

Infinitesimal Calculus

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History

- ▶ Newton & Leibniz formulated calculus using the idea of *infinitesimals*.
- ▶ Infinitesimals are really really small, but not 0.
- ▶ Considered nonsensical, replaced with δ - ϵ .
- ▶ Early 1960's: Abraham Robinson formalizes Nonstandard Analysis.
- ▶ Our formulation of infinitesimals is based off work by Jerzy Łoś.



(a) Abraham Robinson



(b) Jerzy Łoś

Basic Idea

- ▶ We construct a set of *hyperreals*, denoted ${}^*\mathbb{R}$.
- ▶ ${}^*\mathbb{R}$ includes \mathbb{R} , along with (a lot of) infinitesimals.
- ▶ We construct a language of mathematical logic, using symbols like \neg , \wedge , \vee , \forall , etc.
- ▶ We show that any sentence of that language is true in \mathbb{R} iff it is true in ${}^*\mathbb{R}$ (transfer principle).
- ▶ We use transfer to prove things about \mathbb{R} .

Constructing ${}^*\mathbb{R}$

- ▶ Start with \mathbb{R}^∞ , the ring of countable sequences of real numbers.
- ▶ Define our equivalence relation \sim by saying that $\langle r_1, r_2, r_3, \dots \rangle \sim \langle s_1, s_2, s_3, \dots \rangle$ iff $\{n \in \mathbb{N} \mid r_n = s_n\}$ is “big.”
- ▶ Write the equivalence class $[\langle r_1, r_2, r_3, \dots \rangle]$.
- ▶ Define ${}^*\mathbb{R}$ as the quotient ring of \mathbb{R}^∞ under \sim , i.e.
$${}^*\mathbb{R} = \{[\langle r_1, r_2, r_3, \dots \rangle] \mid \langle r_1, r_2, \dots \rangle \in \mathbb{R}^\infty\}.$$
- ▶ Extend any function $f : \mathbb{R} \rightarrow \mathbb{R}$ to a new ${}^*f : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$.
$${}^*f([\langle r_1, r_2, r_3, \dots \rangle]) = [\langle f(r_1), f(r_2), f(r_3), \dots \rangle].$$
- ▶ We can also extend relations, like “ \leq .”
$$[\langle r_1, r_2, r_3, \dots \rangle] \leq [\langle s_1, s_2, s_3, \dots \rangle] \text{ iff } \{n \in \mathbb{N} \mid r_n \leq s_n\} \text{ is big.}$$
- ▶ For any $x \in \mathbb{R}$, we can take $x \in {}^*\mathbb{R}$ to mean $[\langle x, x, x, \dots \rangle]$.

Transfer Principle

- ▶ Our language is made up of:
 - ▶ logical connectives \wedge , \vee , \rightarrow , \leftrightarrow , and \neg ,
 - ▶ quantifiers \forall , \exists ,
 - ▶ parenthesis (and),
 - ▶ variables x, y, z, \dots , and
 - ▶ symbols for every element of \mathbb{R} , every relation on \mathbb{R} , and every function on \mathbb{R} .
- ▶ Sentences are defined recursively.
 $(\forall x \in \mathbb{R})(x < x + 1)$ vs. $5 + \leq v_1 \leftrightarrow \neg \wedge 64$
- ▶ If φ is a sentence that “talks about” \mathbb{R} , we can obtain $^*\varphi$ by replacing each function and relation with its extension in $^*\mathbb{R}$.
- ▶ **Transfer Principle:** Any sentence φ is true iff $^*\varphi$ is true.

Structure of ${}^*\mathbb{R}$

- ▶ We call an element $x \in {}^*\mathbb{R}$:
 - ▶ *infinitesimal* if $|x| < r$ for any $r \in \mathbb{R}^+$,
 - ▶ *unbounded* if $r < |x|$ for any $r \in \mathbb{R}^+$, and
 - ▶ *appreciable* if it is neither infinitesimal nor unbounded.
- ▶ Arithmetic properties of hyperreals are mostly intuitive.
- ▶ We say two elements $x, y \in {}^*\mathbb{R}$ are *infinitely close*, and write $x \simeq y$, if $|x - y|$ is infinitesimal. \simeq is an equivalence relation.
- ▶ Any bounded hyperreal x is infinitely close to a unique real number, called its *standard part* and denoted $\text{st}(x)$.
- ▶ st has most of the nice properties you'd like it to:
 $\text{st}(x + y) = \text{st}(x) + \text{st}(y)$, etc.

Derivatives

- ▶ Say $f : \mathbb{R} \rightarrow \mathbb{R}$. Extend f to ${}^*f : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$
- ▶ Fix $b \in \mathbb{R}$. Let Δx be infinitesimal, and let Δf be ${}^*f(b + \Delta x) - {}^*f(b)$.
- ▶ Then define $f'(b) = \text{st} \left(\frac{\Delta f}{\Delta x} \right)$. So $f'(b) \simeq \frac{\Delta f}{\Delta x}$
- ▶ **Example:** Say $f(x) = x^2$. Then we have

$$\begin{aligned} f'(3) &\simeq \frac{(3 + \Delta x)^2 - 3^2}{\Delta x} = \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ &= \frac{6\Delta x + (\Delta x)^2}{\Delta x} = 6 + \Delta x \simeq 6 \end{aligned}$$

- ▶ So $f'(3) \simeq 6$. But these are both real numbers, so their difference is real. Hence $f'(3) = 6$.

Proof: Chain Rule

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Let Δx be any nonzero infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Then

$$\begin{aligned}(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\&= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\&\simeq f'(g(x)) \cdot g'(x)\end{aligned}$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

When $\Delta g = 0$, $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.

Further, $g'(x) = \text{st} \left(\frac{\Delta g}{\Delta x} \right) = 0$, so $f'(g(x)) \cdot g'(x) = 0$.

Series

- ▶ Recall there are unbounded hypernaturals like $[\langle 1, 2, 3, \dots \rangle] \in {}^*\mathbb{N}$.
- ▶ We take infinite series by extending finite series.
- ▶ Say we have a sequence $\langle r_1, r_2, r_3, \dots \rangle$, and we define $\sum_{i=0}^n r_i$ normally.
- ▶ Let $s : \mathbb{N} \rightarrow \mathbb{R}$ be defined by $s(n) = \sum_{i=0}^n r_i$. Extend s to ${}^*s : {}^*\mathbb{N} \rightarrow {}^*\mathbb{R}$.
- ▶ For any $M \in {}^*\mathbb{N}$, write $\sum_{i=0}^M r_i = s(M)$.
- ▶ If $\text{st} \left(\sum_{i=0}^M r_i \right) = L$ for all unbounded M , write $\sum_{i=0}^{\infty} r_i = L$.

Geometric Series

- ▶ Say $0 < r < 1$. We want to evaluate $\sum_{i=0}^{\infty} r^i$.
- ▶ By difference of powers, $1 - r^{n+1} = (1 - r)(1 + r^2 + \dots + r^n)$.
- ▶ So $(\forall n \in \mathbb{N})(\sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r})$. Transfer this.
- ▶ Plug in an unbounded M . $\sum_{i=0}^M r^i = \frac{1-r^{M+1}}{1-r}$.
- ▶ So $\text{st}\left(\sum_{i=0}^M r^i\right) = \frac{\text{st}(1-r^{M+1})}{\text{st}(1-r)} = \frac{1-\text{st}(r^{M+1})}{1-r} = \frac{1}{1-r}$.
- ▶ So $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$.

- ▶ We define a function $\exp : \mathbb{R} \rightarrow \mathbb{R}$ by $\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.
- ▶ We'd like to show that $\exp'(x) = \exp(x)$.
- ▶ In other words, that $\text{st} \left(\frac{{}^*\exp(x+\Delta x) - {}^*\exp(x)}{\Delta x} \right) = \exp(x)$ for any nonzero infinitesimal Δx .
- ▶ Since both sides are real, it suffices to show they are infinitely close, i.e. that

$$\left| \frac{{}^*\exp(x + \Delta x) - {}^*\exp(x)}{\Delta x} - \exp(x) \right| \text{ is infinitesimal}$$

- ▶ To show that, we will put a bound on

$$\left| \sum_{i=0}^k \frac{(x+d)^i}{i!} - \sum_{i=0}^k \frac{x^i}{i!} - d \cdot \sum_{i=0}^{k-1} \frac{x^i}{i!} \right|$$

and apply transfer several times.