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Leaderless consensus of multi-agent systems with Lipschitz nonlinear dynamics and switching topologies

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ABSTRACT

This paper considers the leaderless consensus problem of multi-agent systems with Lipschitz nonlinearities. The communication topology is assumed to be directed and switching. Based on the property that the graph Laplacian matrix can be factored into the product of two specific matrices, the consensus problem with switching topologies is converted into a stabilization problem of a switched system with lower dimensions by performing a proper variable transformation. Then the consensus problems are solved with two different topology conditions. Firstly, with the assumption that each possible topology contains a directed spanning tree, the consensus problem is solved using the tools from stability analysis of slow switching systems. It is proved that the leaderless consensus can be achieved if the feedback gains matrix is properly designed and the average dwell time larger than a threshold. Secondly, by using common Lyapunov function based method, the consensus problem with arbitrary switching topologies is solved when each possible topology is assumed to be strongly connected and balanced. Finally, numerical simulations are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

Recently, the consensus problem of multi-agent systems has drawn great attention for its broad potential applications in many areas such as cooperative control of vehicle, unmanned air vehicle formation and flocking control [1,2]. Consensus means that all agents will reach a common state in a cooperative fashion throughout distributed controllers. It is called leaderless consensus problem if there is no specified leader in the multi-agent systems, it is called leader-following consensus problem otherwise. Many results have been obtained [3–9]. Note that, all these results are obtained with fixed communication topologies. However, in many applications, the interaction topology among agents may change dynamically. This may happen when the communication links among agents may be unreliable due to disturbance or subject to communication range limitations [10].

Motivated by this, the consensus problems with switching communication topologies have been investigated in [11–17]. Under undirected jointly connected communication topologies, [11] solved the leaderless consensus problem of linear multi-agent systems using extended Barbalat's lemma. When each possible directed topology was balanced, leaderless consensus problem of linear multi-agent systems was solve with common Lyapunov function approach [12]. In [13], by using the multiple Lyapunov function approach, the leaderless consensus problem of linear multi-agent systems was solved with the assumption that each possible topology contained a directed spanning tree. In [14], H_∞ consensus problem of linear multi-agent systems with external disturbance was investigated with slow switching topologies. Based on averaging method, [15,16] solved the leader-following consensus problem of linear multi-agent systems with jointly connected topologies. Under the assumption that each possible topology had a directed spanning tree rooted at the leader, [17] solved the leader-following consensus problem of linear multi-agent systems with switching topologies and occasionally missing control inputs.

At the same time, the consensus problems of multi-agent systems with Lipschitz nonlinearities were investigated with switching communication topologies in [18–23]. In [18], the leader-following consensus problem of nonlinear multi-agent systems was considered with undirected and jointly connected topologies. With the assumption that each possible topology contains a directed spanning tree, the leader-following consensus problem was investigated in [19] with M-matrix theory and the tools from the stability analysis of switched system. In [20], the leader-following consensus problem was investigated with jointly connected topologies using distributed adaptive protocols. In [21], when there were randomly occurring nonlinearities and uncertainties and stochastic disturbances, the leader-following consensus problem was solved in the mean square sense. With the

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assumption that each possible topology contains a directed spanning tree, [22,23] solved the leaderless consensus of the first-order multi-agent systems with Lipschitz nonlinearities.

Note that, when the directed communication topologies are assumed to be time-varying and there are Lipschitz uncertainties in system dynamics, the results of the leaderless consensus problem are obtained with the system dynamics being restricted to be first-order [22,23]. As for high-order dynamics which can include the first-order dynamics as a special case, the existing conclusions are mainly focused on the leader-following case [19–21]. Considering the fact that, for the homogeneous agents, the leaderless consensus problem can include the leader-following consensus problem as special cases, the leaderless case is more challenging than the leader-following case. There are two main reasons. First, in the leader-following case, the consensus problem can be conveniently converted into a stabilization problem of a switched system by constructing the tracking error variables. Then the stability analysis method of switched system and the M-matrix theory can be adopted for analysis directly. As for the leaderless consensus problem, this is no specified leaders and M-matrix theory is not applicable to this case due to the singularity of the Laplacian matrix of the directed topology. Second, in the leader-following case, it is requited that each possible connected (or jointly connected) topology should have the same root. However, in the leaderless case, the roots of all possible topologies are not necessarily the same. This means that the requirement of the topology in the leaderless consensus problem is quite weaker than that in leader-following case. Actually, until now, when the communication topology is assumed to be directed and switching, the leaderless consensus problem of high-order multi-agent systems with Lipschitz nonlinearities has not been solved.

Motivated by above observation, this paper aims to solve the leaderless consensus problem of high-order multi-agent systems with Lipschitz nonlinearities and directed switching topologies. By performing a special kind of matrix decomposition, the graph Laplacian matrix is factored into the product of two specific matrices. Based on this property of the Laplacian matrix and a properly performed variable transformation, the consensus problem with switching topologies is converted into a stabilization problem of a switched system with lower dimensions. Then the leaderless consensus problem is solved with following two different topology conditions. Firstly, we assume that each possible topology contains a directed spanning tree. The consensus problem is solved with restricted switching topologies. The tools from stability analysis of slow switching systems are employed for analysis. It is proved that the leaderless consensus can be achieved if the feedback gains matrix is properly designed and the average dwell time is large than a threshold. Secondly, we assume that each possible topology is strongly connected and balanced. Then the consensus problem is solved with arbitrary switching topologies using the common Lyapunov function based approach.

In summary, the main contributions of the present work are two-fold. Firstly, the system dynamics of the agents is quite general, which can include the agents with first-order dynamics as special cases. Secondly, when the topologies are assumed to be directed and switching, the leaderless consensus problem is solved under two different topology conditions.

The remainder of this paper is organized as follows. In Section 2, some preliminaries and the problem formulation are provided. In Section 3, the leaderless consensus problem is solved with restricted switching topologies. In Section 4, the leaderless consensus problem is solved with arbitrary switching topologies. In Section 5, some simulation examples are presented. Section 6 is the conclusion.

2. Preliminaries and problem formulation

2.1. Preliminaries

In this paper, following notations will be used. $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ denote the set of $n\times n$ real and complex matrices, respectively. $\otimes$ denotes the Kronecker product. For $\mu\in\mathbb{C}$, the real part is $\mathrm{Re}(\mu)$. I_n is the $n\times n$ identity matrix. $\|\cdot\|$ stands for the induced matrix 2-norm. For a square matrix A, $\lambda(A)$ denotes the eigenvalues of matrix A; $rank(A)$ denotes its rank. The inertia of a symmetric matrix A is a triplet of nonnegative integers (m,z,p) where m, z and p are respectively the number of negative, zero and positive elements of $\lambda(A)$, $\max\{\lambda(A)\}$ $(\min\{\lambda(A)\})$ denotes the largest (smallest) eigenvalue of the matrix A. $A>B$ $(A\geq B)$ means that $A-B$ is positive definite (respectively, positive semidefinite). (A,B) is said to be stabilizable if there exists a real matrix K such that $A+BK$ is Hurwitz.

A directed graph $G=(V,\mathscr{E},A)$ contains the vertex set $V=\{1,2,\dots,N\}$, the directed edges set $\mathscr{E}\subseteq\mathcal{V}\times\mathcal{V}$, the weighted adjacency matrix $\mathcal{A}=[a_{ij}]_{N\times N}$ with nonnegative elements a_{ij}. $a_{ij}=1$ if there is a directed edge between vertex i and j, $a_{ij}=0$ otherwise. The set of neighbors of i is defined as $N_i:=\{j\in V:\ a_{ij}=1\}$. A directed path is a sequence of ordered edges of the form $(i_1,i_2),(i_2,i_3),\dots$, where $i_j\in\mathcal{V}$. The Laplacian matrix of the topology $\mathcal{G}$ is defined as $\mathscr{L}=[\mathscr{L}_{ij}]_{N\times N}$, where$\mathscr{L}_{ii}=\sum_{j\neq i}a_{ij}$ and $\mathscr{L}_{ij}=-a_{ij}$. Then 0 is an eigenvalue of $\mathscr{L}$ with 1_N as the eigenvector. A directed graph is called balanced if $\sum_{j=1}^{N}a_{ij}=\sum_{j=1}^{N}a_{ji}$. A directed graph is said to have a spanning tree if there is a vertex called the root such that there is a directed path from this vertex to every other vertex. A directed graph is said be strongly connected if there is a directed path between every pair of distinct vertices.

In this paper, the communication topology is molded by a directed graph and we assume that the communication topology is time-varying. Denote $\hat{\mathcal{G}}=\{\mathcal{G}^1,\mathcal{G}^2,\dots,\mathcal{G}^p\},p\geq1$ be the set of all possible directed topologies. We define the switching signal $\sigma(t)$, where $\sigma(t)$: $[0,+\infty)\rightarrow P=\{1,2,\dots,p\}$. $0=t_0<t_1<t_2<\dots$ denote the switching instants of $\sigma(t)$. Let $\mathcal{G}^{\sigma(t)}\in\hat{\mathcal{G}}$ be the communication topology at time t. Across each time interval $[t_j,t_{j+1})$, $j\in Z$, the graph $\mathcal{G}^{\sigma(t)}$ is fixed.

Lemma 1. *(Ren and Beard [10]). Zero is a simple eigenvalue of $\mathscr{L}$ and all the other nonzero eigenvalues have positive real parts if and only if the graph $\mathcal{G}$has a directed spanning tree, i.e., $0=\lambda_1<\mathrm{Re}(\lambda_2(\mathscr{L}))\leq\dots\leq\mathrm{Re}(\lambda_N(\mathscr{L}))$.*

Lemma 2. *(Yu et al. [3]). Suppose that the graph $\mathcal{G}$ is strongly connected and balanced. Then, $\mathscr{L}+\mathscr{L}^{\mathrm{T}}$ is positive semi-definite with zero being its simple eigenvalue.*

2.2. Problem formulation

Consider a multi-agent system composed of N agents with the following identical dynamics

$$\dot{x}_i(t)=Ax_i(t)+Bu_i(t)+Df(x_i(t),t)\quad i=1,2,\dots,N,\tag{1}$$

where $x_i(t)\in\mathbb{R}^n$, $u_i(t)\in\mathbb{R}^p$are the state and the control input of the $i-$th agent , respectively. A, B and D are constant system matrices with compatible dimensions. $f:\mathbb{R}^n\times[0,\infty)\rightarrow\mathbb{R}^m$ is a continuously differentiable vector-valued function representing the nonlinearities which satisfies Lipschitz condition, i.e.,

$$\|f(x(t),t)-f(y(t),t)\|\leq\rho\|x(t)-y(t)\|,\quad\forall x,y\in\mathbb{R}^n,t\geq0,$$

where $\rho>0$ is a constant scalar.

Definition 1. The consensus of system (1) is said to be achieved with any finite initial value $x_i(0)$, if there exists a controller $u_i(t)$ such that

$\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0,\quad \forall i,j=1,2,\ldots,N.$

In order to achieve consensus, the following distributed consensus controller based on local relative states information of neighbor agents is proposed

$$u_i(t)=cK\sum_{j=1}^{N}a_{ij}^{\sigma(t)}(t)\big(x_j(t)-x_i(t)\big),\ \ i=1,2,\ldots,N, \tag{2}$$

where $K\in\mathbb{R}^{p\times n}$ is the feedback matrix to be designed, c is the coupling strength to be selected, $a_{ij}^{\sigma(t)}(t)$ is the element of the adjacency matrix $\mathcal{A}^{\sigma(t)}$ of the graph $\mathcal{G}^{\sigma(t)}$.

The closed-loop system dynamics of (1) with the controller (2) is

$$\dot{x}(t)=\big(I_N\otimes A-c\mathcal{L}^{\sigma(t)}\otimes BK\big)x(t)+(I_N\otimes D)F(x,t), \tag{3}$$

where $x(t)=[x_1(t)^T,x_2(t)^T,\ldots,x_N(t)^T]^T$, $\mathcal{L}^{\sigma(t)}\in\mathbb{R}^{N\times N}$ is the Laplacian matrix of the graph $\mathcal{G}^{\sigma(t)}$, $F(x,t)=\big[f(x_1(t),t)^T,f(x_2(t),t)^T,\ldots,f(x_N(t),t)^T\big]^T$.

Lemma 3. *(Horn and Johnson [24]). For any given $x,y\in\mathbb{R}^n$, and matrices $P>0$, D and S of appropriate dimensions, one has*

$2x^TDSy\le x^TDPD^Tx+y^TS^TP^{-1}Sy.$

3. Leaderless consensus with restricted switching topologies

In this section, the leaderless consensus problem with restricted switching topologies will be investigated. Following assumption is first introduced.

Assumption 1. In this section, we assume that each possible graph $\mathcal{G}^{\sigma(t)}\in\hat{\mathcal{G}}$ contains a directed spanning tree.

Before moving forward, the following lemma is introduced.

Lemma 4. *[25]. For a Laplacian matrix $\mathcal{L}\in\mathbb{R}^{N\times N}$ of graph $\mathcal{G}$ and a full row rank matrix $E\in\mathbb{R}^{(N-1)\times N}$ defined as*

$$E=\begin{bmatrix}1&-1&0&\cdots&0\\0&1&-1&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&1&-1\end{bmatrix}, \tag{4}$$

there exists a matrix $M\in\mathbb{R}^{N\times(N-1)}$ such that $\mathcal{L}=ME$. Furthermore, if the graph has a directed spanning tree, M is full column rank and $\mathrm{Re}(\lambda(EM))>0$.

Based on Assumption 1 and Lemma 4, we can conclude that, for each $\mathcal{L}^{\sigma(t)}$, $\sigma(t)\in\mathcal{P}$, there exists a full column rank matrix $M^{\sigma(t)}$ such that $\mathcal{L}^{\sigma(t)}=M^{\sigma(t)}E$, $\sigma(t)\in\mathcal{P}$, where E is a matrix defined in (4). The eigenvalues of $EM^{\sigma(t)}$ have positive real parts, i.e., $\mathrm{Re}\big(\lambda\big(EM^{\sigma(t)}\big)\big)>0$, $\sigma(t)\in\mathcal{P}$. Then following conclusion can be obtained.

Lemma 5. *Consider the matrix $EM^{(i)}$, $i\in\mathcal{P}$ defined above. If the graph fulfills Assumption 1, there exist a positive definite matrix $Q^{(i)}$ and a common positive scalar α_0 for $i\in\mathcal{P}$, such that*

$\big(EM^{(i)}\big)^TQ^{(i)}+Q^{(i)}EM^{(i)}>\alpha_0Q^{(i)}.$

Proof. According to Assumption 1 and Lemma 4, one has $\mathrm{Re}\big(\lambda\big(EM^{(i)}\big)\big)>0$, $i\in\mathcal{P}$. Then there exists a positive scalar α_i such that $\mathrm{Re}\big(\lambda\big(EM^{(i)}-\frac{1}{2}\alpha_iI\big)\big)>0$, where $0<\alpha_i<2\min\big\{\mathrm{Re}\big(\lambda\big(EM^{(i)}\big)\big)\big\}$. Thus there exists a positive definite matrix $Q^{(i)}$ such that

$$\big(EM^{(i)}\big)^TQ^{(i)}+Q^{(i)}EM^{(i)}>\alpha_iQ^{(i)} \tag{5}$$

for $i\in\mathcal{P}$. Set $\alpha_0=\min_{i\in\mathcal{P}}\{\alpha_i\}$, then, the proof is straight forward. □

Let $\xi(t)=(E\otimes I_n)x(t)$, where $\xi_i(t)=x_i(t)-x_{i+1}(t)$, E is defined in (4). Then the closed-loop system dynamics (3) can be rewritten as

$$\begin{aligned}\dot{\xi}(t)&=(E\otimes I_n)\big((I_N\otimes A-c\mathcal{L}^{\sigma(t)}\otimes BK)x(t)+(I_N\otimes D)F(x,t)\big)\\&=\Big(I_{N-1}\otimes A-cEM^{\sigma(t)}\otimes BK\Big)\xi(t)+(I_{N-1}\otimes D)\Phi(x,t)\end{aligned} \tag{6}$$

where $\Phi(x,t)=(E\otimes I)F(x,t)=\big[(f(x_1(t),t)-f(x_2(t),t))^T,\ldots,(f(x_{N-1}(t),t)-f(x_N(t),t))^T\big]^T$, $M^{\sigma(t)}$ is the corresponding matrix such that $\mathcal{L}^{\sigma(t)}=M^{\sigma(t)}E$, $\sigma(t)\in\mathcal{P}$.

According to the definition of $\xi(t)$, we can conclude that $\xi(t)=0$ if and only if $x_1(t)=x_2(t)=\ldots=x_N(t)$. Thus, $\xi(t)$ can be seen as the disagreement vector. Until now, the consensus control problem of system (3) with switching topologies has been converted to a stabilization problem of lower dimension switched system (6). Based on above analysis, the following theorem is proposed.

Definition 2. (*Hespanha and Morse* [26]). A positive constant τ_a is called the average dwell time for a switching signal $\sigma(t)$ if

$N_\sigma(t_2,t_1)\le N_0+\frac{t_2-t_1}{\tau_a}$

holds for all $t_2\ge t_1\ge0$ and some scalar $N_0\ge0$, where $N_\sigma(t_2,t_1)$ denotes the number of mode switches of a given switching signal $\sigma(t)$ over the interval (t_1,t_2).

Theorem 1. *Suppose that Assumption 1 holds. The consensus problem of multi-agent system (1) with the controller (2) is solved if there exist real scalars $c>0$, $\beta_0>0$ and a positive definite matrix P such that*

$$\begin{bmatrix}A^TP+PA-c\alpha_0PBB^TP+\beta_0P&PD&I_n\\D^TP&-\frac{1}{\varphi_1}I_n&0\\I_n&0&-\frac{\varphi_2}{\rho^2}I_n\end{bmatrix}<0, \tag{7}$$

where $\varphi_1=\max_{i\in\mathcal{P}}\big\{\lambda\big(Q^{(i)}\big)\big\}$, $\varphi_2=\min_{i\in\mathcal{P}}\big\{\lambda\big(Q^{(i)}\big)\big\}$, $Q^{(i)}$ and α_0 are defined in Lemma 5; the feedback matrix is designed as $K=B^TP$ and the average dwell time satisfies following condition

$\tau_a>\tau_a{}^*=\frac{\ln h_0}{\beta_0},$

where $h_0=\varphi_1/\varphi_2$.

Proof. According to Schur complement lemma [27], linear matrix inequality (LMI) (7) holds if and only if the following inequality holds

$$A^TP+PA-c\alpha_0PBB^TP+\frac{\rho^2}{\varphi_2}I_n+\varphi_1PDD^TP+\beta_0P<0. \tag{8}$$

Consider the following piecewise Lyapunov candidate of the switched system (6)

$$V_i(t)=\xi(t)^T\Big(Q^{(i)}\otimes P\Big)\xi(t), \tag{9}$$

where P is a solution of the inequality (7), $Q^{(i)}$, $i\in\mathcal{P}$, is a feasible solution of (5).

Note that the communication topology is fixed for time interval $t\in\big[t_i,t_{i+1}\big)$. Then, the derivation of this Lyapunov candidate along the trajectory of system (6) is

$$\begin{aligned}\dot{V}_i(t)=&\,2\xi(t)^T\Big(I_{N-1}\otimes A-cEM^{(i)}\otimes BK\Big)^T\Big(Q^{(i)}\otimes P\Big)\xi(t)\\&+2\Phi(x,t)^T\Big(Q^{(i)}\otimes D^TP\Big)\xi(t).\end{aligned} \tag{10}$$

Substituting $K = B^T P$ into (10) yields

$$\dot{V}_i(t) = \xi(t)^T \left(Q^{(i)} \otimes \left(A^T P + PA\right)\right.$$
$$-c\left(\left(\left(EM^{(i)}\right)^T Q^{(i)} + Q^{(i)} EM^{(i)}\right) \otimes PBB^T P\right)\xi(t)$$
$$+2\Phi(x,t)^T \left(Q^{(i)} \otimes D^T P\right)\xi(t). \tag{11}$$

According to Lemma 3 and the Lipschitz condition, one has

$$2\Phi(x,t)^T \left(Q^{(i)} \otimes D^T P\right)\xi(t)$$
$$\le \Phi(x,t)^T \Phi(x,t) + \xi(t)^T \left(\left(Q^{(i)}\right)^2 \otimes PDD^T P\right)\xi(t)$$
$$\le \rho^2 \xi(t)^T \xi(t) + \xi(t)^T \left(\left(Q^{(i)}\right)^2 \otimes PDD^T P\right)\xi(t). \tag{12}$$

In light of Lemma 5 and the fact that $Q^{(i)} \le \varphi_1 I_{N-1}$, $Q^{(i)} \ge \varphi_2 I_{N-1}$, $i \in \mathcal{P}$, where $\varphi_1 = \max_{i \in \mathcal{P}}\left\{\lambda\left(Q^{(i)}\right)\right\}$, $\varphi_2 = \min_{i \in \mathcal{P}}\left\{\lambda\left(Q^{(i)}\right)\right\}$, it then follows from (11) using (12) that

$$\dot{V}_i(t) \le \xi(t)^T \left(Q^{(i)} \otimes \left(A^T P + PA - c\alpha_0 PBB^T P + \frac{\rho^2}{\varphi_2} I_n + \varphi_1 PDD^T P\right)\right)\xi(t). \tag{13}$$

From (8), one has

$$\dot{V}_i(t) < -\beta_0 \xi(t)^T \left(Q^{(i)} \otimes P\right)\xi(t).$$

Thus by (9), one can obtain that

$$V_i(t) < e^{-\beta_0 (t - t_i)} V_i(t_i). \tag{14}$$

Note that the communication topology switches at $t = t_i$, then one can get

$$V_i(t_i) < h_0 V_{i-1}(t_i^-) \tag{15}$$

where $h_0 = \varphi_1 / \varphi_2$.

Thus, when $t \in [t_i, t_{i+1})$, from (14) and (15), one has

$$V_i(t) \le e^{-\beta_0 (t - t_i)} h_0 V_{i-1}(t_i^-)$$
$$\le e^{-\beta_0 (t - t_i)} h_0 e^{-\beta_0 (t_i - t_{i-1})} V_{i-1}(t_{i-1})$$
$$\le e^{-\beta_0 (t - t_0)} h_0^i V_0(t_0)$$

Since $i \le N_0 + \frac{t - t_0}{\tau_a}$ then

$$V_i(t) \le e^{-\left(\beta_0 - \frac{\ln h_0}{\tau_a}\right)(t - t_0)} h_0^{N_0} V_0(t_0). \tag{16}$$

Furthermore, by (9) one can obtain that

$$V_0(t_0) \le \phi_1 \|\xi(t_0)\|^2 \ \text{ and } \ \phi_2 \|\xi(t)\|^2 \le V_i(t). \tag{17}$$

where $\phi_1 = \varphi_1 \max\{\lambda(P)\}$, $\phi_2 = \varphi_2 \min\{\lambda(P)\}$.

According to (16) and (17), one has

$$\|\xi(t)\|^2 \le \frac{\phi_1}{\phi_2} e^{-\left(\beta_0 - \frac{\ln h_0}{\tau_a}\right)(t - t_0)} h_0^{N_0} \|\xi(t_0)\|^2. \tag{18}$$

Note that $\beta_0 - \frac{\ln h_0}{\tau_a} > 0$, then (18) implies that $\xi(t) \to 0$ as $t \to \infty$. This means that the consensus problem of (1) is solved. □

Remark 1. Here, relaying on the special property of the Laplacian matrix in Lemma 4 and the properly performed variable transformation $\xi(t) = (E \otimes I_n)x(t)$, the leaderless consensus problem of system (3) with switching topologies is successfully converted to a stabilization problem of a switched system (6). Then the tools from the switched systems are employed directly to obtain the sufficient conditions for achieving consensus. Since, for the homogeneous agents, the leader-following consensus problem can be seen as a special case of the leaderless problem in which there is no specified leaders, the problem considered and the processing approach used here are quite general.

Remark 2. It is worth mentioning that the same consensus problem was also investigated in [22]. However, the system dynamics is restricted to be the first-order which can be seen as a special case of that used here. Thus, the conclusions obtained here are also applicable to [22].

Remark 3. In [19], the leader-following consensus problem of the same system was solved with switching topologies. It is required that each possible topology should contain a directed spanning tree with the same root. In our paper, we allow the roots of all possible topologies to be different. Thus the requirement for the communication topology is significantly relaxed. What's more, it was proved that in [19] the dwell time should be strictly no smaller than a threshold. In contrast, the lower threshold of the average dwell time is obtained here, which means that the consensus can still be achieved if one occasionally has a smaller dwell time between switching. Thus, the switching scheme in our paper characterizes a larger class of switching signals than that in [19].

4. Leaderless consensus with arbitrary switching topologies

In above section, the leaderless consensus problem is solved with restricted switching topologies. In order to achieve consensus, the average dwell time should be larger than a threshold, which means that the topologies should not switch too fast in order to achieve consensus. In contrast, in this section, we will show that the consensus can be achieved with arbitrarily fast switching topologies if the topology and the feedback matrix in controllers fulfill some conditions.

Assumption 2. In this section, we assume that each possible graph $\mathcal{G}^{\sigma(t)} \in \hat{\mathcal{G}}$ is strongly connected and balanced.

Lemma 6. *(Hespanha and Morse[28]). (Sylvester Law of Inertia) If $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{n \times n}$ is nonsingular, then A and $B^T AB$ have the same inertia.*

Lemma 7. . *Suppose that the graph $\mathcal{G}$ is strongly connected and balanced, then matrix $E(\mathcal{L} + \mathcal{L}^T)E^T$ is positive definite, where E is defined in (4), $\mathcal{L}$ is the Laplacian matrix of graph $\mathcal{G}$.*

Proof. Let $\hat{E} = \begin{bmatrix} E^T & \mathbf{1}_N \end{bmatrix}$, thus $\hat{E}$ is nonsingular. Then one has

$$\hat{E}^T(\mathcal{L} + \mathcal{L}^T)\hat{E} = \begin{bmatrix} E(\mathcal{L} + \mathcal{L}^T)E^T & E(\mathcal{L} + \mathcal{L}^T)\mathbf{1}_N \\ \mathbf{1}_N^T(\mathcal{L} + \mathcal{L}^T) & \mathbf{1}_N^T(\mathcal{L} + \mathcal{L}^T)\mathbf{1}_N \end{bmatrix}$$

Considering the fact that $\mathcal{G}$ is balanced, one has $(\mathcal{L} + \mathcal{L}^T)\mathbf{1}_N = \mathbf{1}_N^T(\mathcal{L} + \mathcal{L}^T) = \mathbf{0}$. Then one can obtain that

$$\hat{E}^T(\mathcal{L} + \mathcal{L}^T)\hat{E} = \begin{bmatrix} E(\mathcal{L} + \mathcal{L}^T)E^T & 0 \\ 0 & 0 \end{bmatrix}.$$

According to the fact that $\hat{E}$ is nonsingular and the Sylvester Law of Inertia, $\hat{E}^T(\mathcal{L} + \mathcal{L}^T)\hat{E}$ and $\mathcal{L} + \mathcal{L}^T$ have the same inertia. Thus, in light of the fact that $\mathcal{L} + \mathcal{L}^T$ is positive semi-definite with zero being its simple eigenvalue, one can obtain that $rank\left(E(\mathcal{L} + \mathcal{L}^T)E^T\right) = N - 1$ and $\lambda\left(E(\mathcal{L} + \mathcal{L}^T)E^T\right) > 0$, that is, $E(\mathcal{L} + \mathcal{L}^T)E^T > 0$.□

In above section, the consensus problem with switching topologies of system (3) has been converted to a stabilization problem of switched system (6). Based on this, the following conclusions are introduced.

Theorem 2. *Suppose that Assumption 2 holds. The consensus problem of multi-agent system (1) is solved with arbitrary switching* *topologies if there exist a real scalars $c>0$and a positive definite matrix P such that*

$$\begin{bmatrix} A^TP+PA-\frac{c\lambda_0}{\tilde{\varphi}_1}PBB^TP & PD & I_n \\ D^TP & -\tilde{\varphi}_2 I_n & 0 \\ I_n & 0 & -\frac{1}{\rho^2\tilde{\varphi}_1}I_n \end{bmatrix}<0 \tag{19}$$

where $\tilde{\varphi}_1=\max\left\{\lambda\left(EE^T\right)\right\}$, $\tilde{\varphi}_2=\min\left\{\lambda\left(EE^T\right)\right\}$, $\lambda_0=\min\limits_{i\in\mathcal{P}}\left\{\lambda\left(E\left(\left(\mathcal{L}^{(i)}\right)^T+\mathcal{L}^{(i)}\right)E^T\right)\right\}$, and the feedback matrix is designed as $K=B^TP$.

Proof. According to Schur complement lemma, LMI (19) holds if and only if following inequality holds

$$A^TP+PA-\frac{c\lambda_0}{\tilde{\varphi}_1}PBB^TP+\rho^2\tilde{\varphi}_1 I_n+\frac{1}{\tilde{\varphi}_2}PDD^TP<0. \tag{20}$$

Consider the following common Lyapunov candidate of the switched system (6)

$$V(t)=\xi(t)^T\left(\left(EE^T\right)^{-1}\otimes P\right)\xi(t),$$

where E is a full row rank matrix defined in (4), P is a positive solution of the inequality (19).

The derivation of this Lyapunov candidate along the trajectory of system (6) is

$$\begin{aligned}\dot{V}(t)=&\,\xi(t)^T\left(I_{N-1}\otimes A-cEM^{(i)}\otimes BK\right)^T\left(\left(EE^T\right)^{-1}\otimes P\right)\xi(t)\\ &+\xi(t)^T\left(\left(EE^T\right)^{-1}\otimes P\right)\left(I_{N-1}\otimes A-cEM^{(i)}\otimes BK\right)\xi(t)\\ &+2\Phi(x,t)^T(I_{N-1}\otimes D)^T\left(\left(EE^T\right)^{-1}\otimes P\right)\xi(t).\end{aligned} \tag{21}$$

Substituting $K=B^TP$ into (21) yields

$$\begin{aligned}\dot{V}(t)=&\,\xi(t)^T\left(\left(EE^T\right)^{-1}\otimes\left(A^TP+PA\right)\right)\xi(t)\\ &-c\xi(t)^T\left(\left(\left(EM^{(i)}\right)^T\left(EE^T\right)^{-1}+\left(EE^T\right)^{-1}EM^{(i)}\right)\otimes PBB^TP\right)\xi(t)\\ &+2\Phi(x,t)^T\left(\left(EE^T\right)^{-1}\otimes D^TP\right)\xi(t).\end{aligned} \tag{22}$$

In light of the fact that $M^{(i)}=\mathcal{L}^{(i)}E^T\left(EE^T\right)^{-1}$, one has

$$\begin{aligned}&\left(EM^{(i)}\right)^T\left(EE^T\right)^{-1}+\left(EE^T\right)^{-1}EM^{(i)}\\ &=\left(EE^T\right)^{-1}E\left(\mathcal{L}^{(i)}\right)^TE^T\left(EE^T\right)^{-1}+\left(EE^T\right)^{-1}E\mathcal{L}^{(i)}E^T\left(EE^T\right)^{-1}\\ &=\left(EE^T\right)^{-1}E\left(\left(\mathcal{L}^{(i)}\right)^T+\mathcal{L}^{(i)}\right)E^T\left(EE^T\right)^{-1}\end{aligned} \tag{23}$$

Since $E\left(\left(\mathcal{L}^{(i)}\right)^T+\mathcal{L}^{(i)}\right)E^T>0$, it follows from (23), one has

$$\left(EM^{(i)}\right)^T\left(EE^T\right)^{-1}+\left(EE^T\right)^{-1}EM^{(i)}\geq\lambda_0\left(\left(EE^T\right)^{-1}\right)^2, \tag{24}$$

where $\lambda_0=\min\limits_{i\in\mathcal{P}}\left\{\lambda\left(E\left(\left(\mathcal{L}^{(i)}\right)^T+\mathcal{L}^{(i)}\right)E^T\right)\right\}$.

Based on the Lipschitz condition and (24), if follows from (22) that

$$\begin{aligned}\dot{V}(t)\leq&\,\xi(t)^T\left(\left(EE^T\right)^{-1}\otimes\left(A^TP+PA\right)\right)\xi(t)-c\lambda_0\xi(t)^T\left(\left(\left(EE^T\right)^{-1}\right)^2\right.\\ &\left.\otimes PBB^TP\right)\xi(t)+\rho^2\xi(t)^T\xi(t)+\xi(t)^T\left(\left(\left(EE^T\right)^{-1}\right)^2\otimes PDD^TP\right)\xi(t).\end{aligned} \tag{25}$$

Since $EE^T\leq\tilde{\varphi}_1 I_{N-1}$, $EE^T\geq\tilde{\varphi}_2 I_{N-1}$, where $\tilde{\varphi}_1=\max\left\{\lambda\left(EE^T\right)\right\}$, $\tilde{\varphi}_2=\min\left\{\lambda\left(EE^T\right)\right\}$, one can obtain that

$$\dot{V}(t)\leq\xi(t)^T\left(\left(EE^T\right)^{-1}\otimes\left(A^TP+PA-\frac{c\lambda_0}{\tilde{\varphi}_1}PBB^TP+\rho^2\tilde{\varphi}_1 I_n+\frac{1}{\tilde{\varphi}_2}PDD^TP\right)\right)\xi(t). \tag{26}$$

According to (20), one can obtain that $\dot{V}(t)<0$. This means that the consensus problem is solved. This completes the proof. □

Remark 4. In Theorem 2, a common Lyapunov function is constructed to solve the leaderless consensus problem with arbitrary switching topologies. In contrast with the case with restricted switching topologies considered in above section, the consensus can be achieved here without any constraint on the switching speed of the topologies. This means that the achievement of consensus can be guaranteed even the topology switches with arbitrarily fast speed. However, constraints on topology are strengthened, which can be seen as the tradeoff.

5. Examples

In this section, we provide some examples to illustrate the effectiveness of the above theoretical results. A multi-agent system consisting five agents is considered. The dynamics is defined as

$$x_i=\begin{bmatrix}x_{i1}\\x_{i2}\end{bmatrix},\quad A=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad B=\begin{bmatrix}0\\1\end{bmatrix},\quad D=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

$f(x_i,t)=[0,0.05sin(x_{i2}(t))]^T$.

We can obtain that $\rho=0.05$.

Example 1. In this example, the leaderless consensus problem with restricted switching topologies is considered. The directed communication topologies $\hat{\mathcal{G}}=\left\{\mathcal{G}^1,\mathcal{G}^2,\mathcal{G}^3,\mathcal{G}^4\right\}$ are given in Fig. 1. Clearly, each topology contains a directed spanning tree.

Thus, we can obtain that $\varphi_1=3.4175$, $\varphi_2=0.2009$, and $h_0=17.0075$. According to Lemma 5, we can set $\alpha_0=0.8$ with $\alpha_1=1.7$, $\alpha_2=1.4$,$\alpha_3=1.1$, and $\alpha_4=0.8$. Solving LMI (7) with

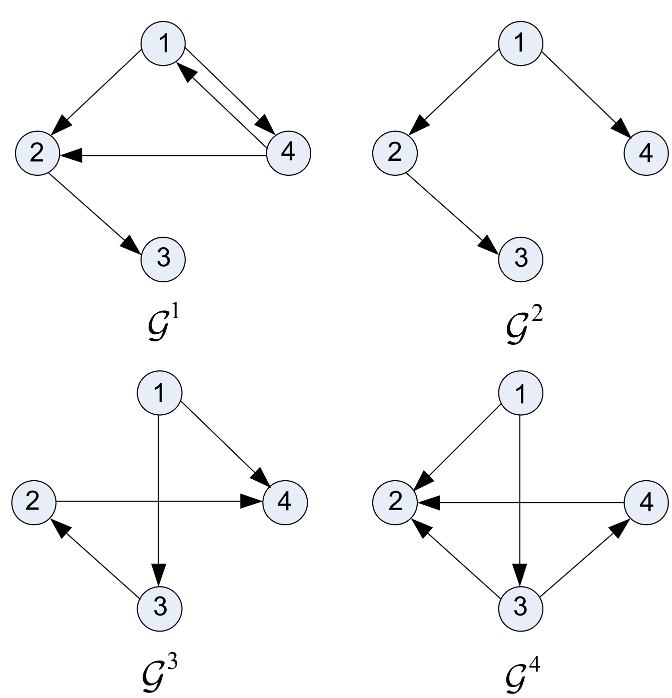

Fig. 1. Communication topologies of Example 1$\left\{\mathcal{G}^1,\mathcal{G}^2,\mathcal{G}^3,\mathcal{G}^4\right\}$.

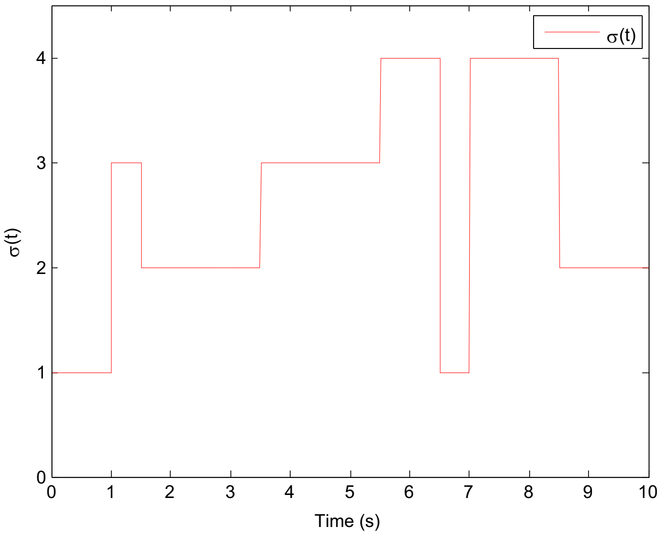

Fig. 2. Switching signal $\sigma(t)$ of Example 1.

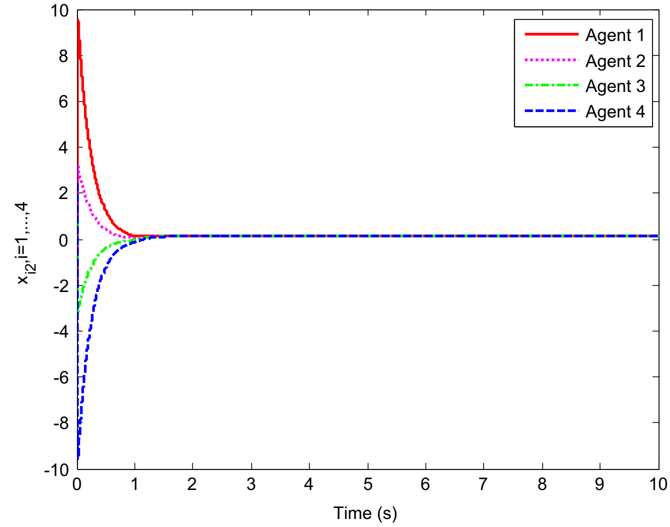

Fig. 4. Trajectories of the second states of Example 1.

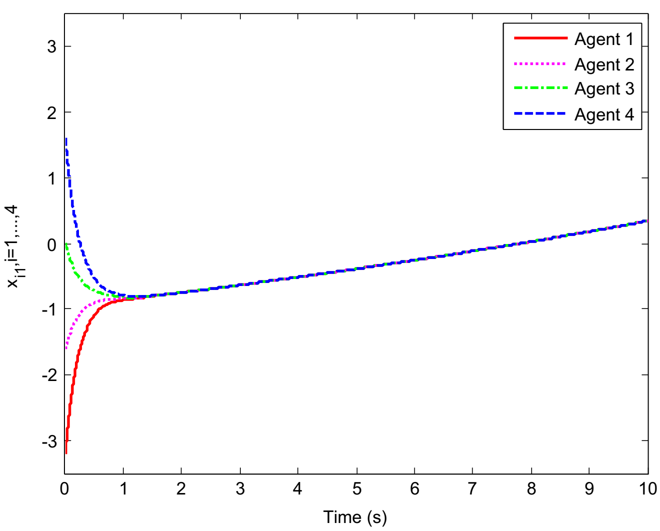

Fig. 3. Trajectories of the first states of Example 1.

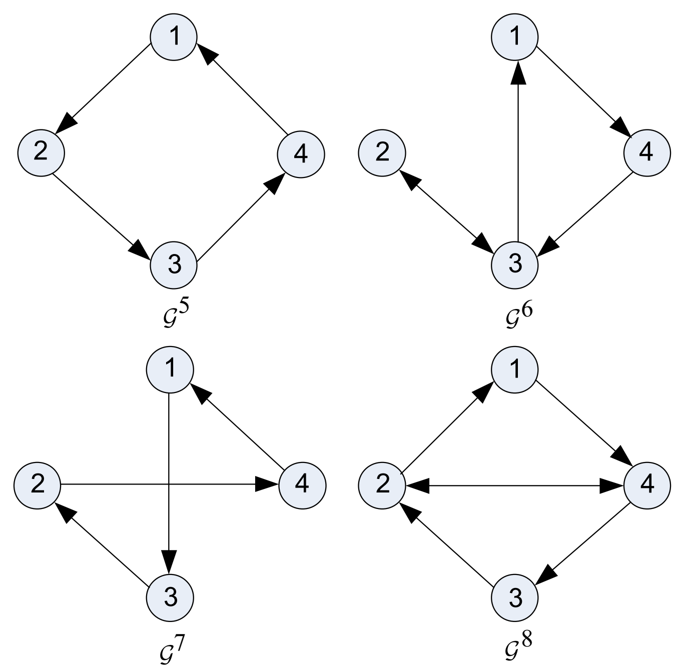

Fig. 5. Communication topologies of Example 2$\left\{\mathcal{G}^5, \mathcal{G}^6, \mathcal{G}^7, \mathcal{G}^8\right\}$.

$\beta_0 = 4.2$, $c = 100$, a feasible solution can be obtain

$$P = \begin{bmatrix} 22.1046 & 4.9546 \\ 4.9546 & 1.1727 \end{bmatrix}.$$

According to Theorem 1, the feedback matrix can be chosen as $K = \begin{bmatrix} 4.9546 & 1.1727 \end{bmatrix}$.

The switching signal is shown in Fig. 2 with the average dwell time satisfying

$$\tau_a > \tau_a{}^* = \frac{\ln h_0}{\beta_0} = 0.6747.$$

Figs. 3 and 4 show the states trajectories of all the agents. It is shown that the consensus is achieved.

Example 2. In this example, the leaderless consensus problem with arbitrary switching topologies is considered. The directed communication topologies $\hat{\mathcal{G}} = \left\{\mathcal{G}^5, \mathcal{G}^6, \mathcal{G}^7, \mathcal{G}^8\right\}$ are given in Fig. 5. Clearly, each directed topology is strongly connected and balanced.

Thus we can get $\lambda_0 = 1.6754$, $\tilde{\varphi}_1 = 3.4142$, and $\tilde{\varphi}_2 = 0.5858$. Solving the LMI (19) with $c = 30$, we get a feasible solution

$$P = \begin{bmatrix} 0.0687 & 0.0776 \\ 0.0776 & 0.1652 \end{bmatrix}.$$

According to Theorem 2, the feedback matrix can be chosen as $K = \begin{bmatrix} 0.0776 & 0.1652 \end{bmatrix}$.

Figs. 6, 7 and 8 show the switching signal and the states trajectories of all the agents. It is shown that the consensus problem is solved.

6. Conclusions

In this paper, the leaderless consensus problem of multi-agent systems with Lipschitz nonlinearities and switching topologies has been investigated. According to a special property of the Laplacian matrix, the consensus problem with switching topologies has been converted into the stabilization problem of a switched system with lower dimensions. Then, under two different topology conditions, the leaderless consensus problems have been solved with

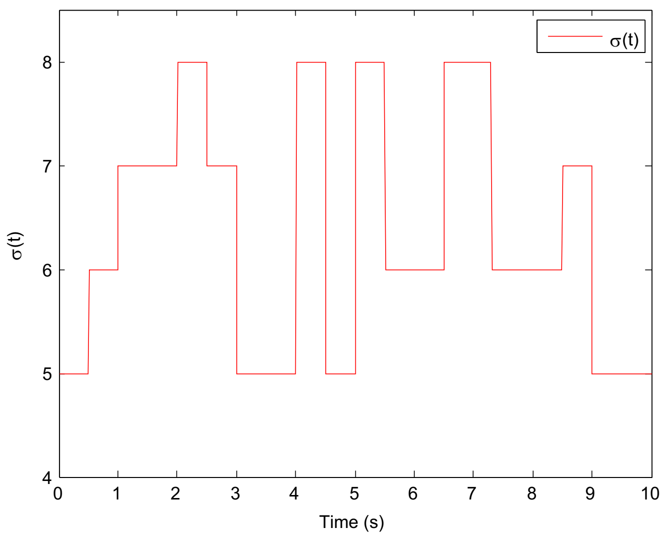

Fig. 6. Switching signal $\sigma(t)$ of Example 2.

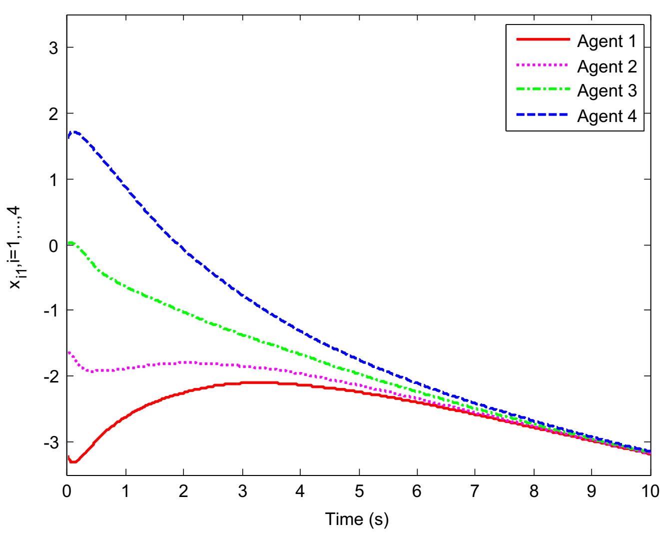

Fig. 7. Trajectories of the first states of Example 2.

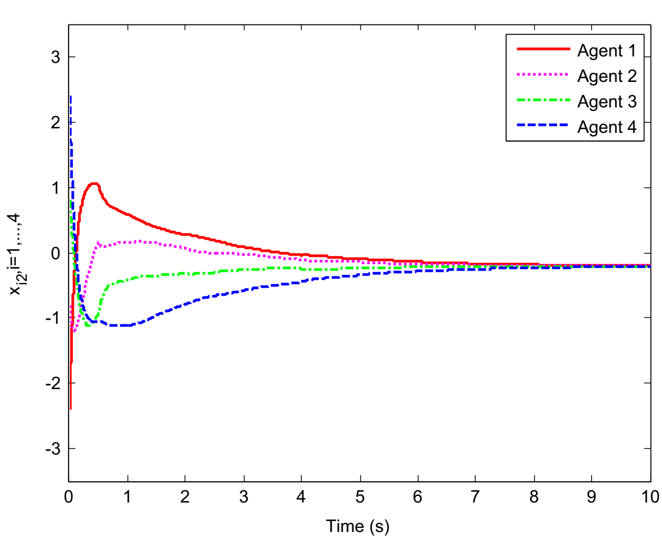

Fig. 8. Trajectories of the second states of Example 2.

restricted and arbitrary switching topologies using the piecewise Lyapunov function based approach and common Lyapunov function based approach, respectively.

References

[1] J.A. Fax, R.M. Murray, Information flow and cooperative control of vehicle formations, IEEE Trans. Autom. Control 49 (9) (2004) 1465–1476.

[2] A. Jadbabaie, J. Lin, A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Trans. Autom. Control 48 (6) (2003) 988–1001.

[3] W. Yu, G. Chen, M. Cao, J. Kurths, Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics, IEEE Trans. Syst. Man Cybern. B: Cybern. 40 (3) (2010) 881–891.

[4] W. Ren, On consensus algorithms for double-integrator dynamics, IEEE Trans. Autom. Control 53 (6) (2008) 1503–1509.

[5] Y. Gao, B. Liu, J. Yu, J. Ma, T. Jiang, Consensus of first-order multi-agent systems with intermittent interaction, Neurocomputing 129 (2014) 273–278.

[6] S.E. Tuna, Synchronizing linear systems via partial-state coupling, Automatica 44 (8) (2008) 2179–2184.

[7] C. Ma, J. Zhang, Necessary and sufficient conditions for consensusability of linear multi-agent systems, IEEE Trans. Autom. Control 55 (5) (2010) 1263–1268.

[8] Z. Li, Z. Duan, G. Chen, Dynamic consensus of linear multi-agent systems, IET Control Theory Appl. 5 (1) (2011) 19–28.

[9] Q. Li, B. Shen, J. Liang, H. Shu, Event-triggered synchronization control for complex networks with uncertain inner coupling, Int. J. Gen. Syst. 44 (2015) 212–225.

[10] W. Ren, R.W. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies, IEEE Trans. Autom. Control 50 (5) (2005) 655–661.

[11] Y. Su, J. Huang, Stability of a class of linear switching systems with applications to two consensus problems, IEEE Trans. Autom. Control 57 (6) (2012) 1420–1430.

[12] G. Wen, W. Yu, J. Cao, G. Hu, G. Chen, Consensus control of switching directed networks with general linear node dynamics, in: Proceedings of the 9th Asian Control Conference, 2013, pp. 1–6.

[13] G. Wen, V. Ugrinovskii, Distributed Consensus of Linear Multi-agent Systems with Switching Directed Topologies ⟨http://arxiv.org/abs/1409.5519⟩, 2014.

[14] I. Saboori, K. Khorasani, H_∞ consensus achievement of multi-agent systems with directed and switching topology networks, IEEE Trans. Autom. Control 59 (11) (2014) 3104–3109.

[15] W. Ni, X. Wang, C. Xiong, Leader-following consensus of multiple linear systems under switching topologies: an averaging method, Kybernetika 48 (6) (2012) 1194–1210.

[16] W. Ni, X. Wang, C. Xiong, Consensus controllability, observability and robust design for leader-following linear multi-agent systems, Automatica 49 (2013) 2199–2205.

[17] G. Wen, G. Hu, W. Yu, J. Cao, G. Chen, Consensus tracking for higher-order multi-agent systems with switching directed topologies and occasionally missing control inputs, Syst. Control Lett. 62 (2013) 1151–1158.

[18] W. Xu, J. Cao, W. Yu, J. Lu, Leader-following consensus of non-linear multi-agent systems with jointly connected topology, IET Control Theory Appl. 8 (6) (2014) 432–440.

[19] G. Wen, Z. Duan, G. Chen, W. Yu, Consensus tracking of multi-agent systems with Lipschitz-type node dynamics and switching topologies, IEEE Trans. Circuits Syst. I: Regul. Papers 61 (2) (2014) 499–511.

[20] X. Mu, X. Xiao, K. Liu, J. Zhang, Leader-following consensus of multi-agent systems with jointly connected topology using distributed adaptive protocols, J. Frankl. Inst. 351 (2014) 5399–5410.

[21] M. Hu, L. Guo, A. Hu, Y. Yang, Leader-following consensus of linear multi-agent systems with randomly occurring nonlinearities and uncertainties and stochastic disturbances, Neurocomputing 149 (2015) 884–890.

[22] J. Wang, K. Chen, Y. Zhang, Consensus of multi-agent nonlinear dynamic systems under slow switching topology, in: Proceedings of the 26th Chinese Control and Decision Conference, 2014, pp. 1147–1152.

[23] K. Chen, J. Wang, Y. Zhang, Z. Liu, Consensus of second-order nonlinear multi-agent systems with restricted switching topology and time delay, Nonlinear Dyn. 78 (2014) 881–887.

[24] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.

[25] S. Zhou, W. Liu, Q. Wu and G. Yin, Leaderless consensus of linear multi-agent systems: matrix decomposition approach, in: Proceedings of the 7th International Conference on Intelligent Human-Machine Systems and Cybernetics, 2015 pp. 327-332.

[26] J.P. Hespanha, A.S. Morse, Stability of switched systems with average dwell-Time, in: Proceedings of the 38th Conferenc on Decision and Control, 1999, pp. 2655–2660.

[27] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in Systems and Control Theory, SIAM, Philadelphia, 1994.

[28] G.H. Golub, C.F. Van Loan, Matrix Computations, third edition, Johns Hopkins University Press, London, 1996.

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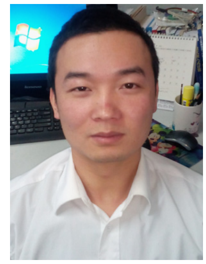

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