CSM 16A (Designing Information Systems and Devices)

Week 1 Worksheet Solutions

Term: **Spring 2020** *Name:*

Problem 1: Vector operations and Matrix-vector multiplication

Learning Goal: Students should be comfortable working with basic vector operations (such as addition) matrix vector multiplications.

Consider the following:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{bmatrix}$$

1. What is the transpose of $\vec{v}_1$?

   Solution:

$$\vec{v}_1' = \begin{bmatrix} 4 & 7 & -5 \end{bmatrix}$$

2. What is $\vec{v}_1 + \vec{v}_2$?

   Solution:

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 4+1 \\ 7+3 \\ -5-1 \end{bmatrix}$$

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 5 \\ 10 \\ -6 \end{bmatrix}$$

3. What is $2\vec{v}_1 - 3\vec{v}_2$?

   Solution:

$$2\vec{v}_1 - 3\vec{v}_2 = \begin{bmatrix} 2*4-3*1 \\ 2*7-3*3 \\ 2*(-5)-3*(-1) \end{bmatrix}$$

$$2\vec{v}_1 - 3\vec{v}_2 = \begin{bmatrix} 5 \\ 5 \\ -7 \end{bmatrix}$$

4. What is $\vec{v}_1 \cdot \vec{v}_2$ (dot product)?

   Solution:

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = 4 * 1 + 7 * 3 + (-5) * (-1)$$
$$\vec{v}_1 \cdot \vec{v}_2 = 30$$

   Dot product of two vectors having the same dimensions is the sum of products of corresponding terms in the vectors. Dot products are undefined for vectors of different dimensions.

5. What is $A\vec{v}_1$?

   Solution:

$$A\vec{v}_1 = \begin{bmatrix} 1 & 3 & -1 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 1 * 4 + 3 * 7 + (-1) * (-5) \\ 7 * 4 + 9 * 7 + 11 * (-5) \\ 13 * 4 + 15 * 7 + 17 * (-5) \end{bmatrix}$$

   Note: Matrix vector multiplication is just stacked vector vector dot products. The first row of the product is the same as the answer to the last problem.

$$A\vec{v}_1 = \begin{bmatrix} 30 \\ 36 \\ 72 \end{bmatrix}$$

Problem 2: Gaussian Eliminations, Span, Pivots and Free Variables

Learning Goal: Students should be comfortable solving a three-variable system of equations using GE with the forward/backward elimination method. Additionally, they should know how to convert a solution with a free variable from equations describing the solution set into vector notation.

Description: Simple mechanical gaussian elimination problem + some insight about span and free variables

1. Consider the following set of linear equations:

$$1x - 3y + 1z = 4$$
$$2x - 8y + 8z = -2$$
$$-6x + 3y - 15z = 9$$

Place these equations into a matrix A, and row reduce A to solve the equations.

Solution:

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

$$R_2 = R_2 - 2 * R_1$$
$$R_3 = R_3 + 6 * R_1$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & -15 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 33 \end{bmatrix}$$

$$R_2 = R_2/2$$
$$R_3 = R_3/3$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 11 \end{bmatrix}$$

$$R_3 = R_3 - 5 * R_2$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 36 \end{bmatrix}$$

$$z = -2$$
$$y = -1$$
$$x = 3$$

2. Consider another set of linear equations:

$$2x + 3y + 5z = 0$$
$$-1x - 4y - 10z = 0$$
$$x - 2y - 8z = 0$$

Place these equations into a matrix A, and row reduce A.

Meta: Note to mentors: When you do Gaussian Elimination – start by making $a_{2,1} = 0$ using some multiple of $a_{1,1}$. Next, make $a_{3,1} = 0$ using some multiple of $a_{1,1}$. Next, make $a_{3,2} = 0$ by using some multiple of $a_{2,2}$. In this last step, when you use row 2's pivot to subtract out row 3, the first element of row 3 will not be affected (it will remain 0). This is because in the previous steps, we got rid of the first element of row 2 as well. This is what I like to call the zig zag method of doing Gaussian Elimination. (Elena: I call this the 'staircase', and I this is the Gaussian Elimination method 16A currently teaches. I think it is officially called forward/backward elimination.) Start at the top left, move down the column. Then start again at the top of the second column and move down.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

$$R_2 = R_2 + \frac{1}{2}R_1$$
$$R_3 = R_3 - \frac{1}{2}R_1$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -2.5 & -7.5 \\ 0 & -3.5 & -10.5 \end{bmatrix}$$

Remember that we can only do row operations without caring about the RHS because the RHS is all zeroes. Hence, any linear row operations won't affect the RHS i.e. it will remain the zero vector.

Make the numbers nicer by dividing row 2 by -2.5, and multiplying row 3 by -2. This is always a good thing to do if you realize your numbers are getting messy! (Also, feel free to keep all the numbers as non-fractional values by finding the least common multiple of the two numbers you are trying to cancel out.)

$$R_2 = \frac{1}{-2.5}R_2$$
$$R_3 = -2R_3$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix}$$

$$R_3 = R_3 - 7R_2$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Convert the row reduced matrix back into equation form.

Solution:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 3y + 5z = 0$$

$$0x + 1y + 3z = 0$$

$$0x + 0y + 0z = 0$$

4. Intuitively, what does the last equation from the previous part tell us?

Meta: It tells us that there are infinite solutions to the equations. $0x + 0y + 0z = 0$ is satisfied by **any** x, y, z.

Solution: If students are confused at this point about why we can infer this, their confusion is well justified. Suppose that there were **4** equations in 3 variables – 3 of them were linearly independent, and the fourth one was $0x + 0y + 0z = 0$, then the system still has just 1 solution. The last equation is never *used* in some sense. Feel free to talk about this with students. Present it as: what if you had 4 equations, you wrote them in matrix form, got pivots in all rows except for one where you got a row of all 0s – are there still infinite solutions? The answer is no.

5. How many pivots are there in the row reduced matrix? What are the free variables?

Meta: It is important to emphasize to the students that for this particular matrix, either y or z can be a free variable. By Gaussian Elimination's convention, however, we always choose in the priority of right to left (in this case, we choose z as our free variable)

Solution: There are 2 pivots in this row reduced matrix, and the corresponding pivot columns (following Gaussian Elimination's convention) are column 1 and 2. There are no more pivots since the third row are all zeros, and we require a non-zero element at the position of the third column (following column 2) and the third row for there to be one more pivot.

The free variables can be y or z in this case, but we choose z as our free variable by Gaussian Elimination's convention.

6. What is the dimension of the span of all the column vectors in A?

Meta: If the students are getting overwhelmed by all the technical terminologies in this question, try to break the question down term by term:

dimension: the number of entries in the vector that can take on infinitely many values (for example, the dimension of a 1D number line is 1, the dimension of a 2D grid is 2, and the dimension of a 3D space is 3).

span: the set of linear combinations of a set of vectors

column vectors: we treat each column in the matrix A as a vector (call it column vector)

It is also helpful to remind them think about the row reduced matrix we have in the end and the number of pivots we have.

Solution: As we can see, in the row reduced form of A, since the third row are all zeros, and there are only 2 pivots with z as the free variable, the dimension of the span of all the column vectors in A is equal to 2 (number of pivots).

Alternatively, we can follow the definition of a span and algebraically write out the linear combinations of all the column vectors in the row reduced form of A, and we can see that the third entry in the resulting linear combination will always be 0 (since all 3 column vectors have 0's in their third entries), hence there are only 2 dimensions (entries) in the resulting vectors whose values we have control over.

7. Now that we've established that this system has infinite solutions, does every possible combination of $x, y, z \in \mathbb{R}$ solve these equations

   Meta: This is supposed to be a quick part. Explain that the existence of infinite solutions doesn't mean that all possible combinations work.

   Solution: No. $x = 1, y = 1, z = 1$ doesn't work, for instance.

8. What is the general form (in the form of a constant vector multiplied by a variable t) of the infinite solutions to the system?

   Meta: Explain why z is the free variable. (Because it is the one that doesn't have a pivot in the corresponding column). Also explain what "general form" means if students are confused.

   Solution: z is a free variable. If $z = t$, then

   $$y = -3z = -3t$$

   $$2x + 3y + 5z = 0 \implies 2x - 9t + 5t = 0 \implies 2x = 4t \implies x = 2t$$

   The general solution is then $t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. What this means is that any multiple of the vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ will satisfy the equations. Try it!

Problem 3: Proof on Consistency of $A\vec{x} = \vec{b}$

Learning Goal: Students should be comfortable working with proofs in linear algebra that stems from definitions and baisc properties.

Let A be an $m \times n$ matrix. Show that the following 4 statements about A are all **logically equivalent**. That is, for a particular A, either these statements are all true or they are all false.

1. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$,
4. A has a pivot position in every row.

Hint: Specifically, **show that the following statements are equivalent**:

- Statement 1 is equivalent to statement 2
- Statement 2 is equivalent to statement 3
- Statement 1 is equivalent to statement 3
- Statement 1 is equivalent to statement 4

Another Hint: In general, to show two statements are equivalent, we can take either of the approaches below:

- Show that the statements carry the same meaning by transforming the definitions/properties in one of the statements into those expressed in the other.
- Show that:
  1. If one statement is true, the other one must also be true.
  2. If one statement is false, the other one must also be false.

  It is important that both cases be justified.

Solution:
Let's first show **tatement 1 is equivalent to statement 2**.

By definitions of the matrix vector product $A\vec{x}$ and span of a set of vectors in $\mathbb{R}^m$, define:

$$A = \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

If the equation $A\vec{x} = \vec{b}$ has a solution, that means **the equality holds**. Substituting the more specific expressions we defined above (replace A with its column vectors $\vec{a}_1$ through $\vec{a}_n$, and replace $\vec{x}$ with its entries x_1 through x_n), we have:

$$\begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \vec{b}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$$

Since $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$ is a linear combination of the column vectors of A by definition, from this equation, we have shown that $\vec{b}$ is expressed as a linear combination of the columns of A if $A\vec{x} = \vec{b}$ has a solution. $\square$

Next, let's show **statement 2 is equivalent to statement 3**.

Since we are looking at all $\vec{b}$ in $\mathbb{R}^m$, and a linear combination of the columns of A has the form $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n$, if every such vector $\vec{b}$ can be expressed as $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n$, that means the linear combination of the columns vectors of A can reach any vector in $\mathbb{R}^m$.

Equivalently, this implies the columns of A span $\mathbb{R}^m$. $\square$

Now, to show **statement 1 is equivalent to statement 3**, notice we have shown that **statement 1 is equivalent to statement 2** and **statement 2 is equivalent to statement 3**, and we know that logical equivalences (just like equalities) are transitive (i.e. If $a = b$, $b = c$, then $a = c$), we conclude that **statement 1 must also be equivalent to statement 3** as well. $\square$

Finally, let's show **statement 1 is equivalent to statement 4**.

Let U be an echelon form of A. Given the vector $\vec{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to an augmented matrix $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ for some different vector $\vec{d}$ in $\mathbb{R}^m$:

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} U & \vec{d} \end{bmatrix}.$$

If statement 4 is indeed true, then each row of U must contain a pivot position and **there can be no pivot in the augmented column**. So $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$, and statement 1 must also be true.

On the other hand, if statement 4 is false, the last row of U will be all zeros. Let $\vec{d}$ be any vector with a 1 in its last entry, then $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ represents an inconsistent system. Since **the row operations we use when reducing a matrix are reversible**, this means $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ can be transformed back into the form $\begin{bmatrix} A & \vec{b} \end{bmatrix}$. Hence, the new system $A\vec{x} = \vec{b}$ is also inconsistent, and statement 1 will be false as well. $\square$

Hence, we have shown that all of the 4 statements are logically equivalent. $\square$

Problem 4: Proof on Linear Dependence/Independence

Learning Goal: Students should be comfortable working with proofs on linear independence and dependence of a set of vectors.

Given that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors, and $\vec{v}_4$ is not in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, show that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ must be linearly independent.

Solution: Consider the following equation with respect to the linear combination of all the vectors in the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$.

Suppose by contradiction, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent, by definition of linear independence, we know that $\vec{v}_4$ can be expressed as a linear combination of the rest of the vectors in the set.

In other words:

$$\vec{v_4} = -\frac{c_1}{c_4}\vec{v}_1 - \frac{c_2}{c_4}\vec{v}_2 - \frac{c_3}{c_4}\vec{v}_3, \;\; c_4 \neq 0$$

However, this also implies that $\vec{v}_4$ is in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, which is a contradiction with the condition given in the problem.

Hence, it must be that $c_4 = 0$, and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ must be a linearly independet set. $\square$