The Binomial Theorem

Pascal's Triangle:

It is well known that

$$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2$$

Consider the expansions of each of the following:

$$(a+b)^2 = 1a^2+2ab+1b^2$$

$$(a+b)^3 = 1a^3+3a^2b+3ab^2+1b^3$$

$$(a+b)^4 = 1a^4+4a^3b+6a^2b^2+4ab^3+1b^4$$

The coefficients of a and b can be displayed in an array as:

$$\begin{array}{ccccccccc} & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The array of coefficients displayed above is called Pascal's triangle, and it is used in determining the co-efficients of the terms of the powers of a binomial expression. Two significant features of a Pascal's triangle are:

(a) each line of coefficients is symmetrical

(b) each line of coefficients can be obtained from the line of coefficients immediately ~~proceding~~ preceding it.

For completeness, it may be observed that

$$(a+b)^0 = 1 \quad \text{and} \quad (a+b)^1 = 1a+1b$$

Therefore the table of coefficients may be written in a triangle as follows:

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

When an expression is written as a series of terms, it is said to be expanded, and the series is called it's expansion. Thus, the expansion of $(a+b)^3$ is $a^3+3a^2b+3ab^2+b^3$. Similarly, we can obtain the coefficients of $(a+b)^4$ from the coefficients of $(a+b)^3$ as follows: (2)

Coefficients of $(a+b)^3$ 1   3   3   1

Coefficients of $(a+b)^4$ 1   4   6   4   1

Hence, the coefficients of $(a+b)^4$ are: 1, 4, 6, 4, 1. Thus,

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

In the expression of $(a+b)^4$ for example, the following features of the expansion can be enumerated:

(a) There are 5 terms.

(b) In each of the terms involved in the expansion, the power of a and b put together is 4. We say that the expansion is homogeneous in a and b.

(c) While the power of a is in decreasing order, the power of b is in increasing order.

These features are characteristic of the general expansion of $(a+b)^n$

Ex1: Using Pascal's triangle, expand and simplify completely: $(2x+3y)^4$

Soln

$$(2x+3y)^4 = (2x)^4+4(2x)^3(3y)+6(2x)^2(3y)^2+4(2x)(3y)^3+(3y)^4$$

$$= 16x^4+96x^3y+216x^2y^2+216xy^3+81y^4.$$

Ex2: Using Pascal's triangle, expand and simplify completely $(x-2y)^5$

Soln

Using Pascal's triangle, the coefficients of $(x+y)^5$ derived from the coefficient of $(x+y)^4$ are: 1, 5, 10, 10, 5, 1

$$(x-2y)^5 = x^5+5x^4(-2y)+10x^3(-2y)^2+10x^2(-2y)^3+5x(-2y)^4+(-2y)^5$$

$$= x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5.$$

Ex3:- Obtain the expansion of $(2x-\frac{1}{2})^4$, in descending powers of x.

Soln

Here $a=2x$ and $b=-\frac{1}{2}$, therefore the five terms of the expansion will involve

$(2x)^4, (2x)^3(-\frac{1}{2}), (2x)^2(-\frac{1}{2})^2, (2x)(-\frac{1}{2})^3, (-\frac{1}{2})^4$

and their coefficients will be respectively

1, 4, 6, 4, 1

$\therefore (2x-\frac{1}{2})^4 = (2x)^4 + 4(2x)^3(-\frac{1}{2}) + 6(2x)^2(-\frac{1}{2})^2 + 4(2x)(-\frac{1}{2})^3 + (-\frac{1}{2})^4$

$= 16x^4 + 4(8x^3)(-\frac{1}{2}) + 6(4x^2)(\frac{1}{4}) + 4(2x)(-\frac{1}{8}) + \frac{1}{16}$

$= 16x^4 - 16x^3 + 6x^2 - x + \frac{1}{16}$ in descending powers of x.

Ex4:- Use Pascal's triangle to obtain the value of $(1.002)^5$, correct to six decimal places.

Soln

1.002 may be written as $1+0.002$, so that the expansion of $(a+b)^5$ may be used, with $a=1$ and $b=0.002$.

The terms of in the expansion will involve

$1, (0.002), (0.002)^2, (0.002)^3, (0.002)^4, (0.002)^5$

and the coefficients will be 1, 5, 10, 10, 5, 1.

$\therefore (1.002)^5 = (1+0.002)^5 = 1+5(0.002)+10(0.002)^2+10(0.002)^3+5(0.002)^4+(0.002)^5$

$= 1+0.010+0.000040$

$= 1.010040$, to 6 d.p.

Ex5:- Using Pascal's triangle, simplify, correct to 5 decimal places $(1.01)^4$.

Soln

$(1.01)^4 = (1+0.01)^4 = 1+4(0.01)+6(0.01)^2+4(0.01)^3+(0.01)^4$

$= 1+0.04+0.0006+0.000004+0.00000001$

$= 1.04060$ to 5 d.p.

Exercises

1. Expand:
(a) $(2x+\frac{1}{3})^5$ (b) $(2x+3y)^4$ (c) $(4z+1)^3$ (d) $(x-\frac{1}{x})^5$ (e) $(a^2-b^2)^5$

2. Write down the expansion of $(2+x)^5$ in ascending powers of x. Taking the first ~~three~~ three terms of the expansion, put $x=0.001$, and find the value of $(2.001)^5$ correct to five places of decimals.

3. Write the expansion of $(1+\frac{1}{4}x)^4$. Taking the first three terms of the expansion, put $x=0.1$, and find the value of $(1.025)^4$, correct to three places of decimals.

4. Expand $(2-x)^6$ in ascending powers of x.

The Binomial Expansion Formula

Consider the expansion of $(x+y)^5$:

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$
$$= x^5 + {}^5C_1 x^4 y + {}^5C_2 x^3 y^2 + {}^5C_3 x^2 y^3 + {}^5C_4 x y^4 + y^5$$

$$\Rightarrow (x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n\text{-factors}}$$

$$(x+y)^n = x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_r x^{n-r} y^r + \cdots + y^n$$

But $${}^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!\,r!}$$
$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Hence, $$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^{n-r}y^r + \cdots y^n.$$

It can be shown that the binomial expansion formula holds for positive, negative, integral or any rational value of n, provided there is a restriction on the values of x and y in the expansion of $(x+y)^n$. We shall however consider only the binomial expansion formula for a positive integral n. (8)

Ex1. Write down the binomial expansion of $(1+\frac{1}{4}x)^6$ simplifying all the terms.

b). Use the expansion in (a) to evaluate $(1.0025)^6$ correct to five significant figures.

Soln

$$(1+\tfrac{1}{4}x)^6 = 1 + {}^6C_1(\tfrac{1}{4}x)^1 + {}^6C_2(\tfrac{1}{4}x)^2 + {}^6C_3(\tfrac{1}{4}x)^3 + {}^6C_4(\tfrac{1}{4}x)^4 + {}^6C_5(\tfrac{1}{4}x)^5 + {}^6C_6(\tfrac{1}{4}x)^6$$

$$= 1 + \frac{6}{1}\cdot\frac{1}{4}x + \frac{6\times5}{1\times2}(\tfrac{1}{4}x)^2 + \frac{6\times5\times4}{1\times2\times3}(\tfrac{1}{4}x)^3 + \frac{6\times5\times4\times3}{1\times2\times3\times4}(\tfrac{1}{4}x)^4 + \frac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5}(\tfrac{1}{4}x)^5 + (\tfrac{1}{4}x)^6$$

$$= 1 + \frac{3}{2}x + \frac{15}{16}x^2 + \frac{5}{16}x^3 + \frac{15}{256}x^4 + \frac{3}{512}x^5 + \frac{1}{4096}x^6$$

(b) $$(1.0025)^6 = (1+0.0025)^6 = (1+\frac{25}{10000})^6 = (1+\frac{1}{400})^6$$

Put $\frac{1}{4}x = \frac{1}{400}$

$$x = \frac{1}{400}\times4 = \frac{1}{100} = 0.01$$

$$\therefore (1.0025)^6 = 1 + \frac{3}{2}(0.01) + \frac{15}{16}(0.01)^2 + \frac{5}{16}(0.01)^3 + \frac{15}{256}(0.01)^4 + \cdots$$

$$= 1 + 0.015 + 0.00009375 + 0.0000003125$$

$$= 1.0150940625$$

$$= 1.0151 \ (5\ s.f).$$