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Sherif I. Ammar
Menoufia University

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Transient analysis of an $M/M/1$ queue with impatient behavior and multiple vacations

Sherif I. Ammar [a,b,*]

[a] Department of Mathematics, Faculty of Science, Menofia University, Shebin El Kom 32511, Egypt
[b] Department of Mathematics, College of Science, Taibah University, Madinah Munawwarah, Saudi Arabia

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ABSTRACT

In this paper, we carry out an analysis for a single server queue with impatient customers and multiple vacations where customers impatience is due to an absentee of servers upon arrival. Customers arrive at the system according to a Poisson process and exponential service times. Explicit expressions are obtained for the time dependent probabilities, mean and variance of the system size in terms of the modified Bessel functions, by employing the generating functions along with continued fractions and the properties of the confluent hypergeometric function. Finally, some numerical illustrations are provided.

1. Introduction

Our main purpose in this presentation, is to study the transient behavior of such Markovian queueing system, namely $M/M/1$ queue with impatient customers and multiple vacations. In recent years, there is a considerable interest in studying queueing systems with server vacations, where many authors have introduced a new class of queueing system with server vacations. Such interesting is backed to the wide range of applications in many practical fields, such as in flexible manufacturing systems, service systems, computer systems, communication networks, production managing and so forth. Moreover, in many real world queueing systems, a server may become unavailable for a random period of time. For the background of such vacations models and their applications to practical situations in everyday life, the reader should refer to many works, among of them are those of Doshi [1], Takagi [2] and the monographs by Tian and Zhang [3].

Queueing systems incorporating impatient customers have attained a lot of attention by many researchers during the past two decades. This is mainly due to the fact that queueing systems with impatient customers arise in the performance analysis of a wide range of systems such as communication systems, call centers, production-inventory systems and other many related areas. Researchers interested in this field can refer to several related literatures and references such as [4] and references therein. In most literatures, the researchers have dealt with queueing system with impatient customers attribute the cause of impatience, can be either a long wait already experienced upon arrival at a queue, or a long wait anticipated by a customer upon arrival.

The emergence of vacation queueing systems have attracted much attention and handed by numerous researchers in order to demonstrate the impatient behavior in queueing systems when the cause of the impatience is owing to a servers, are on vacation and unavailable for service.

Altman and Yechiali [5] have developed a comprehensive analysis of some queueing models such as $M/M/1$, $M/G/1$, and $M/M/c$, queue with server vacations and customer impatience, where customers became impatient only when the servers are

* Department of Mathematics, Faculty of Science, Menofia University, Shebin El Kom 32511, Egypt. Tel.: +20483674082.
E-mail address: sherifamar2000@yahoo.com

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on vacation. They have considered both cases of single and multiple vacations, and consequently obtained various closed-form results. Altman and Yechiali [6] have investigated the $M/M/1$ queueing model with impatient customers and vacations. They have obtained the probability generating function of the number of customers in the system and calculated values of key performance measures.

It is well known that the analytical treatment of the transient solution of queueing systems faces many difficulties even for simple cases. Generally, vacation queueing models have traditionally emphasized steady-state or equilibrium performance. Steady state measures of system performance simply do not make sense for systems that never approach equilibrium. In addition, the steady state results are inappropriate in situations wherein the time horizon of operations is finite.

The transient analysis provides us with a deep understanding of the behavior of a system when the parameters involved are perturbed and it can contribute to the costs and benefits of operating system. Further, such transient analysis enables us to obtain the steady state probabilities and optimal solutions which lead to the control of the system.

Though the great interest in studying of queueing systems with vacations, a few works have dealt with the transient solution of these systems. Kalidass *et al.* [7] have discussed the time dependent behavior of an $M/M/1$ queueing system with multiple vacation scheme and the possibilities of catastrophes. Kalidass and Ramanath [8] have studied the time dependent analysis of a Markovian queue with server vacations and a waiting server. The transient solution of two-dimensional $M/M/1$ queueing model with multiple working vacations and Bernoulli schedule have analyzed by Indra and Renu [9]. Sudhesh and Francis Raj [10] have obtained the time dependent system size probabilities of a $M/M/1$ queue with working vacation. Yang and Wu [11] have investigated the transient behavior of the finite capacity queue with working breakdowns and multiple vacation, while Kalidass and Ramanath [12] have derived the time dependent probabilities of the $M/M/1$ queue with multiple vacations. Recently, Ammar [16] have analyzed the time dependent solution of a two-processor heterogeneous system with catastrophes, server failures and repairs.

It is noted that none of the aforementioned researches deals with getting a time dependent solution for the system under study, where the transient analysis in case of the presence impatient characteristic makes mathematical calculations harder. The main difficulty is due to the dependency on the number of customers in the system. Thus, the most important purpose of the current paper is to carry out the time dependent solution of an $M/M/1$ queue with vacations and impenitence behavior by means of the generating functions along with continued fractions and the properties of the confluent hypergeometric function. Explicit expressions for the time dependent probabilities, mean and variance are derived. From this point of view, the current work may be regarded as an extension of some works, especially that achieved by Altman and Yechiali [5] and Kalidass and Ramanath [12].

The outline of this paper is given as follows: the next section gives a brief background about the confluent hypergeometric function where we will utilize in the definition and properties of this function in our analysis. We have described the $M/M/1$ queue with vacations and impenitence behavior in Section 3, while Section 4 is devoted to developing the transient analysis of the system state and deriving the time dependent probabilities together with mean and variance in terms of the modified Bessel functions. Finally, in Section 5, numerical illustrations are provided in order to exhibit the sensitivity of system performance measures, depending on the various involved parameters and then the concluding remarks are summarized in Section 6.

2. Confluent hypergeometric function

In this section, we display the definition of confluent hypergeometric function and some properties of this function. We obtain an explicit solution for the $M/M/1$ queue with multiple vacations and impatience behavior by using these properties of the confluent hypergeometric function.

The confluent hypergeometric function is denoted by ${}_1F_1(a;c;z)$ and is defined by the power series

$$\begin{aligned} {}_1F_1(a;c;z) &= 1 + \frac{a}{c}\frac{z}{1!} + \frac{a}{c}\frac{(a+1)}{(c+1)}\frac{z^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!} \end{aligned} \tag{2.1}$$

provided that c does not equal $0, -1, -2, \dots$. Here $(\alpha)_k$ is the rising factorial function (the Pochhammer symbol), which is defined by:

$$(\alpha)_n = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}, \qquad n = 0, 1, 2, \dots$$

We observe that

$${}_1F_1(0;c;z) = 1.$$

The recurrence relation for the confluent hypergeometric function is given by

$$c(c-1){}_1F_1(a-1;c-1;z) - az\,{}_1F_1(a+1;c+1;z) = c(c-1-z)\,{}_1F_1(a;c;z)$$

(see Abramowitz and Stegun [13])

The quotient of two hypergeometric functions may be expressed as continued fractions. The following identity from Lorentzen and Waadeland [14].

$$\frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)} = \frac{c}{c-z+}\,\frac{(a+1)z}{c-z+1+}\,\frac{(a+2)z}{c-z+2+}\cdots,$$

which can be rewritten as

$$c\frac{{}_1F_1(a;c;z)}{{}_1F_1(a+1;c+1;z)} - (c-z) = \frac{(a+1)z}{c-z+1+}\,\frac{(a+2)z}{c-z+2+}\cdots \tag{2.2}$$

and

$$\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{y^k}{k!}\,{}_1F_1(a+k;c+k;x) = {}_1F_1(a;c;x+y). \tag{2.3}$$

3. Model description

Consider an $M/M/1$ queueing system with multiple vacations and customers impatience. The assumptions of the system model as follows:

(a) customers arrive according to Poisson process with rate λ, and the service times are exponentially distributed with rate μ.
(b) The server begins vacation after completion of all the services. As before, the server vacation time is a exponential random variable with a parameter γ. After returning from the vacation, if the server finds no customers in the system, the server is permitted to take another vacation.
(c) We assume that inter-arrival times, service times, and vacation times are mutually independent. In addition, the service discipline is first come first served (FCFS).
(d) During the vacation, customers become impatient. That is, each individual customer activates an independent "impatience timer", exponentially distributed with parameter ξ such that, if the customer's service has not been completed before the customer's timer expires, he abandons the system never to return.

Let $\{X(t), t \geq 0\}$ be the number of customers in the system at time t, and let $J(t)$ be the status of the server at time t, which is defined as follows:

$$J(t) = \begin{cases} 1, & \text{if the server is busy at time } t \\ 0, & \text{if the server is on vacation at time } t. \end{cases}$$

Then $\{X(t), J(t), t \geq 0\}$ is a continuous time Markov process on the state space $S = \{(0,0)\} \cup \{(j,n) : j = 0,1;\ n = 1,2,\ldots\}$. Let

$$P_{0,n}(t) = \text{Prob}\{J(t) = 0, X(t) = n\}, \quad n = 0,1,2,\ldots,$$

$$P_{1,n}(t) = \text{Prob}\{J(t) = 1, X(t) = n\}, \quad n = 0,1,2,\ldots$$

Then, the set of forward Kolmogorov differential difference equations are given in the following section.

3.1. Governing equations

The state probabilities $P_{0n}(t)$ and $P_{1n}(t)$ are satisfy the following forward Kolmpgorov differential difference equations:

$$P'_{00}(t) = -\lambda P_{00}(t) + \xi P_{01}(t) + \mu P_{11}(t), \tag{3.1}$$

$$P'_{0n}(t) = \lambda P_{0,n-1}(t) - (\lambda + n\xi + \gamma)P_{0n}(t) + (n+1)\xi P_{0,n+1}(t), \quad n \geq 1, \tag{3.2}$$

$$P'_{11}(t) = \gamma P_{01}(t) - (\lambda + \mu)P_{11}(t) + \mu P_{12}(t), \tag{3.3}$$

$$P'_{1n}(t) = \gamma P_{0n}(t) + \lambda P_{1,n-1}(t) - (\lambda + \mu)P_{1n}(t) + \mu P_{1,n+1}(t), \quad n \geq 2 \tag{3.4}$$

Initially, it is assumed that there is no customer in the system, i.e., $P_{00}(0) = 1$

4. Transient analysis

In this section, we derive the transient solution for the model under consideration by employing generating functions, continued fractions and some properties of confluent hypergeometric function.

4.1. Evaluation for $P_{1n}(t)$

In the analysis coming below, we express $P_{1n}(t),\quad n \geq 1$, in terms of $P_{0n}(t)$.
Define

$$P(z,t) = \sum_{n=1}^{\infty} P_{1n}(t) z^n. \tag{4.1}$$

From (3.3) and (3.4), after some algebraic manipulations, we have

$$\frac{\partial P(z,t)}{\partial t} = \left[\lambda + \mu + \lambda z + \frac{\mu}{z}\right] P(z,t) + \gamma \sum_{n=1}^{\infty} P_{0n}(t) z^n - \mu P_{11}(t).$$

Integrating

$$P(z,t) = \gamma \int_0^t \sum_{m=1}^{\infty} P_{0m}(y) z^m \; e^{[-(\lambda+\mu)(t-u)]} e^{[(\lambda z + \frac{\mu}{z})(t-u)]} du + \mu \int_0^t P_{11}(u) e^{[-(\lambda+\mu)(t-u)]} e^{[(\lambda z + \frac{\mu}{z})(t-u)]} du \tag{4.2}$$

It is well known that if $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$, then

$$\exp\left[(\lambda z + \frac{\mu}{z})t\right] = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t),$$

where $I_n(.)$ is the modified Bessel function of the first kind.
Comparing the coefficients of z^n on both sides of (4.2) for $n = 1, 2, 3, \ldots,$ we have

$$P_{1n}(t) = \gamma \int_0^t \sum_{m=1}^{\infty} P_{0m}(y) \beta^{n-m} I_{n-m}(\alpha(t-u)) e^{[-(\lambda+\mu)(t-u)]} du + \mu \int_0^t P_{11}(y) \beta^n I_n(\alpha(t-u)) e^{[-(\lambda+\mu)(t-u)]} du \tag{4.3}$$

The above equation holds for $n = -1, -2, -3, \ldots,$ with left-hand side replaced by zero.
Using $I_{-n}(.) = I_n(.)$, for $n = 1, 2, 3, \ldots,$

$$0 = \gamma \int_0^t \sum_{m=1}^{\infty} P_{0m}(u) \beta^{-n-m} I_{n+m}(\alpha(t-u)) e^{[-(\lambda+\mu)(t-u)]} du + \mu \int_0^t P_{11}(u) \beta^{-n} I_n(\alpha(t-u)) e^{[-(\lambda+\mu)(t-u)]} du \tag{4.4}$$

From (4.3) and (4.4), for $n = 1, 2, 3, \ldots$

$$P_{1n}(t) = \gamma \int_0^t e^{[-(\lambda+\mu)(t-u)]} \sum_{m=1}^{\infty} \beta^{n-m} P_{0m}(y) \times [I_{n-m}(\alpha(t-u)) - I_{n+m}(\alpha(t-u))] du \tag{4.5}$$

Thus, we have expressed $P_{1n}(t)$ in terms of $P_{0n}(t)$, $n = 1, 2, 3, \ldots$

4.2. Evaluation for $P_{0n}(t)$

Invoking the continued fraction and the well-known identities of confluent hypergeometric function, we can obtain an expression for $P_{0n}(t)$ In the sequel, let $f(s)$ be the Laplace transform of the function $f(t)$. On taking the Laplace transforms of the system (3.1) and (3.2), Eq. (3.2) will give the expression

$$\frac{P_{0n}(s)}{P_{0,n-1}(s)} = \frac{\lambda}{s + \lambda + \gamma + n\xi - (n+1)\xi \frac{P_{0,n+1}(s)}{P_{0n}(s)}},$$

The above equation can be written as a continued fraction as follows:

$$\frac{P_{0n}(s)}{P_{0,n-1}(s)} = \frac{\lambda}{s + \lambda + \gamma + n\xi - \frac{(n+1)\xi\lambda}{s+\lambda+\gamma+(n+1)\xi - \frac{(n+2)\xi\lambda}{s+\lambda+\gamma+(n+2)\xi - \frac{(n+3)\xi\lambda}{s+\lambda+\gamma+(n+3)\xi - \ldots}}}} \tag{4.6}$$

By means of the identity (2.2) of confluent hypergeometric function, Eq. (4.6) will take the following form

$$\frac{P_{0n}(s)}{P_{0,n-1}(s)} = \frac{\lambda}{\xi} \frac{{}_1F_1(n+1; \frac{s+\gamma}{\xi} + n + 1; \frac{-\lambda}{\xi})}{(\frac{s+\gamma}{\xi} + n)\,{}_1F_1(n; \frac{s+\gamma}{\xi} + n; \frac{-\lambda}{\xi})}. \tag{4.7}$$

Invoking of the above equation we can obtain for $n = 1, 2, 3, \ldots,$

$$P_{0n}(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}\left(\frac{s+\gamma}{\xi}+i\right)} \frac{{}_1F_1\left(n+1; \frac{s+\gamma}{\xi}+n+1; \frac{-\lambda}{\xi}\right)}{{}_1F_1\left(1; \frac{s+\gamma}{\xi}+1; \frac{-\lambda}{\xi}\right)} P_{00}(s)$$
$$= \Phi_n(s) P_{00}(s). \tag{4.8}$$

Then

$$P_{0n}(t) = \Phi_n(t) * P_{00}(t). \tag{4.9}$$

where $\Phi_n(t)$ is the inverse Laplace transform of $\Phi_n(s)$ and its formula will be given in Section 4.4, with * denotes convolution.

4.3. Evaluation for $P_{00}(t)$

On taking the Laplace transforms of the system of Eq. (3.1), we obtain

$$P_{00}(s) = \frac{1}{s+\lambda-\xi\frac{P_{01}(s)}{P_{00}(s)}-\mu\frac{P_{11}(s)}{P_{00}(s)}} \tag{4.10}$$

From (4.5), for $n = 1$

$$P_{11}(t) = \gamma \int_0^t e^{[-(\lambda+\mu)(t-u)]} \sum_{m=1}^{\infty} \beta^{1-m} P_{0m}(y) \times [I_{m-1}(\alpha(t-u)) - I_{m+1}(\alpha(t-u))]du \tag{4.11}$$

and from (4.9) for $n = 1$

$$P_{01}(t) = \Phi_1(t) * P_{00}(t). \tag{4.12}$$

On using (4.11) and (4.12) together with (4.10), after some mathematical manipulations, we obtain

$$P_{00}(s) = \sum_{k=0}^{\infty}\sum_{r=0}^{k}(-1)^k\gamma^k\binom{k}{r}\left(\frac{\xi}{\gamma}\right)^r \frac{\Phi_1^r(s)}{(s+\lambda)^{k+1}}\left[\sum_{m=1}^{\infty}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^m \Phi_m(s)\right]^{k-r}$$

where $p = s + \lambda + \mu$

Laplace inversion yields

$$P_{00}(t) = \mu\sum_{k=0}^{\infty}\sum_{r=0}^{k}(-1)^k\gamma^k\binom{k}{r}\left(\frac{\xi}{\gamma}\right)^r e^{-\lambda t}\frac{t^k}{k!} * \Phi_1^r(t) * \left[\sum_{m=1}^{\infty}\beta^{1-m}\left[I_m(\alpha(t-y)) - I_{m+2}(\alpha(t-y))\right]\right.$$
$$\left.\times\; e^{-(\lambda+\mu)t}) * \Phi_m(t)\right]^{*(k-r)} \tag{4.13}$$

where $*$ denotes the convolution, while "$*(k-r)$" stands for the $(k-r)$-fold convolution.

4.4. Expression for $\Phi_n(t)$

Eq. (4.8) enables us to obtain

$$\Phi_n(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}\left(\frac{s+\gamma}{\xi}+i\right)} \frac{{}_1F_1\left(n+1; \frac{s+\gamma}{\xi}+n+1; \frac{-\lambda}{\xi}\right)}{{}_1F_1\left(1; \frac{s+\gamma}{\xi}+1; \frac{-\lambda}{\xi}\right)} \tag{4.14}$$

By using the definition (2.1)

$$\frac{{}_1F_1\left(n+1; \frac{s+\gamma}{\xi}+n+1; \frac{-\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\gamma}{\xi}+i\right)} = \xi^n \sum_{k=0}^{\infty}\frac{\binom{n+k}{k}(-\lambda)^k}{\prod_{i=1}^{n+k}(s+\gamma+i\xi)}$$

By resolving into partial fractions, we have

$$\frac{{}_1F_1\left(n+1; \frac{s+\gamma}{\xi}+n+1; \frac{-\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\gamma}{\xi}+i\right)} = \xi^n \sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-\lambda}{\xi}\right)^k \sum_{i=1}^{n+k}\frac{(-1)^{i-1}}{(i-1)!(n+k-i)!}\frac{1}{s+\gamma+i\xi}. \tag{4.15}$$

Also,

$${}_1F_1\left(1; \frac{s+\gamma}{\xi}+1; \frac{-\lambda}{\xi}\right) = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{\prod_{i=1}^{k}(s+\gamma+i\xi)} = \sum_{k=0}^{\infty}(-\lambda)^k a_k(s), \quad a_0(s) = 1,$$

where

$$a_k(s) = \frac{1}{\prod_{i=1}^{k}(s+\gamma+i\xi)} = \frac{1}{\xi^{k-1}}\sum_{r=1}^{k}\frac{(-1)^{r-1}}{(r-1)!(k-r)!}\frac{1}{s+\gamma+r\xi}, \quad k=1,2,3,\ldots$$

Using the identity given in [15],

$$\left[{}_1F_1\left(1;\frac{s+\gamma}{\xi}+1;\frac{-\lambda}{\xi}\right)\right]^{-1} = \sum_{k=0}^{\infty} b_k(s)\lambda^k, \tag{4.16}$$

where $b_0(s)=1$ and for $k=1,2,3,\ldots$

$$b_k(s) = \begin{vmatrix} a_1(s) & 1 & & \cdots & & \\ a_2(s) & a_1(s) & 1 & \cdots & & \\ a_3(s) & a_2(s) & a_1(s) & \cdots & & \\ \vdots & \vdots & \vdots & & & \\ a_{k-1}(s) & a_{k-2}(s) & a_{k-3}(s) & \cdots & a_1(s) & 1 \\ a_k(s) & a_{k-1}(s) & a_{k-2}(s) & \cdots & a_2(s) & a_1(s) \end{vmatrix}$$

$$= \sum_{i=1}^{k}(-1)^{i-1}a_i(s)b_{k-i}(s).$$

By substituting (4.15) and (4.16) in (4.14), we obtain,

$$\Phi_n(s) = \lambda^n\sum_{j=0}^{\infty}(-\lambda)^j\binom{n+j}{j}a_{n+j}(s)\sum_{k=1}^{\infty}\lambda^k b_k(s).$$

On inversion,

$$\Phi_n(t) = \lambda^n\sum_{j=0}^{\infty}(-\lambda)^j\binom{n+j}{j}a_{n+j}(t)*\sum_{k=1}^{\infty}\lambda^k b_k(t), \tag{4.17}$$

where

$$a_k(t) = \frac{1}{\xi^{k-1}}\sum_{r=1}^{k}\frac{(-1)^{r-1}}{(r-1)!(k-r)!}e^{-(\gamma+r\xi)t}, \quad k=1,2,3,\ldots$$

$$b_k(t) = \sum_{i=1}^{k}(-1)^{i-1}a_i(t)*b_{k-i}(t) \quad k=2,3,4,\ldots;\ b_1(t)=a_1(t)$$

4.5. Performance measures

In this section, we consider the time dependent performance measures of the system.

4.5.1. Mean

Let $N(t)$ denote the number of customers in the system at time t. The average number of customers in the system at time t is given by

$$E(N(t)) = m(t) = \sum_{n=1}^{\infty} n(P_{0n}(t)+P_{1n}(t)) \tag{4.18}$$

$$m(0) = \sum_{n=1}^{\infty} n(P_{0n}(0)+P_{1n}(0)) = 0$$

$$m'(t) = \sum_{n=1}^{\infty} n(P'_{0n}(t)+P'_{1n}(t))$$

From Eqs. (3.1)–(3.4), after considerable mathematical manipulations, the above equation will lead to the following differential equation

$$m'(t) = \lambda - \mu\sum_{n=1}^{\infty}P_{1n}(t) - \xi\sum_{n=1}^{\infty}nP_{0n}(t)$$

Therefore,

$$m(t) = \lambda t - \mu\sum_{n=1}^{\infty}\int_0^t P_{1n}(y)dy - \xi\sum_{n=1}^{\infty}n\int_0^t P_{0n}(y)dy \tag{4.19}$$

where $P_{0n}(t)$ and $P_{1n}(t)$ are given in (4.9) and (4.5).

4.5.2. Variance

Let $X(t)$ denote the number of jobs in the system at time t. The variance number of customers in the system at time t is given by

$$\mathrm{Var}(X(t)) = E(X^2(t)) - (E(X(t)))^2 \tag{4.20}$$

$$\mathrm{Var}(X(t)) = r(t) - (m(t))^2$$

where

$$r(t) = E(X^2(t)) = \sum_{n=1}^{\infty} n^2 (P_{0n}(t) + P_{1n}(t))$$

$$r(0) = \sum_{n=1}^{\infty} n^2 (P_{0n}(0) + P_{1n}(0))$$

Then

$$r'(t) = E(X^2(t)) = \sum_{n=1}^{\infty} n^2 (P'_{0n}(t) + P'_{1n}(t)) \tag{4.21}$$

From Eqs. (3.2)–(3.4) and after considerable mathematical manipulations, the above equation will lead to the following differential equation

$$r'(t) = 2(\lambda - \mu)m(t) + (\lambda + \mu) - \xi \sum_{n=1}^{\infty}(2n^2 - n)P_{0n}(t) + 2\mu \sum_{n=1}^{\infty} nP_{0n}(t) - \mu \sum_{n=1}^{\infty} P_{0n}(t)$$

Therefore,

$$r(t) = (\lambda + \mu)t - \xi \sum_{n=1}^{\infty}(2n^2 - n)\int_0^t P_{0n}(y)dy + 2\mu \sum_{n=1}^{\infty} n \int_0^t P_{0n}(y)dy - \mu \sum_{n=1}^{\infty} \int_0^t P_{0n}(t) + 2(\lambda - \mu)\int_0^t m(y)dy \tag{4.22}$$

where $P_{0n}(t)$ and $P_{1n}(t)$ are given in (4.9) and (4.5).

5. Numerical examples

In this section, we discuss some interesting numerical examples that qualitatively describe the behavior of the queueing model under investigation. To gain understanding of the system behavior, we study the effect of the system parameters on the queueing model. The numerical calculations displayed in Fig. 1 aim to demonstrate the profiles of the time evolution of the mean

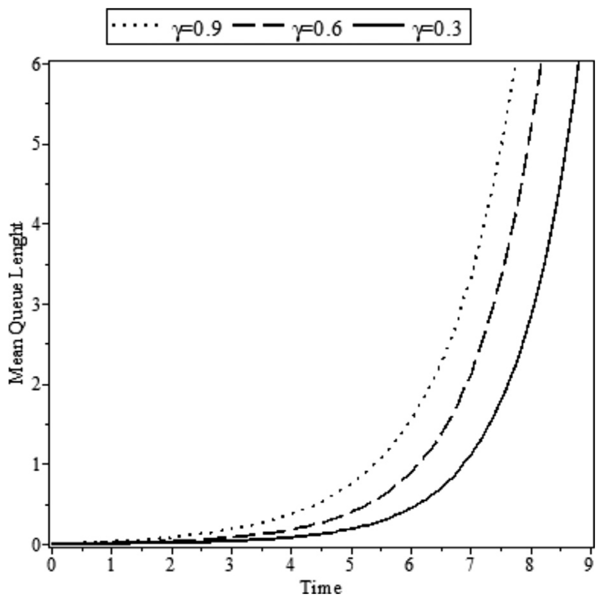

Fig. 1. The influence of both γ and t on the mean system size corresponding to the parameters $\lambda = 5$ and $\mu = 6$ is represented.

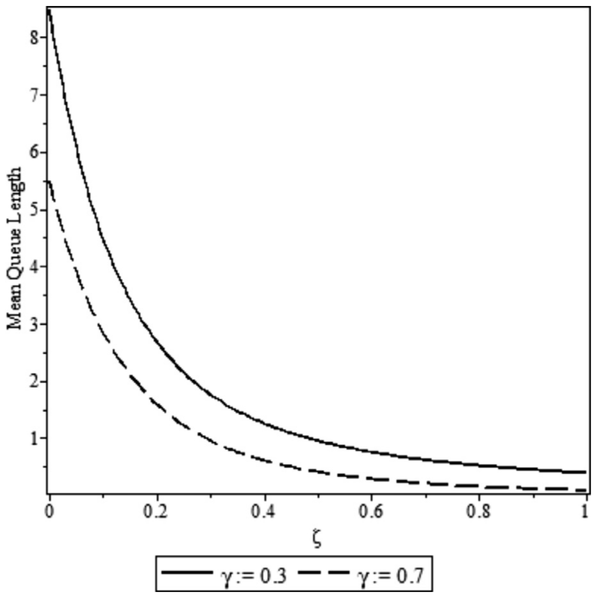

Fig. 2. The mean system size with variation of ξ with the parameters $\lambda = 5$ and $\mu = 6$ is illustrated.

number of customer, $E(N(t))$ dues to the variation of the vacation rate γ. So, we have plotted the value of $E(N(t))$ against the time t at three different values for γ, for the sake of comparison. The graphs reported in Fig. 1 show that the increasing in the values of the rate vacation γ, causes a decreasing in the values of the mean number of customers. Moreover, it is to be noted that the mean number of customers $E(N(t))$ is monotonically increased when the values of time t goes on. Fig. 2, illustrates the behavior of the expected number of customer $E(N(t))$ versus the impatience rate ξ. It is seen that the parameter γ has a same effect in the $E(N(t)) - \xi$ plane to that concluded in the $E(N(t)) - t$ plane. The growth in the values of γ, leads to a decay in these of $E(N(t))$. Further, the rate of decay rapidly decreases corresponds to the large the values of ξ. The results illustrate the evolution of the variance with each of the time and impatient rate are presented in Figs. 3 and 4, respectively. Investigation of the graphs depicted in Figs. 3 and 4 together with that of Figs. 1 and 2, exposes that increasing the parameter γ, has the same influence on variance in the two cases.

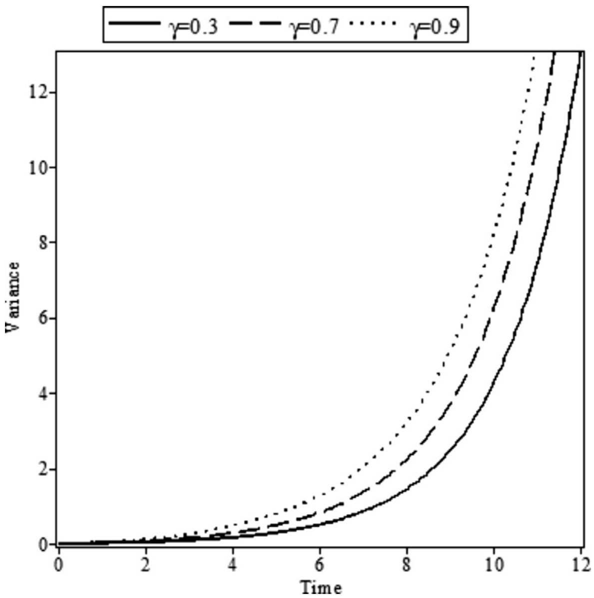

Fig. 3. The behavior of the variance with the variation of γ and t at the values $\lambda = 6$ and $\mu = 7$ is shown.

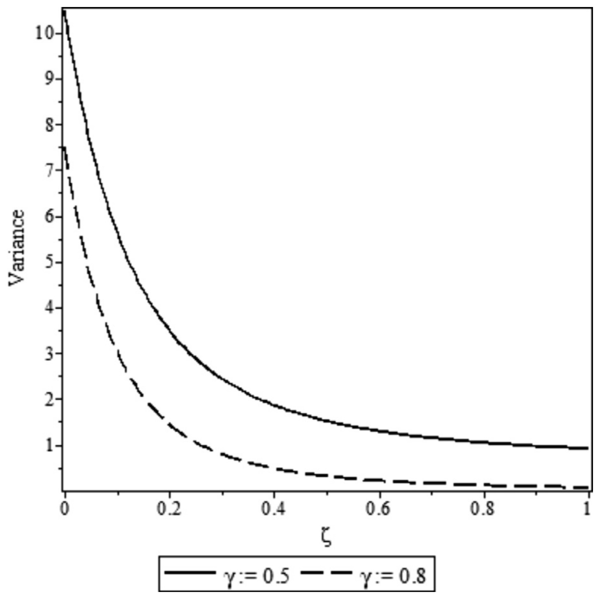

Fig. 4. The variation of the variance with the evolution of the parameter ξ as $\lambda = 6$ and $\mu = 7$ is displayed.

6. Conclusions

In this paper, we investigated an $M/M/1$ vacation queueing system in which customers, whose arrival times are governed by a Markovian arrival process and exponential service times, are subjected to impatience when the system is in vacation. We have derived an explicit solution that can be easily evaluated numerically if desired.

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