

Infinitesimal Calculus

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History

- ▶ Newton & Leibniz formulated calculus using the idea of *infinitesimals*.
- ▶ Infinitesimals are really really small, but not 0.
- ▶ Considered nonsensical, replaced with δ - ϵ .
- ▶ Early 1960's: Abraham Robinson formalizes Nonstandard Analysis.
- ▶ Most modern formulations are based on work by Jerzy Łoś.



(a) Abraham Robinson



(b) Jerzy Łoś

Hyperreals

We construct a set of *hyperreals*, call it ${}^*\mathbb{R}$, such that we know three things:

1. $\mathbb{R} \subseteq {}^*\mathbb{R}$
2. ${}^*\mathbb{R}$ contains at least one infinitesimal δ , such that $0 < \delta$ but $\delta < r$ for any positive real number r
3. Any sentence of first-order logic is true in \mathbb{R} iff it is true* in ${}^*\mathbb{R}$

First-Order Logic

Infinitesimals

- ▶ We construct a set of *hyperreals* ${}^*\mathbb{R} \supseteq \mathbb{R}$.
- ▶ ${}^*\mathbb{R}$ is “like” \mathbb{R} , but it includes *infinitesimals*, elements δ such that $\delta \neq 0$ but $|\delta| < r$ for every $r \in \mathbb{R}^+$.
- ▶ We can add these infinitesimals to other numbers to get things like $1 + \delta$, a number that is “infinitely close to” 1 but not 1.
- ▶ If $|x - y|$ is infinitesimal or 0, we say $x \simeq y$
- ▶ If $x \in {}^*\mathbb{R}$, we denote by $\text{st}(x)$ the *standard part of* x , the unique real number that is infinitely close to x . $\text{st}(1 + \delta) = 1$.
- ▶ We can also take the reciprocals of these infinitesimals to get *unbounded* hyperreals, like $\frac{1}{\delta}$. These have no standard part.
- ▶ We can of course combine all these elements however we’d like. If δ and γ are infinitesimals, we have $\frac{1}{\delta} + 4 + \pi + \gamma \in {}^*\mathbb{R}$.

Derivatives, the way Leibniz intended

- ▶ Say $f : \mathbb{R} \rightarrow \mathbb{R}$. We “extend” f to $*f : *\mathbb{R} \rightarrow *\mathbb{R}$.
- ▶ Fix $b \in \mathbb{R}$. Let Δx be infinitesimal, and let Δf be $*f(b + \Delta x) - *f(b)$.
- ▶ Then define $f'(b) = \text{st} \left(\frac{\Delta f}{\Delta x} \right)$. So $f'(b) \simeq \frac{\Delta f}{\Delta x}$.
- ▶ **Example:** Say $f(x) = x^2$. Then we have

$$\begin{aligned} f'(3) &\simeq \frac{(3 + \Delta x)^2 - 3^2}{\Delta x} = \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ &= \frac{6\Delta x + (\Delta x)^2}{\Delta x} = 6 + \Delta x \simeq 6 \end{aligned}$$

- ▶ So $f'(3) \simeq 6$. But these are both real numbers, so their difference can't be infinitesimal. Hence $f'(3) = 6$.

Proof: Chain Rule

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Let Δx be any infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Since $g'(x) = \text{st}(\Delta g / \Delta x)$ is defined, Δg must be infinitesimal. Then

$$\begin{aligned}(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\&= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\&\simeq f'(g(x)) \cdot g'(x)\end{aligned}$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

In the case where $\Delta g = 0$, we clearly have $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.