What I Threw Out

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y) &= \mathrm{st}\left(\frac{\frac{\partial}{\partial y}f(x+dx,y)-\frac{\partial}{\partial y}f(x,y)}{dx}\right)\\
\text{(This doesn't exist)} &= \mathrm{st}\left(\frac{\mathrm{st}\left(\frac{f(x+dx,y+dy)-f(x+dx,y)}{dy}\right)-\mathrm{st}\left(\frac{f(x,y+dy)-f(x,y)}{dy}\right)}{dx}\right)\\
\text{(This probably doesn't either)} &= \mathrm{st}\left(\frac{f(x+dx,y+dy)-f(x+dx,y)-f(x,y+dy)+f(x,y)}{dxdy}\right)\\
\text{(This still doesn't)} &= \mathrm{st}\left(\frac{\mathrm{st}\left(\frac{f(x+dx,y+dy)-f(x,y+dy)}{dx}\right)-\mathrm{st}\left(\frac{f(x+dx,y)-f(x,y)}{dx}\right)}{dy}\right)\\
&= \mathrm{st}\left(\frac{\frac{\partial}{\partial x}f(x,x+dy)-\frac{\partial}{\partial x}f(x,y)}{dy}\right)\\
&= \frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y)
\end{aligned}$$

Note that we can shift the scope of st around because $\mathrm{st}(x+y)=\mathrm{st}x+\mathrm{st}y$, $\mathrm{st}(x-y)=\mathrm{st}x-\mathrm{st}y$, $\mathrm{st}(xy)=\mathrm{st}x\mathrm{st}y$, and $\mathrm{st}(x/y)=\mathrm{st}x/\mathrm{st}y$ (proofs are fairly trivial). (WEEK 4 NOTE: Excecpt this is wrong because $\mathrm{st}(x/y)=\mathrm{st}(x)/\mathrm{st}(y)$ only when $y\not\approx 0$, because division by 0 is undefined. And in any case x/y has to be finite in order for $\mathrm{st}(x/y)$ to exist). This is easy—too easy. This equality shouldn't hold when the second partial derivatives aren't continuous. Take

$$f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

Standardly, this function has $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}=1$ and $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=-1$. If we use the approach above, we find that

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}=\mathrm{st}\left(\frac{\frac{\partial}{\partial y}f(dx,0)-\frac{\partial}{\partial y}f(0,0)}{dx}\right)$$

Note that $f(0,y)=f(x,0)=0$ for any x,y. Now, $\frac{\partial}{\partial y}f(0,0)=\mathrm{st}\left(\frac{f(0,dy)-f(0,0)}{dy}\right)=0$ and $\frac{\partial}{\partial y}f(dx,0)=\mathrm{st}\left(\frac{f(dx,dy)-f(dx,0)}{dy}\right)=\mathrm{st}\left(\frac{dxdy(dx^2-dy^2)}{dy(dx^2+dy^2)}\right)=\mathrm{st}\left(\frac{dx(dx^2-dy^2)}{dx^2+dy^2}\right)$. So

$$\mathrm{st}\left(\frac{\frac{\partial}{\partial y}f(dx,0)-\frac{\partial}{\partial y}f(0,0)}{dx}\right)=\mathrm{st}\left(\frac{dx^2-dy^2}{dx^2+dy^2}\right)$$

Which clearly depends on our choice of dx and dy, suggesting the second partial doesn't exist here.

Week 4 Note: This doesn't seem like a lot but it's so many LaTeXsymbols it took me a while. I probably should have triple-checked this before TeXing it. Lesson learned. At least I noticed something was wrong in that we proved "too much."

Also, I should note: I took that example of a function that doesn't have symmetrical second partials from Wikipedia, which claims it's "due to Peano." I assume that if I wanted to actually use it, I'd have to find some sort of scholarly source for that claim? Would I even need to cite where the function came from, if it's considered common knowledge? Obviously I can't claim to have come up with it, which seems like what I'm tacitly doing by not citing, but I'm not sure the exact procedure.