Infinitesimal Calculus

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History

- Newton & Leibniz formulated calculus using the idea of *infinitesimals*
- Infinitesimals are really really small, but not 0
- Considered nonsensical, replaced with δ-ϵ
- Early 1960's: Abraham Robinson formalizes Nonstandard Analysis
- Our formulation of infinitesimals is based off work by Jerzy Łoś

(a) Abraham Robinson

(b) Jerzy Łoś

Basic Idea

- We construct a set of *hyperreals*, denoted ${}^*\mathbb{R}$.
- ${}^*\mathbb{R}$ includes $\mathbb{R}$, along with (a lot of) hyperreals.
- We construct a language of mathematical logic, using symbols like $\neg$, $\wedge$, $\vee$, $\forall$, etc.
- We show that any sentence of that language is true in $\mathbb{R}$ iff it is true in ${}^*\mathbb{R}$ (transfer principle).
- We use transfer to prove things about $\mathbb{R}$.

Constructing ${}^*\mathbb{R}$

- We start with the ring $\mathbb{R}^\infty$.
- Identify sequences that are the same "almost everywhere," like $\langle 0, 1, 1, \ldots \rangle \sim \langle 1, 1, 1, \ldots \rangle$.
- Sequences are the same almost everywhere if the set of indices at which they are the same is "big."
- The set of "big" sets of natural numbers is an *ultrafilter* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$.

Constructing ${}^*\mathbb{R}$

- Now we can define our equivalence relation $\sim$ by saying that $\langle r_1, r_2, r_3, \ldots \rangle \sim \langle s_1, s_2, s_3, \ldots \rangle$ iff $\{n \in \mathbb{N} \,|\, r_n = s_n\} \in \mathcal{F}$.
- Write the equivalence class $[\langle r_1, r_2, r_3, \ldots \rangle]$.
- Define ${}^*\mathbb{R}$ as the quotient ring of $\mathbb{R}^\infty$ under $\sim$, i.e. ${}^*\mathbb{R} = \{[\langle r_1, r_2, r_3, \ldots \rangle] \,|\, \langle r_1, r_2, \ldots \rangle \in \mathbb{R}^\infty\}$.
- Extend any function $f : \mathbb{R} \to \mathbb{R}$ to a new ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$. ${}^*f([\langle r_1, r_2, r_3, \ldots \rangle]) = [\langle f(r_1), f(r_2), f(r_3), \ldots \rangle]$.
- We can also extend relations, like "$\leq$." $\langle r_1, r_2, r_3, \ldots \rangle \leq \langle s_1, s_2, s_3, \ldots \rangle$ iff $\{n \in \mathbb{N} \,|\, r_n \leq s_n\} \in \mathcal{F}$.
- For any $x \in \mathbb{R}$, we can take $x \in {}^*\mathbb{R}$ to mean $[\langle x, x, x, \ldots \rangle]$.

Transfer Principle

- Our language is made up of:
  - logical connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$,
  - quantifiers $\forall$, $\exists$,
  - parenthesis (and),
  - variables $x, y, z, \ldots$, and
  - symbols for every element of $\mathbb{R}$, every relation on $\mathbb{R}$, and every function on $\mathbb{R}$.
- Sentences are defined recursively.
  $(\forall x \in \mathbb{R})(x < x + 1)$ vs. $))5+ \leq v_1 \leftrightarrow \neg \wedge 64$
- If φ is a sentence that "talks about" $\mathbb{R}$, we can obtain ${}^*\varphi$ by replacing each function and relation with its extension in ${}^*\mathbb{R}$.
- **Transfer Principle:** Any sentence φ is true iff ${}^*\varphi$ is true.

Structure of ${}^*\mathbb{R}$

- We call an element $x \in {}^*\mathbb{R}$:
  - *infinitesimal* if $|x| < r$ for any $r \in \mathbb{R}^+$,
  - *unbounded* if $r < |x|$ for any $r \in \mathbb{R}^+$, and
  - *appreciable* if it is neither infinitesimal nor unbounded.
- Arithmetic properties of hyperreals are mostly intuitive.
- We say two elements $x, y \in {}^*\mathbb{R}$ are *infinitely close*, and write $x \simeq y$, if $|x - y|$ is infinitesimal. $\simeq$ is an equivalence relation.
- Any bounded hyperreal x is infinitely close to a unique real number, called its *standard part* and denoted $\mathrm{st}(x)$.
- st has most of the nice properties you'd like it to: $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y)$, etc.

Derivatives

- Say $f : \mathbb{R} \to \mathbb{R}$. Extend f to ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$
- Fix $b \in \mathbb{R}$. Let Δx be infinitesimal, and let Δf be ${}^*f(b + \Delta x) - {}^*f(b)$.
- Then define $f'(b) = \mathrm{st}\left(\frac{\Delta f}{\Delta x}\right)$. So $f'(b) \simeq \frac{\Delta f}{\Delta x}$
- **Example:** Say $f(x) = x^2$. Then we have

$$f'(3) \simeq \frac{(3 + \Delta x)^2 - 3^2}{\Delta x} = \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x}$$
$$= \frac{6\Delta x + (\Delta x)^2}{\Delta x} = 6 + \Delta x \simeq 6$$

- So $f'(3) \simeq 6$. But these are both real numbers, so their difference is real. Hence $f'(3) = 6$.

Proof: Chain Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let Δx be any nonzero infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Then

$$\begin{aligned}(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\ &= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\ &\simeq f'(g(x)) \cdot g'(x)\end{aligned}$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

When $\Delta g = 0$, $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.

Further, $g'(x) = \mathrm{st}\left(\frac{\Delta g}{\Delta x}\right) = 0$, so $f'(g(x)) \cdot g'(x) = 0$.

Series

- Recall there are unbounded hypernaturals like $[\langle 1, 2, 3, \ldots \rangle] \in {}^*\mathbb{N}$.
- We take infinite series by extending finite series.
- Say we have a sequences $\langle r_1, r_2, r_3, \ldots \rangle$, and we define $\sum_{i=0}^{n} r_i$ normally.
- Let $s : \mathbb{N} \to \mathbb{R}$ be defined by $s(n) = \sum_{i=0}^{n} r_i$. Extend s to ${}^*s : {}^*\mathbb{N} \to {}^*\mathbb{R}$.
- For any $M \in {}^*\mathbb{N}$, write $\sum_{i=0}^{M} r_i = s(M)$.
- If $\operatorname{st}\left(\sum_{i=0}^{M} r_i\right) = L$ for all unbounded M, write $\sum_{i=0}^{\infty} r_i = L$.

Geometric Series

- Say $0 < r < 1$. We want to evaluate $\sum_{i=0}^{\infty} r^i$.
- By difference of powers, $1 - r^{n+1} = (1-r)(1 + r^2 + \cdots + r^n)$.
- So $(\forall n \in \mathbb{N})(\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r})$. Transfer this.
- Plug in an unbounded M. $\sum_{i=0}^{M} r^i = \frac{1-r^{M+1}}{1-r}$.
- So $\mathrm{st}\left(\sum_{i=0}^{M} r^i\right) = \frac{\mathrm{st}\left(1-r^{M+1}\right)}{\mathrm{st}(1-r)} = \frac{1-\mathrm{st}\left(r^{M+1}\right)}{1-r} = \frac{1}{1-r}$.
- So $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$.

exp

- We define a function $\exp : \mathbb{R} \to \mathbb{R}$ by $\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.
- We'd like to show that $\exp'(x) = \exp(x)$.
- In other words, that $\mathrm{st}\left(\frac{{}^*\exp(x+\Delta x) - {}^*\exp(x)}{\Delta x}\right) = \exp(x)$ for any nonzero infinitesimal Δx.
- Since both sides are real, it suffices to show they are infinitely close, i.e. that

$$\left|\frac{{}^*\exp(x+\Delta x) - {}^*\exp(x)}{\Delta x} - \exp(x)\right| \text{ is infinitesimal}$$

- To show that, we will put a bound on

$$\left|\sum_{i=0}^{k} \frac{(x+d)^i}{i!} - \sum_{i=0}^{k} \frac{x^i}{i!} - d \cdot \sum_{i=0}^{k-1} \frac{x^i}{i!}\right|$$

and apply transfer several times.