Week 9 Reflection

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October 29nd, 2024

1 Our language $\mathcal{L}_{\mathcal{S}}$

We will define a mathematical language $\mathcal{L}_{\mathcal{S}}$ on a given set S.

1.1 Symbols

Our language will contain the following symbols:

- A constant symbol $\dot{x}$ for each $x \in S$.
- A countable number of variables $v_1, v_2, v_3, \ldots$.
- A relation symbol $\dot{R}_n$ for any n-ary relation R_n on S, that is, any $R_n \subseteq S^n$. A 1-ary relation symbol is also called a *set symbol.*
- A function symbol $\dot{f}_n$ for any n-place function f_n on S, that is, any $f_n : S^n \to S$.
- The logical connectives $\wedge$, $\vee$, and $\neg$.
- Parentheses (and).
- The universal quantifier symbol $\forall$.

1.2 Terms

The terms of our language are defined recusively, as follows:

- Any single constant symbol $\dot{x}$ is a term.
- Any single variable v_n is a term.
- If $\dot{f}_n$ is an n-place function symbol, and $t_1, t_2, \ldots, t_n$ are terms, then $\dot{f}_n(t_1, t_2, \ldots, t_n)$ is a term.

A *closed* term is a term that contains no variables.

1.3 Well-Formed Formulae

The well-formed formulae (wffs) of our language are defined recursively, as follows:

- If $\dot{R}_n$ is an n-ary relation symbol, and $t_1, \ldots, t_n$ are terms, then $\dot{R}_n(t_1, \ldots, t_n)$ is a wff. This type of wff is called an *atomic formula*
- If φ and ψ are wffs, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $\neg\varphi$.
- If φ is a wff and v_n is a variable, then $(\forall v_n)\varphi$ is a wff wffs.

1.4 Bound and Free Variables

An occurance of a variable v_n that occurs in a wff φ is called *bound* if it is in the scope of a quantifier $\forall v_n$, and *free* otherwise. For instance, in the wff $(P(v_1) \vee (\forall v_1)(Q(v_1) \wedge Q(v_2)))$, the first occurance of v_1 is free, while the second is bound. The occurance of v_2 is also free, since it's not in the scope of $(\forall v_2)$. A wff with no free variables is called a *sentence.*

1.5 Replacement Notation

If φ is a wff that is not a sentence, we write $\varphi(v_1, v_2, \ldots, v_n)$ to indicate that $v_1, \ldots, v_n$ are all the variables that occur freely in φ. Then $\varphi(\dot{x}_1, \ldots, \dot{x}_n)$ is the sentence we get by replacing each free occurance of v_i by the constant symbol $\dot{x}_i$.

Similarly, we write $\varphi(v_1, \ldots, v_n, \dot{A}_1, \ldots, \dot{A}_m)$ to indicate that $v_1, \ldots, v_n$ are all the variables that occur freely in φ, and $A_1, \ldots, A_m$ are all the set symbols that occur in φ. We then write $\varphi(\dot{x}_1, \ldots, \dot{x}_n, \dot{B}_1, \ldots, \dot{B}_m)$ to mean the sentence we get by replacing v_i by $\dot{x}_i$ and $\dot{A}_i$ by $\dot{B}_i$.

1.6 Simplifications and Abbreviations

- We will omit outer parentheses and other parentheses not necessary to understand a wff, as in $\dot{P}(v_1) \vee \dot{Q}(v_1)$. We will also sometimes use [and] in place of (and), respectively.
- We will write $x \in \dot{A}$ to mean $A(x)$.
- We will write $\varphi \rightarrow \psi$ to mean $\neg\varphi \vee \psi$.
- We will write $(\exists x)\varphi$ to mean $\neg(\forall x)\neg\varphi$.
- We will write $(\forall x \in \dot{A})\varphi$ to mean $(\forall x)(x \in \dot{A} \rightarrow \varphi)$, and similarly for $\exists$.
- We will frequently omit the dots from constant, function, relation, and set symbols, using x to denote the constant symbol associated with $x \in S$.

2 Truth in $\mathcal{L}_\mathcal{S}$

2.1 s functions

Let $s : V \rightarrow S$, where V is the set of variables. We then define $\bar{s} : T \rightarrow S$, where T is the set of terms, by:

- $\bar{s}(v_n) = s(v_n)$ for any variable v_n.
- $\bar{s}(\dot{x}) = x$ for any constant symbol $\dot{x}$.
- $\bar{s}(\dot{f}_n(t_1, \ldots, t_n)) = f_n(\bar{s}(t_1), \ldots, \bar{s}(t_n))$.

2.2 s-satisfaction

We say that $s : V \rightarrow S$ either satisfies or doesn't satisfy any wff φ, defined recursively by:

- s satisfies an atomic formula $R_n(t_1, \ldots, t_n)$ if $(\bar{s}(t_1), \ldots, \bar{s}(t_n)) \in R_n$.
- s satisfies $\neg\varphi$ if it doesn't satisfy φ.
- s satisfies $\varphi \vee \psi$ if it satisfies φ or ψ.
- s satisfies $\varphi \wedge \psi$ if it satisfies φ and ψ.
- s satisfies $(\forall v_n)\varphi$ if every $t : V \rightarrow S$ such that $t(v_i) = s(v_i)$ for all $i \neq n$ satisfies φ.

2.3 Truth and Falsity

If φ is a sentence, it is easy to see that it is either satisfied by every $s : V \rightarrow S$ or not satisfied by every s. If it is satisfied by every s, we say that φ is *true*, and otherwise we say it is *false*.

3 $\mathcal{L}_\mathfrak{R}$, $\mathcal{L}_{^*\mathfrak{R}}$, and Transfer

Let $\mathcal{L}_\mathfrak{R}$ be defined as our language on $\mathbb{R}$, and $\mathcal{L}_{^*\mathfrak{R}}$ as our language on $^*\mathbb{R}$.

3.1 $*$ transforms

The $*$ transform of a term t is defined recursively, as follows:

- If v_n is a variable, ${}^*v_n = v_n$.
- If x is a constant symbol, ${}^*x = [(x, x, x, \ldots)]$.
- ${}^*(f_n(t_1, t_2, \ldots, t_n)) = {}^*f_n({}^*t_1, {}^*t_2, \ldots, {}^*t_n)$, where ${}^*f_n$ is the extension of f_n.

If φ is a wff of $\mathcal{L}_{\mathfrak{R}}$, let ${}^*\varphi$ denote the wff of $\mathcal{L}_{{}^*\mathfrak{R}}$ obtained by replacing each relation symbol R_n by ${}^*R_n$ and each term t by ${}^*t$.

3.2 Transfer Principle

Theorem (Transfer Principle)**.** *Let $\varphi(v_1, \ldots, v_k, A_1, \ldots, A_m)$ be a wff of $\mathcal{L}_{\mathfrak{R}}$. Then*

$$ {}^*\varphi([x_n^1], [x_n^2], \ldots, [x_n^k], [A_n^1], \ldots, [A_n^m]) \iff [[\varphi(x_n^1, \ldots, x_n^k, A_n^1, \ldots, A_n^m)]] \in \mathcal{F} $$

Where $[[P(n)]] = \{n \in \mathbb{N} \mid P(n)\}$.

Proof. First, say ${}^*t([x_n^1], \ldots, [x_n^k])$ is any closed term of $\mathcal{L}_{{}^*\mathfrak{R}}$, containing constant symbols $[x_n^1], \ldots, [x_n^k]$. Let t_n denote $t(x_n^1, \ldots, x_n^k)$. Then we easily see ${}^*t = [t_n]$ by a simple induction on the definition of terms, as by definition ${}^*(f_k(t^1, \ldots, t^k)) = {}^*f_k({}^*t^1, \ldots, {}^*t^n) = {}^*f_k([t_n^1], \ldots, [t_n^k]) = [f_k(t_n^1, \ldots, t_n^k)]$.

Now, we prove the theorem by induction on wffs. For a wff ${}^*\varphi([x_n^1], [x_n^2], \ldots, [x_n^k], [A_n^1], \ldots, [A_n^m])$, let φ_n denote $\varphi(x_n^1, \ldots, x_n^k, A_n^1, \ldots, A_n^m)$.

- ${}^*t \in [A_n]$ iff $[t_n] \in [A_n]$ iff $[[t_n \in A_n]] \in \mathcal{F}$ by the definition of $[A_n]$.
- ${}^*R_k(t^1, \ldots, t^k)$ iff $(t^1, \ldots, t^k) \in {}^*R_k$ iff $([t_n^1], \ldots, [t_n^k]) \in {}^*R_k$ iff $[[(t_n^1, \ldots, t_n^k) \in R_k]] \in \mathcal{F}$ iff $[[R_k(t_n^1, \ldots, t_n^k)]] \in \mathcal{F}$.
- ${}^*(\varphi \wedge \psi)$ iff ${}^*\varphi \wedge {}^*\psi$ iff $([[\varphi_n]] \in \mathcal{F}$ and $[[\psi_n]] \in \mathcal{F})$ iff $[[\varphi_n]] \cap [[\psi_n]] \in \mathcal{F}$ iff $[[\varphi_n \wedge \psi_n]] \in \mathcal{F}$.
- ${}^*(\varphi \vee \psi)$ iff ${}^*\varphi \vee {}^*\psi$ iff $([[\varphi_n]] \in \mathcal{F}$ or $[[\psi_n]] \in \mathcal{F})$ iff $[[\varphi_n]] \cup [[\psi_n]] \in \mathcal{F}$ iff $[[\varphi_n \vee \psi_n]] \in \mathcal{F}$. Note $[[\varphi_n]] \cup [[\psi_n]] \in \mathcal{F}$ implies $[[\varphi_n]]$ or $[[\psi_n]] \in \mathcal{F}$ because, in general, if $A \cup B \in \mathcal{F}$ and $A \notin \mathcal{F}$, then $A^c \in \mathcal{F}$ and so $A^c \cap (A \cup B) \subseteq B \in \mathcal{F}$.
- ${}^*(\neg\varphi)$ iff $\neg\,{}^*\varphi$ iff $[[\varphi_n]] \notin \mathcal{F}$ iff $[[\varphi_n]]^c = [[\neg\varphi_n]] \in \mathcal{F}$.
- ${}^*(\forall v_i)\varphi(v_i)$ iff $(\forall v_i)\,{}^*\varphi(v_i)$ iff ${}^*\varphi([x_n])$ for every $[x_n]$ iff $[[\varphi_n(x_n)]] \in \mathcal{F}$ for every sequence $\langle x_n \rangle$. Now, clearly if $[[(\forall v_i)\varphi_n(v_i)]] \in \mathcal{F}$, then $[[(\forall v_i)\varphi_n(v_i)]] \subseteq [[\varphi_n(x_n)]]$ implies $[[\varphi_n(x_n)]] \in \mathcal{F}$ for any sequence $\langle x_n \rangle$. Conversely, if $[[(\forall v_i)\varphi_n(v_i)]] \notin \mathcal{F}$, then for each $n \in [[\neg(\forall v_i)\varphi_n(v_i)]] \in \mathcal{F}$ we can find x_n such that $\neg\varphi_n(x_n)$, and letting x_n take an arbitrary value for $n \in [[(\forall v_i)\varphi_n(v_i)]]$ we get $[[\varphi_n(x_n)]] = [[(\forall v_i)\varphi_n(v_i)]] \notin \mathcal{F}$. So $[[\varphi_n(x_n)]] \in \mathcal{F}$ for every sequence $\langle x_n \rangle$ iff $[[(\forall v_i)\varphi_n(v_i)]] \in \mathcal{F}$, so ${}^*(\forall v_i)\varphi(v_i)$ iff $[[(\forall v_i)\varphi_n(v_i)]] \in \mathcal{F}$.

$\square$

This result, omitting set symbols and generalizing to any language $\mathcal{L}_{\mathcal{S}}$, is called **Łoś' theorem**. Note, as a special case, that if φ is a sentence, then φ is true iff ${}^*\varphi$ is true.

4 Universes & Nonstandard Frameworks

4.1 Universes

- A set A is *transitive* if $x \in y \in A$ implies $x \in A$, i.e. if $y \in A$ implies $y \subseteq A$ for any set y.
- A set A is *strongly transitive* if for any set $x \in A$, there exists a transitive $y \in A$ such that $x \subseteq y \subseteq A$. Note this implies $x \subseteq A$, so A is transitive.
- A *universe* $\mathbb{U}$ is a set that is strongly transitive, closed under unions (i.e. $A, B \in \mathbb{U}$ implies $A \cup B \in \mathbb{U}$), closed under power sets (i.e. $A \in \mathbb{U}$ implies $\mathcal{P}(A) \in \mathbb{U}$), and closed under pairing (i.e. $a, b \in \mathbb{U}$ implies $\{a, b\} \in \mathbb{U}$).
- A universe $\mathbb{U}$ is called a *universe over* $\mathbb{X}$ if $\mathbb{X} \in \mathbb{U}$ and the elements of $\mathbb{X}$ aren't considered as sets, i.e. $(\forall x \in \mathbb{X})(x \neq \emptyset \wedge (\forall y \in \mathbb{U})(y \notin x))$.

4.2 $\mathcal{L}_{\mathbb{U}}$

4.2.1 Terms

- Each variable $v_1, v_2, \ldots$ is a term.
- Each element of $\mathbb{U}$ is a constant term.
- If $t_1, \ldots, t_m$ are terms, then $\langle t_1, \ldots, t_m \rangle$ is a term (an m-tuple).
- If t and s are terms, then $t(s)$ is a *funciton-value* term.

A closed term is *defined* if it names an element of $\mathbb{U}$. This is more rigorously defined recursively: all constants are defined, tuples $\langle t_1, \ldots, t_m \rangle$ are defined when all their components $t_1, \ldots, t_m$ are defined, and $t(s)$ is defined when t names a function and s names an element of that function's domain.

4.2.2 Wffs

Atomic formulae take the form $t = s$ or $t \in s$ for terms t and s. Note $f(t) = s$ and $\langle t, s \rangle \in f$ have the same meaning.

Other wffs are defined inductively as for $\mathcal{L}_{\mathfrak{R}}$, with the exception that in the case of the universal quantifier the rule is that if φ is a wff then $(\forall v_i \in t)\varphi$ is a wff for any variable v_i and any term t that does not contain v_i.

4.3 Nonstandard Frameworks

Let $* : \mathbb{U} \to \mathbb{U}'$ be a mapping between two universes. Write $*(a) = {}^*a$. Then, for any term t, write ${}^*t$ to mean the term obtained by replacing every constant a in t by ${}^*a$. Then, for any wff φ, write ${}^*\varphi$ to mean the wff obtained by replacing every term t in φ by ${}^*t$.

A *nonstandard framework* on $\mathbb{X}$ is a universe $\mathbb{U}$ over $\mathbb{X}$ and a function $* : \mathbb{U} \to \mathbb{U}'$ such that:

- ${}^*a = a$ for all $a \in \mathbb{X}$.
- ${}^*\emptyset = \emptyset$.
- Every $\mathcal{L}_{\mathbb{U}}$ sentence φ is true iff ${}^*\varphi$ is true.

4.4 Standard & Internal entities

An element $b \in \mathbb{U}'$ is *standard* if $b = {}^*a$ for some $a \in \mathbb{U}$. It is *internal* if $b \in {}^*a$ for some $a \in \mathbb{U}$.

Exercise (13.9.3)**.** *Prove that standard sets are uniquely determined by their standard members.*

Say $A = {}^*C$ and $B = {}^*D$. If $A \neq B$, then $C \neq D$. Let $x \in C - D$ (the proof is similar if $x \in D - C$). Then $x \in C$, so by transfer ${}^*x \in {}^*C$, and $x \notin D$, so by transfer ${}^*x \notin {}^*D$, so ${}^*x \in A - B$ is a standard element that's in A but not B. So if A and B are different sets, they have different standard members—so if A and B have the same standard members, $A = B$.

Exercise (13.8.1, partial)**.** *Verify the following: If a funciton $f : A \to B$ belongs to $\mathbb{U}$, then ${}^*f$ is a function from ${}^*A$ to ${}^*B$, with ${}^*(f(a)) = {}^*f({}^*a)$ for all $a \in A$. Also, f is injective/surjective iff ${}^*f$ is.*

The fact that $f : A \to B$ is represented by the statement

$$\underbrace{(\forall x \in f)(\exists a \in A)(\exists b \in B)(x = \langle a, b \rangle)}_{f \text{ is a set of pairs } \langle a,b \rangle} \wedge \underbrace{(\forall a \in A)(\exists b \in B)(\langle a, b \rangle \in f)}_{\text{for every } a,\ f(a) \text{ is defined}} \wedge$$
$$\underbrace{\neg(\exists a \in A)(\exists b \in B)(\exists c \in B)(b \neq c \wedge \langle a, b \rangle \in f \wedge \langle a, c \rangle \in f)}_{f(a) \text{ is unique}}$$

The transfer of this says that ${}^*f : {}^*A \to {}^*B$. Similarly, for any $a \in A$, if we let $b = f(a)$, we have $\langle a, b \rangle \in f$, and so by transfer $\langle {}^*a, {}^*b \rangle \in {}^*f$, where ${}^*b = {}^*(f(a))$. So ${}^*f({}^*a) = {}^*(f(a))$.

f is injective if $(\forall a \in A)(\forall a' \in A)(a \neq a' \to f(a) \neq f(a'))$, and it is surjective if $(\forall b \in B)(\exists a \in A)(f(a) = b)$, both of which straightforwardly transfer to say that ${}^*f$ is injective and surjective, respectively.

Exercise (13.13.2)**.** *Show that $\mathcal{P}(A)$ is in bijective correspondence with the set of all functions $f : A \to \{0, 1\}$. Adapt this to show that ${}^*\mathcal{P}(A)$ is in bijective correspondence with the set of all internal functions $f : {}^*A \to \{0, 1\}$.*

Let $b : \mathcal{P}(A) \to \{0,1\}^A$ be defined by $[b(S)](a) = 1$ if $a \in S$, and $[b(S)](a) = 1$ if $a \notin S$. Clearly if $S \neq T$ then we can choose $x \in S - T$ (or $x \in T - S$) and have $[b(S)](x) \neq [b(T)](x)$, implying $b(S) \neq b(T)$, so b is injective. Further, if $f : A \to \{0,1\}$, we can take $S = \{a \in A \mid f(a) = 1\}$, and then we have $[b(s)](a) = f(a)$ for all $a \in A$, so $b(s) = f$, so b is surjective.

Now, ${}^*b : {}^*\mathcal{P}(A) \to \{0,1\}^{{}^*A}$ (note ${}^*\{0,1\} = \{0,1\}$), which is bijective by exercise 13.8.1, so we're done.

5 Notes & Goals

5.1 Notes

Chapter 13 is insane. We're way off the rails. As far as I can tell, we don't get back on track (i.e. we don't start talking about how any of this relates to ultrafilters) until section 14.3, where an ultrafilter is used to create an "enlargement" of a universe (which is in turn used to create a nonstandard framework in which ${}^*\mathbb{X} - \mathbb{X} \neq \emptyset$). It is very, very cool though, since you're in some sense doing a higher-order logic (since you can quantify over subsets and functions and things).

Perhaps writing out all the stuff about the language before reading chapter 13 was a mistake, given how some of the quirks of the language start to make more sense in the context of universes (why it allows for undefined terms, why it requires all quantifiers to be bounded, etc.). I also wasn't sure how to cleanly integrate set symbols, since Goldblatt makes absolutely no effort to. I play a little loosey-goosey with notation in a few places, too, in a way I hope isn't too confusing. I'm also torn how much of a stickler to be about separating *symbols* from the underlying constants, relations, etc.—in Mathematical Logic it felt very important, but I think being less strict about it here makes sense since I have less time to explain (Goldblatt identifies constant symbols with the constants they represent, for instance).

I've written some stuff down about integrals, but haven't TeXed it if only because I don't want to have to TeX all the stuff about hyperfinite sums. Alas, TeX it I shall.

5.2 Goals for Next Week

- Read Chapters 13.14—14 (pp. 176-189) in Goldblatt. I'm starting to think this might be mostly for fun at this point. Do a few exercises.
- TeX up the hyperfinite sums definition of integration, explain how it's basically the same.
- Sit down and make a list of everything I want to include in my thesis, and make sure I have enough exercises and new stuff to feel like I'm not just writing a book report.

Things I'm Looking At

[1] Robert Goldblatt. *Lectures on the Hyperreals*. Vol. 188. Graduate Texts in Mathematics. New York, NY: Springer, 1998. ISBN: 978-1-4612-6841-3 978-1-4612-0615-6. DOI: `10.1007/978-1-4612-0615-6`. URL: `http://link.springer.com/10.1007/978-1-4612-0615-6` (visited on 09/25/2024).