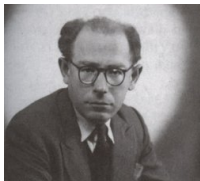


Nonstandard Analysis

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History

- ▶ Newton & Leibniz formulated calculus using the idea of *infinitesimals*
- ▶ Infinitesimals are really really small, but not 0
- ▶ Considered nonsensical, replaced with δ - ϵ
- ▶ 1960s: Abraham Robinson formalizes Nonstandard Analysis
- ▶ Most modern formulations are based on work by Jerzy Łoś



(a) Abraham Robinson



(b) Jerzy Łoś

Hyperreals

We construct a set of *hyperreals*, call it ${}^*\mathbb{R}$, such that we know three things:

1. $\mathbb{R} \subseteq {}^*\mathbb{R}$
2. ${}^*\mathbb{R}$ contains at least one infinitesimal δ , such that $0 < \delta$ but $\delta < r$ for any positive real number r
3. **Transfer Principle:** Any sentence of first-order logic is true in \mathbb{R} iff it is true* in ${}^*\mathbb{R}$

First-Order Logic

- ▶ Our logical language has, in addition to symbols for numbers, sets, and functions on real numbers, the following logical symbols:
 - ▶ \neg for “not”
 - ▶ \rightarrow for “if... then...”
 - ▶ \forall for “for all”
 - ▶ \exists for “there exists”
 - ▶ \in for set membership
- ▶ $5 + 3 = 8$
- ▶ $(\forall x \in \mathbb{R})(5 + x = 8 \rightarrow x = 3)$
- ▶ $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(x < y)$
- ▶ $(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})(x < \frac{1}{n} \rightarrow n < \frac{1}{x})$

Transfer Principle

Every sentence that's true in \mathbb{R} is also “true in ${}^*\mathbb{R}$,” when modified to be talking about ${}^*\mathbb{R}$. For instance:

- ▶ $(\forall x \in {}^*\mathbb{R})(\forall y \in {}^*\mathbb{R})(x + y = y + x)$, i.e. addition is commutative
 - ▶ We can similarly prove other arithmetic properties, so we can do algebra as normal in ${}^*\mathbb{R}$
- ▶ $(\forall x \in {}^*\mathbb{R})(x \neq 0 \rightarrow (\exists y \in {}^*\mathbb{R})(x \cdot y = 1))$, i.e. for any nonzero x there exists a multiplicative inverse $\frac{1}{x}$
 - ▶ So even if δ is an infinitesimal, we know $\frac{1}{\delta}$ has to exist

What is ${}^*\mathbb{R}$ like?

- ▶ Call a hyperreal x *infinitesimal* when $|x| < r$ for every positive real r .
 - ▶ There's exactly one real infinitesimal: 0
- ▶ There are “infinite” hyperreals—take any positive infinitesimal δ , and $\frac{1}{\delta}$ is greater than any real number
- ▶ Two hyperreals x and y are *infinitely close*, denoted $x \simeq y$, when their difference $x - y$ is infinitesimal
 - ▶ No two distinct real numbers are infinitely close to each other
 - ▶ Transitive: if $x \simeq y$ and $y \simeq z$, then $x \simeq z$
- ▶ Any finite hyperreal x is infinitely close to exactly one real number, called its *standard part* $\text{st}(x)$
 - ▶ For instance, $\text{st}(1 + \delta) = 1$

Derivatives, the way Leibniz intended

- ▶ Say $f : \mathbb{R} \rightarrow \mathbb{R}$
- ▶ Fix $b \in \mathbb{R}$, and let Δx be a nonzero infinitesimal. Then

$$f'(b) = \text{st} \left(\frac{f(b + \Delta x) - f(b)}{\Delta x} \right)$$

so that $f'(b) \simeq \frac{f(b + \Delta x) - f(b)}{\Delta x}$.

- ▶ **Example:** Say $f(x) = x^2$. Then we have

$$\begin{aligned} f'(3) &\simeq \frac{(3 + \Delta x)^2 - 3^2}{\Delta x} = \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ &= \frac{6\Delta x + (\Delta x)^2}{\Delta x} = 6 + \Delta x \simeq 6. \end{aligned}$$

- ▶ So $f'(3) \simeq 6$, so their difference is infinitesimal. But they're both *real numbers*, so $f'(3) = 6$.

Continuity

We call $f(x)$ *continuous* at a point c if for every hyperreal $x \simeq c$, we have $f(x) \simeq f(c)$.

Theorem

The composition of continuous functions is continuous.

Proof.

Let c be some real number, and let $x \simeq c$. We want to show $(f \circ g)(x) \simeq (f \circ g)(c)$, given that g is continuous at c and f is continuous at $g(c)$.

Since g is continuous at c , $g(x) \simeq g(c)$. Since f is continuous at $g(c)$, we have $f(g(x)) \simeq f(g(c))$. □

Proof: Chain Rule

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Let Δx be any infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Since $g'(x) = \text{st}(\Delta g / \Delta x)$ is defined, Δg must be infinitesimal. Then

$$\begin{aligned}(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\&= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\&\simeq f'(g(x)) \cdot g'(x)\end{aligned}$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

In the case where $\Delta g = 0$, we clearly have $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.