**CSM 16A** — **Designing Information Systems and Devices**

# Week 3 Worksheet

*Term:* **Spring 2020** *Name:*

## Problem 1: Conceptual Checks

For each of the following statements, determine if they are **TRUE** or **FALSE**. If they are **FALSE**, try to come up with a counterexample; if they are **TRUE**, give a brief explanation.

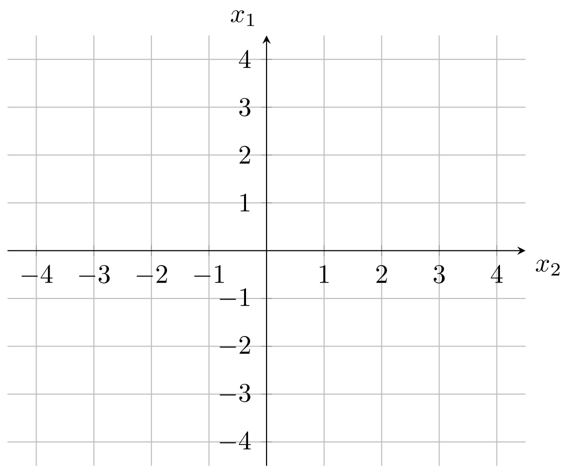
1. If the augmented matrix of the linear system represented by $A\vec{x} = \vec{b}$ has a pivot in the last column, then the matrix vector equation $A\vec{x} = \vec{b}$ has no solution.

2. If A is a $3 \times 3$ matrix such that the matrix vector equation $A\vec{x} = \vec{0}$ has only the trivial solution ($\vec{x} = \vec{0}$), then the matrix vector equation $A\vec{x} = \vec{b}$ is consistent for every vector $\vec{b}$ in $\mathbb{R}^3$

3. If the matrix vector equation $A\vec{x} = \vec{0}$ is true only when $\vec{x} = \vec{0}$, then the matrix $A$ has an inverse ($A$ is invertible).

4. A matrix $A$ is called *symmetric* if it is equal to its transpose: $A = A^T$. If $A$ is an invertible and *symmetric* matrix, $A^{-1}$ must also be *symmetric.*

**Problem 2: Range Intuition**




$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

1. Draw the space on the figure above that is represented by the span of all the column vectors in $A$. Also draw the space covered by the span of all the row vectors in $A$. What dimension are these spaces?

2. Consider some arbitrary vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Write out the product $\mathbf{A}\vec{v}$ in terms of $v_1$, $v_2$, and the columns of $\mathbf{A}$.

3. We have talked about how matrices like $\mathbf{A}$ have no inverse. Give a geometric explanation for why this is the case.

4. Consider all points $\vec{y}$ such that $\mathbf{A}\vec{y} = 0$ Draw the space that the $\vec{y}$'s will make up. What do you notice geometrically? What is the dimension of this space?

**Problem 3: More on Linear Transformation**

1. Consider a matrix $\mathbf{S}$ that transforms a vector $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ to $\vec{y} = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$. Note that $a, b$ can take on any values in $\mathbb{R}$. In other words, $\mathbf{S}\vec{x} = \vec{y}$. Is this transformation linear?

2. What is the matrix $\mathbf{S}$? Is the matrix invertible? Is the transformation invertible?

3. Consider a matrix $\mathbf{S}$ that transforms a vector $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\vec{y} = \begin{bmatrix} a-b-c \\ a-b-c \\ a-b+c \end{bmatrix}$. Note that $a, b, c$ can take on any values in $\mathbb{R}$. In other words, $\mathbf{S}\vec{x} = \vec{y}$. Is this transformation linear?

4. Write out the matrix $\mathbf{S}$ from part 3. Is it invertible? Combining with what you saw in the previous part, what can you say about the relationship between whether a matrix is invertible and whether the matrix transformation is a linear transformation?

**Problem 4: Subsets v.s. Subspaces**

**Learning Goal:** Prereqs: What are vector spaces and subspaces?

Description: Explains how to read set notation, tries to make students really realize that the notation means a set of vectors, and that a subspace is also a set of vectors. And what a subspace intuitively means.

1. Consider the set $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_1, a_2, a_3 \in \mathbb{R} : a_1 + 2a_2 - 3a_3 = 0 \right\}$. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ an element of the set $W$?

2. Write any 3 elements from this set.

3. Is the set $W$ a subspace?

4. How can we now quickly find more elements of this set?

5. Consider the set $X = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_1, a_2, a_3 \in \mathbb{R} : a_1 * a_2 * a_3 = 0 \right\}$. Is $X$? is subspace?

**Problem 5: Null Spaces and Transformations**

Assume that the vector $\vec{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. For each of the following matrices $\mathbf{A} \in \mathbb{R}^{n \times m}$, answer the following:

- Compute the matrix product $\mathbf{A}\vec{x}$. Explain in words how the matrix transforms the vector.
- Suppose you know that $A$ transforms $\vec{x}$ to give $\vec{y}$. Given $\vec{y}$, can you find what the original vector $\vec{x}$ was?
- Is the matrix $\mathbf{A}$ invertible? How do you know? If it is invertible, find the inverse.
- Verify that (dimension of nullspace) + (dimension of column space) = # of columns.

(a). $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(b). $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c). $\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{y} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

(d). $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$