

# Number Theory Notebook

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## 1 Chapter 1

### Divisibility and congruence

**Theorem 1.1.** *Let  $a$ ,  $b$ , and  $c$  be integers. If  $a \mid b$  and  $a \mid c$ , then  $a \mid (b + c)$ .*

*Proof.*

(1.1.1)	$\exists d_b \in \mathbb{Z} \ni b = a \cdot d_b$	because $a \mid b$	
(1.1.2)	$\exists d_c \in \mathbb{Z} \ni c = a \cdot d_c$	because $a \mid c$	
(1.1.3)	$b + c = a \cdot d_b + a \cdot d_c$	by (1.1.1) and (1.1.2)	
(1.1.4)	$b + c = a \cdot (d_b + d_c)$	by distributive property	
(1.1.5)	$d_b + d_c \in \mathbb{Z}$	because $d_b \in \mathbb{Z}$ and $d_c \in \mathbb{Z}$	
	$a \mid (b + c)$	by def'n of divides	□

**Theorem 1.2.** *Let  $a$ ,  $b$ , and  $c$  be integers. If  $a \mid b$  and  $a \mid c$ , then  $a \mid (b - c)$ .*

*Proof.*

(1.2.1)	$\exists d_b \in \mathbb{Z} \ni b = a \cdot d_b$	because $a \mid b$	
(1.2.2)	$\exists d_c \in \mathbb{Z} \ni c = a \cdot d_c$	because $a \mid c$	
(1.2.3)	$b - c = a \cdot d_b - a \cdot d_c$	by (1.2.1) and (1.2.2)	
(1.2.4)	$b - c = a \cdot (d_b - d_c)$	by distributive property	
(1.2.5)	$d_b - d_c \in \mathbb{Z}$	because $d_b \in \mathbb{Z}$ and $d_c \in \mathbb{Z}$	
	$a \mid (b - c)$	by def'n of divides	□

**Theorem 1.3.** *Let  $a$ ,  $b$ , and  $c$  be integers. If  $a \mid b$  and  $a \mid c$ , then  $a \mid bc$ .*

*Proof.*

(1.3.1)	$\exists d_b \in \mathbb{Z} \ni b = a \cdot d_b$	because $a \mid b$	
(1.3.2)	$\exists d_c \in \mathbb{Z} \ni c = a \cdot d_c$	because $a \mid c$	
(1.3.3)	$bc = (a \cdot d_b) \cdot (a \cdot d_c)$	by (1.3.1) and (1.3.2)	
(1.3.4)	$bc = a \cdot (a \cdot d_b \cdot d_c)$	by associativity and commutativity	
(1.3.5)	$a \cdot d_b \cdot d_c \in \mathbb{Z}$	because $d_b \in \mathbb{Z}$ and $d_c \in \mathbb{Z}$	
	$a \mid bc$	by def'n of divides	□

**Question 1.4.** *Can you weaken the hypothesis of the previous theorem and still prove the conclusion? Can you keep the same hypothesis, but replace the conclusion by the stronger conclusion that  $a^2 \mid bc$  and still prove the theorem?*

Yes. You can remove the  $a \mid c$  condition to weaken the hypothesis, or with both  $a \mid b$  and  $a \mid c$  you can show  $a^2 \mid bc$ .

**Question 1.5.** Can you formulate your own conjecture along the lines of the above theorems and then prove it to make it your theorem?

Yes.

**Paul's Conjecture 1.** Let  $a$ ,  $b$ , and  $c$  be integers. If  $a|b$  and  $a|c$ , then  $a^2|bc$ .

*Proof.* First, take lines (1.3.1) through (1.3.4) of the proof of Theorem 1.3. Then,

$$\begin{array}{ll} d_b \cdot d_c \in \mathbb{Z} & \text{because } d_b \in \mathbb{Z} \text{ and } d_c \in \mathbb{Z} \\ a^2|bc & \text{by def'n of divides} \end{array} \quad \square$$

**Theorem 1.6.** Let  $a$ ,  $b$ , and  $c$  be integers. If  $a|b$ , then  $a|bc$ .

*Proof.*

$$\begin{array}{lll} (1.6.1) & \exists d \in \mathbb{Z} \ni ad = b & \text{because } a|b \\ (1.6.2) & bc = adc & \text{by (1.6.1)} \\ (1.6.3) & dc \in \mathbb{Z} & \text{because } d \in \mathbb{Z} \text{ and } c \in \mathbb{Z} \\ & a|bc & \text{by def'n of divides} \end{array} \quad \square$$

**Exercise 1.7.** Answer each of the following questions, and prove that your answer is correct.

1. Is  $45 \equiv 9 \pmod{4}$ ?  
Yes.  $4 \cdot 9 = 36 = 45 - 9$ .
2. Is  $37 \equiv 2 \pmod{5}$ ?  
Yes.  $5 \cdot 7 = 35 = 37 - 2$ .
3. Is  $37 \equiv 3 \pmod{5}$ ?  
No.  $37 - 3 = 34$  which is not a multiple of 5.
4. Is  $37 \equiv -3 \pmod{5}$ ?  
Yes.  $5 \cdot 8 = 40 = 37 - (-3)$ .

**Exercise 1.8.** For each of the following congruences, characterize all the integers  $m$  that satisfy that congruence.

1.  $m \equiv 0 \pmod{3}$   
 $m \in \{3z \mid z \in \mathbb{Z}\}$
2.  $m \equiv 1 \pmod{3}$   
 $m \in \{3z + 1 \mid z \in \mathbb{Z}\}$
3.  $m \equiv 2 \pmod{3}$   
 $m \in \{3z + 2 \mid z \in \mathbb{Z}\}$
4.  $m \equiv 3 \pmod{3}$   
 $m \in \{3z \mid z \in \mathbb{Z}\}$
5.  $m \equiv 4 \pmod{3}$   
 $m \in \{3z + 1 \mid z \in \mathbb{Z}\}$

**Theorem 1.9.** Let  $a$  and  $n$  be integers with  $n > 0$ . Then  $a \equiv a \pmod{n}$ .

*Proof.*

(1.9.1)	$0 \in \mathbb{Z}$	
(1.9.2)	$n \cdot 0 = 0$	
(1.9.3)	$n 0$	By def'n of divides
(1.9.4)	$a - a = 0$	
(1.9.5)	$n (a - a)$	By (1.9.3) and (1.9.4)
	$a \equiv a \pmod{n}$	By def'n of modular congruence <span style="float: right;">□</span>

**Theorem 1.10.** Let  $a$ ,  $b$ , and  $n$  be integers with  $n > 0$ . If  $a \equiv b \pmod{n}$ , then  $b \equiv a \pmod{n}$ .

*Proof.*

(1.10.1)	$a \equiv b \pmod{n}$	Given
(1.10.2)	$\exists d \in \mathbb{Z} \ni nd = a - b$	By def'n of modular congruence
(1.10.3)	$-1nd = -1 \cdot (a - b)$	By multiplicative property of equality
(1.10.4)	$n \cdot (-d) = b - a$	By various algebra
(1.10.5)	$-d \in \mathbb{Z}$	By multiplicative closure of $\mathbb{Z}$
(1.10.6)	$n (b - a)$	By (1.10.4), (1.10.5)
	$b \equiv a \pmod{n}$	By def'n of modular congruence <span style="float: right;">□</span>

**Theorem 1.11.** Let  $a$ ,  $b$ , and  $n$  be integers with  $n > 0$ . If  $a \equiv b \pmod{n}$  and  $b \equiv c \pmod{n}$ , then  $a \equiv c \pmod{n}$ .

*Proof.*

(1.11.1)	$n a - b$	By $a \equiv b \pmod{n}$
(1.11.2)	$n b - c$	By $b \equiv c \pmod{n}$
(1.11.3)	$\exists d_1 \in \mathbb{Z} \ni nd_1 = a - b$	By (1.11.1)
(1.11.4)	$\exists d_2 \in \mathbb{Z} \ni nd_2 = b - c$	By (1.11.2)
(1.11.5)	$nd_1 + nd_2 = (a - b) + (b - c)$	By additive property of equality
(1.11.6)	$n(d_1 + d_2) = a - c$	By various algebra
(1.11.7)	$d_1 + d_2 \in \mathbb{Z}$	By closure of integers under addition
(1.11.8)	$n (a - c)$	By def'n of divides
	$a \equiv c \pmod{n}$	By def'n of modular congruence <span style="float: right;">□</span>

**Theorem 1.12.** Let  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $n$  be integers with  $n > 0$ . If  $a \equiv b \pmod{n}$  and  $c \equiv d \pmod{n}$ , then  $a + c \equiv b + d \pmod{n}$ .

*Proof.*

(1.12.1)	$n (a - b)$	By $a \equiv b \pmod{n}$
(1.12.2)	$\exists d_1 \in \mathbb{Z} \ni nd_1 = a - b$	By def'n divides
(1.12.3)	$n (c - d)$	By $c \equiv d \pmod{n}$
(1.12.4)	$\exists d_2 \in \mathbb{Z} \ni nd_2 = c - d$	By def'n divides
(1.12.5)	$nd_1 + nd_2 = (a - b) + (c - d)$	By additive property of equality
(1.12.6)	$n \cdot (d_1 + d_2) = (a + c) - (b + d)$	By various algebra
(1.12.7)	$d_1 + d_2 \in \mathbb{Z}$	By additive closure of $\mathbb{Z}$
(1.12.8)	$n ((a + c) - (b + d))$	By def'n of divides
	$a + c \equiv b + d \pmod{n}$	By def'n of modular congruence <span style="float: right;">□</span>

**Theorem 1.13.** Let  $a, b, c, d$ , and  $n$  be integers with  $n > 0$ . If  $a \equiv b \pmod{n}$  and  $c \equiv d \pmod{n}$ , then  $a - c \equiv b - d \pmod{n}$ .

*Proof.* Notice  $-c$  and  $-d$  are integers, and  $-c \equiv -d \pmod{n}$  (glossing over the proof of that for now). Then simply cite 1.12 and we're done.  $\square$

**Theorem 1.14.** Let  $a, b, c, d$ , and  $n$  be integers with  $n > 0$ . If  $a \equiv b \pmod{n}$  and  $c \equiv d \pmod{n}$ , then  $ac \equiv bd \pmod{n}$ .

*Proof.*

$$\begin{array}{lll}
 (1.14.1) & n|(a-b) & \text{By } a \equiv b \pmod{n} \\
 (1.14.2) & \exists k_1 \in \mathbb{Z} \ni a-b = nk_1 & \\
 (1.14.3) & a = nk_1 + b & \\
 (1.14.4) & n|(c-d) & \text{By } c \equiv d \pmod{n} \\
 (1.14.5) & \exists k_2 \in \mathbb{Z} \ni c-d = nk_2 & \\
 (1.14.6) & c = nk_2 + d & \\
 (1.14.7) & ac = (nk_1 + b)(nk_2 + d) & \text{By (1.14.3) and (1.14.6)} \\
 (1.14.8) & ac = n^2k_1k_2 + nk_1d + nk_2b + bd & \\
 (1.14.9) & ac - bd = n \cdot (nk_1k_2 + k_1d + k_2b) & \\
 (1.14.10) & n|(ac - bd) & \text{Since } nk_1k_2 + k_1d + k_2b \in \mathbb{Z} \\
 & ac \equiv bd \pmod{n} & \square
 \end{array}$$

**Exercise 1.15.** Let  $a, b$ , and  $n$  be integers with  $n > 0$ . Show that if  $a \equiv b \pmod{n}$ , then  $a^2 \equiv b^2 \pmod{n}$ .

*Proof.*

$$\begin{array}{lll}
 (1.15.1) & a \equiv b \pmod{n} & \text{Given} \\
 (1.15.2) & a \cdot a \equiv b \cdot b \pmod{n} & 1.14 \\
 & a^2 \equiv b^2 \pmod{n} & \square
 \end{array}$$

**Exercise 1.16.** Let  $a, b$ , and  $n$  be integers with  $n > 0$ . Show that if  $a \equiv b \pmod{n}$ , then  $a^3 \equiv b^3 \pmod{n}$ .

*Proof.*

$$\begin{array}{lll}
 (1.16.1) & a \equiv b \pmod{n} & \text{Given} \\
 (1.16.2) & a^2 \equiv b^2 \pmod{n} & 1.15 \\
 (1.16.3) & a \cdot a^2 \equiv b \cdot b^2 \pmod{n} & \text{By 1.14 on (1.16.1) and (1.16.2)} \\
 & a^3 \equiv b^3 \pmod{n} & \square
 \end{array}$$

**Exercise 1.17.** Let  $a, b, k$ , and  $n$  be integers with  $n > 0$  and  $k > 1$ . Show that if  $a \equiv b \pmod{n}$  and  $a^{k-1} \equiv b^{k-1} \pmod{n}$ , then  $a^k \equiv b^k \pmod{n}$ .

*Proof.*

$$\begin{array}{lll}
 (1.17.1) & a \equiv b \pmod{n} & \text{Given} \\
 (1.17.2) & a^{k-1} \equiv b^{k-1} \pmod{n} & 1.15 \\
 (1.17.3) & a \cdot a^{k-1} \equiv b \cdot b^{k-1} \pmod{n} & \text{By 1.14 on (1.17.1) and (1.17.2)} \\
 & a^k \equiv b^k \pmod{n} & \square
 \end{array}$$

**Theorem 1.18.** Let  $a, b, k$ , and  $n$  be integers with  $n > 0$  and  $k > 0$ . If  $a \equiv b \pmod{n}$ , then  $a^k \equiv b^k \pmod{n}$ .

*Proof.* Our base case is 1.9. Our induction hypothesis is " $a, b, k$ , and  $n$  are integers with  $n > 0$  and  $k > 1$  such that  $\forall j \ni 0 < j < k$ , we find  $a^j \equiv b^j \pmod{n}$ ". Notice our induction hypothesis fulfills the criteria for 1.17, and in fact 1.17 covers our induction step.  $\square$

**Exercise 1.19.** Illustrate each of Theorems 1.12 - 1.18 with an example using actual numbers

1.12

$2 \equiv 12 \pmod{10}$  and  $5 \equiv 15 \pmod{10}$  imply  $7 \equiv 27 \pmod{10}$ .

1.13

$7 \equiv 27 \pmod{10}$  and  $12 \equiv 2 \pmod{10}$  imply that  $-5 \equiv 25 \pmod{10}$ .

1.14

$2 \equiv 7 \pmod{5}$  and  $3 \equiv 8 \pmod{5}$  imply that  $6 \equiv 56 \pmod{5}$ .

1.15

$2 \equiv 7 \pmod{5}$  implies that  $4 \equiv 49 \pmod{5}$ .

1.16

$1 \equiv 3 \pmod{2}$  implies that  $1 \equiv 27 \pmod{2}$ .

1.17

$1 \equiv 3 \pmod{2}$  and  $1 \equiv 27 \pmod{2}$  imply that  $1 \equiv 81 \pmod{2}$ .

1.18

$1 \equiv 3 \pmod{2}$  implies that  $1 \equiv 81 \pmod{2}$ .

**Question 1.20.** Let  $a, b, c$ , and  $n$  be integers for which  $ac \equiv bc \pmod{n}$ . Can we conclude that  $a \equiv b \pmod{n}$ ? If you answer "yes", try to give a proof. If you answer "no", try to give a counterexample.

No. Notice  $1 \cdot 0 \equiv 2 \cdot 0 \pmod{5}$  and yet  $1 \not\equiv 2 \pmod{5}$ .

**Theorem 1.21.** Let a natural number  $n$  be expressed in base 10 as

$$n = a_k a_{k-1} \dots a_1 a_0$$

If  $m = a_k + a_{k-1} + \dots + a_1 + a_0$  then  $n \equiv m \pmod{3}$ .

First, a Lemma that will help us later.

**Lemma 1.21.1.** Let  $a$  be an integer and  $j$  a natural number. Then  $a \equiv a \cdot 10^j \pmod{3}$ .

*Proof.* Notice that  $1 \equiv 10 \pmod{3}$ . Then, by 1.18, we find  $1^j \equiv 10^j \pmod{3}$  and thus that  $1 \equiv 10^j \pmod{3}$ . Then, since  $a \equiv a \pmod{3}$  (by 1.9), we invoke 1.14 to find  $a \cdot 1 \equiv a \cdot 10^j \pmod{3}$ , implying that  $a \equiv a \cdot 10^j \pmod{3}$ .  $\square$

Now we begin our proof of the theorem in full.

*Proof.* Notice that  $n$  can be written as  $a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \dots + a_1 \cdot 10 + a_0$ , or more easily as

$$n = \sum_{i=0}^k a_i \cdot 10^i$$

Now notice that

$$m = \sum_{i=0}^k a_i$$

By 1.21.1, we notice that  $\forall i, a_i \equiv a_i \cdot 10^i \pmod{3}$ . Thus,  $n$  and  $m$  are sums of terms that are congruent modulo 3. By repeatedly invoking 1.12, we eventually find that the two strings of congruent sums are themselves congruent, i.e. that  $n \equiv m \pmod{3}$ .  $\square$

**Theorem 1.22.** If a natural number is divisible by 3, then, when expressed in base 10, the sum of its digits is divisible by 3.

*Proof.* Let the natural number be  $n$ , and the sum of its digits  $m$ . We're given by the theorem  $n \equiv 0 \pmod{3}$ , and by 1.21 we know  $n \equiv m \pmod{3}$ , so we can cite 1.11 and conclude  $m \equiv 0 \pmod{3}$ , i.e.  $m$  is divisible by 3.  $\square$

**Theorem 1.23.** *If the sum of the digits of a natural number expressed in base 10 is divisible by 3, then the number is divisible by 3 as well.*

*Proof.* Let the natural number be  $n$ , and the sum of its digits  $m$ . We're given by the theorem  $m \equiv 0 \pmod{3}$ , and by 1.21 we know  $n \equiv m \pmod{3}$ , so we can cite 1.11 and conclude  $n \equiv 0 \pmod{3}$ , i.e.  $n$  is divisible by 3.  $\square$

**Exercise 1.24.** *Devise and prove other divisibility criteria similar to the preceding one.*

A number is divisible by 2 if and only if its last digit is divisible by 2, because any (base 10) number  $n = a_k a_{k-1} \dots a_1 a_0 = a_k a_{k-1} \dots a_1 \cdot 10 + a_0$ , and  $2|10$  so  $2|\dots \cdot 10$ . Thus,  $2|\dots \cdot 10 + a_0$  iff  $2|a_0$ .

Similar proofs can be done for 5 and the last digit, 4 and the last 2 digits, 8 and the last 3 digits, 16 and the last 4 digits, 32 and the last 5 digits, etc.

## The Division Algorithm

**Exercise 1.25.** *Illustrate the division algorithm for:*

1.  $m = 25, n = 7$ .  
 $25 = 7 \cdot 3 + 4$ .
2.  $m = 277, n = 4$ .  
 $277 = 4 \cdot 69 + 1$ .
3.  $m = 33, n = 11$ .  
 $33 = 11 \cdot 3 + 0$ .
4.  $m = 33, n = 45$ .  
 $33 = 44 \cdot 0 + 33$ .

**Theorem 1.26.** *Prove the existence part of the Division Algorithm. In other words, given natural numbers  $n$  and  $m$ , show there exist integers  $q$  and  $r$  such that  $m = nq + r$  and  $0 \leq r \leq n - 1$ .*

*Proof.* Let  $S = \{x \in \mathbb{Z} \mid nx > m\}$ . By the Well-Ordering Axiom,  $S$  has a smallest element: call it  $s$ . Let  $q = s - 1$ . This definition gives us two important properties:

1.  $nq \leq m$ , for if  $nq > m$  then  $q \in S$  with  $q < s$ , which is impossible since  $s$  is the smallest element of  $S$ .
2.  $m < n(q + 1) = nq + n$ , for  $q + 1 = s$  and  $sx > m$  because  $s \in S$ .

Now, we define  $r = m - nq$ , so that by definition  $m = nq + r$ . Since  $nq \leq m$ , we know  $r \geq 0$ . Since  $m < nq + n$ , and yet  $m = nq + r$ , implying  $nq + r < nq + n \implies r < n \implies r \leq n - 1$ .

Thus, we have found  $q, r$  such that  $m = nq + r$  and  $0 \leq r \leq n - 1$ .  $\square$

**Theorem 1.27.** *Prove the uniqueness part of the Division Algorithm. In other words, given natural numbers  $n$  and  $m$ , if there are 4 integers  $q, q', r$ , and  $r'$ , such that  $m = nq + r = nq' + r'$  with  $0 \leq r, r' \leq n - 1$  then  $q = q'$  and  $r = r'$ .*

*Proof.* Notice that  $nq + r = nq' + r'$  implies that  $nq - nq' = r' - r \implies n(q - q') = r' - r$ .

Since  $0 \leq r, r' \leq n - 1$ , we conclude that  $-n + 1 \leq r' - r \leq n - 1$ . By our previous equality, then,  $-n + 1 \leq n(q - q') \leq n - 1 \implies -n < n(q - q') < n$ . Since  $n$  is a natural number, we can divide by  $n$  to get  $-1 < q - q' < 1$ . Since  $q$  and  $q'$  are integers,  $q - q'$  must also be an integer. The only integer between  $-1$  and  $1$  is  $0$ , so we conclude  $q - q' = 0 \implies q = q'$ .

Once we have  $q = q'$ , we see that  $nq + r = nq' + r' \implies nq + r = nq + r' \implies r = r'$ .  $\square$

**Theorem 1.28.** Let  $a$ ,  $b$ , and  $n$  be integers with  $n > 0$ . Then  $a \equiv b \pmod{n}$  if and only if  $a$  and  $b$  have the same remainder when divided by  $n$ . Equivalently,  $a \equiv b \pmod{n}$  if and only if when  $a = nq_1 + r_1$  ( $0 \leq r_1 \leq n - 1$ ) and  $b = nq_2 + r_2$  ( $0 \leq r_2 \leq n - 1$ ) then  $r_1 = r_2$ .

First, we will show that  $a \equiv b \pmod{n} \implies r_1 = r_2$ .

*Proof.* Notice by the definition of modular congruence that  $a \equiv b \pmod{n}$  implies that  $n \mid (b - a)$ , or  $\exists d \in \mathbb{Z} \ni nd = b - a$ . Using  $a = nq_1 + r_1$  and  $b = nq_2 + r_2$  we get  $nd = nq_1 + r_1 - nq_2 - r_2 = n(q_1 - q_2) + r_1 - r_2$ . Then we get  $nd - n(q_1 - q_2) = r_1 - r_2$  or  $n(d - q_1 + q_2) = r_1 - r_2$ .

Since  $0 \leq r_1, r_2 \leq n - 1$  we find that  $-n + 1 \leq r_1 - r_2 \leq n - 1 \implies -n < r_1 - r_2 < n$ . Using our previous equation with  $r_1 - r_2$  we get that  $-n < n(d - q_1 + q_2) < n$ , and dividing by  $n$  (which we can do because  $n > 0$ ) we get  $-1 < d - q_1 + q_2 < 1$ . Since  $d$ ,  $q_1$ , and  $q_2$  are all integers,  $d - q_1 + q_2$  is also an integer, and the only integer between  $-1$  and  $1$  is  $0$  so we find  $d - q_1 + q_2 = 0$ .

Plugging this back in to  $n(d - q_1 + q_2) = r_1 - r_2$ , we find  $n \cdot 0 = r_1 - r_2$ , which implies  $0 = r_1 - r_2$ , or  $r_1 = r_2$ .  $\square$

Second, we will show that  $r_1 = r_2 \implies a \equiv b \pmod{n}$ .

*Proof.* Notice  $a - b = nq_1 + r_1 - (nq_2 + r_2)$ . With some simple rearranging, we obtain  $a - b = n(q_1 - q_2) + r_1 - r_2$ . Since we know  $r_1 = r_2$ , we know  $r_1 - r_2 = 0$ , and plugging this in we obtain  $a - b = n(q_1 - q_2)$ .

Since  $q_1$  and  $q_2$  are integers,  $q_1 - q_2$  is also an integer. Thus,  $n$  times some integer is  $a - b$ : in other words,  $n \mid (a - b)$ .

Then, by the definition of modular congruence, we obtain  $a \equiv b \pmod{n}$ .  $\square$

## Greatest common divisors and linear Diophantine equations

**Question 1.29.** Do every two integers have at least one common divisor?

Yes. For any two integers  $a$  and  $b$ ,  $1 \cdot a = a$  and  $1 \cdot b = b$  so  $1 \mid a$  and  $1 \mid b$ , making  $1$  a common divisor of  $a$  and  $b$ .

**Question 1.30.** Can two integers have infinitely many common divisors?

No, if the two integers are distinct. Any nonzero integer  $n$  can only have finitely many divisors, as any integer  $d$  such that  $d < -|n|$  or  $d > |n|$  cannot be a divisor (since  $1d$  and  $-1d$  have a greater absolute value than  $n$ , and  $0d = 0 \neq n$ ). In other words, only the numbers  $f$  such that  $-n \leq f \leq n$  are “eligible” to be divisors of  $n$ , so there can only be finitely many divisors of  $n$ .

**Exercise 1.31.** Find the following greatest common divisors. Which pairs are relatively prime?

1.  $(36, 22)$   
2
2.  $(45, -15)$   
15
3.  $(-296, -88)$   
8
4.  $(0, 256)$   
256
5.  $(15, 28)$   
1 (relatively prime)
6.  $(1, -2436)$   
1 (relatively prime)

**Theorem 1.32.** Let  $a$ ,  $n$ ,  $b$ ,  $r$ , and  $k$  be integers. If  $a = nb + r$  and  $k \mid a$  and  $k \mid b$ , then  $k \mid r$ .

*Proof.* Let  $a = d_a k$  and  $b = d_b k$ , where  $d_a$  and  $d_b$  are the integers guaranteed by the facts that  $k \mid a$  and  $k \mid b$ . Then, we have  $d_a k = nd_b k + r$ . Isolating  $r$ , we get  $r = d_a k - nd_b k = k(d_a - nd_b)$ . Since  $n$ ,  $d_a$ , and  $d_b$  are all integers, we know  $d_a - nd_b$  is an integer. Thus, we’ve found  $r$  is equal to  $k$  times some integer, so  $k \mid r$ .  $\square$

**Theorem 1.33.** Let  $a, b, n_1$ , and  $r_1$  be integers with  $a$  and  $b$  not both 0. If  $a = n_1b + r_1$ , then  $(a, b) = (b, r_1)$ .

*Proof.* We will show that the common divisors of  $a$  and  $b$  are the same as the common divisors of  $b$  and  $r_1$ , and thus conclude that the greatest element of  $S$  is also the greatest element of  $T$ .

Let  $S$  be the set of common divisors of  $a$  and  $b$ , and let  $T$  be the set of common divisors of  $b$  and  $r_1$ . We will show  $S = T$  by double inclusion.

First, let's show  $S \subset T$ . Take an arbitrary  $s \in S$ . Since  $s|a$  and  $s|b$ , we conclude  $\exists d_a, d_b \in \mathbb{Z} \ni a = sd_a, b = sd_b$ . We can then rearrange  $a = n_1b + r_1$  to read  $r_1 = a - n_1b$ , and then plug in our previous two equations to get  $r_1 = sd_a - n_1sd_b \implies r_1 = s(d_a - n_1d_b)$ . Since  $d_a, d_b$ , and  $n_1$  are all integers, we know  $d_a - n_1d_b$  is an integer, thus implying that  $s|r_1$ . Since we know  $s|b$  since  $s \in S$ , we conclude  $s \in T$ . Thus, any arbitrary  $s \in S$  is an element of  $T$ , so  $S \subset T$ .

Showing that  $T \subset S$  proceeds in much the same way. Take  $t \in T$ , conclude since  $t|b$  and  $t|r_1$  we find  $\exists d_b, d_r \in \mathbb{Z} \ni b = td_b, r_1 = td_r$ , and then plug those in to  $a = n_1b + r_1$  to get  $a = n_1td_b + td_r \implies a = t(n_1d_b + d_r)$ . Since  $n_1, d_b$ , and  $d_r$  are integers, we find  $t|a$ , and since  $t|b$  because  $t \in T$ , we thus conclude  $t \in S$ . Thus any arbitrary  $t \in T$  is an element of  $S$ , so  $T \subset S$ .

Thus, by double inclusion,  $S = T$ . This implies that the greatest element of  $S$ , i.e.  $(a, b)$ , is equal to the greatest element of  $T$ , i.e.  $(b, r_1)$ .  $\square$

**Exercise 1.34.** Use the preceding theorem to show that if  $a = 51$  and  $b = 15$ , then  $(51, 15) = (6, 3) = 3$ .

*Proof.* Since  $51 = 3 \cdot 15 + 6$ , we find  $(51, 15)$ , we cite 1.33 to see  $(51, 15) = (15, 6)$ . Then, since  $15 = 2 \cdot 6 + 3$ , we again cite 1.33 to find  $(15, 6) = (6, 3)$ . We see that  $(6, 3) = 3$  by inspection. Then, since equality is transitive, we conclude  $(51, 15) = (6, 3) = 3$ .  $\square$

**Exercise 1.35.** Using the previous theorem and the Division Algorithm successively, devise a procedure for finding the greatest common divisor of two integers.

Well you kind of gave the game away when you said to use 1.33 and the division algorithm successively huh. If you're trying to find  $(a, b)$ , you simply invoke the division algorithm to get  $a = nb + r$  (assuming WLOG that  $a \geq b$ ), and then rewrite  $(a, b)$  as  $(b, r)$ . Then, you use the division algorithm to get  $b = nr + r'$ , simplifying to  $(r, r')$ , etc., until at some point you have  $(x, 0)$ , which by inspection is equal to  $x$ .

You will always reach  $(x, 0)$  because the division algorithm produces a remainder  $r$  that is strictly less than the smaller input  $b$ , so (informally) the smaller of the two numbers you're working with always gets smaller while never going negative.

**Exercise 1.36.** Use the Euclidean Algorithm to find the following.

1.  $(96, 112)$

$112 = 1 \cdot 96 + 16$ , simplifying the problem to  $(96, 16)$ . Then  $96 = 5 \cdot 16 + 0$ , so we get  $(16, 0) = 16$

2.  $(162, 31)$

$162 = 5 \cdot 31 + 7 \implies (31, 7) \implies 31 = 4 \cdot 7 + 3 \implies (7, 3) \implies 7 = 2 \cdot 3 + 1 \implies (3, 1) = 1$ .

3.  $(0, 256)$

Since everything divides 0, this is trivially 256.

4.  $(-288, -166)$

$-166 = 1 \cdot -288 + 122 \implies (-288, 122) \implies -288 = -3 \cdot 122 + 78 \implies (122, 78) \implies 122 = 1 \cdot 78 + 44 \implies (78, 44) \implies 78 = 1 \cdot 44 + 34 \implies (44, 34) \implies 44 = 1 \cdot 34 + 10 \implies (34, 10) = 2$  by inspection.

5.  $(1, -2436)$

Since the only integers that divide 1 are  $-1, 0$ , and  $1$ , we trivially find 1.

**Exercise 1.37.** Find integers  $x$  and  $y$  such that  $162x + 31y = 1$ .

By division algorithm,  $162 = 5 \cdot 31 + 7 \implies 7 = 1 \cdot 162 + (-5) \cdot 31$ .

By division algorithm,  $31 = 4 \cdot 7 + 3 \implies 3 = 1 \cdot 31 + (-4) \cdot 7 = 1 \cdot 31 + (-4) \cdot (1 \cdot 162 + (-5) \cdot 31) = (-4) \cdot 162 + 21 \cdot 31$ .

By division algorithm,  $7 = 2 \cdot 3 + 1 \implies 1 = 1 \cdot 7 + (-2) \cdot 3 = 1 \cdot (1 \cdot 162 + (-5) \cdot 31) + (-2) \cdot ((-4) \cdot 162 + 21 \cdot 31) = 9 \cdot 162 + (-47) \cdot 31$ .

Thus, we've found our solution  $x = 9$  and  $y = -47$ .



**Theorem 1.38.** *Let  $a$  and  $b$  be integers. If  $(a, b) = 1$ , then there exist integers  $x$  and  $y$  such that  $ax + by = 1$ .*

*Proof.* If either  $a$  or  $b$  is negative, replace it with  $-a$  or  $-b$  for the rest of this proof. At the end, you can replace either  $x$  or  $y$  with  $-x$  or  $-y$  to get an answer; for instance, if  $a = -3$ , we can replace  $a = 3$ , do the proof to obtain  $x_0$  and  $y_0$  such that  $3x_0 + by_0 = 1$  and then realize that  $(-3)(-x_0) + by_0 = 1$ , which since  $-x_0$  is still an integer still suffices. Now we will only be worrying about non-negative  $a$  and  $b$ s.

We will demonstrate an algorithm to find  $x$  and  $y$ . WLOG, assume  $a \geq b$ . Invoke the division algorithm to get  $a = n_1b + r_1$ . Then invoke it again to get  $b = n_2r_1 + r_2$ . Then invoke it again to get  $r_1 = n_3r_2 + r_3$ . Etc. etc. etc.

We will show that the series “remainder” generated by this algorithm eventually has to hit 0: in other words,  $\exists i \in \mathbb{N} \ni r_i = 0$ . To do this, we must notice that for any index  $j$ , since  $r_j$  is generated by calling the division algorithm on  $r_{j-2}$  and  $r_{j-1}$ , we find that  $r_j \leq r_{j-1} - 1$ . Notice, then, that we can apply this to  $r_{j-1}$  to obtain  $r_{j-1} \leq r_{j-2} - 1$ , and then plug that in to our previous inequality to get  $r_j \leq r_{j-1} - 1 \leq r_{j-2} - 2$ .

By inspection (i.e. I’m lazy and don’t want to formalize this), we notice we can continually apply this. We will apply this to  $r_b$ , and notice that  $r_b \leq r_{b-1} - 1 \leq r_{b-2} - 2 \leq \dots \leq r_1 - (b-1) \leq b - b$ . Since  $b - b = 0$ , we find  $r_b \leq 0$ , but since  $r_b$  is a remainder from the division algorithm we know  $r_b \geq 0$ , so we conclude  $r_b = 0$ .

Notice we have *not* proven that  $r_b$  is the *first* 0, only that the remainders must *eventually* reach 0 at *some* point.

Now, keep invoking the division algorithm until the “remainder” generated by the algorithm is 0: we will label that step  $k + 1$ , so that we find  $r_{k+1} = n_{k+1}r_k + 0$ . We will show that  $r_k$  is 1.

By invoking 1.33 repeatedly, we find that  $(a, b) = (b, r_1) = (r_1, r_2) = \dots = (r_{k-1}, r_k) = (r_k, r_{k+1})$ . Since  $r_{k+1} = 0$ , we conclude  $(a, b) = (r_k, 0)$ . Since 0 divides everything,  $(r_k, 0) = r_k$ , so  $(a, b) = r_k$ , and since  $a$  and  $b$  are relatively prime we conclude  $1 = r_k$ .

Now, we take all of our equations and rewrite them to solve for the remainder. For example,  $a = n_1b + r_1$  becomes  $r_1 = a + (-n_1)b$ , and  $b = n_2r_1 + r_2$  becomes  $r_2 = b + (-n_2)r_1$ .

This gives us a bunch of equations of the form  $r_j = \delta_j r_{j-2} + \gamma_j r_{j-1}$ . This includes one for  $r_k$ , namely  $r_k = \delta_k r_{k-2} + \gamma_k r_{k-1}$ . We can then substitute in lower indices of  $r$  for  $r_{k-2}$  and  $r_{k-1}$ , using the generic equation, to get something like  $r_k = \delta_k(\delta_{k-2}r_{k-4} + \gamma_{k-2}r_{k-3}) + \gamma_k(\delta_{k-1}r_{k-3} + \gamma_{k-1}r_{k-2})$ .

That looks horrifying, but the important bit is that we notice if we simplify it we get  $r_k = Ar_{k-4} + Br_{k-3} + Cr_{k-2}$  with  $A, B, C \in \mathbb{Z}$ . That is, *by replacing all  $r_j$ ’s with their respective equations, we have reduced the highest index on an  $r$  in the right hand side by 1*. Previously, the highest index was  $k - 1$ , but now it’s  $k - 2$ , because we had an equation to represent  $r_{k-1}$  in terms of  $r_{k-3}$  and  $r_{k-4}$ .

Notice, though, that not all  $r$ ’s satisfy this property: namely,  $r_1$  and  $r_2$  simplify down to  $a$  and  $b$ , which then don’t have equations of their own. So, we apply the equations for  $r_k$  through  $r_1$  in “reverse” order, pairing down the maximum index of  $k$  each time, until we’re left with only  $r_1$ ’s and  $r_2$ ’s on the left hand side and can apply those equations to get a linear expression in  $a$  and  $b$  on the right hand side.

We’ve been talking a lot about the right hand side, but remember, the left hand side is  $r_k$ , and we’ve shown  $r_k = 1$ , so we’ve just found a linear expression in  $a$  and  $b$  that is equal to 1. In other words,  $1 = ax + by$  for some  $x, y \in \mathbb{Z}$ .  $\square$

**Theorem 1.39.** *Let  $a$  and  $b$  be integers. If there exist integers  $x$  and  $y$  with  $ax + by = 1$ , then  $(a, b) = 1$ .*

*Proof.* Readers of the last proof will be glad to hear this one is much simpler.

By definition,  $(a, b) | a$  and  $(a, b) | b$ . Then,  $(a, b) | ax$  and  $(a, b) | by$  by 1.6. Then,  $(a, b) | ax + by$  by 1.1. Then, since  $ax + by = 1$ , we find  $(a, b) | 1$ . We know  $1 | a$  and  $1 | b$ , so  $(a, b) \geq 1$ . The only number  $\geq 1$  that divides 1 is 1, so since  $(a, b) \geq 1$  and  $(a, b) | 1$  we conclude  $(a, b) = 1$ .  $\square$

**Theorem 1.40.** *For any integers  $a$  and  $b$  not both 0, there are integers  $x$  and  $y$  such that  $ax + by = (a, b)$ .*

*Proof.* Let  $c = a/(a, b)$  and  $d = b/(a, b)$ . Notice that since  $(c, d) | c$  and  $a = c \cdot (a, b)$  we find  $((c, d) \cdot (a, b)) | a$ , and similarly since  $(c, d) | d$  and  $b = d \cdot (a, b)$  we find  $((c, d) \cdot (a, b)) | b$ .

Since  $c$  and  $d$  are integers not both 0,  $(c, d)$  must be a positive integer. Since  $(c, d) \cdot (a, b)$  is a common factor of  $a$  and  $b$ , and  $(a, b)$  is the *greatest* common factor of  $a$  and  $b$ , we find  $(c, d) \cdot (a, b) \leq (a, b) \implies (c, d) = 1$ .

Thus, we invoke 1.38 to find integers  $x$  and  $y$  such that  $cx + dy = 1$ . Then, we multiply both sides by  $(a, b)$  to find that  $(a, b) \cdot cx + (a, b) \cdot dy = (a, b)$ . Since  $a = (a, b) \cdot c$  and  $b = (a, b) \cdot d$  we conclude  $ax + by = (a, b)$ .  $\square$

**Theorem 1.41.** Let  $a$ ,  $b$ , and  $c$  be integers. If  $a|bc$  and  $(a, b) = 1$ , then  $a|c$ .

*Proof.* Since  $(a, b) = 1$ , we can invoke 1.38 to find  $x, y \in \mathbb{Z} \ni ax + by = 1$ .

Now, since  $a|bc$ , we can cite 1.6 to obtain  $a|bcy$ .

Since  $a \cdot 1 = a$  we find  $a|a$ , and then by 1.6 we get  $a|acx$ .

Then, by 1.1 we get  $a|(acx + bcy)$ . We can then do some simple algebraic rearrangement to get  $a|(c \cdot (ax + by)) \implies a|(c \cdot 1) \implies a|c$ .  $\square$

**Theorem 1.42.** Let  $a$ ,  $b$ , and  $n$  be integers. If  $a|n$ ,  $b|n$ , and  $(a, b) = 1$ , then  $ab|n$ .

*Proof.* Since  $a|n$  and  $b|n$  we find integers  $k, j$  such that  $ak = n$  and  $bj = n$ . By the transitive property of equality,  $ak = bj$ . Since  $j$  is an integer, we conclude  $b|ak$ . Since  $(a, b) = 1$ , we invoke 1.41 to find  $b|k$ . Thus, we invoke an integer  $d$  such that  $bd = k$ . Substituting this into  $ak = n$ , we find  $abd = n$ , and since  $d$  is an integer we conclude  $ab|n$ .  $\square$

**Theorem 1.43.** Let  $a$ ,  $b$ , and  $n$  be integers. If  $(a, n) = 1$  and  $(b, n) = 1$ , then  $(ab, n) = 1$ .

*Proof.* Invoking 1.38 twice, we find two pairs of integers,  $x_a, y_a, x_b$ , and  $y_b$  such that  $ax_a + ny_a = 1$  and  $bx_b + ny_b = 1$ . We notice then that  $(ax_a + ny_a) \cdot (bx_b + ny_b) = 1 \cdot 1 = 1$ , and we simplify the left-hand side to  $ax_a bx_b + ax_a ny_b + ny_a bx_b + ny_a ny_b = ab(x_a x_b) + n(ax_a y_b + y_a bx_b + ny_a y_b) = 1$ , and then by closure of the integers and 1.39 we find that  $(ab, n) = 1$ .  $\square$

**Question 1.44.** What hypotheses about  $a$ ,  $b$ ,  $c$ , and  $n$  could be added so that  $ac \equiv bc \pmod{n}$ ? State an appropriate theorem and prove it before reading on.

**I know this from Math Seminar.**  $ac \equiv bc \pmod{n}$  implies  $a \equiv b \pmod{n}$  if and only if  $(c, n) = 1$ .

*Proof.* First, we will show that  $ac \equiv bc \pmod{n}$  and  $(c, n) = 1$  imply that  $a \equiv b \pmod{n}$ .

Notice that  $ac \equiv bc \pmod{n}$  implies  $n|(bc - ac)$ . By distribution, we obtain  $n|(c(b - a))$ . Then, since  $(c, n) = 1$ , we cite 1.41 to obtain  $n|(b - a)$ , which by definition means  $a \equiv b \pmod{n}$ .

Now, we will show that if  $(c, n) > 1$ , then  $ac \equiv bc \pmod{n}$  does *not* imply  $a \equiv b \pmod{n}$ .

We will do this by example. Notice that  $n|(c \cdot (n/(c, n)))$ : the right-hand side can be rearranged to read  $n \cdot (c/(c, n))$  and  $c/(c, n)$  is an integer because  $(c, n)$  is a factor of  $c$ . Then, since  $n|n$ , we cite 1.6 to find  $n|(n \cdot (c/(c, n)))$ . We then cite the facts that  $n|0$  and 1.2 to find  $n|((c \cdot (n/(c, n))) - 0)$ , and we can substitute in  $c \cdot 0$  for 0 to find  $n|((c \cdot (n/(c, n))) - c \cdot 0)$ . We then, by definition, obtain  $c \cdot (n/(c, n)) \equiv c \cdot 0 \pmod{n}$ .

However, since  $n > 0$  (because congruence “modulo  $n$ ” is defined) and  $(c, n) > 1$ , we find that  $0 < n/(c, n) < n$ . This implies that  $n/(c, n) \not\equiv 0 \pmod{n}$ , despite the fact that  $c \cdot (n/(c, n)) \equiv c \cdot 0 \pmod{n}$ , giving us our counterexample.  $\square$

**Theorem 1.45.** Let  $a$ ,  $b$ ,  $c$ , and  $n$  be integers with  $n > 0$ . If  $ac \equiv bc \pmod{n}$  and  $(c, n) = 1$ , then  $a \equiv b \pmod{n}$ .

See 1.44.

**Question 1.46.** Suppose  $a$ ,  $b$ , and  $c$  are integers and that there is a solution to the linear Diophantine equation  $ax + by = c$ . That is, suppose there are integer  $x$  and  $y$  that satisfy the equation  $ax + by = c$ . What condition must  $c$  satisfy in terms of  $a$  and  $b$ ?

Since  $(a, b)|(ax + by)$ , we conclude  $(a, b)|c$ .

**Question 1.47.** Can you make a conjecture by completing the following statement?

**Paul’s Conjecture 2.** Given integers  $a$ ,  $b$ , and  $c$ , there exist integers  $x$  and  $y$  that satisfy the equation  $ax + by = c$  if and only if  $(a, b)|c$ .

*Proof.* Notice that an integer solution to  $ax + by = c$  implies that, since  $(a, b)|a$  and  $(a, b)|b \implies (a, b)|(ax + by)$  (1.6 and 1.1), we conclude  $(a, b)|c$ .

Now, notice  $(a, b)|c \implies \exists d \in \mathbb{Z} \ni d(a, b) = c$ . We invoke 1.40 to find integers  $w$  and  $z$  such that  $aw + bz = (a, b)$ . Then, we can multiply both sides by  $d$  to obtain  $d(aw + bz) = d(a, b)$ , which simplifies to  $awd + bzd = c$ , giving us the solution  $x = wd$  and  $y = zd$ .  $\square$

**Theorem 1.48.** Given integers  $a$ ,  $b$ , and  $c$  with  $a$  and  $b$  not both 0, there exist integers  $x$  and  $y$  that satisfy the equation  $ax + by = c$  if and only if  $(a, b)|c$ .

See Paul’s Conjecture 2.

**Question 1.49.** For integers  $a$ ,  $b$ , and  $c$ , consider the linear Diophantine equation  $ax + by = c$ . Suppose integers  $x_0$  and  $y_0$  satisfy the equation: that is,  $ax_0 + by_0 = c$ . What other values

$$x = x_0 + h \text{ and } y = y_0 + k$$

also satisfy  $ax + by = c$ ? Formulate a conjecture that answers this question. Devise some numerical examples to ground your exploration. For example,  $6(-3) + 15 \cdot 2 = 12$ . Can you find other integers  $x$  and  $y$  such that  $6x + 15y = 12$ ? How many other pairs of integers  $x$  and  $y$  can you find? Can you find infinitely many other solutions?

**Paul's Conjecture 3.** The integers  $x_1 = x_0 + h$  and  $y_1 = y_0 + k$  satisfy the equation  $ax_1 + by_1 = c$  if and only if  $\frac{b}{(a,b)}|h$  and  $k = -\frac{ah}{b}$ .

*Proof.* First, notice  $ax_1 + by_1 = c$  if and only if  $a(x_0 + h) + b(y_0 + k) = c$ . Then with rearrangement, we find this is equivalent to  $ax_0 + by_0 + ah + bk = c \iff c + ah + bk = c \iff ah + bk = 0$ . Then, we find  $bk = -ah \iff k = -(ah/b)$ .

Notice that this “if-and-only-if chain” doesn’t show that  $k$  is an integer. Thus, we will show that  $k$  is an integer if and only if  $(b/(a,b))|h$ , the other condition, to complete our proof.

First, notice  $\frac{b}{(a,b)}|h \implies \exists d \in \mathbb{Z} \ni \frac{b}{(a,b)}d = h$ . Then,  $\frac{b}{(a,b)}da = ah$ . This can be rewritten as  $b \cdot (d\frac{a}{(a,b)}) = ah$ , and since  $a/(a,b)$  is an integer we conclude  $b|ah$ . In other words,  $k = -(ah/b)$  is an integer.

Going the opposite direction is much the same:  $k = -(ah/b)$  being an integer implies  $b|ah$ , implying  $bd = ah$ , implying  $bd/(a,b) = ah/(a,b)$ , implying  $\frac{b}{(a,b)}| \frac{ah}{(a,b)}$ . Then, we notice that since there exist integers  $\gamma$  and  $\delta$  such that  $a\gamma + b\delta = (a,b)$  (1.40), we find  $\frac{a}{(a,b)}\gamma + \frac{b}{(a,b)}\delta = \frac{(a,b)}{(a,b)} = 1$ , which by 1.39 implies that  $(\frac{a}{(a,b)}, \frac{b}{(a,b)}) = 1$ . Thus, we cite 1.41 with  $\frac{b}{(a,b)}| \frac{ah}{(a,b)}$  to find  $\frac{b}{(a,b)}|h$ .  $\square$

**Exercise 1.50.** A farmer lays out the sum of 1,770 crowns in purchasing horses and oxen. He pays 31 crowns for each horse and 21 crowns for each ox. What are the possible numbers of horses and oxen that the farmer bought?

51 horses and 9 oxen is the first situation I found. Using Paul’s Conjecture 3, we can find that further solutions can be found by subtracting 21 from the number of horses while adding 31 to the number of oxen (trust me it makes sense).

30 horses and 40 oxen.

9 horses and 71 oxen.

**Theorem 1.51.** Let  $a$ ,  $b$ ,  $c$ ,  $x_0$ , and  $y_0$  be integers with  $a$  and  $b$  not both 0 such that  $ax_0 + by_0 = c$ . Then the integers

$$x = x_0 + \frac{b}{(a,b)} \text{ and } y = y_0 - \frac{a}{(a,b)}$$

also satisfy the linear Diophantine equation  $ax + by = c$ .

*Proof.* Notice these integers satisfy the requirements for Paul’s Conjecture 3 (I’m too lazy to show how but 1.53 will force me to).  $\square$

**Question 1.52.** If  $a$ ,  $b$ , and  $c$  are integers with  $a$  and  $b$  not both 0, and the linear diophantine equation  $ax + by = c$  has at least one integer solution, can you find a general expression for all the integer solutions to that equation? Prove your conjecture.

**Paul's Conjecture 4.** The set of all pairs of integers  $(x_1, y_1)$  such that  $ax_1 + by_1 = c$  can be written as

$$\left\{ \left( x_0 + \frac{bd}{(a,b)}, y_0 - \frac{ad}{(a,b)} \right) \mid d \in \mathbb{Z} \right\}$$

*Proof.* Paul’s Conjecture 3 can easily be extended here: if we let the integer solution given be  $x_0$  and  $y_0$ , such that  $ax_0 + by_0 = c$ , we want to find a general expression for all integers  $x_1 = x_0 + h$  and  $y_1 = y_0 + k$  where  $\frac{b}{(a,b)}|h$  and  $k = -\frac{ah}{b}$ .

The set of all integers  $h$  such that  $\frac{b}{(a,b)}|h$  can be expressed as  $\{ \frac{bd}{(a,b)} \mid d \in \mathbb{Z} \}$ . The corresponding  $k$  value for any  $h$  is  $-\frac{ah}{b} = -\frac{a(bd/(a,b))}{b} = -\frac{ad}{(a,b)}$ . Thus, any pair of  $x_1 = x_0 + \frac{bd}{(a,b)}$  and  $y_1 = y_0 - \frac{ad}{(a,b)}$  satisfies the Diophantine equation  $ax_1 + by_1 = c$ .  $\square$

**Theorem 1.53.** Let  $a$ ,  $b$ , and  $c$  be integers with  $a$  and  $b$  not both 0. If  $x = x_0$ ,  $y = y_0$  is an integer solution to the equation  $ax + by = c$  (that is,  $ax_0 + by_0 = c$ ) then for every integer  $k$ , the numbers

$$x = x_0 + \frac{kb}{(a, b)} \text{ and } y = y_0 - \frac{ka}{(a, b)}$$

are integers that also satisfy the linear Diophantine equation  $ax + by = c$ . Moreover, every solution to the linear Diophantine equation  $ax + by = c$  is of this form.

*Proof.* This is just a less pretentious way of saying Paul's Conjecture 4 that doesn't involve set notation.  $\square$

**Exercise 1.54.** Find all integer solutions to the equation  $24x + 9y = 33$ .

$$(x, y) \in \{(1 + 3k, 1 - 8k) \mid k \in \mathbb{Z}\}.$$

**Theorem 1.55.** If  $a$  and  $b$  are integers, not both 0, and  $k$  is a natural number, then  $\gcd(ka, kb) = k \cdot \gcd(a, b)$ .

*Proof.* First, we notice that  $k \cdot (a, b)$  is indeed a common factor of  $ka$  and  $kb$ , since  $(a, b) \mid a$  we conclude  $k(a, b) \mid ka$ , and similarly for  $k(a, b) \mid kb$ .

Now, we invoke 1.40 to find integers  $x$  and  $y$  such that  $ax + by = (a, b)$ . We can multiply both sides by  $k$  to find  $k(ax + by) = kax + kby = k(a, b)$ , and then from that invoke 1.48 with  $ka$  and  $kb$  to find that  $(ka, kb) \mid k(a, b)$ . Since  $(ka, kb)$  is the greatest common factor of  $ka$  and  $kb$ , and  $k(a, b)$  is a common factor of  $ka$  and  $kb$ , we conclude  $k(a, b) \leq (ka, kb)$ . However, since  $k(a, b)$  is positive and  $(ka, kb) \mid k(a, b)$ , we conclude  $k(a, b) \geq (ka, kb)$ . Thus, we conclude  $k(a, b) = (ka, kb)$ .  $\square$

**Exercise 1.56.** For natural numbers  $a$  and  $b$ , give a suitable definition for "least common multiple of  $a$  and  $b$ ," denoted  $\text{lcm}(a, b)$ . Construct and compute some examples.

Define  $\text{lcm}(a, b)$  as the smallest positive number  $x$  such that  $a \mid x$  and  $b \mid x$ .

Some examples include  $\text{lcm}(3, 6) = 6$ ,  $\text{lcm}(1, 50) = 50$ , and  $\text{lcm}(2, 5) = 10$ .

**Theorem 1.57.** If  $a$  and  $b$  are natural numbers, then  $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$ .

*Proof.* Notice by 1.55 that  $\gcd(a, b) \cdot \text{lcm}(a, b) = \gcd(a \text{ lcm}(a, b), b \text{ lcm}(a, b))$ .

Since  $a \mid \text{lcm}(a, b)$  we can write  $\text{lcm}(a, b) = ak$  for some integer  $k$ . Similarly, we can write  $\text{lcm}(a, b) = bj$  for an integer  $j$ . Then, we simplify our expression to  $\gcd(a \text{ lcm}(a, b), b \text{ lcm}(a, b)) = \gcd(abj, bak)$ . Then, citing 1.55 again, we find this equal to  $ab \gcd(j, k)$ .

Notice that since  $\gcd(j, k) \mid j$ , we can write  $j = x \gcd(j, k)$ , and likewise we can write  $k = y \gcd(j, k)$ . Then, we notice  $ak = bj \implies ay \gcd(j, k) = bx \gcd(j, k)$ . Since  $ay = bx$  is a common multiple of  $a$  and  $b$ , and  $ay \gcd(j, k)$  is the least common multiple of  $a$  and  $b$ , we find  $ay \gcd(j, k) \leq ay$ , which implies  $\gcd(j, k) = 1$ .

Putting this all together, we find  $\gcd(a, b) \cdot \text{lcm}(a, b) = ab \gcd(j, k) = ab$ .  $\square$

**Corollary 1.58.** If  $a$  and  $b$  are natural numbers, then  $\text{lcm}(a, b) = ab$  if and only if  $a$  and  $b$  are relatively prime.

*Proof.* By 1.57, we find  $ab = \text{lcm}(a, b) \cdot \gcd(a, b) \implies \text{lcm}(a, b) = \frac{ab}{\gcd(a, b)}$ . Thus,  $a$  and  $b$  are relatively prime if and only if  $\gcd(a, b) = 1 \iff \text{lcm}(a, b) = \frac{ab}{\gcd(a, b)} = ab$ .  $\square$