CSM 16A **Designing Information Systems and Devices**

Week 7 Worksheet Metas

Term: **Spring 2020** *Name:*

Problem 1: Linear Algebra Review

Meta: Description: Meant to be a review of eigenvectors, eigenvalues and transformations.

1. Suppose λ is an eigenvalue for the matrix A. Consider the λ-eigenspace of A: the set of all vectors v satisfying the equation $\mathbf{A}\vec{v} = \lambda\vec{v}$. Show that this eigenspace is a subspace by directly checking the three conditions needed to be a subspace.

   Solution: First, we have to check that $\vec{0}$ is in the subspace: this is true because $\mathbf{A}\vec{0} = \lambda\vec{0} = \vec{0}$ (regardless of what the eigenvalue λ is).

   Next, suppose $\vec{u}$ and $\vec{v}$ are in the subspace. This means that:

   $$\mathbf{A}\vec{u} = \lambda\vec{u}$$
   $$\mathbf{A}\vec{v} = \lambda\vec{v}$$

   $$\mathbf{A}(\vec{u} + \vec{v}) = \mathbf{A}\vec{u} + \mathbf{A}\vec{v} = \lambda\vec{u} + \lambda\vec{v} = \lambda(\vec{u} + \vec{v})$$

   This means $\vec{u} + \vec{v}$ is also in the subspace.

   Finally, suppose $\vec{v}$ is in the subspace and r is a scalar. Then,

   $$\mathbf{A}(r\vec{v}) = r(\mathbf{A}\vec{v}) = r(\lambda\vec{v}) = \lambda(r\vec{v})$$

   This means that $r\vec{v}$ is also in the subspace.

   Since the eigenspace satisfies all three conditions of being a subspace, we can say that it is a subspace.

2. Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

   $$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

   Solution: To solve for eigenvalues and eigenvectors, let's go back and review the definition of eigenvectors and eigenvalues:
   If $\vec{x}$ and λ are the eigenvector and eigenvalue of $\mathbf{A}$, respectively, then the following equation holds:

   $$\mathbf{A}\vec{x} = \lambda\vec{x}$$

   Since the (appropriately sized) identity matrix is analogous to multiplying by 1 in arithmetic, we can say:

$$\mathbf{A}\vec{x} = \lambda\mathbf{I}\vec{x}$$

Rearranging, we get:

$$\mathbf{A}\vec{x} - (\lambda\mathbf{I})\vec{x} = \vec{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$

What does this look like? It looks similar to solving for the nullspace of $(\mathbf{A} - \lambda\mathbf{I})$!
Assuming that there is a nontrivial nullspace, that also means that $\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = 0$!
Let's solve for λ first:

$$\begin{aligned}(\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix}4 & 1\\3 & 2\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\\ &= \begin{bmatrix}4-\lambda & 1\\3 & 2-\lambda\end{bmatrix}\\ \mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) &= (4-\lambda)(2-\lambda) - 3\\ &= 5 - 6\lambda + \lambda^2\\ &= (\lambda-5)(\lambda-1)\end{aligned}$$

By factoring:

$$\lambda = 5, 1$$

Let's check: We've just solved for the eigenvalues. But what about the eigenvectors?
To do that, we plug in λ into $(\mathbf{A} - \lambda\mathbf{I})$ and solve for the nullspace!
For $\lambda = 5$:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$
$$\begin{bmatrix}-1 & 1\\3 & -3\end{bmatrix}\vec{x} = \vec{0}$$

We can see that eigenvector $\begin{bmatrix}1\\1\end{bmatrix}$ spans the nullspace of the above matrix.
So the first pair is

$$\lambda = 5, \begin{bmatrix}1\\1\end{bmatrix}$$

Repeating for $\lambda = 1$,

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$
$$\begin{bmatrix}3 & 1\\3 & 1\end{bmatrix}\vec{x} = \vec{0}$$

We can see that eigenvector $\begin{bmatrix}-1\\3\end{bmatrix}$ spans the nullspace of the above matrix.
So, the second pair is

$$\lambda = 1, \begin{bmatrix}-1\\3\end{bmatrix}$$

3. Projection of a vector $\vec{u}$ onto $\vec{v}$ is given by:

$$\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$

Prove that projection onto a vector $\vec{v}$ is a linear transformation.

Solution: Let us represent this transformation using P.

$$P(\vec{a}) = \frac{\vec{a} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$

Let's check if it satisfies the condition of linearity.

$$P(\vec{a} + \vec{b}) = \frac{(\vec{a} + \vec{b}) \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$
$$P(\vec{a} + \vec{b}) = \frac{\vec{a} \cdot \vec{v}}{||\vec{v}||^2}\vec{v} + \frac{\vec{b} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$
$$P(\vec{a} + \vec{b}) = P(\vec{a}) + P(\vec{b})$$

Hence, the projection transformation satisfies additivity. Let's check if it satisfies the condition of scalar multiplication.

$$P(r\vec{a}) = \frac{(r\vec{a}) \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$
$$P(r\vec{a}) = r \cdot \frac{(\vec{a}) \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$
$$P(r\vec{a}) = r \cdot P(\vec{a})$$

Hence, the projection transformation is a linear transformation as it satisfies both the conditions - vector addition and scalar multiplication.