CSM 16A Designing Information Systems and Devices

Week 2 Worksheet Metas

Term: **Spring 2020** *Name:*

Problem 1: Matrix * Matrix

Learning Goal: Students should be comfortable working with basic vector operations (such as addition) matrix vector multiplications.

Meta: Make sure to keep all parts of 1 on the board, so that when the last part is reached, the patterns are easier to see. Ensure the students understand the structure of vector-vector, vector-matrix, matrix-vector, and matrix-matrix multiplication. Make sure to explain how the dimensions of the vectors/matrices must agree with one another in order for the multiplication to work out. For example, a 1x2 matrix multiplied by a 2x2 matrix should equal a 1x2 matrix.

$$\vec{a}_1 = \begin{bmatrix} 1 & 3 \end{bmatrix}$$
$$\vec{a}_2 = \begin{bmatrix} 7 & 9 \end{bmatrix}$$
$$\vec{b}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
$$\vec{b}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$
$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

1. What is $A\vec{b}_1$?

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$
$$\vec{b}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Solution:

$$A\vec{b}_1 = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
$$A\vec{b}_1 = \begin{bmatrix} 1*5+3*3 \\ 7*5+9*3 \end{bmatrix}$$
$$A\vec{b}_1 = \begin{bmatrix} 14 \\ 62 \end{bmatrix}$$

2. What is $A\vec{b}_2$?

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$
$$\vec{b}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Solution:

$$A\vec{b}_2 = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 1*1+3*6 \\ 7*1+9*6 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 19 \\ 61 \end{bmatrix}$$

3. What is $\vec{a}_1^T$ B?

$$\vec{a}_1^T = \begin{bmatrix} 1 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

Solution:

$$\vec{a}_1^T B = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

$$\vec{a}_1^T B = \begin{bmatrix} 1*5+3*3 & 1*1+3*6 \end{bmatrix}$$

$$\vec{a}_1^T B = \begin{bmatrix} 14 & 19 \end{bmatrix}$$

4. What is $\vec{a}_2^T$ B?

$$\vec{a}_2^T = \begin{bmatrix} 7 & 9 \end{bmatrix}$$
$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

Solution:

$$\vec{a}_2^T B = \begin{bmatrix} 7 & 9 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

$$\vec{a}_2^T B = \begin{bmatrix} 7*5+9*3 & 7*1+9*6 \end{bmatrix}$$

$$\vec{a}_2^T B = \begin{bmatrix} 62 & 61 \end{bmatrix}$$

5. What is AB? Do you notice something?

Solution:

$$AB = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1*5+3*3) & (1*1+3*6) \\ (7*5+9*3) & (7*1+9*6) \end{bmatrix}$$

$$AB = \begin{bmatrix} 14 & 19 \\ 62 & 61 \end{bmatrix}$$

Note that $AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}$ and $AB = \begin{bmatrix} \vec{a}_1^T B \\ \vec{a}_2^T B \end{bmatrix}$

Imagine A is a matrix made of row vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ and B is a matrix made of column vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$. Then,

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \ldots & A\vec{b}_n \end{bmatrix}$$

And also,

$$AB = \begin{bmatrix} \vec{a}_1^T B \\ \vec{a}_2^T B \\ \ldots \\ \vec{a}_n^T B \end{bmatrix}$$

.

Problem 2: More Proof on Spans

Learning Goal: Proof on spans and linear dependence/independence

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a set of vectors V. Prove that if the set of vectors is linearly dependent, then at least one vector can be deleted from the set without diminishing its span.

Meta: Make sure to emphasize that v is just some arbitrary vector in the span of the set, while v1 is a linearly dependent vector in the set. This will make it more clear when the substitution (for v1) step happens.

Solution: The general form of a vector $\vec{v}$ in the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

Without loss of generality, let us assume $\vec{v}_1$ can be written as the linear combination of the remaining vectors as $a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n$.

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

If we substitute $\vec{v}_1$ for the above value $(a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n)$ in the general form of $\vec{v}$, we get:

$$\vec{v} = c_1 * (a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n) + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$\vec{v} = c_1a_2\vec{v}_2 + c_1a_3\vec{v}_3 + \cdots + c_1a_n\vec{v}_n + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$\vec{v} = (c_1a_2 + c_2)\vec{v}_2 + (c_1a_3 + c_3)\vec{v}_3 + \cdots + (c_1a_n + c_n)\vec{v}_n$$

We can see that any vector we could represent as a linear combination of the vectors in V can be represented without using $\vec{v}_1$ using the new parameters we got in the above equation.

Hence, if the set of vectors is linearly dependent, then at least one vector can be deleted from the set without diminishing its span.

Problem 3: Step-by-step Inverse

In this question, we will learn about the underlying transformations that allow us to find the inverse of a given matrix by exploring how matrices can be used to represent different types of row operations.

Learning Goal: Students should understand matrix multiplication, linear transformations, Gaussian elimination

1. What matrix B can we left multiply by a 3×3 matrix M to get a new matrix M' that is the same as M but with row 2 scaled by 1/5?

   Meta: Go through the steps of matrix multiplication to illustrate how putting a scalar a in ith column of A grabs elements from the ith row of M multiplied by a.

   Solution: Let $M = \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix}$ and $B = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ then $BM = \begin{bmatrix} a_1\vec{m_1}^T \\ a_2\vec{m_2}^T \\ a_3\vec{m_3}^T \end{bmatrix}$ so if we want to scale row 2 by 1/5 and leave the other rows unchanged, then we should set $a_1 = 1$, $a_2 = \frac{1}{5}$, and $a_3 = 1$.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. What matrix A can we left multiply by a 3×3 matrix M to get a new matrix M' that is the same as M but with the row 1 and row 3 swapped?

   Meta: Go through the steps of matrix multiplication to illustrate how putting a 1 in ith column of A grabs elements from the ith row of M.

   Solution: Let $M = \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix}$ and $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$ then $AM = \begin{bmatrix} a_{1,1}\vec{m_1}^T + a_{1,2}\vec{m_2}^T + a_{1,3}\vec{m_3}^T \\ a_{2,1}\vec{m_1}^T + a_{2,2}\vec{m_2}^T + a_{2,3}\vec{m_3}^T \\ a_{3,1}\vec{m_1}^T + a_{3,2}\vec{m_2}^T + a_{3,3}\vec{m_3}^T \end{bmatrix}$ so if we want to swap row 1 and row 3 and leave row 2 unchanged, then we should make $a_{1,3} = 1$ since it will put one of row 3 in row 1, $a_{3,1} = 1$ since it will put one of row 1 in row 3, $a_{2,2} = 1$ to keep row 2 the same, and the remaining $a_{i,j} = 0$.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

3. What matrix A can we left multiply by a 3×3 matrix M to get a new matrix M' that is the same as M but with the 3 times row 1 added to the row 2?

   Meta: Go through the steps of matrix multiplication again to show how the values from row 1 and row 2 of M are summed to create row 2 of M'.

   Solution: To get this matrix, you can use the method from the solution to part (b), but make $a_{2,1} = 3$ to put 3 times row 1 in row 2, $a_{1,1} = a_{2,2} = a_{3,3} = 1$ to keep the original rows except for what we added to row 2, and the remaining $a_{i,j} = 0$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. What is the the multiplicative inverse of 2? What is the multiplicative identity? What is the additive inverse of 1? What is the additive identity? What is the identity in matrix/vector multiplication?

Meta: The point of this question is to get students to understand what an inverse and identity are. Multiplying something by or adding it to its inverse should give you the identity. Multiplying anything by or adding it to the identity should leave it unchanged. This is why multiplying by the identity matrix has no effect.

Solution: The multiplicative inverse of 2 is $\frac{1}{2}$. The multiplicative identity is 1. The additive inverse of 1 is -1. The additive identity is 0. The identity for matrix multiplication is $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

5. In what order should we apply the transformations described in parts (a), (b), and (c) to the matrix $M = \begin{bmatrix}0 & 0 & 1\\ -15 & 5 & 0\\ 1 & 0 & 0\end{bmatrix}$ to get the identity matrix?

   Meta: Make sure students understand that these would be the steps taken in Gaussian elimination.

   It is also very crucial to let the students see how if you left multiply the 3 matrices from the previous questions (in that order) with M, you will get exactly the matrix we are seeking in this part.

   Take some time to do the computation, and reinforce the concept that

   Elementary row operations can be represented as individual matrices!

   This is a crucial conjecture in the correctness of the algorithm to find the inverse of a matrix!!!

   Solution: Swap row 1 and row 3, then scale row 2 by 1/5, then add 3 times row 1 to row 2.

6. Multiply the matrices for each transformation in the order determined in part (d). What happens when you multiply M by this matrix? What is this matrix called?

   Meta: This part of the question ties together the transformations from the earlier parts with the concept of matrix inverses. If you want, show how to find an inverse using Gaussian elimination and how that is the same as keeping track of each transformation as done above.

   Solution:

$$\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & \frac{1}{5} & 3\\ 1 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 1\\ 0 & \frac{1}{5} & 3\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ -15 & 5 & 0\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

   This matrix is the inverse of M.

7. Are there a set of transformations we can apply to $M = \begin{bmatrix}1 & -2 & -1\\ 0 & 2 & 2\\ 1 & 0 & 1\end{bmatrix}$ to make it the identity? If so, what are they? If not, why is is not possible?

   Meta: This part is meant to help students understand how a matrix with linearly dependent rows is not invertible.

   Solution: No, there are not a set of transformations. It is not possible because the rows are linearly dependent, so you end up with a row of 0s.

8. Can you find the inverse of a non-square matrix (e.g. a 2×3 matrix)?

   Meta: This question forces the students to think about the requirements and properties of an inverse. To supplement with some extended concepts (not in scope for this course), you can provide students with the following definitions:

For any $m \times n$ real matrix A

- A has a left inverse if there exists some $n \times m$ matrix B such that $BA = I_n$.
- A has a right inverse if there exists somne $n \times m$ matrix B such that $AB = I_m$.

As you can see, 16A's definition of matrix inverse only holds if a matrix has both a left inverse and a right inverse!

The concept of left and right inverses is actually well connected to set theory: the study of functional relations among sets (usually finitely/infinitely countable sets). If you want to explore more, here's an important theorem that establishes this connection (the concepts/implications are covered in CS70):

- A matrix A has a left inverse if and only if it is an **injective transformation**. This means that:

$$\forall \vec{v}_1, \vec{v}_2, A\vec{v}_1 = A\vec{v}_2 \Rightarrow \vec{v}_1 = \vec{v}_2.$$

  Equivalently, we also have the contrapositive:

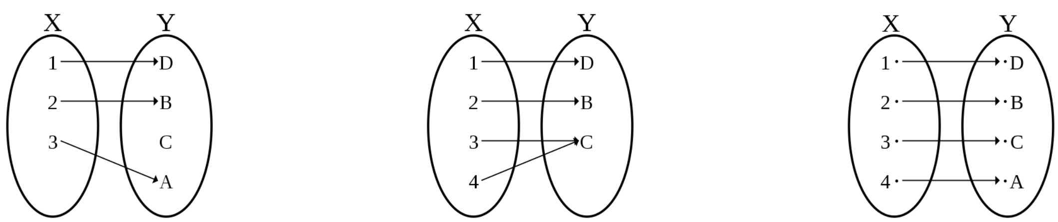
$$\forall \vec{v}_1, \vec{v}_1 \neq \vec{v}_2 \Rightarrow A\vec{v}_1 \neq A\vec{v}_2.$$

- A matrix A has a right inverse if and only if it is a **surjective transformation**. This means that:

$$\forall \vec{u} \in Col(A), \exists \vec{v} \text{ s.t. } A\vec{v} = \vec{u}.$$

- As a corollary, we can see that a matrix A has an inverse if it is both an injective and surjective transformation (we call such a transformation **bijective**).

Here're some visual illustrations that might help with understanding:



An **Injective** Function (Injection) A **Surjective** Function (Surjection) A **Bijective** Function (Bijection)

Solution: No. Recall from lecture, an $n \times n$ square matrix A has an inverse only if $AA^{-1} = A^{-1}A = I$, where I is the identity matrix, and A is an $n \times n$ square matrix. This is not possible with rectangular matrices because their row count and column count differ.

Problem 4: Round and Round

In discussion, we talked about rotation matrix as an example of transformation on a given vector.

A rotation matrix is a matrix that takes a vector and rotates it by some number of degrees (counter-clockwise). That matrix looks like:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

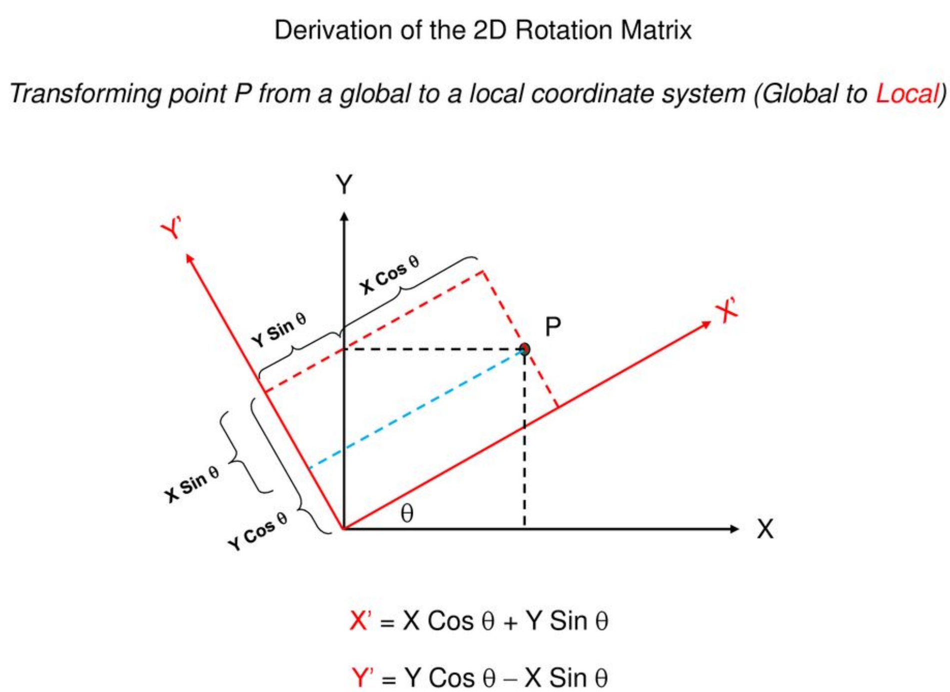
for some angle θ.

In this question, we will explore some of the properties a rotation matrix has in more depth and see how the algebra behind is deeply connected to the geometric transformation we see.

Meta: Before proceeding, please show the students how the rotation matrix is actually derived. Especially show them how it is a linear transformation on the two following vectors in the standard basis of $\mathbb{R}^2$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Here's an accompanying figure that helps illustrate the process:



1. Given the following vector:

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If we want to rotate $\vec{v}$ counter-clockwise by $\theta = 45^o$, what would the rotation matrix corresponding to this transformation be? What would the resulting vector be?

Meta: For this question, it might be helpful to plot out the vector on a 2D coordinate system on the board and first discuss with the students on what the rotation process would look like. It is also a good idea to review some trigonometry with the students in case they weren't as familiar with the notation.

Solution: Since $\theta = 45$ degrees, we can plug θ directly into the given rotation matrix above.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

To find the resulting vector, we can either plot the rotated vector out on a 2D grid and use trigonometry to determine the new coordinates of the vector; or we can directly multiply $\vec{v}$ by the rotation matrix (as a transformation):

$$\vec{v_{rotated}} = R\vec{v} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

2. In this part, we will explore the relationship between a series of counter-clockwise rotations applied on a given vector and how the rotation matrix is represented correspondingly.

   Given that we have found the rotation matrix R for $\theta = 45^o$ in the previous part, now find the rotation matrix for $\theta = 90^o$, $\theta = 135^o$. At the same time, evaluate the matrix product R^2, R^3. What pattern did you see?

   Meta: For this question, since we are dealing with matrix multiplication with entries involving square roots, make sure to walk the students through a demo of the calculation. At the same time, encourage the students to think visually in terms of repeatedly rotating the vector on the 2D grid.

   Solution: Following the same steps in part (a), we can directly plug $\theta = 90^o$ and $\theta = 135^o$ into the given rotation matrix expression. Let the 90-degrees rotation matrix be R_2, and let the 135 degrees rotation matrix be R_3. We have:

$$R_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

   At the same time, if we evaluate R^2 and R^3, where R is the 45-degrees rotation matrix, we will find out that $R_2 = R^2$ and $R_3 = R^3$!

   What this really means, from an intuitive standpoint, is that we can represent the rotation transformation with the rotation matrix. What's more, a 90-degrees rotation can be replaced with two 45-degrees rotation, and a 135-degrees rotation can be replaced with three 45-degrees rotation, and so forth! The multiplication of the rotation matrix continuously applies the rotation transformation on the actual vector.

3. Generalizing from the previous part, if we are given a rotation matrix M that rotates a given vector $\vec{v}$ by k degrees. What would the resulting vector $\vec{v}'$ be if we rotate $\vec{v}$ by a total of $N \times k$ degrees? (N is a positive integer).

   Meta: To help students cope with the more generalized and symbolic notation, it might be helpful to begin again with numerical examples from the previous part and then extend to $\theta = 180^o$, $\theta = 225^o$, ... to show more concrete patterns.

   Solution: As we have seen from the previous part, every time a rotation is applied, we can equivalent multiply the original vector by the corresponding rotation matrix.

   In this case, since we are rotating by a total of $N \times k$ degrees, and we know of a rotation matrix that rotates a given vector by k degrees, we can view it as applying the k-degrees rotation a total of N times.

   Hence, this will lead us to the following expression for the resulting vector:

$$\vec{v}' = M^N\vec{v}$$

4. Backing off from counter-clockwise rotation for a bit, let's now explore **clockwise rotation** instead on a given vector. Given the **counter-clockwise rotation** matrix R_c we provided at the beginning of the question, consider the rotation matrix $R_{c'}$ for a **clockwise rotation** of θ degrees. What is the relationship between $R_{c'}$ and R_c?

   Meta: Encourage the students to think intuitively about this question. Direct them especially at the key point where clockwise rotation is a reverse process of the counter-clockwise rotation. Bonus points if the students also realize that a clockwise rotation by θ is simply a counter-clockwise rotation with $360n - \theta$ degrees (complements of each other)

   Solution: As we can see, clockwise rotation is counter-clockwise rotation but backwards! What that really means is that, if we apply a counter-clockwise rotation matrix to a given vector, we can apply a clockwise rotation matrix to **reverse** that process.

   With that being said, we have learned in class that a transformation matrix that can **undo** a previous transformation T is its inverse T^{-1}.

   Hence, $R_{c'}$ and R_c are inverses of each other!

5. Now that we have learned about the intrinsic connections between rotation matrix multiplication/inverse and the geometric transformation, in a few sentences, explain why the multiplication of rotation matrices is commutative. i.e.: Explain why given two rotation matrices A and B (A and B are both $N \times N$) , $AB = BA$.

   Meta: Encourage the students to think in terms of the overall resulting vector and contrast that with the order of our rotation.

   It is **EXTREMELY CRUCIAL** to emphasize to the students that the reason we can conclude $AB = BA$ in the end is because $\vec{v}$ can be any vector from $\mathbb{R}^2$.

   The implication (out of scope for this class) is because we can see that

   $$Nul(AB - BA) = \mathbb{R}^2,$$

   and by the fundamental theorem of linear algebra,

   $$Col(AB - BA) \oplus Nul(AB - BA) = \mathbb{R}^2,$$

   this tells us that

   $$Col(AB - BA) = \{\vec{0}\}.$$

   The justification above is **NON-TRIVIAL**, and you cannot conclude $AB - BA = 0$ without having $\vec{v}$ being any vector from $\mathbb{R}^2$!!!

   Solution: Geometrically, since we are rotating the vector by the sum of all the given degrees from the rotation matrices. It doesn't matter how many degrees we rotate the vector first during the process.

   Hence, we can conclude that for any given vector $\vec{v}$:

   $$AB\vec{v} = BA\vec{v}$$

   $$(AB - BA)\vec{v} = 0$$

   Since $\vec{v}$ could be any vectors in $\mathbb{R}^2$, we know that it must be true that $AB - BA = 0$, and therefore $AB = BA$.

6. Finally, for one additional nice property that results from rotation matrix multiplication, we know that if we rotate a vector each time by 30 degrees for a total of 12 times, eventually it will be a total of 360 degrees rotation, which puts the vector right back to where it was originally! Utilizing this fact, find a 2×2 matrix M such that $M^7 = I$, where I is the identity matrix.

   Meta: Encourage the students to directly associate matrix multiplication with rotation transformation. Also, utilize the fact that a rotation of a multiple of 360 degrees is equivalent to not rotating at all, which returns the **same and original** vector.

Solution: We can treat the identity matrix as geometrically not touching a given vector at all (just leaving it as it is). Since we know that a 360 degrees rotation will reset the vector back to itself, we know that it is also equivalent to applying the identity matrix to that vector. Given this information, we can now reword our question as follows:

Given a rotation matrix M that rotates a vector $\vec{v}$ by a total of 7 times (this comes from M^7), each time by x degrees, what should x be such that $7x$ is equal to some multiples of 360?

Now, this becomes a much simpler algebra question. For ease of computation, we can pick 360 as one of the multiples. This will give us the equation:

$$7x = 360^\circ$$

Solving for x, we have $x = 360/7°$. Hence, we know that if we apply a rotation matrix that rotates a vector by 360/7 degrees for a total of 7 times, it will be equivalent to leaving the vector as it is!

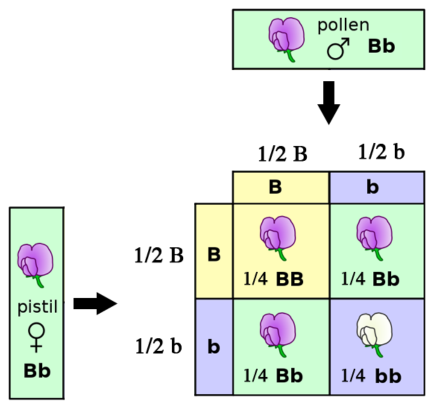
Hence, $M =$ the rotation matrix with $\theta = 360/7^\circ$.

Problem 5: (Challenging Exam-level Question) Gen(e) Z

Living things like you and me inherit from our parents many of their physical characteristics. In the study of population genetics, there are several types of inheritance; one of them is the **autosomal type**, where each heritable trait is assumed to be governed by a single gene.

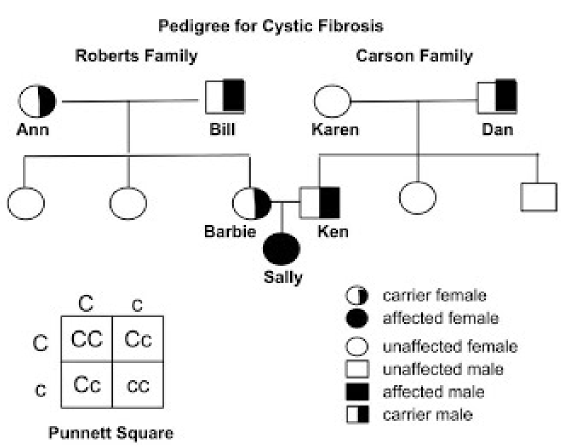
Typically, there are two different forms of genes denoted by A and a. Each individual in a population carries a pair of genes; the pairs are called the individual's genotype. This gives three possible genotypes for each inheritable trait: AA, Aa, and aa (aA is genetically the same as Aa, or in other words, the order of the genes in the genotypes doesn't matter).

As some of you might have recalled from a high school biology class, the Mendel's experiment is one of the earliest genetics studies that explores the possible genotypes and variation in the inheritable traits from crossing different individuals in the population:



Mendel's Experiment using Punnet Square

As you can see in the figure above, each cell in the square represents the chance that you will get a specific genotype for the flower after crossing. **The reason we care to calculate such chances** is because among the majority purple flowers, you can find a white flower (which fully manifests a recessive trait). Unfortunately, recessive traits will sometimes show in the form of disorders or diseases. Here's an example of how studying the likelihood of genotypes on the genotype that can cause *Cystic Fibrosis* (a very serious neural degenerative disease).



Now that you have some background in how popular genetics works, let's dive back to this problem! Suppose we have just discovered a new population of animals on a hypothetical Planet 16A, and our biologist-in-residence Kevin has found that an autosomal model of inheritance controls eye coloration (what colors the eyes have). Here is what Kevin has found:

- Genotypes AA and Aa have brown eyes.
- Genotype aa has blue eyes.

Kevin believes that the A gene dominates the a gene, and he further classifies an animal as **dominant** if it carries AA genes, **hybrid** if it carries Aa genes, and **recessive** if it carries aa genes. We can see that in this case, the dominant and hybrid genes are indistinguishable in appearance.

To further investigate how the distribution of the eye-color genes of this animal change over time, as a leading engineer on the research team, you are tasked with simulating the distribution of genotypes over multiple generations for this animal.

Note: for all of the following parts, we assume that each offspring inherits one gene from each parent in a completely random manner.

1. Given the genotypes of the parents, we can determine the distribution of the genotypes for the offspring. Suppose that in the original sample of 200 animals, 50 of them carry the **dominant** genes, 120 of them carry the **hybrid** genes, and the rest carries the **recessive** genes. We want to represent this distribution as a vector $\vec{v_p}^{(0)}$, where each entry $v_{p,i}$ in $\vec{v_p}^{(0)}$ represents the chance that a randomly selected animal from our sample population carries the genotype i. Find $\vec{v_p}^{(0)}$ (Entries should be in order of **dominant**, **hybrid**, and **recessive**)

   Meta: When explaining this problem, focus on how we can relate the process of choosing an animal randomly and discovering its genotype to the overall proportion of each genotype in the population. It might be a good idea to start with a much smaller number for the sample population so you can actually draw out each genotype in the sample population on the board.

   Solution: The chance that a randomly selected animal carries the genotype i can be represented by the proportion of the genotype i with respect to the total sample population.

$$\vec{v_p}^{(0)} = \begin{bmatrix} 50/200 \\ 120/200 \\ (200-50-120)/200 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/5 \\ 3/20 \end{bmatrix}$$

2. Now, you woud like to consider a series of simulated experiments where we continuously breed all animals in our sample population **only** with animals that carry a **dominant** genotype. Suppose after 1 round of breeding, the distribution of the genotypes in our population becomes $\vec{v_p}^{(1)}$. Find $\vec{v_p}^{(1)}$.

   Note: For ease of computation, for all later parts of this question, we will assume that the original distribution vector (for the genome)

$$\vec{v_p}^{(0)} = \begin{bmatrix} Pr(AA \text{ at } t=0) \\ Pr(Aa \text{ at } t=0) \\ Pr(aa \text{ at } t=0) \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

   Meta: For this question, make sure to explain how gene crossing works for the students: especially the part where an offspring inherits exactly one randomly chosen gene from both of its parents.

   Solution: Let us consider this 1 round of breeding in 3 different scenarios (based on what initial genotype the animal has).

   - AA with AA (dominant + dominant): Since the offspring will have one gene from each parent, it will be of type AA as well. Thus the probabilities of AA, Aa, and aa are 1, 0 and 0 respectively.
   - Aa with AA (hybrid + dominant): Taking one gene from each parent, we have the possibilities of AA, AA (taking A from the first parent and each A in turn from the second parent), aA, and aA (taking a from the first parent and each A in turn from the second parent). Thus the probabilities of AA, Aa, and aa, respectively, are 1/2, 1/2 and 0.

- aa with AA (recessive + dominant): There is only one possibility, namely aA. Thus the probabilities of AA, Aa, and aa are 0, 1 and 0 respectively.

For those more comfortable with probability notations, we can rephrase the calculations above as the following:

For each of the genotypes (AA, Aa, aa), the respective probability of us getting a particular genotype T (i.e. T could potentially be AA, Aa, or aa) by crossing with a **dominant** genotype is:

$$Pr(AA \text{ CROSS } AA \Rightarrow T) = \frac{Pr(Ti \in \{A, A\} \times \{A, A\})}{4}$$
$$Pr(Aa \text{ CROSS } AA \Rightarrow T) = \frac{Pr(Ti \in \{A, a\} \times \{A, A\})}{4}$$
$$Pr(aa \text{ CROSS } AA \Rightarrow T) = \frac{Pr(T \in \{a, a\} \times \{A, A\})}{4}$$

Here, $\times$ represents the **cross product** between 2 sets of elements: it creates a set containing all possible pairwise combinations of the elements from both sets.

Now that we know the new distribution of genotypes given any one of the initial genotypes (**dominant**, **hybrid**, **recessive**), we can calculate the new overall distributions for the genotypes:

- **Dominant**: $1(1/3) + 1/2(1/3) + 0(1/3) = 1/2$
- **Hybrid**: $0(1/3) + 1/2(1/3) + 1(1/3) = 1/2$
- **Recessive**: $0(1/3) + 0(1/3) + 0(1/3) = 0$

Hence, we know that:

$$\vec{v_p}^{(1)} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

3. Now that you have completed one round of breeding with the **dominant** genotype, you are eager to continue more rounds of simulation. However, before going about doing this, you would like to know if you can re-represent this one round of breeding in a more concise and matrix-oriented way. In other words, you would like to see if there exists a matrix A such that it can predict what the $(T + 1)$st (next) round's distribution of genotypes will be given the Tth (current) round. Mathematically, we can represent this as the equation below:

$$\vec{v_p}^{(T+1)} = A\vec{v_p}^{(T)}$$

Does A exist? If so, find A; if not, explain why.

Meta: As a great place to start, encourage the students to look through the previous part when we calculated the distribution of genotypes after the first round of breeding. Specifically, motivate the students to look at how the new distribution is calculated individually for each genotype (as a linear combination)

Solution: Based on the previous part, we can observe how we calculate the distribution of genotypes after the first round of breeding. Specifically, given:

- **Dominant**: $1(1/3) + 1/2(1/3) + 0(1/3) = 1/2$
- **Hybrid**: $0(1/3) + 1/2(1/3) + 1(1/3) = 1/2$
- **Recessive**: $0(1/3) + 0(1/3) + 0(1/3) = 0$

We can interpret the new distribution of each genotype as follows (here, we use **dominant** genotype as an example): The new distribution of the dominant genotype is equal to the sum of all the followings:

(a) $Pr(AA^{(T+1)} \mid AA^{(T)})P(AA^{(T)})$
The probability of a dominant genotype given that the original genotype is also **dominant** $\times$ the original distribution of the dominant genotype

(b) $Pr(AA^{(T+1)} \mid Aa^{(T)})P(Aa^{(T)})$
The probability of a dominant genotype given that the original genotype is **hybrid** $\times$ the original distribution of the dominant genotype

(c) $Pr(AA^{(T+1)} \mid aa^{(T)})P(aa^{(T)})$
The probability of a dominant genotype given that the original genotype is **recessive** $\times$ the original distribution of the dominant genotype

For those who are interested and familiar with some probability theory, this is **no more than an application of Bayes' Theorem on conditional probabilities**!

$$
\begin{aligned}
&Pr(AA^{(T+1)}) \\
&\quad= \sum_{trait^{(T)} \in \{AA^{(T)}, Aa^{(T)}, aa^{(T)}\}} Pr\left[AA^{(T+1)},\, trait^{(T)}\right] \\
&\quad= \sum_{trait^{(T)} \in \{AA^{(T)}, Aa^{(T)}, aa^{(T)}\}} Pr\left[AA^{(T+1)} \mid trait^{(T)}\right] Pr\left[trait^{(T)}\right] \\
&\quad= Pr\left[AA^{(T+1)} | AA^{(T)}\right] P\left[AA^{(T)}\right] + Pr\left[AA^{(T+1)} | Aa^{(T)}\right] P\left[Aa^{(T)}\right] + Pr\left[AA^{(T+1)} | aa^{(T)}\right] P\left[aa^{(T)}\right]
\end{aligned}
$$

From a matrix-vector multiplication standpoint, we can see that we are actually computing the linear combination of all the respective chances of acquiring a dominant genotype given different initial genotypes with respect to the original genotype (as in the first entry of our distribution vector $\vec{v_p}^{(0)}$).

Applying the same observations for other vectors, we can deduce our matrix A from the following matrix-vector decomposition:

$$
\vec{v_p}^{(1)} = \begin{bmatrix} 1(1/3) + 1/2(1/3) + 0(1/3) \\ 0(1/3) + 1/2(1/3) + 1(1/3) \\ 0(1/3) + 0(1/3) + 0(1/3) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\begin{bmatrix}1\\0\\0\end{bmatrix} & \frac{1}{3}\begin{bmatrix}1/2\\1/2\\0\end{bmatrix} & \frac{1}{3}\begin{bmatrix}0\\1\\0\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = A\vec{v_p}^{(0)}
$$

Therefore, we can see that:

$$
A = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

4. **Without explicitly solving for the inverse or row reducing the matrix**, determine if the transformation matrix A is invertible or not. Provide your explanations in a few sentences.

Meta: Encourage the students to think in the context of the question, specifically on the previous example of how one round of breeding is done and what happens to the distribution of each genotype. It might be helpful to point out to students that the distribution of the receissive traits comes from the bottom row of A.

Solution: A is not invertible. We can tell this by looking at specifically the breeding of an initially **recessive** genotype (aa) with the **dominant** (AA) genotype. Over 1 round of breeding, the new distribution of aa has dropped to 0, meaning that it no longer exists in our distribution. In a geometric way, we can interpret this as we have "collapsed" one dimension of the distribution vector. From an invertibility standpoint, given the current distribution of genomes, where there are 0 recessive genotypes, it is **impossible** for us to actually tell what the distribution for the recessive genotypes from the previous round will be (since everything is dropped to 0 regardless). Since we cannot determine what the previous distribution is given the current one, we cannot find the inverse of A, and thereby, A is not invertible.