

# Week 10 Reflection

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## 1 General Results about Limits

**Theorem.**  $\lim_{x \rightarrow h} f(x) = L$  iff  $\text{st}(f(\gamma)) = L$  for all  $\gamma \simeq h$

*Proof.* If  $\lim_{x \rightarrow h} f(x) = L$ , then for every  $\epsilon > 0$  there is a  $\delta > 0$  such that  $(\forall x)(|x - h| < \delta \rightarrow |f(x) - L| < \epsilon)$ . Transferring this, we notice that  $|\gamma - h| < \delta \rightarrow |f(\gamma) - L| < \epsilon$ . But  $|\gamma - h| < \delta$  for every real  $\delta > 0$ , and so  $|f(\gamma) - L| < \epsilon$  for every real  $\epsilon > 0$ . Hence  $f(\gamma) \simeq L$ .

Next, say any  $\gamma \simeq h$  satisfies  $f(\gamma) \simeq L$  (equivalently,  $\text{st}(f(\gamma)) = L$ ). Then, for any  $\epsilon > 0$ , we can take any positive infinitesimal  $\theta$  and have  $(\forall x)(|x - h| < \theta \rightarrow |f(x) - L| < \epsilon)$ , since if  $|x - h| < \theta$  we have  $x \simeq h$  and so  $f(x) \simeq L$ . Thus, in the hyperreals we have  $(\exists \delta > 0)(\forall x)(|x - h| < \delta \rightarrow |f(x) - L| < \epsilon)$ , and we can then transfer that statement to the reals. Since this is true for any real  $\epsilon > 0$ , we find  $\lim_{x \rightarrow h} f(x) = L$ .  $\square$

## 2 Integration as Hyperfinite Sums

### 2.1 Hyperfinite Sums

**Definition.** An internal set  $A = [A_n]$  is *hyperfinite* if  $A_n$  is finite for all  $n$ .

Since  ${}^* \varphi(A)$  iff  $[[\varphi(A_n)]] \in \mathcal{F}$ , we can transfer a lot of properties of finite sets to hyperfinite ones. In particular, if  $\varphi(X)$  for any finite set  $X$ , then  ${}^* \varphi(A)$  for any hyperfinite set  $A$ . For instance, every hyperfinite set has a maximum element by  $\varphi(X) = (\exists x \in X)(\forall y \in X)(y \leq x)$ .

**Definition.** If  $A = [A_n]$  is hyperfinite and  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we define the *hyperfinite sum*

$$\sum_{x \in [A_n]} {}^* f(x) = \left[ \sum_{x \in A_n} f(x) \right]$$

Note that  $\sum_{x \in A_n} f(x)$  is just the sum of a finite number of real numbers.

### 2.2 Integration

Now, let  $f : [a, b] \rightarrow \mathbb{R}$  be an integrable function. To take  $\int_a^b f(x) dx$ , we want to divide  $[a, b]$  into infinitely many segments of infinitesimal width, and then add up the area of the rectangles above or below those segments. Hyperfinite sums give us a way to do this.

To divide  $[a, b]$  into infinitely many segments of infinitesimal width, we will construct a hyperfinite partition where the segments are of infinitesimal width  $dx > 0$ . Say  $dx = [\langle \Delta x_1, \Delta x_2, \dots \rangle]$ . Let  $P_n \cup \{b\}$  be partition of  $[a, b]$  into segments of width  $\Delta x_n$  (plus a final “remainder” segment of length  $\leq \Delta x_n$ ). So

$$P_n = \left\{ a + k\Delta x_n \mid 0 \leq k < \frac{b-a}{\Delta x_n}, k \in \mathbb{N} \right\}$$

Let  $N_n$  denote  $|P_n| = \frac{b-a}{\Delta x_n}$ , and let  $r_n$  denote the length of the “remainder” segment  $r_n = b - (a + k(N_n - 1))$ . Notice  $r_n \leq \Delta x_n$ . Then, the left Riemann sum of  $f$  on  $[a, b]$  with partition  $P_n \cup \{b\}$  is

$$S_a^b(f, \Delta x_n) = \sum_{x \in P_n} f(x) \Delta x_n - f(a + k(N_n - 1))(\Delta x_n - r_n)$$

Intuitively, we would hope that the integral  $\int_a^b f(x)dx$  is the sum of the infinitely many rectangles of infinitesimal width

$$\sum_{x \in P} f(x)dx = \left[ \sum_{x \in P_n} f(x)\Delta x_n \right]$$

as well as the extension

$$^*S_a^b(f, dx) := \left[ \sum_{x \in P_n} f(x)\Delta x_n - f(a + k(N_n - 1))(\Delta x_n - r_n) \right]$$

The difference between these two hyperreals is  $[f(a + k(N_n - 1))(\Delta x_n - r_n)] = [f(a + k(N_n - 1))][(\Delta x_n - r_n)]$ . Since  $f$  is integrable on  $[a, b]$ , it is bounded, and so  $[f(a + k(N_n - 1))]$  is bounded. Meanwhile,  $[\Delta x_n]$  and  $[r_n]$  are infinitesimal, so  $[\Delta x_n - r_n] = [\Delta x_n] - [r_n]$  is infinitesimal. Since the product of an infinitesimal and a bounded number is infinitesimal, we get that  $[f(a + k(N_n - 1))][(\Delta x_n - r_n)]$  is infinitesimal, so

$$^*S_a^b(f, dx) \simeq \sum_{x \in P} f(x)dx$$

and so

$$\int_a^b f(x)dx = \text{st} \left( ^*S_a^b(f, dx) \right) = \text{st} \left( \sum_{x \in P} f(x)dx \right)$$

In our hyperfinite sum, we neglected to account for the last interval perhaps being shorter than  $dx$ , but as we see here the difference it would make is infinitesimal and so can safely be ignored.

### 2.3 Improper Integrals

Say  $f : [a, b) \rightarrow \mathbb{R}$  is integrable on every interval  $[a, c]$  for  $a < c < b$ . Standardly, we take the *improper integral* (where defined) to be

$$\int_a^b f(x)dx := \lim_{c \rightarrow b} \int_a^c f(x)dx$$

By an easy modification of the theorem in section 1, we have that

$$\int_a^b f(x)dx = \text{st} \left( \int_a^\gamma f(x)dx \right)$$

Where  $b \simeq \gamma < b$  and  $\int_a^t f(x)dx$  indicates the extension  $^*g(t)$  of  $g(t) = \int_a^t f(x)dx$ . Similarly, we have

$$\int_a^\infty f(x)dx = \text{st} \left( \int_a^\kappa f(x)dx \right)$$

Where  $\kappa$  is a positive unbounded hyperreal. To be clear, there is no guarantee that these standard parts exist, or that they are the same across all potential  $\gamma$ 's or  $\kappa$ 's—in those cases, the improper integral is undefined.

**Example.** Say we want to evaluate  $\int_0^1 \frac{1}{\sqrt{x}} dx$ .

## 3 Exponentiation

Let  $e_n : \mathbb{R} \rightarrow \mathbb{R}$  be the function  $e_n(x) = \sum_{k \in \{0, 1, 2, \dots, n\}} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$ . We define the internal function  $\text{inx} = [e_n]$  by  $[e_n]([x_n]) = [e_n(x_n)]$ . Note that for any real number  $x$ , we have  $\text{inx}(x) = [e_n(x)] = [\sum_{k \in \{0, 1, \dots, n\}} \frac{x^k}{k!}] = \sum_{k \in \mathbf{N}} \frac{x^k}{k!}$ , where  $\mathbf{N}$  is the hyperfinite set  $[\{0, 1\}, \{0, 1, 2\}, \dots]$ .

Let  $\exp : \mathbb{R} \rightarrow \mathbb{R}$  be  $\exp(x) = \text{st}(\text{inx}(x))$ . Then we have its extension  $^*\exp([x_n]) = [\text{st}(\text{inx}(x_n))]$ . It's worth being careful here: we may be tempted to expand this to  $[\text{st}([e_n](x_n))] = [\text{st}([e_n(x_n)])]$ , but this is mixing indices. The proper expansion, using  $[x_k]_k$  to denote  $[\langle x_1, x_2, \dots \rangle]$ , is  $^*\exp([x_n]_n) = [\text{st}([e_k(x_n)]_k)]_n$ , by which we mean  $[\text{st}([e_k(\langle x_n, x_n, x_n, \dots \rangle)]_k)]_n$ .

### 3.1 Derivative of exp

Let  $\delta = [\delta_n]$  be a nonzero infinitesimal. We want to evaluate

$$\frac{d}{dx} \exp(x) = \text{st} \left( \frac{{}^*\exp(x + \delta) - \exp(x)}{\delta} \right)$$

Consider

$$\begin{aligned} \frac{{}^*\exp(x + \delta) - \exp(x)}{\delta} &= \frac{[\exp(x + \delta_n)] - [\exp(x)]}{[\delta_n]} \\ &= \left[ \frac{\exp(x + \delta_n) - \exp(x)}{\delta_n} \right] \end{aligned}$$

Peeling back another layer,

$$\begin{aligned} \frac{\exp(x + \delta_n) - \exp(x)}{\delta_n} &\simeq \frac{\text{inx}(x + \delta_n) - \text{inx}(x)}{\delta_n} \\ &= \frac{[e_k(x + \delta_n)]_k - [e_k(x)]_k}{\delta_n} \\ &= \left[ \frac{e_k(x + \delta_n) - e_k(x)}{\delta_n} \right]_k \end{aligned}$$

Now, we know

$$\begin{aligned} e_k(x + \delta_n) &= 1 + x + \delta_n + \frac{x^2 + 2x\delta_n + \delta_n^2}{2} + \frac{x^3 + 3x^2\delta_n + \delta_n^2 \cdot (\dots)}{3 \cdot 2!} + \dots + \frac{x^k + kx^{k-1}\delta_n + \delta_n^2(\dots)}{k!} \\ &= e_k(x) + \delta_n e_{k-1}(x) + \delta_n^2 \cdot (\dots) \end{aligned}$$

So

$$\frac{e_k(x + \delta_n) - e_k(x)}{\delta_n} = e_{k-1}(x) + \delta_n \cdot (\dots)$$

And so

$$\begin{aligned} \frac{\exp(x + \delta_n) - \exp(x)}{\delta_n} &\simeq \left[ \frac{e_k(x + \delta_n) - e_k(x)}{\delta_n} \right]_k \\ &= [e_{k-1}(x)]_k + \delta_n \cdot [(\dots)]_k \end{aligned}$$

We will assume, for now, that  $[e_{k-1}(x)] \simeq \text{inx}(x)$ . Then we have

$$\frac{\exp(x + \delta_n) - \exp(x)}{\delta_n} \simeq \text{inx}(x) + \delta_n \cdot R_n$$

But the left side of this is real, and so must be equal to the standard part of the right side

$$\frac{\exp(x + \delta_n) - \exp(x)}{\delta_n} = \text{st}(\text{inx}(x) + \delta_n \cdot R_n) = \text{st}(\text{inx}(x)) + \text{st}(\delta_n \cdot R_n) = \exp(x) + \text{st}(\delta_n R_n)$$

And after all that, we have

$$\begin{aligned} \frac{{}^*\exp(x + \delta) - \exp(x)}{\delta} &= \left[ \frac{\exp(x + \delta_n) - \exp(x)}{\delta_n} \right] \\ &= [\exp(x) + \delta_n \text{st}(R_n)] = \exp(x) + [\delta_n] \text{st}(R_n) \end{aligned}$$

And so, assuming for now that  $[\delta_n \text{st}(R_n)]$  is infinitesimal, we finally have

$$\begin{aligned} \frac{d}{dx} \exp(x) &= \text{st} \left( \frac{{}^*\exp(x + \delta) - \exp(x)}{\delta} \right) \\ &= \text{st}(\exp(x) + [\delta_n] \text{st}(R)) \\ &= \exp(x) \end{aligned}$$

## 3.2 Lemmas for 3.1

**Lemma.**  $[e_{n-1}(x)] \simeq \text{inx}(x)$

*Proof.* We want to show that  $\text{inx}(x) - [e_{n-1}(x)] \simeq 0$ , i.e. that  $[e_n(x) - e_{n-1}(x)] \simeq 0$ , i.e. that  $\left[\frac{x^n}{n!}\right] \simeq 0$ , which is true for any choice of  $x$  by the same reasoning that shows  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ .  $\square$

**Lemma.** Given an infinitesimal  $\delta = [\delta_n]$ , define  $R_n = [(e_k(x + \delta_n) - e_k(x) - \delta_n e_{k-1}(x))/\delta_n]_k$ . Show that  $[\text{st}(R_n)]_n$  is infinitesimal.

We have changed notation here slightly, so that what we are now calling  $R_n$  we were before calling  $\delta_n R_n$ .

*Proof.* Let  $R_n = [R_n^k]_k$  (that's an upper index, not an exponent). We'll assume that both  $x$  and  $\delta_n$  are positive—if not, we can just as easily replace  $R_n^k$  by  $|R_n^k|$ ,  $x$  by  $|x|$ , and  $\delta_n$  by  $|\delta_n|$  and still get a proof that  $[|R_n|]$  is infinitesimal, which will suffice. From 3.1, we have

$$R_n^k = \frac{\delta_n}{2} + \frac{3x\delta_n + \delta_n^2}{3!} + \frac{6x^2\delta_n + 4x\delta_n^2 + \delta_n^3}{4!} + \cdots + \frac{1}{k!} \sum_{q=0}^{k-2} \binom{k}{q} x^q \delta_n^{(k-1-q)}$$

In index notation,

$$\begin{aligned} R_n^k &= \sum_{i=2}^k \sum_{q=0}^{i-2} \binom{i}{q} \frac{x^q \delta_n^{(i-1-q)}}{i!} \\ &= \sum_{i=2}^k \sum_{q=0}^{i-2} \frac{x^q \delta_n^{(i-1-q)}}{(i-q)!q!} \\ &= \delta_n \sum_{i=2}^k \sum_{q=0}^{i-2} \frac{x^q \delta_n^{(i-2-q)}}{(i-q)!q!} \\ &= \delta_n \sum_{q=0}^{k-2} \sum_{i=q+2}^k \frac{x^q \delta_n^{(i-2-q)}}{(i-q)!q!} \\ &\leq \delta_n \sum_{q=0}^{k-2} \sum_{p=0}^{k-2-q} \frac{x^q \delta_n^p}{2q!} \\ &\leq \delta_n \sum_{q=0}^{k-2} \frac{x^q}{2q!(1-\delta_n)} = \frac{\delta_n}{(1-\delta_n)} \sum_{q=0}^{k-2} \frac{x^q}{2q!} \end{aligned}$$

Note that the geometric series  $\sum_{p=0}^{k-2-q} a \delta_n^p \leq \frac{a}{1-\delta_n}$ , a fact I am using from Calc II. We know  $\sum_{q=0}^{\infty} \frac{x^q}{2q!}$  converges in the standard definition by the ratio test (also Calc II), and so we know there is some upper bound  $B$  such that  $\sum_{q=0}^{k-2} \frac{x^q}{2q!} < B$  for all  $k$ .

Now, take any positive real number  $r$ .  $\delta = [\delta_n]$  is infinitesimal, so  $\frac{\delta}{1-\delta} = \left[\frac{\delta_n}{1-\delta_n}\right]$  is too. So  $\left[\frac{\delta_n}{1-\delta_n}\right] < r/B$ , and so  $\frac{\delta_n}{1-\delta_n} < r/B$  for  $\mathcal{F}$ -almost all  $n$ . When  $\frac{\delta_n}{1-\delta_n} < r/B$ , we have  $R_n^k \leq \frac{\delta_n}{1-\delta_n} \cdot B \leq \frac{r}{B} \cdot B = r$  for all  $k$ , and so  $[R_n^k]_k \leq r$ .

So, given any positive real number  $r$ , we know that for  $\mathcal{F}$ -almost all  $n$ , we have  $[R_n^k]_k \leq r$ , so  $\text{st}(R_n) \leq r$ . So for any positive real number  $r$ , we have  $[\text{st}(R_n)]_n \leq r$ , and hence  $[\text{st}(R_n)]_n$  is infinitesimal.  $\square$

How did this happen. I thought this would be easy. I guess I have to talk about geometric series and the ratio test next week. I don't even know if there is a nonstandard proof of the ratio test.

## 4 Notes & Goals

### 4.1 Notes

This wasn't supposed to happen. I spend so much time on exponentiation. It was supposed to be a quick detour.

Anyways, unless I've made a critical error and it's all nonsense this is definitely going in, if only because it's all original work. Defining  $\exp$  in terms of hyperfinite summation is a cool idea, I think. I do still have to prove that

that...is bounded, and hence has a standard part, for all real inputs. Actaully, if I do that properly then I won't have to bring it up in the middle of the lemma.

I also spent a lot of time this week worrying about the “remainder” section in integration, before I realized that I *can* assume that if  $f$  is integrable on  $[a, b]$  it is bounded. I forgot that improper integrals weren't real integrals. Whoops.

## 4.2 Goals

- *Actually for real* sit down and list everything that's going in the thesis.
- Figure out if I can prove the properties of geometric series and the ratio test nonstandardly.
- Read Chapters 13.14—14 (pp. 176-189) in Goldblatt. This is almost certainly not going to be concluded, but dang it, I wanna make sure I've read it by the end of the semester anyways.

## Things I'm Looking At

- [1] Robert Goldblatt. *Lectures on the Hyperreals*. Vol. 188. Graduate Texts in Mathematics. New York, NY: Springer, 1998. ISBN: 978-1-4612-6841-3 978-1-4612-0615-6. DOI: 10.1007/978-1-4612-0615-6. URL: <http://link.springer.com/10.1007/978-1-4612-0615-6> (visited on 09/25/2024).