

1. **Ultrafilters, Ultrapowers, ${}^*\mathbb{R}$.** A definition of an ultrafilter, and the proof that one exists on \mathbb{N} . The ultrapower construction of ${}^*\mathbb{R}$, including the extension of any function or relation. The definition of internal sets and functions.
2. **First-Order Logic & Transfer.** A definition of our language on \mathbb{R} , including set symbols (sort of new). The proof of the alteration of Loś' theorem (also sort of new).
3. **Structure of ${}^*\mathbb{R}$.** Introduction of basic terms such as infinitesimal, appreciable, and unbounded. Standard parts and infinite closeness. Density of the hyperrationals, archimedean property, things like that. Maybe saturation (intuitive explanation). Likely light on proofs, since I don't have any new work here.
4. **Differentiation.** One of the motivations of nonstandard analysis is its intuitiveness, and that is demonstrated here. Proof of the chain rule, because it's short. Partial derivatives. Series. \exp as its own derivative.
5. **Integration.** Integrals as hyperfinite sums. Maybe some of the technical worries about the ends of partitions worked out (which would be new), although that sounds a bit dull and like it would take more space than it really should.
6. **Topology.** Questionable inclusion. Maybe some proofs that the topologies Goldblatt defines are *actually* topologies, if possible.

1 Ultrafilters, Ultrapowers, ${}^*\mathbb{R}$

2 First Order Logic & Transfer

3 Structure of ${}^*\mathbb{R}$

4 Differentiation

4.1 Definition

One of the primary motivations for infinitesimal calculus is that it allows us to access the more intuitive, less roundabout conceptualizations of derivatives and integrals. Unlike integrals, which still take some work to define nonstandardly, the nonstandard derivative is almost exactly what we would first guess it to be.

Let Δx be a nonzero infinitesimal, and let $\Delta f(x, \Delta x) = f(x + \Delta x) - f(x)$. Intuitively, Δx is an infinitesimal change in x , and Δf is the corresponding change in f caused by “moving” Δx along the x -axis. Then we have:

$$f'(x) = \frac{\Delta f(x, \Delta x)}{\Delta x}.$$

One small problem: we'd like f' to be a real-valued function on the reals. Luckily, we have a tool to do that:

$$f'(x) = \text{st} \left(\frac{\Delta f(x, \Delta x)}{\Delta x} \right).$$

And we have our definition. Well, this might also not be well-defined: we get around *that* problem by definition. We only consider $f'(x)$ to exist when $\text{st} \left(\frac{\Delta f(x, \Delta x)}{\Delta x} \right)$ is the same for *any* nonzero infinitesimal Δx . In this case, we say that f is *differentiable* at x . Note that this definition of differentiability and the derivative is equivalent to the standard definition.

Theorem 4.1 ([1, p. 8.1.1]). *If f is defined at $x \in \mathbb{R}$, then $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = L$ iff for every nonzero infinitesimal ϵ , $f(x + \epsilon)$ is defined and $\frac{f(x + \epsilon) - f(x)}{\epsilon} \simeq L$.*

Definition. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a real number x , we say that f is *differentiable* at x if there is some constant $f'(x)$ such that, for any nonzero infinitesimal Δx ,

$$f'(x) = \text{st} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right).$$

4.2 Simple Proofs Using Infinitesimals

A number of proofs of basic calculus results can be easily accomplished by infinitesimals. Perhaps most striking is the chain rule.

Theorem 4.2 (Chain Rule). *Given differentiable functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$,*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Proof adapted from [1, ch. 8.4]. Let Δx be any nonzero infinitesimal, and let $\Delta g = g(x + \Delta x) - g(x)$. If $\Delta g = 0$, then $g(x + \Delta x) = g(x)$, so $(f \circ g)'(x) = \text{st} \left(\frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \right) = 0$ and clearly $g'(x) = \text{st} \left(\frac{\Delta g}{\Delta x} \right) = 0$, so we are done. If $\Delta g \neq 0$, then since $g'(x) = \text{st} \left(\frac{\Delta g}{\Delta x} \right)$ is defined, we conclude $\frac{\Delta g}{\Delta x}$ is bounded and (since Δx is infinitesimal) that Δg is infinitesimal. Thus

$$\begin{aligned} (f \circ g)'(x) &= \text{st} \left(\frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \right) \\ &= \text{st} \left(\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \right) \\ &= \text{st} \left(\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \right) \cdot \text{st} \left(\frac{\Delta g}{\Delta x} \right) \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

□

4.3 Partial Derivatives

This section is all individual work without the guidance of any texts.

Definition. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$, then we define

$$f_{x_k}(b_1, b_2, \dots, b_n) = \text{st} \left(\frac{f(b_1, \dots, b_k + \Delta x, \dots, b_n) - f(b_1, \dots, b_k, \dots, b_n)}{\Delta x} \right)$$

for any infinitesimal Δx . This is only defined when it doesn't depend on our choice of Δx .

This gets complicated when we want to deal with repeated partial derivatives. Say $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, and denote the inputs of f by $f(x, y)$. Then we might want to write

$$\begin{aligned} f_{yx}(a, b) &= \text{st} \left(\frac{f_y(a + \Delta x, b) - f_y(a, b)}{\Delta x} \right) \\ &= \text{st} \left(\frac{\text{st} \left(\frac{f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b)}{\Delta y} \right) - \text{st} \left(\frac{f(a, b + \Delta y) - f(a, b)}{\Delta y} \right)}{\Delta x} \right), \end{aligned}$$

which would allow us to easily get $f_{yx}(a, b) = f_{xy}(a, b)$. However, this isn't right. Firstly, the numerator is the difference of two real numbers, and so is real, which would mean $f_{yx}(a, b)$ is either 0 or undefined (as a real divided by an infinitesimal is either 0 or unbounded). The mistake here is that $f_y(a + \Delta x, b) \neq \text{st} \left(\frac{f(a + \Delta x, b + \Delta y) - f(a + \Delta x, b)}{\Delta y} \right)$. Since in this case f_y is taking a nonreal input, we have to use the extension $*f_y(a + \Delta x, b)$. If $\Delta x = [\Delta x_n]$, then this is $[f_y(a + \Delta x_n, b)] = \left[\text{st} \left(\frac{f(a + \Delta x_n, b + \Delta y) - f(a + \Delta x_n, b)}{\Delta y} \right) \right]$. This is the equivalence class of a sequence of real numbers, but it needn't be a real number itself.

Theorem 4.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Let \hat{f}_{x_k} denote the standard definition of the partial derivative, namely*

$$\hat{f}_{x_k}(b_1, \dots, b_n) = \lim_{h \rightarrow 0} \frac{f(b_1, \dots, b_k + h, \dots, b_n) - f(b_1, \dots, b_k, \dots, b_n)}{h}.$$

Then $\hat{f}_{x_k}(b_1, \dots, b_n)$ exists iff $f_{x_k}(b_1, \dots, b_n)$ does. If they both exist, then $\hat{f}_{x_k}(b_1, \dots, b_n) = f_{x_k}(b_1, \dots, b_n)$.

Proof. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x) = f(b_1, \dots, b_{k-1}, x, b_{k+1}, \dots, b_n)$. Let $\hat{g}'(x)$ denote the standard derivative of g and $g'(x)$ denote the nonstandard derivative of g . Clearly $\hat{g}'(x) = \hat{f}_{x_k}(x)$ and $g'(x) = f_{x_k}(x)$, and Theorem 4.1 we have $\hat{g}'(b_k)$ is defined iff $g'(b_k)$, and $\hat{g}'(b_k) = g'(b_k)$ if both exist, so we are done. □

4.4 Series

The definitions in this section are thanks to Goldblatt, and most of the results are exercises.

Let $a_n : \mathbb{N} \rightarrow \mathbb{R}$. We can define $s_n : \mathbb{N} \rightarrow \mathbb{R}$ by

$$s_n = \sum_{i=0}^n a_i.$$

Now, for any unbounded $N \in {}^*\mathbb{N}$, we can define

$$\sum_{i=0}^N a_i = {}^*s_N.$$

Finally, if for any unbounded $N, M \in {}^*\mathbb{N}$ we have $\sum_{i=0}^N a_i \simeq \sum_{i=0}^M a_i$, then we say $\sum_{i=0}^\infty a_i$ converges and define

$$\sum_{i=0}^\infty a_i = \text{st} \left(\sum_{i=0}^N a_i \right).$$

When $n > m$, we write $\sum_{i=m}^n a_i$ to mean $\sum_{i=0}^n a_i - \sum_{i=0}^{m-1} a_i$. This gives the expected result when using finite naturals, but also extends to unbounded hypernaturals. Then $\sum_{i=0}^\infty a_i$ converges when $\sum_{i=M+1}^N a_i \simeq 0$ for any unbounded $N, M \in {}^*\mathbb{N}$ such that $N > M$.

Theorem 4.4 (Cauchy Convergence Criterion for Series). *If for any unbounded $N, M \in {}^*\mathbb{N}$, $\sum_{i=0}^N a_i \simeq \sum_{i=0}^M a_i$, then $\sum_{i=0}^\infty a_i$ converges.*

Proof adapted from [1, thm. 6.5.2]. The issue here is that $\sum_{i=0}^N a_i$ might be unbounded for all unbounded $N \in {}^*\mathbb{N}$. Let $s(n) = \sum_{i=0}^n a_i$.

First, assume that (in \mathbb{R}) $\neg(\exists r \in \mathbb{R})(\forall n \in \mathbb{N})(|s(n)| < r)$. This says the sequence $s(n)$ is not bounded. By transfer, this is true in ${}^*\mathbb{R}$ as well. Take an unbounded $N \in {}^*\mathbb{N}$. Since $|s(n)|$ has no upper bound in ${}^*\mathbb{R}$, there is some $M \in {}^*\mathbb{N}$ such that $|s(N)| \leq 2 \cdot |s(M)| < |s(M)|$. Since $|s(N)| \simeq |s(M)|$, this implies $|s(N)| \simeq 2 \cdot |s(M)|$, implying $|s(N)| \simeq 0$, and so for any unbounded $M \in {}^*\mathbb{N}$ we have $|s(M)| \simeq |s(N)| \simeq 0$ and so $\sum_{i=0}^\infty a_i = 0$.

Now, say $(\exists r \in \mathbb{R})(\forall n \in \mathbb{N})(|s(n)| < r)$. Whatever this r is, call it b . Then $(\forall n \in \mathbb{N})(|s(n)| < b)$, and so by transfer $(\forall n \in {}^*\mathbb{N})(|s(n)| < b)$. So if we take an unbounded $N \in {}^*\mathbb{N}$, we know $s(N)$ is bounded and hence $\sum_{i=0}^\infty a_i = \text{st}(s(N))$. \square

Theorem 4.5 (Geometric Series). *Let $0 < r < 1$. Then*

$$\sum_{i=0}^\infty r^i = \frac{1}{1-r}.$$

Proof [1, ex. 6.7]. By difference of powers, we know that $1 - r^{n+1} = (1 + r + r^2 + \dots + r^n)(1 - r)$. So, we conclude that $\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}$. Hence, by transfer, for any unbounded $N \in {}^*\mathbb{N}$ we have $\sum_{i=0}^N r^i = \frac{1 - r^{N+1}}{1 - r}$.

Consider r^N . Assume for a contradiction $\text{st}(r^N) > 0$. Then as $0 < r < 1$, we have $r^{N+1} = r \cdot r^N \simeq r \cdot \text{st}(r^N) < \text{st}(r^N)$. Hence, in ${}^*\mathbb{R}$, $(\exists M \in {}^*\mathbb{N})(r^M < \text{st}(r^N))$. By transfer, $(\exists m \in \mathbb{N})(r^m < \text{st}(r^N))$. But r^m is real, so $r^m < r^N$, even though $0 < r < 1$ and $m < N$. This is a contradiction, so we find $\text{st}(r^N) \leq 0$, and since $0 < r < 1$ we conclude $\text{st}(r^N) = 0$. Hence,

$$\sum_{i=0}^\infty r^i = \text{st} \left(\sum_{i=0}^N r^i \right) = \text{st} \left(\frac{1 - r^{N+1}}{1 - r} \right) = \frac{1}{1 - r}.$$

\square

Theorem 4.6 (Absolute Convergence Implies Convergence). *If $\sum_{i=0}^\infty |a_i|$ converges, then $\sum_{i=0}^\infty a_i$ converges.*

Proof. Take any two unbounded $N, M \in {}^*\mathbb{N}$, say $N > M$. In \mathbb{R} , the triangle inequality implies

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N}) \left(n > m \rightarrow \left| \sum_{i=m+1}^n a_i \right| \leq \sum_{i=m+1}^n |a_i| \right),$$

and so this is true in ${}^*\mathbb{R}$ too. So, we have

$$0 < \left| \sum_{i=M+1}^N a_i \right| \leq \sum_{i=M+1}^N |a_i| \simeq 0,$$

since $\sum_{i=0}^{\infty} |a_i|$ converges. This implies $\sum_{i=M+1}^N a_i \simeq 0$, so $\sum_{i=0}^{\infty} a_i$ converges. \square

Theorem 4.7 (Ratio Test). *Let $a_i : \mathbb{N} \rightarrow \mathbb{R}$ be a sequence. If for every unbounded $M \in {}^*\mathbb{N}$ we have $\left| \text{st} \left(\frac{a_{M+1}}{a_M} \right) \right| = L$ for some $L < 1$, then $\sum_{i=0}^{\infty} a_i$ converges.*

Proof [1, ex. 6.8]. Assume that $a_i \geq 0$ —if not, apply the theorem to $|a_i|$ and use Theorem 4.6. Now, take $r \in \mathbb{R}$ such that $L < r < 1$, so that $\left| \frac{a_{M+1}}{a_M} \right| < r < 1$ for any unbounded hypernatural M .

Take any unbounded $N \in {}^*\mathbb{N}$. For any $M \in {}^*\mathbb{N}$, if $N \leq M$, then M is also unbounded, and so $\frac{a_{M+1}}{a_M} < r$. So we have the sentence

$$(\exists k \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N}) \left(n \leq m \rightarrow \frac{a_{m+1}}{a_m} < r \right).$$

If we transfer this to the reals, we find a $k \in \mathbb{N}$ such that for any $m \geq k$ we have $\frac{a_{m+1}}{a_m} < r$. We will show that for any $n \in \mathbb{N}$, $a_{k+n} \leq r^n a_k$. As our base case, $a_k = r^0 a_k$. Next, if $a_{k+n} \leq r^n a_k$, then since $k+n \geq k$ we know $\frac{a_{k+n+1}}{a_{k+n}} < r$ and so $a_{k+n+1} < r \cdot r^n a_k = r^{n+1} a_k$.

Now, in the reals, for any $n, m \in \mathbb{N}$ with $n \geq m$, we have

$$\sum_{i=k+m}^{k+n} a_i \leq \sum_{i=k+m}^{k+n} r^{i-k} a_k = \left(\sum_{i=m}^n r^i \right) a_k.$$

Transferring this to the hyperreals, if we have two unbounded $N, M \in {}^*\mathbb{N}$ with $N > M$, we get

$$\sum_{i=M+1}^N a_i = \sum_{i=k+(M+1-k)}^{k+(N-k)} a_i \leq \left(\sum_{i=M+1-k}^{N-k} r^i \right) a_k.$$

Note that $M = 1 - k$ and $N - k$ are both unbounded hyperreals, and so $\sum_{i=M+1-k}^{N-k} r^i$ is infinitesimal by Theorem 4.5 (as $\sum_{i=0}^{N-k} r^i \simeq \sum_{i=0}^{M+1-k} r^i \simeq \frac{1}{1-r}$). Since a_k is appreciable, this means the product is infinitesimal, hence $\sum_{i=M+1}^N a_i$ is infinitesimal, hence $\sum_{i=0}^{\infty} a_i$ converges. \square

4.5 The exp function

This entire section is individual work without the guidance of any texts. We define the exponential function

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

For convenience, we will write $e(x, k) = \sum_{i=0}^k \frac{x^i}{i!}$. For any unbounded $N \in {}^*\mathbb{N}$, we have $\exp(x) = \text{st}(e(x, N))$.

Theorem 4.8. *For any real x , $\exp(x)$ exists.*

Proof. We use the ratio test (Theorem 4.7) to prove that $\sum_{i=0}^{\infty} \frac{x^i}{i!}$ converges. Let M be an unbounded hypernatural. Then

$$\frac{x^{M+1}}{(M+1)!} \div \frac{x^M}{M!} = \frac{x^{M+1} M!}{x^M (M+1)!} = \frac{x}{M+1}.$$

Since x is real and $M+1$ is unbounded, $\frac{x}{M+1}$ is infinitesimal, and hence has standard part 0. Since $0 < 1$, and since this holds for any unbounded hypernatural, the conditions of Theorem 4.7 are met and we are done. \square

We now want to prove that $\exp'(x) = \exp(x)$. This will involve the following lemma.

Lemma 4.9. *Let $R_k = e(x + d, k) - e(x, k) - d \cdot e(x, k-1)$ for some $x, d \in \mathbb{R}$ with $|d| < 1$ and $k \in \mathbb{N}$. Then*

$$|R_k| < \frac{|d|^2}{1 - |d|} \cdot \exp(|x|).$$

Proof. First, we will find a more explicit formula for R_k . We have

$$e(x+d, k) = 1 + x + d + \frac{x^2 + 2xd + d^2}{2!} + \cdots + \frac{x^k + kx^{k-1}d + \cdots + d^k}{k!},$$

and so

$$\begin{aligned} e(x+d, k) - e(x, k) &= d + \frac{2xd + d^2}{2!} + \frac{3x^2d + 3xd^2 + d^3}{3!} + \cdots + \frac{kx^{k-1}d + \cdots + d^k}{k!} \\ &= d \left(1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{kx^{k-1}}{k!} \right) + \frac{d^2}{2!} + \frac{3xd^2 + d^3}{3!} + \cdots + \frac{\binom{k}{2}x^{k-2}d^2 + \cdots + d^k}{k!} \\ &= d \cdot e(x, k-1) + \frac{d^2}{2!} + \frac{3xd^2 + d^3}{3!} + \cdots + \frac{\binom{k}{2}x^{k-2}d^2 + \cdots + d^k}{k!}. \end{aligned}$$

So we find that

$$R_k = \frac{d^2}{2!} + \frac{3xd^2 + d^3}{3!} + \cdots + \frac{\binom{k}{2}x^{k-2}d^2 + \cdots + d^k}{k!} = \sum_{i=1}^k \sum_{q=0}^{i-2} \binom{i}{q} \frac{x^q d^{(i-q)}}{i!}.$$

Using the fact that $\binom{i}{q} = \frac{i!}{q!(i-q)!}$, we have

$$\begin{aligned} R_k &= \sum_{i=2}^k \sum_{q=0}^{i-2} \binom{i}{q} \frac{x^q d^{(i-q)}}{i!} \\ &= \sum_{i=2}^k \sum_{q=0}^{i-2} \frac{x^q d^{(i-q)}}{(i-q)!q!} \\ &= \sum_{q=0}^{k-2} \sum_{i=q+2}^k \frac{x^q d^{(i-q)}}{(i-q)!q!}. \end{aligned}$$

So

$$|R_k| = \sum_{q=0}^{k-2} \sum_{i=q+2}^k \frac{|x|^q \cdot |d|^{(i-q)}}{(i-q)!q!}.$$

Now, $i-q \geq 2$ in every term, and so $(i-q)! \geq 2! \geq 1$ and so $\frac{|x|^q \cdot |d|^{(i-q)}}{(i-q)!q!} \leq \frac{|x|^q \cdot |d|^{(i-q)}}{q!}$. Combining this move with a change of index, setting $p = i - q$, we get

$$|R_k| \leq \sum_{q=0}^{k-2} \sum_{i=q+2}^k \frac{|x|^q \cdot |d|^{(i-q)}}{q!} = \sum_{q=0}^{k-2} \sum_{p=2}^{k-q} \frac{|x|^q \cdot |d|^p}{q!} = \sum_{q=0}^{k-2} \left(\frac{|x|^q}{q!} \cdot \sum_{p=2}^{k-q} |d|^p \right).$$

Since $|d| < 1$, Theorem 4.5 tells us that for any unbounded hypernatural N , $\sum_{p=0}^N |d|^p \simeq \frac{1}{1-|d|}$. Since each term of $|d|^p$ is positive, we have

$$\sum_{p=2}^{k-q} |d|^p = \sum_{p=0}^{k-q} |d|^p - \sum_{p=0}^1 |d|^p \leq \frac{1}{1-|d|} - (1 + |d|) = \frac{1 - 1(1-|d|) - |d|(1-|d|)}{1-|d|} = \frac{1 - 1 + |d| - |d| + |d|^2}{1-|d|} = \frac{|d|^2}{1-|d|}.$$

Then

$$|R_k| \leq \sum_{q=0}^{k-2} \left(\frac{|x|^q}{q!} \cdot \frac{|d|^2}{1-|d|} \right) = \frac{|d|^2}{1-|d|} \cdot \sum_{q=0}^{k-2} \frac{|x|^q}{q!} \leq \frac{|d|^2}{1-|d|} \cdot \exp(|x|).$$

□

Theorem 4.10. For any real x , $\exp'(x) = \exp(x)$.

Proof. By Lemma 4.9, we have that for any $k \in \mathbb{N}$ and $x, d \in \mathbb{R}$ with $|d| < 1$, we have

$$|e(x + d, k) - e(x, k) - d \cdot e(x, k - 1)| \leq \frac{|d|^2}{1 - |d|} \cdot \exp(|x|),$$

and so

$$(\forall k \in \mathbb{N}) \left(\left| \frac{e(x + d, k) - e(x, k)}{d} - e(x, k - 1) \right| \leq \frac{|d|}{1 - |d|} \cdot \exp(|x|) \right).$$

We then transfer this statement to ${}^*\mathbb{R}$, pick an unbounded $N \in {}^*\mathbb{N}$ to plug in for k , and we get

$$\left| \frac{e(x + d, N) - e(x, N)}{d} - e(x, N - 1) \right| \leq \frac{|d|}{1 - |d|} \cdot \exp(|x|).$$

Taking the standard part of both sides (which does nothing to the right side since it's real), we get

$$\begin{aligned} \text{st} \left(\left| \frac{e(x + d, N) - e(x, N)}{d} - e(x, N - 1) \right| \right) &= \left| \frac{\text{st}(e(x + d, N)) - \text{st}(e(x, N))}{d} - \text{st}(e(x, N - 1)) \right| \\ &= \left| \frac{\exp(x + d) - \exp(x)}{d} - \exp(x) \right| \leq \frac{|d|}{1 - |d|} \cdot \exp(|x|). \end{aligned}$$

So we know the following sentence is true in \mathbb{R}

$$(\forall d \in \mathbb{R}) \left((|d| < 1 \wedge d \neq 0) \rightarrow \left| \frac{\exp(x + d) - \exp(x)}{d} - \exp(x) \right| \leq \frac{|d|}{1 - |d|} \cdot \exp(|x|) \right).$$

Now, transfer this to the hyperreals and plug in any nonzero infinitesimal δ in place of d . Now we know

$$\left| \frac{\exp(x + \delta) - \exp(x)}{\delta} - \exp(x) \right| \leq \frac{|\delta|}{1 - |\delta|} \cdot \exp(|x|).$$

The right side of this is infinitesimal, since $|\delta|$ is infinitesimal while $1 - |\delta|$ and $\exp(|x|)$ are appreciable (the latter by Theorem 4.8). Hence $\left| \frac{\exp(x + \delta) - \exp(x)}{\delta} - \exp(x) \right|$ is infinitesimal. Hence $\frac{\exp(x + \delta) - \exp(x)}{\delta} \simeq \exp(x)$. Since $\exp(x)$ is real, this implies

$$\text{st} \left(\frac{\exp(x + \delta) - \exp(x)}{\delta} \right) = \exp'(x) = \exp(x).$$

□

5 Integration

5.1 Hyperfinite Sets

Definition. An internal set $A = [A_n]$ is *hyperfinite* if every A_n is finite.

The hyperfinite sets are “internally finite.” They share a lot of properties with finite sets.

Theorem 5.1. If $\varphi(A)$ holds for every finite $A \subseteq \mathbb{R}$, then ${}^*\varphi(X)$ holds for every hyperfinite $X \subseteq {}^*\mathbb{R}$.

Proof. Say $X = [A_n]$, with each A_n finite. Then $[[\varphi(A_n)]] = \mathbb{N}$, and so by transfer ${}^*\varphi(X)$. □

This lets us easily get a lot of nice properties about hyperfinite sets. For instance, $(\exists x \in A)(\forall y \in A)(x \geq y)$ ensures that every hyperfinite set has a maximum element.

5.2 Hyperfinite Sums

For any finite set A_n and function $f_n : \mathbb{R} \rightarrow \mathbb{R}$, we can easily define the sum $\sum_{x \in A_n} f_n(x)$. This is, after all, just a sum of a finite collection of numbers. Using this, however, we can easily extend our summation to hyperfinite sets:

Definition. If $A = [A_n]$ is a hyperfinite set, and $f = [f_n]$ is an internal function, we define the *hyperfinite sum*

$$\sum_{x \in [A_n]} f(x) = \left[\sum_{x \in A_n} f_n(x) \right].$$

5.3 Integrability & Integrals

If f is a function bounded on $[a, b]$ and $P = \{x_0 = a, x_1, \dots, x_n = b\}$ a partition on $[a, b]$, let M_i and m_i be the supremum and infimum respectively of f on $[x_{i-1}, x_i]$, and $\Delta x_i = x_i - x_{i-1}$. We then define:

- The upper Reimann sum $U_a^b(f, P) = \sum_{i=1}^n M_i \Delta x_i$
- The lower Reimann sum $L_a^b(f, P) = \sum_{i=1}^n m_i \Delta x_i$
- The ordinary Riemann sum $S_a^b(f, P) = \sum_{i=1}^n f(x_{i-1}) \Delta x_i$

For any positive real Δx , let $P_{\Delta x} = \{a, a + \frac{b-a}{\Delta x}, a + 2\frac{b-a}{\Delta x}, \dots, a + n\frac{b-a}{\Delta x}, b\}$. Now, for any positive real Δx , let $U_a^b(f, \Delta x) = U_a^b(f, P_{\Delta x})$. Similarly for $L_a^b(f, \Delta x)$ and $S_a^b(f, \Delta x)$.

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *integrable* on $[a, b]$ if $L_a^b(f, \Delta x) \simeq U_a^b(f, \Delta x)$ for every infinitesimal Δx . This is equivalent to the standard definition of integrability [1, p. 110].

If f is integrable on $[a, b]$, then we define $\int_a^b f(x) dx = \text{st}(S_a^b(f, \Delta x))$ for some positive infinitesimal Δx .

Lemma 5.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $[a, b] \subseteq \mathbb{R}$. Given any two partitions P_1 and P_2 of $[a, b]$, we find $L_a^b(f, P_1) \leq U_a^b(f, P_2)$.*

Proof hinted at in [1]. □

Corollary 5.3. *If $0 < \Delta x_1, \Delta x_2$, then $L_a^b(f, \Delta x_1) \leq U_a^b(f, \Delta x_2)$.*

Theorem 5.4. *If f is integrable on $[a, b]$, then $\int_a^b f(x) dx$ is well-defined.*

Proof adapted heavily from [1, ch. 9.2]. The issue here is that $\int_a^b f(x) dx = \text{st}(S_a^b(f, \Delta x))$ might depend on our choice of positive infinitesimal Δx . Let Δx_1 and Δx_2 be two positive infinitesimals. We want to show that $S_a^b(f, \Delta x_1) \simeq S_a^b(f, \Delta x_2)$.

Note that in the reals, $(\forall \Delta x)(S_a^b(f, \Delta x) \leq U_a^b(f, \Delta x))$. This is because $\sum_{i=0}^n f(a + (i-1)\frac{b-a}{\Delta x}) \Delta x \leq \sum_{i=0}^n M_i \Delta x$, as M_i is a maximum of f on $[a + (i-1)\frac{b-a}{\Delta x}, a + i\frac{b-a}{\Delta x}]$ and so $f(a + (i-1)\frac{b-a}{\Delta x}) \leq M_i$. We can transfer this statement to the hyperreals to conclude that $S_a^b(f, \Delta x_1) \leq U_a^b(f, \Delta x_1)$, and by a similar line of reasoning we conclude $L_a^b(f, \Delta x_1) \leq S_a^b(f, \Delta x_1)$. Since $L_a^b(f, \Delta x_1) \simeq U_a^b(f, \Delta x_1)$, this implies $S_a^b(f, \Delta x_1) \simeq U_a^b(f, \Delta x_1)$ as well. All of this equally applies to Δx_2 , of course.

Let $L_1 = L_a^b(f, \Delta x_1)$, $L_2 = L_a^b(f, \Delta x_2)$, and similarly for U_1 and U_2 . We know by Corollary 5.3 that $L_1, L_2 \leq U_1, U_2$, and so the possible orderings are $L_1 \leq L_2 \leq U_1 \leq U_2$, $L_1 \leq L_2 \leq U_2 \leq U_1$, or either of those with the indices swapped. In any case, the fact that $L_1 \simeq U_1$ and $L_2 \simeq U_2$ implies $L_1 \simeq L_2 \simeq U_1 \simeq U_2$. But we have $S_a^b(f, \Delta x_1) \simeq U_1 \simeq U_2 \simeq S_a^b(f, \Delta x_2)$, so we are done. □

5.4 Integrals as Hyperfinite Sums

Now, let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function. To take $\int_a^b f(x) dx$, we want to divide $[a, b]$ into infinitely many segments of infinitesimal width, and then add up the area of the rectangles above or below those segments. Hyperfinite sums give us a way to do this.

To divide $[a, b]$ into infinitely many segments of infinitesimal width, we will construct a hyperfinite partition where the segments are of infinitesimal width $dx > 0$. Say $dx = [\langle \Delta x_1, \Delta x_2, \dots \rangle]$. Let $P_n \cup \{b\}$ be partition of $[a, b]$ into segments of width Δx_n (plus a final “remainder” segment of length $\leq \Delta x_n$). So

$$P_n = \left\{ a + k\Delta x_n \mid 0 \leq k < \frac{b-a}{\Delta x_n}, k \in \mathbb{N} \right\}.$$

Let N_n denote $|P_n| = \frac{b-a}{\Delta x_n}$, and let r_n denote the length of the “remainder” segment $r_n = b - (a + \Delta x_n(N_n - 1))$. Notice $r_n \leq \Delta x_n$. Then, the left Riemann sum of f on $[a, b]$ with partition $P_n \cup \{b\}$ is

References

- [1] Robert Goldblatt. *Lectures on the Hyperreals*. Vol. 188. Graduate Texts in Mathematics. New York, NY: Springer, 1998. ISBN: 978-1-4612-6841-3 978-1-4612-0615-6. DOI: 10.1007/978-1-4612-0615-6. URL: <http://link.springer.com/10.1007/978-1-4612-0615-6> (visited on 09/25/2024).
- [2] James M. Henle and Eugene M. Kleinberg. *Infinitesimal Calculus*. Cambridge, Mass: MIT Press, 1979. 135 pp. ISBN: 978-0-262-08097-2.