Generalized Predicate Completion

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Abstract

Circumscription, proposed by McCarthy, is a formalism of non-monotonic reasoning. Predicate completion is an approach, proposed by Clark, to closed world reasoning which assumes that given sufficient conditions on a predicate are also necessary. Reiter has shown that for clausal sentences which are Horn in a predicate p, the circumscription of p logically subsumes the predicate completion of p. In this paper, we present a *generalized completion* of a predicate p, which is appropriate for clausal sentences which are not Horn in p. The main results of this paper are: (1) for *non-overlapping* clausal sentences (which may not necessarily be Horn in the predicate p), the circumscription of p logically subsumes the generalized completion of p; (2) for *non-overlapping* clausal sentences which are *collapsible wrt* (with respect to) the predicate p, the generalized completion of p is even logically equivalent to the circumscription of p.

1. Introduction

Non-monotonic reasoning is an area of growing significance to artificial intelligence [Clark, 1978; McCarthy, 1980; McCarthy, 1986; Lifschitz, 1985; Reiter, 1978; Reiter, 1980]. Especially, McCarthy's *circumscription* [McCarthy, 1980; McCarthy, 1986; Lifschitz, 1985] turns out to be an influential formalism which attempts to characterize a rule of conjecture - the objects that can be shown to have a certain property from certain facts are only those that satisfy this property. Prior to McCarthy's circumscription, Reiter has proposed the *closed world assumption* (CWA) [Reiter, 1978], which says that the implicit representation of negative facts presumes total knowledge. Circumscription is very similar to CWA in terms of "minimal entailment" and "minimal inference" which captures some characters of human plausible reasoning. As we consider clausal sentences, CWA can efficiently be implemented via Clark's *negation as failure* [Clark, 1978]. Furthermore, that can be proved with negation as failure inference rule from a clausal sentence is a logical consequence of the predicate completion of this sentence [Clark, 1978]. Predicate completion simply states that the given sufficient conditions on a predicate are also necessary.

Since both predicate completion and circumscription attempt to capture some similar phenomena in the aspect of non-monotonic character, it is therefore important to achieve better understanding of the relationship between them. Recently, Reiter has shown that for clausal sentences which are Horn in a predicate p, Clark's predicate completion is implied by McCarthy's circumscription [Reiter, 1982]. Clearly, the completion is a non-trivial logical consequence of circumscription. That predicate completion is subsumed by circumscription for a wide class of clausal sentences is of some theoretical and computational interests. From these points of view, we shall enlarge the class of first-order clausal sentences for which predicate completion can be subsumed by circumscription. In this paper, we shall present a *generalized completion* of a predicate p, which refines on the definition of Clark's predicate completion. The generalized predicate completion is appropriate for clausal sentences which are not Horn in p. Clark's predicate completion and Reiter's result mentioned above are covered by our generalized predicate completion and results. Our main results of this paper are: (1) for *non-overlapping* clausal sentences (which may not necessarily be Horn in the predicate p), the circumscription of p subsumes the generalized completion of p; (2) for non-overlapping clausal sentences which are *collapsible wrt* (with respect to) the predicate p, the generalized completion of p is even logically equivalent to the circumscription of p.

This paper is organized as follows. In *Section* 2 we recall the definition of McCarthy's circumscription and some of its useful characterizations; In *Section* 3 we recall the definition of Clark's predicate completion and Reiter's result; In *Section* 4 we make some investigation on Clark's predicate completion, propose a generalized predicate completion and show our results. This paper is concluded in *Section* 5.

2. Circumscription and Minimal Entailment

Circumscription [McCarthy, 1980; McCarthy, 1986; Lifschitz, 1985] is an approach to the problem of non-monotonic reasoning, which augments formulas with a refinement of minimal inference. In this paper, we are particularly interested in clausal sentences. A *clausal sentence* is a conjunction of clauses (equivalently, a finite set of clauses). A *clause* is a universally quantified disjunction of literals, written as $l_1 \vee \cdots \vee l_n$, which is logically identified with $\forall x. (l_1 \vee \cdots \vee l_n)$, where x is the tuple of variables appearing in the clause and l_i is a *literal* (an *atom* or the negation of an atom) for i, $1 \leq i \leq n$.

Definition 1. Let T be a clausal sentence, p and z distinct predicate symbols. The *circumscription* of p in T with parameter z, denoted by Circum(T; p; z), is defined as the second-order formula:

$$T \wedge \forall p', z'. [\, T(p', z') \wedge \forall x. (\, p'(x) \supset p(x)\,) \supset \forall x. (\, p(x) \supset p'(x)\,)\,],$$

where p' and z' are predicate variables of the same arity as p and z, $T(p', z')$ is the result of T substituting p' and z' for each occurrence of p and z, and x is the tuple of variables. □

This formula states that p has a minimal possible *extension* under the assumption that $T(p, z)$ holds and the extension of z is allowed to vary in the process of minimalization. If no z will be involved, Circum(T; p) is used for Circum(T; p; z).

In the following, we assume that the reader is familiar with the notions of *structures, Herbrand structures, models, logical consequence*, and so on. See [Chang and Lee, 1973], for instance. Note that a Herbrand structure can simply be represented as a set S of ground atoms. Since we are concerned with clausal sentences it is reasonable to consider Herbrand structures as structures. Thus, unless noted otherwise, by a structure we mean a Herbrand structure.

Definition 2. Let M and N be structures, p and z distinct predicate symbols. M is a *substructure* of N *wrt* $\leqq_{p;\,z}$, written as $M \leqq_{p;\,z} N$, *iff* $|M| = |N|$; $M[q] = N[q]$ for any symbol $q \notin \{p, z\}$; and $M[p^+] \subseteq N[p^+]$, where $|M|$ denotes the *domain* of M, $M[q]$ is the *interpretation* of q in M, and $M[q^+] = \{\, a \in |M|^n \mid M[q](a) = \mathsf{True} \,\}$. □

Thus, $M \leqq_{p;z} N$ if M and N differ only in how they interpret the predicate symbols p and z, and the extension $M[p^+]$ of p in M is a subset of its extension $N[p^+]$ in N. A structure M is *minimal* in a class S of structures if $M \in S$ and there is no structure $M' \in$ S such that $M' <_{p;z} M$. We write $M <_{p;z} N$ if $M \leqq_{p;z} N$ but not $N \leqq_{p;z} M$. If M is minimal in the class of all models for a clausal sentence T, we simply say that M is a *model of T minimal wrt* $\leqq_{p;z}$, denoted by $M \models_{p;z} T$. $T \models_{p;z} C$ indicates that a clause C is true in very model of T minimal *wrt* $\leqq_{p;z}$. In this case, C is said to be *minimally entailed* by T *wrt* $\leqq_{p;z}$. If no z will be involved, $\leqq_p$ and $\models_p$ are used for $\leqq_{p;\,z}$ and $\models_{p;\,z}$, respectively.

Theorem 1. [McCarthy, 1980; Lifschitz, 1985] *Let T be a sentence, p and z distinct predicate symbols, and M any structure. M is a model of* Circum(T; p; z) *iff M is a model of T minimal wrt* $\leqq_{p;\,z}$. □

3. Predicate Completion

Predicate completion is an approach, proposed by Clark [Clark, 1978; Lloyd, 1984], to closed world reasoning which assumes that given sufficient conditions on a predicate are also necessary. Let p be a distinguished predicate symbol. According to Clark's definition, a clause written of the following implication form is called a clause *about p*

$$l_1 \wedge \cdots \wedge l_m \supset p(t_1, \cdots, t_n), \tag{3.1}$$

where $l_1, \cdots, l_m$ are literals and $t_1, \cdots, t_n$ are terms. Let $x_1, \cdots, x_n$ be variables not appearing in the clause (3.1) and L be $l_1 \wedge \cdots \wedge l_m$. The clause (3.1) can then be equivalently transformed into the clause

$$x_1 = t_1 \wedge \cdots \wedge x_n = t_n \wedge L \supset p(x_1, \cdots, x_n).$$

Finally, if $y_1, \cdots, y_r$ are the variables in (3.1), the clause (3.1) is itself equivalent to

$$\exists y_1, \cdots, y_r. [\, x_1 = t_1 \wedge \cdots \wedge x_n = t_n \wedge L \,] \supset p(x_1, \cdots, x_n). \quad (3.2)$$

We call (3.2) the *general form* of the clause (3.1). Suppose there are exactly $k \geqq 0$ clauses about the predicate symbol p. Let

$$\begin{array}{l} E_1 \supset p(x_1, \cdots, x_n) \\ \quad \cdots\cdots \\ E_k \supset p(x_1, \cdots, x_n) \end{array} \quad (3.3)$$

be k general forms of these clauses. Each E_i will be an existentially quantified conjunction of literals as the left hand side of (3.2), The *completed definition of* p, implicitly given by those k clauses, is

$$\forall x_1, \cdots, x_n. [\, E_1 \vee E_2 \vee \cdots \vee E_k \equiv p(x_1, \cdots, x_n) \,]. \quad (3.4)$$

When there is no clause about p, *i.e.*, $k = 0$, the completed definition of p is

$$\forall x_1, \cdots, x_n. [\, \mathsf{False} \equiv p(x_1, \cdots, x_n) \,].$$

Let T be a clausal sentence. The *completion* of p in T, denoted by Comp(T; p), is the sentence T along with the completed definition of p and the *equality axioms* (E1)~(E8) by Clark in [Clark, 1978; Lloyd, 1984].

$$\begin{array}{rl} \mathrm{Comp}(T; p) \equiv & T \wedge \forall x_1, \cdots, x_n. [\, E_1 \vee E_2 \vee \cdots \vee E_k \equiv p(x_1, \cdots, x_n) \,] \\ & \wedge (\mathrm{E1}) \wedge \cdots \wedge (\mathrm{E8}) \end{array} \quad (3.5)$$

A clause is said to be *Horn* in a predicate p *iff* it contains at most one positive literal on p. A clausal sentence T is said to be Horn in p *iff* every clause in T is Horn in p.

Theorem 2. [Reiter, 1982] *Let T be a clausal sentence Horn in a predicate symbol p. Then* Circum(T; p) $\models$ Comp(T; p), *i.e., the completion of p in T is implied by the circumscription of p in T.* □

4. Generalized Predicate Completion

As discussed previously, we know that for clausal sentences Horn in p, the completion of p is implied by the circumscription of p. By the investigation below, we understand this is not always the case for clausal sentences.

4.1 Investigation on Predicate Completion

Example 1. Let T be a sentence consisting of the single clause (4.1).

$$\neg p(a) \supset p(b) \quad (4.1)$$

We have the following completed definition of p in T.

$$\forall x. [\, x = b \wedge \neg p(a) \equiv p(x) \,] \tag{4.2}$$

Let $M_1 = \{ p(a) \}$. M_1 is a model of T minimal *wrt* $\leqq_p$. Since $M_1 \models p(a)$, the *only-if-half* of the expression (4.2) is not true in M_1, hence M_1 Comp(T; p). Therefore, Circum(T; p) Comp(T; p). □

Proposition 1. *The predicate completion is not always implied by the circumscription for any clausal sentence.* □

In this example, the predicate designated by p is actually defined by the clauses $\neg p(a) \supset p(b)$ and $\neg p(b) \supset p(a)$, while Comp(T; p) specifies only one part of the definition given by $\neg p(a) \supset p(b)$. Thus the minimal model M_1 of T, which satisfies the part of the definition given by $\neg p(b) \supset p(a)$, does not satisfy Comp(T; p). We attempt to compensate for the loss of the partial definition given by $\neg p(b) \supset p(a)$. To illustrate how to refine on the completion of p, let us continue observing this example.

Let split T into the two parts.

$$\neg p(a) \supset p(b) \tag{4.1}$$
$$\neg p(b) \supset p(a) \tag{4.3}$$

Note that (4.3) is logically identical with (4.1). Then Comp(T; p) is the expression (4.4) and the equality axioms (E1)~(E8).

$$\forall x. [\, (x = b \wedge \neg p(a)) \vee (x = a \wedge \neg p(b)) \equiv p(x) \,] \tag{4.4}$$

All models of T minimal *wrt* $\leqq_p$ are $M_1 = \{ p(a) \}$ and $M_2 = \{ p(b) \}$. Hence, Circum(T; p) $\models$ Comp(T; p).

4.2 Generalized Predicate Completion

In [Clark, 1978], Clark pays attention upon only one positive literal on p. Therefore, he has imposed the condition that a given clausal sentence is Horn in p to get the completed definition on p. Here we shall focus on each positive literal on p respectively. Therefore we write a clause about p in the form of (4.5), which puts all positive literals on p explicitly on the right hand side of $\supset$. Suppose that

$$l_1 \wedge \cdots \wedge l_m \supset p(t_{11}, \cdots, t_{1n}) \vee \cdots \vee p(t_{h1}, \cdots, t_{hn}) \tag{4.5}$$

is a clause about p, where l_i is any literal which is not negative one on p for any i, $1 \leqq i \leqq m$. The tuple $(t_{j1}, \cdots, t_{jn})$ is simply denoted by t_j, for j, $1 \leqq j \leqq h$, and the conjunction $l_1 \wedge \cdots \wedge l_m$ by L. Then the clause (4.5) can be equivalently transformed into each of the following h clauses about a predicate p.

$$L \wedge \neg p(t_2) \wedge \cdots \wedge \neg p(t_h) \supset p(t_1)$$
$$\cdots\cdots$$
$$L \wedge \neg p(t_1) \wedge \cdots \wedge \neg p(t_{j-1}) \wedge \neg p(t_{j+1}) \wedge \cdots \wedge \neg p(t_h) \supset p(t_j)$$

$$\cdots\cdots$$

$$L \wedge \neg p(t_1) \wedge \cdots \wedge \neg p(t_{h-1}) \supset p(t_h)$$

Let x_1 , $\cdots$, x_n be variables not appearing in (4.5), simply denoted by x. Then the above clauses are equivalent to the following ones, respectively.

$$x = t_1 \wedge L \wedge \neg p(t_2) \wedge \cdots \wedge \neg p(t_h) \supset p(x)$$

$$\cdots\cdots$$

$$x = t_j \wedge L \wedge \neg p(t_1) \wedge \cdots \wedge \neg p(t_{j-1}) \wedge \neg p(t_{j+1}) \wedge \cdots \wedge \neg p(t_h) \supset p(x)$$

$$\cdots\cdots$$

$$x = t_h \wedge L \wedge \neg p(t_1) \wedge \cdots \wedge \neg p(t_{h-1}) \supset p(x)$$

If y_1 , $\cdots$, y_r are all variables appearing in (4.5), simply denoted by y, those clauses can be equivalently transformed into the follows.

$$\exists y. [\, x = t_1 \wedge L \wedge \neg p(t_2) \wedge \cdots \wedge \neg p(t_h) \,] \supset p(x)$$

$$\cdots\cdots$$

$$\exists y. [x = t_j \wedge L \wedge \neg p(t_1) \wedge \cdots \wedge \neg p(t_{j-1}) \wedge \neg p(t_{j+1}) \wedge \cdots \wedge \neg p(t_h)] \supset p(x) \tag{4.6}$$

$$\cdots\cdots$$

$$\exists y. [\, x = t_h \wedge L \wedge \neg p(t_1) \wedge \cdots \wedge \neg p(t_{h-1}) \,] \supset p(x)$$

We call (4.6) the *general forms* of the clause (4.5).

Let T be a clausal sentence. Suppose there are exactly $k \geqq 0$ clauses in T about the predicate p. Let

$$\begin{array}{l} E_{11} \supset p(x) \\ \cdots\cdots \\ E_{1h_1} \supset p(x) \\ \cdots\cdots \\ E_{k1} \supset p(x) \\ \cdots\cdots \\ E_{kh_k} \supset p(x) \end{array} \tag{4.7}$$

be $h_1 + h_2 + \cdots + h_k$ general forms of these k clauses. Each E_{ij} will be an existentially quantified conjunction of literals.

$$\exists y_i. [x = t_{ij} \wedge L_i \wedge \neg p(t_{i1}) \wedge \cdots \wedge \neg p(t_{ij-1}) \wedge \neg p(t_{ij+1}) \wedge \cdots \wedge \neg p(t_{ih_i})]$$

The *generalized completed definition* of p in T is the expression (4.8).

$$\forall x. [\, E_{11} \vee \cdots \vee E_{1h_1} \vee \cdots \vee E_{k1} \vee \cdots \vee E_{kh_k} \equiv p(x) \,] \tag{4.8}$$

When there is no clause about p, $k = 0$, the generalized completed definition of p is

$$\forall x. [\, \text{false} \equiv p(x) \,].$$

The *generalized completion* of p in T, $\mathrm{Comp_G}(T; p)$, is the sentence T along with the generalized completed definition of p in T and the equality axioms (E1)~(E8).

$$\mathrm{Comp_G}(T; p) \equiv T \wedge \forall x. [\, E_{11} \vee \cdots \vee E_{1h_1} \vee \cdots \vee E_{k1} \vee \cdots \vee E_{kh_k} \equiv p(x) \,] \\ \wedge (\mathrm{E1}) \wedge \cdots \wedge (\mathrm{E8})$$

For a clause not Horn in a predicate p, Clark's predicate completion gives only one component of the completed definition of p. Different from his, for each positive literal P on p in the clause we give the completed definition of p and group them. The grouped completed definition of p is the generalized completed definition of p. If a clausal sentence is Horn in p the generalized completion of p is apparently identical with Clark's completion.

Proposition 2. *Let T be a clausal sentence and p a predicate symbol. If T is Horn in p, then* $\mathrm{Comp}_G(T;p) = \mathrm{Comp}(T;p)$. □

4.3 Circumscription implies Generalized Predicate Completion (Sometimes)

The generalized predicate completion is not always a logical consequence of the circumscription. Let us consider the clause

$$p(a,y) \vee p(x,y). \tag{4.9}$$

We know $p(a, y)$ is an instance of $p(x, y)$, and the clause (4.9) is logically equivalent to $p(x, y)$, where x and y are variables and a is a constant. The generalized completed definition of p is

$$\begin{aligned}&\forall x_1, x_2. [\exists x, y. (x_1 = x \wedge x_2 = y \wedge \neg p(a,y)) \vee \\ &\quad \exists x, y. (x_1 = a \wedge x_2 = y \wedge \neg p(x,y)) \equiv p(x_1, x_2)].\end{aligned} \tag{4.10}$$

Let consider the structure $M = \{ p(a,a) \}$ with its domain $\{ a \}$, which is a model of (4.9) minimal *wrt* $\leqq_p$. Let $x_1 = a$ and $x_2 = a$. Then

$$[\exists x, y. (a = x \wedge a = y \wedge \neg p(a,y)) \vee \exists x, y. (a = y \wedge \neg p(x,y)) \equiv p(a,a)] \tag{4.11}$$

will be induced from (4.10). Since there is only a single individual a in the domain, (4.11) is always false in M. Hence, M is not a model of the generalized completed definition of p in T (4.10). Note that the distinct positive literals on p in the clause are unifiable.

Proposition 3. *The generalized predicate completion is not always implied by the circumscription for any clausal sentence.* □

Lemma 3.1 *Let F be a formula without positive occurrences of p. If F is true in a structure M then it is true in every substructure of M wrt $\leqq_p$.* □

Lemma 3.2 *Let F be a formula without negative occurrences of p and M a structure. If F is true in some substructure of M wrt $\leqq_p$, then F is true in M.* □

Definition 3. Let C be a clause and p a predicate symbol. C is *non-overlapping wrt p iff* for any distinct positive literals P and P' on p in C, P is not unifiable with P'. If every clause in a clausal sentence T is non-overlapping *wrt p*, T is said to be *non-overlapping wrt p*. □

Theorem 3. *Let T be a clausal sentence and p a predicate symbol. If T is non-overlapping wrt p then* Circum(T; p) $\models$ Comp$_G$(T; p), *i.e., the generalized completion of p in T is implied by the circumscription of p in T.*

Proof. It is sufficient to show the the *only if* half (4.12) of the generalized completed definition of p in T is true in every model of T minimal *wrt* $\leqq_p$.

$$\forall x. [\, p(x) \supset E_{11} \vee \cdots \vee E_{kh_k} \,] \tag{4.12}$$

Here, x is the tuple of variables and each E_{ij} is the expression of the form

$$\exists y_i. [x = t_{ij} \wedge L_i \wedge \neg p(t_{i1}) \wedge \cdots \wedge \neg p(t_{ij-1}) \wedge \neg p(t_{ij+1}) \wedge \cdots \wedge \neg p(t_{ih_i})] \tag{4.13}$$

Let M be any model of T minimal *wrt* $\leqq_p$. Suppose (4.12) is false in M. Then there is at least a ground substitution θ such that $p(x)\theta$ is true in M but

$$(\, E_{11} \vee \cdots \vee E_{kh_k})\theta \tag{4.14}$$

is false in M. Assume $p(x)\theta = p(t)$, where t is the tuple of ground terms. Construct a proper substructure M_0 of M *wrt* $\leqq_p$ in the following way:

$$M_0[q] = M[q] \qquad \text{if } q \neq p, \text{ and}$$
$$M_0[p+] = M[p+] - \{\, t \,\}.$$

Now, we shall prove $M_0 \models T$. By *Lemma* 3.1, every clause in T without positive occurrences of p is true in M_0 since M is a model of T. Then it is sufficient to show the *if* half (4.15) of the generalized completed definition of p in T is true in M_0.

$$\forall x. [\, E_{11} \vee \cdots \vee E_{kh_k} \supset p(x) \,] \tag{4.15}$$

It is clear from the construction of M_0 that $(E_{11} \vee \cdots \vee E_{kh_k} \supset p(x))\sigma$ is true in M_0 for every ground substitution $\sigma \neq \theta$. Thus, it remains to show that

$$(E_{11} \vee \cdots \vee E_{kh_k} \supset p(x))\theta \tag{4.16}$$

is true in M_0. Note that $p(x)\theta$ is false in M_0. Suppose $M_0 \models E_{ij}\theta$, for some i, j. That is,

$$[t = t_{ij} \wedge L_i \wedge \neg p(t_{i1}) \wedge \cdots \wedge \neg p(t_{ij-1}) \wedge \neg p(t_{ij+1}) \wedge \cdots \wedge \neg p(t_{ih_i})]\sigma \tag{4.17}$$

is true in M_0 for some ground substitution σ which replaces variables in y_i by ground terms. $L_i\sigma$ is true in M by *Lemma* 3.2. Because there is no negative occurrences of p in L_i. $t = t_{ij}\sigma$ must be true in M since it is true in M_0. Since T is non-overlapping, $t = t_{ij}\sigma \neq t_{ij'}\sigma$, for very $j' \neq j$. Hence, $[\neg p(t_{i1}) \wedge \cdots \wedge \neg p(t_{ij-1}) \wedge \neg p(t_{ij+1}) \wedge \cdots \wedge \neg p(t_{ih_i})]\sigma$ is true in M since it is true in M_0. Thus, (4.17) is true in M. This means that $E_{ij}\theta$ is true in M, which contradicts that (4.14) is false in M. Therefore, M_0 $E_{ij}\theta$, for every i, j. Hence, (4.16) is true in M_0.

Hence, we can conclude $M_0 \models T$. Since M_0 is a proper substructure of M *wrt* $\leqq_p$, $M_0 \models T$ contradicts the minimality of M. Thus, (4.12) is true in M. **Q.E.D.**

Corollary 3.1 *Let T be a clausal sentence non-overlapping wrt a predicate symbol p. Then* Th(T) $\subseteq$ Th(Comp$_G$(T; p)) $\subseteq$ Th(Circum(T; p)), *where* Th(T) *denotes the set of all logical consequences form T.* □

By *Proposition* 2, *Theorem* 2 is apparently covered by *Theorem* 3. *Theorem* 3 can also be easily extended to the circumscription with parameter.

Corollary 3.2 *Let T be a clausal sentence, p and z distinct predicate symbols. If T is non-overlapping wrt p then* $\mathrm{Circum}(T; p; z) \models \mathrm{Comp}_G(T; p)$, *i.e., the generalized completion of p in T is implied by the circumscription of p in T with parameter z.*

Proof. Because every minimal model of T *wrt* $\leqq_{p;z}$ is also minimal *wrt* $\leqq_p$.
Q.E.D.

It is clear that the converse of *Theorem* 3 is not always true. Now we shall figure out the cases in which the converse of *Theorem* 3 is true.

Definition 4. Let T be a clausal sentence and p a predicate symbol. T is *collapsible wrt p* if it consists of clauses containing no positive occurrences of p or clauses containing no negative occurrences of p. □

Theorem 4. *Let T be a clausal sentence and p a predicate symbol. If T is non-overlapping and collapsible wrt p, then the generalized completion of p in T is logically equivalent to the circumscription of p in T.*

Proof. It is sufficient to show that any model of $\mathrm{Comp}_G(T; p)$ is a minimal model of T *wrt* $\leqq_p$. Let M be a model of $\mathrm{Comp}_G(T; p)$. We shall prove M is a minimal model of T *wrt* $\leqq_p$. It is clear $M \models T$. Let M_0 be any model of T such that $M_0 \leqq_p M$. Since T is collapsible *wrt* p, then E_{ij} in (4.8) contains no positive occurrences of p for any i, j, $1 \leqq i \leqq k$, $1 \leqq j \leqq h_i$. For any $t \in M[p+]$, *i.e.*, t such that $p(t)$ is true in M,

$$(E_{11} \vee \cdots \vee E_{kh_k})\sigma \tag{4.18}$$

is also true in M, for σ such that $p(x)\sigma = p(t)$ since $M \models \mathrm{Comp}_G(T; p)$. Then (4.18) is true in M_0 by *Lemma* 3.1. Hence, $M_0 \models p(t)$, *i.e.*, $t \in M_0[p+]$. Otherwise M_0 T. Therefore $M[p+] \subseteq M_0[p+]$. Since $M_0 \leqq_p M$, then $M_0[p+] \subseteq M[p+]$, hence $M[p+] = M_0[p+]$. Thus M must be a minimal model of T *wrt* $\leqq_p$. **Q.E.D.**

Corollary 4.1 *Let T be a clausal sentence and p a predicate symbol. If T is Horn in p and collapsible wrt p, then the predicate completion of p in T is logically equivalent to the circumscription of p in T.* □

5. Conclusion

In this paper, we have presented a generalized completion of a predicate p, which is appropriate for clausal sentences not Horn in p. It is clear that the generalized predicate completion presented in this paper is exactly same as the predicate completion proposed by Clark, if the relative sentence is Horn in the predicate p. The class of clausal sentences whose generalized predicate completion can be subsumed by the circumscription is obviously wider than the class of clausal sentences which are Horn in p. The result obtained by Reiter could be covered by *Theorem* 3. In some cases the generalized predicate completion is rather close to the circumscription. As shown in *Theorem* 4 and *Corollary* 4.1, sometimes both are completely logically equivalent. As the generalized predicate completion can be constructed in a heuristic way, it may be an effective approach to circumscription within the framework of first-order logic before invoking the full power of the circumscription.

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