CSM: EECS 16A **(Designing Information Devices and Systems I)**

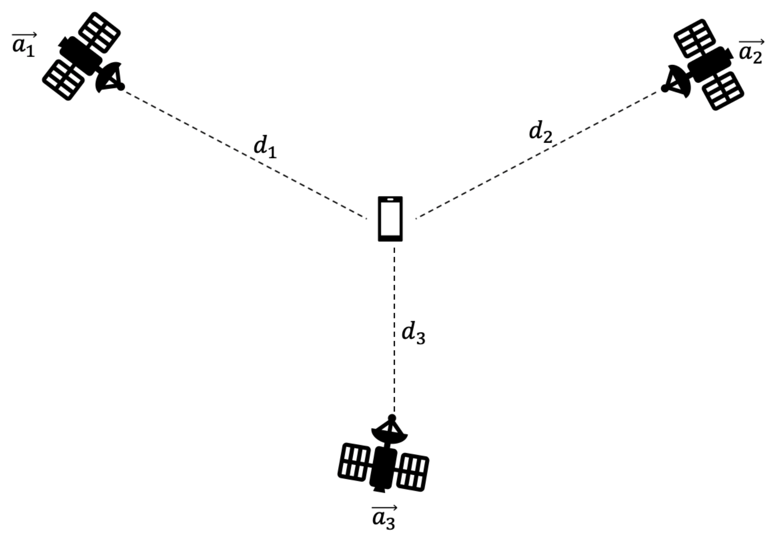
Worksheet #12

Term: **Fall 2019** *Name:*

Problem 1: GPS - Global Positioning System

Learning Goal: Understanding how trilateration works and how it is derived, working with norms and inner products, and applying cross correlation to find delays in received signals.

Suppose that you are the engineer tasked with the job of making Google Maps. For this, you want to be able to determine the position of a user using satellite information. In particular, assume that you know d_1, d_2 and d_3, the distances from the user's cellphone to 3 satellites. You know the positions of these satellites to be $\vec{a_1}, \vec{a_2}$, and $\vec{a_3}$. Here's a simplified figure demonstrating what's been given so far:



Note: What does it mean when we say "position"? You can assume that these positions are taken relative to some common origin. Say, the Google HQ - Mountain View, CA.

1. Suppose the user's location (or the phone's location) is given by the vector $\vec{x}$, write out a system of equations representing the distances from the user to all 3 satellites (Express your answer in terms of $\vec{x}, \vec{a_1}, \vec{a_2}, \vec{a_3}, d_1, d_2$, and d_3).

2. Rewrite these equations in terms of inner products of vectors. Are these equations linear with respect to $\vec{x}$?

3. Are there any non-linear terms in the equations from the previous part? Using **elimination of variables**, rewrite everything as a system of **linear** equations.

4. **Using the system of linear equations we have from the previous part**, if the location of the user (i.e. $\vec{x}$ is a 3-dimensional vector), do we have sufficient information to solve for $\vec{x}$? If not, then how many satellites do you need to locate the user?

5. Suppose now in more generalized terms, we want to not only triangulate the user's position, but also keep track of other information about the user to make more customized analysis. Given that the vector representing

the user location now contains a total of n entries, what is the minimum number of satellites we need to find that vector?

> In real life, we won't actually be given the distances from the user to the satellites, either. In other words, we also need to figure out how far away the satellites are from us! Fortunately, as we have already learned in class, **cross correlation** is something that might come in handy for us to figure out the distances. For all the remaining parts of this question, we will use what we have learned about **cross correlation** to figure out what the distances from the user to the satellites are.

6. To figure out how far away the satellites are from us, we can use our phone to receive radio signals from the satellites in the orbit. Once we have received the signals, we can then compare them with a reference signal on our phone to figure out the time it takes for the signal to reach us. Given our original reference signal:

$$\vec{s} = \begin{bmatrix} -1 & -1 & -1 & 1 & -1 \end{bmatrix}^T,$$

   and the three signals we received, each having a period of 4 (we will only show one period of each signal):

$$\vec{r}_1 = \begin{bmatrix} -1 & -1 & -1 & 1 \end{bmatrix}^T$$

$$\vec{r}_2 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T$$

$$\vec{r}_3 = \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix}^T$$

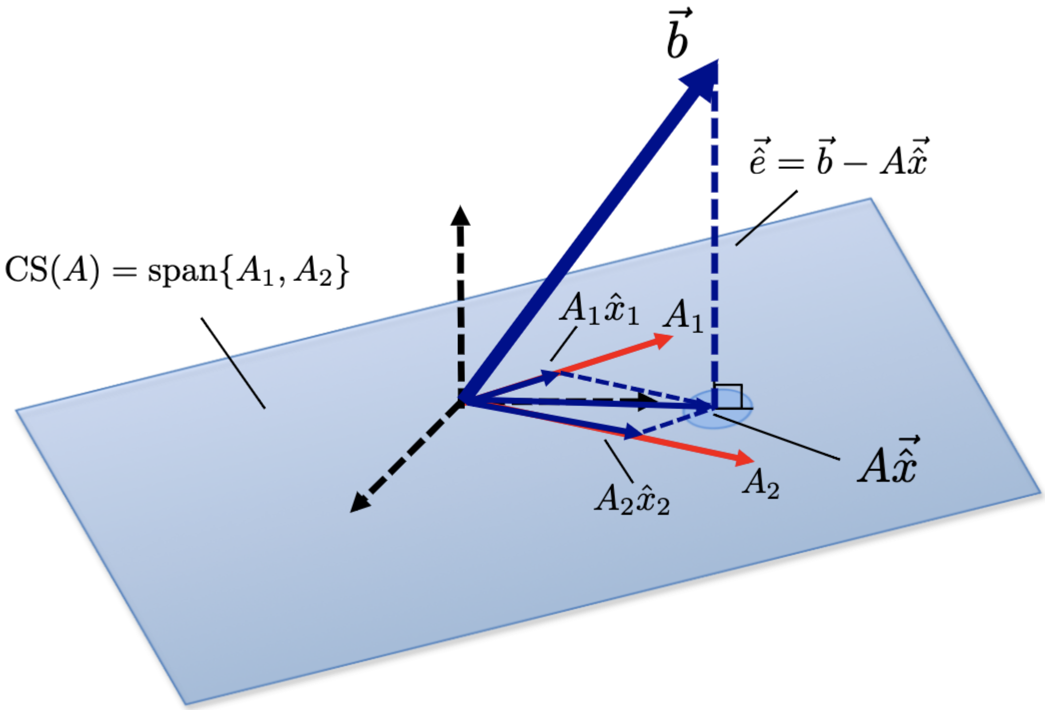
   Find the cross correlations $\text{corr}_{\vec{r}_1}(\vec{s})$, $\text{corr}_{\vec{r}_2}(\vec{s})$, and $\text{corr}_{\vec{r}_3}(\vec{s})$ between $\vec{s}$ and all three received signals respectively, and plot them out below.

7. Based on the cross-correlated signals, determine the delays (in seconds) for all 3 received signals $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

8. Given that the radio signal has a transmission speed of v, and assume all delays are relative to the source signal $\vec{s}$ (this means we assume $\vec{s}$ is received at time $t = 0$), find the distance d_1, d_2, and d_3 between the user location and the 3 satellites in orbit.

Problem 2: Least Squares - Geometric Intuitions

Learning Goal: Understanding of the mechanics and interpretations of squared norms, describing and deriving the least squares problem and solution, first geometrically, and then using calculus.



1. Consider that you have some equations of the form $\mathbf{A}\vec{x} = \vec{b}$, however, that there is no solution $\vec{x}$ that solves the equations. What does this tell us about $\vec{b}$ with respect to $\mathbf{A}_1, \mathbf{A}_2$ (the columns of $\mathbf{A}$)?

2. We know that there is no $\vec{x}$ that satisfies the equations exactly, but we still want to solve the equations to get a solution as close as possible.

   Let's say you had 3 choices, $\vec{x}_i, \vec{x}_j, \vec{x}_k$ (these are not drawn on the image). What could you compute in order to determine which of these would be the best choice instead of $\vec{x}$

3. Suppose that the real $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and you have two close $\vec{x}_1, \vec{x}_2$, which result in possible $\vec{b}_1 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$ and another $\vec{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. How do we know which one is *closer*? What if we define *closer* to mean the sum of the components of the difference of the vectors.

4. What is a different approach to solve the issue discussed above?

5. Using this definition, how close is $\vec{b}_1$ to $\vec{b}$? How close is $\vec{b}_2$ to $\vec{b}$?

6. More generally, we actually don't have a choice of just two or three $\vec{x}$s to pick to get as close to $\vec{b}$ as possible. We have an infinite number of choices. How can we tell which one is the best? Look at the image given at the top of the question, and decide something about $\vec{\hat{x}}$ and $\vec{\hat{e}}$.

7. Let's begin by recalling three facts. Recall that if we want to minimize some quantity squared, it is enough to minimize the quantity itself. Also, recall that given a point and a plane, the shortest line that one can possibly get starting at the point and ending at the plane, is a perpendicular line from the point to the plane.

Finally, recall that if a vector $\vec{v}$ is orthogonal to another vector u, their inner product $\langle \vec{v}, \vec{u} \rangle = \vec{v}^T \vec{u} = 0$. Using this information, come up with a method to minimize the norm-squared of $\vec{\hat{e}}$.