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LOGIC OF MANY-SORTED THEORIES

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1. Introduction. Certain axiomatic systems involve more than one category of fundamental objects; for example, points, lines, and planes in geometry; individuals, classes of individuals, etc. in the theory of types or in predicate calculi of orders higher than one. It is natural to use variables of different kinds with their ranges respectively restricted to different categories of objects, and to assume as substructure the usual quantification theory (the restricted predicate calculus) for each of the various kinds of variables together with the usual theory of truth functions for the formulas of the system. An axiomatic theory set up in this manner will be called many-sorted.[1] We shall refer to the theory of truth functions and quantifiers in it as its (many-sorted) elementary logic,[2] and call the primitive symbols and axioms (including axiom schemata) the proper primitive symbols and proper axioms of the system. Our purpose in this paper is to investigate the many-sorted systems and their elementary logics.

Among the proper primitive symbols of a many-sorted[3] system T_n ($n = 2, \cdots, \omega$) there may be included symbols of some or all of the following kinds: (1) predicates denoting the properties and relations treated in the system; (2) functors denoting the functions treated in the system; (3) constant names for certain objects of the system. We may either take as primitive or define a predicate denoting the identity relation in T_n. In any case, it is usually desirable to include in T_n the usual theory of identity for the objects of the system. We shall assume that T_n contains the usual theory of identity[4] as a part. Then we know we can introduce descriptions by contextual definitions such as

$$—(\imath x)\phi x— \textit{ for } (\exists y)((x)(x = y \equiv \phi x) \,.\, —y—).$$

But we also know that once we have descriptions at hand, we can make use of additional predicates to get rid of the primitive names[5] and functors.[6] On this ground we shall assume, for simplicity, that the systems T_n which we shall consider contain neither names nor functors. In other words, we shall assume that the primitive symbols of T_n are just the truth-functional connectives, the quantifiers, the brackets, and the predicates.

We can describe each theory T_n as follows. There is at least one predicate. There are variables of different kinds: x_1, y_1, z_1, $\cdots$ (variables of the first

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[1] A term introduced in [2] as a translation of the word *mehrsortig* used in [1]. I wish to thank Professor Alonzo Church for first calling my attention to [1].

[2] A (one-sorted) elementary logic is the usual complete theory of truth functions and quantifiers (e.g. as it is formulated on p. 88 of [3]) with its formulas as specified in a one-sorted axiomatic system.

[3] n refers to the number of kinds of variables in the system. We assume that n may be 2, 3, 4, $\cdots$, or ω. T_ω will be a theory with denumerably many kinds of variables.

[4] That amounts to the law of reflexivity and the principle of substitutivity for the variables of the system.

[5] See [3], pp. 149–152.

[6] See [4], vol. 1, pp. 460–462.

kind); x_2, y_2, z_2, $\cdots$; $\cdots$; x_n, y_n, z_n, $\cdots$. Each k-placed ($k = 1, 2, \cdots$) predicate with its places filled up by variables of the *proper* kinds is a formula (an atomic formula); and if ϕ and ψ are any formulas and α is a variable of any kind, then $(\alpha)\phi$ and $\phi \downarrow \psi$ are formulas.[7] In general, for each place of a predicate more than one kind of variable may be proper. However, to simplify our considerations, we shall always assume that each place of every predicate is to be filled up by one and only one kind of variable. Free and bound variables and occurrences will be understood as having been defined in the usual manner. A statement is a formula containing no free variables. A closure of a formula ϕ is a statement formed from ϕ by prefixing distinct general quantifiers to all the free variables of ϕ in an arbitrary order. We write $\vdash\phi$ to mean that the closures of ϕ are theorems. Then a many-sorted elementary logic L_n is determined in the following manner. The formulas of L_n (ϕ, ψ, ϕ', etc.) are just those given above and the theorems of L_n are defined by the principles 1_n–5_n:[8]

1_n. If ϕ is a truth-functional tautology, then $\vdash\phi$.

2_n. $\vdash(\alpha)(\phi \supset \psi) \supset ((\alpha)\phi \supset (\alpha)\psi)$.

3_n. If α is not free in ϕ, then $\vdash\phi \supset (\alpha)\phi$.

4_n. If α and α' are variables of the same kind, and ϕ' is like ϕ except for containing free occurrences of α' whenever ϕ contains free occurrences of α, then $\vdash(\alpha)\phi \supset \phi'$.

5_n. If $\phi \supset \psi$ and ϕ are theorems of L_n, so is ψ.

By adding certain proper axioms (or also axiom schemata) to L_n, we obtain a system T_n.

As an alternative way, we may also formulate a system involving several categories of fundamental objects by using merely one kind of variables which have the sum of all the categories as their range of values. The simplest way to bring in the distinction of categories is to introduce n one-place predicates S_1, S_2, $\cdots$, S_n such that x belongs to the i-th category if and only if $S_i(x)$. We can then set up a one-sorted theory $T_1^{(n)}$ corresponding to T_n in the following manner. In $T_1^{(n)}$ the atomic formulas are determined by the predicates of T_n plus S_1, $\cdots$, S_n with their places all filled up by general variables. Formulas, etc. can be defined in $T_1^{(n)}$ in the usual way. And $T_1^{(n)}$ contains a usual one-sorted elementary logic L_1 determined by[9] five principles 1_1–5_1 which are similar to 1_n–5_n but are concerned with formulas and variables of $T_1^{(n)}$. Then we understand by the elementary logic $L_1^{(n)}$ the system obtained from L_1 by adding the following additional principle:

6_1. For every i ($i = 1, \cdots, n$), $(\exists\alpha)S_i(\alpha)$ is a theorem.

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[7] We shall follow [3] in using Greek letters as syntactical variables for expressions. The letters ϕ, ψ, χ and their accented and subscripted variants will be used to refer to formulas, and the letters α, β, γ, δ, and their variants to variables (cf. [3], p. 75). Indeed, in formulating the system T_n, we are following closely the pattern set up in [3]. We shall omit the corners used in [3].

[8] These principles answer to *100, *102–*105 of [3]. A principle answering to *101 can be dropped just as in L_1; see [3], p. 89.

[9] Compare the preceding footnote. This time ϕ, ψ refer to formulas of T_1^n, and α, α' to variables of T_1^n.

And we introduce a rule for translating between statements of L_n and those of $L_1^{(n)}$:

RT. A statement ϕ' in $L_1^{(n)}$ and a statement ϕ in L_n are translations of each other if and only if ϕ' is the result obtained from ϕ by substituting simultaneously, for each expression of the form $(x_i)(\text{—}x_i\text{—})$ in ϕ $(i = 1, \cdots, n)$, an expression of the form $(x)(S_i(x) \supset (\text{—}x\text{—}))$ (with the understanding that differferent variables in ϕ are replaced by different variables in ϕ').

By using this rule, we see that every statement of L_n has a translation in $L_1^{(n)}$, and some (although not all) statements of $L_1^{(n)}$ have translations in L_n. In particular, the proper axioms of T_n all have translations in $L_1^{(n)}$, and $T_1^{(n)}$ is just $L_1^{(n)}$ plus the translations of these proper axioms of T_n.

The main purpose of the present paper is to investigate the relations between any T_n (or L_n) and its corresponding $T_1^{(n)}$ (or $L_1^{(n)}$). By a comparative study of L_n and L_1, we shall also indicate that many known metamathematical results about a usual elementary logic L_1 have counterparts for L_n.

Preparatory to stating the results of this paper, we first make a few historical remarks. In [**5**] Herbrand states a theorem which amounts to the following (see [**5**], p. 64):

(I) A statement of any system T_n is provable in T_n if and only if its translation in the corresponding system $T_1^{(n)}$ is provable in $T_1^{(n)}$.

However, the proof he gives there is inadequate, failing to take into account that there are certain reasonings which can be carried out in $L_1^{(n)}$ but not in L_n. In [**1**], Arnold Schmidt points this out and devotes his paper to giving a careful proof of the theorem. Then Langford puts forward in [**2**] (a review of [**1**]) the problem whether the following is true in general:

(II) If a system T_n is consistent, then the corresponding system $T_1^{(n)}$ is also consistent.

This, as Professor Bernays has communicated to us in conversation, can be answered positively by the following argument. Obviously there exists a statement ϕ of $T_1^{(n)}$ such that both ϕ and $\sim\phi$ are translatable into T_n. Assume that $T_1^{(n)}$ is inconsistent. Then every statement in $T_1^{(n)}$ is provable, and therefore ϕ and $\sim\phi$ are both provable in $T_1^{(n)}$. Hence, by (I), their translations ψ and $\sim\psi$ according to RT are both provable in T_n, Hence, T_n is inconsistent.

In this paper, we shall first indicate that in L_n we can easily prove counterparts of theorems in L_1 and that about L_n we can prove counterparts of the metamathematical theorems of completeness, etc. about L_1. We shall then show that from these the theorem (I) (and therewith the theorem (II)) follows. We shall also show that, conversely, given (I) and the metamathematical theorems about L_1, we can prove certain similar theorems about L_n as corollaries. In passing, we may mention here that the following converse of (II) is obviously true:

(III) If $T_1^{(n)}$ is consistent, then T_n is.

It would then seem that, merely for the purpose of proving (I), we could dispense with Schmidt's rather involved arguments. However, Schmidt actually proves in his paper the following more interesting theorem:

(IV) Given a statement of T_n and a proof for it in T_n, there is an effective way of finding a proof in $T_1^{(n)}$ for its translation in $T_1^{(n)}$; and, conversely, given a statement of $T_1^{(n)}$ which has a translation in T_n, and given a proof for it in $T_1^{(n)}$, there is an effective way of finding a proof in T_n for its translation in T_n.

Although we can prove (I) by considering the completeness of L_1 and L_n, it does not seem possible to prove (IV) similarly, for (IV) depends on syntactical considerations about the proofs in $L_1^{(n)}$ and L_n. We shall, following a suggestion of Professor Bernays, give a simpler alternative proof for (IV) by application of Herbrand's theorem. (See [4], vol. 2, pp. 149–163.)

From the results (I), (II), and (III), we see that for purposes of questions concerned with the consistency of T_n, we may consider $T_1^{(n)}$ instead which is simpler in that it contains only one kind of variables. However, $T_1^{(n)}$ is more complicated than T_n in that it contains new predicates $S_1, S_2, \cdots, S_n$. We contend that in many cases, given a system T_n, we can find a corresponding system which contains only one kind of variables and no new predicates, and which can serve the same purposes both for the study of consistency questions and for the development of theory. Whether we can find such a corresponding system depends on whether we can express membership in the different categories by the following means: general variables (whose range of value is the sum of all the special domains), the quantifiers and truth-functional connectives, the brackets, plus the predicate letters of the given many-sorted theory re-construed as having their argument places filled up by general variables. It seems that in most cases we can. The simple theory of types will afford an example of T_n for which we can give a corresponding theory relatively consistent to it, with one kind of variable and no new predicates, and essentially as rich. This example is of special interest if we want to compare the theory of types with Zermelo's set theory.

2. The many-sorted elementary logics L_n. In this section we shall sketch how theorems in and about L_n can be proved in a similar manner to theorems in and about L_1.

We first observe that in L_n we can prove from 1_n–5_n all the usual quantificational theorems of L_1 for each kind of variables. For example, we can prove in L_n all theorems which fall under principles notationally the same as *110–*171 of [3] with nearly the same proofs.[10]

Thus[11] we can define prenex normal form and Skolem normal form for L_n and prove the laws of them for L_n just as for L_1. We can prove the deduction theorem and the consistency theorem for L_n just as for L_1.

[10] In 4_n and theorems answering to cases of *134 of [3] which are concerned with the relation between free and bound variables, we need the condition that the variables are of the same kind.

As we come to the proofs, the only places where we need take somewhat seriously into consideration the different kinds of variables are in the proofs of the generalized modus ponens answering to *111 of [3] and the principles of generalization answering to *112 of [3]. But in both cases, proofs for these principles in L_n are easily obtainable by slightly changing the proofs of *111 and *112 in [3]. In particular, in L_n we can prove $\vdash (\alpha)(\beta)\phi \equiv (\beta)(\alpha)\phi$, no matter whether α, β are of the same kind or not.

[11] Cf. [6], pp. 59–61, pp. 68–72, pp. 45–46, pp. 42–44.

Likewise we can define valid and satisfiable formulas of L_n just as those of L_1:

2.1. A value assignment for a predicate or its corresponding atomic formula $fx_{n_1}^{(k_1)} \cdots x_{n_j}^{(k_j)}$ of L_n over a set of n non-empty domains is a function from the predicate or its corresponding atomic formula to a j-adic relation whose i-th place takes the individuals of the n_i-th domain.

2.2. A formula ϕ of L_n with no free variables is valid in a particular set of n non-empty domains if all value assignments for all the atomic formulas occurring in ϕ are such that, under the normal interpretation of the truth-functional connectives and quantifiers, ϕ becomes true. ϕ is valid if it is valid in all sets of n non-empty domains.

2.3. ϕ is satisfiable in a particular set of n non-empty domains if $\sim\phi$ is not valid in it. ϕ is satisfiable if it is satisfiable in some set of n non-empty domains.

With these definitions we can prove the following theorems[12] for L_n just as for L_1.

2.4. If $\vdash\phi$ in L_n, then the closure of ϕ is valid.

2.5. If the closure of ϕ is valid in a set of n denumerable domains, then $\vdash\phi$ in L_n.

2.6. If the closure of ϕ is valid, then $\vdash\phi$ in L_n.

2.7. If $\phi_1, \phi_2, \cdots$ are statements of L_n and the system T_n obtained from L_n by adding $\phi_1, \phi_2, \cdots$ as proper axioms is consistent, then $\phi_1, \phi_2, \cdots$ are simultaneously satisfiable in a set of n denumerable domains.

We merely outline a proof for the following theorem 2.8 from which 2.5 follows immediately.

2.8. If the statement $\sim\phi$ is not provable in L_n, then ϕ is satisfiable in a set of n denumerable domains.

Suppose that the variables of the p-th kind ($p = 1, \cdots, n$) in L_n are $v_p^{(1)}$, $v_p^{(2)}, \cdots$ and ϕ is the statement $(v_{n_1}^{(k_1)}) \cdots (v_{n_t}^{(k_t)})(\exists v_{m_1}^{(j_1)}) \cdots (\exists v_{m_s}^{(j_s)})\psi(v_{n_1}^{(k_1)}, \cdots, v_{n_t}^{(k_t)}; v_{m_1}^{(j_1)}, \cdots, v_{m_s}^{(j_s)})$. Let ψ_i ($i = 1, 2, \cdots$) be $\psi(v_{n_1}^{\tau(i,1)}, \cdots, v_{n_t}^{\tau(i,t)}; v_{m_1}^{\sigma(i,1)}, \cdots, v_{m_s}^{\sigma(i,s)})$, where $(\tau(i, 1), \cdots, \tau(i, t))$ is the i-th term of the sequence of all the t-tuples of positive integers ordered according to the sum of the t integers and, for those with the same sum, lexicographically; and the sequence of the s-tuples $(\sigma(i, 1), \cdots, \sigma(i, s))(i = 1, 2, \cdots)$ is such that, if among $m_1, \cdots, m_s$, m_{r_1} is identical with $m_{r_2}, \cdots, m_{r_q}$ and with no others, then $\sigma(1, r_1), \cdots, \sigma(1, r_q), \sigma(2, r_1), \cdots, \sigma(2, r_q), \sigma(3, r_1), \cdots$ coincide with $1, \cdots, q, (q + 1), \cdots, 2q, (2q + 1), \cdots$.

In order to prove 2.8, we observe first that we can prove just as in the case of L_1 the following two propositions.

2.8.1. If $\sim\phi$ is not provable in L_n, then none of $\sim\psi_1, \sim\psi_1 \vee \sim\psi_2, \cdots$ is a tautology.

2.8.2. If none of $\sim\psi_1, \sim\psi_1 \vee \sim\psi_2, \cdots$ is a tautology, then $\psi_1, \psi_2, \cdots$ are simultaneously satisfiable.

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[12] The proof for 2.4 is easy and 2.6 follows from 2.5 as an immediate corollary. 2.7 can be proved by using arguments resembling those for 2.5. (Compare the proof of its counterpart for L_1 on pp. 357–359 of [7].) The proof for 2.5 sketched below resembles that for the completeness of L_1 (cf. [6], pp. 73–79) except for certain minor complications in connection with the ordering of variables and the assignment of truth values to atomic formulas.

Therefore, by correlating each variable $v_j^{(k)}$ in $\psi_1, \psi_2, \cdots$ with the j-th power of the k-th prime number, we can, similarly as in the case of L_1, provide a true interpretation for ϕ in the set of the domains $D_1, D_2, \cdots$ such that D_j is the set of the j-th powers of all the prime numbers. Hence, 2.8 and 2.5 can be proved.

We note in passing that we can also avoid the complications regarding the definitions of $\psi_1, \psi_2, \cdots$ and prove 2.8 more simply by treating, for any i, j, k, $v_i^{(k)}$ and $v_j^{(k)}$ as the same in our considerations. Then we can use almost completely the arguments for L_1 to give a true interpretation for ϕ in a set of n identical domains, each being the set of positive integers.

Since in many cases we want the different categories (e.g., points, lines, and planes, etc.) to be mutually exclusive, we might think that in such cases there should be no satisfying assignments with all the domains identical. However, the possibility just indicated shows that this is not the case. Indeed, it becomes clear that there is no means to express in L_n explicitly the requirement that the domains of any satisfying assignment for ϕ must be different. Such a requirement is merely one of the implicitly understood conditions which we want a normal interpretation of the theory to fulfill. But there is nothing in the definitions of the satisfying assignments of values to preclude cases where such informal conditions are not fulfilled. In a one-sorted theory we can add axioms such as $\sim(\exists x)(S_i(x) \,.\, S_j(x))$ to make the demand explicit (compare Langford [2]), because in the value assignments we insist that the truth-functional and quantificational operators retain their normal interpretations.

3. The theorem *(I)* **and the completeness of** L_n. From the completeness of L_n, we can derive the theorem (I) stated in section 1.

Let us consider a statement ϕ in T_n and its translation ϕ' in $T_1^{(n)}$. Suppose that the variables in ϕ are all among the m_1-th, $\cdots$, and the m_k-th kinds. If A is a value assignment for $\sim\phi$ in a set D of domains, then there is an associated assignment A' for $\sim\phi'$ in the sum D' of all the domains of the set D, such that $(\exists x)S_{m_1} \,.\, \cdots \,.\, (\exists x)S_{m_k}(x)$ receives the value truth and that all the other predicate letters in $\sim\phi'$ receive, for those entities of D' which belong to the proper domains of D, the same values as within A and, for all the other entities of D', receive (say) the value falsehood. Conversely, given an assignment A' for $\sim\phi'$ in a domain D' such that $(\exists x)S_{m_1}(x) \,.\, \cdots \,.\, (\exists x)S_{m_k}(x)$ receives the value truth, there is an associated A for $\sim\phi$ such that the m_i-th ($i = 1, \cdots, k$) domain consists of the things x such that $S_{m_i}(x)$ receives the value truth in A' and all the predicate letters of $\sim\phi$ receive the same values as in A'. Obviously in either case, A satisfies $\sim\phi$ if and only if A' satisfies $\sim\phi'$. Hence, we have: $\sim\phi$ is satisfiable if and only if $(\exists x)S_{m_1}(x) \,.\, \cdots \,.\, (\exists x)S_{m_k}(x) \,.\, \sim\phi'$ is. Therefore, we have:

3.1. ϕ is valid if and only if $(\exists x)S_{m_1}(x) \,.\, \cdots \,.\, (\exists x)S_{m_k}(x) \,.\supset \phi'$ is.

Therefore, we can prove:

3.2. ϕ is provable in L_n if and only if ϕ' is provable in $L_1^{(n)}$.

Proof. If ϕ is provable in L_n, then, by 2.4, it is valid. Hence, by 3.1, $(\exists x)S_{m_1}(x) \,.\, \cdots \,.\, (\exists x)S_{m_k}(x) \,.\supset \phi'$ is valid and therefore, by the completeness of L_1, provable in L_1. Hence, by 6_1, ϕ' is provable in $L_1^{(n)}$.

Conversely, if ϕ' is provable in $L_1^{(n)}$, we can assume that all the (finitely many) cases of 6_1 used in the proof for ϕ' in $L_1^{(n)}$ are among $(\exists x)S_{m_1}(x), \cdots, (\exists x)S_{m_k}(x)$, for we can so choose $m_1, \cdots, m_k$. Therefore, by the deduction theorem for L_1, $(\exists x)S_{m_1}(x) . \cdots . (\exists x)S_{m_k}(x) . \supset \phi'$ is provable in L_1 and therefore valid. Hence, by 3.1, ϕ is valid and, by 2.6, provable in L_n.

From 3.2, the theorem (I) follows immediately by the deduction theorems for L_1 and L_n. Conversely, given (I) we can also derive 3.2. Moreover, as noted in section 1, the theorem (II) stated there is a corollary of (I). Now we prove that 2.6 and 2.7 can be inferred, with the help of (I), from their corresponding theorems for L_1.

Proof of 2.6. If ϕ is valid, then by 3.1, $(\exists x)S_{m_1}(x) . \cdots . (\exists x)S_{m_k}(x) . \supset \phi'$ is valid and therefore, by the completeness of L_1, provable in L_1. Hence, by 6_1, ϕ' is provable in $L_1^{(n)}$. Hence, by 3.2, ϕ is provable in L_n.

Proof 2.7. Assume that the system T_n obtained from L_n by adding the statements $\phi_1, \phi_2, \cdots$ of L_n as proper axioms is consistent. By (II), the system $T_1^{(n)}$ corresponding to T_n is consistent. Hence, by the theorem for L_1 corresponding to the theorem 2.7 for L_n, all the axioms of $T_1^{(n)}$ are simultaneously satisfiable in a denumerable domain. But the axioms of $T_1^{(n)}$ are just those of L_1, the axioms $(\exists x)S_1(x), \cdots, (\exists x)S_n(x)$, and the translations $\phi_1', \phi_2', \cdots$ of $\phi_1, \phi_2, \cdots$. Hence we can divide the domain into n domains such that the i-th domain consists of all the individuals x such that $S_i(x)$ is true. In this way, we obtain a set of n non-empty domains each either finite or denumerable in which both ϕ_1, $\phi_2, \cdots$ and the axioms of L_n are satisfiable (compare the arguments in the proof of 3.1). Consequently, we can find a set of n denumerable domains in which T_n is satisfiable. And the proof of 2.7 is completed.

4. Proof of the theorem (IV). We may break up the theorem (IV) into two parts.

4.1. There is an effective process by which, given any proof in T_n for a statement ϕ of T_n, we can find a proof in $T_1^{(n)}$ for the translation ϕ' of ϕ in $T_1^{(n)}$.

4.2. There is an effective process by which, given any proof in $T_1^{(n)}$ for a statement ϕ' of $T_1^{(n)}$ which has a translation ϕ in T_n, we can find a proof in T_n for ϕ.

First, we prove 4.1. In the proof of ϕ, we employ only a finite number of the proper axioms of T_n. Let the conjunction of these axioms be Φ. By the deduction theorem, we have an effective process by which, given the proof of ϕ in T_n, we can find a proof of $\Phi \supset \phi$ in L_n. And if its translation $\Phi' \supset \phi'$ has a proof in $L_1^{(n)}$, then we have immediately a proof in $T_1^{(n)}$ for ϕ' by modus ponens and the proper axioms of $T_1^{(n)}$, because Φ' is the translation of the conjunction of certain proper axioms of T_n. Hence, we need only prove that there is an effective process by which, given a proof in L_n for a formula ψ of T_n, we can find a proof in $L_1^{(n)}$ for its translation ψ' in $T_1^{(n)}$.

By arguments like those used in proving *100′, *102′–*105′ in [8], we can prove as metatheorems in $L_1^{(n)}$ the translations of 1_n–5_n for each kind of variables in $L_1^{(n)}$. Since in each proof of L_n, we use only a finite number of special cases of 1_n–5_n, given any proof in L_n for a formula ψ of L_n, we have a proof for its translation ψ' in $L_1^{(n)}$ which consists of the proofs of the translations in $L_1^{(n)}$ of these special cases together with a translation in $L_1^{(n)}$ of the proof for ψ in L_n. Hence, 4.1 is proved.

The proof of 4.2 is more complex. We note that it is sufficient to prove the following theorem.

4.3. There is an effective process by which, given any proof in $L_1^{(n)}$ for a statement χ' of $T_1^{(n)}$ which has a translation χ in T_n, we can find a proof in L_n for χ.

Thus, let ϕ' be a statement of $T_1^{(n)}$ with a proof in $T_1^{(n)}$, then, by the deduction theorem for L_1, we have a proof for $\Phi' \supset \phi'$ in $L_1^{(n)}$, Φ' being the conjunction of the proper axioms of $T_1^{(n)}$ used in the given proof of ϕ'. Hence, by 4.3, we have a proof in L_n for the translation $\Phi \supset \phi$ of $\Phi' \supset \phi'$ in L_n, and thereby also a proof for ϕ in T_n.

Consequently, given 4.3, we can prove 4.2. We shall prove 4.3.

By hypothesis, a proof Δ' in $L_1^{(n)}$ is given for a statement χ' of $T_1^{(n)}$ which has a translation χ in T_n. Our problem is to find a proof Δ in L_n for the translation χ of χ' in T_n. In what follows, we shall assume that χ' has been given in such a form that its translation χ is in the prenex normal form. Accordingly, since for each variable α and each formula ψ of $T_1^{(n)}$ we can substitute $(\exists\alpha)(S_i(\alpha) \,.\, \psi)$ for $\sim(\alpha)(S_i(\alpha) \supset \sim\psi)$, each quantification in χ' is either of the form $(\alpha)(S_i(\alpha) \supset \phi)$ or of the form $(\exists\beta)(S_j(\beta) \,.\, \phi')$, where α and β are variables in $T_1^{(n)}$, ϕ and ϕ' are formulas in $T_1^{(n)}$, and i and j are among $1, \cdots, n$. Moreover, every formula $S_i(\alpha)$ occurs, if at all in χ', in one and only one context either of the form $(\alpha)(S_i(\alpha) \supset \phi)$ or of the form $(\exists\alpha)(S_i(\alpha) \,.\, \phi)$; and every variable α occurs, if at all in χ', in one unique part either of the form $(\alpha)(S_i(\alpha) \supset \phi)$ or of the form $(\exists\alpha)(S_i(\alpha) \,.\, \phi)$. Such an assumption as to the form of χ' does not restrict our result in any way, because we know that each statement of $T_1^{(n)}$ which has a translation in T_n can be converted into such a form by procedures analogous to those for transforming a statement into the prenex normal form.

Therefore, if we associate each occurrence of a variable α with the number i when there is a formula $(\alpha)(S_i(\alpha) \supset \phi)$ or a formula $(\exists\alpha)(S_i(\alpha) \,.\, \phi)$ occurring in χ', we see that each occurrence of a variable in χ' is associated with a unique number, and two occurrences of the same variable in χ' always have the same number.

Consider now the formula χ_1 obtained from χ' by dropping all parts of the forms $S_i(\alpha) \supset$ and $S_i(\alpha)\,.$, or, in other words, by replacing each quantification of the form $(\alpha)(S_i(\alpha) \supset \phi)$ by $(\alpha)\phi$, and each quantification of the form $(\exists\alpha)(S_i(\alpha) \,.\, \phi)$ by $(\exists\alpha)\phi$. We see that χ_1 no longer contains occurrences of atomic formulas of the form $S_i(\alpha)$, and that χ_1 is like the translation χ of χ' in T_n except for containing occurrences of variables (say) $x, y, \cdots, z$ which are associated with the numbers $i, j, \cdots, k$ where χ contains occurrences of $x_i, y_j, \cdots, z_k$. Moreover, χ_1 is also in the prenex normal form. From now on we understand that each occurrence of any variable in χ_1 is associated with the number which was given to its corresponding occurrence in χ'.

Let us say that an occurrence of a variable (in a proof of L_1) is associated with the proper number if its number is exactly the number for the kind of variable which is to fill up the place in question of the predicate of T_n that occurs with the variable. For example, an occurrence of α in a context $P\alpha\beta \cdots \gamma$ is said to be associated with the proper number, if α is associated with i and the first argument place of P is to be filled up by the i-th kind of variable in T_n. From this definition and the way numbers are associated with variable occurrences, we have, since χ' has a translation in T_n, the next theorem.

4.4. Each occurrence in χ_1 of any variable is associated with the proper number.

We prove another theorem.

4.5. Given the proof Δ' in $L_1^{(n)}$ for χ', we can actually write out a proof Δ_1 in L_1 for χ_1.

Proof. In Δ' each line is either a case of 1_1–4_1 or 6_1, or a consequence by 5_1 of two previous lines. Let us replace throughout Δ' all occurrences of all formulas of the form $S_i(\alpha)$ by those of formulas of the form $S_i(\alpha) \vee \sim S_i(\alpha)$. Then, in the result Δ'', each line which was a case of 6_1 becomes an easy consequence of 1_1–5_1. If we add the easily obtainable proofs for these cases of 6_1 at the top of Δ'', then we obtain a proof in L_1 for a conclusion χ'' which is like χ' except for containing occurrences of formulas of the form $S_i(\alpha) \vee \sim S_i(\alpha)$ instead of those of the form $S_i(\alpha)$. But, it is then easy to see that from a proof for χ'' in L_1, we can obtain a proof in L_1 for χ_1 by 1_1 and the principle of the substitutivity of biconditionals. Hence, we obtain a proof Δ_1 in L_1 for χ_1.

Now let us apply Herbrand's theorem (see [4], vol. 2, pp. 149–163, especially p. 158; cf. also bottom of p. 135) which for our purpose can be stated thus:

HT. There is an effective method which, for any given proof of L_1 for a statement ψ in prenex normal form, yields a new proof Π for ψ (ψ being therefore the last line of Π) whose first line is a truth-functional tautology and each of whose other lines is obtained from its immediate predecessor by applying one of the following three rules: (1) Given a formula of L_1 which has the form of an alternation (disjunction), we can replace an alternation clause $\phi\beta$ by $(\exists\alpha)\phi\alpha$ where α is an arbitrary variable; (2) Given a formula of L_1 which has the form of an alternation, we can replace an alternation clause $\phi\beta$ by $(\alpha)\phi\alpha$ where β is a variable not free in any other parts of the formula; (3) Given a formula of L_1 which has the form of an alternation, we can omit repetitions of an alternation clause.

It is easy to convince ourselves that the proof Π for ψ as specified in HT is again a proof in L_1 or, more exactly, that from Π (as given) we can easily construct a proof of L_1 with ψ as the last line. Let us refer to proofs for an arbitrary statement ψ which are of the kind as described in HT, as proofs of L_1 in the Herbrand normal form. Then the content of HT says simply that every proof of L_1 for a statement in the prenex normal form can be transformed into one in the Herbrand normal form.

By 4.5 and HT, since χ_1 is in the prenex normal form, we can actually find a proof Π of L_1 for χ_1 in the Herbrand normal form. Suppose given such a proof Π. Our problem is to construct from Π a proof Δ of L_n with χ as its last line.

As was mentioned above, each occurrence in χ_1 of any variable is associated with a definite number, which is, moreover, according to 4.4, the proper number. Using these correlations, we can now associate every occurrence in Π of any variable with a definite number in the following manner.

4.6. If the occurrence is in a line ϕ which is followed by a line ϕ', then it is associated with the same number as the corresponding occurrence of the same variable in ϕ' except for the following special cases:

4.6.1. If ϕ' is obtained from ϕ by substituting $(\alpha)\psi\alpha$ for an alternation clause $\psi\beta$ and the occurrence in ϕ is one of the variable β in the clause $\psi\beta$, then it is associated with the same number as the corresponding occurrence of the variable α in the part $\psi\alpha$ of ϕ'.

4.6.2. Similarly for the case with a particular quantification $(\exists\alpha)\psi\alpha$ in ϕ'.

4.6.3. If ϕ' is obtained from ϕ by omitting repetitions of an alternation clause ϕ_1 and the occurrence in ϕ is in some occurrence of ϕ_1, then it is associated with the same number as the corresponding occurrence in the alternation clause ϕ_1 of ϕ'.

Let us replace every occurrence in Π of a variable associated with the number i by an occurrence of a corresponding variable of the i-th kind in T_n (for instance, if an occurrence of x is associated with i in Π, replace it by an occurrence of x_i) and refer to the result as Δ_2. We easily see that the last line of Δ_2 is exactly χ, the translation of χ' in T_n. Moreover, each line of Δ_2 is a formula of T_n which is either a truth-functional tautology or follows from its immediately preceding line by a quantificationally valid rule of inference (a rule of inference derivable in L_n). Therefore, from Δ_2 we can easily construct a proof Δ of L_n for the conclusion χ.

This completes the proof of 4.3. Therefore, 4.2 and theorem (IV) (using 4.1) are all proved.

5. The simple theory of types. We consider the system P which Gödel uses in [**9**].

Roughly, P contains as primitives the truth-functional operators, the quantifiers, the membership predicate ϵ, the symbol 0 for zero, the symbol f for the successor function, and infinitely many kinds of variables: $x_1, y_1, \cdots; x_2, y_2, \cdots; \cdots$. The predicate ϵ occurs only in contexts of the form $x_n \epsilon y_{n+1}$, etc. ($n = 1, 2, \cdots$). The axioms and rules of inference of P may be stated as follows ($x_n = y_n$ standing for $(z_{n+1})(x_n \epsilon z_{n+1} \equiv y_n \epsilon z_{n+1})$).

A. The principles 1_ω–5_ω of elementary logic for the infinitely many kinds of variables.

B. Axioms for the individuals.

1. $\vdash \sim fx_1 = 0.$

2. $\vdash fx_1 = fy_1 \supset x_1 = y_1.$

3. $\vdash 0 \epsilon x_2 \,.\, (x_1)(x_1 \epsilon x_2 \supset fx_1 \epsilon x_2) \,.\supset y_1 \epsilon x_2.$

C. Principles of extensionality ($n = 1, 2, \cdots$).

$$\vdash (z_n)(z_n \epsilon x_{n+1} \equiv z_n \epsilon y_{n+1}) \supset x_{n+1} = y_{n+1}.$$

D. Principles of class existence. Let ϕ be any formula in which y_{n+1} is not free, then $\vdash (\exists y_{n+1})(x_n)(x_n \epsilon y_{n+1} \equiv \phi)$ ($n = 1, 2, \cdots$).

We want to show that if P is consistent, then the following system Q is also consistent. Q contains merely one kind of variable x, y, z, $\cdots$. In Q we can introduce different kinds of variables corresponding to those of P:

$$
\begin{array}{ll}
x = y & \text{for } (z)(x \in z \equiv y \in z). \\
t1(x) & \text{for } x = 0 \vee (\exists y)(x = fy). \\
\left.\begin{array}{ll} t(n+1)(x) & \text{for } (y)(y \in x \supset tn(y)) \,.\, \sim t1(x). \\ (x_n)\phi x_n & \text{for } (x)(tn(x) \supset \phi x). \end{array}\right\} & (n = 1, 2, \cdots).
\end{array}
$$

The axioms of Q are:

A′. The principles 1_1–5_1 of elementary logic for the variables and the formulas of the system.

B′–D′ are notationally the same as B–D of the system P.

We shall not attempt to provide a formal proof. For example, we shall retain the numeral 0 and the functor f instead of replacing them by descriptions and speak of models for them as well as those for predicates and theories. However, our arguments below, we hope, will make it clear that there is no difficulty in the way of rendering the proof more rigorous.

If the system P is consistent then, by theorem 2.7, it is satisfiable in a set of denumerably many denumerable domains. Assume that such a set $M_1 = \{D_1, D_2, \cdots\}$ is given, where D_i contains the models of the objects of the type i. Obviously D_1 must contain the models 0^*, $(f0)^*$, $(ff0)^*$, $\cdots$ of the terms 0, $f0$, $ff0$, $\cdots$. Of course D_1 may also contain other things besides them. Let E_1 be the subset $\{0^*, (f0)^*, (ff0)^*, \cdots\}$ of D_1 and F_1 be the set consisting of all members of D_1 not belonging to E_1.

Let E_2 be the subset of D_2 such that if a belongs to E_2, then every member b of a (i.e., every b such that in the model $b \in^* a$ receives the value truth) belongs to E_1 and F_2 be its complement in D_2. Similarly let E_3 be the subset of D_3 consisting of all those elements of D_3 which are subsets of E_2 and F_3 be its complement in D_3. And so on.

We delete the sets F_1, F_2, $\cdots$ from M_1 and keep merely the domains E_1, E_2, $\cdots$ together with value assignments in M_1 which merely relate to these domains. It is not hard to see that the result $M_2 = \{E_1, E_2, \cdots\}$ is again a model for P. For, as we can easily check, if M_1 satisfies the axioms of groups B–D, then M_2 also satisfies them.

Moreover, since in the system P, $\alpha \in \beta$ is meaningful only when β is of one type higher than α, we may also, for instance so choose $\in^*$ that $a \in^* b$ can be true only when a and b are of two domains E_k and E_{k+1} respectively. For, since the axioms of P only involve meaningful formulas, a model for P remains one for it when we change the truth values which $a \in^* b$ may take for a and b in other domains. Let us assume that we have given such a model M_3 for the system P.

Then, if we take the sum class K of the domains E_1, E_2, $\cdots$ of M_3 as the range of values of the variables x, y, z, $\cdots$ of the system Q and use the same relations $\in^*$ as in M_3, then we have a model for Q. Thus, the variables x_1, y_1, $\cdots$, x_2,

y_2, $\cdots$ of the various types introduced in Q by the contextual definitions can easily be seen to have the same ranges of values E_1, E_2, $\cdots$ as the variables of the system P. Hence, we obtain a model for Q because all the axioms of Q except those of group A′ remain notationally the same as in the system P, and obviously the axioms A′ are satisfied by the model. Therefore, if P is consistent, then Q is.

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