CSM 16A — Designing Information Systems and Devices

Week 3 Worksheet Metas

Term: **Spring 2020** — *Name:*

Problem 1: Conceptual Checks

For each of the following statements, determine if they are **TRUE** or **FALSE**. If they are **FALSE**, try to come up with a counterexample; if they are **TRUE**, give a brief explanation.

Meta: When walking through every statement in this question, be sure to encourage the students to think from the ground up (beginning with definitions). It is very tempting to try to fit brute-force counterexample; but in some cases, it might be very hard to directly come up with them without understanding the concept first.

1. If the augmented matrix of the linear system represented by $A\vec{x} = \vec{b}$ has a pivot in the last column, then the matrix vector equation $A\vec{x} = \vec{b}$ has no solution.

   Solution: TRUE. A pivot in the last column of an augmented matrix is equivalent to having a row of the following form $\begin{bmatrix} 0 & 0 & 0 & \dots & | & b \end{bmatrix}$, where $b \neq 0$. In this case, there is no possible solution that would make the equivalent equation for this row hold.

2. If A is a 3×3 matrix such that the matrix vector equation $A\vec{x} = \vec{0}$ has only the trivial solution ($\vec{x} = \vec{0}$), then the matrix vector equation $A\vec{x} = \vec{b}$ is consistent for every vector $\vec{b}$ in $\mathbb{R}^3$

   Meta: Make sure to clarify the definition of a null space and how the dimension of the null space and the dimension of the column space of A $\in \mathbb{R}^{m \times n}$ are related to each other:

   $$\dim(\text{Col}(\mathbf{A})) + \dim(\text{Nul}(\mathbf{A})) = n$$

   Solution: TRUE. If $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$, then it implies that the dimension of the null space in A must be 0 (or equivalently, A can be reduced to a row echelon form such that there is a pivot in every column). In this case, the column vectors in A must span all of $\mathbb{R}^3$, and consequently, the matrix vector equation $A\vec{x} = \vec{b}$ is consistent for every vector $\vec{b}$ in $\mathbb{R}^3$.

3. If the matrix vector equation $A\vec{x} = \vec{0}$ is true only when $\vec{x} = \vec{0}$, then the matrix A has an inverse (A is invertible).

   Solution: FALSE. A must also be a square matrix. Consider the matrix

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   This matrix has only $\vec{0}$ as the solution to

   $$A\vec{x} = \vec{0}$$

   ; however, since it is not square, we cannot find an inverse for it.

4. A matrix A is called *symmetric* if it is equal to its transpose: $A = A^T$. If A is an invertible and *symmetric* matrix, A^{-1} must also be *symmetric*.

   Solution: TRUE. We want to show that $A^{-1} = (A^{-1})^T$. $(A^{-1})^T = (A^T)^{-1} = A^{-1}$. Hence, A^{-1} is also *symmetric*.

Note: To show why $(A^{-1})^T = (A^T)^{-1}$, consider:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

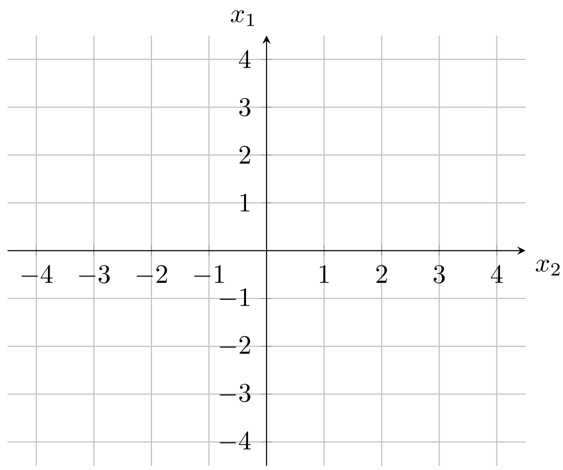
$$A^T(A^T)^{-1} = I$$

Hence,

$$A^T(A^{-1})^T = A^T(A^T)^{-1}, A^T((A^{-1})^T - (A^T)^{-1}) = 0$$

Since A^T can be any matrix, it must true that $(A^{-1})^T - (A^T)^{-1} = 0 \longrightarrow (A^{-1})^T = (A^T)^{-1}$

Problem 2: Range Intuition



$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

1. Draw the space on the figure above that is represented by the span of all the column vectors in A. Also draw the space covered by the span of all the row vectors in A. What dimension are these spaces?

   Solution: The 1 dimensional space for the column space is a line on the $x_2 = x_1$ axis (so a diagonal line that goes perfectly northeast, intersecting the origin along the way). The 1 dimensional space for the row space is the line $x_2 = 2x_1$

   Meta: Remember that the lines that are drawn are infinite, make sure to make a point of that.

2. Consider some arbitrary vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Write out the product $\mathbf{A}\vec{v}$ in terms of v_1, v_2, and the columns of $\mathbf{A}$.

   Solution:

   $$\mathbf{A}\vec{v} = v_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

   Meta: Make sure students are aware that in general, any matrix-vector product $\mathbf{A}\vec{v}$ can always be written as a linear combination of the columns of $\mathbf{A}$. As such, $\mathbf{A}\vec{v}$ is ALWAYS in the span of the column vectors of $\mathbf{A}$.

3. We have talked about how matrices like $\mathbf{A}$ have no inverse. Give a geometric explanation for why this is the case.

   Solution: If we are given some point on the line for the colspace from part (a), we do not know where it came from. For example, if you gave the point given by $\mathbf{A}\vec{x}$, you have no way of knowing that it came from $\vec{x}$. For example, $\vec{x} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ also works.

   Meta: Make sure students see the ties between independence/invertibility and the geometry. Geometrically, if we pick a point described by vector $\vec{b}$ on the graph, invertibility would imply that we can always find the unique $\vec{x}$ such that $\mathbf{A}\vec{X} = \vec{b}$.

4. Consider all points $\vec{y}$ such that $\mathbf{A}\vec{y} = 0$ Draw the space that the $\vec{y}$'s will make up. What do you notice geometrically? What is the dimension of this space?

   Solution: This line should be a straight line that is perpendicular to the line for the row space from part (a). It is a space of dimension 1

Meta: This is a good time to note the relationship between the dimension of the column space and the dimension of the null space. You or your students may have heard this referred to as the rank-nullity theorem: for an $m \times n$ matrix $\mathbf{A}$,

$$\dim(\mathrm{Col}(\mathbf{A})) + \dim(\mathrm{Nul}(\mathbf{A})) = n$$

always holds.

Problem 3: More on Linear Transformation

1. Consider a matrix $\mathbf{S}$ that transforms a vector $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ to $\vec{y} = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$. Note that a, b can take on any values in $\mathbb{R}$. In other words, $\mathbf{S}\vec{x} = \vec{y}$. Is this transformation linear?

   Meta: If students are stuck, prompt the students by asking what three things a transformation needs to be considered linear.

   Description: Please note that this might be the first time students are thinking of matrices as transformations. Let this settle in. The fact that a matrix is essentially a function that takes one vector and makes it a different vector. This is no different from a real-valued function like $f(x) = x^2$, except the only difference is that x is a vector, and f is a matrix. It might also be useful to show simple (non-matrix) examples of linear and non-linear transformations. A simple example of a non-linear transformation is something that squares each component of the vector. A simple example of a linear transformation is the 0 transformation.

   Solution: To prove whether a transformation is linear, we must check whether it preserves scalar multiplication, addition and the zero vector.

   Scalar multiplication
   Let $\alpha \in \mathbb{R}$. Is $\mathbf{S}(\alpha\vec{x}) = \alpha\vec{y}$?
   $\mathbf{S}\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix} = \alpha \begin{bmatrix} a+b \\ a-b \end{bmatrix}$. Try it!

   Addition
   Is $\mathbf{S}(\vec{x}_1 + \vec{x}_2) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$?
   Let $\vec{x}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$. Then $\mathbf{S}(\vec{x}_1 + \vec{x}_2) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$. Try it out!
   Zero vector
   Is $\mathbf{S} \cdot \vec{0} = \vec{0}$? Yes.

   This proves that $\mathbf{S}$ is indeed a linear transformation.

2. What is the matrix $\mathbf{S}$? Is the matrix invertible? Is the transformation invertible?

   Solution: The matrix $\mathbf{S} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
   We can see that the matrix is invertible as all rows and columns are linearly independent. It means we can uniquely recover the input (initial) vector by multiplying the inverse of the matrix with the output (transformed) vector. Try doing it yourself!

   This problem reduced to whether we can uniquely identify a and b given the values of $a + b$ and $a - b$. (Yes, we can!)

   Since we can uniquely recover our input vector by using the inverse of the matrix, there exists a one-to-one mapping between all possible input and output vectors. Hence, this linear transformation is invertible.

3. Consider a matrix $\mathbf{S}$ that transforms a vector $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\vec{y} = \begin{bmatrix} a-b-c \\ a-b-c \\ a-b+c \end{bmatrix}$. Note that a, b, c can take on any values in $\mathbb{R}$. In other words, $\mathbf{S}\vec{x} = \vec{y}$. Is this transformation linear?

   Solution: To prove whether a transformation is linear, we must check whether it preserves scalar multiplication, addition and the zero vector.

Scalar multiplication
Let $\alpha \in \mathbb{R}$. Is $\mathbf{S}(\alpha\vec{x}) = \alpha\vec{y}$?
$\mathbf{S}\begin{bmatrix}\alpha a\\ \alpha b\\ \alpha c\end{bmatrix} = \alpha\begin{bmatrix}a-b-c\\ a-b-c\\ a-b+c\end{bmatrix}$. Try it!

Addition
Is $\mathbf{S}(\vec{x}_1 + \vec{x}_1) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$?
Let $\vec{x}_1 = \begin{bmatrix}a_1\\ b_1\\ c_1\end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix}a_2\\ b_2\\ c_2\end{bmatrix}$. Then $\mathbf{S}(\vec{x}_1 + \vec{x}_2) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$. Try it out!

Zero vector
Is $\mathbf{S}\cdot\vec{0} = \vec{0}$? Yes.

This proves that $\mathbf{S}$ is indeed a linear transformation.

Meta: At the end of this, get the students to ask you "well... but... since matrix-vector multiplication is linear, of course every matrix is a linear operator!!" This should be the next question they ask. Bonus: If every matrix transformation is a linear transformation, can we also say that every linear transformation is a matrix transformation?

4. Write out the matrix **S** from part 3. Is it invertible? Combining with what you saw in the previous part, what can you say about the relationship between whether a matrix is invertible and whether the matrix transformation is a linear transformation?

Solution: $\mathbf{S} = \begin{bmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & -1 & 1\end{bmatrix}$. This matrix is not invertible, but it was still linear! What does this tell us? There is no definitive relationship between invertibility and linear transformation! One does not lead to the other.

Meta: Go back to the definition of invertibility and prove to students why this transformation is non invertible i.e. show them that there exist multiple vectors that can output the same vector when transformed by this matrix.

Problem 4: Subsets v.s. Subspaces

Learning Goal: Prereqs: What are vector spaces and subspaces?

Description: Explains how to read set notation, tries to make students really realize that the notation means a set of vectors, and that a subspace is also a set of vectors. And what a subspace intuitively means.

1. Consider the set $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_1, a_2, a_3 \in \mathbb{R} : a_1 + 2a_2 - 3a_3 = 0 \right\}$. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ an element of the set W?

   Solution: The way set notation works is that for an element to be a part of the set, $a_1 + 2a_2 - 3a_3$ must be satisfied. We can plug in numbers for a_1, a_2, a_3 from the vector and see if the equation is satisfied. $1 + 4 - 9 \neq 0$, so this element is not a member of the set.

2. Write any 3 elements from this set.

   Meta: The purpose here is really to make sure the students realize that W is indeed a set. It has infinite number of elements, but it is still a set. Let the students come up with whatever they want. Ultimately though, it will be super helpful if the final 3 elements you get are the same as the ones in the answer below. We will be using these again! So verify a couple of the elements that the students put forth, but ultimately write these on the board.

   Solution: There are many possibilities. For instance, we could set the values of a_2 and a_3 to be 0 and then see what value of a_1 works. $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in the set.

   Another element can be obtained by setting the values of a_1 and a_2 to be 1, and then we get $1 + 2 - 3a_3 = 0$, or $a_3 = 1$. Therefore $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is also in the set.

   Another element can be obtained by setting the values of $a_1 = 3$ and $a_3 = 2$ and then we get $3 + 2a_2 - 6 = 0 \implies a_2 = \frac{3}{2}$. Therefore $\vec{v}_3 = \begin{bmatrix} 3 \\ \frac{3}{2} \\ 2 \end{bmatrix}$ is in the set too.

3. Is the set W a subspace?

   Meta: Meta: If students attempt an exhaustive proof of all the axioms of a vector space, be sure to note that many are satisfied by nature of the superset R3. Explain what closure under (+, *) means and why it matters. A fast way to disprove a subspace is to show the non-existence of the zero vector. Explain how set notation is defined and how to read it Subset != subspace

   Solution: Step 1: Claim that W is a sub*set* of, say, X.
   W is clearly a subset of $\mathbb{R}^3$. This can be seen because the elements of W contain 3 elements, but W is not equal to $\mathbb{R}^3$ since some elements from $\mathbb{R}^3$ are not in W.

   Step 2: Claim that X is a vector space
   $\mathbb{R}^3$ is a vector space that we have seen in lecture.

   Step 3: If X is a known vector space, and W is a subset of X, only 3 axioms must be proven.

   a: Prove closure under addition

   Consider two arbitrary elements from the set $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Since these elements are a part of the set, it is true that $b_1 + 2b_2 - 3b_3 = 0$ and that $c_1 + 2c_2 - 3c_3 = 0$.

Consider the sum of these elements. $\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix}$. Is this element a part of the set too? In other words, is $(b_1 + c_1) + 2(b_2 + c_2) - 3(b_3 + c_3) = 0$?

$$(b_1 + c_1) + 2(b_2 + c_2) - 3(b_3 + c_3) \stackrel{?}{=} 0$$

$$\implies (b_1 + 2b_2 - 3b_3) + (c_1 + 2c_2 - 3c_3) \stackrel{?}{=} 0$$

Clearly, the left hand side equals the right hand side. Therefore,

$$0 + 0 = 0$$

. Thus, the element $\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix}$ is a part of the set and we have proven closure under addition.

b: Prove closure under scalar multiplication

Consider an arbitrary element from the set $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$. This means that $d_1 + 2d_2 - 3d_3 = 0$ is true.

Consider some scalar s. Is $s\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ in the set W? I.e., is $\begin{bmatrix} sd_1 \\ sd_2 \\ sd_3 \end{bmatrix}$ in the set W? I.e. is $sd_1 + 2sd_2 - 3sd_3 = 0$?

$$sd_1 + 2sd_2 - 3sd_3 \stackrel{?}{=} 0$$

$$\implies s \cdot (d_1 + 2d_2 - 3d_3) \stackrel{?}{=} 0$$

Indeed the left hand side of the equation equals the right hand side, i.e.,

$$s \cdot 0 = 0$$

Therefore, $\begin{bmatrix} sd_1 \\ sd_2 \\ sd_3 \end{bmatrix}$ is in the set W and the set W is closed under scalar multiplication.

c: Prove existence of 0 element

We need to check whether $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ exists in the set. This is easy to check because we just need to check whether $0 + 2 \cdot 0 - 3 \cdot 0 = 0$, which it is. So the 0 element exists in the set.

Therefore, the set W is a subspace of $\mathbb{R}^3$.

4. How can we now quickly find more elements of this set?

Solution: Since we know the set is closed under scalar multiplication and under addition, we can easily find more elements. Previously, we found $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to be an element. Now we know that any multiple of this is in the set too!

We also found that $\vec{v}_2 = \begin{bmatrix} 3 \\ \frac{3}{2} \\ 2 \end{bmatrix}$ was in the set. Now we can add the 2 elements we found, and $\begin{bmatrix} 4 \\ \frac{5}{2} \\ 3 \end{bmatrix}$ is also in the set. In fact, any $s\vec{v}_1 + r\vec{v}_2$, where $s, r \in \mathbb{R}$ are in the set!

5. Consider the set $X = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_1, a_2, a_3 \in \mathbb{R} : a_1 * a_2 * a_3 = 0 \right\}$. Is X? is subspace?

   Solution: We can quickly see that this set would have a zero vector.

   This set also satisfies the scalar multiplication property; if $a_1 * a_2 * a_3 = 0$, one of the elements of the vector has to be zero and scaling the vector will not change that.

   However, this set is not closed under addition. We can verify this:

   $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ belongs to the set as $1 * 1 * 0 = 0$.

   $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ also belongs to the set as $0 * 0 * 1 = 0$.

   However, the sum of the two vectors, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ doesn't belong to the set as $1 * 1 * 1 = 1$.

   Hence, set X is not a subspace.

Problem 5: Null Spaces and Projections

Assume that the vector $\vec{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. For each of the following matrices $\mathbf{A} \in \mathbb{R}^{n \times m}$, answer the following:

- Compute the matrix product $\mathbf{A}\vec{x}$. Explain in words how the matrix transforms the vector.
- Suppose you know that A transforms $\vec{x}$ to give $\vec{y}$. Given $\vec{y}$, can you find what the original vector $\vec{x}$ was?
- Is the matrix $\mathbf{A}$ invertible? How do you know? If it is invertible, find the inverse.
- Verify that (dimension of nullspace) + (dimension of column space) = min(n, m)

Meta:

- In general, when trying to conceptualize the dimension of a vector space it can be helpful to write out the space in terms of basis vectors, and show how the number of basis vectors corresponds to how many degrees of freedom when have within that space which corresponds to its dimension.
- It is very IMPORTANT to mention to the students that when verifying the sum of dimension of nullspace and dimension of column space, the matrix DOES NOT have to be a square matrix (even though all the examples in the question are square matrices).
- Get the students to intuitively understand what each of these transformations mean. These are all fundamental transformations, so it is important that students understand what they mean physically
- Another main point is that matrices are invertible when they represent 'reversible' operations. Try to get students to understand why the examples in (a) and (b) are not invertible, while those in (c) and (d) are.
- For part (b), DO NOT mention the determinant, because students will have not covered that yet. Instead, just talk about how the rows are linearly dependent.

(a). $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Meta: Clarify what it means to project a vector: What component of this vector is in the direction of the x-axis?

Solution:

- $\mathbf{A}\vec{x} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$. We can see that this matrix keeps x_0 and turns x_1 to 0. In other words, the matrix **projects** the vector $\vec{x}$ to the x-axis.
- No. We can figure out x_0, since the value does not change, but the transformation converts all the x_1 values to 0. Given 0 as an output, x_1 could have been any value, so we cannot determine the original vector $\vec{x}$. As a rule of thumb, if there isn't a one-to-one mapping of inputs to outputs for the elements of $\vec{x}$, we cannot find the original vector after the transformation.
- No, $\mathbf{A}$ is not invertible, since the columns are linearly dependent, or there is no way to turn it into an upper triangular matrix using Gaussian elimination. Intuitively, we cannot retrieve a vector to its original state after applying the matrix, and therefore we cannot "invert" the operation.
- The dimension of the nullspace is 1, or the number of elements that cannot be retrieved after the matrix transformation. The column space is dimension 1, since there is only one pivot in the matrix. Adding both together, we get 1 + 1 = 2, which is equal to min(2, 2) = 2.

(b). $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution:

- $\mathbf{A}\vec{x} = \frac{1}{2}\begin{bmatrix} x_0 + x_1 \\ x_0 + x_1 \end{bmatrix}$ The first and second entries of the resulting vector have the same value; $x_0 = x_1$. We can consider the first entry to be equivalent to "x" and the second to "y", Hence, it is a projection onto y = x. Note that the constants are $\frac{1}{2}$ so that the x vector is not scaled.

- No, we cannot retrieve the original values because when we write out the equations represented by the transformation, the two equations that are the same (linearly dependent), so we only have one equation to work with. This is not enough information to retrieve the original information; we are trying to solve for two variables with one equation.

- No; this goes hand-in-hand with the whether or not we can retrieve the original values; the matrix has linearly dependent columns so we cannot invert it. We also know that we cannot invert matrices with determinant equal to 0, and it is clear that this transformation matrix has determinant 0.

- The column space of the matrix is 1 since we retrieved one linearly independent equation from earlier. To calculate the nullspace, we first row reduce the matrix, giving us $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then we solve for $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ This gives us the equation $x_0 + x_1 = 0$. Solving this equation, we get $x_0 = -x_1$, so the nullspace is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, we have one vector in the column space and one vector in the nullspace; $1 + 1 = 2$, the rank of the original transformation, so everything is accounted for.

(c). $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{y} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

Meta: Emphasize that if a transformation is "undoable," it is invertible, as with the rotation matrix. Hammer it in.

Solution:

- $\mathbf{A}\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} x_0 - x_1 \\ x_0 + x_1 \end{bmatrix}$. However, this does not offer us much insight into what the matrix actually **does**. For that, we notice that this is actually the rotation matrix corresponding to an angle of $\theta = \frac{\pi}{4}$. $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$ So, we can interpret this matrix as being a transformation that takes a vector and shifts it counterclockwise by 45°.

- Yes. We know that $x_0 - x_1 = 2$ and $x_0 + x_1 = 0$. Solving these, we get $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This makes sense, since we can think of the vector $\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$ as being a 45-degree-rotated version of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- Yes, the matrix is invertible. We know this because the columns are linearly independent (replace row 2 with row 2 - row 1 to get an upper triangular matrix). This also makes sense intuitively, because we know that we can reverse a rotation by applying its inverse rotation (a clockwise rotation by 45 degrees). So, in this case, the inverse matrix will be $\mathbf{A}^{-1} = \begin{bmatrix} \cos -\frac{\pi}{4} & -\sin -\frac{\pi}{4} \\ \sin -\frac{\pi}{4} & \cos -\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. You can verify that $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Because this matrix is invertible, it has a nullspace of dimension 0. And since it is full rank (rows are lin. ind.), its column space has dimension 2. Their sum is equal to min(2, 2) = 2. Thus the equation is verified.

(d). $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Solution:

- $\mathbf{A}\vec{x} = \begin{bmatrix} x_0 \\ 2x_1 \end{bmatrix}$. This is a diagonal matrix, and performs component-wise scaling. We can interpret this as a transformation that scales the first component by 1, and the second component by 2.
- Yes. We know that $x_0 = 2$ and $2x_1 = 4$. Solving these, we get $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. This makes sense, because the components are getting scaled by 1 and 2 respectively.
- Yes, the matrix is invertible. We know this because the columns are linearly independent (the matrix is already in upper triangular form). We also know that all diagonal matrices with nonzero entries are invertible. This also makes sense intuitively, because we know that we can reverse each component's scaling by applying its inverse scaling. So, in this case, the inverse matrix will be $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. You can verify that $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Because this matrix is invertible, it has a nullspace of dimension 0. And since it is full rank (rows are lin. ind.), its column space has dimension 2. Their sum is equal to $\min(2, 2) = 2$. Thus the equation is verified.