# Nonstandard Analysis

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# History

- Newton & Leibniz formulated calculus using the idea of *infinitesimals*.
- Infinitesimals are really really small, but not 0.
- Considered nonsensical, replaced with $\delta$-$\epsilon$.
- Early 1960's: Abraham Robinson formalizes Nonstandard Analysis.
- Most modern formulations are based on work by Jerzy Łoś.

(a) Abraham Robinson

(b) Jerzy Łoś

# Hyperreals

We construct a set of *hyperreals*, call it ${}^*\mathbb{R}$, such that we know three things:

1. $\mathbb{R} \subseteq {}^*\mathbb{R}$
2. ${}^*\mathbb{R}$ contains at least one infinitesimal $\delta$, such that $0 < \delta$ but $\delta < r$ for any positive real number $r$
3. **Transfer Principle:** Any sentence of first-order logic is true in $\mathbb{R}$ iff it is true* in ${}^*\mathbb{R}$

# First-Order Logic

- Our logical language has, in addition to symbols for numbers, sets, and functions on real numbers, the following logical symbols:
  - $\neg$ for "not"
  - $\rightarrow$ for "if...then..."
  - $\forall$ for "for all"
  - $\exists$ for "there exists"
  - $\in$ for set membership
- $5 + 3 = 8$
- $(\forall x \in \mathbb{R})(5 + x = 8 \rightarrow x = 3)$
- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{N})(x < y)$
- $(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})(x < \frac{1}{n} \rightarrow n < \frac{1}{x})$

# Transfer Principle

Every sentence that's true in $\mathbb{R}$ is also "true in ${}^*\mathbb{R}$," when modified to be talking about ${}^*\mathbb{R}$. For instance:

- $(\forall x \in {}^*\mathbb{R})(\forall y \in {}^*\mathbb{R})(x + y = y + x)$, i.e. addition is commutative
  - We can similarly prove other arithmetic properties, so we can do algebra as normal in ${}^*\mathbb{R}$
- $(\forall x \in {}^*\mathbb{R})\,(x \neq 0 \to (\exists y \in {}^*\mathbb{R})(x \cdot y = 1))$, i.e. for any nonzero $x$ there exists a multiplicative inverse $\frac{1}{x}$
  - So even if $\delta$ is an infinitesimal, we know $\frac{1}{\delta}$ has to exist

# What is ${}^*\mathbb{R}$ like?

- Call a hyperreal $x$ *infinitesimal* when $|x| < r$ for every positive real $r$.
  - There's exactly one real infinitesimal: 0
- There are "infinite" hyperreals—take any nonzero infinitesimal $\delta$, and $\frac{1}{\delta}$ is greater than any real number
- Two hyperreals $x$ and $y$ are *infinitely close*, denoted $x \simeq y$, when their difference $x - y$ is infinitesimal
  - No two distinct real numbers are infinitely close to each other
  - Transitive: if $x \simeq y$ and $y \simeq z$, then $x \simeq z$
- Any finite hyperreal $x$ is infinitely close to exactly one real number, called its *standard part* $\operatorname{st}(x)$
  - For instance, $\operatorname{st}(1 + \delta) = 1$

## Derivatives, the way Leibniz intended

- Say $f : \mathbb{R} \to \mathbb{R}$
- Fix $b \in \mathbb{R}$, and let $\Delta x$ be a nonzero infinitesimal. Then

$$f'(b) = \mathrm{st}\left(\frac{f(b+\Delta x) - f(b)}{\Delta x}\right)$$

  so that $f'(b) \simeq \frac{f(b+\Delta x)-f(b)}{\Delta x}$.
- **Example:** Say $f(x) = x^2$. Then we have

$$\begin{aligned} f'(3) &\simeq \frac{(3+\Delta x)^2 - 3^2}{\Delta x} = \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ &= \frac{6\Delta x + (\Delta x)^2}{\Delta x} = 6 + \Delta x \simeq 6. \end{aligned}$$

- So $f'(3) \simeq 6$, so their difference is infinitesimal. But they're both *real numbers*, so $f'(3) = 6$.

We call $f(x)$ *continuous* at a point $c$ if for every hyperreal $x \simeq c$, we have $f(x) \simeq f(c)$.

**Theorem**

*The composition of continuous functions is continuous.*

**Proof.**

Let $c$ be some real number, and let $x \simeq c$. We want to show $(f \circ g)(x) \simeq (f \circ g)(c)$, given that $g$ is continuous at $c$ and $f$ is continuous at $g(c)$.
Since $g$ is continuous at $c$, $g(x) \simeq g(c)$. Since $f$ is continuous at $g(c)$, we have $f(g(x)) \simeq f(g(c))$. $\square$

# Proof: Chain Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $\Delta x$ be any infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Since $g'(x) = \mathrm{st}(\Delta g / \Delta x)$ is defined, $\Delta g$ must be infinitesimal. Then

$$
\begin{aligned}
(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\
&= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\
&\simeq f'(g(x)) \cdot g'(x)
\end{aligned}
$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

In the case where $\Delta g = 0$, we clearly have $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.