Week 10 Reflection

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November 5th, 2024

1 General Results about Limits

Theorem. $\lim_{x\to h} f(x) = L$ *iff* $\mathrm{st}(f(\gamma)) = L$ *for all* $\gamma \simeq h$

Proof. If $\lim_{x\to h} f(x) = L$, then for every $\epsilon > 0$ there is a $\delta > 0$ such that $(\forall x)(|x-h| < \delta \to |f(x) - L| < \epsilon)$. Transferring this, we notice that $|\gamma - h| < \delta \to |f(\gamma) - L| < \epsilon$. But $|\gamma - h| < \delta$ for every real $\delta > 0$, and so $|f(\gamma) - L| < \epsilon$ for every real $\epsilon > 0$. Hence $f(\gamma) \simeq L$.

Next, say any $\gamma \simeq h$ satisfies $f(\gamma) \simeq L$ (equivalently, $\mathrm{st}(f(\gamma)) = L$). Then, for any $\epsilon > 0$, we can take any positive infinitesmal θ and have $(\forall x)(|x-h| < \theta \to |f(x)-L| < \epsilon)$, since if $|x-h| < \theta$ we have $x \simeq h$ and so $f(x) \simeq L$. Thus, in the hyperreals we have $(\exists \delta > 0)(\forall x)(|x-h| < \delta \to |f(x) - L| < \epsilon)$, and we can then transfer that statement to the reals. Since this is true for any real $\epsilon > 0$, we find $\lim_{x\to h} f(x) = L$. $\square$

2 Integration as Hyperfinite Sums

2.1 Hyperfinite Sums

Definition. An internal set $A = [A_n]$ is *hyperfinite* if A_n is finite for all n.

Since ${}^*\varphi(A)$ iff $[[\varphi(A_n)]] \in \mathcal{F}$, we can transfer a lot of properties of finite sets to hyperfinite ones. In particular, if $\varphi(X)$ for any finite set X, then ${}^*\varphi(A)$ for any hyperfinite set A. For instance, every hyperfinite set has a maximum element by $\varphi(X) = (\exists x \in X)(\forall y \in X)(y \leq x)$.

Definition. If $A = [A_n]$ is hyperfinite and $f : \mathbb{R} \to \mathbb{R}$, we define the *hyperfinite sum*

$$\sum_{x\in[A_n]}{}^* f(x) = \left[\sum_{x\in A_n} f(x)\right]$$

Note that $\sum_{x\in A_n} f(x)$ is just the sum of a finite number of real numbers.

2.2 Integration

Now, let $f : [a,b] \to \mathbb{R}$ be an integrable function. To take $\int_a^b f(x)dx$, we want to divide $[a,b]$ into infinitely many segments of infinitesimal width, and then add up the area of the rectangles above or below those segments. Hyperfinite sums give us a way to do this.

To divide $[a,b]$ into infinitely many segments of infinitesimal width, we will construct a hyperfinite partition where the segments are of infinitesimal width $dx > 0$. Say $dx = [\langle \Delta x_1, \Delta x_2, \ldots\rangle]$. Let $P_n \cup \{b\}$ be partition of $[a,b]$ into segments of width Δx_n (plus a final"remainder" segment of length $\leq \Delta x_n$). So

$$P_n = \left\{ a + k\Delta x_n \,|\, 0 \leq k < \frac{b-a}{\Delta x_n},\ k \in \mathbb{N} \right\}$$

Let N_n denote $|P_n| = \frac{b-a}{\Delta x_n}$, and let r_n denote the length of the "remainder" segment $r_n = b - (a + k(N_n - 1))$. Notice $r_n \leq \Delta x_n$. Then, the left Riemann sum of f on $[a,b]$ with partition $P_n \cup \{b\}$ is

$$S_a^b(f, \Delta x_n) = \sum_{x\in P_n} f(x)\Delta x_n \ -\ f\big(a + k(N_n - 1)\big)(\Delta x_n - r_n)$$

Intuitively, we would hope that the integral $\int_a^b f(x)dx$ is the sum of the infinitely many rectangles of infinitesimal width

$$\sum_{x\in P} f(x)dx = \left[\sum_{x\in P_n} f(x)\Delta x_n\right]$$

as well as the extension

$${}^*S_a^b(f,dx) := \left[\sum_{x\in P_n} f(x)\Delta x_n \;-\; f\big(a+k(N_n-1)\big)(\Delta x_n - r_n)\right]$$

The difference between these two hyperreals is $[f\big(a+k(N_n-1)\big)(\Delta x_n - r_n)] = [f\big(a+k(N_n-1)\big)][(\Delta x_n - r_n)]$. Since f is integrable on $[a,b]$, it is bounded, and so $[f\big(a+k(N_n-1)\big)]$ is bounded. Meanwhile, $[\Delta x_n]$ and $[r_n]$ are infinitesimal, so $[\Delta x_n - r_n] = [\Delta x_n] - [r_n]$ is infinitesimal. Since the product of an infinitesimal and a bounded number is infinitesimal, we get that $[f\big(a+k(N_n-1)\big)][(\Delta x_n - r_n)]$ is infinitesimal, so

$${}^*S_a^b(f,dx) \simeq \sum_{x\in P} f(x)dx$$

and so

$$\int_a^b f(x)dx = \operatorname{st}\left({}^*S_a^b(f,dx)\right) = \operatorname{st}\left(\sum_{x\in P} f(x)dx\right)$$

In our hyperfinite sum, we neglected to account for the last interval perhaps being shorter than dx, but as we see here the difference it would make is infinitesimal and so can safely be ignored.

2.3 Improper Integrals

Say $f : [a,b) \to \mathbb{R}$ is integrable on every interval $[a,c]$ for $a < c < b$. Standardly, we take the *improper integral* (where defined) to be

$$\int_a^b f(x)dx := \lim_{c\to b}\int_a^c f(x)dx$$

By an easy modification of the theorem in section 1, we have that

$$\int_a^b f(x)dx = \operatorname{st}\left({}^*\!\int_a^\gamma f(x)dx\right)$$

Where $b \simeq \gamma < b$ and ${}^*\!\int_a^t f(x)dx$ indicates the extension ${}^*g(t)$ of $g(t) = \int_a^t f(x)dx$. Similarly, we have

$$\int_a^\infty f(x)dx = \operatorname{st}\left({}^*\!\int_a^\kappa f(x)dx\right)$$

Where κ is a positive unbounded hyperreal. To be clear, there is no guarantee that these standard parts exist, or that they are the same across all potential γ's or κ's—in those cases, the improper integral is undefined.

Example. Say we want to evaluate $\int_0^1 \frac{1}{\sqrt{x}}dx$.

3 Exponentiation

Let $e_n : \mathbb{R} \to \mathbb{R}$ be the function $e_n(x) = \sum_{k\in\{0,1,2,\ldots,n\}} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ We define the internal function $\operatorname{inx} = [e_n]$ by $[e_n]([x_n]) = [e_n(x_n)]$. Note that for any real number x, we have $\operatorname{inx}(x) = [e_n(x)] = [\sum_{k\in\{0,1,\ldots,n\}} \frac{x^k}{k!}] = \sum_{k\in\mathbf{N}} \frac{x^k}{k!}$, where $\mathbf{N}$ is the hyperfinite set $[\langle\{0,1\},\{0,1,2\},\ldots\rangle]$.

Let $\exp : \mathbb{R} \to \mathbb{R}$ be $\exp(x) = \operatorname{st}(\operatorname{inx}(x))$. Then we have its extension ${}^*\exp([x_n]) = [\operatorname{st}(\operatorname{inx}(x_n))]$. It's worth being careful here: we may be tempted to expand this to $[\operatorname{st}([e_n](x_n))] = [\operatorname{st}([e_n(x_n)])]$, but this is mixing indices. The proper expansion, using $[x_k]_k$ to denote $[\langle x_1, x_2, \ldots\rangle]$, is ${}^*\exp([x_n]_n) = [\operatorname{st}([e_k(x_n)]_k)]_n$, by which we mean $[\operatorname{st}([e_k([\langle x_n, x_n, x_n, \ldots\rangle])]_k)]_n$.

3.1 Derivative of exp

Let $\delta = [\delta_n]$ be a nonzero infinitesimal. We want to evaluate

$$\frac{d}{dx}\exp(x) = \mathrm{st}\left(\frac{{}^*\exp(x+\delta) - \exp(x)}{\delta}\right)$$

Consider

$$\begin{aligned}\frac{{}^*\exp(x+\delta) - \exp(x)}{\delta} &= \frac{[\exp(x+\delta_n)] - [\exp(x)]}{[\delta_n]} \\ &= \left[\frac{\exp(x+\delta_n) - \exp(x)}{\delta_n}\right]\end{aligned}$$

Peeling back another layer,

$$\begin{aligned}\frac{\exp(x+\delta_n) - \exp(x)}{\delta_n} &\simeq \frac{\mathrm{inx}(x+\delta_n) - \mathrm{inx}(x)}{\delta_n} \\ &= \frac{[e_k(x+\delta_n)]_k - [e_k(x)]_k}{\delta_n} \\ &= \left[\frac{e_k(x+\delta_n) - e_k(x)}{\delta_n}\right]_k\end{aligned}$$

Now, we know

$$\begin{aligned}e_k(x+\delta_n) &= 1 + x + \delta_n + \frac{x^2 + 2x\delta_n + \delta_n^2}{2} + \frac{x^3 + 3x^2\delta_n + \delta_n^2 \cdot (\ldots)}{3 \cdot 2!} + \cdots + \frac{x^k + kx^{k-1}\delta_n + \delta_n^2(\cdots)}{k!} \\ &= e_k(x) + \delta_n e_{k-1}(x) + \delta_n^2 \cdot (\ldots)\end{aligned}$$

So

$$\frac{e_k(x+\delta_n) - e_k(x)}{\delta_n} = e_{k-1}(x) + \delta_n \cdot (\ldots)$$

And so

$$\begin{aligned}\frac{\exp(x+\delta_n) - \exp(x)}{\delta_n} &\simeq \left[\frac{e_k(x+\delta_n) - e_k(x)}{\delta_n}\right]_k \\ &= [e_{k-1}(x)]_k + \delta_n \cdot [(\ldots)]_k\end{aligned}$$

We will assume, for now, that $[e_{k-1}(x)] \simeq \mathrm{inx}(x)$. Then we have

$$\frac{\exp(x+\delta_n) - \exp(x)}{\delta_n} \simeq \mathrm{inx}(x) + \delta_n \cdot R_n$$

But the left side of this is real, and so must be equal to the standard part of the right side

$$\frac{\exp(x+\delta_n) - \exp(x)}{\delta_n} = \mathrm{st}\left(\mathrm{inx}(x) + \delta_n \cdot R_n\right) = \mathrm{st}(\mathrm{inx}(x)) + \mathrm{st}(\delta_n \cdot R_n) = \exp(x) + \mathrm{st}(\delta_n R_n)$$

And after all that, we have

$$\begin{aligned}\frac{{}^*\exp(x+\delta) - \exp(x)}{\delta} &= \left[\frac{\exp(x+\delta_n) - \exp(x)}{\delta_n}\right] \\ &= [\exp(x) + \delta_n\,\mathrm{st}(R_n)] = \exp(x) + [\delta_n]\,\mathrm{st}(R_n)\end{aligned}$$

And so, assuming for now that $[\delta_n\,\mathrm{st}(R_n)]$ is infinitesimal, we finally have

$$\begin{aligned}\frac{d}{dx}\exp(x) &= \mathrm{st}\left(\frac{{}^*\exp(x+\delta) - \exp(x)}{\delta}\right) \\ &= \mathrm{st}\left(\exp(x) + [\delta_n]\,\mathrm{st}(R)\right) \\ &= \exp(x)\end{aligned}$$

3.2 Lemmas for 3.1

Lemma. $[e_{n-1}(x)] \simeq \operatorname{inx}(x)$

Proof. We want to show that $\operatorname{inx}(x) - [e_{n-1}(x)] \simeq 0$, i.e. that $[e_n(x) - e_{n-1}(x)] \simeq 0$, i.e. that $\left[\frac{x^n}{n!}\right] \simeq 0$, which is true for any choice of x by the same reasoning that shows $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. □

Lemma. *Given an infitesimal $\delta = [\delta_n]$, define $R_n = [(e_k(x+\delta_n) - e_k(x) - \delta_n e_{k-1}(x))/\delta_n]_k$. Show that $[\operatorname{st}(R_n)]_n$ is infinitesimal.*

We have changed notation here slightly, so that what we are now calling R_n we were before calling $\delta_n R_n$.

Proof. Let $R_n = [R_n^k]_k$ (that's an upper index, not an exponent). We'll assume that both x and δ_n are positive—if not, we can just as easily replace R_n^k by $|R_n^k|$, x by $|x|$, and δ_n by $|\delta_n$ and still get a proof that $[|R_n|]$ is infinitesimal, which will suffice. From 3.1, we have

$$R_n^k = \frac{\delta_n}{2} + \frac{3x\delta_n + \delta_n^2}{3!} + \frac{6x^2\delta_n + 4x\delta_n^2 + \delta_n^3}{4!} + \cdots + \frac{1}{k!}\sum_{q=0}^{k-2}\binom{k}{q}x^q\delta_n^{(k-1-q)}$$

In index notation,

$$\begin{aligned}
R_n^k &= \sum_{i=2}^{k}\sum_{q=0}^{i-2}\binom{i}{q}\frac{x^q\delta_n^{(i-1-q)}}{i!} \\
&= \sum_{i=2}^{k}\sum_{q=0}^{i-2}\frac{x^q\delta_n^{(i-1-q)}}{(i-q)!q!} \\
&= \delta_n\sum_{i=2}^{k}\sum_{q=0}^{i-2}\frac{x^q\delta_n^{(i-2-q)}}{(i-q)!q!} \\
&= \delta_n\sum_{q=0}^{k-2}\sum_{i=q+2}^{k}\frac{x^q\delta_n^{(i-2-q)}}{(i-q)!q!} \\
&\le \delta_n\sum_{q=0}^{k-2}\sum_{p=0}^{k-2-q}\frac{x^q\delta_n^p}{2q!} \\
&\le \delta_n\sum_{q=0}^{k-2}\frac{x^q}{2q!(1-\delta_n)} = \frac{\delta_n}{(1-\delta_n)}\sum_{q=0}^{k-2}\frac{x^q}{2q!}
\end{aligned}$$

Note that the geometric series $\sum_{p=0}^{k-2-q} a\delta_n^p \le \frac{a}{1-\delta_n}$, a fact I am using from Calc II. We know $\sum_0^\infty \frac{x^q}{2q!}$ converges in the standard definition by the ratio test (also Calc II), and so we know there is some upper bound B such that $\sum_{q=0}^{k-2}\frac{x^q}{2q!} < B$ for all k.

Now, take any positive real number r. $\delta = [\delta_n]$ is infinitesimal, so $\frac{\delta}{1-\delta} = \left[\frac{\delta_n}{1-\delta_n}\right]$ is too. So $\left[\frac{\delta_n}{1-\delta_n}\right] < r/B$, and so $\frac{\delta_n}{1-\delta_n} < r/B$ for $\mathcal{F}$-almost all n. When $\frac{\delta_n}{1-\delta_n} < r/B$, we have $R_n^k \le \frac{\delta_n}{1-\delta_n}\cdot B \le \frac{r}{B}\cdot B = r$ for all k, and so $[R_n^k]_k \le r$.

So, given any positive real number r, we know that for $\mathcal{F}$-almost all n, we have $[R_n^k]_k \le r$, so $\operatorname{st}(R_n) \le r$. So for any positive real number r, we have $[\operatorname{st}(R_n)]_n \le r$, and hence $[\operatorname{st}(R_n)]_n$ is infinitesimal. □

How did this happen. I thought this would be easy. I guess I have to talk about geometric series and the ratio test next week. I don't even know if there *is* a nonstandard proof of the ratio test.

4 Notes & Goals

4.1 Notes

This wasn't supposed to happen. I spend so much time on exponentiation. It was supposed to be a quick detour.

Anyways, unless I've made a critical error and it's all nonsense this is definitely going in, if only because it's all original work. Defining exp in terms of hyperfinite summation is a cool idea, I think. I do still have to prove that

that. . . is bounded, and hence has a standard part, for all real inputs. Actaully, if I do that properly then I won't have to bring it up in the middle of the lemma.

I also spent a lot of time this week worrying about the "remainder" section in integration, before I realized that I *can* assume that if f is integrable on $[a, b]$ it is bounded. I forgot that improper integrals weren't real integrals. Whoops.

4.2 Goals

- *Actually for real* sit down and list everything that's going in the thesis.
- Figure out if I can prove the properties of geometric series and the ratio test nonstandardly.
- Read Chapters 13.14—14 (pp. 176-189) in Goldblatt. This is almost certianly not going to be concluded, but dang it, I wanna make sure I've read it by the end of the semester anyways.

Things I'm Looking At

[1] Robert Goldblatt. *Lectures on the Hyperreals*. Vol. 188. Graduate Texts in Mathematics. New York, NY: Springer, 1998. ISBN: 978-1-4612-6841-3 978-1-4612-0615-6. DOI: `10.1007/978-1-4612-0615-6`. URL: `http://link.springer.com/10.1007/978-1-4612-0615-6` (visited on 09/25/2024).