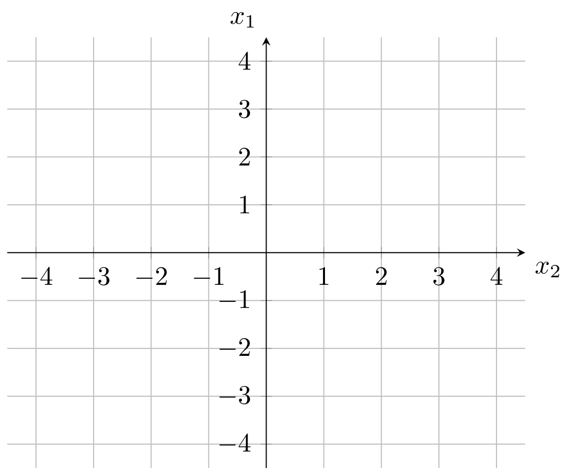
CSM 16A **Designing Information Systems and Devices**

Week 12 Worksheet Metas

Term: **Spring 2020** *Name:*

Problem 1: Projections



1. Consider the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Draw it on the graph provided. Also draw the vectors $\vec{x}$ with the vector $\vec{y_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, find the inner product of $\vec{x}$ with $\vec{y_1}$ and $\vec{y_2}$.

   Solution:

   $$\langle x, y_1 \rangle = 2 \cdot 1 + 4 \cdot 0 = 2$$
   $$\langle x, y_2 \rangle = 2 \cdot 0 + 4 \cdot 1 = 4$$

   Interestingly, we notice that the inner products of $\vec{x}$ with each of the unit vectors in the x and y directions gives us the components of the vector in those directions. This is not a coincidence. If we drop perpendiculars from the vector $\vec{x}$ to tne x and y axis, the resulting vectors are just y_1 and y_2. This 'dropping a perpendicular' is what we mean by projection.

2. Now, find the inner product of $\vec{x}$ with the vector $\vec{y_3} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Is this the same as with $\vec{y_1}$? How can we find the projection of $\vec{x}$ onto $\vec{y_3}$?

   Solution:

   $$\langle x, y_3 \rangle = 2 \cdot 5 + 4 \cdot 0 = 10$$

   However, by dropping the perpendicular from $\vec{x}$ to $\vec{y_1}$, we can see that the projection should be the same as it was for $\vec{x_1}$. So, we have to divide this inner product by 5 to get the correct length of the projection (since $10/5 = 2$). It is not a coincidence that 5 is also the length of $\vec{x_3}$! We have to divide the inner product by the magnitude of the vector we are projecting onto, because while the inner product scales with both of its inputs, the projection should only scale with $\vec{x}$, and it should only depend on the direction of $\vec{y_3}$. That is, the projection of $\vec{x}$ onto any vector of the form $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for nonzero α should be the same.

So, we have that the length of the projection of $\vec{x}$ onto $\vec{y}$ equals
$\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|}$.
Another way of interpreting this is that by dividing by $\|\vec{y}\|$, we are converting $\vec{y}$ into a unit vector. That is, $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|} = \left\langle \vec{x}, \frac{\vec{y}}{\|\vec{y}\|} \right\rangle$

3. Now, let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find the projection of $\vec{x}$ onto $\vec{y}$. Also find the projection of $\vec{y}$ onto $\vec{x}$.

   Solution: The length of the projection of $\vec{x}$ onto $\vec{y}$ will be $\frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{y}||} = \frac{1 \cdot 1 + 0 \cdot 2 + 3 \cdot -1}{\sqrt{1^2 + 0^2 + 1^2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$.

   But we don't just want the length of the projection, we want the actual projection vector. We know the projection's length, and we know that it must lie along the direction characterized by $\vec{y}$. So to get the projection, we can simply scale the unit vector in the direction of $\vec{y}$ by the length we found above. So, the projection is $-\sqrt{2} \cdot \frac{\vec{y}}{||\vec{y}||} = -\sqrt{2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

4. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ Suppose we know that $A^T \vec{x} = \vec{0}$. Based on your knowledge of inner products and projections, what does this tell you about the vector $\vec{x}$

   Solution: It tells you that the vector $\vec{x}$ is orthogonal to the columns of $\mathbf{A}$. Or, in other words, the projection of $\vec{x}$ onto each of the columns of $\mathbf{A}$ is of length 0. This also means that the projection of $\vec{x}$ onto the subspace spanned by the columns of $\mathbf{A}$, also known as the range of $\mathbf{A}$, is the zero vector.

Problem 2: Cross Correlation

You are a Google Maps engineer, trying to calculate the distance from a user's cell phone to a bunch of satellites. Have each of the m satellites emit a signal $\vec{s_i}$, which repeats with a period of N.

You might have an intuitive idea of what a signal is already. But how can we represent a signal in linear algebraic terms? We will do it using vectors. For example, consider the vector

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 \end{bmatrix}^T$$

These components of the vector can be used to represent the amplitude of a signal at various points of time. So at time 0 the signal has amplitude 1, at time 2 it has amplitude 2, and so on. Perhaps you are used to thinking of signals as waves travelling through space? In that case, you can interpret the components of the vector as being the height of the wave at various points in time. But what about periodicity? In order to account for that, we can use this vector to refer to one period of our signal. So, the actual signal would take values 1, 2, 1, -3, 0, 1, 2, 1, ... at times 0, 1, 2, 3, ... In general, in order to represent a signal with period N, we will use a vector of length N.

These signals are chosen in such a way that they are approximately orthogonal, i.e. $\langle \vec{s_i}, \vec{s_j} \rangle$ is very close to 0, if $i \neq j$. These signals reach the user's cell phone. But by the time it reaches the phone, they have been modified in the following ways:

- They have been attenuated, because of travelling the long distance from the satellite to the cell phone.
- They have been time-shifted. That is, the user sees the signal with some delay.

1. Give an expression for the signal $y[k]$ that the user sees, in terms of the signals $s_i[k]$, attenuation factors α_i, and delays τ_i (Hint: There will be a linear combination involved).

   Solution: $y[k] = \sum_{i=1}^{m} \alpha_i s_i[k - \tau_i]$ Where α_i is the attenuation factor of the ith signal, and $s_i[k - \tau_i]$ is a delayed version of the original signal $s_i[k]$, which has been shifted by τ_i. It is important to note that when we index into vectors that represent signals, we are working "mod N". That is, if the index is negative, or if it exceeds N, it "wraps around" the vector and keeps going.

2. First, consider the case when there is only one satellite, with a signal s (which is known to the user). And assume that y can consist of various shifts of s. That is, $y[k] = \sum_{i=1}^{N} \beta_i s[k - i]$. Intuitively, this might happen if the signal bounces off multiple surfaces, resulting in a superposition of s with different shifts and attenuations. How can the user recover the attenuation and the time shifts of the constituent s signals? Formulate this problem as a system of equations.

   Solution: We can write this as a matrix-vector equation. First, define

   $$C_s = s$$

   Then, we have that $C_s \vec{\beta} = \vec{y}$ Solving for β, we have $\beta = C_s^{-1} \vec{y}$

3. Now, consider the case when there are multiple satellites. Will the same method as above work? Why/Why not?

   Solution: No, because we will have more than N unknowns, but still only N equations. Notice that the equation for $y[k] = \sum_{i=1}^{m} y_i[k] = \sum_{i=1}^{m} \alpha_i s[k - \tau_i]$ has $2m$ unknowns, but they are coupled nonlinearly.

   We could try to linearize the problem by introducing extra variables. In order to write this as a matrix-vector equation and solve for the unknowns $\tau_1 \dots \tau_m, \alpha_1 \dots \alpha_m$, we would first have to rewrite $y_i \dots y_m$ as we did in part (b). That gives us $y[k] = \sum_{i=1}^{m} \sum_{j=1}^{N} \alpha_i \beta_j s[k - j]$, with the understanding that we want all β_j to be 0 except the one where $j = \tau_i$. This equation can be treated as linear with mN unknowns (if we lump α and β together). We have N equations from the N points $y[0] \dots y[N-1]$, and it is possible to *approximate* the "0 at all but τ_i" constraint with m more equations. Regardless, this linearized system is still underdetermined, and we definitely cannot apply the same method as in part (b).

4. Recall that the inner product is a measure of "closeness" or "similarity" of 2 vectors. With this in mind, how could you find the approximate attenuation and shift of s_i in the general setting, with multiple satellites?