Delay Analysis for a TDMA Channel with Contiguous Output and Poisson Message Arrival

KING-TIM KO, MEMBER, IEEE, AND BRUCE R. DAVIS, MEMBER, IEEE

Abstract—**An analysis of the exact delay of a TDMA or a loop communication system with a contiguous output and a Poisson message arrival process is presented. The nonlinear relationship of a contiguous output with arbitrary arrival process can be expressed by an infinite series. In the case of a Poisson message arrival process, the series can be summed to a closed form. The derived result, which is validated by simulation, is a vast improvement over the previously used approximate model of multiple frames with single output.**

I. Introduction

TIME division multiple access (TDMA) is a fixed capacity assignment scheme where the channel is divided into equal duration time slots. The slots are allocated to all the stations in the network in a cyclic manner. TDMA is used in digital satellite communication systems and also has application in computer data networks.

In this paper, the incoming traffic or messages into a station are assumed to have a length of one slot each and are served with a first-come-first-served discipline. The station is scheduled to transmit in a predetermined time interval. For the case where the allocated output is equal to one slot, the delay performance of a system with an infinite buffer has been analyzed by Lam [1] and Rubin [2], [3].

If the allocated output is greater than 1 [i.e., multiple contiguous output, Fig. 1(a)], it can be approximated by a single output model with a multiple number of frames, each of which has a reduced framelength [Fig. 1(b)] [2], [3]. However, in order to obtain an exact delay relation, we extend the unity output model of a TDMA or a loop communication system to a true multiple contiguous output model.

II. Delay Analysis

The message delay (d), which is defined as the duration between the message arrival and its transmission, consists of two components—the fraction of a frame remaining after the last frame (w), and the delay in number of frames (r) due to all previous message arrivals prior to the point under consideration (point X in Fig. 2). The possible transmission times are A, B, C, or D as shown in Fig. 2. The framelength for a particular station is defined as the time between the start of adjacent blocks of contiguous transmission slots. The value of r and, hence, d are functions of w. If w is long, the probability of other message arrivals, within the current frame and prior to X, is small. On the other hand, a shorter w gives a higher probability of other arrivals prior to X. Ignoring small variations within the slot itself, the delay in framelength units is given by

$$d = r + w. \tag{1}$$

The value of r depends upon the assigned number of contiguous slots per frame (u), the number of leftover messages (l) from all previous frames, and the number of messages (y) arriving in the interval AX:

$$r = \left[\frac{l+y}{u}\right] \tag{2}$$

where $[x]$ is the integer part of x.

Now l and w are independent variables, and y depends on w but will usually be independent of l. We can write

$$r = \frac{l+y-i}{u} \tag{3}$$

where i is an integer $0 \leqslant i < u$ chosen such that r is an integer.

To find the average delay we need the expected values of l, y, and i. The first two can be found from the message arrival process. If w and l are fixed, then the expected value of i is determined by y alone, and given by summing the product of i and the probabilities that y takes values $mu + i - l$, where m is an integer.

$$E(i \mid w, l) = \sum_{i=0}^{u-1} i \sum_{m=0}^{\infty} P(y = mu + i - l \mid w). \tag{4}$$

For a Poisson message arrival process, the infinite series of (4) can be summed in closed form. By then averaging over first w and then l, the average delay is obtained (see the Appendix). The result in framelengths is

$$E(d) = \tfrac{1}{2} + \frac{1}{u}\left\{\sum_{n=1}^{u-1} \frac{1}{1-z_n} - \frac{u(u-1)-\lambda^2}{2(u-\lambda)} + \frac{\lambda}{2} - \frac{u-1}{2} + \frac{u-\lambda}{\lambda}\sum_{s=1}^{u-1}\frac{a^s}{1-a^s}\prod_{n=1}^{u-1}\frac{a^s-z_n}{1-z_n}\right\} \tag{5}$$

where

$$\lambda = \text{average arrival rate} \tag{6}$$

$$a = \exp(j2\pi/u) \tag{7}$$

and z_n are the roots inside the unit circle of

$$1 - z^u \exp \lambda(1-z) = 0. \tag{8}$$

Note that the minimum average delay is one-half a framelength, as might be expected intuitively.

Paper approved by the Editor for Satellite and Space Communication of the IEEE Communications Society for publication without oral presentation. Manuscript received April 26, 1983; revised August 3, 1983. This work was supported by the University of Adelaide.

K.-T. Ko is with Telecom Australia Research Laboratories, Clayton North, Vic. 3168, Australia.

B. R. Davis is with the Department of Electrical Engineering, University of Adelaide, Adelaide, Australia.

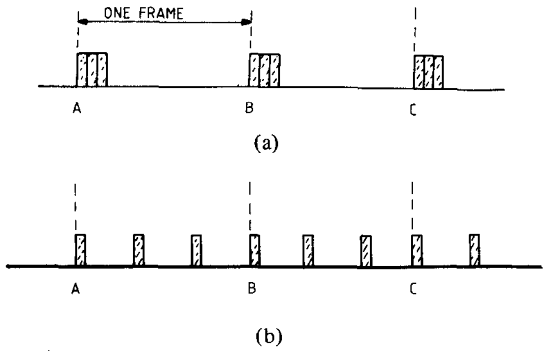

Fig. 1. (a) TDMA channel with multiple contiguous output. (b) Approximate model with multiple subframes and single output.

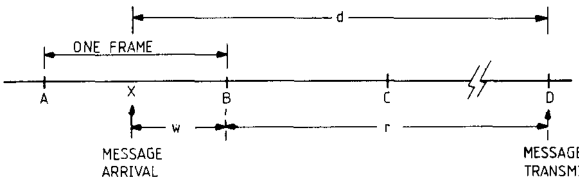

Fig. 2. Components of message delay for multiple contiguous output system.

III. Discussion of Results

One method of validating the theoretical results is to compare them to a computer simulation of the system. In order to achieve this objective, a precise description of the network by the simulation program, a procedure for generating traffic to the network, and a procedure for recording the desired performance characteristics are required. A discrete event-oriented simulation programming package, SPURT-76 [4], was used to simplify many of the above tasks. Due to the detailed description of the network, a typical simulation for an operation of 40 frames takes 100–200 s of CPU time on a CDC Cyber 173.

The calculated values of the average delay versus channel utilization ($\rho = \lambda/u$) are shown in Fig. 3 for four different values of contiguous output per frame (u). For the simplest case of $u = 1$, $E(r)$ simplifies to the well-known Pollaczek-Khintchine result of $0.5\lambda/(1 - \lambda)$ frame for a single deterministic server [5]. The total delay of $0.5/(1 - \lambda)$ frame consists of $E(r)$ and $E(w)$ which has the value of half a frame. The result for the approximate model of multiple frames with single output is $E(d) = 0.5/(u - \lambda)$ frames. As shown in Fig. 3, the calculated average delay for the cases where the output is greater than 1 ($u > 1$) are substantially different from those predicted by the approximate model. The approximate model assumes frames with a reduced framelength, each of which has a single allocated output. This assumption leads to the incorrect minimum value of half a reduced framelength. The discrepancy is particularly noticeable when the channel utilization is low. A better approximation to the exact result can be obtained by taking the larger of half a framelength or $0.5/(u - \lambda)$.

The simulation results are also shown in Fig. 3, and excellent agreement with the theoretical results is obtained. It can be seen that when u is large, the average delay is not significantly greater than half a framelength. This is because of the sum of the leftover messages (l) and recent arrivals (y) is more likely to fit in one frame, thus giving an average frame delay r approximately zero.

IV. Conclusion

In this paper, we have extended the unity output model of a TDMA communication channel to a multiple contiguous output by considering an accurate model. The theoretical results are verified by simulation results and indicate a vast improvement over the previously used approximation. The improvement is particularly significant when the framelength is long or the channel utilization is low. The analysis allows the exact delay calculation of a TDMA or a loop communication system with a variable capacity assignment.

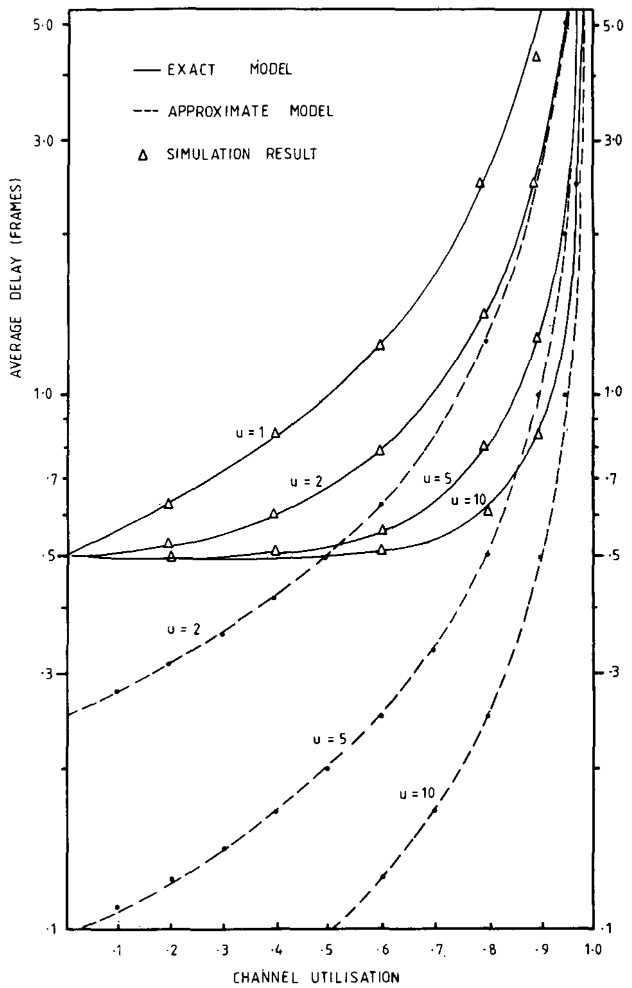

Fig. 3. Comparisons of the delay performance between the exact and the approximate model.

Appendix

For a Poisson message arrival process,

$$P(y = n) = \frac{x^n}{n!} \exp(-x) \qquad n \geqslant 0 \tag{A1}$$

where

$x = \lambda(1 - w)$

$\lambda =$ average number of message arrivals per frame.

Now,

$$\sum_{m=0}^{\infty} P(y = mu + i - l) = \sum_{m=0}^{\infty} \frac{x^{mu+i-l}}{(mu + i - l)!} \exp(-x) = \frac{1}{u} \sum_{s=0}^{u-1} a^{s(l-i)} \exp x(a^s - 1) \tag{A2}$$

where

$a = \exp(j 2\pi/u)$.

This result is easily verified by expanding the second expression in powers of x and using the fact that $\Sigma_{s=0}^{u-1} a^{sk}$ is zero unless k is zero or a multiple of u.

Hence, (4) becomes

$$E(i \mid w, l) = \frac{1}{u} \sum_{s=0}^{u-1} \sum_{i=0}^{u-1} i a^{s(l-i)} \exp x(a^s - 1)$$

$$= \frac{u-1}{2} + \sum_{s=1}^{u-1} \frac{a^{s(l+1)}}{1-a^s} \exp x(a^s - 1). \quad \text{(A3)}$$

We now average over w, assuming w is uniformly distributed over (0, 1). This yields

$$E(i \mid l) = \frac{u-1}{2} + \frac{1}{\lambda} \sum_{s=1}^{u-1} \frac{a^{s(l+1)}}{(1-a^s)^2} \{1 - \exp \lambda(a^s - 1)\}. \quad \text{(A4)}$$

Now

$$E(y \mid w) = x = \lambda(1-w)$$

so

$$E(y) = \lambda/2.$$

Hence, for the delay d,

$$d = w + \frac{l + y - i}{u}$$

$$E(d \mid l) = \tfrac{1}{2} + \frac{1}{u}\left\{ l + \frac{\lambda}{2} - E(i \mid l)\right\}. \quad \text{(A5)}$$

Finally, the average over l only requires $E(l)$ and $E(a^{sl})$ which are found in [3] and [6].

$$E(z^l) = \hat{l}(z) = \frac{(u-\lambda)(1-z)}{\exp \lambda(z-1) - z^u} \prod_{n=1}^{u-1} \frac{z - z_n}{1 - z_n} \quad \text{(A6)}$$

where z_n are the roots inside the unit circle of

$$1 - z^u \exp \lambda(1-z) = 0.$$

These roots can be found by the Newton-Raphson method.

From (A6), we have

$$E(a^{sl}) = \hat{l}(a^s) = \frac{(u-\lambda)(1-a^s)}{\exp \lambda(a^s - 1) - 1} \prod_{n=1}^{u-1} \frac{a^s - z_n}{1 - z_n} \quad \text{(A7)}$$

$$E(l) = \frac{d}{dz} \hat{l}(z)\big|_{z=1} = \sum_{n=1}^{u-1} \frac{1}{1-z_n} - \frac{u(u-1) - \lambda^2}{2(u-\lambda)}. \quad \text{(A8)}$$

Hence, we obtain

$$E(d) = \tfrac{1}{2} + \frac{1}{u}\left\{ \sum_{n=1}^{u-1} \frac{1}{1-z_n} - \frac{u(u-1)-\lambda^2}{2(u-\lambda)} + \frac{\lambda}{2} - \frac{u-1}{2} + \frac{u-\lambda}{\lambda} \sum_{s=1}^{u-1} \frac{a^s}{1-a^s} \prod_{n=1}^{u-1} \frac{a^s - z_n}{1-z_n} \right\}. \quad \text{(A9)}$$

Acknowledgment

The comments and assistance of Dr. J. C. Ellershaw are gratefully acknowledged. The support of The University of Adelaide and the permission of the Director of the Telecom Australia Research Laboratories to publish this paper are also acknowledged.

References

[1] S. S. Lam, "Delay analysis of a time division multiple access (TDMA) channel," *IEEE Trans. Commun.*, vol. COM-25, pp. 1489–1494, Dec. 1977.

[2] I. Rubin, "Message delays in FDMA and TDMA communication channels," *IEEE Trans. Commun.*, vol. COM-27, pp. 769–777, May 1979.

[3] ——, "Access-control disciplines for multi-access communication channels: Reservation and TDMA schemes," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 516–536, Sept. 1979.

[4] C. Cohen and B. Robinson, *Simulation Modelling and Programming with SPURT-76*, Northwestern Univ., Evanston, IL, 1976.

[5] S. Syski, *Congestion Theory in Telephone System.* London, England: Oliver and Boyd, 1958, p. 302.

[6] K. T. Ko and B. R. Davis, "A space division multiple-access protocol for spot beam antenna and satellite-switched communication network," *IEEE J. Select. Areas Commun.*, Special Issue on Digital Satellite Commun., vol. SAC-1, pp. 126–132, Jan. 1983.

★

King-Tim Ko (M'83) was born in Hong Kong in 1954. He received the B.E. (hons.) and Ph.D. degrees from the University of Adelaide, Adelaide, Australia in 1978 and 1983, respectively. His postgraduate research is in the area of multiple access data protocols for a communication satellite network.

Since 1982, he has been with the Telecom Australia Research Laboratories, Melbourne, Australia. His current research interests are in the areas of modeling and interworking of packet switching networks.

★

Bruce R. Davis (M'78) was born in Adelaide, Australia, on July 17, 1939. He received the B.E. (first class honors), B.Sc., and Ph.D. degrees from the University of Adelaide, Adelaide, in 1960, 1963, and 1969, respectively.

He has been with the University of Adelaide since 1964, and at present is a Senior Lecturer in Electrical Engineering. His research interests have been in the field of communication systems. During 1970 he was with Bell Laboratories, Holmdel, NJ, studying various aspects of mobile communiations, and again in 1977 when he was involved in satellite systems research.