CSM 16A **Designing Information Systems and Devices I**

Week 4 Worksheet Solutions

Term: **Spring 2020** *Name:*

Problem 1: Intersection of Subspaces

Suppose U and V are both subspaces of a vector space S, is the intersection of U and V (notation wise, we can represent it as $U \cap V$) also a subspace of S?

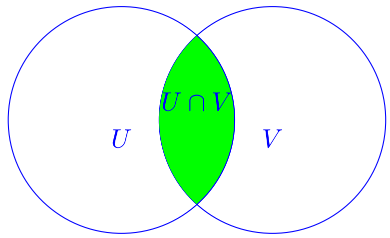
Solution:

1. *Vector Addition*: Consider 2 vectors $\vec{x}, \vec{y} \in U \cap V$. We want to show that $\vec{x} + \vec{y} \in U \cap V$.

   To show that this is true, it seems like a direct approach might be a bit hard since it seems unclear what exactly $U \cap V$ contains in terms of their properties. However, making use of the fact that $U \cap V \subset U, V$ (**The**

intersection of U and V is a subset of U and V) will be crucial to the proof.

It may help to consider this graphically:



We've marked in green the intersection between sets U and V.

If $\vec{x}, \vec{y} \in U \cap V$, both fall in the center region of the venn diagram. This means $\vec{x}, \vec{y} \in U$ as well as V! The implication goes both ways; if some vector $\vec{z}$ falls in both U and V (the left and right circles), then it necessarily falls into their intersection, $U \cap V$.

We're already told that V and U are vector spaces, meaning $\vec{x} + \vec{y} \in U$ and V separately, so $\vec{x} + \vec{y} \in U \cap V$.

2. *Scalar Multiplication:*

   Consider a vector $\vec{x} \in U \cap V$, and a real-number scalar $c \in \mathbb{R}$.

   Again, since $\vec{x} \in U \cap V \implies \vec{x} \in V$, and V is already a vector subspace, so we know that by the property of scalar multiplication for a subspace, it must be true that $c\vec{x} \in V$.

   Applying the same logic again for U, we can see $c\vec{x} \in U$.

Since we've shown that the set is closed under vector addition and scalar multiplication, $U \cap V$ is also a vector subspace of S! $\square$

Problem 2: Eigenvalues and Eigenvectors

Consider a square matrix $\mathbf{A}$ that is $n \times n$. Recall that we say λ is an eigenvalue of $\mathbf{A}$ if there exists a **non-zero** vector $\vec{v}$ such that:

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

We call $\vec{v}$ the eigenvector associated with λ.

1. What is the one eigenvalue and eigenvector of the matrix that you can see without solving any equations?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution: Since this matrix is clearly not-invertible, it must have an eigenvalue 0.

$$\begin{aligned} \mathbf{A}\vec{x} &= \lambda\vec{x} \\ \mathbf{A}\vec{x} &= 0\vec{x} \\ \mathbf{A}\vec{x} &= \vec{0} \end{aligned}$$

This equation is precisely the equation for computing the nullspace of $\mathbf{A}$. Therefore, any $\vec{x} \in \text{Nullspace}(\mathbf{A})$ works.

For example, the vector

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is a valid answer.

2. What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: This is a scaling matrix. It scales any vector by a factor of 3. What this means is that any vector $\vec{x} \in \mathbb{R}^3$ when post-multiplied by $\mathbf{A}$ will output $3\vec{x}$. This matrix has only one eigenvalue, $\lambda = 3$ and any $\vec{x} \in \mathbb{R}^3$ is an eigenvector.

3. What are the eigenvalues of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}?$$

Solution: A non-square matrix (say $m \times n$) maps a vector of dimension n to a vector of dimension m. So, it is impossible for a non-square matrix to have eigenvalues, because the output cannot be a scaled version of the input. In fact, eigenvalues are defined only for square matrices. For similar reasons, the determinant of a matrix is only well-defined if the matrix is square.

4. Consider a matrix that rotates a vector in $\mathbb{R}^2$ by $45°$ counterclockwise. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. Find its eigenvalues and corresponding eigenvectors. This matrix is given as

$$\mathbf{D} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution: Remember that the equation $\mathbf{A}\vec{x} = \lambda\vec{x}$ geometrically means that for the matrix $\mathbf{A}$, there exist some special vectors $\vec{x}$ that are merely scaled by λ when post-multiplied by $\mathbf{A}$. For a matrix that takes a vector and rotates it by $45°$, there are no real-valued vectors that it can simply scale. This means that there are no real eigenvalues for this matrix either.

5. What are the eigenvalues of the following matrix?

$$\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Solution: Remember that for upper triangular matrices, the eigenvalues can be read from the diagonal. $1, \frac{1}{2}, \frac{1}{3}$ are the three eigenvalues.

6. Can you find an eigenvalue of the following matrix without solving any equations?

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Solution: This is a matrix whose rows sum to 1, therefore, it has an eigenvalue 1.

This is proven by letting $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be a potential eigenvector of the matrix $\mathbf{F}$. Looking at the column view of matrix-vector multiplication –

$$\mathbf{F} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$
$$\mathbf{F}\vec{x} = 1 \cdot \vec{x}$$

since the rows sum to one.

Therefore, 1 is an eigenvalue with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

7. Show that a matrix and its transpose have the same eigenvalues

   Hint: The determinant of a matrix is the same as the determinant of its transpose

Solution: For any matrix $\mathbf{M}$,

$$det(\mathbf{M}) = det(\mathbf{M}^T)$$

Eigenvalues are found by solving the equation $det(\mathbf{M} - \lambda \mathbf{I}) = 0$.

Note that $(\mathbf{M} - \lambda \mathbf{I})^T = \mathbf{M}^T - \lambda \mathbf{I}^T = \mathbf{M}^T - \lambda \mathbf{I}$.

Let $\mathbf{M} - \lambda \mathbf{I} = \mathbf{G}$.

$$det(\mathbf{G}) = det(\mathbf{G}^T)$$

$$det(\mathbf{M} - \lambda \mathbf{I}) = det(\mathbf{M}^T - \lambda \mathbf{I})$$

If we set the left hand side to 0 to solve for the lambdas, we also extract the lambdas corresponding to the right hand side. Therefore, $\mathbf{M}$ and its transpose have the same eigenvalues.

8. Consider a matrix whose columns sum to one. What is one possible eigenvalue of this matrix?

Solution: We showed that for any matrix like $\mathbf{F}$ whose rows sum to 1, one eigenvalue is 1. We also showed that a matrix and its transpose have the same eigenvalues. Consider $\mathbf{F}^T$. It has columns summing to 1. Therefore, 1 is an eigenvalue of $\mathbf{F}^T$ too, and by extension of all matrices whose columns sum to one.