CSM 16A | **Designing Information Systems and Devices**

Week 1 Worksheet Metas

Term: **Spring 2020** *Name:*

Problem 1: Vector operations and Matrix-vector multiplication

Learning Goal: Students should be comfortable working with basic vector operations (such as addition) matrix vector multiplications.

Consider the following:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

1. What is the transpose of $\vec{v}_1$?

   Solution:

$$\vec{v}_1' = \begin{bmatrix} 4 & 7 & -5 \end{bmatrix}$$

2. What is $\vec{v}_1 + \vec{v}_2$?

   Solution:

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 4+1 \\ 7+3 \\ -5-1 \end{bmatrix}$$

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 5 \\ 10 \\ -6 \end{bmatrix}$$

3. What is $2\vec{v}_1 - 3\vec{v}_2$?

   Solution:

$$2\vec{v}_1 - 3\vec{v}_2 = \begin{bmatrix} 2*4-3*1 \\ 2*7-3*3 \\ 2*(-5)-3*(-1) \end{bmatrix}$$

$$2\vec{v}_1 - 3\vec{v}_2 = \begin{bmatrix} 5 \\ 5 \\ -7 \end{bmatrix}$$

4. What is $\vec{v}_1^T \vec{v}_2$?

   Solution:

$$\vec{v}_1^T \vec{v}_2 = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = 4 * 1 + 7 * 3 + (-5) * (-1)$$
$$\vec{v}_1 \cdot \vec{v}_2 = 30$$

   As some of you might have recognized, the expression you just evaluate is in fact the same as the dot product between two vectors. For two vectors with the **same dimensions**, we can calculate the sum of products of corresponding terms in the vectors.

5. What is $A\vec{v}_3$?

   Meta: Make sure students internalize the structure of matrix vector multiplication (maybe replace the matrix with row vector variables to show where things end up)

   Also, students should take note of the orientation of the vectors, whether they're row vectors or column vectors. A row vector multiplied by a column vector would equal a scalar, but a column vector multiplied by a row vector would equal a matrix.

   Solution:

$$A\vec{v}_3 = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 1 * 2 + 3 * 4 \\ 7 * 2 + 9 * 4 \end{bmatrix}$$

   Note: Matrix vector multiplication is just stacked vector vector dot products. The first row of the product is the same as the answer to the last problem.

$$A\vec{v}_1 = \begin{bmatrix} 14 \\ 50 \end{bmatrix}$$

6. What is AB?

   Solution:

$$AB = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1 * 5 + 3 * 3) & (1 * 1 + 3 * 6) \\ (7 * 5 + 9 * 3) & (7 * 1 + 9 * 6) \end{bmatrix}$$
$$AB = \begin{bmatrix} 14 & 19 \\ 62 & 61 \end{bmatrix}$$

Problem 2: Gaussian Eliminations, Span, Pivots and Free Variables

Learning Goal: Students should be comfortable solving a three-variable system of equations using GE with the forward/backward elimination method. Additionally, they should know how to convert a solution with a free variable from equations describing the solution set into vector notation.

Description: Simple mechanical gaussian elimination problem + some insight about span and free variables

1. Consider the following set of linear equations:

$$1x - 3y + 1z = 4$$
$$2x - 8y + 8z = -2$$
$$-6x + 3y - 15z = 9$$

Place these equations into a matrix A, and row reduce A to solve the equations.

Meta: This semester, 16A is making the distinction between a matrix in row-echelon form (REF) and reduced row-echelon form (RREF). Per Note 1, an REF matrix looks something like

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

whereas an RREF matrix looks like

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

the difference being that in an RREF matrix has only 1's or 0's in a column with a pivot.

Solution:

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

$$R_2 = R_2 - 2 * R_1$$
$$R_3 = R_3 + 6 * R_1$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & -15 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 33 \end{bmatrix}$$

$$R_2 = R_2/2$$
$$R_3 = R_3/3$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 11 \end{bmatrix}$$

$$R_3 = R_3 - 5 * R_2$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 36 \end{bmatrix}$$

$$R_2 = R_2 * -1$$

$$R_3 = R_3 / -18$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

This form of the matrix is called the row echelon form or the REF.

$$R_2 = R_2 + 3 * R_3$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

$$R_1 = R_1 + 3 * R_2 - R_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Now, we have reduced the matrix to the reduced row echelon form or the RREF.

$$z = -2$$

$$y = -1$$

$$x = 3$$

2. Consider another set of linear equations:

$$2x + 3y + 5z = 0$$

$$-1x - 4y - 10z = 0$$

$$x - 2y - 8z = 0$$

Place these equations into a matrix A, and row reduce A.

Meta: Note to mentors: When you do Gaussian Elimination, start by making $a_{2,1} = 0$ using some multiple of $a_{1,1}$. Next, make $a_{3,1} = 0$ using some multiple of $a_{1,1}$. Next, make $a_{3,2} = 0$ by using some multiple of $a_{2,2}$. In this last step, when you use row 2's pivot to subtract out row 3, the first element of row 3 will not be affected (it will remain 0). This is because in the previous steps, we got rid of the first element of row 2 as well. This is what I like to call the zig zag method of doing Gaussian Elimination. (Elena: I call this the 'staircase', and this is the Gaussian Elimination method 16A currently teaches. I think it is officially called forward/backward elimination.) Start at the top left, move down the column. Then start again at the top of the second column and move down.

Another way of thinking about this process is that when you go forward, you're putting the matrix into REF form, and going backwards puts it into RREF form.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

$$R_2 = R_2 + \frac{1}{2}R_1$$

$$R_3 = R_3 - \frac{1}{2}R_1$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -2.5 & -7.5 \\ 0 & -3.5 & -10.5 \end{bmatrix}$$

Remember that we can only do row operations without caring about the RHS because the RHS is all zeroes. Hence, any linear row operations won't affect the RHS i.e. it will remain the zero vector.

Make the numbers nicer by dividing row 2 by -2.5, and multiplying row 3 by -2. This is always a good thing to do if you realize your numbers are getting messy! (Also, feel free to keep all the numbers as non-fractional values by finding the least common multiple of the two numbers you are trying to cancel out.)

$$R_2 = \frac{1}{-2.5}R_2$$

$$R_3 = -2R_3$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix}$$

$$R_3 = R_3 - 7R_2$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = R_1/2$$

$$A = \begin{bmatrix} 1 & 1.5 & 2.5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the REF of the equation matrix.

$$R_1 = R_1 - 1.5 * R_2$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the RREF of the equation matrix.

3. Convert the row reduced matrix back into equation form.

   Meta: Note that although the equations have infinite solutions, there are still some constraints on x, y, and z: for example, choosing $x = y = z = 1$ wouldn't work, since plugging it into the first equation would give us $2 \cdot 1 + 3 \cdot 1 + 5 \cdot 1 = 10 \neq 0$. This is shown concretely later in the problem, but be sure to keep this in mind in case a student says something along the lines of "any numbers work."

   Solution:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1x + 0y - 2z = 0$$

$$0x + 1y + 3z = 0$$

$$0x + 0y + 0z = 0$$

4. Intuitively, what does the last equation from the previous part tell us?

   Meta: It tells us that there are infinite solutions to the equations. $0x + 0y + 0z = 0$ is satisfied by **any** x, y, z.

   Solution: If students are confused at this point about why we can infer this, their confusion is well justified. Suppose that there were **4** equations in 3 variables – 3 of them were linearly independent, and the fourth one was $0x + 0y + 0z = 0$, then the system still has just 1 solution. The last equation is never *used* in some sense. Feel free to talk about this with students. Present it as: what if you had 4 equations, you wrote them in matrix form, got pivots in all rows except for one where you got a row of all 0s – are there still infinite solutions? The answer is no.

5. How many pivots are there in the row reduced matrix? What are the free variables?

   Meta: It is important to emphasize to the students that for this particular matrix, either y or z can be a free variable. By Gaussian Elimination's convention, however, we always choose in the priority of right to left (in this case, we choose z as our free variable)

   Solution: There are 2 pivots in this row reduced matrix, and the corresponding pivot columns (following Gaussian Elimination's convention) are column 1 and 2. There are no more pivots since the third row are all zeros, and we require a non-zero element at the position of the third column (following column 2) and the third row for there to be one more pivot.

   The free variables can be y or z in this case, but we choose z as our free variable by Gaussian Elimination's convention.

6. What is the dimension of the span of all the column vectors in A?

   Meta: If the students are getting overwhelmed by all the technical terminologies in this question, try to break the question down term by term:

   dimension: the number of entries in the vector that can take on infinitely many values (for example, the dimension of a 1D number line is 1, the dimension of a 2D grid is 2, and the dimension of a 3D space is 3).

   span: the set of linear combinations of a set of vectors

   column vectors: we treat each column in the matrix A as a vector (call it column vector)

   It is also helpful to remind them think about the row reduced matrix we have in the end and the number of pivots we have.

   Solution: As we can see, in the row reduced form of A, since the third row are all zeros, and there are only 2 pivots with z as the free variable, the dimension of the span of all the column vectors in A is equal to 2 (number of pivots).

Alternatively, we can follow the definition of a span and algebraically write out the linear combinations of all the column vectors in the row reduced form of A, and we can see that the third entry in the resulting linear combination will always be 0 (since all 3 column vectors have 0's in their third entries), hence there are only 2 dimensions (entries) in the resulting vectors whose values we have control over.

7. Now that we've established that this system has infinite solutions, does every possible combination of $x, y, z \in \mathbb{R}$ solve these equations

   Meta: This is supposed to be a quick part. Explain that the existence of infinite solutions doesn't mean that all possible combinations work.

   Solution: No. $x = 1, y = 1, z = 1$ doesn't work, for instance.

8. What is the general form (in the form of a constant vector multiplied by a variable t) of the infinite solutions to the system?

   Meta: Explain why z is the free variable. (Because it is the one that doesn't have a pivot in the corresponding column). Also explain what "general form" means if students are confused.

   Students might be confused by the idea of a "general form" for the solution: try to convey that the solution is an equation that describes all possible solutions. If we assign some fixed value for z, then we can solve for a single pair of values for x, y. In the general solution, we replace z with a parameter variable t. The equations in the solution come from setting $z = t$ (our free variable), and plugging these back into the equations from our row-reduced matrix.

   Solution: z is a free variable. If $z = t$, then

$$y = -3z = -3t$$

$$x - 2z = 0 \implies x = 2z = 2t$$

   The general solution is then $t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. What this means is that any multiple of the vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ will satisfy the equations. Try it!

Problem 3: Proof on Consistency of $A\vec{x} = \vec{b}$

Learning Goal: Students should be comfortable working with proofs in linear algebra that stems from definitions and baisc properties.

Let A be an $m \times n$ matrix. Show that the following 4 statements about A are all **logically equivalent**. That is, for a particular A, either these statements are all true or they are all false.

1. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$,
4. A has a pivot position in every row.

Hint: Specifically, **show that the following statements are equivalent**:

- Statement 1 is equivalent to statement 2
- Statement 2 is equivalent to statement 3
- Statement 1 is equivalent to statement 3
- Statement 1 is equivalent to statement 4

Another Hint: In general, to show two statements are equivalent, we can take either of the approaches below:

- Show that the statements carry the same meaning by transforming the definitions/properties in one of the statements into those expressed in the other.
- Show that:
  1. If one statement is true, the other one must also be true.
  2. If one statement is false, the other one must also be false.

  It is important that both cases be justified.

1. Show that statement 1 is equivalent to statement 2.

   Meta:
   Students might be intimidated by proofs and may be unsure where to start. Encourage them to focus on just trying to show how statement 1 is equivalent to statement 2.

   Solution:
   Let's first show **tatement 1 is equivalent to statement 2**.

   By definitions of the matrix vector product $A\vec{x}$ and span of a set of vectors in $\mathbb{R}^m$, define:

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

   If the equation $A\vec{x} = \vec{b}$ has a solution, that means **the equality holds**. Substituting the more specific expressions we defined above (replace A with its column vectors $\vec{a}_1$ through $\vec{a}_n$, and replace $\vec{x}$ with its entries

x_1 through x_n), we have:

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \vec{b}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{b}$$

Since $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$ is a linear combination of the column vectors of A by definition, from this equation, we have shown that $\vec{b}$ is expressed as a linear combination of the columns of A if $A\vec{x} = \vec{b}$ has a solution. □

2. Show that statement 2 is equivalent to statement 3.

Solution:
Next, let's show **statement 2 is equivalent to statement 3**.

Since we are looking at all $\vec{b}$ in $\mathbb{R}^m$, and a linear combination of the columns of A has the form $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n$, if every such vector $\vec{b}$ can be expressed as $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n$, that means the linear combination of the columns vectors of A can reach any vector in $\mathbb{R}^m$.

Equivalently, this implies the columns of A span $\mathbb{R}^m$. □

3. Show that statement 1 is equivalent to statement 3.

Meta:
The solution uses the argument that because we proved statement 1 is equivalent to statement 2 and statement 2 is equivalent to statement 3 we can conclude that statement 1 is equivalent to statement 3. Make a note that this technique isn't constrained to this order. You can also use this fact to show why we only have to prove these four equivalences to demonstrate that all the statements are logically equivalent to each other.

Solution:
Now, to show **statement 1 is equivalent to statement 3**, notice we have shown that **statement 1 is equivalent to statement 2** and **statement 2 is equivalent to statement 3**, and we know that logical equivalences (just like equalities) are transitive (i.e. If $a = b$, $b = c$, then $a = c$), we conclude that **statement 1 must also be equivalent to statement 3** as well. □

4. Finally, show statement 1 is equivalent to statement 4.

Meta:
The first three proofs will probably feel much more comfortable/familiar than this one.

Prompt students by asking them to make the connection of what it means to have a consistent system and how that relates to the echelon form of A. Then what it means to have an inconsistent system and how that relates to the echelon form of a matrix.

Solution:
Finally, let's show **statement 1 is equivalent to statement 4**.

Let U be an echelon form of A. Given the vector $\vec{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to an augmented matrix $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ for some different vector $\vec{d}$ in $\mathbb{R}^m$:

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} U & \vec{d} \end{bmatrix}.$$

If statement 4 is indeed true, then each row of U must contain a pivot position and **there can be no pivot in the augmented column**. So $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$, and statement 1 must also be true.

On the other hand, if statement 4 is false, the last row of U will be all zeros. Let $\vec{d}$ be any vector with a 1 in its last entry, then $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ represents an inconsistent system. Since **the row operations we use when reducing a matrix are reversible**, this means $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ can be transformed back into the form $\begin{bmatrix} A & \vec{b} \end{bmatrix}$. Hence, the new system $A\vec{x} = \vec{b}$ is also inconsistent, and statement 1 will be false as well. $\square$

Hence, we have shown that all of the 4 statements are logically equivalent. $\square$

Problem 4: Proof on Linear Dependence/Independence

Prove that a subset of a linear independent set of vectors is linearly independent.

Hint 1: A subset intuitively means part of something bigger. If you have a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $\{\vec{v}_1, \vec{v}_2\}$ is a subset of S.

Hint 2: Recall the definition of linear independence. If a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent, and there exist a set of constants $c_1, c_2, \ldots, c_n$ such that:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0},$$

then it must be true that $c_1 = c_2 = \ldots = c_n = 0$.

Learning Goal:
Students should know what linear independence and dependence are
Make sure that students are solid on both the (mathematical) definitions of linear (in)dependence and span. Also explain why we might attempt a proof by contradiction; if they are confused, try showing the difficulties in a direct proof.

Most students learn the definition of linear dependence as $c_1\vec{v}1 + c_2\vec{v}2 + ... = \vec{0}$ for nonzero c_i's, but may not yet understand the alternate interpretation of having one vector is expressible as a linear combination of the others. Explicitly doing the algebra for this will be helpful.

Meta:
This is probably pretty early for when students will see proof. Very carefully introduce general proving techniques. Take the question, write down what is given in mathematical notation, and write out what needs to be proven in mathematical notation. A proof is essentially going from the 'given' to the 'to prove'.

Another note is that remember to assume that students have not taken CS70. Assume that they do not know proof techniques such as proof by contradiction, direct proof, induction, etc. This question is a proof by contradiction, so introduce it as such.

Proof by contradiction is not taught in 16A, so it is a good idea to go over the general structure of a proof in this format - assuming the negation of the statement you are trying to prove, and then using reductions to show an impossible scenario/contradiction.

Final note: explain the 'without loss of generality' in the 'To Prove' section.
Solution:
This problem can be tackled from two different approaches: **Direct Proof** and **Proof By Contradiction**. If you are not sure what the second approach means, we have also provided an explanation at the beginning of the second approach.

First Approach:
The problem can be translated mathematically to be the following:
Given a set of n vectors $N = \{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ that are linearly independent, we want to show that any subset of vectors in N is also linearly independent.
Since N is a linearly independent set, we know that if there exists a set of constants $c_1, c_2, \ldots, c_n$ such that:

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = 0$$

then it must be true that $c_1 = c_2 = \cdots = c_n = 0$.
To also symbolically represent a subset of N, since we can shuffle the array or arrange the vectors as much as we want, the following form generalizes to any susbet of vectors in N:

We want to show that for some $k \leq n$, $\{\vec{v_1}, \vec{v_2}, \dots \vec{v_k}\}$ is also a set of linearly independent vectors.
Following the same setup based on the definition of linear independence, suppose there exists a set of constants $b_1, b_2, \dots, b_k$ such that:

$$b_1\vec{v_1} + b_2\vec{v_2} + \cdots + b_k\vec{v_k} = 0$$

Now, we can add more "zeros" to both sides of the equations such that we are extending the linear combinations all the way from v_k to v_n:

$$b_1\vec{v_1} + b_2\vec{v_2} + \cdots + b_k\vec{v_k} + (0)\vec{v_{k+1}} + \cdots + (0)\vec{v_n} = 0$$

Now, since we know that the set $N = \{\vec{v_1}, \vec{v_2}, \dots \vec{v_n}\}$ is linearly independent, we know that all coefficients of the given vectors must be equal to 0 (per definition of linear independence). This means that $b_1 = b_2 = \cdots = b_k = 0$ as well.
Hence, it follows that $\{\vec{v_1}, \vec{v_2}, \dots \vec{v_k}\}$ is also a set of linearly independent vectors.

Second Approach:
This problem can also be approached from another direction using a technique called **Proof by Contradiction**.

More Explanation on Proof by Contradiction: In essence, the technique assumes the opposite of what we are trying to prove (so if the property we are proving is called P, we want to prove *not* P is true), and then reaches two **mutually contradictory** assertions (statements) (i.e., Property A is true and also false at the same time). Since both statements can't be simultaneously true, this leads us to conclude the property not P is in fact wrong, so P must be true.

Given: $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent. This, by definition of linear independence, means that if there exist $\alpha_1, \alpha_2, \dots, \alpha_n$, such that:

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n = 0$$

then

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

In other words, the only solution to the above αs is that the αs are all 0.
To Prove: $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0 \implies \beta_1 = \beta_2 = \ldots = \beta_k = 0$.
Note that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are a subset of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.
Assume that $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0$ is true but not $\beta_1 = \beta_2 = \ldots = \beta_k = 0$.
Consider $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n$. If $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0$ then

$$\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n = 0$$

However, since we assumed that not all $\beta_1, \beta_2, \dots, \beta_k$ are 0, this means that the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is not linearly independent, which is a contradiction because it is given that the set is linearly independent. Therefore, $\beta_1 = \beta_2 = \ldots = \beta_k = 0$ must have been true.