CSM 16A **Designing Information Systems and Devices**

Week 12 Worksheet Metas

Term: **Spring 2020** *Name:*

Problem 1: Least Squares

Meta:
Prereqs: Understanding of Least Squares
Solution:
Description: Describe and derive the least squares problem and solution, first geometrically, and then using calculus.

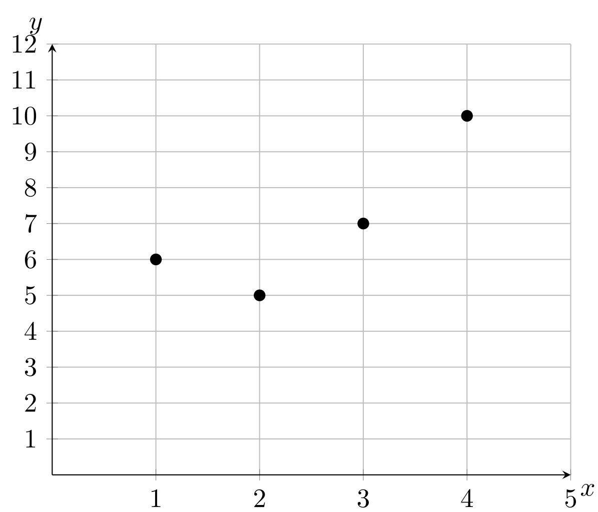
1. Consider the basic problem of finding some $\vec{x}$ such that $\mathbf{A}\vec{x} = \vec{b}$. If we have an equal number of equations as unknowns, then we are pretty happy.

   In many cases, however, we cannot find a solution, as the set of equations that are described by $\mathbf{A}$ and $\vec{b}$ are overdetermined (i.e. there are more equations than there are unknowns).

   In general, least squares involves finding the best approximate solution to these overdetermined systems.

   Looking at this graphically, we can think of the problem of data-fitting. That is to say that we have many samples, and we want to draw a straight line that goes as close to each point as possible.

   Looking at the plot below, setup a system of linear equations describing a line going through each point.



   Solution: Recall that the generalized equation for a line is $y = mx + c$, where m is the slope of the line and c is a constant shift. Since we have 4 points, we have 4 equations with known (x,y) values, and unknown slope and constants (m and c).

$$6 = (1)m + c$$

$$5 = (2)m + c$$
$$7 = (3)m + c$$
$$10 = (4)m + c$$

2. Put this system of linear equations into matrix-vector form.

   Solution:

   $\mathbf{A}$: A matrix whose rows contain coefficients for our x_i.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$$

   $\vec{x}$: A vector containing the parameters that we want to be optimizing

$$\vec{x} = \begin{bmatrix} m \\ c \end{bmatrix}$$

   $\vec{b}$: A vector containing the "true" y values of our original points, which we will want our $\hat{x}$ to approximate

$$\vec{b} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ 10 \end{bmatrix}$$

   Putting this into the standard form $\mathbf{A}\vec{x} = \vec{b}$, we get:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ 10 \end{bmatrix}$$

3. Given that $(\mathbf{A}^T\mathbf{A})^{-1} = \begin{bmatrix} 0.2 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$, use the linear least squares technique you learned earlier on this overdetermined system to solve for a line of best fit

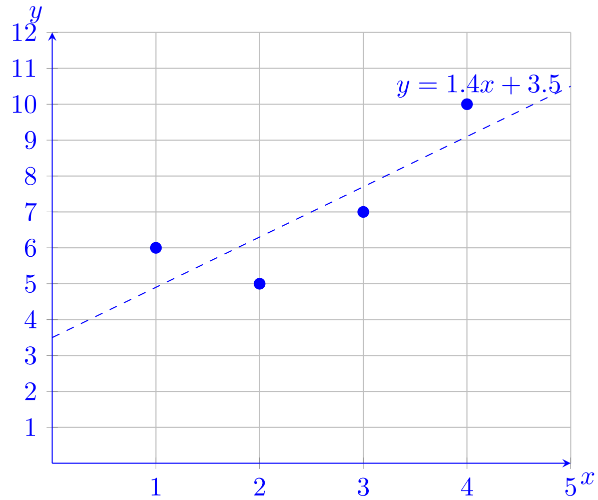
   Solution:

$$\hat{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$$
$$= \begin{bmatrix} 0.2 & -0.5 \\ -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 7 \\ 10 \end{bmatrix}$$
$$= \begin{bmatrix} 0.2 & -0.5 \\ -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 77 \\ 28 \end{bmatrix}$$
$$\hat{x} = \begin{bmatrix} 1.4 \\ 3.5 \end{bmatrix}$$

   Thus, we see that the line of best fit has a slope of 1.4 and a constant shift of 3.5, and the equation of the line is $y = 1.4x + 3.5$

4. Plot this line in the plot above.

   Solution:



5. Gauss used Least Squares to predict where certain planets would be in their orbit. A scientist named Piazzi made 19 observations over the period of a month in regards to the orbit of Ceres (can be viewed as equations). Gauss used some of these observations. He also knew the general shape of the orbit of planets due to Kepler's laws of planetary motion. Gauss set up equations like so:

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y = 1$$

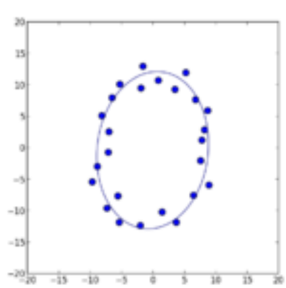
So, if we make observations on the positions of the planetary bodies, how can we use least squares to find the orbit of the planet?

Solution: The question we need to ask is: what are the variables we need to solve for? In this case, we have points in space: thus, we have values of x and y. Using these two values, we can also easily find x^2, xy, and y^2. So, the unknowns in this story are the coefficients of these variables, or the "weights" of each term. Thus we can set this problem for n measurements in the form of $\mathbf{A}\vec{x} = \vec{b}$:

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 \\ & & \vdots & & \\ x_n^2 & x_ny_n & y_n^2 & x_n & y_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

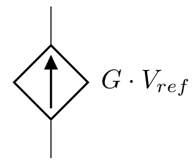
In this case, we have an overdetermined system, which (as we've explored earlier) can be solved using least squares. Simply use the least squares formula to find the best approximation to your weights $\hat{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$

Then, you can plot an ellipse of best fit that models the path of the planetary body.



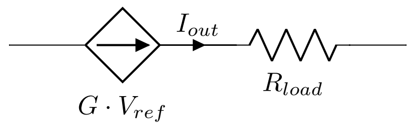
Problem 2: Circuit Design

Meta: Description: Show how to construct a VCCS using two op-amps and some resistors
A Voltage Controlled Current Source looks like this:



A VCCS is a Dependent Current Source that's controlled by a voltage V_{ref}, and produces a current based on that V_{ref}, which will follow the equation $I_{out} = G \cdot V_{ref}$, for some constant G.

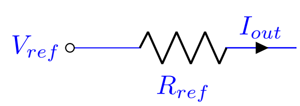
It can be connected to any load resistor (it has a resistance of R_{load}), and guarantees that the current $G \cdot Vref$ will flow through that resistor.



1. In order to create a VCCS, we'll need some way to turn an input voltage into a current. What's the simplest way we can accomplish this?
   Hint: Think about the relation between voltage and current.

   Meta: If they need more help, mention Ohm's Law.

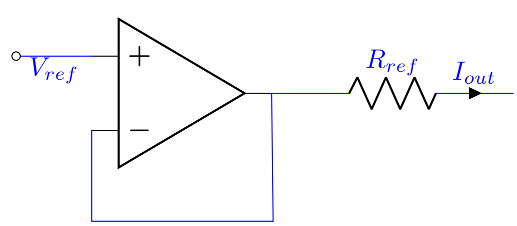
   Solution: Use a resistor! The circuit would look like this, where $I_{out} = V_{ref}/R_{ref}$:



2. We'll also need some way to isolate the input voltage from the previous parts of the circuit that produces our input voltage. In other words, if the input voltage to our VCCS has some resistors or other components connected to it, we don't want that to affect the relation between V_{ref} and I_{out} in our VCCS. What design component can we use to do this?

   Meta: Adding a buffer to make sure the input or output of a circuit is not messed with is a pretty common thing in design problems. Might be a good thing to emphasize.

   Solution: Place a buffer between the input voltage V_{ref} and the resistor R_{ref}.



3. Now we have V_{ref} converted to I_{out} using the formula $G \cdot V_{ref}$. What is G in terms of R_{ref}?

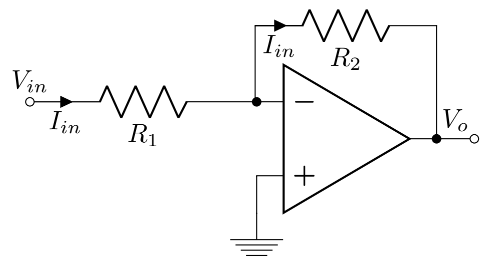
   Meta: Make sure students understand the connection between the resistance and the gain.

   Solution: By Ohm's Law, the voltage dropped across the resistor is the current through it times the resistance: $I_{out} = \dfrac{V_{ref}}{R_{ref}} = GV_{ref} \implies G = \dfrac{1}{R_{ref}}$

4. Are we done? Are there any problems with the current that we're producing?
(Hint: What happens when we place our R_{load} at I_{out}?)

Solution: The value of the current produced changes from $\frac{V_{ref}}{R_{ref}}$ to $\frac{V_{ref}}{R_{ref}+R_{load}}$, so our current source doesn't work the way it should.

5. We can keep the current at the value we want it to be at by using one of the properties of an inverting op amp: the fact that no matter what, the current flowing through the $V-$ to V_{out} branch is equal to the input current to $V-$ (This follows from the fact that no current flows into $V-$).



Where can we place our previous circuit and R_{load} to take advantage of this? Draw the entire circuit configuration for your VCCS.

Meta: Make sure that students understand how this works for an arbitrary G. Basically, if you want a gain of G, then place a resistor of value $\frac{1}{G}$ in place of R_{ref}. Also make sure that students understand that the output terminal of this VCCS is the output of the second op-amp.

Solution: