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A sub-optimal consensus design for multi-agent systems based on hierarchical LQR[☆]

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ABSTRACT

This paper presents a new and systematic procedure to design sub-optimal hierarchical feedback controllers for the leader–follower consensus problem in homogeneous multi-agent systems. First, the given multi-agent system is treated as a two-layer hierarchical system where the agents perform local actions in the lower layer and interact with others in the upper layer to achieve some global goals. Then the consensus controller design is formulated as a hierarchical state feedback control problem. Employing LQR approach with an appropriately selected performance index, an optimal hierarchical state feedback controller is derived which includes two terms namely local and global terms. Consequently, by removing the local term, the remaining global term is proved to make the multi-agent system consensus which results in a sub-optimal hierarchical consensus controller. Finally, some numerical examples are introduced to illustrate the effectiveness of the proposed method.

1. Introduction

Hierarchical dynamical systems universally appear in the real world, such as world wide web, power grids, social networks (Fortunato, 2010; Girvan & Newman, 2002; Ravasz & Barabasi, 2003), and gene networks (Alon, 2007; Zhdanov, 2010). Since the hierarchical structure provides an ability to achieve a global target in the network by designing the local control strategies in the layers, the dimension of the network design problem can be significantly reduced. Furthermore, the network can be designed in a decentralized fashion which is usually required in many practical systems. Due to those advantages, hierarchical networks have been extensively investigated in a variety of research fields including control.

The research in Smith, Broucke, and Francis (2005) developed a hierarchical network for increasing the rate of consensus among vehicles but the structure in each layer is limited to be cyclic. Recently, Hamilton and Broucke (2010) introduced a concept of patterned linear systems but the class of structures is restrictive. Moreover, all information in the lower layer is sent to the upper layer. Since the hierarchical networks in the real world usually have dense interactions inside the sub-networks and sparse communication between them Fortunato (2010), the proposed frameworks in those studies failed to describe this characteristic.

This motivated (Hara, Shimizu, & Kim, 2009; Shimizu & Hara, 2008, 2009) to generalize the hierarchical cyclic pursuit scheme and emphasize the effect of low rank interlayer interactions by aggregation and distribution operations in the network to achieve the rapid consensus. Then continuing this line of research, Fujimori, Liu, Hara, and Tsubakino (2011) and Tsubakino and Hara (2012) presented a new class of low rank intergroup connection namely eigenconnection to analyze and design hierarchical networks such that only some specific eigenvalues of the local interconnection matrices are selectively affected.

However, many of existing results on hierarchical networked control so far are for the analysis and only a few works deal with systematic synthesis. One way to develop a systematic procedure for control system design is to set up an LQR optimal control problem. There have been several works which investigated the mechanism of preserving certain desirable hierarchical structures in the LQR framework, e.g., Fardad (2009) and Motee, Jadbabaie, and Bamieh (2008). The work (Tsubakino, Yoshioka, & Hara, 2013) generalized the results in Motee et al. (2008) and introduced more general classes of structured matrices that preserve their structures under the LQR setting. Borrelli and Keviczky (2008) investigated identical decoupled systems and proposed a way to design sub-optimal controllers based on the LQR approach. For the consensus

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in the network of integrators, Cao and Ren (2010) designed an optimal Laplacian matrix utilizing two cost functions in the LQR problem where the communications between agents were taken into account in the weighting matrices. On the other hand, Zhang, Lewis, and Das (2011) and Zhang, Lewis, and Qu (2012) introduced the LQR-based consensus designs for leader–follower networks in which only local LQR problems were solved and no global LQR problem was considered.

This paper proposes a new method to design sub-optimal consensus controllers for leader–follower homogeneous multi-agent systems based on a hierarchical LQR approach. The proposed consensus controller composes of two components. The first component is a feedback term required to capture a desired information structure in the multi-agent system. The second component deals with the interaction among the followers (agents) and the leader (reference system).

Accordingly, the main contributions in the proposed method are twofold. First, a sub-optimal hierarchical consensus controller is proposed which preserves a given, desirable constraint on the information structure of the multi-agent system. To do so, we employ a unified hierarchical LQR approach as in the design of an optimal hierarchical stabilizing controller in Nguyen and Hara (2014) for leaderless multi-agent systems and prove that the sub-optimal hierarchical consensus controller for leader–follower multi-agent systems can be achieved by removing the local term in that optimal hierarchical stabilizing controller. We also point out that the consensus does not occur in the leader–follower multi-agent systems if this local term is kept. Furthermore, the controller gains are computed just based on a local, low-dimension Riccati equation which obviously saves much computational cost. Second, the dynamics of agents is not restrictive to any special classes.

The organization of the paper is as follows. Section 2 introduces the model of two-layer hierarchical networks for leader–follower multi-agent systems and explains the goal of this paper. Then Section 3 proposes a systematic design procedure for sub-optimal hierarchical consensus state feedback controller. Next, we introduce in Section 4 a systematic design for hierarchical consensus output feedback controller where a decentralized state observer is proposed together with the state feedback controller obtained in Section 3. Finally, some concluded remarks are given in Section 5.

The following notations and symbols will be used in the paper. $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the set of real and complex $n \times 1$ vectors. Next, $\mathbf{1}_n$ represents the $n \times 1$ vector with all elements equal to 1, and I_n denotes the $n \times n$ identity matrix. Moreover, $\otimes$ stands for the Kronecker product. Finally, $\succ$ and $\succeq$ denote the positive definiteness and positive semi-definiteness of a matrix.

2. Problem formulation

Consider a multi-agent dynamical system including of N identical agents whose model is described by

$$\begin{aligned} \dot{x}_k &= Ax_k + Bu_k, \\ y_k &= Cx_k, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, 0 < m \le n, C \in \mathbb{R}^{p\times n}, x_k \in \mathbb{R}^n$ is the state vector of the kth agent, and $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$ are the vectors containing all the inputs and measured outputs of the kth agent, respectively.

The initial multi-agent system without controller therefore can be represented by

$$\begin{aligned} \dot{x} &= (I_N \otimes A)x + (I_N \otimes B)u, \\ y &= (I_N \otimes C)x, \end{aligned} \tag{2}$$

where $x = [x_1^T, \dots, x_N^T]^T, u = [u_1^T, \dots, u_N^T]^T, y = [y_1^T, \dots, y_N^T]^T$.

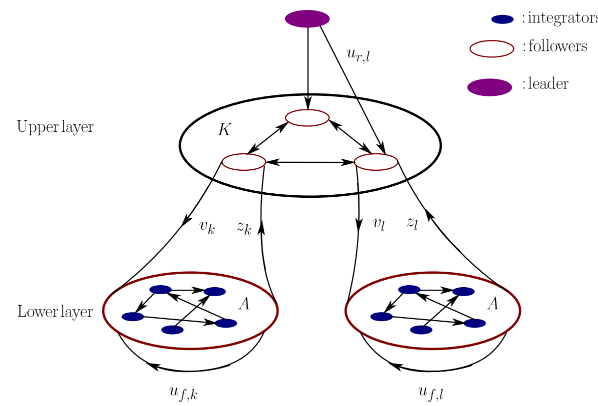

Fig. 1. Demonstration for information exchange in a leader–follower network from a theoretical viewpoint.

In a general setting, the agents are required to connect and exchange information with others to realize some specific goals of which each agent can have its own (local) objectives and the whole multi-agent system should achieve some cooperative purposes namely global objectives. In addition, to capture real situations, the information exchange among agents is required to be decentralized in the following senses: (i) Each agent sends out a unique aggregated signal to collaborate with other connected agents to attain the specified objectives. (ii) Simultaneously, each agent is able to receive the signals sent by other connected agents individually.

From a theoretical point of view, this multi-agent system can be cast as a two-layer hierarchical network where the agents perform to achieve their local objectives in the lower layer and exchange information to collaborate with others to attain global objectives in the upper layer. One of the global objectives for the multi-agent system is the tracking of all agents to a reference system namely a leader whose model is described as follows,

$$\begin{aligned} \dot{x}_r &= Ax_r, \\ y_r &= Cx_r, \end{aligned} \tag{3}$$

where $x_r \in \mathbb{R}^n, y_r \in \mathbb{R}^p$ are the vectors containing all the states and measured outputs of the reference system. Then the agents that track the leader are usually called the followers. We consider in this paper the scenario that the leader only sends information to the followers and does not receive information from them. Hence, there is no input in the leader model (3).

Consequently, the overall leader–follower network information exchange is depicted in Fig. 1.

Denote $\mathcal{G}$ the graph representing the information structure in our multi-agent system, in which each node in $\mathcal{G}$ represents an agent and each edge in $\mathcal{G}$ represents the interconnection between two agents. In this paper, we assume that the communications between agents are bidirectional and symmetric, i.e., $\mathcal{G}$ is undirected. Moreover, $\mathcal{G}$ is assumed to be connected. Then, the consensus of the multi-agent system to the reference system is defined as follows.

Definition 1. The multi-agent system with dynamics of agents described by (1) and the information exchange among agents represented by $\mathcal{G}$ is said to track the reference system (3) or equivalently the multi-agent system reaches consensus with respect to the reference system (3) if

$$\lim_{t\to\infty} \|y_k(t) - y_r(t)\| = 0 \quad \forall k = 1, \dots, N. \tag{4}$$

In order to achieve consensus, we aim at designing a hierarchical feedback controller for the multi-agent system model at initial state (2) such that the agents cooperatively track the leader (3) and

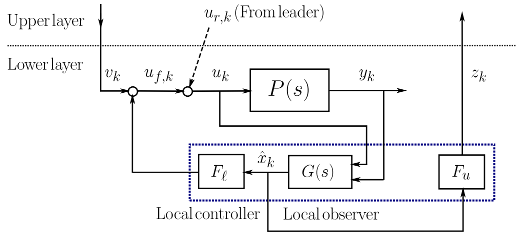

Fig. 2. Block diagram of the kth agent equipped with a hierarchical controller.

the information exchange among agents satisfies a constraint represented by graph $\mathcal{G}$. Therefore, the control input vector u to the network includes two components,

$$u = u_f + u_r, \tag{5}$$

where u_f is the input vector from the hierarchical feedback controller having the form $u_f = -F_c x$, F_c is a hierarchical, structured feedback gain; and u_r is the input vector from the leader. In this work, by saying that two agents are interconnected, we mean that they receive their relative output information. Hence, the input vector from the leader to the followers has the following form,

$$u_r = -(\Sigma \otimes L)(x - \mathbf{1}_N \otimes x_r), \tag{6}$$

where $\Sigma = \mathrm{diag}\{\sigma_k\}_{k=1,\dots,N}, \sigma_k > 0$ if the kth follower is connected to the leader, otherwise $\sigma_k = 0$; L is a common coupling matrix from the leader to the followers. On the other hand, to satisfy the constraint on the information exchange among agents, the hierarchical feedback controller gain F_c must belong to the following class

$$\mathcal{F}_K := \{F_c \in \mathbb{R}^{(mN)\times(Nn)} \mid F_c = I_N \otimes F_\ell + K \otimes F_u\}, \tag{7}$$

where $F_\ell, F_u \in \mathbb{R}^{m\times n}$, $I_N \otimes F_\ell$ and $K \otimes F_u$ represents the local and global feedback terms, respectively; $K \in \mathbb{R}^{N\times N}$ represents the information exchange among agents in the upper layer.

Fig. 2 clearly shows the structure of each agent ($P(s)$) in the lower layer and the information exchange between the layers as well as between the leader and each agent. The dash line represents the signal from the leader which may or may not exist since the leader is only connected to several agents. Meanwhile, the feedback controller is of output feedback type.

There are two different design scenarios depending on what information of agents can be measured. If all the states of agents are measurable, i.e., $C = I_N$ then a hierarchical state feedback controller needs to be designed. On the other hand, if only partial states of agents can be measured, i.e., $C \neq I_N$ then a hierarchical output feedback controller including a state observer should be designed. In the following, we first consider the problem of hierarchical state feedback controller design in Section 3 then the hierarchical output feedback controller design will be presented in Section 4.

3. Hierarchical state feedback consensus controller design

In this section, we assume that all states of agents and the reference are measurable. Consequently, the consensus in the network is achieved if the states of all agents in the network asymptotically converge to the state of the reference.

State feedback design problem: Given a multi-agent system with dynamics of agents in (1) of which (A, B) is stabilizable and the information exchange among agents represented by $\mathcal{G}$, design a hierarchical state feedback controller $u_f = -F_c x$ and matrices Σ, L such that all agents reach consensus to the reference system (3) in the sense of (4).

To capture an overview of our proposed controller design, we first introduce in Section 3.1 a controller design procedure followed by an illustrated example in Section 3.2. Then the details of controller derivation will be presented in Section 3.3.

3.1. Design procedure

This subsection proposes a systematic design procedure for hierarchical state feedback consensus controllers which consists of four steps.

- **Step 1 (Local LQR Design):**
  Select a matrix $Q_1 \in \mathbb{R}^{n\times n}$, $Q_1 \succeq 0$ and a matrix $R_1 \in \mathbb{R}^{m\times m}$, $R_1 \succ 0$. Subsequently, the Riccati equation (17) has a unique positive definite solution P_1.
- **Step 2 (Network Interactions Setting):**
  Choose a positive semidefinite matrix $K \in \mathbb{R}^{N\times N}$ as in condition C1 of Theorem 3 in Section 3.3.
- **Step 3 (Weighting Matrices Setting):**
  Choose $\Sigma \in \mathbb{R}^{N\times N}$, $\Sigma \succeq 0$ and $R_2 \in \mathbb{R}^{m\times m}$, $R_2 \succ 0$ as in conditions C2 and C3 of Theorem 3 in Section 3.3, respectively. Then compute $L = R_2 B^T P_1$.
- **Step 4 (Controller Calculation):**
  Let

  $$u_f = -[K \otimes (R_2 B^T P_1)]x,$$
  $$u_r = -(\Sigma \otimes L)(x - \mathbf{1}_N \otimes x_r),$$

  then the hierarchical state feedback consensus controller is computed by

  $$u = u_f + u_r.$$

3.2. Example 1 (Synchronization design of an oscillator network)

To illustrate the proposed consensus controller design, we consider a network of 3 identical linear oscillators with the model described as follows,

$$\dot{x}_k = Ax_k + Bu_k, \quad k = 1, 2, 3, \tag{8}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{9}$$

and the initial conditions of the oscillators are $(1, -1)$; $(0.5, -2)$; $(-3, 0.8)$, respectively. Then our goal is to design a state feedback consensus controller for the network such that all oscillators track the following reference oscillator

$$\dot{x}_r = Ax_r, \tag{10}$$

with initial conditions $(-0.5, 0.1)$. Moreover, only the first oscillator is connected to the reference oscillator, i.e., $\Sigma = \mathrm{diag}\{1, 0, 0\}$.

Fig. 3 shows the oscillating behaviors of the oscillators in the network as well as the reference oscillator. It can be seen that the states of oscillators are completely different and asynchronous.

Now applying the proposed design method, the weighting matrices are chosen as follows: $Q_1 = I_2$, $R_1 = 1$, we then obtain the solution P_1 of the local Riccati equation (17) to be

$$P_1 = \begin{bmatrix} 1.9123 & 0.4142 \\ 0.4142 & 1.3522 \end{bmatrix}.$$

Subsequently, employing Theorem 3, we utilize a hierarchical state feedback consensus controller (18) with

$$K = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{11}$$

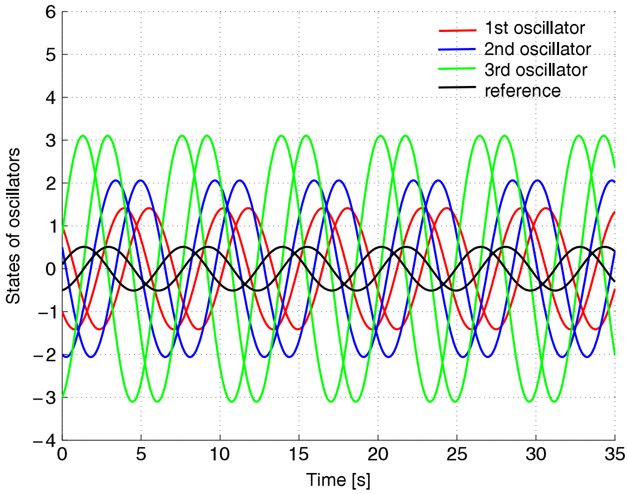

Fig. 3. States of oscillators in the network and the reference oscillator without interconnection.

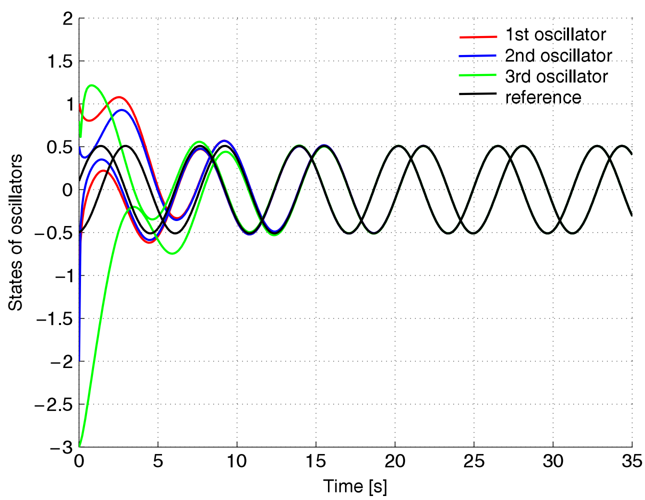

Fig. 4. Consensus of the given oscillator network to the reference oscillator.

Here, $\lambda_{\min}(\Sigma + K) = 0.1981$ so we choose

$$R_2 = 10, \qquad L = R_2 B^T P_1 = \begin{bmatrix} 4.1421 & 13.5219 \end{bmatrix}. \tag{12}$$

The simulation result for the oscillator network with the designed hierarchical consensus controller is displayed in Fig. 4. Obviously, the states of all oscillators in the network converge to the states of the reference oscillator, i.e., the consensus in the oscillator network to the reference is achieved.

3.3. Controller derivation

In this subsection, we first introduce how to design a hierarchical state feedback controller described in Fig. 2 then obtain a sub-optimal hierarchical consensus controller for the consensus state feedback design problem. Following the idea in Nguyen and Hara (2014), we consider the following performance index to specify both local and global objectives in the multi-agent system,

$$J = J_{x,\mathcal{L}} + J_{x,\mathcal{G}} + J_u = \int_0^\infty (x^T Q x + u^T R u) dt, \tag{13}$$

where

$$J_{x,\mathcal{L}} = \int_0^\infty x^T (I_N \otimes Q_1) x dt \quad \text{(local performance index)},$$

$$J_{x,\mathcal{G}} = \int_0^\infty x^T (K \otimes Q_2) x dt \quad \text{(global performance index)},$$

$$J_u = \int_0^\infty u^T R u dt \quad \text{(control input penalty)},$$

and $K \in \mathbb{R}^{N\times N}$ is a symmetric matrix; $Q_1 \in \mathbb{R}^{n\times n}$, $Q_2 \in \mathbb{R}^{n\times n}$, $Q_1 \succeq 0$, $Q_2 \succeq 0$; $R \in \mathbb{R}^{(Nm)\times(Nm)}$, $R \succ 0$. While the agents locally adjust themselves through the local performance index $J_{x,\mathcal{L}}$, they simultaneously cooperate with the others to achieve a global target through the global performance index $J_{x,\mathcal{G}}$. The matrix K here is associated with the communication structure in the network, i.e., $K_{ij} = K_{ji} \neq 0$ if the ith agent and the jth agent are connected and $K_{ij} = K_{ji} = 0$ if the ith agent and the jth agent are unconnected. Hence, the information structure of the network is taken into account in the global performance index $J_{x,\mathcal{G}}$. In this work, K is a Laplacian matrix since we have assumed that the agents exchange their relative information.

From (13), the first weighting matrix is selected by

$$Q = I_N \otimes Q_1 + K \otimes Q_2. \tag{14}$$

Consequently, as shown in Nguyen and Hara (2014), by selecting R as follows,

$$R^{-1} = I_N \otimes R_1 + K \otimes R_2. \tag{15}$$

$R_1, R_2 \in \mathbb{R}^{m\times m}$, $R_1 \succ 0$, $R_2 \succ 0$ and $Q_2 = P_1 B R_2 B^T P_1$, we obtain an optimal hierarchical state feedback controller minimizing J as $u = -Fx$ with F calculated by

$$F = I_N \otimes (R_1 B^T P_1) + K \otimes (R_2 B^T P_1), \tag{16}$$

where P_1 is the solution of the local Riccati equation

$$P_1 A + A^T P_1 - P_1 B R_1 B^T P_1 + Q_1 = 0. \tag{17}$$

It can be seen that the optimal controller (16) belongs to the set $\mathcal{F}_K$ in (7) with $F_\ell = R_1 B^T P_1$ and $F_u = R_2 B^T P_1$. *However, the hierarchical optimal controller (16) will not make our leader–follower multi-agent system consensus as point out in Remark 1.* Therefore, to achieve the consensus, we propose to utilize a sub-optimal hierarchical controller by removing the first term in the formula of the hierarchical optimal controller (16). Now, our sub-optimal hierarchical state feedback controller has the following form which still belongs to the class $\mathcal{F}_K$,

$$F_c = K \otimes (R_2 B^T P_1). \tag{18}$$

We will prove that this controller indeed makes the agents consensus to the reference.

Remark 1. The gap between the sub-optimal solution and the optimal solution of the minimization for the LQR cost (13) can be treated in a similar way as in Borrelli and Keviczky (2008), so we ignore here due to space limitation.

On the other hand, the effect of the first term in the optimal controller (16) to the consensus of the whole leader–follower multi-agent system can be demonstrated in Fig. 5 by an additional simulation with Example 1. It can be seen that all following oscillators seem to be consensus but not to the leading oscillator.

Denote

$$\zeta = x - \mathbf{1}_N \otimes x_r, \tag{19}$$

and using (6), we obtain

$$u = -F_c x - (\Sigma \otimes L)\zeta. \tag{20}$$

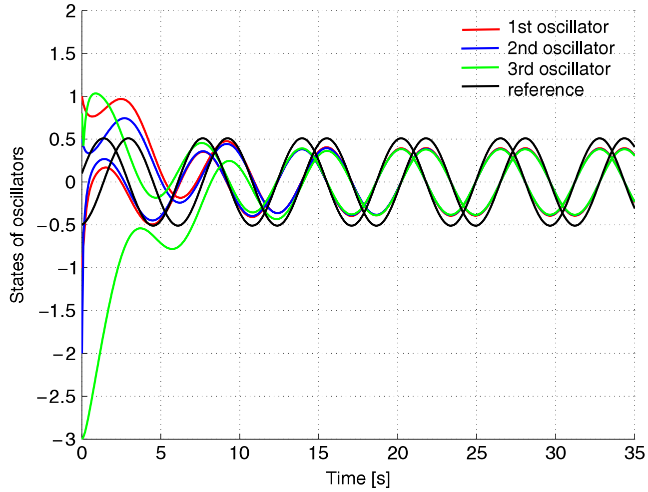

Fig. 5. No consensus of the given oscillator network to the reference oscillator if the first term in (16) is used.

Subsequently,

$$\begin{aligned}\dot{\zeta} &= (I_N \otimes A)\zeta + (I_N \otimes B)u,\\ &= (I_N \otimes A)\zeta - (I_N \otimes B)[K \otimes (R_2B^TP_1)]x\\ &\quad - (I_N \otimes B)(\Sigma \otimes L)\zeta,\\ &= [I_N \otimes A - \Sigma \otimes (BL) - K \otimes (BR_2B^TP_1)]\zeta\\ &\quad - [K \otimes (BR_2B^TP_1)](\mathbf{1}_N \otimes x_r),\\ &= [I_N \otimes A - \Sigma \otimes (BL) - K \otimes (BR_2B^TP_1)]\zeta, \end{aligned} \tag{21}$$

since $K\mathbf{1}_N = 0$ due to a fact that K is a Laplacian matrix. Let us denote

$$A_\zeta = I_N \otimes A - \Sigma \otimes (BL) - K \otimes (BR_2B^TP_1), \tag{22}$$

then (21) can be rewritten as follows,

$$\dot{\zeta} = A_\zeta \zeta. \tag{23}$$

Therefore, the consensus in the network is achieved if Σ, L and K are designed such that A_ζ is stable. The following lemma provides an important step toward it.

Lemma 2. *Let K be a Laplacian matrix associated with a connected, undirected graph with N vertexes and there exists at least one non-zero element on the diagonal of Σ then all eigenvalues of matrix $\Sigma+K$ have positive real parts.*

Proof. The proof of a similar result for directed graphs can be found in Li, Duan, Chen, and Huang (2010) and it is applicable for undirected graph, so we do not duplicate it here for brevity.

With this tool in hand, we are now ready to state the main result of this subsection.

Theorem 3. *Let $L = R_2B^TP_1$ then the consensus of the subsystems in the network to the reference is achieved if the sub-optimal controller (18) is employed and the following conditions are satisfied.*

C1. $K \in \mathbb{R}^{N\times N}$ *is a Laplacian matrix associated with a connected, undirected graph $\mathcal{G}$.*
C2. $\Sigma \neq 0^{N\times N}$.
C3. $R_2 = \mu R_1 + S$ *with $S \succeq 0$ and*

$$\mu > \frac{1}{2\lambda_{\min}(\Sigma + K)}, \tag{24}$$

where $\lambda_{\min}(\Sigma + K)$ is the minimal eigenvalue of $\Sigma + K$.

Proof. With $L = R_2B^TP_1$, A_ζ is equal to

$$A_\zeta = I_N \otimes A - (\Sigma + K) \otimes (BR_2B^TP_1). \tag{25}$$

Consequently,

$$\begin{aligned}&(I_N \otimes P_1)A_\zeta + A_\zeta^T(I_N \otimes P_1)\\ &\quad = I_N \otimes (P_1A + A^TP_1) - 2(\Sigma + K) \otimes (P_1BR_2B^TP_1)\\ &\quad = -I_N \otimes Q_1 + I_N \otimes (P_1BR_1B^TP_1)\\ &\qquad - 2(\Sigma + K) \otimes (P_1BR_2B^TP_1). \end{aligned} \tag{26}$$

Thus, if we choose R_2 as in condition C3 then Eq. (26) becomes

$$\begin{aligned}&(I_N \otimes P_1)A_\zeta + A_\zeta^T(I_N \otimes P_1)\\ &\quad = -I_N \otimes Q_1 - [2\mu(\Sigma + K) - I_N] \otimes (P_1BR_1B^TP_1)\\ &\qquad - 2(\Sigma + K) \otimes (P_1BSB^TP_1). \end{aligned} \tag{27}$$

Suppose that condition C2 is satisfied then we obtain from Lemma 2 that $\lambda_{\min}(\Sigma + K) > 0$. Hence, if $\mu > \frac{1}{2\lambda_{\min}(\Sigma+K)}$ then obviously $2\mu(\Sigma+K) - I_N$ is a symmetric, positive definite matrix. In addition, with $S \succeq 0$, $2(\Sigma + K) \otimes (P_1BSB^TP_1)$ is a symmetric, positive semi-definite matrix. As a result, the right hand side of (27) is negative definite, i.e.,

$$(I_N \otimes P_1)A_\zeta + A_\zeta^T(I_N \otimes P_1) \prec 0. \tag{28}$$

From Lyapunov stability theory, we can immediately deduce that A_ζ is stable and hence $\lim_{t\to\infty} \zeta(t) = 0$, i.e., all agents asymptotically converge to the reference. □

4. Hierarchical output feedback consensus controller design

In the previous section, we have assumed that all the states of agents and of the reference are measurable. Nevertheless, in other situations some states may not be available, instead we can measure some partial states $y_k = Cx_k$, $k = 1, \ldots, N$ and the agents receive the information of the reference through its output $y_r = Cx_r$. Consequently, we need to design an output feedback controller such that all agents asymptotically track the reference.

4.1. Output feedback consensus controller

To design a hierarchical consensus output feedback controller, we first assume that (C, A) is observable and then design local Luenberger-type state observers for all agents as well as for the reference. More specifically, let us represent the local state observer for each agent as follows,

$$\begin{aligned}\dot{\hat{x}}_k &= A\hat{x}_k + Bu_k + H(y_k - \hat{y}_k),\\ \hat{y}_k &= C\hat{x}_k,\ k = 1, \ldots, N, \end{aligned} \tag{29}$$

and the following state observer for the reference

$$\begin{aligned}\dot{\hat{x}}_r &= A\hat{x}_r + H_r(y_r - \hat{y}_r),\\ \hat{y}_r &= C\hat{x}_r. \end{aligned} \tag{30}$$

By combining the local observers in (29) and (30), we obtain a *decentralized* observer for the whole network as follows,

$$\begin{aligned}\dot{\hat{x}} &= (I_N \otimes A)\hat{x} + (I_N \otimes B)u + (I_N \otimes H)(y - \hat{y}),\\ \hat{y} &= (I_N \otimes C)\hat{x},\\ \dot{\hat{x}}_r &= A\hat{x}_r + H_r(y_r - \hat{y}_r),\\ \hat{y}_r &= C\hat{x}_r. \end{aligned} \tag{31}$$

Denote $e = x - \hat{x}$, $e_r = x_r - \hat{x}_r$ which are the error vector between the real state x and the estimated state $\hat{x}$ of agents, and the error

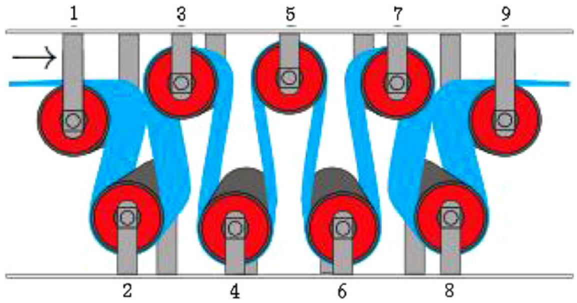

Fig. 6. Drying section of a paper converting machine with 9 rolls (adopted with modification from Mosebach & Lunze, 2013).

between the real and observed states of the reference, respectively. Then by subtracting (1) and (3) with (31), we obtain the following error model,

$$\begin{aligned}\dot{e} &= [I_N \otimes (A - HC)]e,\\ \dot{e}_r &= (A - H_r C)e_r.\end{aligned} \tag{32}$$

As a result, by selecting the observer gains H and H_r such that $A - HC$ and $A - H_r C$ are stable, the error vectors e and e_r will asymptotically converge to 0, i.e., the state vectors $\hat{x}$ and $\hat{x}_r$ will asymptotically converge to the real states of agents and the reference. This allows us to employ the proposed hierarchical sub-optimal state feedback controller with the approximated states $\hat{x}$ and $\hat{x}_r$ such that the outputs of agents reach consensus to the output of the reference. In this scenario, the hierarchical feedback controller becomes

$$u = -[(\Sigma + K) \otimes (R_2 B^T P_1)][\hat{x} - \mathbf{1}_N \otimes \hat{x}_r], \tag{33}$$

where $\hat{x}$ and $\hat{x}_r$ are obtained from the local observers (29) and (30).

4.2. Example 2 (Consensus design for a drying section in a paper machine)

Consider a drying section of a paper converting machine including of 9 rolls as drawn in Fig. 6 which is adopted from Mosebach and Lunze (2013) but the numbering of rolls is re-ordered and a black arrow is added to show the direction of the coming paper. In order to make the paper run smoothly without tearing, the angles of the rolls should be synchronized (Mosebach & Lunze, 2013). Then the 1st roll can be cast as a leader and all other rolls are followers. We also assume that the 1st roll only sends its information to the 2nd roll and each of other rolls just communicates with the rolls in front of and behind it.

The model of the kth roll, $k = 1, \ldots, N$, is

$$\begin{aligned}\dot{x}_k &= Ax_k + Bu_k,\\ y_k &= Cx_k,\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.01 & 0.2 \\ 0 & 0 & -125 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T.$$

Let us first assume that all states of rolls can be measured. Employing our design procedure in Section 3.1, we choose

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

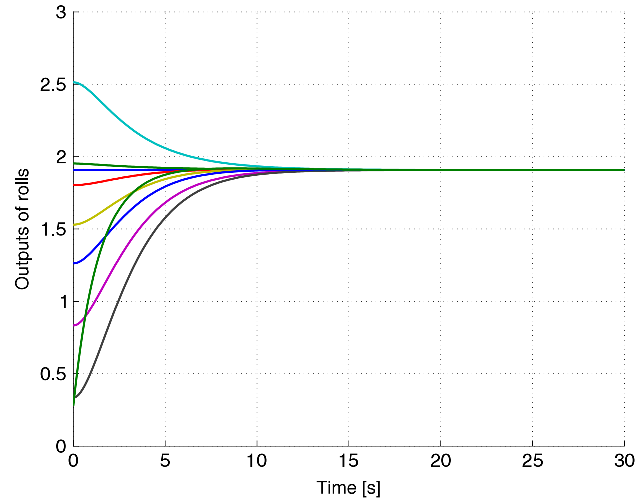

Fig. 7. Consensus in the drying section of a paper converting machine by state feedback consensus controller.

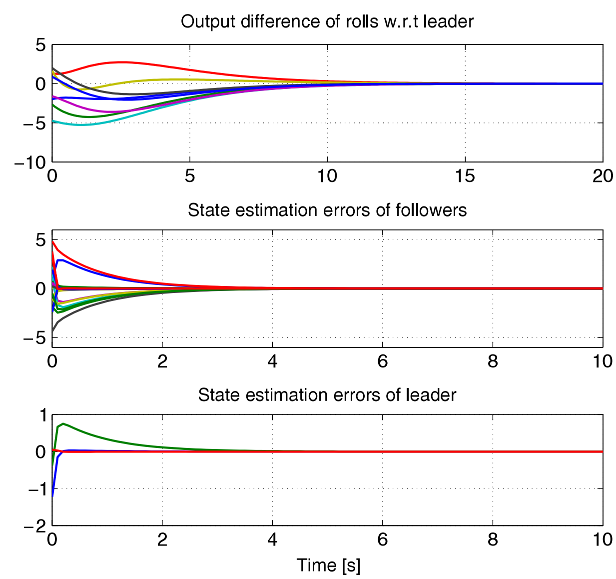

Fig. 8. Consensus in the drying section of a paper converting machine by output feedback consensus controller.

since K must have a tri-diagonal form due to the constraint on the information exchange among the rolls assumed above. Furthermore, select $R_2 = 100$, $R_1 = 1$, $Q_1 = 100I_3$. Fig. 7 exhibits the consensus of the followers' outputs to the leader's output.

Now, suppose that only the outputs of rolls are measurable. Choose the state observer gains to be $H = H_r = 20$. The simulation results with random initial condition of rolls are displayed in Fig. 8. The first subplot reveals that the outputs of the following rolls track the output of the leading roll since the tracking errors come to zero. Moreover, the second and third subplots show that the state estimation errors of all rolls asymptotically converge to zero.

5. Conclusion

This paper has presented a systematic design procedure to obtain a sub-optimal hierarchical consensus feedback controller for homogeneous leader–follower multi-agent systems. The important features of the proposed controller are as follows. First, it preserves the desirable information structure of the multi-agent system and its gains are computed just based on a local, low-dimension Riccati equation. Second, it is derived in a unified way with optimal hierarchical stabilizing feedback controller based on LQR method.

The next research is to design optimal hierarchical consensus controllers for leaderless multi-agent systems.

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