Infinitesimal Calculus

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History

- Newton & Leibniz formulated calculus using the idea of *infinitesimals*
- Infinitesimals are really really small, but not 0
- Considered nonsensical, replaced with δ-ϵ
- Early 1960's: Abraham Robinson formalizes Nonstandard Analysis
- Our formulation of infinitesimals is based off work by Jerzy Łoś

(a) Abraham Robinson

(b) Jerzy Łoś

Basic Idea

- We construct a set of *hyperreals*, denoted ${}^*\mathbb{R}$
- ${}^*\mathbb{R}$ includes $\mathbb{R}$, along with (a lot of) hyperreals
- We construct a language of mathematical logic, using symbols like $\neg$, $\wedge$, $\vee$, $\forall$, etc.
- We show that any sentence of that language is true in $\mathbb{R}$ iff it is true in ${}^*\mathbb{R}$ (transfer principle)
- We use transfer to prove things about $\mathbb{R}$

Constructing ${}^*\mathbb{R}$

- We start with the ring $\mathbb{R}^\infty$
- Identify sequences that are the same "almost everywhere," like $\langle 0, 1, 1, \ldots \rangle \sim \langle 1, 1, 1, \ldots \rangle$
- Sequences are the same almost everywhere if the set of indices at which they are the same is "big"
- The set of "big" sets of natural numbers is an *ultrafilter* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$

Constructing ${}^*\mathbb{R}$

- Now we can define our equivalence relation $\sim$ by saying that $\langle r_1, r_2, r_3, \ldots \rangle \sim \langle s_1, s_2, s_3, \ldots \rangle$ iff $\{n \in \mathbb{N} \mid r_n = s_n\} \in \mathcal{F}$
- Write the equivalence class $[\langle r_1, r_2, r_3, \ldots \rangle]$
- Define ${}^*\mathbb{R}$ as the quotient ring of $\mathbb{R}^\infty$ under $\sim$, i.e. ${}^*\mathbb{R} = \{[\langle r_1, r_2, r_3, \ldots \rangle] \mid \langle r_1, r_2, \ldots \rangle \in \mathbb{R}^\infty\}$
- Extend any function $f : \mathbb{R} \to \mathbb{R}$ to a new ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ ${}^*f([\langle r_1, r_2, r_3, \ldots \rangle]) = [\langle f(r_1), f(r_2), f(r_3), \ldots \rangle]$
- We can also extend relations, like "$\leq$" $\langle r_1, r_2, r_3, \ldots \rangle \leq \langle s_1, s_2, s_3, \ldots \rangle$ iff $\{n \in \mathbb{N} \mid r_n \leq s_n\} \in \mathcal{F}$
- For any $x \in \mathbb{R}$, we can take $x \in {}^*\mathbb{R}$ to mean $[\langle x, x, x, \ldots \rangle]$

Transfer Principle—Language

- Our language is made up of:
  - logical connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$
  - quantifiers $\forall$, $\exists$
  - parenthesis (and)
  - variables $v_1, v_2, v_3, \ldots$
  - symbols for every element of $\mathbb{R}$, every relation on $\mathbb{R}$, and every function on $\mathbb{R}$

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Infinitesimals

- We construct a set of *hyperreals* ${}^*\mathbb{R} \supseteq \mathbb{R}$.
- ${}^*\mathbb{R}$ is "like" $\mathbb{R}$, but it includes *infinitesimals*, elements δ such that $\delta \neq 0$ but $|\delta| < r$ for every $r \in \mathbb{R}^+$.
- We can add these infinitesimals to other numbers to get things like $1 + \delta$, a number that is "infinitely close to" 1 but not 1.
- If $|x - y|$ is infinitesimal or 0, we say $x \simeq y$
- If $x \in {}^*\mathbb{R}$, we denote by $\mathrm{st}(x)$ the *standard part of* x, the unique real number that is infinitely close to x. $\mathrm{st}(1 + \delta) = 1$.
- We can also take the reciprocals of these infinitesimals to get *unbounded* hyperreals, like $\frac{1}{\delta}$. These have no standard part.
- We can of course combine all these elements however we'd like. If δ and γ are infinitesimals, we have $\frac{1}{\delta} + 4 + \pi + \gamma \in {}^*\mathbb{R}$.

Derivatives, the way Leibniz intended

- Say $f : \mathbb{R} \to \mathbb{R}$. We"extend" f to ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$.
- Fix $b \in \mathbb{R}$. Let Δx be infinitesimal, and let Δf be ${}^*f(b + \Delta x) - {}^*f(b)$.
- Then define $f'(b) = \operatorname{st}\left(\frac{\Delta f}{\Delta x}\right)$. So $f'(b) \simeq \frac{\Delta f}{\Delta x}$.
- **Example:** Say $f(x) = x^2$. Then we have

$$\begin{aligned} f'(3) &\simeq \frac{(3+\Delta x)^2 - 3^2}{\Delta x} = \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ &= \frac{6\Delta x + (\Delta x)^2}{\Delta x} = 6 + \Delta x \simeq 6 \end{aligned}$$

- So $f'(3) \simeq 6$. But these are both real numbers, so their difference can't be infinitesimal. Hence $f'(3) = 6$.

Proof: Chain Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let Δx be any infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Since $g'(x) = \mathrm{st}(\Delta g / \Delta x)$ is defined, Δg must be infinitesimal. Then

$$\begin{aligned}(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\ &= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\ &\simeq f'(g(x)) \cdot g'(x)\end{aligned}$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

In the case where $\Delta g = 0$, we clearly have $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.