1 Ultrafilters, Ultrapowers, ${}^*\mathbb{R}$

2 First Order Logic & Transfer

3 Structure of ${}^*\mathbb{R}$

4 Differentiation

4.1 Definition

One of the primary motivations for infinitesimal calculus is that it allows us to access the more intuitive, less roundabout conceptualizations of derivatives and integrals. Unlike integrals, which still take some work to define nonstandardly, the nonstandard derivative is almost exactly what we would first guess it to be.

Let Δx be a nonzero infinitesimal, and let $\Delta f(x, \Delta x) = f(x + \Delta x) - f(x)$. Intuitively, Δx is an infinitesimal change in x, and Δf is the corresponding change in f caused by "moving" Δx along the x-axis. Then we have:

$$f'(x) = \frac{\Delta f(x, \Delta x)}{\Delta x}.$$

One small problem: we'd like f' to be a real-valued function on the reals. Luckily, we have a tool to do that:

$$f'(x) = \operatorname{st}\left(\frac{\Delta f(x, \Delta x)}{\Delta x}\right).$$

And we have our definition. Well, this might also not be well-defined: we get around *that* problem by definition. We only consider $f'(x)$ to exist when $\operatorname{st}\left(\frac{\Delta f(x,\Delta x)}{\Delta x}\right)$ is the same for *any* nonzero infinitesimal Δx. In this case, we say that f is *differentiable* at x. Note that this definition of differentiability and the derivative is equivalent to the standard definition.

Theorem 4.1 ([1, p. 8.1.1]). *If f is defined at $x \in \mathbb{R}$, then $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = L$ iff for every nonzero infinitesimal ϵ, $f(x + \epsilon)$ is defined and $\frac{f(x+\epsilon) - f(x)}{\epsilon} \simeq L$.*

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$ and a real number x, we say that f is *differnetiable* at x if there is some constant $f'(x)$ such that, for any nonzero infinitesimal Δx,

$$f'(x) = \operatorname{st}\left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right).$$

4.2 Simple Proofs Using Infinitesimals

A number of proofs of basic calculus results can be easily accomplished by infinitesimals. Perhaps most striking is the chain rule.

Theorem 4.2 (Chain Rule). *Given differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$,*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Proof adapted from [1, ch. 8.4]. Let Δx be any nonzero infinitesimal, and let $\Delta g = g(x + \Delta x) - g(x)$. If $\Delta g = 0$, then $g(x+\Delta x) = g(x)$, so $(f\circ g)'(x) = \operatorname{st}\left(\frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x}\right) = 0$ and clearly $g'(x) = \operatorname{st}\left(\frac{\Delta g}{\Delta x}\right) = 0$, so we are done. If $\Delta g \neq 0$, then since $g'(x) = \operatorname{st}\left(\frac{\Delta g}{\Delta x}\right)$ is defined, we conclude $\frac{\Delta g}{\Delta x}$ is bounded and (since Δx is infinitesimal) that Δg is infinitesimal. Thus

$$\begin{aligned}
(f \circ g)'(x) &= \operatorname{st}\left(\frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x}\right) \\
&= \operatorname{st}\left(\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x}\right) \\
&= \operatorname{st}\left(\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}\right) \cdot \operatorname{st}\left(\frac{\Delta g}{\Delta x}\right) \\
&= f'(g(x)) \cdot g'(x).
\end{aligned}$$

$\square$

4.3 Partial Derivatives

This section is all individual work without the guidance of any texts.

Definition. If $f : \mathbb{R}^n \to \mathbb{R}$, then we define

$$f_{x_k}(b_1, b_2, \ldots, b_n) = \mathrm{st}\left(\frac{f(b_1, \ldots, b_k + \Delta x, \ldots, b_n) - f(b_1, \ldots, b_k, \ldots, b_n)}{\Delta x}\right)$$

for any infinitesimal Δx. This is only defined when it doesn't depend on our choice of Δx.

This gets complicated when we want to deal with repeated partial derivatives. Say $f : \mathbb{R}^2 \to \mathbb{R}$, and denote the inputs of f by $f(x, y)$. Then we might want to write

$$\begin{aligned} f_{yx}(a,b) &= \mathrm{st}\left(\frac{f_y(a+\Delta x, b) - f_y(a,b)}{\Delta x}\right) \\ &= \mathrm{st}\left(\frac{\mathrm{st}\left(\frac{f(a+\Delta x, b+\Delta y) - f(a+\Delta x, b)}{\Delta y}\right) - \mathrm{st}\left(\frac{f(a, b+\Delta y) - f(a,b)}{\Delta y}\right)}{\Delta x}\right), \end{aligned}$$

which would allow us to easily get $f_{yx}(a,b) = f_{xy}(a,b)$. However, this isn't right. Firstly, the numerator is the difference of two real numbers, and so is real, which would mean $f_{yx}(a,b)$ is either 0 or undefined (as a real divided by an infinitesimal is either 0 or unbounded). The mistake here is that $f_y(a+\Delta x, b) \neq \mathrm{st}\left(\frac{f(a+\Delta x, b+\Delta y) - f(a+\Delta x)}{\Delta y}\right)$. Since in this case f_y is taking a nonreal input, we have to use the extension ${}^*f_y(a+\Delta x, b)$. If $\Delta x = [\Delta x_n]$, then this is $[f_y(a+\Delta x_n, b)] = \left[\mathrm{st}\left(\frac{f(a+\Delta x_n, b+\Delta y) - f(a+\Delta x_n, b)}{\Delta y}\right)\right]$. This is the equivalence class of a sequence of real numbers, but it needn't be a real number itself.

Theorem 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Let $\hat{f}_{x_k}$ denote the standard definition of the partial derivative, namely*

$$\hat{f}_{x_k}(b_1, \ldots, b_n) = \lim_{h \to 0} \frac{f(b_1, \ldots, b_k + h, \ldots, b_n) - f(b_1, \ldots, b_k, \ldots, b_n)}{h}.$$

Then $\hat{f}_{x_k}(b_1, \ldots, b_n)$ exists iff $f_{x_k}(b_1, \ldots, b_n)$ does. If they both exist, then $\hat{f}_{x_k}(b_1, \ldots, b_n) = f_{x_k}(b_1, \ldots, b_n)$.

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = f(b_1, \ldots, b_{k-1}, x, b_{k+1}, \ldots, b_n)$. Let $\hat{g}'(x)$ denote the standard derivative of g and $g'(x)$ denote the nonstandard derivative of g. Clearly $\hat{g}'(x) = \hat{f}_{x_k}(x)$ and $g'(x) = f_{x_k}(x)$, and Theorem 4.1 we have $\hat{g}'(b_k)$ is defined iff $g'(b_k)$, and $\hat{g}'(b_k) = g'(b_k)$ if both exist, so we are done. $\square$

4.4 Series

The definitions in this section are thanks to Goldblatt, and most of the results are exercises.

Let $a_n : \mathbb{N} \to \mathbb{R}$. We can define $s_n : \mathbb{N} \to \mathbb{R}$ by

$$s_n = \sum_{i=0}^{n} a_i.$$

Now, for any unbounded $N \in {}^*\mathbb{N}$, we can define

$$\sum_{i=0}^{N} a_i = {}^*s_N.$$

Finally, if for any unbounded $N, M \in {}^*\mathbb{N}$ we have $\sum_{i=0}^{N} a_i \simeq \sum_{i=0}^{M} a_i$, then we say $\sum_{i=0}^{\infty} a_i$ *converges* and define

$$\sum_{i=0}^{\infty} a_i = \mathrm{st}\left(\sum_{i=0}^{N} a_i\right).$$

When $n > m$, we write $\sum_{i=m}^{n} a_i$ to mean $\sum_{i=0}^{n} a_i - \sum_{i=0}^{m-1} a_i$. This gives the expected result when using finite naturals, but also extends to unbounded hypernaturals. Then $\sum_{i=0}^{\infty} a_i$ converges when $\sum_{i=M+1}^{N} a_i \simeq 0$ for any unbounded $N, M \in {}^*\mathbb{N}$ such that $N > M$.

Theorem 4.4 (Cauchy Convergence Criterion for Series)**.** *If for any unbounded $N, M \in {}^*\mathbb{N}$, $\sum_{i=0}^{N} a_i \simeq \sum_{i=0}^{M} a_i$, then $\sum_{i=0}^{\infty} a_i$ converges.*

Proof adapted from [1, thm. 6.5.2]. The issue here is that $\sum_{i=0}^{N} a_i$ might be unbounded for all unbounded $N \in {}^*\mathbb{N}$. Let $s(n) = \sum_{i=0}^{N} a_i$.

First, assume that (in $\mathbb{R}$) $\neg(\exists r \in \mathbb{R})(\forall n \in \mathbb{N})(|s(n)| < r)$. This says the sequence $s(n)$ is not bounded. By trasnfer, this is true in ${}^*\mathbb{R}$ as well. Take an unbounded $N \in {}^*\mathbb{N}$. Since $|s(n)|$ has no upper bound in ${}^*\mathbb{R}$, there is some $M \in {}^*\mathbb{N}$ such that $|s(N)| \leq 2 \cdot |s(N)| < |s(M)|$. Since $|s(N)| \simeq |s(M)|$, this implies $|s(N)| \simeq 2 \cdot |s(N)|$, implying $|s(N)| \simeq 0$, and so for any unbounded $M \in {}^*\mathbb{N}$ we have $|s(M)| \simeq |s(N)| \simeq 0$ and so $\sum_{i=0}^{\infty} a_i = 0$.

Now, say $(\exists r \in \mathbb{R})(\forall n \in \mathbb{N})(|s(n)| < r)$. Whatever this r is, call it b. Then $(\forall n \in \mathbb{N})(|s(n)| < b)$, and so by transfer $(\forall n \in {}^*\mathbb{N})(|s(n)| < b)$. So if we take an unbounded $N \in {}^*\mathbb{N}$, we know $s(N)$ is bounded and hence $\sum_{i=0}^{\infty} a_i = \operatorname{st}(s(N))$. □

Theorem 4.5 (Geometric Series)**.** *Let $0 < r < 1$. Then*

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}.$$

Proof [1, ex. 6.7]. By difference of powers, we know that $1 - r^{n+1} = (1 + r + r^2 + \cdots + r^n)(1 - r)$. So, we conclude that $\sum_{i=0}^{n} r^i = \dfrac{1 - r^{n+1}}{1 - r}$. Hence, by transfer, for any unbounded $N \in {}^*\mathbb{N}$ we have $\sum_{i=0}^{N} r^i = \dfrac{1 - r^{N+1}}{1 - r}$.

Consider r^N. Assume for a contradiction $\operatorname{st}\left(r^N\right) > 0$. Then as $0 < r < 1$, we have $r^{N+1} = r \cdot r^N \simeq r \cdot \operatorname{st}\left(r^N\right) < \operatorname{st}\left(r^N\right)$. Hence, in ${}^*\mathbb{R}$, $(\exists M \in {}^*\mathbb{N})(r^M < \operatorname{st}\left(r^N\right))$. By transfer, $(\exists m \in \mathbb{N})(r^m < \operatorname{st}\left(r^N\right))$. But r^m is real, so $r^m < r^N$, even though $0 < r < 1$ and $m < N$. This is a contradiction, so we find $\operatorname{st}\left(r^N\right) \leq 0$, and since $0 < r < 1$ we conclude $\operatorname{st}\left(r^N\right) = 0$. Hence,

$$\sum_{i=0}^{\infty} r^i = \operatorname{st}\left(\sum_{i=0}^{N} r^i\right) = \operatorname{st}\left(\frac{1 - r^{N+1}}{1 - r}\right) = \frac{1}{1-r}.$$

□

Theorem 4.6 (Absolute Convergence Implies Convergence)**.** *If $\sum_{i=0}^{\infty} |a_i|$ converges, then $\sum_{i=0}^{\infty} a_i$ converges.*

Proof. Take any two unbounded $N, M \in {}^*\mathbb{N}$, say $N > M$. In $\mathbb{R}$, the triangle inequality implies

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})\left(n > m \to \left|\sum_{i=m+1}^{n} a_i\right| \leq \sum_{i=m+1}^{n} |a_i|\right),$$

and so this is true in ${}^*\mathbb{R}$ too. So, we have

$$0 < \left|\sum_{i=M+1}^{N} a_i\right| \leq \sum_{i=M+1}^{N} |a_i| \simeq 0,$$

since $\sum_{i=0}^{\infty} |a_i|$ converges. This implies $\sum_{i=M+1}^{N} a_i \simeq 0$, so $\sum_{i=0}^{\infty} a_i$ converges. □

Theorem 4.7 (Ratio Test)**.** *Let $a_i : \mathbb{N} \to \mathbb{R}$ be a sequence. If for every unbounded $M \in {}^*\mathbb{N}$ we have $\left|\operatorname{st}\left(\frac{a_{M+1}}{a_M}\right)\right| = L$ for some $L < 1$, then $\sum_{i=0}^{\infty} a_i$ converges.*

Proof [1, ex. 6.8]. Assume that $a_i \geq 0$—if not, apply the theorem to $|a_i|$ and use Theorem 4.6. Now, take $r \in \mathbb{R}$ such that $L < r < 1$, so that $\left|\frac{a_{M+1}}{a_M}\right| < r < 1$ for any unbounded hypernatural M.

Take any unbounded $N \in {}^*\mathbb{N}$. For any $M \in {}^*\mathbb{N}$, if $N \leq M$, then M is also unbounded, and so $\frac{a_{M+1}}{a_M} < r$. So we have the sentence

$$(\exists k \in {}^*\mathbb{N})(\forall m \in {}^*\mathbb{N})\left(n \leq m \to \frac{a_{m+1}}{a_m} < r\right).$$

If we transfer this to the reals, we find a $k \in \mathbb{N}$ such that for any $m \geq k$ we have $\frac{a_{m+1}}{a_m} < r$. We will show that for any $n \in \mathbb{N}$, $a_{k+n} \leq r^n a_k$. As our base case, $a_k = r^0 a_k$. Next, if $a_{k+n} \leq r^n a_k$, then since $k + n \geq k$ we know $\dfrac{a_{k+n+1}}{a_{k+n}} < r$ and so $a_{k+n+1} < r \cdot r^n a_k = r^{n+1} a_k$.

Now, in the reals, for any $n, m \in \mathbb{N}$ with $n \geq m$, we have

$$\sum_{i=k+m}^{k+n} a_i \leq \sum_{i=k+m}^{k+n} r^{i-k} a_k = \left(\sum_{i=m}^{n} r^i \right) a_k.$$

Transferring this to the hyperreals, if we have two unbounded $N, M \in {}^*\mathbb{N}$ with $N > M$, we get

$$\sum_{i=M+1}^{N} a_i = \sum_{i=k+(M+1-k)}^{k+(N-k)} a_i \leq \left(\sum_{i=M+1-k}^{N-k} r^i \right) a_k.$$

Note that $M = 1-k$ and $N-k$ are both unbounded hyperreals, and so $\sum_{i=M+1-k}^{N-k} r^i$ is infinitesimal by Theorem 4.5 (as $\sum_{i=0}^{N-k} r^i \simeq \sum_{i=0}^{M+1-k} \simeq \frac{1}{1-r}$). Since a_k is appreciable, this means the product is infinitesimal, hence $\sum_{i=M+1}^{N} a_i$ is infinitesimal, hence $\sum_{i=0}^{\infty} a_i$ converges. □

4.5 The exp function

This entire section is individual work without the guidance of any texts. We define the exponential function

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

For convenience, we will write $e(x,k) = \sum_{i=0}^{k} \frac{x^i}{i!}$. For any unbounded $N \in {}^*\mathbb{N}$, we have $\exp(x) = \operatorname{st}(e(x,N))$.

Theorem 4.8. *For any real x, $\exp(x)$ exists.*

Proof. We use the ratio test (Theorem 4.7) to prove that $\sum_{i=0}^{\infty} \frac{x^i}{i!}$ converges. Let M be an unbounded hypernatural. Then

$$\frac{x^{M+1}}{(M+1)!} \div \frac{x^M}{M!} = \frac{x^{M+1} M!}{x^M (M+1)!} = \frac{x}{M+1}.$$

Since x is real and $M+1$ is unbounded, $\frac{x}{M+1}$ is infinitesimal, and hence has standard part 0. Since $0 < 1$, and since this holds for any unbounded hypernnatural, the conditions of Theorem 4.7 are met and we are done. □

We now want to prove that $\exp'(x) = \exp(x)$. This will involve the following lemma.

Lemma 4.9. *Let $R_k = e(x+d,k) - e(x,k) - d \cdot e(x,k-1)$ for some $x, d \in \mathbb{R}$ with $|d| < 1$ and $k \in \mathbb{N}$. Then*

$$|R_k| < \frac{|d|^2}{1-|d|} \cdot \exp(|x|).$$

Proof. First, we will find a more explicit formula for R_k. We have

$$e(x+d,k) = 1 + x + d + \frac{x^2 + 2xd + d^2}{2!} + \cdots + \frac{x^k + kx^{k-1}d + \cdots + d^k}{k!},$$

and so

$$\begin{aligned} e(x+d,k) - e(x,k) &= d + \frac{2xd + d^2}{2!} + \frac{3x^2 d + 3xd^2 + d^3}{3!} + \cdots + \frac{kx^{k-1}d + \cdots + d^k}{k!} \\ &= d\left(1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{kx^{k-1}}{k!}\right) + \frac{d^2}{2!} + \frac{3xd^2 + d^3}{3!} + \cdots + \frac{\binom{k}{2}x^{k-2}d^2 + \cdots + d^k}{k!} \\ &= d \cdot e(x,k-1) + \frac{d^2}{2!} + \frac{3xd^2 + d^3}{3!} + \cdots + \frac{\binom{k}{2}x^{k-2}d^2 + \cdots + d^k}{k!}. \end{aligned}$$

So we find that

$$R_k = \frac{d^2}{2!} + \frac{3xd^2 + d^3}{3!} + \cdots + \frac{\binom{k}{2}x^{k-2}d^2 + \cdots + d^k}{k!} = \sum_{i=1}^{k} \sum_{q=0}^{i-2} \binom{i}{q} \frac{x^q d^{(i-q)}}{i!}.$$

Using the fact that $\binom{i}{q} = \dfrac{i!}{q!(i-q!)}$, we have

$$\begin{aligned} R_k &= \sum_{i=2}^{k} \sum_{q=0}^{i-2} \binom{i}{q} \frac{x^q d^{(i-q)}}{i!} \\ &= \sum_{i=2}^{k} \sum_{q=0}^{i-2} \frac{x^q d^{(i-q)}}{(i-q)!q!} \\ &= \sum_{q=0}^{k-2} \sum_{i=q+2}^{k} \frac{x^q d^{(i-q)}}{(i-q)!q!}. \end{aligned}$$

So

$$|R_k| = \sum_{q=0}^{k-2} \sum_{i=q+2}^{k} \frac{|x|^q \cdot |d|^{(i-q)}}{(i-q)!q!}.$$

Now, $i-q \geq 2$ in every term, and so $(i-q)! \geq 2! \geq 1$ and so $\dfrac{|x|^q \cdot |d|^{(i-q)}}{(i-q)!q!} \leq \dfrac{|x|^q \cdot |d|^{(i-q)}}{q!}$. Combining this move with a change of index, setting $p = i - q$, we get

$$|R_k| \leq \sum_{q=0}^{k-2} \sum_{i=q+2}^{k} \frac{|x|^q \cdot |d|^{(i-q)}}{q!} = \sum_{q=0}^{k-2} \sum_{p=2}^{k-q} \frac{|x|^q \cdot |d|^p}{q!} = \sum_{q=0}^{k-2} \left(\frac{|x|^q}{q!} \cdot \sum_{p=2}^{k-q} |d|^p \right).$$

Since $|d| < 1$, Theorem 4.5 tells us that for any unbounded hypernatural N, $\sum_{p=0}^{N} |d|^p \simeq \frac{1}{1-|d|}$. Since each term of $|d|^p$ is positive, we have

$$\sum_{p=2}^{k-q} |d|^p = \sum_{p=0}^{k-q} |d|^p - \sum_{p=0}^{1} |d|^p \leq \frac{1}{1-|d|} - (1 + |d|) = \frac{1 - 1(1-|d|) - |d|(1-|d|)}{1-|d|} = \frac{1 - 1 + |d| - |d| + |d|^2}{1-|d|} = \frac{|d|^2}{1-|d|}.$$

Then

$$|R_k| \leq \sum_{q=0}^{k-2} \left(\frac{|x|^q}{q!} \cdot \frac{|d|^2}{1-|d|} \right) = \frac{|d|^2}{1-|d|} \cdot \sum_{q=0}^{k-2} \frac{|x|^q}{q!} \leq \frac{|d|^2}{1-|d|} \cdot \exp(|x|).$$

□

Theorem 4.10. *For any real x, $\exp'(x) = \exp(x)$.*

Proof. By Lemma 4.9, we have that for any $k \in \mathbb{N}$ and $x, d \in \mathbb{R}$ with $|d| < 1$, we have

$$|e(x+d,k) - e(x,k) - d \cdot e(x,k-1)| \leq \frac{|d|^2}{1-|d|} \cdot \exp(|x|),$$

and so

$$(\forall k \in \mathbb{N}) \left(\left| \frac{e(x+d,k) - e(x,k)}{d} - e(x,k-1) \right| \leq \frac{|d|}{1-|d|} \cdot \exp(|x|) \right).$$

We then transfer this statement to ${}^*\mathbb{R}$, pick an unbounded $N \in {}^*\mathbb{N}$ to plug in for k, and we get

$$\left| \frac{e(x+d,N) - e(x,N)}{d} - e(x,N-1) \right| \leq \frac{|d|}{1-|d|} \cdot \exp(|x|).$$

Taking the standard part of both sides (which does nothing to the right side since it's real), we get

$$\begin{aligned} \operatorname{st}\left(\left| \frac{e(x+d,N) - e(x,N)}{d} - e(x,N-1) \right| \right) &= \left| \frac{\operatorname{st}\left(e(x+d,N)\right) - \operatorname{st}\left(e(x,N)\right)}{d} - \operatorname{st}\left(e(x,N-1)\right) \right| \\ &= \left| \frac{\exp(x+d) - \exp(x)}{d} - \exp(x) \right| \leq \frac{|d|}{1-|d|} \cdot \exp(|x|). \end{aligned}$$

So we know the following sentence is true in $\mathbb{R}$

$$(\forall d \in \mathbb{R})\left(\big(|d| < 1 \land d \neq 0\big) \to \left|\frac{\exp(x+d)-\exp(x)}{d} - \exp(x)\right| \leq \frac{|d|}{1-|d|} \cdot \exp(|x|)\right).$$

Now, transfer this to the hyperreals and plug in any nonzero infinitesimal δ in place of d. Now we know

$$\left|\frac{\exp(x+\delta)-\exp(x)}{\delta} - \exp(x)\right| \leq \frac{|\delta|}{1-|\delta|} \cdot \exp(|x|).$$

The right side of this is infinitesimal, since $|\delta|$ is infinitesimal while $1-|\delta|$ and $\exp(|x|)$ are appreciable (the latter by Theorem 4.8). Hence $\left|\frac{\exp(x+\delta)-\exp(x)}{\delta} - \exp(x)\right|$ is infinitesimal. Hence $\frac{\exp(x+\delta)-\exp(x)}{\delta} \simeq \exp(x)$. Since $\exp(x)$ is real, this implies

$$\operatorname{st}\left(\frac{\exp(x+\delta)-\exp(x)}{\delta}\right) = \exp'(x) = \exp(x).$$

□

5 Integration

5.1 Integrability & Integrals

If f is a function bounded on $[a,b]$ and $P = \{x_0 = a, x_1, \ldots, x_n = b\}$ a partition on $[a,b]$, let M_i and m_i be the supremum and infimum respectively of f on $[x_{i-1}, x_i]$, and $\Delta x_i = x_i - x_{i-1}$. We then define:

- The upper Reimann sum $U_a^b(f,P) = \sum_{i=1}^n M_i \Delta x_i$
- The lower Reimann sum $L_a^b(f,P) = \sum_{i=1}^n m_i \Delta x_i$
- The ordinary Riemann sum $S_a^b(f,P) = \sum_{i=1}^n f(x_{i-1}) \Delta x_i$

For any positive real Δx, let $P_{\Delta x} = \{a, a + \frac{b-a}{\Delta x}, a + 2\frac{b-a}{\Delta x}, \ldots, a + n\frac{b-a}{\Delta x}, b\}$. Now, for any positive real Δx, let $U_a^b(f, \Delta x) = U_a^b(f, P_{\Delta x})$. Similarly for $L_a^b(f, \Delta x)$ and $S_a^b(f, \Delta x)$.

A function $f : \mathbb{R} \to \mathbb{R}$ is *integrable* on $[a,b]$ if $L_a^b(f, \Delta x) \simeq U_a^b(f, \Delta x)$ for every infinitesimal Δx. This is equivalent to the standard definition of integrability [1, p. 110].

If f is integrable on $[a,b]$, then we define $\int_a^b f(x)\, dx = \operatorname{st}\big(S_a^b(f, \Delta x)\big)$ for some positive infinitesimal Δx. Note that we can't properly speaking extend S_a^b, as S_a^b has a function as an argument and we can only extend functions that take in real numbers as arguments. Really, we're extending the funciton $g(x) = S_a^b(f, x)$, and writing $S_a^b(f, \Delta x) = {}^*g(\Delta x)$. To show this is well-defined, we need a lemma.

Lemma 5.1 ([1, p. 106])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $[a,b] \subseteq \mathbb{R}$. Given any two partitions P_1 and P_2 of $[a,b]$, we find $L_a^b(f, P_1) \leq U_a^b(f, P_2)$.*

Corollary 5.2. *If $0 < \Delta x_1, \Delta x_2$, then $L_a^b(f, \Delta x_1) \leq U_a^b(f, \Delta x_2)$.*

Theorem 5.3. *If f is integrable on $[a,b]$, then $\int_a^b f(x)dx$ is well-defined.*

Proof adapted heavily from [1, Chapter 9.2]. The issue here is that $\int_a^b f(x)\, dx = \operatorname{st}\big(S_a^b(f, \Delta x)\big)$ might depend on our choice of positive infintiesimal Δx. Let Δx_1 and Δx_2 be two positive infinitesimals. We want to show that $S_a^b(f, \Delta x_1) \simeq S_a^b(f, \Delta x_2)$.

Note that in the reals, $(\forall \Delta x)(S_a^b(f, \Delta x) \leq U_a^b(f, \Delta x))$. This is because $\sum_{i=0}^n f(a + (i-1)\frac{b-a}{\Delta x})\Delta x \leq \sum_{i=0}^n M_i \Delta x$, as M_i is a maximum of f on $[a + (i-1)\frac{b-a}{\Delta x}, +i\frac{b-a}{\Delta x}]$ and so $f(a + (i-1)\frac{b-a}{\Delta x}) \leq M_i$. We can transfer this statement to the hyperreals to conclude that $S_a^b(f, \Delta x_1) \leq U_a^b(f, \Delta x_1)$, and by a similar line of reasoning we conclude $L_a^b(f, \Delta x_1) \leq S_a^b(f, \Delta x_1)$. Since $L_a^b(f, \Delta x_1) \simeq U_a^b(f, \Delta x_1)$, this implies $S_a^b(f, \Delta x_1) \simeq U_a^b(f, \Delta x_1)$ as well. All of this equally applies to Δx_2, of course.

Let $L_1 = L_a^b(f, \Delta x_1)$, $L_2 = L_a^b(f, \Delta x_2)$, and similarly for U_1 and U_2. We know by Corollary 5.2 that $L_1, L_2 \leq U_1, U_2$, and so the possible orderings are $L_1 \leq L_2 \leq U_1 \leq U_2$, $L_1 \leq L_2 \leq U_2 \leq U_1$, or either of those with the indices swapped. In any case, the fact that $L_1 \simeq U_1$ and $L_2 \simeq U_2$ implies $L_1 \simeq L_2 \simeq U_1 \simeq U_2$. But we have $S_a^b(f, \Delta x_1) \simeq U_1 \simeq U_2 \simeq S_a^b(f, \Delta x_2)$, so we are done. □

We now state the Fundamental Thoerem of Calculus, proven for nonstandard objects in [1, pp. 111–112].

Theorem 5.4 (Fundamental Theorem of Calculus, [1, Theorem 9.4.2])**.** *If a function G has a continuous derivative f on $[a,b]$, then $\int_a^b f(x)dx = G(b) - G(a)$.*

5.2 Improper Integrals

Say $f : [a, b) \to \mathbb{R}$ is integrable on every interval $[a, c]$ for $a < c < b$. Standardly, we take the *improper integral* (where defined) to be

$$\int_a^b f(x)dx := \lim_{c \to b} \int_a^c f(x)dx$$

Nonstandardly, we instead take

$$\int_a^b f(x)dx = \operatorname{st}\left({}^*\!\int_a^\gamma f(x)dx\right)$$

Where $b \simeq \gamma < b$ and ${}^*\!\int_a^t f(x)dx$ indicates the extension ${}^*g(t)$ of $g(t) = \int_a^t f(x)dx$. Similarly, we have

$$\int_a^\infty f(x)dx = \operatorname{st}\left({}^*\!\int_a^\kappa f(x)dx\right)$$

Where κ is a positive unbounded hyperreal. To be clear, there is no guarantee that these standard parts exist, or that they are the same across all potential γ's or κ's—in those cases, the improper integral is undefined.

Say we want to take $\int_0^1 \frac{1}{\sqrt{x}}dx$. Since $\frac{1}{\sqrt{x}}$ isn't bounded on $[0, 1]$, we can't take a proper integral. So we instead venture to take $\int_\delta^1 \frac{1}{\sqrt{x}}dx$ for some positive infinitesimal δ.

By 5.4, we have the sentence

$$(\forall a \in \mathbb{R})\left(0 < a < 1 \to \int_a^1 \frac{1}{\sqrt{x}}dx = 2 - 2\sqrt{a}\right).$$

This transfers to ${}^*\mathbb{R}$, and so we conclude $\int_\delta^1 \frac{1}{\sqrt{x}}dx = 2 - 2\sqrt{\delta}$. Then $\int_0^1 \frac{1}{\sqrt{x}} = \operatorname{st}\left(2 - 2\sqrt{\delta}\right) = 2 - 2 \cdot \operatorname{st}\left(\sqrt{\delta}\right) = 2.$

5.3 Hyperfinite Sets & Sums

Definition. An internal set $A = [A_n]$ is *hyperfinite* if every A_n is finite.

The hyperfinite sets are "internally finite." They share a lot of properties with finite sets.

Theorem 5.5. *If $\varphi(A)$ holds for every finite $A \subseteq \mathbb{R}$, then ${}^*\varphi(X)$ holds for every hyperfinite $X \subseteq {}^*\mathbb{R}$.*

Proof. Say $X = [A_n]$, with each A_n finite. Then $[[\varphi(A_n)]] = \mathbb{N}$, and so by transfer ${}^*\varphi(X)$. □

This lets us easily get a lot of nice properties about hyperfinite sets. For instance, $(\exists x \in A)(\forall y \in A)(x \geq y)$ ensures that every hyperfinite set has a maximum element.

Now, for any finite set A_n and function $f_n : \mathbb{R} \to \mathbb{R}$, we can easily define the sum $\sum_{x \in A_n} f_n(x)$. This is, after all, just a sum of a finite collection of numbers. Using this, however, we can easily extend our summation to hyperfinite sets:

Definition. If $A = [A_n]$ is a hyperfinite set, and $f = [f_n]$ is an internal function, we define the *hyperfinite sum*

$$\sum_{x \in [A_n]} f(x) = \left[\sum_{x \in A_n} f_n(x)\right].$$

5.4 Integrals as Hyperfinite Sums

Now, let $f : [a, b] \to \mathbb{R}$ be an integrable function. To take $\int_a^b f(x)dx$, we want to divide $[a, b]$ into infinitely many segments of infinitesimal width, and then add up the area of the rectangles above or below those segments. Hyperfinite sums give us a way to do this.

To divide $[a, b]$ into infinitely many segments of infinitesimal width, we will construct a hyperfinite partition where the segments are of infinitesimal width $dx > 0$. Say $dx = [\langle \Delta x_1, \Delta x_2, \ldots \rangle]$. Let $P_n \cup \{b\}$ be partition of $[a, b]$ into segments of width Δx_n (plus a final"remainder" segment of length $\leq \Delta x_n$). So

$$P_n = \left\{a + k\Delta x_n \,|\, 0 \leq k < \frac{b-a}{\Delta x_n},\ k \in \mathbb{N}\right\}.$$

Let c_n denote the greatest element of P_n, the second-to-last element of our partition (the last element is b). So when Δx_n doesn't "evenly divide" $b-a$, we have $c_n = a + \lfloor \frac{b-a}{\Delta x_n} \rfloor \cdot \Delta x_n$. Let r_n denote the length of the "remainder" segment $r_n = b - c_n$. Notice $r_n \leq \Delta x_n$. Then, the ordinary Riemann sum of f on $[a,b]$ with partition $P_n \cup \{b\}$ is

$$S_a^b(f, \Delta x_n) = \sum_{x \in P_n} f(x)\Delta x_n + f(c_n) \cdot (r_n - \Delta x_n).$$

This is because $\sum_{x \in P_n} f(x)\Delta x_n$ includes a term corresponding to $f(c_n)\Delta x_n$, while $S_a^b(f, \Delta x_n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$ includes as a term $f(c_n)r_n$ (as the width of the segment of the partition from c_n to b is r_n).

Now, let $P = [P_n]$ be our "hyperfinite partition" of $[a,b]$ into hyperfinitely many intervals of infinitesimal width dx. Intuitively, we would hope that $\int_a^b f(x)dx$ is equal to the hyperfinite "Reimann sum" over this partition, i.e.

$$\mathrm{st}\left(\sum_{x \in P} f(x) \cdot dx\right) = \int_a^b f(x)dx = \mathrm{st}\left(S_a^b(f, dx)\right).$$

Note that we're taking the hyperfinite sum over the internal function $g(x) = f(x) \cdot dx$. This is internal because we let $g = [g_n]$, where $g_n(x) = f(x) \cdot \Delta x_n$. Then $g(x) = [f(x) \cdot \Delta x_n] = [f(x)] \cdot [\Delta x_n] = f(x) \cdot dx$. To show this, we can write (adapted from [1, Chapter 12.7]):

$$\mathrm{st}\left(\sum_{x \in P} f(x) \cdot dx\right) = \mathrm{st}\left(\left[\sum_{x \in P_n} f(x) \cdot \Delta x_n\right]\right).$$

Now, letting N_n be such that $a + N_n \frac{b-a}{\Delta x_n} = c_n$ and letting $x_i = a + i \cdot \frac{b-a}{\Delta x_n}$, we have:

$$\begin{aligned}\mathrm{st}\left(\left[\sum_{x \in P_n} f(x) \cdot \Delta x_n\right]\right) &= \mathrm{st}\left(\left[\sum_{i=0}^{N_n+1} f(x_{i-1}) \cdot \Delta x_n\right]\right) \\ &= \mathrm{st}\left(\left[S_a^b(f, \Delta x_n)\right]\right) \\ &= \mathrm{st}\left({}^*S_a^b(f, dx)\right) = \int_a^b f(x)dx.\end{aligned}$$

Intuitively, this is enough to see that $\sum_{x \in P} f(x) \cdot dx$ and $hrS_a^b(f, dx)$ really are doing the same thing, in that they're taking the hyperreal corresponding to a sequences of real numbers that are closer and closer approximations of $\int_a^b f(x)dx$ by Reimann sums.

The issue here is that $\sum_{i=0}^{N_n+1} f(x_{i-1}) \cdot \Delta x_n$ isn't necessarily equal to $S_a^b(f, \Delta x_n)$, due to our concerns about the width of the final segment of our partition r_n. In order to make this reasoning rigorous, we need to deal with this "last segment" problem (not addressed in [1]):

Lemma 5.6.

$$\sum_{x \in P} f(x) \cdot dx \simeq S_a^b(f, dx)$$

Proof. We have that

$$\sum_{x \in P} f(x) \cdot dx - S_a^b(f, dx) = [\sum_{x \in P_n} (f(x) \cdot \Delta x_n) - S_a^b(f, \Delta x)]$$

and our earlier observation that

$$S_a^b(f, \Delta x_n) = \sum_{x \in P_n} f(x)\Delta x_n + f(c_n) \cdot (r_n - \Delta x_n).$$

and so

$$\sum_{x \in P} f(x) \cdot dx - S_a^b(f, dx) = [f(c_n) \cdot (\Delta x_n - r_n)] = [f(c_n)] \cdot [\Delta x_n - r_n].$$

We want to show that this is infinitesimal. Now, $0 < r_n \leq \Delta x_n$, so $|[\Delta x_n - r_n]| = [|\Delta x_n - r_n|] \leq [|2 \cdot \Delta x_n] = 2 \cdot |[\Delta x_n] = 2 \cdot dx \simeq 0$, and so $[\Delta x_n - r_n]$ is infinitesimal. Since f is integrable on $[a,b]$, it is (by definition) bounded on $[a,b]$, and so $[f(c_n)]$ is bounded. (We know that $(\forall x \in \mathbb{R})(a \leq x \leq b \to |f(x) < L)$ for some L, and so by transfer this holds in ${}^*\mathbb{R}$ too, and $a \leq c_n \leq b_n$. So $|f(c_n)| < L$ for all n, and so $|[f(c_n)]| < L$.) So since a bounded number times an infinitesimal is infinitesimal, $[f(c_n)] \cdot [\Delta x_n - r_n]$ is infinitesimal, and we're done. □

With this lemma, we can write:

$$\int_a^b f(x)dx = \mathrm{st}\left(S_a^b(f, dx)\right) = \sum_{x \in P} f(x) \cdot dx$$

References

[1] Robert Goldblatt. *Lectures on the Hyperreals*. Vol. 188. Graduate Texts in Mathematics. New York, NY: Springer, 1998. ISBN: 978-1-4612-6841-3 978-1-4612-0615-6. DOI: 10.1007/978-1-4612-0615-6.

[2] James M. Henle and Eugene M. Kleinberg. *Infinitesimal Calculus*. Cambridge, Mass: MIT Press, 1979. 135 pp. ISBN: 978-0-262-08097-2.