Infinitesimal Calculus

Paul Schulze

History

- Newton & Leibniz formulated calculus using the idea of *infinitesimals*.
- Infinitesimals are really really small, but not 0.
- Considered nonsensical, replaced with δ-ϵ.
- Early 1960's: Abraham Robinson formalizes Nonstandard Analysis.
- Most modern formulations are based on work by Jerzy Łoś.

(a) Abraham Robinson

(b) Jerzy Łoś

Hyperreals

We construct a set of *hyperreals*, call it ${}^*\mathbb{R}$, such that we know three things:

1. $\mathbb{R} \subseteq {}^*\mathbb{R}$
2. ${}^*\mathbb{R}$ contains at least one infinitesimal δ, such that $0 < \delta$ but $\delta < r$ for any positive real number r
3. Any sentence of first-order logic is true in $\mathbb{R}$ iff it is true* in ${}^*\mathbb{R}$

First-Order Logic

- Our logical language has the following logical symbols:
  - $\&$ for "and"
  - $\vee$ for "or"
  - $\rightarrow$ for "if...then..."
  - $\leftrightarrow$ for "if and only if"
  - $\forall$ for "for all"
  - $\exists$ for "there exists"
  - $\in$ for set membership
- $5 + 3 = 8 \; \& \; 2^3 = 8$
- $1 + 1 = 1 \vee 5 + 7 = 12$
- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{N})(x < y)$
- $(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})(x < \frac{1}{n} \rightarrow n < \frac{1}{x})$

Transfer Principle

Every sentence that's true in $\mathbb{R}$ is also "true in ${}^*\mathbb{R}$," when modified to be talking about ${}^*\mathbb{R}$. For instance:

- $(\forall x \in {}^*\mathbb{R})(\forall y \in {}^*\mathbb{R})(x + y = y + x)$, i.e. addition is commutative
  - We can similarly prove other arithmetic properties, so we can do algebra as normal in ${}^*\mathbb{R}$
- $(\forall x \in \mathit{hreals})(\exists y \in {}^*\mathbb{R})(x \neq 0 \to x \cdot y = 1)$, i.e. for any x there exists a multiplicative inverse $\frac{1}{x}$
  - So if δ is an infinitesimal, then $\frac{1}{\delta}$ has to exist
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Infinitesimals

- We construct a set of *hyperreals* ${}^*\mathbb{R} \supseteq \mathbb{R}$.
- ${}^*\mathbb{R}$ is "like" $\mathbb{R}$, but it includes *infinitesimals*, elements δ such that $\delta \neq 0$ but $|\delta| < r$ for every $r \in \mathbb{R}^+$.
- We can add these infinitesimals to other numbers to get things like $1 + \delta$, a number that is "infinitely close to" 1 but not 1.
- If $|x - y|$ is infinitesimal or 0, we say $x \simeq y$
- If $x \in {}^*\mathbb{R}$, we denote by $\mathrm{st}(x)$ the *standard part of* x, the unique real number that is infinitely close to x. $\mathrm{st}(1 + \delta) = 1$.
- We can also take the recipricals of these infinitesimals to get *unbounded* hyperreals, like $\frac{1}{\delta}$. These have no standard part.
- We can of course combine all these elements however we'd like. If δ and γ are infinitesimals, we have $\frac{1}{\delta} + 4 + \pi + \gamma \in {}^*\mathbb{R}$.

Derivatives, the way Leibniz intended

- Say $f : \mathbb{R} \to \mathbb{R}$. We"extend" f to ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$.
- Fix $b \in \mathbb{R}$. Let Δx be infinitesimal, and let Δf be ${}^*f(b + \Delta x) - {}^*f(b)$.
- Then define $f'(b) = \operatorname{st}\left(\frac{\Delta f}{\Delta x}\right)$. So $f'(b) \simeq \frac{\Delta f}{\Delta x}$.
- **Example:** Say $f(x) = x^2$. Then we have

$$\begin{aligned} f'(3) \simeq \frac{(3+\Delta x)^2 - 3^2}{\Delta x} &= \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ = \frac{6\Delta x + (\Delta x)^2}{\Delta x} &= 6 + \Delta x \simeq 6 \end{aligned}$$

- So $f'(3) \simeq 6$. But these are both real numbers, so their difference can't be infinitesimal. Hence $f'(3) = 6$.

Proof: Chain Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let Δx be any infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Since $g'(x) = \mathrm{st}(\Delta g / \Delta x)$ is defined, Δg must be infinitesimal. Then

$$\begin{aligned}(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\ &= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\ &\simeq f'(g(x)) \cdot g'(x)\end{aligned}$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

In the case where $\Delta g = 0$, we clearly have $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.