$, the weighted average value of all the states of the agents with Laplacian matrix L_{11}, is the final group consensus value for all the N agents. That is, the consensus value is decided by all the agents which can be connected to all the other nodes by directed paths.

Remark 2: Note that in the leader-following framework, as in [8], the leader, indexed as agent 0, is modeled by $\dot{x}_0 = Ax_0$, and the control law for agent i is designed as:

$$u_i = K\sum_{j\in N_i} a_{ij}(x_j - x_i) + Kd_i(x_0 - x_i), \ \ i = 1,2,\dots,N, \tag{15}$$

where $d_i > 0$ whenever the agent i can receive information from the leader and $d_i = 0$ otherwise.

Let $a_{i0} = d_i$ and $a_{0i} = 0$, $i = 1,\dots,N$, then the control laws for all the agents as well as the leader take in the following form

$$u_i = K\sum_{j=0, j\ne i}^{N} a_{ij}(x_j - x_i), \ \ i = 0,1,\dots,N. \tag{16}$$

This implies that the leader-following consensus problem is transformed to the consensus problem of $N+1$ agents. Let $\bar{G}$ denote the digraph consisting of G, leader 0 and the directed edges from leader 0 to the agents in G which have access to the leader. Therefore, as a corollary of Theorem 1, we directly obtain that all the agents can follow the leader exponentially fast by assigning an appropriate K to control law (15) if $\bar{G}$ has a spanning tree.

When G is undirected, this is just the main result considered in [8] for the fixed topology. Thus the result in Theorem 1 extends the result in [8] regarding the case with fixed topology to a very general setting.

Example 1: Let $A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ and $B = \left[\begin{array}{ccc} 0 & 1 & 0 \end{array}\right]^{\mathrm{T}}$. Obviously (A,B) is stabilizable. Assume the interaction topology is as shown in Figure 1. By some manipulation, we can get $\delta = 1$. According to (14), P can be chosen as $P = \begin{bmatrix} 1.3322 & -1.5792 & -0.2676 \\ -1.5792 & 5.0263 & 0.8785 \\ -0.2676 & 0.8785 & 0.4975 \end{bmatrix}$ and thus $K = B^{\mathrm{T}}P = \left[\begin{array}{ccc} -1.5792 & 5.0263 & 0.8785 \end{array}\right]$.

It can be seen from Figure 2, which specifies the trajectories of the error states $e_i = x_i - \frac{1}{3}(x_1 + x_2 + x_3)$, $i = 1,\dots,6$, that the states of all the agents converge to the weighted average of the states of the three "leader" agents, i.e. agents 1, 2, and 3.

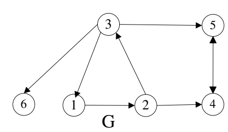

Fig. 1. Example of a interaction topology having a directed spanning tree.

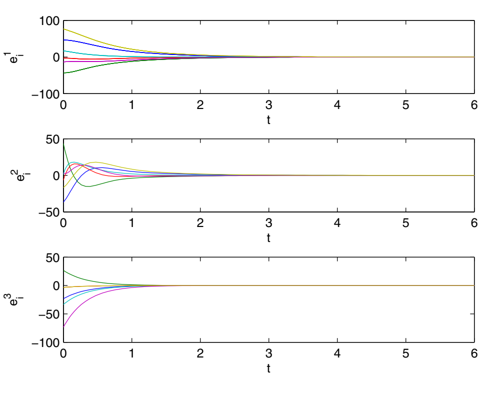

Fig. 2. Error states e_i, $i = 1,2,\dots,6$.

IV. CONSENSUS UNDER SWITCHING TOPOLOGY

Let $\sigma\colon [0,+\infty) \to \Gamma_N$ be a switching signal that determines the weighted interaction topology, where Γ_N is the index set of all the possible weighted interaction topologies. Different from most of the existing literature (see, e.g. [9], [8], [15]) in which Γ_N is a finite set and thus weighting factors are constant for each interaction topology, Γ_N here is an infinite set as the weighting factors are allowed to be dynamically changing.

The control law for agent i is designed as:

$$u_i = K \sum_{j \in N_i(t)} a_{ij}(t)(x_j - x_i), \ i = 1, 2, \ldots, N,$$

where $N_i(t)$ is the set of neighbors of agent i at time t. We assume that all the nonzero and hence positive weighting factors are both uniformly lower and upper bounded, i.e., $a_{ij}(t) \in [\underline{\alpha}, \bar{\alpha}]$, where $0 < \underline{\alpha} < \bar{\alpha}$, if $j \in N_i(t)$.

Denote the state error between agent i and $\frac{1}{N}\sum_{k=1}^{N} x_k(t)$ as $e_i(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N} x_k(t), i = 1, 2, \ldots, N$. Let $e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_N^{\mathrm{T}}]^{\mathrm{T}}$. Then, by observing that $\mathbf{1}_N^{\mathrm{T}} L_\sigma = 0$, where L_σ is the Laplacian matrix associated with G_σ, then similar to the deriving of (7), one can obtain the following compact form of the error dynamical system:

$$\dot{e} \ = \ [I_N \otimes A - L_\sigma \otimes BK]e. \tag{17}$$

Lemma 5: Let Υ be the set of all possible Laplacian matrices with which the associated digraph are balanced and weakly connected, i.e.

$$\Upsilon = \{L = [\ell_{ij}] | L \text{ is a graph Laplacian };-\ell_{ij} \in \{0\} \cup [\underline{\alpha}, \bar{\alpha}], \\ i, j = 1, \ldots, N, i \neq j; \mathbf{1}^{\mathrm{T}} L = 0; \ G(L) \text{ is weakly connected}\},$$

then Υ is a compact set in $\mathbb{R}^{N^2}$.

Proof: See the appendix. ■

Theorem 2: Consider a group of agents modeled by a directed graph with switching topology G_σ which is kept weakly connected and balanced. Then, for any arbitrary switching signal $\sigma(\cdot)$, all the agents reach average consensus exponentially fast by using control law (5).

Proof: According to Theorem 7 in [9], $\frac{L_\sigma + L_\sigma^T}{2}$ is a Laplacian matrix of $\hat{G}_\sigma$, the undirected mirror graph of G_σ. Since G_σ is weakly connected, $\hat{G}_\sigma$ is connected. Thus $\lambda_2(\frac{L_\sigma + L_\sigma^T}{2}) > 0$ for any $\sigma(\cdot)$. Define the following multivariate function:

$$g: \ \Upsilon \to \mathbb{R}^1; \ g(L) := \lambda_2(\frac{L + L^T}{2}), \forall L \in \Upsilon.$$

It is clear from the well-known fact that the eigenvalues of any matrix are continuous functions of the elements of the matrix that $g(\cdot)$ is a continuous function. This, together with Lemma 5 that the set Υ is compact, implies that there exists a positive number $\bar{\lambda}_2$ such that $\bar{\lambda}_2 = \min\{\lambda_2(\frac{L+L^T}{2}) | L \in \Upsilon\} > 0$. It then follows from the fact that (A, B) is stabilizable that there exists a solution $P > 0$ to the following Riccati inequality

$$PA + A^{\mathrm{T}}P - 2\bar{\lambda}_2 PBB^{\mathrm{T}}P + 2\bar{\lambda}_2 I_n < 0. \tag{18}$$

Consider the following Lyapunov function candidate

$$V(t) = e^{\mathrm{T}}(t)[I_N \otimes P]e(t).$$

Differentiating $V(t)$ along the trajectories of (17) yields

$$\begin{aligned}
&\dot{V}(t) \\
&= e^{\mathrm{T}}(t)[I_N \otimes (A^{\mathrm{T}}P + PA) - (L_\sigma + L_\sigma^T) \otimes PBB^{\mathrm{T}}P]e(t) \quad (19) \\
&\le e^{\mathrm{T}}(t)[I_N \otimes (A^{\mathrm{T}}P + PA) - 2\bar{\lambda}_2 I_N \otimes PBB^{\mathrm{T}}P]e(t) \quad (20) \\
&= \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)[A^{\mathrm{T}}P + PA - 2\bar{\lambda}_2 PBB^{\mathrm{T}}P]e_i(t) \\
&\le -2\bar{\lambda}_2 \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) < 0,
\end{aligned}$$

for any $e(t) \neq 0$, where (20) is obtained by observing the fact that $\sum_{i=1}^{N} e_i(t) = 0$ and then applying Corollary 1 to the second term in (19).

Note that $V(t) \le \lambda_{max}(P)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)$, which, together with the above inequality, implies that $\dot{V}(t) \le -\frac{2\bar{\lambda}_2}{\lambda_{max}(P)}V(t)$ and thus $V(e(t)) \le V(e(0))exp(-\frac{2\bar{\lambda}_2}{\lambda_{max}(P)})$. This means that the error state $e(t)$ approaches 0 exponentially fast with a least speed of $\gamma = -\frac{\bar{\lambda}_2}{\lambda_{max}(P)}$ ■

V. Conclusion

In this paper, we have studied the leaderless consensus control for a group of agents under fixed or switching directed interaction topology. In the framework of fixed topology, we have shown that if the interaction topology has a spanning tree, final consensus can be reached by assigning an appropriate feedback matrix which extend some of the existing results to a very general case; while for the case with switching topology, we have investigated the consensus control problem in a widely used balanced graph context, but being more general in that the weighting factors are allowed to change dynamically to model more practical dynamics.

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APPENDIX

Proof of Lemma 2: First, we prove the following inequality

$$x^{\mathrm{T}}(\hat{L}\otimes I_n)x \geq a(L)x^{\mathrm{T}}(\Xi\otimes I_n)x \tag{21}$$

holds for any x satisfying (2).

For the proof's convenience, denote x as $x=[x_1^{\mathrm{T}},x_2^{\mathrm{T}},\ldots,x_N^{\mathrm{T}}]^{\mathrm{T}}$, where each $x_i=[x_i^1,\ldots,x_i^n]^{\mathrm{T}}$, $i=1,2,\ldots,N$ is an $n\times 1$ column vector. With this notation, the equation in (2) holds if and only if $\sum_{i=1}^N \beta_i x_i = 0$. Then,

$$x^{\mathrm{T}}(\hat{L}\otimes I_n)x \;=\; \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{N}\sum_{j=1}^{N}\hat{\ell}_{ij}(x_i^k-x_j^k)^2, \tag{22}$$

where $\hat{L}=[\hat{\ell}_{ij}]_{N\times N}$.

Let $\tilde{x}^k=[x_1^k,x_2^k(t),\ldots,x_N^k]^{\mathrm{T}}$, $k=1,2,\ldots,n$. It then follows from (2) that $\beta^{\mathrm{T}}\tilde{x}^k=0$, $k=1,2,\ldots,n$, which together with (22) and the fact that

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\hat{\ell}_{ij}(x_i^k-x_j^k)^2=(\tilde{x}^k)^{\mathrm{T}}\hat{L}x^k,$$

yields $x^{\mathrm{T}}(\hat{L}\otimes I_n)x=\sum_{k=1}^n(\tilde{x}^k)^{\mathrm{T}}\hat{L}\tilde{x}^k$.

Note that $\beta^{\mathrm{T}}\tilde{x}^k=0, k=1,\ldots,n,$ it then follows from (1) that

$$x^{\mathrm{T}}(\hat{L}\otimes I_n)x \;\geq\; \sum_{k=1}^{n}a(L)(\tilde{x}^k)^{\mathrm{T}}\Xi\tilde{x}^k=a(L)x^{\mathrm{T}}(\Xi\otimes I_n)x. \tag{23}$$

Now we proceed to prove the theorem. Since matrix B is symmetric positive semidefinite, there exists a matrix $E\in\mathbb{R}^{m\times n}$ such that $B=E^{\mathrm{T}}E$. Thus,

$$\begin{aligned} x^{\mathrm{T}}(\hat{L}\otimes B)x &= x^{\mathrm{T}}[\hat{L}\otimes(E^{\mathrm{T}}I_mE)]x \\ &= [(I_N\otimes E)x]^{\mathrm{T}}(\hat{L}\otimes I_m)[(I_N\otimes E)x] \end{aligned}$$

Let $y=(I_N\otimes E)x$ and ς_i be the m by 1 column vector with the ith entry being 1 and 0 elsewhere. By observing that

$$(\beta^{\mathrm{T}}\otimes\varsigma_i^{\mathrm{T}})y=(\beta^{\mathrm{T}}\otimes\varsigma_i^{\mathrm{T}}E)x=\varsigma_i^{\mathrm{T}}E\sum_{j=1}^{N}\beta_j x_j=0,$$

and the inequality in (21), we have

$$y^{\mathrm{T}}(\hat{L}\otimes I_m)y\geq a(L)y^{\mathrm{T}}(\Xi\otimes I_m)y=a(L)x^{\mathrm{T}}(\Xi\otimes B)x,$$

thereby completing the proof. ∎

Proof of Lemma 5: Note that the set of all $N\times N$ matrices can be viewed as the metric space $\mathbb{R}^{N^2}$. Each $L=[\ell_{ij}]$ in Υ can be viewed as a vector $[\ell_{1,1},\ldots,\ell_{1,N},\ell_{2,1},\ldots,\ell_{2,N},\ell_{N,1},\ldots,\ell_{N,N}]$ in $\mathbb{R}^{N^2}$. To prove that Υ is compact in Euclidean Space $\mathbb{R}^{N^2}$, it is equivalent to prove that Υ is a closed and bounded set. Let

$$\Upsilon_1=\left\{[\ell_{ij}]\middle| -\ell_{ij}\in\{0\}\cup[\underline{\alpha},\bar{\alpha}],\ i,j=1,\ldots,N,\ i\neq j;\right.$$
$$\left.\ell_{ii}\in[0,N\bar{\alpha}],\ i=1,\ldots,N\right\},\ \Upsilon_2=\left\{[\ell_{ij}]\middle|\sum_{j=1}^{N}\ell_{ij}=0,\right.$$
$$\left. i=1,\ldots,N\right\}\text{and}\Upsilon_3=\left\{[\ell_{ij}]\middle|\sum_{i=1}^{N}\ell_{ij}=0,\ j=1,\ldots,N\right\}.$$

It can be easily derived from the definition of Laplacian matrix that $\Upsilon_1\cap\Upsilon_2\cap\Upsilon_3$ is the set consisting of all graph Laplacian L satisfying $\mathbf{1}^{\mathrm{T}}L=0$ and the off-diagonal elements of L are chosen from the set $0\cup[\underline{\alpha},\bar{\alpha}]$. We first prove that $\Upsilon_1\cap\Upsilon_2\cap\Upsilon_3$ is closed and bounded in $\mathbb{R}^{N^2}$. In fact, Υ_1 is closed and bounded; and the sets Υ_2,Υ_3 are closed but unbounded. The former argument holds since it is the product space of N^2 closed and bounded sets in $\mathbb{R}^1$. For the latter argument,we only prove in the following that the set Υ_2 is closed, similar proof can be derived for that of Υ_3. let

$$S_i=\left\{[\ell_{i,1},\ldots,\ell_{i,N}]\middle|[\ell_{i,1},\ldots,\ell_{i,N}]\text{ is the vector taken from}\right.$$
$$\left.\text{the } i\text{-th row of } [\ell_{ij}]\in\Upsilon_2\right\},\ i=1,2,\ldots,N.$$

Then, $\Upsilon_2=S_1\times S_2\times\cdots\times S_N$. It is clear that Υ_2 is a closed set in $\mathbb{R}^{N^2}$ if each S_i, $i=1,2,\ldots,N$ is closed in $\mathbb{R}^N$. In order to prove that S_i is closed, we introduce the following continuous multivariate function:

$$f:\ \mathbb{R}^N\to\mathbb{R}^1,\ f(x):=\sum_{i=1}^{N}x_i,\ \forall x=[x_1,x_2,\ldots,x_N]\in\mathbb{R}^N.$$

Since f is continuous and $\{1\}$ is a closed set in $\mathbb{R}^1$, $f^{-1}(\{1\})$ is closed in $\mathbb{R}^N$, i.e. each set S_i, $i=1,\ldots,N$, is closed in $\mathbb{R}^N$. Therefore, $\Upsilon_1\cap\Upsilon_2\cap\Upsilon_3$ is closed and bounded.

On the other hand, denote by Ω the set of all $N\times N$ nonnegative matrices with zero diagonal elements, it is clear that there are only finite different types of matrices (two nonnegative matrices P_1 and P_2 are said of the same type [7], $P_1\sim P_2$, if they have zero elements and positive elements in the same places) in Ω and all the digraphs associated with the matrices having the same type in Ω are also with the same topological structure. Note that Ω can be partitioned by the equivalence relation $\sim$. Let $[A]:=\{B\in\Omega|B\sim A\}$ denote the equivalence to which A ($A\in\Omega$) belongs. Without loss of generality, denote by $[A_1],[A_2],\ldots,[A_m]$ all the equivalence classes with which the associated digraph are weakly connected and let

$$\Upsilon_4=\bigcup_{k=1}^{m}\Upsilon_4^k=\bigcup_{k=1}^{m}\left\{[\ell_{ij}]\middle| -\ell_{ij}\in\{0\}\cup[\underline{\alpha},\bar{\alpha}],\ \ell_{ii}\in[0,N\bar{\alpha}],\right.$$
$$\left. i,j=1,\ldots,N,\ i\neq j;\ \text{and} -L+diag\{\ell_{11},\ldots,\ell_{NN}\}\sim A_k\right\}$$

It is clear from the definition of Υ that $\Upsilon=\Upsilon_1\cap\Upsilon_2\cap\Upsilon_3\cap\Upsilon_4$.

Similar to the discussion above, it can be easily obtained that Υ_4 is also closed and bounded since each Υ_4^k is a product space of N^2 closed and bounded set in $\mathbb{R}^1$. This combined with the above analysis imply that Υ is compact in $\mathbb{R}^{N^2}$. ∎