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Technical Notes

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Improving Aircraft Collision Risk Estimation Using the Cross-Entropy Method

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I. Introduction

COMPREHENSIVE risk assessments are required before introducing changes to air traffic control procedures or equipment. For example, rigorous safety studies were conducted before the certification and mandate of the Traffic Alert and Collision Avoidance System (TCAS) [1,2] in both the United States and worldwide. These safety studies involved estimating the risk of collision, generally obtained through large numbers of Monte Carlo simulations. Due to the rarity of collision events, millions of Monte Carlo simulations were run at significant computational cost. The computation can be especially problematic when parametric studies for a collision avoidance system are required. An efficient simulation method for estimating midair collision risk is important to speed up safety studies of the TCAS and next-generation collision avoidance systems for manned and unmanned aircraft [3]. The efficiency of simulations can be improved by carefully selecting which scenarios to simulate.

The analysis of rare events in simulation has been well studied because of its importance to many fields. Applications include error estimation in telecommunication systems [4], reliability analysis of transportation systems [5], failure analysis of nuclear plants [6], simulation of molecular reactions in systems biology [7], and collision risk estimation [8,9]. Importance sampling is a rare event simulation technique that involves biasing the sampling distribution toward the rare events of interest [10]. Importance sampling often achieves a faster convergence rate than direct (Monte Carlo) sampling, but it requires an appropriate sampling distribution. The cross-entropy method [11,12] is an iterative method for finding an approximately optimal sampling distribution. It has been widely applied to general problems such as buffer allocation [13], control and navigation [14], scheduling and vehicle routing [15], reinforcement learning [16], and combinatorial optimization [17].

Importance sampling was applied recently to the safety analysis of the TCAS [8,18]. In the safety analysis, a statistical model of aircraft encounters in the U.S. airspace was developed from a radar feed. The model was represented as Bayesian networks [19] that captured the statistical relationships between a large set of variables, such as altitude, climb rate, and turn rate. The objective was to estimate the probability of near-midair collision (NMAC) with the TCAS, defined to have occurred when two aircraft came within 500 ft horizontally and 100 ft vertically of each other. Since NMACs are exceedingly rare in the airspace, directly sampling from the distribution represented by the Bayesian networks almost always results in encounters without NMACs. Importance sampling, however, focuses the sampling distribution on encounters that are more likely to result in a NMAC, reweighting the results to provide an estimate of NMAC probability. The fundamental challenge with importance sampling is choosing the sampling distribution. The choice of distribution can have a significant impact on the rate of convergence. For the TCAS analysis, a distribution was selected by hand such that most of the encounters would have low horizontal and vertical miss distances in the absence of the TCAS, but the particular choice of distribution was not optimized in any way [18].

This Note applies the cross-entropy method to the estimation of collision risk in the airspace model. The cross-entropy method iteratively attempts to estimate the parameters of the optimal importance sampling distribution. However, there are few challenges discovered when applying the cross-entropy method to collision risk estimation. These challenges are addressed in this Note.

The Note proceeds as follows. Section II describes approaches to generating samples in simulation and estimating metrics such as collision probability. Section III presents experimental results. Direct sampling, importance sampling, and the cross-entropy method are applied to an airspace encounter model. The concept of limiting sample weights is introduced to prevent highly weighted samples from deteriorating the convergence rate of simulations. This Note analyzes which variables are best to bias and which families of sampling distributions are most effective for collision risk estimation. The results are validated using an independent high-fidelity simulation framework.

II. Methods

This Note uses an airspace encounter model developed previously [18] for estimating the probability that an encounter will lead to a NMAC: $P(\text{NMAC}|\text{encounter})$. The airspace encounter model is represented as Bayesian networks that specify a joint distribution over variables. The variables in the model determine the encounter geometry (induced, in part, by airspace structure and air traffic control procedures) and how the aircraft maneuver involved in the encounter maneuver over time in the absence of collision avoidance. A particular instance of these state variables is denoted as x, and the probability density the model associates with x is denoted as $p(x)$.

A simulator can convert the encounter parameters x into a pair of aircraft trajectories, against which different metrics can be evaluated. Of particular interest in this Note is the probability of a NMAC. The simulators used in this Note incorporate models of the TCAS [1], the pilot response to advisories [20], and aircraft dynamics. Given the input encounter parameters, the simulators produce aircraft

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trajectories at 1 Hz. These trajectories can be checked for NMAC events.

This Note compares three general methods for arriving at an unbiased estimate of $P(\text{NMAC}|\text{encounter})$ from the airspace encounter model. The methods differ in how they select encounters for simulation and how they weight their results, as outlined in the following. Their performance is measured in terms of relative error, which is the sample standard deviation divided by the sample mean.

A. Direct Sampling

Direct samples from Bayesian networks may be obtained efficiently using logic sampling [21], which involves sampling from the conditional probability distributions associated with the nodes in the Bayesian network in topological order. If $x^{(1)}, \ldots, x^{(N)}$ are the samples drawn from p, the probability of a NMAC may be estimated as follows:

$$P(\text{NMAC}|\text{encounter}) = \int P(\text{NMAC}|x)p(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} P(\text{NMAC}|x^{(i)}) \tag{1}$$

The preceding $P(\text{NMAC}|x^{(i)})$ can be evaluated in simulation with or without TCAS. If the two aircraft come within 500 ft horizontally and 100 ft vertically in encounter $x^{(i)}$, then $P(\text{NMAC}|x^{(i)}) = 1$; otherwise, it is equal to zero. The problem with direct sampling is that $P(\text{NMAC}|x^{(i)}) = 0$ for almost all encounters, leading to a high variance estimate of the NMAC probability, even with a large number of samples.

B. Importance Sampling

Importance sampling is a variance-reduction technique [10]. Instead of sampling from a distribution with density $p(x)$, it samples from an alternative distribution $q(x)$, called a proposal or sampling distribution, chosen to bias the samples toward more informative regions. If $x^{(1)}, \ldots, x^{(N)}$ come from the proposal distribution, then

$$\begin{aligned} P(\text{NMAC}|\text{encounter}) &= \int P(\text{NMAC}|x)p(x)\frac{q(x)}{q(x)}\,\mathrm{d}x \\ &= \int P(\text{NMAC}|x)\frac{p(x)}{q(x)}q(x)\,\mathrm{d}x \approx \frac{1}{N}\sum_{i=1}^{N} P(\text{NMAC}|x^{(i)})\frac{p(x^{(i)})}{q(x^{(i)})} \end{aligned} \tag{2}$$

It can be shown that the optimal choice of proposal distribution has $q(x) \propto P(\text{NMAC}|x)p(x)$. Hence, the evaluation of different metrics and collision-avoidance systems may require different proposal distributions. However, computing this optimal proposal distribution is just as hard as computing $P(\text{NMAC}|\text{encounter})$, and so the use of this method requires a guess of an appropriate proposal distribution.

The experiments in this Note used the proposal distribution from prior studies [8,18]. The proposal distribution biased only two variables in the Bayesian network: 1) hmd, which is the horizontal miss distance between the aircraft in the absence of collision avoidance and 2) vmd, which is the vertical miss distance at the time of minimum horizontal distance. Prior studies generated hmd from a piecewise uniform proposal distribution such that a miss distance was less than 500 ft with a probability of 0.95 and between 500 ft and 3 nmi with a probability of 0.05. The proposal distribution for vmd followed an exponential distribution with a mean of 500 ft. These parameters from prior studies were chosen in an ad hoc fashion in an attempt to speed convergence of the estimate of the NMAC probability.

C. Cross-Entropy Method

The cross-entropy method is an adaptive algorithm that searches for the optimal parameters of a proposal distribution [11]. The method is often used for estimating probabilities of rare events. It is assumed that the original and proposal distributions are parameterized by u and v, respectively. Algorithm 1 iteratively searches for the optimal parameter v to reduce the cross-entropy distance between $q(x; v)$ and $P(\text{NMAC}|x)p(x; u)$. The cross-entropy distance is also known as the Kullback–Leibler divergence, which defines a distance between two probability density functions. Random samples are generated from the proposal distribution, and the parameters are updated based on the samples to produce better samples in the next iteration. Once the parameters minimizing the cross-entropy distance are found, they are used to produce an estimate using importance sampling.

Algorithm 1 Cross-entropy method

$\hat{v}_0 \leftarrow u$
$t \leftarrow 0$
repeat
  $t \leftarrow t + 1$
  Generate samples $x^{(1)}, \ldots, x^{(N_1)}$ from the density $q(\cdot; \hat{v}_{t-1})$
  Sort $S(x^{(1)}), \ldots, S(x^{(N_1)})$ to $S_{(1)} \le S_{(2)} \le \cdots \le S_{(N_1)}$
  $\hat{\gamma} \leftarrow S_{(\lceil(1-\rho)N_1\rceil)}$
  if $\hat{\gamma} \ge \gamma$
    $\hat{\gamma} \leftarrow \gamma$
  $\hat{v}_t \leftarrow \arg\max_v (1/N_1)\sum_{i=1}^{N_1} I_{\{S(x^{(i)})\ge\hat{\gamma}\}}$
   $W(x^{(i)}; u, \hat{v}_{t-1}) \ln q(x^{(i)}; v)$,
    where $W(X; u, v) = p(x; u)/q(x; v)$
until $\hat{\gamma} = \gamma$
Generate samples $x^{(1)}, \ldots, x^{(N)}$ from the density $q(\cdot; \hat{v}_t)$
$\hat{l} = 1/N\sum_{i=1}^{N} I_{\{S(x^{(i)})\ge\gamma\}} W(x^{(i)}; u, \hat{v}_t)$

In the algorithm, S is the sample performance function, N_1 is the number of samples in a single iteration (often 1000), ρ is the rarity parameter (usually between 0.01 and 0.1), and N is the number of importance samples used at the end to estimate the probability of the rare event. The sample performance function S needs to be formulated such that the probability estimate of an event is equal to $\mathbb{E}I_{\{S(x)\ge\gamma\}}$. The indicator function $I_{\{S(x)\ge\gamma\}}$ is equal to one if the sample performance exceeds a threshold γ for sample x. In this Note, we define the sample performance function

$$S(x) = -\min_{0\le t\le T} \max\{hsd(t)/5, vsd(t)\} \tag{3}$$

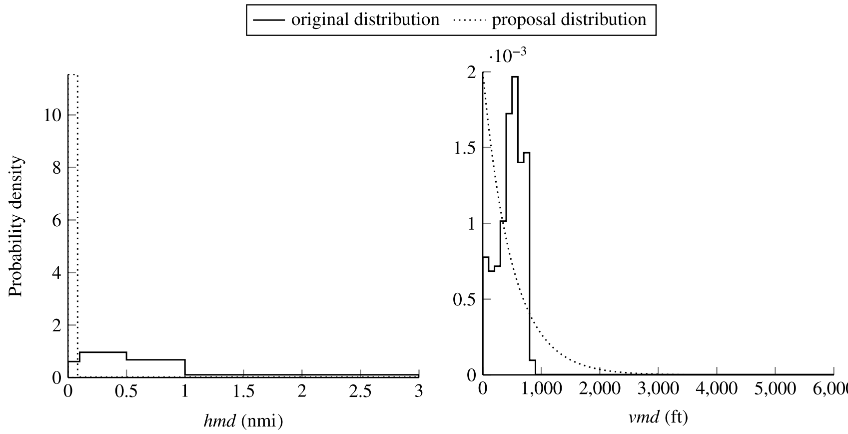
where $hsd(t)$ and $vsd(t)$ are horizontal and vertical separation distances at time t as determined through simulation, and T is the time that the simulation ends. With the performance sample function and $\gamma = -100$ ft, the indicator function allows us to count NMAC events. The choice of the rarity parameter ρ influences the convergence of the algorithm and is domain specific.

III. Results

Simulation results are compared using methods introduced in the previous section. A few modifications to the cross-entropy method are introduced with experimental results. Finally, the results are validated with a high-fidelity simulation framework. Encounters generated by the airspace encounter model are simulated using the Stanford Intelligent Systems Laboratory Encounter Simulator (SISLES). The SISLES includes a simplified version of the TCAS that only provides initial corrective advisories and does not coordinate between aircraft to ensure complementary advisories.

A. Simulations when Biasing Model Variables from Prior Study

Our first experiments involved comparing simulation results using direct sampling, importance sampling, and the cross-entropy method when biasing the model variables hmd and vmd chosen in prior studies. Figure 1 shows the proposal distributions used in prior studies for importance sampling along with the original distributions. As can be seen in the figure, the proposal distribution is highly biased to lower miss distances. As mentioned in Sec. II.B, these proposal distribution parameters were chosen by hand without careful optimization. For the cross-entropy method, instead of keeping the proba-



a) Horizontal miss distance

b) Vertical miss distance

Fig. 1 Original and proposal distributions for model variables hmd and vmd.

bility of $hmd < 500$ ft fixed at 0.95 and vmd coming from an exponential distribution with a mean of 500 ft, we allowed the method to tune two parameters: the probability of $hmd < 500$ ft, and the mean of the exponential distribution. We set $N_1 = 1000$ and $\rho = 0.01$.

Table 1 and Fig. 2 show simulation results and convergence plots, respectively, without and with the TCAS. In simulations using the cross-entropy method, parameters were learned within several iterations. The number of samples in the table includes the number of samples generated from learning iterations. As shown in the simulation results without the TCAS, all methods converge to approximately the same NMAC probability. Importance sampling requires far fewer samples to achieve the same relative error, even though parameters for proposal distributions were chosen without optimization. The cross-entropy method improves the rate of convergence further and requires even fewer samples than importance sampling to achieve the same relative error. The cross-entropy method optimizes parameters for proposal distributions so that it generates more informative samples than direct sampling and importance sampling, allowing for faster convergence.

However, in the case when the TCAS is turned on, the cross-entropy method does not converge to the NMAC probability to which direct sampling and importance sampling converge. Figure 2d shows that the cross-entropy method converges slower than importance sampling. The convergence curve shows a few sudden jumps in relative error. These jumps are caused by a few highly weighted samples. A sample can receive a high weight if the probability of the proposal distribution is much lower than the probability of the original distribution. It is known that importance sampling does not work well if the tail of the proposal distribution decays much faster than the original distribution [10].

B. Variance Reduction Using Weight Limits

To address the issue of highly weighted samples, we forced the proposal distribution to be such that no weights exceed a certain threshold. This introduces bounds for parameters of the proposal distribution. If a parameter exceeds its bound, the parameter is set to the bound and other parameters are adjusted accordingly.

We ran a series of experiments to determine a suitable threshold for the weights. If the weight limit is too high, jumps appear in the convergence curves. If the weight limit is too low, it is difficult to realize benefits from using the cross-entropy method. Figure 3 shows convergence as the weight threshold is varied for the hmd piecewise-constant proposal distribution. As shown in the figure, setting the weight limit to 10 leads to acceptable convergence. Setting the weight limits for an exponential distribution is less straightforward because it has infinite support. In our experiments, we set the weight limit at $vmd = 900$ ft to be 10.

Table 2 compares simulation results between importance sampling and the cross-entropy method with the weight limits. The cross-entropy method with the weight limits requires fewer samples than both importance sampling and the cross-entropy method without weight limits to achieve the same relative error.

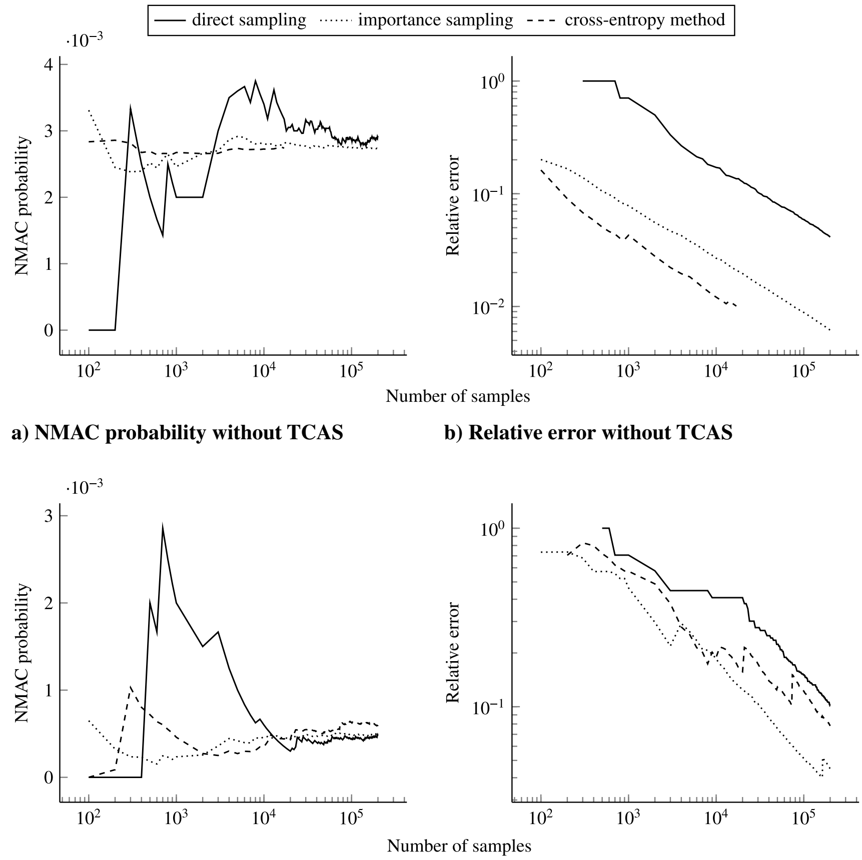
C. Variable and Distribution Selection

Model variables hmd and vmd are biased for the cross-entropy method in the previous section, and significant benefit is shown. It can be hypothesized that biasing additional model variables would bring more benefit. In these experiments, 10 additional model variables are biased such as airspace class A, altitude layer L, approach angle χ, bearing β, aircraft categories (C_1, C_2), initial airspeeds (v_1, v_2), and accelerations ($\dot{v}_1$, $\dot{v}_2$) [18]. The same families of proposal distributions in the previous section are used for model variables hmd and vmd. Piecewise uniform distributions are used for other model variables. The cut points of the piecewise uniform distributions are kept the same as the original model [18]. Without the TCAS, the cross-entropy method converges to the same NMAC probability from direct sampling and importance sampling. However, with the TCAS, it fails to converge.

The cross-entropy method with the TCAS fails to find the optimal parameters for the proposal distributions due to insufficient NMAC samples and the inclusion of model variables that do not influence NMAC sample generation. The cross-entropy method learns parameters based on samples generated in its learning step. If too few NMAC samples are generated in the learning step, parameters are fit to only a small number of samples and are unlikely to be close to

Table 1 Simulation results using direct sampling, importance sampling, and cross-entropy method

Method	TCAS	$P(\text{NMAC}\mid\text{encounter})$	Relative error	Number of samples
Direct sampling	No	2.851×10^{-3}	0.01870	1,000,000
Importance sampling	No	2.800×10^{-3}	0.01863	22,000
Cross-entropy method	No	2.736×10^{-3}	0.01831	5000
Direct sampling	Yes	4.900×10^{-4}	0.1010	200,000
Importance sampling	Yes	4.967×10^{-4}	0.1006	76,000
Cross-entropy method	Yes	6.158×10^{-4}	0.1002	130,000



a) NMAC probability without TCAS **b) Relative error without TCAS**

c) NMAC probability with TCAS **d) Relative error with TCAS**

Fig. 2 Convergence plots of simulations using direct sampling, importance sampling, and cross-entropy method.

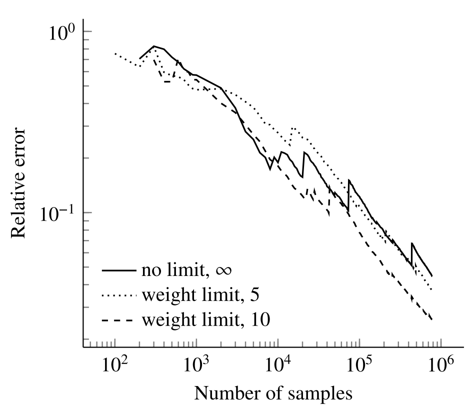


Fig. 3 Convergence plot of simulations with different weight limits.

optimal. In addition, as more noninfluential model variables are included in simulations, we discover that the cross-entropy method becomes harder to converge.

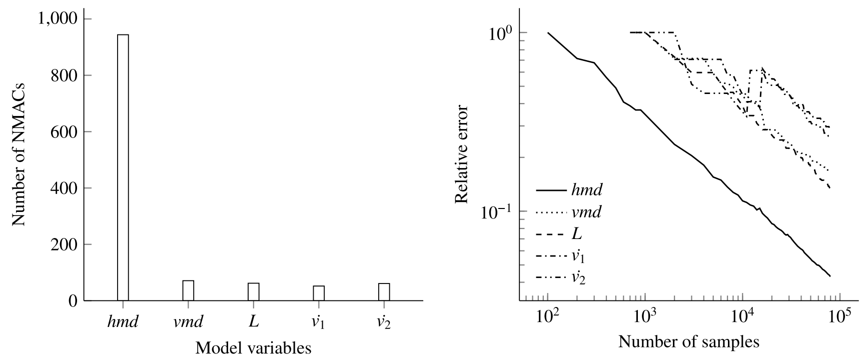
To address these issues, we select the most influential model variables and choose proposal distribution families generating more NMAC samples. To determine how much each variable contributes to generating NMAC samples, the cross-entropy method is applied to learn the proposal distribution for each model variable in isolation. Figure 4a shows the number of NMAC samples generated from simulation with each variable. One-hundred-thousand samples are generated for each simulation. The bar plot shows the top five variables generating the most NMAC samples. Figure 4b shows convergence curves. In the figure, it is found that model variable hmd contributes the most in terms of NMAC sample generation and convergence rate. The next most influential model variables are vmd and L. Other variables do not significantly influence the convergence rate.

Figure 5 shows convergence curves from different families of proposal distributions. Model variables hmd and vmd are evaluated with piecewise uniform distributions with 4 and 10 segments, respectively. In the figure, the piecewise uniform distribution and the exponential distribution show better convergence for hmd and vmd, respectively. In the case of hmd, the piecewise uniform distribution has more parameters and can better approximate the optimal proposal distribution. However, the exponential distribution with a single parameter works better for vmd. The cross-entropy method fails to learn the parameters of the piecewise uniform distribution because only a small number of NMAC samples are generated.

Table 3 shows simulation results with the most influential model variables, the corresponding proposal distribution families, and the weight limits. The cross-entropy method with model variables hmd, vmd, and L is slightly better than the cross-entropy method with model variables hmd and vmd. The cross-entropy method with the selected variables and weight limits requires fewer samples than direct sampling and importance sampling.

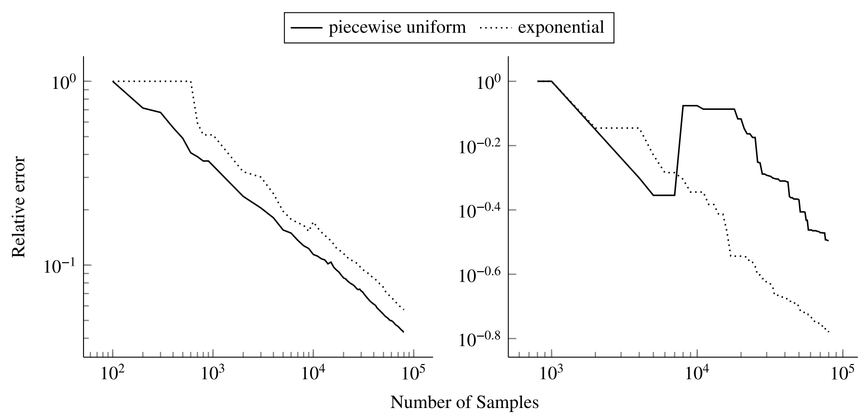
Table 2 Simulation results using cross-entropy method with weight limits

Method	TCAS	P(NMAC|encounter)	Relative error	Number of samples
Importance sampling	Yes	5.024×10^{-4}	0.02001	940,000
Cross-entropy method (without limits)	Yes	5.098×10^{-4}	0.03869	1,000,000
Cross-entropy method (with limits)	Yes	4.958×10^{-4}	0.02003	680,000



a) Number of NMACs **b) Convergence**

Fig. 4 Simulation results using cross-entropy method for each model variable.



a) Convergence for model variable *hmd* **b) Convergence for model variable *vmd***

Fig. 5 Convergence plots of simulations using cross-entropy method with different proposal distribution families.

D. Validation Using a High-Fidelity Simulator

To validate the results found using the SISLES and prove the usefulness of the cross-entropy method with the most influential variables and weight limits, simulations are performed using a high-fidelity simulation framework from Lincoln Laboratory, Massachusetts Institute of Technology [22]. The simulation framework includes a certified implementation of the TCAS with coordination, the standard International Civil Aviation Organization pilot response model [20], more realistic flight dynamics, and sensor error characteristics.

Table 4 and Fig. 6 show simulation results without and with the TCAS using the high-fidelity simulator, respectively. In the experiments, the same airspace encounter model used in the SISLES is used for the simulator. The cross entropy method used a piecewise uniform distribution for hmd and an exponential distribution for vmd. For both distributions, a weight limit of 10 was used.

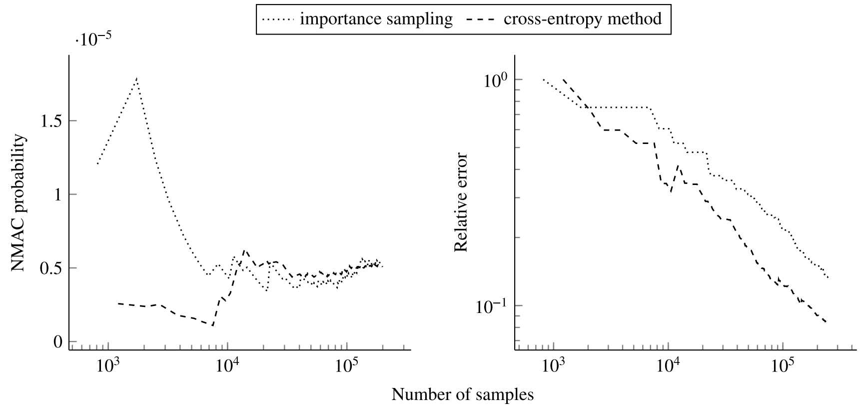
As shown in Table 4, the NMAC probability without the TCAS using each method converges to the same value, and the probability is the same as the NMAC probability from the SISLES given in Table 3. In Fig. 6a, the NMAC probability with the TCAS using each method converges to the same value, but the probability is much lower than that from the SISLES given in Table 3, since the version of the TCAS implemented in the high-fidelity simulator supports coordination between aircraft and strengthening advisories [1]. In both cases, the cross-entropy method requires fewer samples than direct sampling or importance sampling for the same relative error.

Table 3 Simulation results with most influential variables and weight limits

Method	TCAS	P(NMAC|encounter)	Relative error	Number of samples
Direct sampling	Yes	4.950×10^{-4}	0.04494	1,000,000
Importance sampling (with hmd and vmd)	Yes	5.024×10^{-4}	0.02001	940,000
Cross-entropy method (with hmd and vmd)	Yes	4.962×10^{-4}	0.02007	680,000
Cross-entropy method (with hmd, vmd and L)	Yes	4.944×10^{-4}	0.02012	660,000

Table 4 Simulation results without TCAS using the high-fidelity simulator

Method	P(NMAC|encounter)	Relative error	Number of samples
Direct sampling	2.966×10^{-3}	0.01852	980,000
Importance sampling	2.877×10^{-3}	0.01005	76,000
Cross-entropy method	2.898×10^{-3}	0.01010	55,000



a) NMAC probability **b) Relative error**

Fig. 6 Convergence plots of simulations with TCAS using the high-fidelity simulator.

IV. Conclusions

This Note has shown how to improve aircraft collision risk estimation using the cross-entropy method. Since near-midair collisions are rare events, relatively few near-midair collisions are found among millions of simulations with direct sampling. Prior work used importance sampling to generate more stressing encounters, weighting the results appropriately to accurately estimate collision risk with fewer samples. Finding an appropriate proposal distribution to use for importance sampling is not straightforward. The cross-entropy method provides a systematic way of finding approximately optimal parameters of the proposal distribution, allowing estimation of collision risk with fewer samples compared to other methods.

Although the cross-entropy method offers relief from having to tune the parameters of the proposal distribution by hand, the choice still needs to be made of which model variables to bias and which distribution families to employ. The variables and distribution families were identified that most influence collision probability by evaluating the number of NMAC samples generated using the cross-entropy method and the rate of convergence. In addition, weight limits were set to avoid proposal distributions that increased variance in the estimates. It was discovered that the cross-entropy method did not work well with a large number of parameters. The impact of the number of distribution parameters on convergence needs to be investigated further.

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Associate Editor