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Linear Encodings for Polytope Containment Problems

Sadra Sadraddini and Russ Tedrake

Abstract— **The polytope containment problem is deciding whether a polytope is a contained within another polytope. The complexity heavily depends on how the polytopes are represented. While there exists efficient necessary and sufficient conditions for polytope containment when their hyperplanes are available (H-polytopes), the case when polytopes are represented by affine transformations of H-polytopes, which we refer to as AH-polytopes, is known to be co-NP-complete. In this paper, we provide a sufficient condition for AH-polytope in AH-polytope problem that can be cast as a linear set of constraints with size that grows linearly with the number of hyperplanes of each polytope. These efficient encodings enable us to designate certain components of polytopes as decision variables, and incorporate them into a convex optimization problem. We present the usefulness of our results on applications to the zonotope containment problem, computing polytopic Hausdorff distances, and finding inner approximations to orthogonal projections of polytopes. Illustrative examples are included.**

I. INTRODUCTION

We are interested in establishing the conditions for the following relation to hold:

$$\mathbb{X} \subseteq \mathbb{Y}, \tag{1}$$

where $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ are polytopes. We call (1) the *polytope containment problem*, which is a subfamily of set containment problems (SCPs) [1], [2]. We refer to $\mathbb{X}$ in (1) as the *inbody*, and $\mathbb{Y}$ as the *circumbody*. The decision problem (1) appears in applications such as computational geometry [3], machine learning [4], and control theory [5]. For example, the inbody can represent the reachable states of a dynamical system, while the circumbody is the target set of states. In this context, the reachability verification problem becomes a polytope containment problem. When the inbody is characterized by the parameters of a controller, synthesizing the controller requires finding the parameters such that (1) holds. The focus of this paper is not only providing a Boolean answer to (1), but finding an efficient linear encoding, so (1) can be added to the constraints of a (potentially mixed-integer) linear/quadratic program.

The complexity of writing (1) as linear constraints heavily depends on how the inbody and the circumbody are represented. In general, there are two fundamental ways to represent a polytope: representation by hyperplanes (H-polytope), or representation by vertices (V-polytope). H-polytopes are almost always preferred in high dimensions as the number of vertices is often very large. For example, a box in n dimensions has $2n$ hyperplanes, but 2^n vertices. There exist several algorithms for conversions between H-polytopes and V-polytopes [6], [7], but their worst-case complexities are exponential in the number of dimensions, making it often impractical to navigate between H-polytopes and V-polytopes beyond 2 or 3 dimensions. Therefore, we do not focus on V-polytopes in this paper.

A. Problem Statement

We consider the following generic form of (1), where

$$\mathbb{X} = \bar{x} + X\mathbb{P}_x, \mathbb{Y} = \bar{y} + Y\mathbb{P}_y, \tag{2}$$

where $\bar{x}, \bar{y} \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times n_x}, Y \in \mathbb{R}^{n \times n_y}$, and $\mathbb{P}_x \subset \mathbb{R}^{n_x}, \mathbb{P}_y \subset \mathbb{R}^{n_y}$ are given H-polytopes. In other words, $\mathbb{X}$ and $\mathbb{Y}$ are affine transformations of H-polytopes, which we refer to as AH-polytopes. A similar notion was introduced in [8]. AH-polytopes are very expressive. For example, one can write Minkowski sums and convex hulls of multiple H-polytopes as an AH-polytope. Zonotopes, which are widely used in estimation and control theory [9], [10], are affine transformations of boxes. While one can use a quantifier elimination method, such as the Fourier-Motzkin elimination method [11], to find the H-polytope form of an AH-polytope, it may lead to an exponential number of hyperplanes. We desire to cast (1) using (2) without explicitly computing H-polytope forms of $\mathbb{X}$ or $\mathbb{Y}$.

B. Organization and Main Contributions

In Section II, we provide the necessary background. The main contributions of this paper are:

- we provide a set of linear constraints as sufficient conditions such that (1) holds for (2) (Section III). We show that necessity holds when Y in (2) has a left-inverse. We present interpretations for the conservativeness.
- we show applications of our results on zonotope containment problems, computing Hausdorff distance between polytopes, and inner approximation of orthogonal projections. We empirically find that the associated conservativeness is very small for zonotope containment problems. (Section IV)

Software: The scripts of our results are available in a python package called `pypolycontain` [1].

C. Related Work

Containment problems for convex sets are closely related to their dual characterization, support functions, and polar sets [12], and S-lemma (or S-procedure) [13], which is

The authors are with Computer Science and Artificial Intelligence Laboratory (CSAIL) at Massachusetts Institute of Technology, 32 Vassar st, Cambridge, MA 02139, {`sadra,russt`}`@mit.edu`. This work was partially supported by ONR Award No.: N00014-17-1-2699.

[1]Available in `github.com/sadraddini/pypolycontain`. The extended version of this paper is available at `https://arxiv.org/abs/1903.05214`.

often used for implications of quadratic inequalities. Deciding if a H-polytope is contained within a V-polytope is co-NP-complete [1], while H-polytope inside H-polytope, V-polytope inside H-polytope, and V-polytope inside V-polytope can be decided in polynomial time. The authors in [14] extended the work in [1] for containment problems of spectrahedra - the feasible set of semidefinite programs. Tiwary [15] showed that the deciding whether an H-polytope is equivalent to the convex hull or the Minkowski sum of two H-polytopes is co-NP-complete. The proof relied on the fact that such a decision problem must involve vertex/facet enumeration. More recently, Kellner [16] proved that polytope containment problem for projections of H-polytopes is co-NP-complete, and cast (2) as a bilinear optimization problem, which can be solved using sequential semidefinite programs. This paper provides linear sufficient conditions for this problem. Various version of (2) have been studied to prove correctness of controllers in design of robust control invariant sets and tube model predictive controllers [17], [18]. The authors in [5] proposed a sufficient condition for a special case of zonotope containment problems, which were used to compute backward reachable sets of dynamical systems. For this special case, we show that necessary and sufficient conditions actually exist.

II. Preliminaries

The set of real and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. Given matrices A_1, A_2 of appropriate dimensions, we use $[A_1, A_2]$, (A_1, A_2), and $\mathrm{blk}(A_1, A_2)$ to denote the matrices obtained by stacking A_1 and A_2 vertically, horizontally, and block-diagonally, respectively. Given $\mathbb{S} \subset \mathbb{R}^n$ and $A \in \mathbb{R}^{n_A \times n}$, we interpret $A\mathbb{S}$ as $\{As | s \in \mathbb{S}\}$. Given two sets $\mathbb{S}_1, \mathbb{S}_2 \subset \mathbb{R}^n$, their Minkowski sum is denoted by $\mathbb{S}_1 \oplus \mathbb{S}_2 = \{s_1 + s_2 | s_1 \in \mathbb{S}_1, s_2 \in \mathbb{S}_2\}$. Given $s \in \mathbb{R}^n$, $s + \mathbb{S}$ is interpreted as $\{s\} \oplus \mathbb{S}$.

Given matrix A, we use $\mathrm{range}(A)$ and $\ker(A)$ to denote its column-space and null-space, and A' and $A^\dagger$ to denote its transpose and Moore-Penrose inverse, respectively. The matrix $|A|$ is the matrix obtained by taking the absolute values of A, element-wise. The infinity norm of matrix A is denoted by $\|A\|_\infty$, which is the maximum absolute row sum. The identity matrix and the vector of all ones are denoted by I and $\underline{1}$, where the dimension is unambiguously interpretable from the context. All matrix inequality relations are interpreted element-wise in this paper.

An *H-polyhedron* [19] $\mathbb{P} \subset \mathbb{R}^n$ is the intersection of a finite number of closed half-spaces in the form $\mathbb{P} = \{x \in \mathbb{R}^n | Hx \leq h\}$, where $H \in \mathbb{R}^{n_H \times n}, h \in \mathbb{R}^{n_H}$ define the hyperplanes. A bounded H-polyhedron is called a *H-polytope*. An *AH-polytope* $\mathbb{X} \subset \mathbb{R}^n$ is a polytope that is given as an affine transformation of an H-polytope $\mathbb{P} \subset \mathbb{R}^m$, $\mathbb{X} = \bar{x} + X\mathbb{P}$, where $X \in \mathbb{R}^{n \times m}, \bar{x} \in \mathbb{R}^n$. The *n-dimensional unit box*, or the unit ball corresponding to L_∞ norm, denoted by $\mathbb{B}_n$, is defined as $\mathbb{B}_n := \{x \in \mathbb{R}^n | \|x\|_\infty \leq 1\}$. Its H-polytope form is $\mathbb{B}_n = \{x \in \mathbb{R}^n | [I, -I]x \leq \underline{1}\}$. A H-polytope $\mathbb{P} \subset \mathbb{R}^n$ is *full-dimensional* if there exists $\bar{x} \in \mathbb{P}$, $\epsilon > 0$, such that $\bar{x} + \epsilon\mathbb{B}_n \subset \mathbb{P}$. A *zonotope* $\mathbb{Z}$ is a polytope that can be written as an affine transformation of the unit box $\mathbb{Z} := \langle \bar{x}, X \rangle = \bar{x} + X\mathbb{B}_m$, where $\bar{x} \in \mathbb{R}^n$ is the center and $X \in \mathbb{R}^{n \times m}$ is the generator matrix. The zonotope order is defined as $\frac{m}{n}$. Zonotopes are a special case of AH-polytopes. An appealing feature of zonotopes is its operational convenience with Minkowski sums: $\langle \bar{x}_1, X_1 \rangle \oplus \langle \bar{x}_2, X_2 \rangle = \langle \bar{x}_1 + \bar{x}_2, (X_1, X_2) \rangle$. In practice, most zonotopes have order greater than one. Finding the H-polytope version of a zonotope requires facet enumeration, which its worst-case complexity is exponential in n and m [20], the number of rows and columns of the generator, respectively.

As mentioned earlier, converting an AH-polytope to its equivalent H-polytope may have an exponential complexity. The special case in which conversion is simple is when X has a left inverse, in which case we have:

$$\begin{aligned} &\{\bar{x} + Xx | Hx \leq h\} \\ &= \{y \in \mathbb{R}^n | HX^\dagger y \leq h + HX^\dagger \bar{x}\}, \quad X^\dagger X = I. \end{aligned} \tag{3}$$

However, the H-polytope form of $\mathbb{P}_1 \oplus \mathbb{P}_2$ is not easy to obtain. Unlike H-polytopes, AH-polytopes are suitable to represent affine transformations and Minkowski sums, while the case of intersections is less trivial but still possible. Let $\mathbb{X}_i = \bar{x}_i + X_i\mathbb{P}_i, \mathbb{P}_i = \{z \in \mathbb{R}^{n_i} | H_i z \leq h_i\}, i = 1, 2, X_i \in \mathbb{R}^{n \times n_i}, \bar{x}_i \in \mathbb{R}^n$, be two AH-polytopes. Note that, similar to H-polytopes, multiple AH-polytopes can represent the same polytope. Given $g \in \mathbb{R}^q, G \in \mathbb{R}^{q \times n}$, we have: $G(\bar{x} + X\mathbb{P}) + g = (G\bar{x} + g) + GX\mathbb{P}$. For Minkowski sums, We have the following relation:

$$(\bar{x}_1 + X_1\mathbb{P}_1) \oplus (\bar{x}_2 + X_2\mathbb{P}_2) = \bar{x}_1 + \bar{x}_2 + (X_1, X_2)\mathbb{P}_\oplus, \tag{4a}$$

$$\mathbb{P}_\oplus = \{z \in \mathbb{R}^{n_1 + n_2} | \mathrm{blk}(H_1, H_2)z \leq [h_1, h_2]\}. \tag{4b}$$

III. Main Result

In this section, we provide the main result of this paper in Theorem 1. First, we revisit the well-known result on H-polytope in H-polytope containment.

Lemma 1 (H-Polytope in H-Polytope): Let $\mathbb{X} = \{x \in \mathbb{R}^n | H_x x \leq h_x\}, \mathbb{Y} = \{y \in \mathbb{R}^n | H_y y \leq h_y\} \subset \mathbb{R}^n$, $H_x \in \mathbb{R}^{q_x \times n}, H_y \in \mathbb{R}^{q_y \times n}$. Then $\mathbb{X} \subseteq \mathbb{Y}$ if and only if

$$\exists \Lambda \in \mathbb{R}_+^{q_y \times q_x} \text{ such that } \Lambda H_x = H_y, \Lambda h_x \leq h_y. \tag{5}$$

The proof of Lemma 1 is based on checking q_y inequalities:

$$\max_{x \in \mathbb{X}} H_{y,i} x \quad \leq \quad h_{y,i}, i = 1, \cdots, q_y, \tag{6}$$

where $H_{y,i}$ is the i'th row of H_y (the same for $h_{y,i}$).

Theorem 1 (AH-polytope in AH-polytope): Let $\mathbb{X} = \bar{x} + X\mathbb{P}_x, \mathbb{Y} = \bar{y} + Y\mathbb{P}_y$, where $\mathbb{P}_x = \{z \in \mathbb{R}^{n_x} | H_x z \leq h_x\}$ is a full-dimensional polytope, $\mathbb{P}_y = \{z \in \mathbb{R}^{n_y} | H_y z \leq h_y\}$, where q_x, q_y are number of rows in H_x and H_y, respectively. Then we have $\mathbb{X} \subseteq \mathbb{Y}$ if:

$$\exists \Gamma \in \mathbb{R}^{n_y \times n_x}, \exists \beta \in \mathbb{R}^{n_y}, \exists \Lambda \in \mathbb{R}_+^{q_y \times q_x} \tag{7}$$

such that the following relations hold:

$$X = Y\Gamma, \bar{y} - \bar{x} = Y\beta, \tag{8a}$$

$$\Lambda H_x = H_y \Gamma, \Lambda h_x \leq h_y + H_y \beta. \tag{8b}$$

Proof: Since the hyperplanes of the circumbody are not available, we need to specify the argument similar to (6) for all directions in $\mathbb{R}^n$. Therefore, $\mathbb{X} \subseteq \mathbb{Y}$ is equivalent to:

$$\forall c \in \mathbb{R}^n, \max_{x\in\mathbb{P}_x} c'(\bar{x} + Xx) \leq \max_{y\in\mathbb{P}_y} c'(\bar{y} + Yy). \quad (9)$$

We write the dual of the right hand side to arrive at:

$$\max_{x\in\mathbb{P}_x} c'(\bar{x} + Xx) \leq \min_{u\in\mathbb{R}_+^{q_y}, u'H_y = c'Y} u'h_y + c'\bar{y}. \quad (10)$$

Since minimum of the right-hand side set is greater than the maximum of the left-hand side set, it implies that any element of the right-hand side set is greater than any element of the left-hand side set. Therefore, $\forall c \in \mathbb{R}^n, \forall u \in \mathbb{R}_+^{q_y}, u'H_y = c'Y, \forall x \in \mathbb{P}_x$, we must have the following relation:

$$u'h_y + c'\bar{y} \geq c'Xx + c'\bar{x}. \quad (11)$$

First, we show that the parametrization in (8a) is always possible when $\mathbb{X} \subseteq \mathbb{Y}$ and $\mathbb{P}_x$ is full dimensional. There exists $p_x^0 \in \mathbb{P}_x, \epsilon > 0, p_y^0 \in \mathbb{P}_y$ such that

$$\bar{x} + Xp_x^0 = \bar{y} + Yp_y^0, \bar{x} + X(p_x^0 + \epsilon\mathbb{B}_{n_x}) \subset \mathbb{Y},$$

which implies that $\bar{x} + X(p_x^0 + \epsilon e_i) = \bar{y} + Y(p_y^0 + \gamma_i)$, for some $\gamma_i \in \mathbb{R}^{n_y}, i = 1, \cdots, n_x$, where e_i is the unit vector in the i'th Cartesian direction. Therefore, $Xe_i = Y\gamma_i, i = 1, \cdots, n_x$. Thus, all columns of X lie in range(Y). Moreover, $\bar{y} - \bar{x} = Xp_x^0 - Y\bar{x}$, which also implies that $\bar{y} - \bar{x} \in$ range(Y). Now substitute $\bar{y} - \bar{x}$ with $Y\beta$ and X with $Y\Gamma$, and finally $c'Y$ with $u'H_y$ in (11) to obtain:

$$\begin{gathered}\forall u \in \mathbb{R}_+^{q_y}, \forall c \in \mathbb{R}^n, u'H_y = c'Y, \forall x \in \mathbb{P}_x, \\ u'(h_y + H_y\beta - H_y\Gamma x) \geq 0.\end{gathered} \quad (12)$$

Until this point, every relation is necessary and sufficient for $\mathbb{X} \subseteq \mathbb{Y}$. The conditions in (12) is bilinear in u and x. Furthermore, $H_y\Gamma$ is not necessarily positive definite, so we can not efficiently find the minimum in (12) and check if it is non-negative. Notice that c does not appear directly in the bilinear expression, but we have $u'H = c'Y$. Even though c is allowed to take all values in $\mathbb{R}^n$, u is restricted - we postpone its characterization to the next theorem.

By dropping $u'H = c'Y$, u becomes only constrained to be non-negative, which means that $H_y + H_y\beta - H_y\Gamma x \geq 0$ for all $x \in \mathbb{P}_x$. Define H-polytope $\mathbb{Q} := \{z \in \mathbb{R}^{n_y} | H_y\Gamma z \leq h_y + H_y\beta\}$. This means that any point in $\mathbb{P}_x$ is also in $\mathbb{Q}$, or $\mathbb{P}_x \subseteq \mathbb{Q}$. Therefore, from Lemma 1 we have $\Lambda H_x = H_y\Gamma, \Lambda H_x \leq H_y + H_y\beta, \Lambda \geq 0$, and the proof is complete. ∎

It is worth to note that Theorem 1 has the following geometrical interpretation. It implies that

$$-\beta + \Gamma\mathbb{P}_x \subseteq \mathbb{P}_y. \quad (13)$$

By left multiplying both sides in (13) by Y, $\mathbb{X} \subseteq \mathbb{Y}$ is established. Before moving to the necessary conditions, we remark that (12) is a generalized form of the conditions reported in [16], where the authors considered containment problems for orthogonal projections of polytopes. Note that u is restricted to a cone that is the intersection of positive orthant and the linear subspace given by $\{u \in \mathbb{R}^{q_y} | H_y'u \in \text{range}(Y)\}$. Enumerating all the extreme rays of this cone is not possible in polynomial time. The main idea in Theorem 1 is avoiding the bilinear terms using parameterization in (8a) and then relaxing u to be in the positive orthant. The conservativeness can also be interpreted as replacing a cone in the positive orthant by the positive orthant itself (a larger cone that does not depend on H_y and Y). Now we state a sufficient condition in which (8) becomes necessary.

Corollary 1: The conditions in (8) is necessary if

$$\text{range}(H_y'^{\dagger}Y') \oplus \ker(H_y') = \mathbb{R}^{q_y}. \quad (14)$$

We now provide a sufficient condition for (14) to hold.

Corollary 2: The conditions in (8) are necessary and sufficient if Y has a left-inverse.

Corollary 3: The AH-polytope in H-polytope containment problem can be decided in polynomial time.

Corollary 2 is not surprising as it was already mentioned that if Y has a left inverse, one can replace $Y\mathbb{P}_y$ by H-polytope $\mathbb{Q} = \{z \in \mathbb{R}^n | H_yY^{\dagger}z \leq h_y\}$. Corollary 3 is a known result in the literature. A version was derived in [16]. It can also be proved using other techniques in basic convex analysis [12]. The authors in [17] also derived linear encodings for a specific version of AH-polytope in H-polytope containment.

IV. Applications

Here we demonstrate the usefulness of Theorem 1 on applications in computational convexity and problems encountered in formal methods approach to control theory.

A. Zonotope Containment Problems

Zonotopes are popular in estimation and characterizing reachable sets in control theory [9], [10], [21]. In this section, we provide a version of Theorem 1 for the zonotope containment problem and empirically test its conservativeness.

Corollary 4 (Zonotope in Zonotope Containment): We have$\langle\bar{x}, X\rangle \subseteq \langle\bar{y}, Y\rangle$, $X \in \mathbb{R}^{n\times n_x}, Y \in \mathbb{R}^{n\times n_y}$, if there exists $\Gamma \in \mathbb{R}^{n_y\times n_x}, \beta \in \mathbb{R}^{n_y}$ such that:

$$X = Y\Gamma, \bar{y} - \bar{x} = Y\beta, \|(\Gamma, \beta)\|_\infty \leq 1. \quad (15)$$

Example 1: Consider two zonotopes in $\mathbb{R}^2$:

$$\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -3 \end{pmatrix},$$

$$\bar{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 & -2 \end{pmatrix}.$$

The containment of $\mathbb{Z}_x \subseteq \mathbb{Z}_y$ is verified by Theorem (4) and is also illustrated in Fig. 1 [Left]. If we drop the last column from Y, containment no longer holds, which follows from the infeasibility of (15) and is also shown in Fig. 1 [Right].

On the Conservativeness of Corollary 4: Surprisingly, we found that Corollary 4 is often lossless. We found manually searching for a counterexample, where containment holds but Corollary 4 fails to verify it, to be non-trivial. It remains an open problem whether a lossless condition for zonotope containment without relying on vertex/hyperplane enumeration exists. In order to characterize conservativeness,

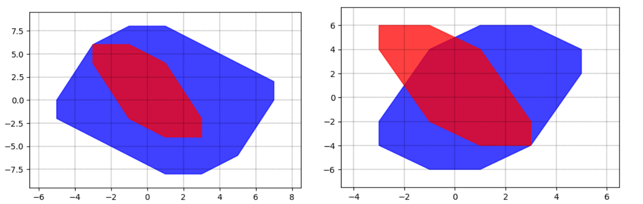

Fig. 1. Example 1: Zonotope Containment Problem: [left] $\mathbb{Z}_x \subseteq \mathbb{Z}_y$, [Right] $\mathbb{Z}_x \not\subseteq \mathbb{Z}_y^*$, where the last column of Y is dropped.

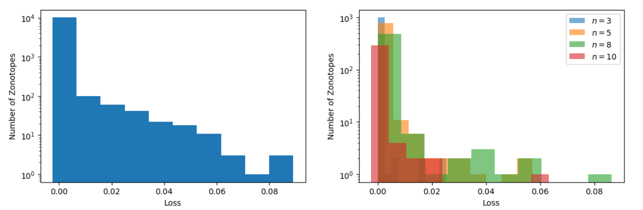

Fig. 2. Histograms of Loss values for Corollary 4

given zonotopes $\mathbb{Z}_x$ and $\mathbb{Z}_y$, we enumerate the vertices of $\mathbb{Z}_x$ - there are at most 2^{n_x} of them, where n_x is the number of columns in X. Then we solve two linear programs: I) the maximum $\lambda_{\text{lossless}}$ such that all the vertices of $\lambda_{\text{lossless}}\mathbb{Z}_x$ are inside $\mathbb{Z}_y$, and II) the maximum $\lambda_{\text{Theorem 4}}$ such that (15) holds for $\lambda_{\text{Theorem 4}}\mathbb{Z}_x \subseteq \mathbb{Z}_y$. We introduce the loss function

$$\text{loss} = (\lambda_{\text{lossless}} - \lambda_{\text{Corollary 4}})/\lambda_{\text{lossless}}. \tag{16}$$

If loss $= 0$ for a certain zonotope containment problem, then Corollary 4 does not introduce any conservativeness. We randomly generated over 10000 zonotopes in $\mathbb{R}^n, n = 3, \cdots, 10$, with random number of generator columns (uniformly sampled between n and 12) for the inbody and the circumbody, and random values of generator matrix entries uniformly and independently chosen between -1 and 1, and performed a statistical analysis. The loss was smaller than 0.01 for nearly 98% of the zonotopes. The histogram of loss values for all zonotopes is shown in Fig. 2 [Left], and for different dimensions in Fig. 2 [Right]. The histogram is so skewed toward large numbers of zonotopes with small loss values that we used logarithmic scale for meaningful illustration. We observed a trend, albeit not very strong, of loss values getting larger with zonotope dimension n. We never observed a loss greater than 0.1. We intuitively expect that at very high dimensions ($n > 10$), the loss may be significant, but verifying this fact requires zonotope vertex/hyperplane enumeration, which is not possible for very large values of n.

With a randomized search over rational generators, we found the following counterexample in $n = 3$. We never found a counterexample with $n = 2$.

Example 2: Consider two zonotopes in $\mathbb{R}^3$ with centroids at origin:

$$X = \begin{pmatrix} 5 & -1 & 2 \\ -4 & -2 & 2 \\ 4 & -1 & -4 \end{pmatrix}, Y = \begin{pmatrix} 4 & 0 & -4 & 1 & 0 \\ -3 & 0 & 0 & 4 & 1 \\ 1 & -4 & -5 & -1 & -3 \end{pmatrix}$$

One can verify $\mathbb{Z}_x \subseteq \mathbb{Z}_y$ through checking all the vertices of $\mathbb{Z}_x$. However, Corollary 4 fails to establish containment as (15) is infeasible. However, feasibility is gained by scaling $\mathbb{Z}_x$ by 0.9915. The loss for this case is 0.0085.

Remark 1: The authors in [5] provided a sufficient condition based on linear matrix inequalities (LMI) for the zonotope containment problem in which the zonotopes were represented as $\mathbb{Z} = \{x \in \mathbb{R}^n | \|H(x-\bar{x})\|_\infty \leq 1\}$. However, in this case, the H-polytope form of zonotopes is already available as $\mathbb{Z} = \{x \in \mathbb{R}^n | [I - I]H(x-\bar{x}) \leq \underline{1}\}$. Therefore, Lemma 1 provides necessary and sufficient conditions for this restricted classes of zonotopes and a sufficient condition based on an LMI characterization is not required.

B. Polytopic Hausdorff Distance

Hausdorff distance is a popular metric to compute the dissimilarity between two volumetric objects and has applications in computer vision [22] and set-valued control [23]. In this section, based on Theorem 1, we provide an optimization-based method to compute an upper-bound for Hausdorff distance between two polytopes. To the best of our knowledge, all existing algorithms for the exact computation of Hausdorff distances between polytopes rely on vertex enumeration. While this is not a significant problem in computer vision, where the dimensions are not greater than 2 or 3, it introduces computational bottlenecks in high-dimensional control applications. Using Theorem 1, we provide a linear programming approach to compute an upper-bound for the Hausdorff distance between two polytopes.

Given a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the Hausdorff distance provides a metric for subsets of $\mathbb{R}^n$ as follows:

$$d_H(\mathbb{S}_1, \mathbb{S}_2) := \max\{D_{12}, D_{21}\}, \tag{17}$$

where D_{12}, D_{21} are given as:

$$\begin{array}{l} D_{12} := \sup_{s_2 \in \mathbb{S}_2} \inf_{s_1 \in \mathbb{S}_1} d(s_1, s_2), \\ D_{21} := \sup_{s_1 \in \mathbb{S}_1} \inf_{s_2 \in \mathbb{S}_2} d(s_1, s_2). \end{array} \tag{18}$$

D_{12} and D_{21} are known as directed distances, which do not satisfy metric properties as it is possible to have $D_{21} \neq D_{12}$. Note that $D_{ij} = 0$ if and only if $\mathbb{S}_i \subseteq \mathbb{S}_j$, $(i,j) \in \{(1,2),(2,1)\}$. For compact sets, the sup and inf in (18) can be replaced by max and min, respectively, and $d_H(\mathbb{S}_1, \mathbb{S}_2) = 0$ if and only if $\mathbb{S}_1 = \mathbb{S}_2$.

It is straightforward to show that, if $\mathbb{S}_1, \mathbb{S}_2$ are compact sets, (17) can be written as:

$$d_H(\mathbb{S}_1, \mathbb{S}_2) = \max\{\min_{\mathbb{P}_1 \subseteq \mathbb{P}_2 \oplus d_1 \mathbb{P}_{\text{Ball}}} d_1, \min_{\mathbb{P}_2 \subseteq \mathbb{P}_1 \oplus d_2 \mathbb{P}_{\text{Ball}}} d_2\}, \tag{19}$$

where $\mathbb{P}_{\text{Ball}}$ is the unit ball in $\mathbb{R}^n$ corresponding to the underlying norm. For 1-norm and ∞-norm, this ball is a polytope. For Euclidean norm and other norms, the balls have to be approximated by polytopes. There are methods to approximate Euclidean norm ball by zonotopes. The approximation can be made arbitrarily precise by increasing the zonotope order [24]. We denote the polytope corresponding to the unit ball in $\mathbb{R}^n$ by $\mathbb{P}_{\text{Ball}} = \{x \in \mathbb{R}^n | H_{\text{Ball}}x \leq h_{\text{Ball}}\}$. We use Theorem 1 to convert (19) into a linear program. The norm used in the examples in this paper is ∞-norm.

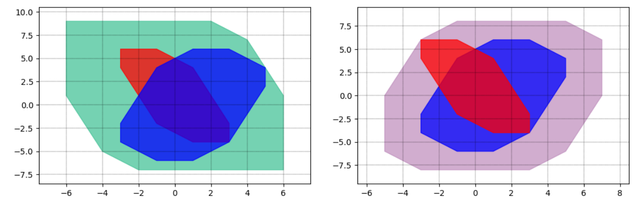

Fig. 3. Example 3: Computing Hausdorff distance between zonotopes: [left] minimal ball added to $\mathbb{Z}_x$ to contain $\mathbb{Z}_{y^*}$ [Right] minimal ball added to $\mathbb{Z}_{y^*}$ to contain $\mathbb{Z}_x$. The Hausdorff distance is upper bounded by $\max(2,3)=3$.

Corollary 5: Given two AH-polytopes $\mathbb{X}_1, \mathbb{X}_2 \subset \mathbb{R}^n$, where $\mathbb{X}_i = \bar{x}_i + X_i\mathbb{P}_i, \mathbb{P}_i = \{z \in \mathbb{R}^{n_i} | H_i z \leq h_i\}, i = 1,2, X_i \in \mathbb{R}^{n\times n_i}, \bar{x}_i \in \mathbb{R}^n$, the following linear program provides an upper-bound for their Hausdorff distance:

$$
\begin{array}{ll}
\min & D \\
\text{s.t.} & \Lambda_1 H_1 = H_2\Gamma_1, \Lambda_2 H_1 = H_{\text{Ball}}\Gamma_2, \\
& \Lambda_1 h_1 \leq h_2 + H_2\beta_1, \Lambda_2 h_1 \leq D h_{\text{Ball}} + H_{\text{Ball}}\beta_2, \\
& \Lambda_3 H_2 = H_1\Gamma_3, \Lambda_3 H_4 = H_{\text{Ball}}\Gamma_4, \\
& \Lambda_3 h_2 \leq h_2 + H_1\beta_3, \Lambda_4 h_2 \leq D h_{\text{Ball}} + H_{\text{Ball}}\beta_4 \\
& \Lambda_i \geq 0, i = 1,2,3,4, \\
& \bar{x}_2 - X_2\beta_1 - \beta_2 = \bar{x}_1, X_2\Gamma_1 + \Gamma_2 = X_1, \\
& \bar{x}_1 - X_1\beta_3 - \beta_4 = \bar{x}_2, X_1\Gamma_3 + \Gamma_4 = X_2.
\end{array} \tag{20}
$$

The tightness of the upper bound provided by Corollary 5 is as good as the necessity of Theorem 1. We have $\mathbb{X}_1 = \mathbb{X}_2$ if and only if (20) returns 0. Note that if lines corresponding to $\mathbb{X}_2 \subseteq \mathbb{X}_1 \oplus D\mathbb{P}_{\text{Ball}}$ in (20) are removed, we arrive in an upper-bound for the directed distance D_{12}, which becomes zero if and only if $\mathbb{X}_1 \subseteq \mathbb{X}_2$.

Example 3: Consider the zonotopes in Example 1. If the last column of Y is dropped (see Fig. 1 [Right]). Using Corollary 5, we obtain the following upper bounds $D_{12} \leq 2$ and $D_{21} \leq 3$, so $d_H(\mathbb{Z}_y^*, \mathbb{Z}_x) \leq 3$. Augmented zonotopes $\mathbb{Z}_x \oplus D_{12}\mathbb{B}_2$ and $\mathbb{Z}_y^* \oplus D_{21}\mathbb{B}_2$ are shown in Fig. 3.

C. Orthogonal Projections

Computing orthogonal projections of polytopes is a central problem in many applications such as computing feasible regions of model predictive controllers. The exact computation of projections requires variable elimination, which is a costly procedure. Vertex-based projections are more convenient to implement but do not scale well in high dimensions. Moreover, it is often preferred to have the H-polytope rather than a V-polytope of the projection. Not only the projection procedure is computationally intense, but the number of hyperplanes in the projected polytope itself may be inevitably too large, in particular when a high-dimensional polytopes are projected into low dimensional spaces.

We use the results of this paper to introduce an output-sensitive inner-approximation alternative to orthogonal projection that is based on a single optimization problem. Consider the set $\mathbb{F} = \{(x,u) \in \mathbb{R}^{n+m} | Hx + Fu \leq g\}$. The projection into x-space is given by $\mathbb{F}_{\text{proj}} = \{x | \exists u \in \mathbb{R}^m, (x,u) \in \mathbb{F}\}$. We desire to find H_x and h_x in $\mathbb{X} = \{x \in \mathbb{R}^n | H_x x \leq h_x\}$ such that $\mathbb{X} \subseteq \mathbb{F}_{\text{proj}}$ and $d_H(\mathbb{X}, \mathbb{F}_{\text{proj}})$ is minimized. The choice of the number of rows in H_x is made by the user - we expect $d_H(\mathbb{X}, \mathbb{F}_{\text{proj}})$ to decrease with the number of rows. While there has been theoretical results on the quality of approximating projections of high dimensional polytopes by polytopes with controlled number of facets [25], to the best of our knowledge, the following approach is unique in the respect that it is optimization-based, and provides a guaranteed upper bound on the Hausdorff distance between the approximated projected polytope and the actual one, which does not need to be computed.

We parameterize $\mathbb{X}$ by $\bar{x} + \{x \in \mathbb{R}^n | H_x x \leq \underline{1}\}$. Note that by design, $\mathbb{X}$ contains $\bar{x}$. The optimal values in H_x are given by the following optimization problem:

$$
\begin{array}{rl}
H_x = & \arg\min \epsilon \\
\text{s.t.} & \mathbb{F} = \{(x,u) \in \mathbb{R}^{n+Nm} | Hx + Fu \leq g\} \\
& \mathbb{X} = \{x \in \mathbb{R}^n | H_x x \leq \underline{1}\}, \\
& \bar{x} + \mathbb{X} \subseteq (I,0)\mathbb{F}, (I,0)\mathbb{F} \subseteq \bar{x} + \mathbb{X} \oplus \epsilon\mathbb{B}
\end{array} \tag{21}
$$

Corollary 6: Given $\mathbb{F} = \{(x,u) \in \mathbb{R}^{n+m} | Hx + Fu \leq g\}$, let $\mathbb{F}_{\text{proj}} = \{x | \exists u \in \mathbb{R}^m, (x,u) \in \mathbb{F}\}$. Consider the following optimization problem:

$$
\begin{array}{rl}
H_x = & \arg\min \epsilon, \\
\text{s.t.} & \Lambda_0 H_x = H + F\Gamma, \Lambda_0 \underline{1} \leq g - H\bar{x} + F\beta_u, \\
& \Lambda_1 H = H_x X_1, \Lambda_1 F = H_x X_2, \\
& \Lambda_2 H = H_B - H_{\text{Ball}} X_1, \Lambda_2 F = -H_{\text{Ball}} X_2, \\
& \Lambda_1 g \leq h_{\text{Ball}} - H_x \beta_x, \\
& \Lambda_2 g \leq \epsilon h_{\text{Ball}} - H_{\text{Ball}}\bar{x} + H_{\text{Ball}}\beta_x, \\
& \Lambda_0, \Lambda_1, \Lambda_2 \geq 0,
\end{array} \tag{22}
$$

where $\{x \in \mathbb{R}^n | H_{\text{Ball}} x \leq h_{\text{Ball}}\}$ is the unit ball of the underlying norm. Let H_x be a feasible solution with cost ϵ, and $\mathbb{X} = \bar{x} + \{x \in \mathbb{R}^n | H_x x \leq \underline{1}\}$. Then we have $\mathbb{X} \subseteq \mathbb{F}_{\text{proj}}$ and $d_H(\mathbb{X}, \mathbb{F}_{\text{proj}}) \leq \epsilon$.

The optimization problem (22) has some bilinear terms, therefore it is difficult to solve it to global optimality. Nevertheless, we can use local optimization methods using successive linear programming to find suboptimal solutions.

Example 4: In this example, we compute an inner-approximation of an MPC feasible set. Consider the linear system $x_{t+1} = Ax_t + Bu_t$, where

$$
A = \begin{pmatrix} 1 & 0.1 \\ -0.1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}.
$$

We impose hard constraints $x \in [-1,1]^2$ and $u \in [-1,1]$. We wish to compute the set of states that can be steered into the origin in $N = 20$ steps, while satisfying the box constraints. We have: $\mathbb{F} = \{x_0, u_0, \cdots, u_{N-1} | x_N = 0\}$, and $\mathbb{F}_{\text{proj}} = \{x_0 | \exists u_0, \cdots, u_{N-1}, \text{such that } x_N = 0\}$, where $x_n = A^n x_0 + \sum_{\tau=0}^{n} A^{n-\tau-1} B u_\tau$. We need to find $\mathbb{F}_{proj}$, which is provided by projecting $N+2 = 22$ dimensional polytope of joint state and control sequence space into 2-dimensional state-space. By writing the constraints above, $\mathbb{F}$ has 128 hyperplanes (some may be redundant).

We may use Fourier-Motzkin elimination method to find the exact projection. Its computation is costly as 20 variables are eliminated, and at each iteration, many linear programs are required to remove redundant hyperplanes. The method

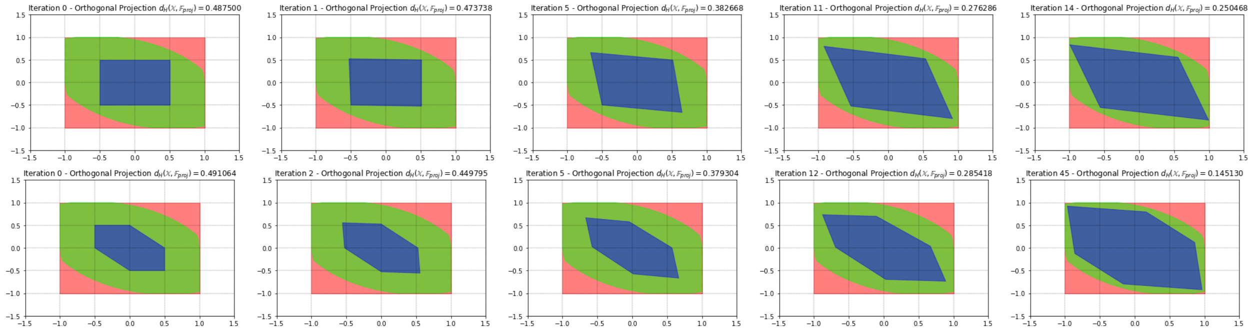

Fig. 4. Example 4: Optimization-based iterative orthogonal projection from $\mathbb{R}^{22}$ to $\mathbb{R}^2$ to obtain the feasible set of a model predictive controller. The state box constraint is red, the exact $\mathbb{F}_{\text{proj}}$ is green, and the inner-approximation $\mathbb{X}$ is blue. [Top] 4 hyperplanes [Bottom] 6 hyperplanes.

returns the exact $\mathbb{F}_{\text{proj}}$ with 28 irreducible hyperplanes. As an alternative, we use the method described in Proposition 6. We consider two initializations: H_x with 4 hyperplanes (a box), and H_x with 6 hyperplanes. We consider maximum step size of 0.05 in each entry of matrix variables. We let $\bar{x} = 0$. Snapshots of the gradient decent iterations are shown in Fig. 4. It is observed that we are closely inner-approximate the MPC feasible set using user-defined number of hyperplanes, while not encountering any potential exponential blow up due to vertex elimination or the number of required hyperplanes. Note that the reported Hausdorff distances are the values of ϵ in (22), which are guaranteed to be upper-bounds and are computed without explicitly knowing $\mathbb{F}_{\text{proj}}$. ∎

V. Conclusion and Future Work

We provided sufficient conditions for containment of a polytope inside another polytope, where both polytopes are represented by affine transformations of hyperplane-represented polytopes. The method does not require computing the hyperplanes of the affine transformations of the polytopes, which can be computationally prohibitive. Instead, the encoding provides a set of linear constraints with size growing linearly in the problem size. We provided interpretations for the sufficiency, the conditions for necessity, and demonstrated the usefulness on problems encountered in control applications. Future work will focus more deeply on applications to verification and control of hybrid systems.

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