

Week 4 Worksheet

Term: **Spring 2020***Name:***Problem 1: Eigenvalues and Eigenvectors**

Consider a square matrix \mathbf{A} that is $n \times n$. Recall that we say λ is an eigenvalue of \mathbf{A} if there exists a **non-zero** vector \vec{v} such that:

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

We call \vec{v} the eigenvector associated with λ .

1. What is the one eigenvalue and eigenvector of the matrix that you can see without solving any equations?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

2. What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3. What are the eigenvalues of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}?$$

4. Consider a matrix that rotates a vector in \mathbb{R}^2 by 45° counterclockwise. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. Find its eigenvalues and corresponding eigenvectors. This matrix is given as

$$\mathbf{D} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

5. What are the eigenvalues of the following matrix?

$$\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

6. Can you find an eigenvalue of the following matrix without solving any equations?

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

7. Show that a matrix and its transpose have the same eigenvalues

Hint: The determinant of a matrix is the same as the determinant of its transpose

8. Consider a matrix whose columns sum to one. What is one possible eigenvalue of this matrix?

Problem 2: Eigenvalue Calculations

1. Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

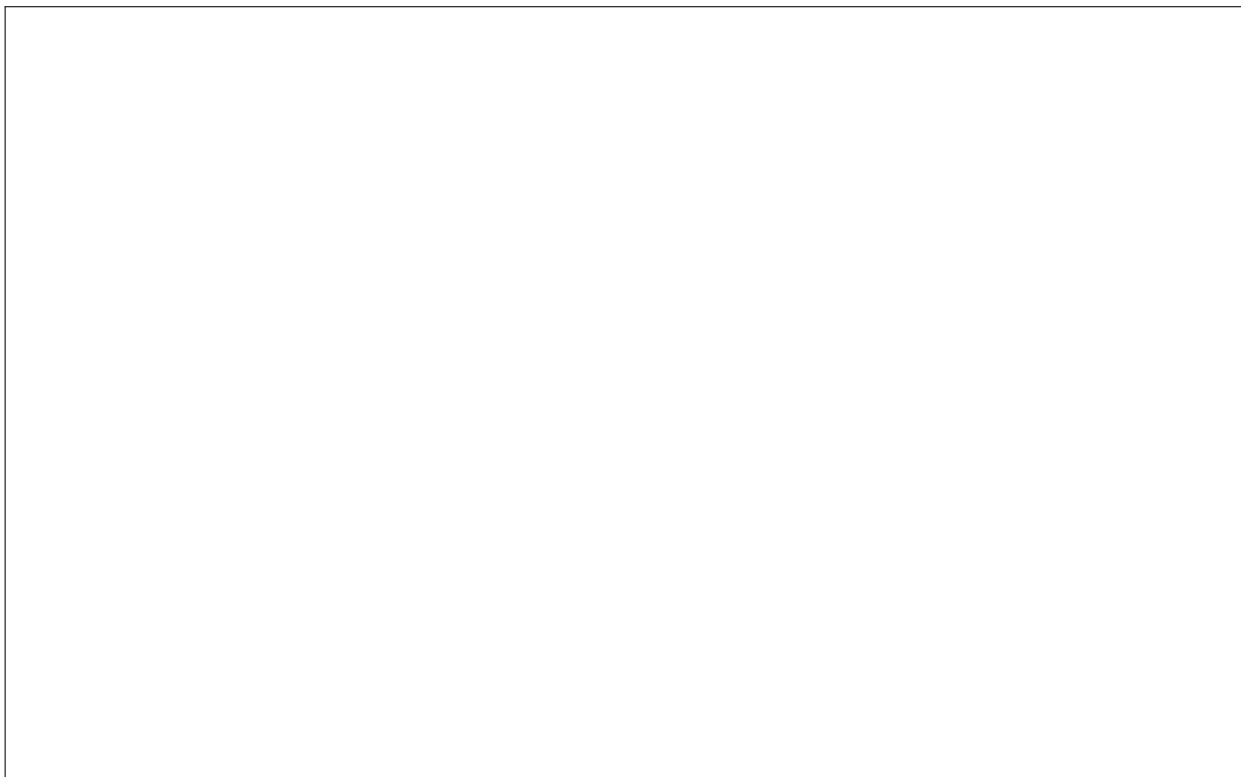
2. Find the eigenvectors for matrix \mathbf{A} given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

3. Find the eigenvalues for matrix \mathbf{A} given that we know that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are the

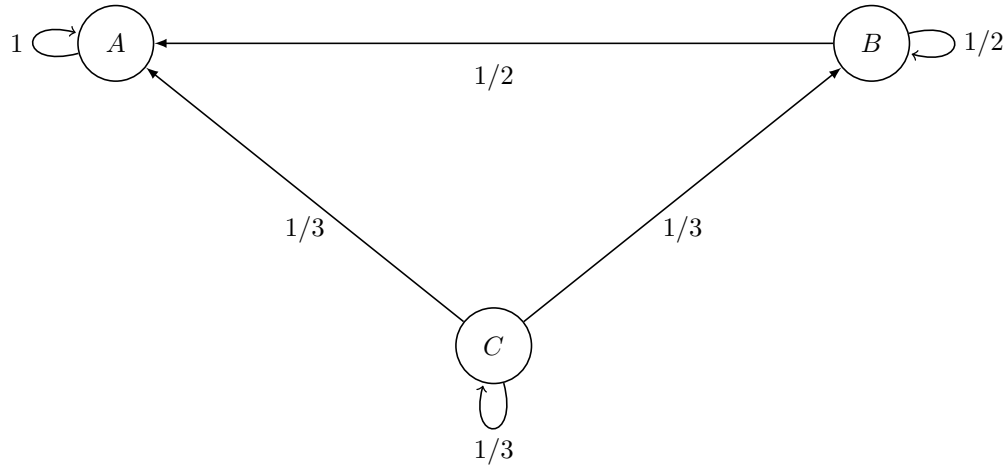
eigenvectors of \mathbf{A} , and that

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$



Problem 3: Mechanical PageRank

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



1. Write down the probability transition matrix for this graph, and call it \mathbf{P} . Can you say something about the eigenvalues/eigenvectors of \mathbf{P}^T ? (*Hint: Try to recall the properties of transition matrices*).

2. We want to rank these webpages in order of importance. But first, find the eigenvector of \mathbf{P} corresponding to eigenvalue 1.

3. Now looking at the matrix \mathbf{P} , can you identify what its other eigenvalues are?

4. Suppose that we start with 90 users evenly distributed among the websites. What is the steady-state number of people who will end up at each website?