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Finite-Time Consensus Problems for Networks of Dynamic Agents

Long Wang and Feng Xiao

***Abstract*—In this note, we discuss finite-time state consensus problems for multi-agent systems and present one framework for constructing effective distributed protocols, which are continuous state feedbacks. By employing the theory of finite-time stability, we investigate both the bidirectional interaction case and the unidirectional interaction case, and prove that if the sum of time intervals, in which the interaction topology is connected, is sufficiently large, the proposed protocols will solve the finite-time consensus problems.**

***Index Terms*—Distributed control, finite-time consensus, multi-agent systems, time-varying topologies.**

I. Introduction

The consensus theory of multi-agent systems has emerged as a challenging new area of research in recent years [1]. It is a basic and fundamental research topic in decentralized control of networks of dynamic agents and has attracted great attention of researchers. This is partly due to its broad applications in cooperative control of unmanned air vehicles, formation control of mobile robots, control of communication networks, design of sensor networks, flocking of social insects, swarm-based computing, etc.

In the analysis of consensus problems, convergence rate is an important performance indicator for the proposed consensus protocol. It was shown that the second smallest eigenvalue of the interaction graph Laplacian, called *algebraic connectivity*, quantifies the convergence rate under the typical protocol presented in [2]. To get high convergence rate, several researchers endeavored to find proper interaction graphs with larger algebraic connectivity. In [3], Kim and Mesbahi considered the problem of finding the best vertex positional configuration so that the algebraic connectivity of the associated interaction graph is maximized, where the weight for the edge between any two vertices was assumed to be a function of the distance between the two corresponding agents. In [4], Xiao and Boyd considered and solved the problem of weight design by using semi-definite convex programming, and the convergence rate is also increased. Simulation results showed that the interaction graph with small-world property possesses large algebraic connectivity [5]. However, it can be observed that all those efforts were to choose proper interaction graphs, but not to find available protocols with high performance. On the other hand, although by maximizing the algebraic connectivity of interaction graph, we can increase convergence rate with respect to the linear protocol proposed in [2], the state consensus can never occur in finite time. In practice, it is often required that the consensus be reached in a finite time. And there are a number of situations, in which finite-time convergence is

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desirable, such as when the control accuracy is crucial. Besides faster convergence, other advantages of finite-time consensus include better disturbance rejection and robustness against uncertainties [6].

Now, we summarize some results, closely related to finite-time consensus. In [7], Sundaram and Hadjicostis gave a finite-time consensus algorithm for discrete-time systems, by using the property of minimal polynomial of weight matrix. However, the design of this protocol depends on the global information of interaction topology, though the decentralized calculation of minimal polynomial was discussed. In [8], Cortés proposed two finite-time consensus protocols for continuous-time systems, under either of which, the differential equations of the overall systems have *discontinuous* right-hand sides.

The main contribution of this note is threefold. First, it provides an effective way to construct finite-time consensus protocols, which are continuous state feedbacks and bridge the gap between asymptotical consensus protocols and discontinuous finite-time consensus protocols. This technique has its potential application in the finite-time stability analysis. Second, the theory of finite-time Lyapunov stability is successfully applied to the finite-time consensus theory and the analysis method developed in this note is of itself interest. This is partly motivated by the work of [9], in which continuous finite-time differential equations were investigated for the design of fast accurate controllers of dynamical systems, and partly by the results of finite-time stability of autonomous systems [6]. Third, the requirement for the possible interaction topologies is not restrictive. They may be time-varying weighted directed graphs. Moreover, within this framework, we can construct another class of finite-time consensus protocols, which are valid for the case when the interaction topology has a spanning tree.

This note is organized as follows. In Section II, the problem is formulated. Convergence results are presented in Section III. Concluding remarks are stated in Section IV. Finally, in the Appendix, the preliminary graph notions and necessary lemmas in deriving the main results are assembled.

Notations: Let $\mathcal{I}_n = \{1, 2, \cdots, n\}$, $\mathbf{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$, $\mathrm{span}(\mathbf{1}) = \{\boldsymbol{\xi} \in \mathbb{R}^n : \boldsymbol{\xi} = r\mathbf{1}, r \in \mathbb{R}\}$, and let $\|\cdot\|_\infty$ denote the l_∞-norm on $\mathbb{R}^n$. If $\boldsymbol{\omega} = [\omega_1, \omega_2, \ldots, \omega_n]^T \in \mathbb{R}^n$, then $\mathrm{diag}(\boldsymbol{\omega})$ denotes the diagonal matrix with ω_i as the (i, i) entry.

II. Problem Formulation

The distributed dynamic system studied in this note consists of n autonomous agents, e.g. particles or robots, labeled 1 through n. All these agents share a common state space $\mathbb{R}$. The state of agent i is denoted by x_i, $i \in \mathcal{I}_n$, and the column vector $[x_1, x_2, \ldots, x_n]^T$ is denoted by $\boldsymbol{x}$.

Suppose that agent i is with the following dynamics

$$\dot{x}_i(t) = u_i(t) \tag{1}$$

where $u_i(t)$ is the *protocol* to be designed.

In this multi-agent system, each agent can communicate with some other agents, which are defined as its neighbors. Protocol u_i is a state feedback, which is designed based on the local information received by agent i from its neighbors. We use weighted directed graph $\mathcal{G}(\boldsymbol{A})$ to represent the interaction topology, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Vertex v_i represents agent i, the existence of edge $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(\boldsymbol{A}))$ implies the existence of an available information channel from agent i to agent j, and edge weight a_{ji} represents the reliability of the corresponding information channel. The index set of the neighbors of agent i is denoted by $\mathcal{N}_i = \{j : (v_j, v_i) \in \mathcal{E}(\mathcal{G}(\boldsymbol{A})), j \neq i\}$. Notice that the diagonal entries of $\boldsymbol{A}$ are assumed to be zeros in this note.

Given protocol u_i, $i \in \mathcal{I}_n$, u_i or this multi-agent system is said to *solve a consensus problem asymptotically* if for any given initial states, there exists an asymptotical stable equilibrium x^* such that for any j, $x_j(t) \to x^*$ as $t \to \infty$, and it is said to *solve a finite-time consensus problem* if it solves a consensus problem, and given any initial states, there exist a finite time t^* and a real number x^* such that $x_j(t) = x^*$ for all $t \geq t^*$ and all $j \in \mathcal{I}_n$. If the final consensus state is a function of initial states, namely, $x_j(t) \to \chi(\boldsymbol{x}(0))$ as $t \to \infty$ for all $j \in \mathcal{I}_n$, where $\chi : \mathbb{R}^n \to \mathbb{R}$ is a function, then we say that it *solves the χ-consensus problem* [2]. Specially, if $\chi(\boldsymbol{x}(0)) = \sum_{j=1}^n x_j(0)/n$, the system is said to *solve the average-consensus problem.*

With the above preparations, we are now in a position to present the class of consensus protocols, which will be shown to solve finite-time consensus problems:[1]

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}\mathrm{sign}(x_j - x_i)|x_j - x_i|^{\alpha_{ij}} \tag{2}$$

where $0 < \alpha_{ij} < 1$, $|\cdot|$ represents the absolute value, and $\mathrm{sign}(\cdot)$ is the sign function defined by

$$\mathrm{sign}(r) = \begin{cases} 1, & r > 0 \\ 0, & r = 0 \\ -1, & r < 0 \end{cases}$$

For simplicity, function $\mathrm{sign}(r)|r|^\alpha$ is denoted by $\mathrm{sig}(r)^\alpha$.

Remark: $\mathrm{sig}(r)^\alpha$, $\alpha > 0$, is a continuous function with respect to r, which leads to the continuity of protocol (2) with respect to state variables (provided that the interaction topology is time-invariant). Protocol (2) represents a class of protocols. If we set $\alpha_{ij} = 1$, $i, j \in \mathcal{I}_n$, in the above protocol, then it will become the typical linear consensus protocol studied in [2] and [10], and in this case it solves an asymptotical consensus problem provided that $\mathcal{G}(\boldsymbol{A})$ has a spanning tree. If we set $\alpha_{ij} = 0$, it will become discontinuous. The discontinuous protocols were studied by Cortés [8].

In the next section, we will investigate the mathematical conditions that guarantee protocol (2) to solve a finite-time consensus problem. The remaining part of this section lists some base properties of system (1) under protocol (2).

By Peano's Existence Theorem and Extension Theorem [11], it can be obtained that

Property 1: If the interaction topology is time-invariant, then under protocol (2), differential equations (1) have continuous right-hand side, and there exists at least one solution on $[0, \infty)$ for any initial state $\boldsymbol{x}(0)$. Furthermore, $\|\boldsymbol{x}(t)\|_\infty$ is non-increasing and $\|\boldsymbol{x}(t)\|_\infty \leq \|\boldsymbol{x}(0)\|_\infty$ for all $t \geq 0$.

One notices that protocol (2) is not Lipschitz at some points. As all solutions will reach subspace $\mathrm{span}(\mathbf{1})$ in finite time, there exists nonuniqueness of solutions in backwards time. This, of course, violates the uniqueness condition for solutions of Lipschitz differential equations.

The following property characterizes the equilibrium point set of system (1).

Property 2: Under protocol (2), the equilibrium point set of the differential equations $\dot{x}_i = u_i$, $i \in \mathcal{I}_n$, is $\mathrm{span}(\mathbf{1})$, provided that $\mathcal{G}(\boldsymbol{A})$ has a spanning tree.

Proof: Since $\mathcal{G}(\boldsymbol{A})$ has a spanning tree, there exists at most one agent, whose neighbor set is empty. Let $\boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_n]^T$ satisfy the equations $u_i|_{\boldsymbol{x}=\boldsymbol{\xi}} = 0$, $i \in \mathcal{I}_n$. We prove $\boldsymbol{\xi} \in \mathrm{span}(\mathbf{1})$ by contradiction. Assume that $\min_i \xi_i \neq \max_i \xi_i$. Let $\mathcal{M} = \{j : \xi_j = \min_i \xi_i\}$ and let $\mathcal{H} = \{j : \xi_j = \max_i \xi_i\}$. Then $\mathcal{M} \cap \mathcal{H} = \phi$. Suppose that vertex v_j is the root vertex. Then $j \notin \mathcal{M}$ or $j \notin \mathcal{H}$ or both hold. Without loss of generality, assume that $j \notin \mathcal{M}$. For any $k \in \mathcal{M}$, there exists a path $v_j = v_{i_1}, v_{i_2}, \ldots, v_{i_s} = v_k$, connecting v_j to v_k. Let l be the number, such that $i_l \in \mathcal{M}$ but $i_{l-1} \notin \mathcal{M}$. Obviously, $\xi_{i_{l-1}} > \xi_{i_l} = \min_i \xi_i$ and $i_{l-1} \in \mathcal{N}_{i_l}$ and thus $u_{i_l} > 0$, which is a contradiction. therefore $\min_i \xi_i = \max_i \xi_i$, namely, $\boldsymbol{\xi} \in \mathrm{span}(\mathbf{1})$. ■

[1]If $\mathcal{N}_i = \emptyset$, then $u_i = 0$.

III. Convergence Results

A. Networks Under Time-Varying Bidirectional Interaction Topologies

In many practical situations, the information channel between any two agents may not be always available because of failure of physical devices, limited sensing range or existence of obstacles. Therefore, it is reasonable to assume that the interaction topology is dynamically changing.

To describe the time-varying topology, suppose that a_{ij} are piecewise constant right continuous functions with respect to time and take values in a finite set, such that $a_{ii}(t)=0$ and $a_{ij}(t)\geq 0$ for all i,j,t. In addition, for protocol (2), parameters α_{ij} can also be changing to reflect the reliability of information as weights a_{ij}. We also assume that all α_{ij} are piecewise-constant right continuous functions and take values in a finite set.

The following theorem is one of the main results.

Theorem 1: Suppose that $\mathcal{G}(\boldsymbol{A}(t))$ is undirected and the sum of time-intervals, in which $\mathcal{G}(\boldsymbol{A}(t))$ is connected, is sufficient large. If $\alpha_{ij}(t)=\alpha_{ji}(t)$ and $0<\alpha_{ij}(t)<1$ for all i,j,t, then protocol (2) solves the finite-time average-consensus problem.

Proof: Step 1: We first study the case when the interaction topology $\mathcal{G}(\boldsymbol{A}(t))$ and protocol parameters $\alpha_{ij}(t)$ are time-invariant.

Since $a_{ij}=a_{ji}$ and $\alpha_{ij}=\alpha_{ji}$ for all $i,j\in\mathcal{I}_n$, it can be obtained that

$$\sum_{i=1}^{n}\dot{x}_i(t)=0. \tag{3}$$

Let $x^*=(1/n)\sum_{i=1}^{n}x_i(t)$, $x_i(t)=x^*+\delta_i(t)$, and let $\boldsymbol{\delta}(t)=[\delta_1(t),\delta_2(t),\ldots,\delta_n(t)]^T$, where $\boldsymbol{\delta}(t)$ is referred to as the *group disagreement vector* in [2]. It follows from (3) that x^* is time-invariant, and thus $\dot{\delta}_i(t)=\dot{x}_i(t)$. Choose Lyapunov candidate

$$V_1\left(\boldsymbol{\delta}(t)\right)=\frac{1}{2}\sum_{i=1}^{n}\delta_i^2(t).$$

It will be shown that $V_1(t)$ satisfies the conditions of Lemma 3 (in the Appendix) for some K and α and hence $V_1(t)$ will reach zero in finite time, which yields that all agents' states will reach an agreement in finite time. Moreover, the final state is x^*, namely, the average of the initial states.

Differentiate $V_1(t)$ with respect to t

$$\begin{aligned}\frac{dV_1(t)}{dt}&=\sum_{i=1}^{n}\delta_i(t)\dot{\delta}_i(t)\\&=\sum_{i=1}^{n}\delta_i(t)\sum_{j=1}^{n}a_{ij}\mathrm{sig}(\delta_j-\delta_i)^{\alpha_{ij}} \qquad (4)\\&=\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(\delta_i-\delta_j)\mathrm{sig}(\delta_j-\delta_i)^{\alpha_{ij}}\\&=-\frac{1}{2}\sum_{i,j=1}^{n}\left(a_{ij}^{\frac{2}{1+\alpha_0}}\left((\delta_j-\delta_i)^2\right)^{\frac{1+\alpha_{ij}}{1+\alpha_0}}\right)^{\frac{1+\alpha_0}{2}} \qquad (5)\end{aligned}$$

where $\alpha_0=\max_{ij}\alpha_{ij}$ and (4) follows from that $\delta_j-\delta_i=x_j-x_i$. Clearly, $1/2<(1+\alpha_0)/2<1$.

Suppose that $V_1(t)\neq 0$. By Lemma 2 (in the Appendix)

$$\begin{aligned}\frac{dV_1(t)}{dt}&\leq-\frac{1}{2}\left(\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}\left((\delta_i-\delta_j)^2\right)^{\frac{1+\alpha_{ij}}{1+\alpha_0}}\right)^{\frac{1+\alpha_0}{2}}\\&=-\frac{1}{2}\left(\frac{\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}\left((\delta_i-\delta_j)^2\right)^{\frac{1+\alpha_{ij}}{1+\alpha_0}}}{\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}(\delta_i-\delta_j)^2}\times\frac{\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}(\delta_i-\delta_j)^2}{V_1(t)}V_1(t)\right)^{\frac{1+\alpha_0}{2.}}\end{aligned} \tag{6}$$

Next, we estimate the lower bounds of the first two quantities between the big brackets in (6).

By Property 1, $\max_i x_i-\min_i x_i$ is non-increasing and thus for any $i,j\in\mathcal{I}_n$, $|\delta_i(t)-\delta_j(t)|\leq\max_k x_k(t)-\min_k x_k(t)\leq\max_k x_k(0)-\min_k x_k(0)$. Let

$$K_1=\frac{1}{\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}}\times\min_{\substack{i,j\in\mathcal{I}_n\\a_{ij}\neq 0}}a_{ij}^{\frac{2}{1+\alpha_0}}\left(\max_k x_k(0)-\min_k x_k(0)\right)^{2\left(\frac{1+\alpha_{ij}}{1+\alpha_0}-1\right)}$$

which is positive, and let $(i_0,j_0)=\arg\max_{\substack{i,j\in\mathcal{I}_n\\a_{ij}\not\equiv 0}}(\delta_i-\delta_j)^2$. It follows from $(1+\alpha_{ij})/(1+\alpha_0)-1\leq 0$ that

$$\begin{aligned}K_1&\leq\frac{a_{i_0j_0}^{\frac{2}{1+\alpha_0}}\left((\delta_{i_0}-\delta_{j_0})^2\right)^{\frac{1+\alpha_{i_0j_0}}{1+\alpha_0}}}{\left(\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}\right)(\delta_{i_0}-\delta_{j_0})^2}\\&\leq\frac{\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}\left((\delta_i-\delta_j)^2\right)^{\frac{1+\alpha_{ij}}{1+\alpha_0}}}{\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}(\delta_i-\delta_j)^2}.\end{aligned} \tag{7}$$

Now consider the second quantity. Denote $\boldsymbol{B}=[a_{ij}^{2/(1+\alpha_0)}]\in\mathbb{R}^{n\times n}$. Then by Lemma 1 (in the Appendix) $\sum_{i,j=1}^{n}a_{ij}^{2/(1+\alpha_0)}(\delta_i-\delta_j)^2=2\boldsymbol{\delta}^TL(\boldsymbol{B})\boldsymbol{\delta}$, and noticing that $\boldsymbol{\delta}\perp\mathbf{1}$, we have

$$\frac{\sum_{i,j=1}^{n}a_{ij}^{\frac{2}{1+\alpha_0}}(\delta_i-\delta_j)^2}{V_1(t)}=\frac{2\boldsymbol{\delta}^TL(\boldsymbol{B})\boldsymbol{\delta}}{\frac{1}{2}\boldsymbol{\delta}^T\boldsymbol{\delta}}\geq 4\lambda_2\left(L(\boldsymbol{B})\right)>0$$

where $L(\boldsymbol{B})$ is the graph Laplacian of $\mathcal{G}(\boldsymbol{B})$ and $\lambda_2(L(\boldsymbol{B}))$ is the algebraic connectivity of $\mathcal{G}(\boldsymbol{B})$. Recall (6), and we have that

$$\frac{dV_1(t)}{dt}\leq-\frac{1}{2}\left(4K_1\lambda_2\left(L(\boldsymbol{B})\right)\right)^{\frac{1+\alpha_0}{2}}V_1(t)^{\frac{1+\alpha_0}{2}}.$$

Thus, by Lemma 3, system (1) solves the finite-time average-consensus problem.

Step 2: Now consider the general case. Clearly x^* is also time-invariant. Let $\alpha_0=\max_{i,j,t}\alpha_{ij}(t)$ and also consider the Lyapunov candidate $V_1(t)$. By (5), $V_1(t)$ is nonincreasing. At any time t when $\mathcal{G}(\boldsymbol{A}(t))$ is connected, we have an estimation $K_1(t)$, which is dependent on the interaction topology, protocol parameters and the initial states. Since all possible $\boldsymbol{A}(t)$ and $\alpha_{ij}(t)$ are finite

$$K_2=\min\left\{K_1(t)\lambda_2\left(L\left(\boldsymbol{B}(t)\right)\right):\mathcal{G}\left(\boldsymbol{A}(t)\right)\text{ is connected}\right\}$$

exists and is larger than 0. Then if $\mathcal{G}(\boldsymbol{A}(t))$ is connected

$$\frac{dV_1(t)}{dt}\leq-2^{\alpha_0}K_2^{\frac{1+\alpha_0}{2}}V_1(t)^{\frac{1+\alpha_0}{2}}.$$

By Comparison Principle of differential equations, $V_1(t)$ will reach zero in finite time if the sum of time intervals, in which $\mathcal{G}(\boldsymbol{A}(t))$ is connected, is larger than $t^*=2^{1-\alpha_0}V_1(0)^{(1-\alpha_0)/2}/((1-$

$\alpha_0)K_2^{(1+\alpha_0)/2}$), and the switching system solves the finite-time average-consensus problem. ■

In the above analysis, the interaction topology is assumed to be undirected and time-varying. The following results are focused on the fixed topology case, in which we will prove the validity of protocol (2) under more relaxable topology conditions.

B. Networks Under Time-Invariant Unidirectional Interaction Topologies

The notion of detail-balanced graph is needed in this section. Weighted directed graph $\mathcal{G}(\boldsymbol{A})$ is said to satisfy the *detail-balanced condition* in weights if there exist some scalars $\omega_i > 0, i \in \mathcal{I}_n$, such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_n$ [12].

The next result characterizes the consensus property of protocol (2) under the time-invariant interaction topology.

Theorem 2: If the interaction topology $\mathcal{G}(\boldsymbol{A})$ is time-invariant and contains a spanning tree, each strongly connected component of the topology is detail-balanced, and parameters $\alpha_{ij} = \alpha_{ji}$ and $0 < \alpha_{ij} < 1$ for all $i, j \in \mathcal{I}_n$, then protocol (2) solves a finite-time consensus problem.

Proof: This theorem is proved by induction through the following three steps.

Step 1: Suppose that $\mathcal{G}(\boldsymbol{A})$ is strongly connected and detail-balanced.

In this case, there exists a positive column vector $\boldsymbol{\omega} = [\omega_1, \omega_2, \dots, \omega_n]^T$, satisfying the property that $\omega_i a_{ij} = \omega_j a_{ji}$ for all i, j. It is easy to check that $\mathrm{diag}(\boldsymbol{\omega})\boldsymbol{A}$ is symmetric and $\sum_{i=1}^n \omega_i \dot{x}_i(t) = 0$, which yields that $x^* = (1/\sum_{i=1}^n \omega_i)\sum_{i=1}^n \omega_i x_i(t)$ is time-invariant. Let $\delta_i(t) = x_i(t) - x^*$ and then $\boldsymbol{w}^T\boldsymbol{\delta}(t) = 0$. Consider Lyapunov candidate $V_2(t) = (1/2)\sum_{i=1}^n \omega_i \delta_i(t)^2$ and differentiate it with respect to time

$$\frac{dV_2(t)}{dt} = -\frac{1}{2}\sum_{i,j=1}^n \omega_i a_{ij}|\delta_j - \delta_i|^{1+\alpha_{ij}}.$$

It can be shown that

$$\min_{\boldsymbol{\delta}\perp\boldsymbol{\omega}} \frac{\boldsymbol{\delta}^T L(\boldsymbol{B})\boldsymbol{\delta}}{\boldsymbol{\delta}^T \mathrm{diag}(\boldsymbol{\omega})\boldsymbol{\delta}} > 0$$

where $\boldsymbol{B} = [(\omega_i a_{ij})^{2/(1+\alpha_0)}]$, $\alpha_0 = \max_{ij}\alpha_{ij}$. With the same argument as in the proof of Theorem 1, there exists $K_3 > 0$ such that

$$\frac{dV_2(t)}{dt} \le -K_3 V_2(t)^{\frac{1+\alpha_0}{2}}$$

And thus this system solves a finite-time consensus problem and the final state is $(1/\sum_{i=1}^n \omega_i)\sum_{i=1}^n \omega_i x_i(0)$, which can be viewed as a *weighted-average-consensus function*.

Step 2: Suppose that the system consists of m, $m < n$, followers,[2] and the local interaction topology among the followers is strongly connected and detail-balanced.

Without loss of generality, suppose that agents $1, 2, \dots, m$ are the followers and by the result of the first step, the leaders will reach an agreement in finite time t^*. Denote the final state of leaders by x^*, let $\alpha_0 = \max_{ij}\alpha_{ij}$, and let $\bar{\boldsymbol{\omega}} = [\omega_1, \omega_2, \dots, \omega_m]^T$ such that $\omega_i > 0$ and $\omega_i a_{ij} = \omega_j a_{ji}$ for $i, j \in \mathcal{I}_m$. Rewrite protocol (2) as

$$u_i = \sum_{j=1}^m a_{ij}\mathrm{sig}(x_j - x_i)^{\alpha_{ij}} + \sum_{j=m+1}^n a_{ij}\mathrm{sig}(x^* - x_i)^{\alpha_{ij}}, \quad t \ge t^*, i \in \mathcal{I}_m.$$

[2]In this note, if the interaction topology $\mathcal{G}(A)$ contains spanning trees, then all root agents are defined as leaders and other agents are defined as followers.

Let $\delta_i = x_i - x^*$, $i \in \mathcal{I}_n$. Then $\delta_i \equiv 0$ for $i = m+1, m+2, \dots, n$, and

$$\dot{\delta}_i = \dot{x}_i = \sum_{j=1}^m a_{ij}\mathrm{sig}(\delta_j - \delta_i)^{\alpha_{ij}} - \sum_{j=m+1}^n a_{ij}\mathrm{sig}(\delta_i)^{\alpha_{ij}}, \ i \in \mathcal{I}_m.$$

Also consider the Lyapunov candidate $V_2(t)$, $t \ge t^*$, with ω_i, $i = m+1, \dots, n$, all being zeros. We have that

$$\begin{aligned}\frac{dV_2(t)}{dt} &= \sum_{i=1}^m \omega_i\delta_i\left(\sum_{j=1}^m a_{ij}\mathrm{sig}(\delta_j - \delta_i)^{\alpha_{ij}} - \sum_{j=m+1}^n a_{ij}\mathrm{sig}(\delta_i)^{\alpha_{ij}}\right)\\
&= -\frac{1}{2}\sum_{i,j=1}^m\left((\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}|\delta_j - \delta_i|^{\frac{2(1+\alpha_{ij})}{1+\alpha_0}}\right)^{\frac{1+\alpha_0}{2}}\\
&\quad - \sum_{i=1}^m\sum_{j=m+1}^n\left((\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}|\delta_i|^{\frac{2(1+\alpha_{ij})}{1+\alpha_0}}\right)^{\frac{1+\alpha_0}{2}}\\
&\le -\frac{1}{2}\left(\sum_{i,j=1}^m(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}|\delta_j - \delta_i|^{\frac{2(1+\alpha_{ij})}{1+\alpha_0}} + 2\sum_{i=1}^m\sum_{j=m+1}^n(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}|\delta_i|^{\frac{2(1+\alpha_{ij})}{1+\alpha_0}}\right)^{\frac{1+\alpha_0}{2}}\end{aligned}$$

where the last inequality follows from Lemma 2.

For simplicity, let

$$G_1(t) = \sum_{i,j=1}^m(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}|\delta_j - \delta_i|^{\frac{2(1+\alpha_{ij})}{1+\alpha_0}} + 2\sum_{i=1}^m\sum_{j=m+1}^n(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}|\delta_i|^{\frac{2(1+\alpha_{ij})}{1+\alpha_0}}$$

and

$$G_2(t) = \sum_{i,j=1}^m(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}(\delta_j - \delta_i)^2 + 2\sum_{i=1}^m\sum_{j=m+1}^n(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}\delta_i^2.$$

With the same arguments as in the proof of inequality (7), if $G_2(t) \ne 0$, then

$$\frac{G_1(t)}{G_2(t)} \ge \frac{1}{\sum_{i,j=1}^m(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}} + 2\sum_{i=1}^m\sum_{j=m+1}^n(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}} \times \min_{\substack{i\in\mathcal{I}_m, j\in\mathcal{I}_n\\ a_{ij}\ne 0}}(\omega_i a_{ij})^{\frac{2}{1+\alpha_0}}\left(\max_k x_k(0) - \min_k x_k(0)\right)^{2\left(\frac{1+\alpha_{ij}}{1+\alpha_0}-1\right)}.$$

Denote the quantity on the right side of the above equation by K_4.

Let $\bar{\boldsymbol{B}} = [(\omega_i a_{ij})^{2/(1+\alpha_0)}]_{1\le i,j\le m}$ and let $\bar{\boldsymbol{b}} = \sum_{j=m+1}^n[(\omega_1 a_{1j})^{2/(1+\alpha_0)}, (\omega_2 a_{2j})^{2/(1+\alpha_0)}, \dots, (\omega_m a_{mj})^{2/(1+\alpha_0)}]^T$. It follows from Corollary 2 (in the Appendix) that $L(\bar{\boldsymbol{B}}) + \mathrm{diag}(\bar{\boldsymbol{b}})$ is positive definite. Denote its smallest eigenvalue by $\lambda_1(L(\bar{\boldsymbol{B}}) + \mathrm{diag}(\bar{\boldsymbol{b}}))$. Then

$$\begin{aligned}\frac{G_2(t)}{V_2(t)} &= \frac{2[\delta_1, \delta_2, \dots, \delta_m]\left(L(\bar{\boldsymbol{B}}) + \mathrm{diag}(\bar{\boldsymbol{b}})\right)[\delta_1, \delta_2, \dots, \delta_m]^T}{\frac{1}{2}[\delta_1, \delta_2, \dots, \delta_m]\mathrm{diag}(\bar{\boldsymbol{\omega}})[\delta_1, \delta_2, \dots, \delta_m]^T}\\
&\ge \frac{4}{\omega_0}\lambda_1\left(L(\bar{\boldsymbol{B}}) + \mathrm{diag}(\bar{\boldsymbol{b}})\right)\end{aligned}$$

where $\omega_0 = \max_{i\in\mathcal{I}_m}\omega_i$. Thus

$$\frac{dV_2(t)}{dt} \le -2^{\alpha_0}\left(\frac{1}{\omega_0}K_4\lambda_1\left(L(\bar{\boldsymbol{B}}) + \mathrm{diag}(\bar{\boldsymbol{b}})\right)\right)^{\frac{1+\alpha_0}{2}} V_2(t)^{\frac{1+\alpha_0}{2}}.$$

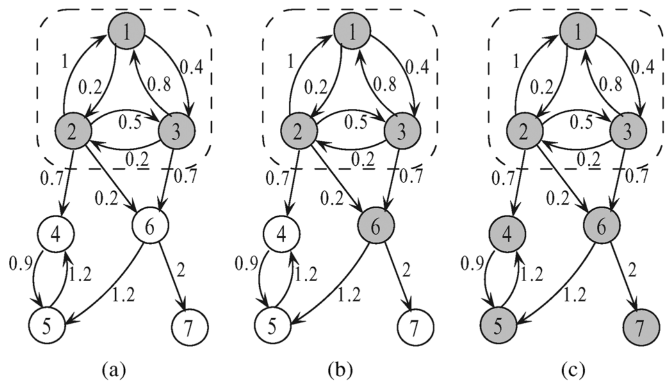

Fig. 1. Demonstration of the proof of Theorem 2.

By Lemma 3, $V_2(t)$ will reach zero in finite time, which implies that followers' states will reach an agreement with the leaders' in finite time.

Step 3: Finally, we investigate the general case. Consider the dynamics of followers in a common strongly connected component of $\mathcal{G}(\boldsymbol{A})$ and suppose that all the other agents that can be connected to those followers have already reached an agreement on states with each other. By the same discussion as in the second step, we can conclude that those followers will agree with the leaders in finite time. By induction, the system solves a finite-time consensus problem. The illustration of the proof is shown in Fig. 1. ■

Suppose that the above directed graph (Fig. 1) is the underlying interaction topology, where the numbers in the circles are the indices of agents, and the numbers near the edges are the weights. Clearly, the vertices in the dashed square are the leaders and their local interaction topology is detail-balanced. After a period of time, the states of them will agree. Paint the agents in consensus state in grey, as shown in (a). After that, by the conclusion of the second step, agent 6 will agree with the leaders in a finite time, as shown in (b). And finally, all agents will reach an agreement on states, as shown in (c).

C. Discussions

First, it can be seen from the proof of Theorem 2 that the condition in Theorem 1 can be relaxed in several ways. In fact, $\mathcal{G}(\boldsymbol{A}(t))$ is not necessarily undirected. This condition can be replaced by that $\mathcal{G}(\boldsymbol{A}(t))$ is always detail-balanced with a time-invariant positive column vector $\boldsymbol{\omega}$ such that $\mathrm{diag}(\boldsymbol{\omega})^T\boldsymbol{A}(t)$ is symmetric.

Second, Lyapunov candidates $V_1(t)$ and $V_2(t)$ adopted in the proof of the main results measure how much agents' states differ from each other. Undoubtedly, larger initial value of $V_1(t)$ or $V_2(t)$ will result in longer convergence time. This fact is also reflected in the estimations about K_1, K_2, K_3, and K_4. And it is not surprising that larger algebraic connectivity of $\mathcal{G}(\boldsymbol{B})$ will result in shorter convergence time to some extent. Furthermore, by some straightforward arguments, it can be shown that smaller.[3] α_{ij} can lead to a higher convergence rate when agents' states differ a little from each other, and larger α_{ij} can lead to a higher convergence rate when agents states differ a lot from each other.

Finally, within the proposed framework for the design and analysis of finite-time consensus protocols, we can prove that the following protocol is valid when the interaction topology is time-invariant and contains a spanning tree

$$u_i = \mathrm{sig}\left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\right)^{\alpha_i}, \quad i \in \mathcal{I}_n, 0 < \alpha_i < 1.$$

[3]Its explicit threshold was given in [13].

IV. Conclusion

We have discussed the finite-time consensus problems for multi-agent systems and presented the framework for constructing effective continuous finite-time consensus protocols. The work of this note is the first step toward the finite-time consensus analysis of multi-agent systems, and there are still some other interesting and important topics needed to be addressed. For example, does protocol (2) still work when the interaction topology has a spanning tree but without the detail-balanced property as stated in Theorem 2? If the system evolves under switching topology with communication time-delays, do there exist similar results? These problems are currently under investigations.

Appendix I
Related Notions in Graph Theory

Directed graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G})$, where $\mathcal{E}(\mathcal{G}) \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(\mathcal{G})\}$. A *path* in directed graph $\mathcal{G}$ is a finite sequence $v_{i_1}, \ldots, v_{i_j}$ of vertices such that $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{V}(\mathcal{G})$ for $k = 1, 2, \ldots, j-1$. If there exists a special vertex, which can be connected to any other vertex though pathes, then $\mathcal{G}$ is said to have a *spanning tree* and the special vertex is called the *root* vertex. If between every pair of distinct vertices v_i and v_j, there exists a path that begins at v_i and ends at v_j, then directed graph $\mathcal{G}$ is said to be *strongly connected*. A *subgraph* $\mathcal{G}_s$ of directed graph $\mathcal{G}$ is a directed graph such that the vertex set $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$ and the edge set $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$. If for any $v_i, v_j \in \mathcal{V}(\mathcal{G}_s)$, $(v_i, v_j) \in \mathcal{E}(\mathcal{G}_s) \iff (v_i, v_j) \in \mathcal{E}(\mathcal{G})$, $\mathcal{G}_s$ is called an *induced subgraph*. In this case, $\mathcal{G}_s$ is also said to be induced by $\mathcal{V}(\mathcal{G}_s)$. A *strongly connected component* of a directed graph is an induced subgraph that is maximal, subject to being strongly connected. Since any subgraph consisting of one vertex is strongly connected, it follows that each vertex lies in a strongly connected component, and therefore the strongly connected components of a directed graph partition its vertices.

Weighted directed graph $\mathcal{G}(\boldsymbol{A})$ is a directed graph $\mathcal{G}$ together with a nonnegative square matrix $\boldsymbol{A} = [a_{ij}]$, such that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \iff a_{ji} > 0$, where a_{ji} is called the *weight* of edge (v_i, v_j) and $\boldsymbol{A}$ is called *weight matrix*. If $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \iff (v_j, v_i) \in \mathcal{E}(\mathcal{G})$, $\mathcal{G}$ is called to be *undirected*. If $\boldsymbol{A}^T = \boldsymbol{A}$, then $\mathcal{G}(\boldsymbol{A})$ is called a weighted undirected graph. Clearly, for undirected graph, having a spanning tree is equivalent to being strongly connected. Those strongly connected undirected graphs are usually called to be *connected*.

Appendix II
Preliminary Lemmas

To establish the main results, the following several lemmas are needed.

Lemma 1 ([2], [10], [13]): Denote the *graph Laplacian* of $\mathcal{G}(\boldsymbol{A})$ by $L(\boldsymbol{A}) = [l_{ij}]$, where $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ and

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases}.$$

We have that:

(i) 0 is an eigenvalue of $L(\boldsymbol{A})$ and $\mathbf{1}$ is the associated eigenvector;

(ii) if $\mathcal{G}(\boldsymbol{A})$ has a spanning tree, then eigenvalue 0 is algebraically simple and all other eigenvalues are with positive real parts;

(iii) if $\mathcal{G}(\boldsymbol{A})$ is strongly connected, then there exists a positive column vector $\boldsymbol{\omega} \in \mathbb{R}^n$ such that $\boldsymbol{w}^T L(\boldsymbol{A}) = 0$;

if $\mathcal{G}(\boldsymbol{A})$ is undirected and connected, then $L(\boldsymbol{A})$ has the following properties:

(iv) $\boldsymbol{\xi}^T L(\boldsymbol{A})\boldsymbol{\xi} = (1/2)\sum_{i,j=1}^{n} a_{ij}(\xi_j - \xi_i)^2$ for any $\boldsymbol{\xi} = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{R}^n$, and therefore $L(\boldsymbol{A})$ is positive semi-definite;

(v) the second smallest eigenvalue of $L(\boldsymbol{A})$, which is called the *algebraic connectivity* of $\mathcal{G}(\boldsymbol{A})$ and denoted by $\lambda_2(L(\boldsymbol{A}))$, is larger than zero;

(vi) the algebraic connectivity of $\mathcal{G}(\boldsymbol{A})$ is equal to $\min_{\boldsymbol{\xi}\not\in 0, \mathbf{1}^T\boldsymbol{\xi}=0} \boldsymbol{\xi}^T L(\boldsymbol{A})\boldsymbol{\xi}/\boldsymbol{\xi}^T\boldsymbol{\xi}$, and thus, if $\mathbf{1}^T\boldsymbol{\xi} = 0$, then $\boldsymbol{\xi}^T L(\boldsymbol{A})\boldsymbol{\xi} \geq \lambda_2(L(\boldsymbol{A}))\boldsymbol{\xi}^T\boldsymbol{\xi}$.

Corollary 1: Suppose $\mathcal{G}(\boldsymbol{A})$ is strongly connected and $\boldsymbol{\omega}$ is positive column vector such that $\boldsymbol{w}^T L(\boldsymbol{A}) = 0$. Then $\text{diag}(\boldsymbol{\omega})L(\boldsymbol{A}) + L(\boldsymbol{A})^T \text{diag}(\boldsymbol{\omega})$ is the graph Laplacian of the undirected weighted graph $\mathcal{G}(\text{diag}(\boldsymbol{\omega})\boldsymbol{A} + \boldsymbol{A}^T \text{diag}(\boldsymbol{\omega}))$. And it is positive semi-definite, 0 is its algebraically simple eigenvalue and $\mathbf{1}$ is the associated eigenvector.

Corollary 2: For any nonnegative column vector $\boldsymbol{b}$ with compatible dimensions, if $\boldsymbol{b} \neq 0$ and $\mathcal{G}(\boldsymbol{A})$ is undirected and connected, then $L(\boldsymbol{A}) + \text{diag}(\boldsymbol{b})$ is positive definite.

Lemma 3 ([14], Lemma 1): If $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $0 < p \leq 1$, then

$$\left(\sum_{i=1}^{n} \xi_i\right)^p \leq \sum_{i=1}^{n} \xi_i^p.$$

Lemma 3 (cf. [6], Theorem 1): Suppose that function $V(t) : [0, \infty) \to [0, \infty)$ is differentiable (the derivative of $V(t)$ at 0 is in fact its right derivative) and

$$\frac{dV(t)}{dt} \leq -KV(t)^\alpha$$

where $K > 0$ and $0 < \alpha < 1$. Then $V(t)$ will reach zero at finite time $t^* \leq V(0)^{1-\alpha}/(K(1-\alpha))$ and $V(t) = 0$ for all $t \geq t^*$.

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Characterization of Majorization Monotone Quantum Dynamics

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***Abstract*—In this technical note, the author studies the dynamics of open quantum system in Markovian environment. The author gives necessary and sufficient conditions for such dynamics to be majorization monotone, which are those dynamics always mixing the states.**

***Index Terms*—Majorization, open quantum dynamics.**

I. INTRODUCTION

In the last two decades, control theory has been applied to an increasingly wide number of problems in physics and chemistry whose dynamics are governed by the time-dependent Schrödinger equation (TDSE), including control of chemical reactions [1]–[8], state-to-state population transfer [9]–[13], shaped wavepackets [14], NMR spin dynamics [15]–[19], Bose-Einstein condensation [20]–[22], quantum computing [23]–[27], oriented rotational wavepackets [28]–[30], etc. More recently, there has been vigorous effort in studying the control of open quantum systems which are governed by Lindblad equations, where the central object is the density matrix, rather than the wave function [31]–[37]. The Lindblad equation is an extension of the TDSE that allows for the inclusion of dissipative processes. In this article, the author will study those dynamics governed by Lindblad equations and give necessary and sufficient conditions for the dynamics to be majorization monotone, which are those dynamics always mixing the states. This study suggests that majorization may serve as time arrow under these dynamics in analog to entropy in second law of thermal dynamics.

The article is organized as follows: Section II gives a brief introduction to majorization; Section III gives the definition of majorization monotone quantum dynamics; then in Section IV, necessary and sufficient conditions for majorization monotone quantum dynamics are given.

II. BRIEF INTRODUCTION TO MAJORIZATION

In this section, the author gives a brief introduction on majorization, most stuff in this section can be found in the second chapter of Bhatia's book [42].

For a vector $x = (x_1, \ldots, x_n)^T$ in $\mathbb{R}^n$, we denote by $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)^T$ a permutation of x so that $x_i^\downarrow \geq x_j^\downarrow$ if $i < j$, where $1 \leq i, j \leq n$.

Definition 1 (Majorization): A vector $x \in \mathbb{R}^n$ is majorized by a vector $y \in \mathbb{R}^n$ (denoted by $x \prec y$), if

$$\sum_{j=1}^{d} x_j^\downarrow \leq \sum_{j=1}^{d} y_j^\downarrow \tag{1}$$

for $d = 1, \ldots, n-1$, and the inequality holds with equality when $d = n$.

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