1. **Ultrafilters, Ultrapowers,** $^*\mathbb{R}$. A definition of an ultrafilter, and the proof that one exists on $\mathbb{N}$. The ultrapower construction of $^*\mathbb{R}$, inlcuding the extension of any function or relation. The definition of internal sets and functions.

2. **First-Order Logic & Transfer**. A definition of our language on $\mathbb{R}$, including set symbols (sort of new). The proof of the alteration of Łoś' theorem (also sort of new).

3. **Structure of** $^*\mathbb{R}$. Introduction of basic terms such as infinitesimal, appreciable, and unbounded. Standard parts and infinite closeness. Density of the hyperrationals, archimedian property, things like that. Maybe saturation (intuitive explanation). Likely light on proofs, since I don't have any new work here.

4. **Differentiation**. One of the motivations of nonstandard analysis is its intuitiveness, and that is demonstrated here. Proof of the chain rule, because it's short. Maybe some thoughts on partial derivatives and the care you have to take there, restricting the derivative to the reals and re-extending it.

5. **Hyperfinite Sums**. A treatment of series, including the proof of the ratio test. exp as a hyperfinite sum, and the proof that it is its own derivative. Lots of new stuff, but I'm not sure if it's any good.

6. **Integration**. Integrals as hyperfinite sums. Maybe some of the technical worries about the ends of partitions worked out (which would be new), although that sounds a bit dull and like it would take more space than it really should.

7. **Topology**. Questionable inclusion. Maybe some proofs that the topologies Goldblatt defines are *actually* topologies, if possible.

1 Ultrafilters, Ultrapowers, $^*\mathbb{R}$

2 First Order Logic & Transfer

3 Differentiation

3.1 Definition

One of the primary motivations for infinitesimal calculus is that it allows use to access the more intuitive, less roundabout conceptualizations of derivatives and integrals. Unlike integrals, which still take some work to define nonstandardly, the nonstandard derivative is almost exactly what we would first guess it to be.

Let Δx be a nonzero infinitesimal, and let $\Delta f(x, \Delta x) = f(x + \Delta x) - f(x)$. Intuitively, Δx is an infinitesimal change in x, and Δf is the corresponding change in f caused by "moving" Δx along the x-axis. Then we have:

$$f'(x) = \frac{\Delta f(x, \Delta x)}{\Delta x}$$

One small problem: we'd like f' to be a real-valued function on the reals. Luckily, we have a tool to do that:

$$f'(x) = \operatorname{st}\left(\frac{\Delta f(x, \Delta x)}{\Delta x}\right)$$

And we have our definition. Well, this might also not be well-defined: we get around *that* problem by definition. We only consider $f'(x)$ to exist when $\operatorname{st}\left(\frac{\Delta f(x,\Delta x)}{\Delta x}\right)$ is the same for *any* nonzero infinitesimal Δx. In this case, we say that f is *differentiable* at x.

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$ and a real number x, we say that f is *differnetiable* at x if there is some constant $f'(x)$ such that, for any nonzero infinitesimal Δx,

$$f'(x) = \operatorname{st}\left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right)$$

3.2 Simple Proofs Using Infinitesimals

A number of proofs of basic calculus results can be easily accomplished by infinitesimals. Perhaps most striking is the chain rule.

Theorem 3.1 (Chain Rule)**.** *Given differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$,*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Proof. Let Δx be any nonzero infinitesimal, and let $\Delta g = g(x + \Delta x) - g(x)$. If $\Delta g = 0$, then $g(x + \Delta x) = g(x)$, so $(f \circ g)'(x) = \operatorname{st}\left(\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}\right) = 0$ and clearly $g'(x) = \operatorname{st}(\Delta g/\Delta x) = 0$, so we are done. If $\Delta g \neq 0$, then since $g'(x) = \operatorname{st}(\Delta g/\Delta x)$ is defined, we conclude $\Delta g/\Delta x$ is bounded and (since Δx is infinitesimal) that Δg is infinitesimal. Thus

$$\begin{aligned}
(f \circ g)'(x) &= \operatorname{st}\left(\frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x}\right) \\
&= \operatorname{st}\left(\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x}\right) \\
&= \operatorname{st}\left(\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}\right) \cdot \operatorname{st}\left(\frac{\Delta g}{\Delta x}\right) \\
&= f'(g(x)) \cdot g'(x)
\end{aligned}$$

□

4 Hyperfinite Sums

4.1 Hyperfinite Sets

Definition. An internal set $A = [A_n]$ is *hyperfinite* if every A_n is finite.

The hyperfinite sets are "internally finite." They share a lot of properties with finite sets.

Theorem 4.1. *If $\varphi(A)$ holds for every finite $A \subseteq \mathbb{R}$, then $^*\varphi(X)$ holds for every hyperfinite $X \subseteq {}^*\mathbb{R}$.*

Proof. Say $X = [A_n]$, with each A_n finite. Then $[[\varphi(A_n)]] = \mathbb{N}$, and so by transfer $^*\varphi(X)$. □

This lets us easily get a lot of nice properties about hyperfinite sets. For instance, $(\exists x \in A)(\forall y \in A)(x \geq y)$ ensures that every hyperfinite set has a maximum element.

4.2 Hyperfinite Sums

For any finite set A_n and function $f_n : \mathbb{R} \to \mathbb{R}$, we can easily define the sum $\sum_{x \in A_n} f(x)$. This is, after all, just a sum of a finite collection of numbers. Using this, however, we can easily extend our summation to hyperfinite sets:

Definition. If $A = [A_n]$ is a hyperfinite set, and $f = [f_n]$ is an internal function, we define the *hyperfinite sum*

$$\sum_{x \in [A_n]} f(x) = \left[\sum_{x \in A_n} f_n(x) \right]$$

We'll see this general form used later for integration, but for now we will focus on the type of hyperfinite sums that corresponds to series in standard calculus. Fist, some notation. Let $\{0 \ldots n\}$ denote $\{0, 1, 2, \ldots, n\}$ for any $n \in \mathbb{N}$. Then, let $\mathcal{N} = [\langle 0, 1, 2, \ldots \rangle]$, and let $\{0 \ldots \mathcal{N}\}$ denote $[\{0 \ldots n\}]$.

For a sequence $a_i : \mathbb{N} \to \mathbb{R}$, we have $\sum_{i \in \{0 \ldots n\}} a_i = \sum_{i=0}^{n} a_i$. So we denote $\sum_{i \in \{0 \ldots \mathcal{N}\}} a_i = \sum_{i=0}^{\mathcal{N}} a_i = [\sum_{i=0}^{n} a_i]$.

Theorem 4.2 (Geometric Series)**.** *Let $0 < r < 1$. Then*

$$\sum_{i=0}^{\mathcal{N}} r^i \simeq \frac{1}{1 - r}$$

Proof. Let $S = \sum_{i=0}^{\mathcal{N}} r^i$. Recall $\sum_{i=0}^{\mathcal{N}} r^i = [\sum_{i=0}^{n} r^i] = [\langle 1, 1{+}r, 1{+}r{+}r^2, \ldots\rangle]$. Then $r \cdot S = [\langle r, r{+}r^2, r{+}r^2{+}r^3, \ldots\rangle]$, so $1 + r \cdot S = [\langle 1 + r, 1 + r + r^2, 1 + r + r^2 + r^3, \ldots\rangle]$. We conclude that $(1 + r \cdot S) - S = [\langle r, r^2, r^3, \ldots\rangle] \simeq 0$ (recall $0 < r < 1$). So we have $S \simeq 1 + r \cdot S$, and so $S \cdot (1 - r) \simeq 1$, and so since $1 - r$ is appreciable $S \simeq \frac{1}{1-r}$. ☐

Corollary 4.3. *For any $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{n} r^i \leq \frac{1}{1-r}$$

Proof. Let $f(n) = \sum_{i=0}^{n} r^i$. Then the extension $f(\mathcal{N}) = [\sum_{i \in \{0 \ldots n\}} r^i] = \sum_{i \in [\{0 \ldots n\}]} r^i = \sum_{i=0}^{\mathcal{N}} r^i$.

Since $0 < r^i$, we have in the reals $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(n < m \to f(n) < f(m))$. If we transfer this to the hyperreals and plug in $\mathcal{N}$ for m, we get $(\forall n \in {}^*\mathbb{N})(n < \mathcal{N} \to f(n) < f(\mathcal{N}))$. Any $n \in \mathbb{N}$ is in the hypernaturals and less than $\mathcal{N}$, and so for any $n \in \mathbb{N}$ we have $f(n) < f(\mathcal{N}) \simeq \frac{1}{1-r}$. Since both $f(n)$ and $\frac{1}{1-r}$ are real, this implies $f(n) \leq \frac{1}{1-r}$. ☐

Theorem 4.4 (Absolute Convergence Implies Convergence)**.** *If $\sum_{i=0}^{\mathcal{N}} |a_i|$ is bounded, then $\sum_{i=0}^{\mathcal{N}} a_i$ is bounded.*

Proof. Say $\sum_{i=0}^{\mathcal{N}} |a_i| < R$ for some real R. For any $n \in \mathbb{N}$, we have $|\sum_{i=0}^{n} a_i| \leq \sum_{i=0}^{n} |a_i| < R$, and so $\left|\sum_{i=0}^{\mathcal{N}} a_i\right| = |[\sum_{i=0}^{n} a_i]| < R$. ☐

Theorem 4.5 (Ratio Test)**.** *Let $a_i : \mathbb{N} \to \mathbb{R}$ be a sequence. If for every unbounded $M \in {}^*\mathbb{N}$ we have $\left|\mathrm{st}\left(\frac{a_{M+1}}{a_M}\right)\right| = L$ for some $L < 1$, then $\sum_{i=0}^{\mathcal{N}} a_i$ is bounded.*

Proof. Assume that $a_i \geq 0$—if not, apply the theorem to $|a_i|$ and use 4.4. Now, take $r \in \mathbb{R}$ such that $L < r < 1$, so that $\left|\frac{a_{M+1}}{a_M}\right| < r < 1$ for any unbounded hypernatural M.

Take any unbounded $N \in {}^*\mathbb{N}$. For any $M \in {}^*\mathbb{N}$, if $N \leq M$, then M is also unbounded, and so $\frac{a_{M+1}}{a_M} < r$. So we have the sentence

$$(\exists n \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})(n \leq m \to \frac{a_{m+1}}{a_m} < r)$$

If we transfer this over to the reals, we find that there is some natural number n such that for any natural $m \geq n$ we have $\frac{a_{m+1}}{a_m} < r$. Then clearly, for any $m > n$, we have

$$\begin{aligned}\sum_{i=0}^{m} a_i &= \sum_{i=0}^{n} a_i + \sum_{i=n+1}^{m} a_i \\ &\leq \sum_{i=0}^{n} a_i + \sum_{i=n+1}^{m} a_n \cdot r^{i-n} \\ &\leq \sum_{i=0}^{n} a_i + a_n \cdot \sum_{i=1}^{m-n} r^i \leq \sum_{i=0}^{n} a_i + a_n \cdot \frac{r}{1-r}\end{aligned}$$

Where that last inequality comes from 4.3.

So, we find that there is some n such that for any $m \geq n$, we have $\sum_{i=0}^{m} a_i \leq \sum_{i=0}^{n} a_i + a_n \cdot \frac{r}{1-r}$. We conclude that $\sum_{i=0}^{\mathcal{N}} a_i \leq \sum_{i=0}^{n} a_i + a_n \cdot \frac{r}{1-r}$, and hence that $\sum_{i=0}^{\mathcal{N}} a_i$ is bounded.

☐