Hierarchical Consensus Problem via Group Information Exchange

Jian Hou and Ronghao Zheng

***Abstract*—This paper presents a hierarchical structure to solve the consensus problem of multiagent systems. The new scheme divides the agents into several groups, with each group containing a value concerning all of the intragroup agents' states, which we call *group information*. For each single agent, it receives not only the *agent information* from its intragroup neighbors, but also the group information from its neighboring groups. It is then shown that global consensus can be achieved under the proposed scheme in both discrete time and continuous time. Moreover, a sufficient condition to achieve average consensus is provided. This hierarchical model can be well used in the PageRank algorithm to reduce the communication loads, and to reveal the attractors for Boolean networks by reducing the computational complexity.**

***Index Terms*—Consensus, hierarchical structure, multiagent system, robust consensus.**

I. Introduction

IN RECENT years, the consensus problem for multiagent systems has received compelling attention from various scientific communities, for its potential applications in broad areas, such as spacecraft formation flying [1], sensor networks [2], and cooperative surveillance [3].

The consensus problem dates back in management science and statistics in 70th [4], but the theoretical analysis and proof is proposed until the 21st century [5], [6], with a system frame and available mathematical tool setting up later [7]–[9]. It has been obtained that a quasi-strongly connected topology, i.e., at least one special node (agent) in the topology with information flow to any other nodes, is not only sufficient but necessary graphical condition to guarantee the system consensus in either fixed or switching topologies.

This paper proposes a novel aggregation-based hierarchical consensus strategy. All the agents are divided into several groups with each group containing a value concerning all the intragroup agents' states, which we call *group information*. In this strategy, each agent constructs its control input from two networks: 1) the intragroup topology network and 2) the intergroup topology network. It has to be pointed out that in this paper the interaction among the groups utilizes the group information instead of single agent state in cluster consensus [10], [11] or some other existing hierarchical consensus problems [12], [13]. Moreover, the hierarchical consensus strategy achieves global consensus with the states of all agents in all groups converging to the same value, rather than local consensus of different convergence values for different groups in cluster consensus.

This network aggregation approach is motivated by [14] where the Web is aggregated into groups to reduce the computation and communication loads in PageRank. The aggregated groups communicate with each other in a decentralized manner by exchanging group PageRank values (i.e., group information) that represent the integral value of the group members. Afterward, this group PageRank value is distributed among the group members to determine their individual values. Another application by using aggregation approach is to reveal the attractors of Boolean networks. Since the computational cost to reveal the number of attractors increases exponentially as the number of nodes in the networks increases, the work [15] divides the Boolean networks into several subnetworks, and reveals the structure of the attractors by the composition of the input-state cycles of subnetworks. It turns out to reduce the computational complexity in finding attractors of Boolean networks in many scenarios.

As far as we know, the scheme using group information to solve the consensus problem is first introduced in [16], where the group information is defined as the average value of all the agents' states inside the corresponding group. The work [16] discusses a stochastic network that at each step all the agents are randomly partitioned into two groups with the relative group information used as the control input, resulting in almost sure consensus. In this paper, we focus on the analysis of topological properties. A hierarchical structure of two-level network is obtained by partitioning the agents into multiple groups, where the relative agent state exchanges inside each group at the low level, and the relative group information interacts among groups at the high level. The group information is defined in this paper as a random convex combination of all the agents in each group. In addition, as the network connectedness cannot always be guaranteed in practical application due to environment influence, node power, node mobility, *et al.*,

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we consider here the network topology among the groups is directed and time-varying. The concept of "robust consensus" is brought in to verify the convergence of the system.

The contributions of this paper are as follows.

1) A novel aggregation-based hierarchical consensus strategy is provided to solve the consensus problem with both relative group and individual information. By using general group information instead of individual information between different groups to achieve consensus, this strategy can reduce the communication loads or computational complexity in many applications, such as PageRank, Boolean networks and so on. The group information proposed is a general idea which offers more possibility to solve the consensus problem.
2) The strategy proposed can be applied under relatively mild conditions with time-varying directed topology among the groups and random convex combination group information which happen in many real applications. In addition, both discrete-time and continuous-time systems are considered as well as average consensus problem.

This paper is organized as follows. Section II first introduces several necessary preliminaries on graph theory and especially, on robust consensus. The consensus problem based on a two-level hierarchical structure is presented in Section III. Section IV gives the main result of hierarchical consensus with corresponding analysis. Numerical simulations and the conclusion are proposed in Sections V and VI, respectively.

Notations: Throughout this paper, we use x and boldface $\mathbf{x}$ to represent scalar and vector, respectively. R and $\mathbf{R}^{|*|}$ are used to indicate real number set of scalar and ($*$-order) vector. Let $R_{>0}$ and $R_{\geq 0}$ be the scalar set of positive real number and non-negative real number, respectively. We denote $\|\cdot\|$ as the standard Euclidean norm and $|\cdot|$ as the cardinality of a set. Moreover, the capital letter I (or J) indicates group index, and lower-case i (or j) represents single agent.

II. Preliminaries

A. Directed Graphs

A directed graph (digraph) $G = (V, E, A)$ consists of a finite node set V, an edge set E and a weighted adjacency matrix A with non-negative elements a_{ij}. An edge from node i to node j exists if and only if the adjacency element $a_{ij} > 0$. A path from node i to j is a sequence of distinct nodes $i_0, i_1, \ldots, i_m$, where $i_0 = i$ and $i_m = j$ with $(i_l, i_{l+1}) \in E, 0 \leq l \leq m-1$. A digraph is said to be quasi-strongly connected (or called having a spanning tree in some papers) if and only if there exists a node $i \in V$, called root, such that there is a path from root i to any other nodes. The union of a collection of digraph with the same node set V is a digraph with node set V and edge set equaling the union of the edge sets of all of the digraphs in the collection. We say the node i of a digraph $G = (V, E, A)$ is balanced if and only if its in-degree $\sum_j a_{ji}$ and out-degree $\sum_j a_{ij}$ are equal. The digraph is called balanced only if all its nodes are balanced.

B. System Model With Noise

In linear discrete-time system with noise

$$\begin{aligned} x_i(k+1) &= x_i(k) + u_i(k) + \omega_i(k), \qquad i = 1, \ldots, n \\ \text{or} \quad \mathbf{x}(k+1) &= P(k)\mathbf{x}(k) + \omega(k) \end{aligned} \tag{1}$$

where $\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbf{R}^n$ is the states of agents, $u_i(k) \in R$ represents the control input of agent i. $\omega(k) = [\omega_1(k), \omega_2(k), \ldots, \omega_n(k)]^T \in \mathbf{R}^n$ indicates the noise. The initial state of the n agents is denoted by $\mathbf{x}(0) = [x_1(0), \ldots, x_n(0)]^T$. $\{P(k), k \geq 1\}$ is a sequence of stochastic matrix. We say a stochastic matrix is a real non-negative matrix, with each row summing to 1.

In linear continuous-time system with noise

$$\dot{x}_i(t) = \sum_{j \in N_i(\sigma(t))} a_{ji}(t)(x_j(t) - x_i(t)) + \omega_i(t), \quad i = 1, \ldots, n \tag{2}$$

in which the time-varying graph $G_{\sigma(t)} = (V, E_{\sigma(t)}, A(t))$ is known as the communication topology, with $\sigma : [0, \infty) \to \mathcal{Q}$ as a piecewise constant function, where $\mathcal{Q}$ is a finite set indicating all possible graphs. Agent j is called a neighbor of agent i at time t only if there is an edge $(j, i) \in E_{\sigma(t)}$, and $N_i(\sigma(t))$ represents the neighbor set for agent i at time t. In addition, we denote the joint graph $G_{\sigma(t)}$ in time interval $[t_1, t_2)$ with $t_1 < t_2$ as $G([t_1, t_2)) = (V, \cup_{t \in [t_1, t_2)} E_{\sigma(t)})$. Then we have the following definition.

Definition 1: $G_{\sigma(t)}$ is said to be uniformly (jointly) quasi-strongly connected (UQSC) if there exists a constant $T > 0$ such that $G([t, t+T])$ is quasi-strongly connected (QSC) for any $t \geq 0$.

C. Robust Consensus

Robust consensus [17]–[19] is presented to describe the consensus of systems with noise influence.

1) Discrete-Time Case: We define the distance d between a vector $\mathbf{x}$ and a subspace U by

$$d(\mathbf{x}, U) = \inf_{\mathbf{y} \in U} d(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{y} \in U} \|\mathbf{x} - \mathbf{y}\| \tag{3}$$

where $\|\cdot\|$ is the standard Euclidean norm. To describe the consensus property, we take U as the space spanned by the vector $[1, 1, \ldots, 1]^T$ with appropriate order, i.e., each dimension takes the same value. In the following, a function set and a noise set are presented, respectively:

$$\begin{aligned} D_0 = \{\eta(\cdot) | \eta : R_{>0} \to R_{>0}, \eta(0) = 0 \\ \eta(\delta) \text{ decreases to } 0 \text{ as } \delta \to 0\} \end{aligned}$$

$$B(\delta) = \left\{ \{\omega(k)\} | \sup_{k \geq 0} d(\omega(k), U) \leq \delta \right\}.$$

Definition 2: The system (1) is said to reach robust consensus with respect to noise, if for any $\delta > 0$, $\mathbf{x}(0) \in \mathbf{R}^n$, and any sequence $\{\omega(k)\} \in B(\delta)$, there always exists a function $\eta(\cdot) \in D_0$ and a constant $K > 0$ such that

$$d(\mathbf{x}(k), U) \leq \eta(\delta), \quad \forall k \geq K. \tag{4}$$

Theorem 1 [17]: Suppose that the matrix sequence $\{P(k)\}_{k=1}^{\infty}$ satisfies the following assumptions.

1) For each k, $P(k)$ is a stochastic matrix with positive diagonal entries.
2) The nonzero entries of $P(k)$ have the following unanimous low bound:

$$\min_{P_{ij}(k)>0} P_{ij}(k) \geq \alpha > 0 \ \ \forall k \geq 1. \tag{5}$$

Then, the system (1) is robust consensus if and only if there exists a constant $q > 0$ such that for any $k \geq 0$ the union of digraphs associated with matrices $\{P(k+1), P(k+2), \ldots, P(k+q)\}$ is QSC.

2) Continuous-Time Case: We define a function $\gamma : R_{\geq 0} \to R_{\geq 0}$ to be a $\mathcal{K}$-class function if it is continuous, strictly increasing and $\gamma(0) = 0$. Moreover, a function $\beta : R_{\geq 0} \times R_{\geq 0} \to R$ is a $\mathcal{KL}$-class function if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to zero as $t \to \infty$.

Denote

$$\|z\|_{\infty} \triangleq \sup\{|z(t)|, t \geq 0\}$$

with $|z(t)| \triangleq \max_i |z_i(t)|$. Then we define $\mathcal{F} \triangleq \{z : R_{\geq 0} \to \mathbf{R}^n |\, z(t)$ is continuous except for a set with measure zero, $\|z\|_{\infty} < \infty\}$. Let $H(\mathbf{x}(t))$ be the difference between the maximum and minimum agent state value at time t, i.e.,

$$H(\mathbf{x}(t)) = \max_{i\in V}\{x_i(t)\} - \min_{j\in V}\{x_j(t)\}.$$

Definition 3: The system (2) is said to be global robust consensus, if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$H(\mathbf{x}(t)) \leq \beta(H(\mathbf{x}(0)), t) + \gamma(\|\omega\|_{\infty}) \tag{6}$$

for all $\omega \in \mathcal{F}$.

Remark 1: This definition is a nonlinear generalization of the bound $|H(\mathbf{x}(t))| \leq |H(\mathbf{x}(0))|e^{-\hat{\alpha}t} + \hat{c}\|\omega\|_{\infty}$ for linear systems $\dot{x} = \hat{A}x + \hat{B}u$ with asymptotically stable matrix $\hat{A}$.

Definition 4: A global consensus is achieved for system (2) if

$$\lim_{t\to\infty} H(\mathbf{x}(t)) = 0 \tag{7}$$

for any initial conditions $\mathbf{x}(0)$.

Theorem 2 [19]: Suppose that the system (2) satisfies the following assumptions.

1) There is a lower bound $\tau_D > 0$ between two consecutive switching time instants of $\sigma(t)$.
2) There are two constants $0 < \underline{\alpha} \leq \overline{\alpha}$ such that $\underline{\alpha} \leq a_{ij}(t) \leq \overline{\alpha}$ for $t > 0$.
3) $\omega(t) \in \mathcal{F}$.

Then system (2) achieves global robust consensus if and only if $G_{\sigma(t)}$ is UQSC.

In addition, we define a set

$$\mathcal{F}_1 \triangleq \left\{z(t) \in \mathcal{F} : \lim_{t\to\infty} z(t) = 0\right\}.$$

Then the system reaches global consensus for any $\omega \in \mathcal{F}_1$ if $G_{\sigma(t)}$ is UQSC.

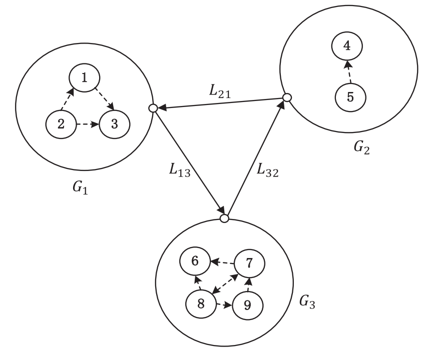

Fig. 1. Example of nine agents partitioned into three groups. Agents in the same group interact with each other, and communication also happens among groups (with coefficient L_{IJ}).

III. Problem Formulation

We consider $\widehat{n}$ agents with discrete-time system model

$$x_i(k+1) = x_i(k) + u_i(k), \quad i = 1, \ldots, \widehat{n} \tag{8}$$

or continuous-time system model

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, \widehat{n}. \tag{9}$$

All the agents are partitioned into n groups $\mathbf{G} = [G_1, \ldots, G_n]^T$ $(\widehat{n} \geq n)$ with the agent number in group G_I is $|G_I| \neq 0$ and $\sum_{I=1}^{n} |G_I| = \widehat{n}$. An agent should belong to only one group, i.e., $G_I \bigcap G_J = \emptyset$. Therefore, an aggregated digraph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{A})$ can be constructed with node set $\widetilde{V} \triangleq \{1, 2, \ldots, n\}$, edge set $\widetilde{E}$ and weighted group adjacency matrix $\widetilde{A}$ with non-negative elements L_{IJ} representing whether there is information flow from group G_I to group G_J. In need of special note is that in this paper the capital letter I (or J) indicates group index, and lower-case i (or j) means single agent.

In our system, each agent could communicate with its neighbors in the same group. However, different from the cluster consensus, there is no direct interaction between agents in different groups. Instead, a piece of alternative information $f_{G_I} : \mathbf{R}^{|G_I|} \to R$ is available to each agent, that we call group information. An example is shown in Fig. 1, in which the interagent relative state is available inside each group, and intergroup relative information exchanges among the groups. Therefore, a two-level hierarchical structure is obtained.

Now, we are ready to introduce the concept of group information, which is a random state lying inside the convex hull spanned by the states of the agents in the corresponding group. The formal definition is given below.

Definition 5: Let $f_{G_I} : \mathbf{R}^{|G_I|} \to R$ be the convex combinations of all the agents in group G_I. That is,

$$f_{G_I} = \sum_{j\in G_I} b_j^I x_j \tag{10}$$

where b_j^I is the (time-varying) non-negative coefficient satisfying $\sum_{j\in G_I} b_j^I = 1$, $I = 1, \ldots, n$. x_{ave}^I represents the average value of all the states of agents in group G_I.

Objective: This paper aims to achieve consensus in the proposed hierarchical networks by utilizing the interagent relative states inside each group and the relative group information among groups.

IV. Main Results

In this section, we discuss hierarchical consensus in which the communication topology among the groups is directed and time-varying.

A. Discrete-Time Case

For each agent $i \in G_I$, we present the control law

$$u_i(k) = g_i^I(k) + \sum_{J \in N_I(k)} L_{JI}(k)\big(f_{G_J}(k) - f_{G_I}(k)\big) \tag{11}$$

where $g_i^I(k)$ represents a local control law used by the agents in group G_I, and L_{JI} indicates whether there is information flow from group G_J to group G_I. To be convenient, $g_i^I(k)$ could be some existing consensus strategies, either linear or nonlinear. $N_I(k)$ indicates the group neighbor set of group G_I at time k. Due to the practical situation, we consider the communication topology among the groups directed and time-varying. In addition, we assume the weighted coefficients satisfy the following assumptions.

Assumption 1: $\sum_{J \in N_I(k)} L_{JI}(k) < 1, \quad \forall k \geq 1$.

Assumption 2: $L_{JI}(k) \geq \alpha > 0, \quad \forall k \geq 1$ when group G_I could obtain the information from group G_J. Here α is a constant.

Remark 2: Assumption 1 is used to guarantee that when (11) is transformed by matrix P combining all groups, the diagonal entries are positive. Assumption 2 defines a lower bound. Therefore, the conditions in Theorem 1 are satisfied.

Now, we give our first main result.

Theorem 3: When the group information is given as (10), the system (8) achieves consensus by control law (11) if and only if there exists a constant $q > 0$ such that for any $k \geq 0$ the union of digraphs $\widetilde{G}$ associated with matrices $\{P(k+1), P(k+2), \ldots, P(k+q)\}$ is QSC.

Proof: From the control law (11), it is obtained that for agents in the same group G_I, they share a common group information $\sum_{J \in N_I(k)} L_{JI}(k)(f_{G_J}(k) - f_{G_I}(k))$, and different intragroup information $g_i^I(k)$ which determines the convergence property inside the group. Therefore, by using the existing consensus strategy $g_i^I(k)$, the agent states in the same group converge. When we consider the linear communication protocol among groups, and group information f_{G_I} in (10)

$$\begin{aligned} x_i(k+1) &= x_i(k) + g_i^I(k) + \sum_{J \in N_I(k)} L_{JI}(k)(f_{G_J}(k) - f_{G_I}(k)) \\ &= x_i(k) + g_i^I(k) + \sum_{J \in N_I(k)} L_{JI}(k) \\ &\quad \times \left(\sum_{j \in G_J(k)} b_j^J(k) x_j(k) - \sum_{j' \in G_I(k)} b_{j'}^I(k) x_{j'}(k) \right) \; i \in G_I. \end{aligned} \tag{12}$$

We add the value $x_{\text{ave}}^I(k+1) + x_{\text{ave}}^I(k) + \sum_{J \in N_I(k)} L_{JI}(k)(x_{\text{ave}}^J(k) - x_{\text{ave}}^I(k))$ to both sides, the equation above is

$$\begin{aligned} x_{\text{ave}}^I(k+1) &= x_{\text{ave}}^I(k) + \sum_{J \in N_I(k)} L_{JI}(k)\big(x_{\text{ave}}^J(k) - x_{\text{ave}}^I(k)\big) \\ &\quad + \big(x_{\text{ave}}^I(k+1) - x_i(k+1)\big) + \big(x_i(k) - x_{\text{ave}}^I(k)\big) \\ &\quad + g_i^I(k) + \sum_{J \in N_I(k)} L_{JI}(k)\big(f_{G_J}(k) - x_{\text{ave}}^J(k)\big) \\ &\quad + \sum_{J \in N_I(k)} L_{JI}(k)\big(x_{\text{ave}}^I(k) - f_{G_I}(k)\big) \;\; I = 1, \ldots, n. \end{aligned} \tag{13}$$

Let

$$\begin{aligned} \omega^I(k) &= \big(x_{\text{ave}}^I(k+1) - x_i(k+1)\big) + \big(x_i(k) - x_{\text{ave}}^I(k)\big) \\ &\quad + g_i^I(k) + \sum_{J \in N_I(k)} L_{JI}(k)\big(f_{G_J}(k) - x_{\text{ave}}^J(k)\big) \\ &\quad + \sum_{J \in N_I(k)} L_{JI}(k)\big(x_{\text{ave}}^I(k) - f_{G_I}(k)\big) \quad I = 1, \ldots, n. \end{aligned}$$

Equation (13) can be rewritten as

$$\begin{aligned} x_{\text{ave}}^I(k+1) &= x_{\text{ave}}^I(k) + \sum_{J \in N_I(k)} L_{JI}(k)\big(x_{\text{ave}}^J(k) - x_{\text{ave}}^I(k)\big) \\ &\quad + \omega^I(k) \qquad I = 1, \ldots, n. \end{aligned} \tag{14}$$

As $g_i^I(k)$ guarantees the convergence inside each group G_I, we have $\lim_{k\to\infty} g_i^I(k) = 0$, $\lim_{k\to\infty} x_i^I(k) = \lim_{k\to\infty} x_j^I(k) = \lim_{k\to\infty} x_{\text{ave}}^I(k)$. Moreover, it is obtained that $\lim_{k\to\infty} \sum_{j\in G_J} b_j^J(k) x_j(k) = x_{\text{ave}}^J(k)$. $\omega^I(k)$ converges to 0 as $k \to \infty$.

Combing groups G_1 to G_n in (14), we have

$$\mathbf{x}_{\text{ave}}(k+1) = P(k)\mathbf{x}_{\text{ave}}(k) + \omega(k) \tag{15}$$

where $\omega(k) = [\omega^1(k), \ldots, \omega^n(k)]^T$. This equation is just similar to the system (1) with $\omega(k)$ functions as the noise to the group interaction. It is obtained that $P(k)$ is a stochastic matrix with positive diagonal entries due to Assumption 1. Moreover, the nonzero entries of $P(k)$ have an unanimous low bound as a result of Assumption 2. Thus, by Theorem 1, the system (15) is robust consensus if and only if there exists a constant $q > 0$ such that for any $k \geq 0$ the union of digraphs $\widetilde{G}$ associated with matrices $\{P(k+1), P(k+2), \ldots, P(k+q)\}$ is QSC.

Finally, we prove the global consensus of the system. For any $\delta > 0$, because function $\omega(k)$ converges to 0, there exists $K(\delta) > 0$ such that $d(\omega(k), U) \leq \delta$, $\forall k \geq K(\delta)$. Thus, according to Definition 2 with initial time $K(\delta)$ instead of 0, we obtain that there exists a function $\eta(\cdot) \in D_0$ and a constant $K' > K(\delta)$ such that

$$d(\mathbf{x}(k), U) \leq \eta(\delta), \quad \forall k \geq K'. \tag{16}$$

Since δ can be arbitrarily small, $\eta(\delta) \to 0$ as $\delta \to 0$, the global consensus follows immediately by taking $k \to \infty$. In addition, the states consensus in the same group is achieved, thus, the whole system (8) achieves consensus. ■

Remark 3: All the analysis above is based on the fact of asymptotic convergence of $g_i^I(k)$. If specially, $g_i^I(k)$ is a finite time convergence control law, which means all the agents states inside each group is synchronized after certain time, then

$\omega^I(k) = 0$ and (14) turns to be a noiseless linear consensus system which is obviously convergent.

B. Continuous-Time Case

For continuous-time case, the control law is given by

$$\dot{x}_i(t) = u_i(t) = g_i^I(t) + \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(f_{G_J}(t) - f_{G_I}(t)\big) \quad (17)$$

where $\sigma : [0,\infty) \to \mathcal{Q}$ as a piecewise constant function, with $\mathcal{Q}$ is a finite set indicating all possible graphs of groups. $N_I(\sigma(t))$ indicates the group neighbor set of group G_I at time t. $g_i^I(t)$ and $L_{JI}(t)$ also have the similar definitions as those in discrete-time case except for the following assumptions.

Assumption 3: There is a lower bound $\tau_D > 0$ between two consecutive switching time instants of $\sigma(t)$.

Assumption 4: There are two constants $0 < \underline{\alpha} \le \overline{\alpha}$ such that $\underline{\alpha} \le L_{JI}(t) \le \overline{\alpha}$ for $t > 0$.

Now, we give our main result in continuous-time system.

Theorem 4: When the group information is given as (10), the system (9) achieves consensus by control law (17) if $G_{\sigma(t)}$ is UQSC.

Proof: Refer to the proof part of Theorem 3, as $g_i^I(t)$ guarantees the convergence inside each group G_I, we have $\lim_{t\to\infty} g_i^I(t) = 0$, $\lim_{t\to\infty} x_i^I(t) = \lim_{t\to\infty} x_j^J(t) = \lim_{t\to\infty} x_{\text{ave}}^I(t)$. Moreover, it is obtained that $\lim_{t\to\infty}\sum_{j\in G_J} b_j^J(t)x_j(t) = x_{\text{ave}}^J(t)$. By using the average value $x_{\text{ave}}^I(t) = (1/|G_I(t)|)\sum_{i\in G_I(t)} x_i(t)$, (17) is rewritten as

$$\begin{aligned}
\dot{x}_{\text{ave}}^I(t) &= \frac{1}{|G_I(t)|}\sum_{i\in G_I(t)} \dot{x}_i(t) \\
&= \frac{1}{|G_I(t)|}\sum_{i\in G_I(t)} g_i^I(t) + \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(f_{G_J}(t) - f_{G_I}(t)\big) \\
&= \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(x_{\text{ave}}^J(t) - x_{\text{ave}}^I(t)\big) \\
&\quad + \frac{1}{|G_I(t)|}\sum_{i\in G_I(t)} g_i^I(t) \\
&\quad + \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(f_{G_J}(t) - x_{\text{ave}}^J(t)\big) \\
&\quad + \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(x_{\text{ave}}^I(t) - f_{G_I}(t)\big) \\
&\quad I = 1,\dots,n.
\end{aligned} \quad (18)$$

We let

$$\begin{aligned}
\omega^I(t) &= \frac{1}{|G_I(t)|}\sum_{i\in G_I(t)} g_i^I(t) + \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(f_{G_J}(t) - x_{\text{ave}}^J(t)\big) \\
&\quad + \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(x_{\text{ave}}^I(t) - f_{G_I}(t)\big).
\end{aligned} \quad (19)$$

Equation (18) can thus be

$$\begin{aligned}
\dot{x}_{\text{ave}}^I(t) &= \sum_{J\in N_I(\sigma(t))} L_{JI}(t)\big(x_{\text{ave}}^J(t) - x_{\text{ave}}^I(t)\big) + \omega^I(t) \\
&\quad I = 1,\dots,n.
\end{aligned} \quad (20)$$

Since $\omega^I(t)$ is clearly continuous, bounded, and converging to 0, and thus belong to $\mathcal{F}_1$. By Theorem 2, the system (20) is global consensus. Besides of the fact that agents states inside each group is synchronized, we get that the system (9) achieves consensus. ■

C. Average Consensus

In this section, we consider the average consensus problem. Average consensus is a special case with each state converging to the average value of all the agents initial states, i.e., $\lim_{k\to\infty} x_i(k) = (1/\widehat{n})\sum_i x_i(0)$. In this paper, we consider only g_i^I an average consensus control law. Take the discrete-time system as example, for all agents in (11), we get that

$$\begin{aligned}
\sum_I\sum_i u_i(k) &= \sum_I\sum_i g_i^I(k) \\
&\quad + \sum_I |G_I(k)| \sum_{J\in N_I(k)} L_{JI}(k)\big(f_{G_J}(k) - f_{G_I}(k)\big).
\end{aligned} \quad (21)$$

Clearly, if the equation above is 0, and consensus is achieved, then average consensus is obtained.

Theorem 5: When the group information is given as (10), and there exists a constant $q > 0$ such that for any $k \ge 0$ the union of digraphs associated with matrices $\{P(k+1), P(k+2),\dots,P(k+q)\}$ is QSC. If at any time k, $g_i^I(k)$ is an average consensus control law, and $\sum_J |G_I|L_{JI} = \sum_J |G_J|L_{IJ}$ for each group G_I, then the system (8) achieves average consensus by control law (11).

Proof: From the Theorem 3, it is obtained that the system (8) achieves consensus by control law (11). In addition, $g_i^I(k)$ is an average consensus control law for any time k, then $\sum_i g_i^I(k) = 0$. In (21), the coefficient of each group information $f_{G_I}(k)$ is $\sum_J |G_J(k)|L_{IJ}(k) - |G_I(k)|\sum_{J\in N_I(k)} L_{JI}(k)$. So when we take $\sum_J |G_I|L_{JI} = \sum_J |G_J|L_{IJ}$ into (21), the sum of control variable $\sum_I\sum_i u_i(k)$ is 0, which means that the whole system (8) achieves average consensus. ■

V. Simulation Results

In this section, several numerical simulations are presented to verify our theoretical results. In the simulations, we consider totally $\widehat{n} = 15$ agents. The initial state of each agent is a random value between -100 and 100. The agents are divided into $n = 4$ groups with each group containing agents number $3, 3, 4, 5$. To make simple description, we assume g_i^I fully connected network with linear control law ensuring states consensus achieved inside each group. Four possible inter group topologies are considered as Fig. 2 with random coefficients $0.1 \le L_{IJ} \le 0.3$. Moreover, we let these four topologies be ergodic every certain time, which guarantees QSC of the system.

A. Discrete-Time Case

In this section, we first consider the discrete-time case, for which f_{G_I} is a time-varying convex combination of all the agents states in group G_I. The simulation results is presented

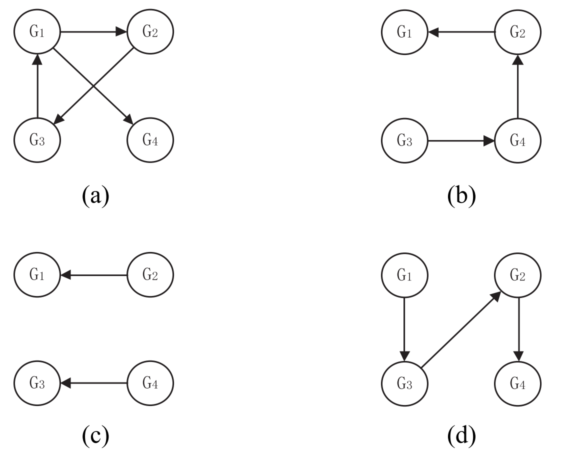

Fig. 2. Four possible inter group topologies with random coefficients greater than 0.01. In addition, ergodicity of the topologies leading to system QSC is guaranteed.

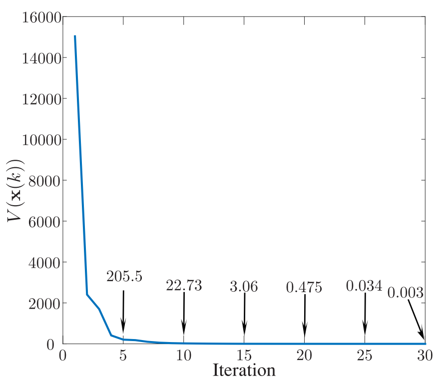

Fig. 3. Reach consensus in discrete-time under control law (11) and the union of digraphs is QSC.

in Fig. 3 where $V(\mathbf{x}(k)) = \sum_{i=1}^{14}(x_i(k) - x_{i+1}(k))^2$ is used as a metric. The simulation results validate our theoretic result of hierarchical consensus problem.

Second, besides the consensus condition, we let the communication topology inside each group balanced such that average consensus inside each group is achieved. In addition, $\sum_J |G_I|L_{JI} = \sum_J |G_J|L_{IJ}$ for each group G_I is required. The simulation results is presented in Fig. 4 in which all the agents states converging to the red line average value quite soon.

B. Continuous-Time Case

In the following, a simulation result for the continuous-time case is given. The graph switching is made about every 0.001 second such that UQSC is guaranteed. The simulation results is presented in Fig. 5. The simulation also confirms our result for the continuous case.

C. Rendezvous Application

At last, we apply this hierarchical model into the rendezvous problem. Eight agents divided into three groups are placed initially randomly at a 20 ∗ 20 square. By using the control law (17) on position coordinates, all the agents finally achieve rendezvous as Fig. 6.

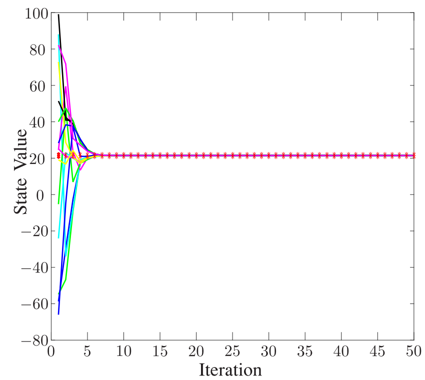

Fig. 4. Reach average consensus in discrete-time with additional requirements that $g_i^I(k)$ is an average consensus control law, and $\sum_J |G_I|L_{IJ} = \sum_J |G_J|L_{JI}$ for each group G_I.

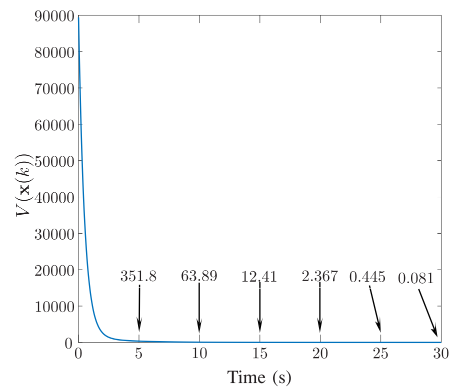

Fig. 5. Reach consensus in continuous-time under control law (17) and $G_{\sigma(t)}$ is UQSC.

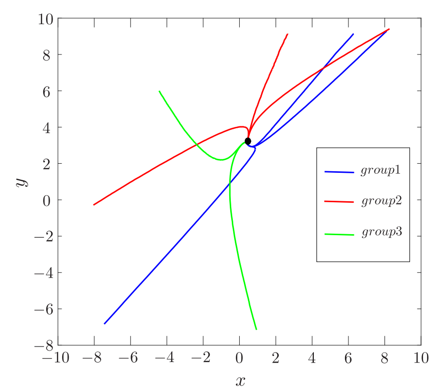

Fig. 6. Eight agents achieve rendezvous under control law (17).

VI. Conclusion

In this paper, we present a novel hierarchical strategy to reach consensus for multiagent systems. The new scheme considers agents divided into several groups, and then uses both relative agent state in the same group, and relative group information among groups to update agents' states. It is shown that in both discrete-time and continuous-time cases, global consensus can be achieved under the proposed scheme with QSC group communication topology. Moreover, average consensus is discussed as a special case. Numerical simulations also demonstrate the effectiveness of the proposed scheme.

The proposed hierarchical strategy provides a new point of view to solve the consensus problem and many relevant questions are still open. First, we take some parts of the systems regarding as noise of the systems after system transformations in this paper, the environment noise and time-delay [20] influence before transformations should be taken into account for practical application. Second, the relationship between the convergence rate and network topology of both intragroup and intergroup should be discussed. And also, we will consider the nonlinear group information f_{G_I}.

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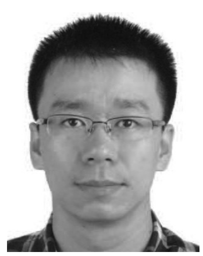

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