Infinitesimal Calculus

Paul Schulze

History

- Newton & Leibniz formulated calculus using the idea of *infinitesimals*.
- Infinitesimals are really really small, but not 0.
- Considered nonsensical, replaced with δ-ϵ.
- Early 1960's: Abraham Robinson formalizes Nonstandard Analysis.
- Most modern formulations are based on work by Jerzy Łoś.

(a) Abraham Robinson

(b) Jerzy Łoś

Hyperreals

We construct a set of *hyperreals*, call it ${}^*\mathbb{R}$, such that we know three things:

1. $\mathbb{R} \subseteq {}^*\mathbb{R}$
2. ${}^*\mathbb{R}$ contains at least one infinitesimal δ, such that $0 < \delta$ but $\delta < r$ for any positive real number r
3. Any sentence of first-order logic is true in $\mathbb{R}$ iff it is true* in ${}^*\mathbb{R}$

First-Order Logic

- Our logical language has the following logical symbols:
  - & for "and"
  - $\vee$ for "or"
  - $\rightarrow$ for "if...then..."
  - $\leftrightarrow$ for "if and only if"
  - $\forall$ for "for all"
  - $\exists$ for "there exists"
  - $\in$ for set membership
- $5 + 3 = 8 \,\&\, 2^3 = 8$
- $1 + 1 = 1 \vee 5 + 7 = 12$
- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{N})(x < y)$
- $(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})(x < \frac{1}{n} \rightarrow n < \frac{1}{x})$

Transfer Principle

Every sentence that's true in $\mathbb{R}$ is also "true in ${}^*\mathbb{R}$," when modified to be talking about ${}^*\mathbb{R}$. For instance:

- $(\forall x \in {}^*\mathbb{R})(\forall y \in {}^*\mathbb{R})(x + y = y + x)$, i.e. addition is commutative
  - We can similarly prove other arithmetic properties, so we can do algebra as normal in ${}^*\mathbb{R}$
- $(\forall x \in {}^*\mathbb{R})\,(x \neq 0 \rightarrow (\exists y \in {}^*\mathbb{R})(x \cdot y = 1))$, i.e. for any nonzero x there exists a multiplicative inverse $\frac{1}{x}$
  - So even if δ is an infinitesimal, we know $\frac{1}{\delta}$ has to exist
- $(\forall x \in {}^*\mathbb{R})(\exists y \in {}^*\mathbb{N})(x < y)$
  - So ${}^*\mathbb{N}$ has to contain "infinite" elements, and thus ${}^*\mathbb{N} \neq \mathbb{N}$

What is ${}^*\mathbb{R}$ like?

- Call a hyperreal x *infinitesimal* when $|x| < r$ for every positive real r.
  - There's exactly one real infinitesimal: 0
- There are "infinite" hyperreals—take any nonzero infinitesimal δ, and $\frac{1}{\delta}$ is greater than any real number
- Two hyperreals x and y are *infinitely close*, denoted $x \simeq y$, when their difference $x - y$ is infinitesimal
  - No two distinct real numbers are infinitely close to each other
- Any finite hyperreal x is infinitely close to exactly one real number, called its *standard part* $\operatorname{st}(x)$
  - For instance, $\operatorname{st}(1 + \delta) = 1$

Derivatives, the way Leibniz intended

- Say $f : \mathbb{R} \to \mathbb{R}$
- Fix $b \in \mathbb{R}$, and let Δx be a nonzero infinitesimal. Then

  $$f'(b) = \operatorname{st}\left(\frac{f(b+\Delta x) - f(b)}{\Delta x}\right)$$

  so that $f'(b) \simeq \frac{f(b+\Delta x)-f(b)}{\Delta x}$.
- **Example:** Say $f(x) = x^2$. Then we have

  $$\begin{aligned} f'(3) &\simeq \frac{(3+\Delta x)^2 - 3^2}{\Delta x} = \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ &= \frac{6\Delta x + (\Delta x)^2}{\Delta x} = 6 + \Delta x \simeq 6 \end{aligned}$$

- So $f'(3) \simeq 6$, so their difference is infinitesimal. But they're both *real numbers*, so their difference has to be real. The only real infinitesimal is 0, so their difference is 0. Thus $f'(3) = 6$.

Proof: Chain Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let Δx be any infinitesimal, and $\Delta g = g(x + \Delta x) - g(x)$. Since $g'(x) = \mathrm{st}(\Delta g / \Delta x)$ is defined, Δg must be infinitesimal. Then

$$\begin{aligned}
(f \circ g)'(x) &\simeq \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\
&= \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g} \cdot \frac{\Delta g}{\Delta x} \\
&\simeq f'(g(x)) \cdot g'(x)
\end{aligned}$$

So $(f \circ g)'(x) \simeq f'(g(x)) \cdot g'(x)$. But since both of these numbers are real, they must be identical.

In the case where $\Delta g = 0$, we clearly have $f(g(x) + \Delta g) - f(g(x)) = 0$ and so $(f \circ g)'(x) = 0$.