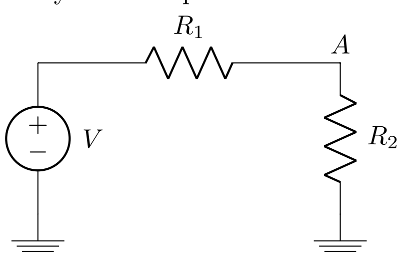
CSM 16A **Designing Information Systems and Devices**

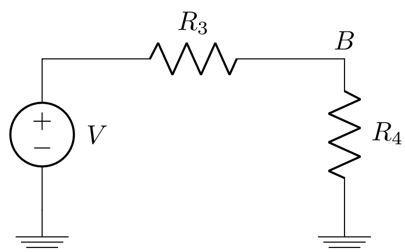
Week 6 Worksheet Metas

Term: **Spring 2020** *Name:*

Problem 1: Kirchoff's Laws

1. Here, you can see two standard voltage dividers. Find the voltage at the points A and B using standard nodal analysis techniques.





Solution: We can apply KCL at node A. We get:

$$\frac{V_A - 0}{R_2} = \frac{V - V_A}{R_1}$$

$$\implies \frac{R_1}{R_2} V_A = V - V_A$$

$$\implies V_A\left(1 + \frac{R_1}{R_2}\right) = V$$

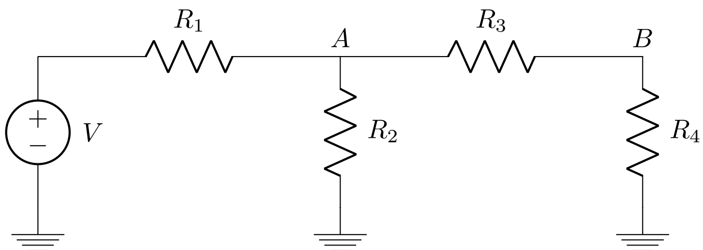
$$\implies V_A \frac{R_1 + R_2}{R_2} = V$$

$$\implies V_A = V \frac{R_2}{R_1 + R_2}$$

Similarly, for the second circuit, we have

$$V_B = V \frac{R_4}{R_3 + R_4}$$

2. Now, let us modify this circuit to add a second voltage divider stage starting at A. You can think of this as cascading 2 voltage dividers: one which which has an input of V and an output of V_A, and a second one that has an input of V_A and an output of V_B. Find the voltage at B using nodal analysis.



Solution: We can apply KCL at node A and at node B. We get the equations:

$$\frac{V - V_A}{R_1} + \frac{0 - V_A}{R_2} = \frac{V_A - V_B}{R_3}$$

$$\frac{V_A - V_B}{R_3} = \frac{V_B - 0}{R_4}$$

We can simplify these equations to give us

$$\frac{V}{R_1} - \frac{V_A}{R_1} - \frac{V_A}{R_2} = \frac{V_A}{R_3} - \frac{V_B}{R_3}$$

$$\implies \frac{V_B}{R_3} + \frac{V}{R_1} = \frac{V_A}{R_1} + \frac{V_A}{R_2} + \frac{V_A}{R_3}$$

$$\implies \frac{V_B}{R_3} + \frac{V}{R_1} = V_A(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3})$$

$$\frac{V_A}{R_3} - \frac{V_B}{R_3} = \frac{V_B}{R_4}$$

$$\implies \frac{V_A}{R_3} = \frac{V_B}{R_3} + \frac{V_B}{R_4}$$

$$\implies V_A = V_B R_3(\frac{1}{R_3} + \frac{1}{R_4})$$

Combining these 2 equations, we get:

$$\frac{V_B}{R_3} + \frac{V}{R_1} = V_B R_3(\frac{1}{R_3} + \frac{1}{R_4})(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3})$$

$$\implies \frac{V_B}{R_3} + \frac{V}{R_1} = V_B R_3(\frac{1}{R_3} + \frac{1}{R_4})(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3})$$

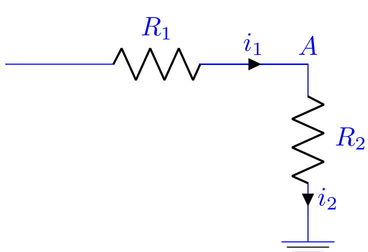
$$\implies \frac{V}{R_1} = V_B R_3(\frac{1}{R_3} + \frac{1}{R_4})(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}) + \frac{V_B}{R_3} = V_B(R_3(\frac{1}{R_3} + \frac{1}{R_4})(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}) + \frac{1}{R_3})$$

$$\implies V_B = \frac{V}{R_1}/(R_3(\frac{1}{R_3} + \frac{1}{R_4})(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}) + \frac{1}{R_3})$$

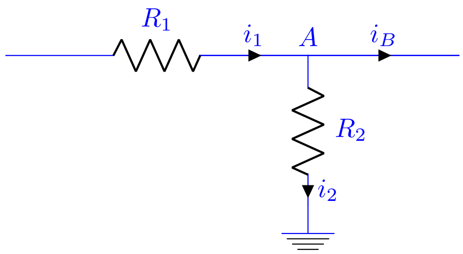
3. Compare your answer with what you get when you multiplying the amplification factors of each individual voltage divider. Are they the same or different? Does this surprise you? Why or why not?

Solution: They are clearly not the same! The next part explains why.

What we see here is called "loading". Ideally, we would like to be able to stack voltage dividers like this in order to cascade their effects. However this does not work because by adding the second circuit, we are changing what the first one does. In particular, the second circuit will draw some current from the first circuit. We can see this difference if we zoom in on the node A. In part (a), we had the following picture:



And we were able to say that $i_1 = i_2$, because they are the only currents at node A. But after adding the second voltage divider stage, the picture now looks like this:

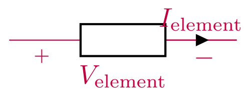


And so the KCL equation we wrote at node A is no longer valid.
Voltage dividers "divide" the voltage between two resistors in series. However, when we add the second voltage divider, R_1 and R_2 can't be said to be in series anymore.

Think about the conditions under which the impact of cascaded voltage dividers can be calculated by simply multiplying the two ratios. We will learn how to do that later in lecture with a special circuit element which helps us "modularize" the circuit.

Problem 2: Power

Meta: For an arbitrary circuit element

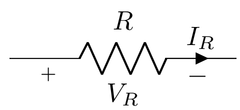


we can calculate the power it dissipates (i.e. the power it consumes) with the expression

$$P_{\text{element}} = I_{\text{element}} V_{\text{element}}$$

(a) For resistors (and resistors only), we can relate the voltage drop across the resistor and the current passing through the resistor with Ohm's Law:

$$V_R = I_R R$$



Find an expression for the power dissipated by the resistor in terms of the following:

i. V_R and I_R
ii. V_R and R
iii. I_R and R

Meta:
Starting with the expression for power

$$P_R = I_R V_R$$

we manipulate Ohm's Law to find appropriate substitutions for the terms we can't use. In particular,

$$V_R = I_R R$$
$$I_R = \frac{V_R}{R}$$

From this we can also get

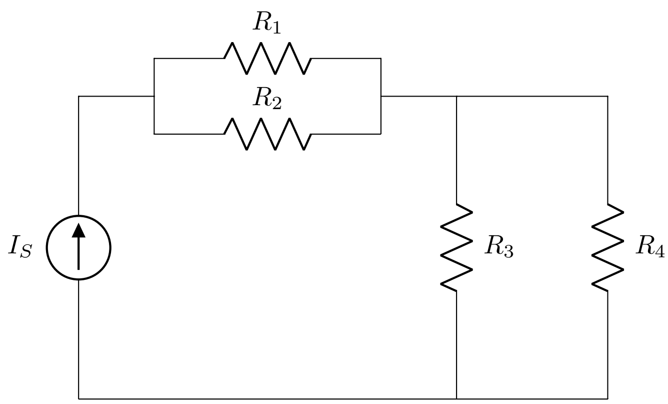
$$R = \frac{V_R}{I_R}$$

but in this case we don't actually need it.

Solution:

i. $I_R V_R$
ii. $\frac{V_R^2}{R}$
iii. $I_R^2 R$

For the rest of this question, use the circuit below:



(b) Which individual components have the same magnitude voltage drop across them?

Solution:

R_1 and R_2 are in parallel and so see the same voltage drop as one another.

R_3 and R_4 are also in parallel and experience the same voltage drop as one another.

(c) Under what condition(s) will R_1 and R_2 dissipate the same amount of power?

Solution:

We know

$$P = IV$$

and that R_1 and R_2 have the same voltage drop across them. In order to dissipate the same amount of power, the two need to have equal current flowing through them in the same direction. Manipulating Ohm's Law, we get

$$I_1 = \frac{V}{R_1}, I_2 = \frac{V}{R_2}$$

For these to be equal, R_1 must be equal to R_2. Alternatively, we can go straight to the expression for resistor power

$$P_R = \frac{V_R^2}{R}$$

and see from here that the resistances must be the same for the two to dissipate the same amount of power.

(d) Use the following values and calculate the amount of power consumed by each of the resistors R_1, R_2, R_3, and R_4.

Component	Value	Units
R_1, R_2	2	Ω
R_3, R_4	6	Ω
I_S	1	A

Meta:

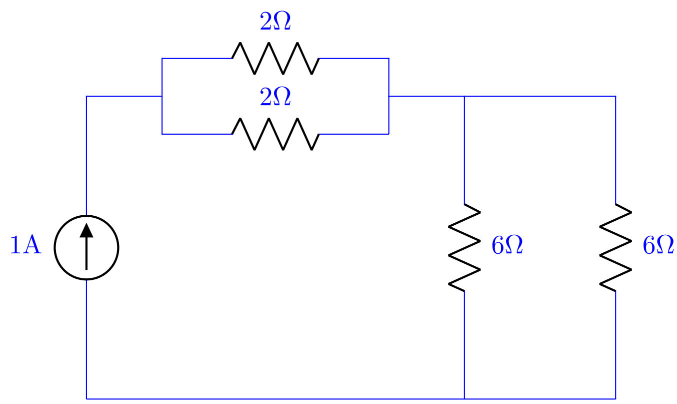
Ideally by this point students are comfortable with parallel/series combinations of resistors. You should be prepared to work out the problem in full rigor if your students ask.

Solution:

To find the power dissipated by a resistor, we can use any of the following:

$$\begin{aligned} P &= IV \\ &= \frac{V^2}{R} \\ &= I^2R \end{aligned}$$

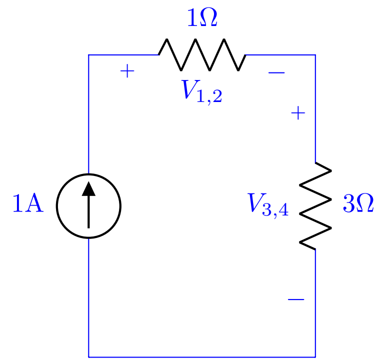
Because we're already given the resistance, we can either find the voltage drop across each of the resistors or the current flowing through them. Redrawing the circuit with the numerical values labeled:



Current Solution: We know from part 3c that R_1 and R_2 will have the same amount of current flowing through them, i.e. the 1A is evenly divided between them. The same goes for R_3 and R_4.

$$\begin{aligned} P_{R_1,R_2} &= \left(\frac{1}{2}\text{A}\right)^2 \cdot 2\Omega \\ &= 0.5\text{W} \\ P_{R_3,R_4} &= \left(\frac{1}{2}\text{A}\right)^2 \cdot 6\Omega \\ &= 1.5\text{W} \end{aligned}$$

Voltage Solution: Using parallel resistor combinations, we can combine the two 2Ω and combine the two 6Ω resistors to get the following:



We're given a current source, so we can use Ohm's law to find $V_{1,2}$ and $V_{3,4}$:

$$\begin{aligned} V_{1,2} &= 1\text{A} \cdot 1\Omega \\ &= 1\text{V} \\ V_{3,4} &= 1\text{A} \cdot 3\Omega \\ &= 3\text{V} \end{aligned}$$

and from here

$$\begin{aligned}
P_{R_1,R_2} &= \frac{V_{1,2}^2}{2\Omega} \\
&= \frac{(1\text{V})^2}{2\Omega} \\
&= 0.5\text{W} \\
P_{R_3,R_4} &= \frac{V_{3,4}^2}{6\Omega} \\
&= \frac{(3\text{V})^2}{6\Omega} \\
&= 1.5\text{W}
\end{aligned}$$

(e) How much power does the current source consume? *Hint: Consider the conservation of energy.*

Meta: Again, be prepared to work this out in full rigor if your students ask.

Solution: Because of the conservation of energy (and by proxy power because $P = \frac{dE}{dt}$), we know all the power the resistors dissipate must be generated by the current source. Using our answers from part 3d:

$$\begin{aligned}
P_{I_S} + P_{R_1} + P_{R_2} + P_{R_3} + P_{R_4} &= 0\text{W} \\
P_{I_S} &= -P_{R_1} - P_{R_2} - P_{R_3} - P_{R_4} \\
&= -(0.5\text{W} + 0.5\text{W} + 1.5\text{W} + 1.5\text{W}) \\
&= -4\text{W}
\end{aligned}$$

Note the sign! Negative power indicates that a component is dissipating negative power, i.e. that it's generating power.