CSM 16A **Designing Information Systems and Devices**

Week 4 Worksheet Metas

Term: **Spring 2020** *Name:*

Problem 1: Intersection of Subspaces

Suppose U and V are both subspaces of a vector space S, is the intersection of U and V (notation wise, we can represent it as $U \cap V$) also a subspace of S?

Meta: Please make sure to explain to the students that $A \cap B$ means the intersection between sets A and B. This is one of the more abstract examples on vector subspaces that also involve some understanding on simple set relations. Make sure to show the students that the subspace test process still remains mostly unchanged.

Solution:

1. *Vector Addition*: Consider 2 vectors $\vec{x}, \vec{y} \in U \cap V$. We want to show that $\vec{x} + \vec{y} \in U \cap V$.

   To show that this is true, it seems like a direct approach might be a bit hard since it seems unclear what exactly $U \cap V$ contains in terms of their properties. However, making use of the fact that $U \cap V \subset U, V$ (**The intersection of U and V is a subset of U and V**) will be crucial to the proof.

   It may help to consider this graphically: TO BE INSERTED

   We've marked in green the intersection between sets U and V.

   If $\vec{x}, \vec{y} \in U \cap V$, both fall in the center region of the venn diagram. This means $\vec{x}, \vec{y} \in U$ as well as V! The implication goes both ways; if some vector $\vec{z}$ falls in both U and V (the left and right circles), then it necessarily falls into their intersection, $U \cap V$.

   We're already told that V and U are vector spaces, meaning $\vec{x} + \vec{y} \in U$ and V separately, so $\vec{x} + \vec{y} \in U \cap V$.

2. *Scalar Multiplication:*

   Consider a vector $\vec{x} \in U \cap V$, and a real-number scalar $c \in \mathbb{R}$.

   Again, since $\vec{x} \in U \cap V \implies \vec{x} \in V$, and V is already a vector subspace, so we know that by the property of scalar multiplication for a subspace, it must be true that $c\vec{x} \in V$.

   Applying the same logic again for U, we can see $c\vec{x} \in U$.

Since we've shown that the set is closed under vector addition and scalar multiplication, $U \cap V$ is also a vector subspace of S! $\square$

Problem 2: Eigenvalues and Eigenvectors

Consider a square matrix $\mathbf{A}$ that is $n \times n$. Recall that we say λ is an eigenvalue of $\mathbf{A}$ if there exists a **non-zero** vector $\vec{v}$ such that:

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

We call $\vec{v}$ the eigenvector associated with λ.

Meta: Prereq: All of linear algebra basically, including page rank etc.
Description: Meant to be an intuition problem on eigenvalues and eigenvectors.

1. What is the one eigenvalue and eigenvector of the matrix that you can see without solving any equations?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution: Since this matrix is clearly not-invertible, it must have an eigenvalue 0.

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$
$$\mathbf{A}\vec{x} = 0\vec{x}$$
$$\mathbf{A}\vec{x} = \vec{0}$$

This equation is precisely the equation for computing the nullspace of $\mathbf{A}$. Therefore, any $\vec{x} \in \text{Nullspace}(\mathbf{A})$ works.

For example, the vector

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is a valid answer.

Meta: This problem is relatively fast to do so please try to go through it. The point of this problem is not to find the eigenvalues mechanically, but instead use properties of the matrix that you can eyeball to figure out some eigenvalues and eigenvectors. Don't spend time mechancially computing the eigenvalues.

2. What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: This is a scaling matrix. It scales any vector by a factor of 3. What this means is that any vector $\vec{x} \in \mathbb{R}^3$ when post-multiplied by $\mathbf{A}$ will output $3\vec{x}$. This matrix has only one eigenvalue, $\lambda = 3$ and any $\vec{x} \in \mathbb{R}^3$ is an eigenvector.

3. What are the eigenvalues of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}?$$

Solution: A non-square matrix (say $m \times n$) maps a vector of dimension n to a vector of dimension m. So, it is impossible for a non-square matrix to have eigenvalues, because the output cannot be a scaled version of the input. In fact, eigenvalues are defined only for square matrices. For similar reasons, the determinant of a matrix is only well-defined if the matrix is square.

4. Consider a matrix that rotates a vector in $\mathbb{R}^2$ by $45°$ counterclockwise. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. Find its eigenvalues and corresponding eigenvectors. This matrix is given as

$$\mathbf{D} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Meta: Please draw a picture to show what the matrix does to a vector. Also remember we are only considering real eigenvalues, as written in the prompt of the problem.

Solution: Remember that the equation $\mathbf{A}\vec{x} = \lambda\vec{x}$ geometrically means that for the matrix $\mathbf{A}$, there exist some special vectors $\vec{x}$ that are merely scaled by λ when post-multiplied by $\mathbf{A}$. For a matrix that takes a vector and rotates it by $45°$, there are no real-valued vectors that it can simply scale. This means that there are no real eigenvalues for this matrix either.

5. What are the eigenvalues of the following matrix?

$$\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Solution: Remember that for upper triangular matrices, the eigenvalues can be read from the diagonal. $1, \frac{1}{2}, \frac{1}{3}$ are the three eigenvalues.

6. Can you find an eigenvalue of the following matrix without solving any equations?

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Solution: This is a matrix whose rows sum to 1, therefore, it has an eigenvalue 1.

This is proven by letting $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be a potential eigenvector of the matrix $\mathbf{F}$. Looking at the column view of matrix-vector multiplication –

$$\mathbf{F}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$
$$\mathbf{F}\vec{x} = 1 \cdot \vec{x}$$

since the rows sum to one.

Therefore, 1 is an eigenvalue with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Meta: Make sure students see why this works generally. Essentially $\mathbf{A}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where $\vec{v}_i$ are the columns of $\mathbf{A}$, and the sum equals $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ because each row sums to one.

7. Show that a matrix and its transpose have the same eigenvalues

Hint: The determinant of a matrix is the same as the determinant of its transpose

Solution: For any matrix $\mathbf{M}$,

$$det(\mathbf{M}) = det(\mathbf{M}^T)$$

Eigenvalues are found by solving the equation $det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

Note that $(\mathbf{M} - \lambda\mathbf{I})^T = \mathbf{M}^T - \lambda\mathbf{I}^T = \mathbf{M}^T - \lambda\mathbf{I}$.

Let $\mathbf{M} - \lambda\mathbf{I} = \mathbf{G}$.

$$det(\mathbf{G}) = det(\mathbf{G}^T)$$
$$det(\mathbf{M} - \lambda\mathbf{I}) = det(\mathbf{M}^T - \lambda\mathbf{I})$$

If we set the left hand side to 0 to solve for the lambdas, we also extract the lambdas corresponding to the right hand side. Therefore, $\mathbf{M}$ and its transpose have the same eigenvalues.

8. Consider a matrix whose columns sum to one. What is one possible eigenvalue of this matrix?

   Solution: We showed that for any matrix like $\mathbf{F}$ whose rows sum to 1, one eigenvalue is 1. We also showed that a matrix and its transpose have the same eigenvalues. Consider $\mathbf{F}^T$. It has columns summing to 1. Therefore, 1 is an eigenvalue of $\mathbf{F}^T$ too, and by extension of all matrices whose columns sum to one.

Problem 3: Eigenvalue Calculations

Meta: This problem is supposed to be straightforward, so make sure to stress the technique used in part (a), and let the students work amongst themselves for the rest of the question. [Notice]: Mentors, please go through this question quickly as there are a lot of other questions you will need to cover.

1. Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Solution: To solve for eigenvalues and eigenvectors, let's go back and review the definition of eigenvectors and eigenvalues:
If $\vec{x}$ and λ are the eigenvector and eigenvalue of $\mathbf{A}$, respectively, then the following equation holds:

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

Since the (appropriately sized) identity matrix is analogous to multiplying by 1 in arithmetic, we can say:

$$\mathbf{A}\vec{x} = \lambda\mathbf{I}\vec{x}$$

Rearranging, we get:

$$\mathbf{A}\vec{x} - (\lambda\mathbf{I})\vec{x} = \vec{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$

What does this look like? It looks similar to solving for the nullspace of $(\mathbf{A} - \lambda\mathbf{I})$!

Assuming that there is a nontrivial nullspace, that also means that $\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = 0$!

Let's solve for λ first:

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}$$
$$\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(4-\lambda) - 2$$
$$= 10 - 7\lambda + \lambda^2$$
$$= (\lambda - 5)(\lambda - 2)$$

By factoring:

$$\lambda = 5, 2$$

Let's check: We've just solved for the eigenvalues. But what about the eigenvectors?

To do that, we plug in λ into $(\mathbf{A} - \lambda\mathbf{I})$ and solve for the nullspace!

For $\lambda = 5$:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$
$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

By row reduction:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

So the first pair is

$$\lambda = 5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Repeating for $\lambda = 2$,

$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$x_1 = -2x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

So, the second pair is

$$\lambda = 2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2. Find the eigenvectors for matrix $\mathbf{A}$ given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Solution: Once the eigenvalues of a matrix have been found, we can find the eigenvectors by Gaussian Elimination.

Step 1: For each eigenvalue λ, we have

$$(\mathbf{A} - \lambda \mathbf{I})\vec{x} = 0$$

where $\vec{x}$ is the eigenvector associated with eigenvalue λ.

Step 2: Find $\vec{x}$ in the nullspace of $(\mathbf{A} - \lambda \mathbf{I})$ by plugging in a value of λ and using Gaussian elimination to solve.

Case 1: $\lambda = 4$. First, form the matrix $\mathbf{A} - 4\mathbf{I}$:

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

To make our numbers nicer, first let's divide our first row by -3

$$R_1 = R_1 \cdot \frac{-1}{3}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

$$R_2 = R_2 - 3 \cdot R_1$$

$$R_3 = R_3 - 6 \cdot R_1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix}$$

$$R_3 = R_3 - R_2$$

$$R_2 = R_2 \cdot \frac{1}{6}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we see that we have reached a row of 0s, which means that our last variable x_3 is the free variable in our system. Now, we can expand this matrix by putting it into a system of linear equations and solving for all the variables in terms of our free variable x_3

$$x_1 + x_2 - x_3 = 0$$

$$-2x_2 + x_3 = 0$$

$$x_2 = \frac{x_3}{2}$$

$$x_1 + \frac{x_3}{2} - x_3 = 0$$

$$x_1 = \frac{x_3}{2}$$

$$\vec{x} = \begin{bmatrix} \frac{x_3}{2} \\ \frac{x_3}{2} \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, \forall x_3 \in \mathbb{R}$$

So the eigenvector for when $\lambda = 4$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Now, let's use this same technique to find the eigenvector for $\lambda = -2$

Meta: Here might be a good time to ask your students how many eigenvectors the next value of lambda yields considering that there are two lambda values that are equal to it

Solution: Case 2: Now let's plug in $\lambda = -2$ into $\mathbf{A} - \lambda\mathbf{I}$ to get

$$\mathbf{A} + \mathbf{2I} = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$$

And, just like before, let's use Gaussian elimination to reduce the matrix. We can see that this will only take a few steps.

$$R_2 = R_2 - R_1$$
$$R_3 = R_3 - 2 \cdot R_1$$
$$R_1 = R_1 \cdot \frac{1}{3}$$
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As we can see here, we have two rows of 0s, which means that we have two free variables (x_2 and x_3). Now we can take this matrix and write it as a linear system to get

$$x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = x_3 - x_2$$

Thus,

$$\vec{x} = \begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Which are the two eigenvectors associated with $\lambda = -2$

3. Find the eigenvalues for matrix $\mathbf{A}$ given that we know that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are the eigenvectors of $\mathbf{A}$, and that

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution: There are 2 ways to go about solving this problem. Either you can plug each eigenvector $\vec{v}_i$ into $\mathbf{A}v = \lambda v$ or the nullspace equation to come up with 3 equations and solve. As you have had a lot of practice with the latter, we will use the former to try to answer this question.

Let's plug in the first eigenvector and solve for the first eigenvalue.

$$\mathbf{A}\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, we can see that $\lambda_1 = 1$. Similarly, we can do this for the other two eigenvectors.

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

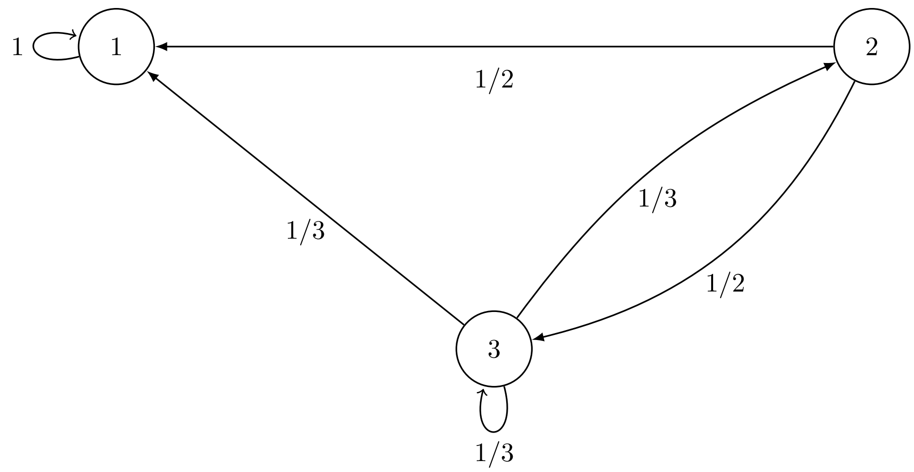
So, we can see that $\lambda_2 = 2$.

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

So, we can see that $\lambda_3 = 3$.

Problem 4: Mechanical PageRank

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



1. Write down the probability transition matrix for this graph, and call it $\mathbf{P}$. Can you say something about the eigenalues/eigenvectors of $\mathbf{P}^T$? (*Hint: Try to recall the properties of transition matrices*).

   Meta: Mentors: Explain how $\mathbf{P}$ being a transition matrix relates to $\mathbf{P}^T$ as a transition matrix, and depending on how comfortable students are with eigenvalues, mention that the eigenvalues of transposed matrices are always the same as original matrices.

   Solution: The transition matrix is:
$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

   We know that the columns of a probability transition matrix must sum to 1. This means that the rows of $\mathbf{P}^T$ must sum to 1. So, we have that the matrix-vector product $\mathbf{P}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This means that 1 must be an eigenvalue of the matrix $\mathbf{P}^T$, and therefore from part (a), it must also be an eigenvalue of $\mathbf{P}$. This is true for any probability transition matrix.

2. We want to rank these webpages in order of importance. But first, find the eigenvector of $\mathbf{P}$ corresponding to eigenvalue 1.

   Meta: This is largely a mechanical question, so ensure that students understand (i) the purpose of this calculation, and (ii) the techniques involved in it.

   Solution:
$$\begin{aligned} P - I &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} \\ &\xrightarrow[R1 \leftrightarrow R2, R2 \leftrightarrow R3]{R1 \to R1 + R2 + R3} \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can see that the pivots lie in the second and third columns. So, we want to solve the equation

$$\begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-\frac{1}{2}x_2 + \frac{1}{3}x_3 = 0 \text{ and } -\frac{2}{3}x_3 = 0$$

$$\implies x_3 = 0 \text{ and } x_2 = 0$$

This means that the eigenvector is of the form $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1$. And since x_1 is a free variable, the eigenvectors corresponding to eigenvalue 1 must belong in $\text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$

3. Now looking at the matrix $\mathbf{P}$, can you identify what its other eigenvalues are?

   Meta: Mentors: Please clarify to the students that the eigenvalues of an upper triangular matrix can be read off its diagonal is a known fact for EECS 16A. Refrain from proving why it is true algebraically since calculating the determinant of a 3 by 3 matrix is not in scope anymore this semester.

   Solution: $\mathbf{P}$ is an upper-triangular matrix, which means that the diagonal elements are the eigenvalues. So, the eigenvalues are $1, \frac{1}{2}, and \frac{1}{3}$ (we already found the eigenvalue 1 in part (b) through a different method).

4. Suppose that we start with 90 users evenly distributed among the websites. What is the steady-state number of people who will end up at each website?

   Meta: Ensure that the students understand why components with smaller eigenvalues will die out. Optionally, also explain why in any transition matrix no eigenvalue can be greater than 1, and so every component without eigenvalue one will die out.

   Solution: The initial vector of people is $\vec{x} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$. We know that since the other eigenvalues are less than 1, those components will die out as we keep applying $\mathbf{P}$ to $\vec{x}$. So we only care about the component of $\vec{x}$ that is in the direction of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This is just the first component of the vector, which is $\begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}$. However, the total number of people must be conserved, so we multiply by 3 so that the total is 90, the same as before. So, the steady-state distribution is $\begin{bmatrix} 90 \\ 0 \\ 0 \end{bmatrix}$