CSM 16A (Designing Information Systems and Devices)

Week 1 Worksheet Solutions

Term: **Spring 2020** *Name:*

Problem System of Equations, Pivots, and Free Variables

Learning Goal: Students should be comfortable solving a three-variable system of equations using GE with the forward/backward elimination method. Additionally, they should know how to convert a solution with a free variable from equations describing the solution set into vector notation.

Description: Simple mechanical gaussian elimination problem + some insight about span and free variables

1. Consider the following set of linear equations:

$$1x - 3y + 1z = 4$$
$$2x - 8y + 8z = -2$$
$$-6x + 3y - 15z = 9$$

Place these equations into a matrix A, and row reduce A to solve the equations.

Solution:

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

$$R_2 = R_2 - 2 * R_1$$
$$R_3 = R_3 + 6 * R_1$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & -15 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 33 \end{bmatrix}$$

$$R_2 = R_2/2$$
$$R_3 = R_3/3$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 11 \end{bmatrix}$$

$$R_3 = R_3 - 5 * R_2$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 36 \end{bmatrix}$$

$$z = -2$$
$$y = -1$$
$$x = 3$$

2. Consider another set of linear equations:

$$2x + 3y + 5z = 0$$
$$-1x - 4y - 10z = 0$$
$$x - 2y - 8z = 0$$

Place these equations into a matrix A, and row reduce A.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

$$R_2 = R_2 + \frac{1}{2}R_1$$
$$R_3 = R_3 - \frac{1}{2}R_1$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -2.5 & -7.5 \\ 0 & -3.5 & -10.5 \end{bmatrix}$$

Remember that we can only do row operations without caring about the RHS because the RHS is all zeroes. Hence, any linear row operations won't affect the RHS i.e. it will remain the zero vector.

Make the numbers nicer by dividing row 2 by -2.5, and multiplying row 3 by -2. This is always a good thing to do if you realize your numbers are getting messy! (Also, feel free to keep all the numbers as non-fractional values by finding the least common multiple of the two numbers you are trying to cancel out.)

$$R_2 = \frac{1}{-2.5}R_2$$
$$R_3 = -2R_3$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix}$$

$$R_3 = R_3 - 7R_2$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Convert the row reduced matrix back into equation form.

Solution:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$2x + 3y + 5z = 0$$
$$0x + 1y + 3z = 0$$
$$0x + 0y + 0z = 0$$

4. Intuitively, what does the last equation from the previous part tell us?

   Solution: If students are confused at this point about why we can infer this, their confusion is well justified. Suppose that there were **4** equations in 3 variables – 3 of them were linearly independent, and the fourth one was $0x + 0y + 0z = 0$, then the system still has just 1 solution. The last equation is never *used* in some sense. Feel free to talk about this with students. Present it as: what if you had 4 equations, you wrote them in matrix form, got pivots in all rows except for one where you got a row of all 0s – are there still infinite solutions? The answer is no.

5. How many pivots are there in the row reduced matrix? What are the free variables?

   Solution: There are 2 pivots in this row reduced matrix, and the corresponding pivot columns (following Gaussian Elimination's convention) are column 1 and 2. There are no more pivots since the third row are all zeros, and we require a non-zero element at the position of the third column (following column 2) and the third row for there to be one more pivot.

   The free variables can be y or z in this case, but we choose z as our free variable by Gaussian Elimination's convention.

6. What is the dimension of the span of all the column vectors in A?

   Solution: As we can see, in the row reduced form of A, since the third row are all zeros, and there are only 2 pivots with z as the free variable, the dimension of the span of all the column vectors in A is equal to 2 (number of pivots).

   Alternatively, we can follow the definition of a span and algebraically write out the linear combinations of all the column vectors in the row reduced form of A, and we can see that the third entry in the resulting linear combination will always be 0 (since all 3 column vectors have 0's in their third entries), hence there are only 2 dimensions (entries) in the resulting vectors whose values we have control over.

7. Now that we've established that this system has infinite solutions, does every possible combination of $x, y, z \in \mathbb{R}$ solve these equations

   Solution: No. $x = 1, y = 1, z = 1$ doesn't work, for instance.

8. What is the general form (in the form of a constant vector multiplied by a variable t) of the infinite solutions to the system?

   Solution: z is a free variable. If $z = t$, then

   $$y = -3z = -3t$$

   $$2x + 3y + 5z = 0 \implies 2x - 9t + 5t = 0 \implies 2x = 4t \implies x = 2t$$

   The general solution is then $t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. What this means is that any multiple of the vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ will satisfy the equations. Try it!