Linking Resource Quantifiers & Operational Tasks in Quantum Info$^{\underline{n}}$ Theory

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joint work with: Noah Linden & Andres Ducuara

Based on PS & N. Linden Phys. Rev. Lett. 122 140403 (2019)

A. F. Ducuara & PS Phys. Rev. Lett. 125 110401 (2020)

University of BRISTOL

THE ROYAL SOCIETY

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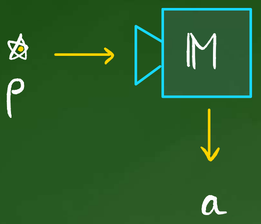
Outline

1. Resource theory of measurement informativeness
2. Quantifiers of measurement informativeness
3. Operational significance of quantifiers
4. Complete sets of monotones for measurement simulation
5. More general resource theories?
6. Conclusions

Resource theory of measurement informativeness

Informativeness of a measurement



$$\mathbb{M} = \{M_a\}$$

$$a = 1, \dots, o$$

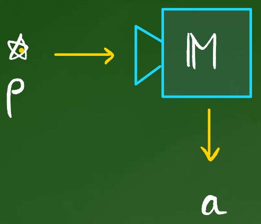
$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

Q: How informative is a measurement?

→ take a resource-theory approach

- Resource : 'informative' measurement
- Free : 'uninformative' measurement
- Operations: 'measurement simulation'

Informativeness of a measurement



$$\mathbb{M} = \{M_a\}$$

$$a = 1, \dots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

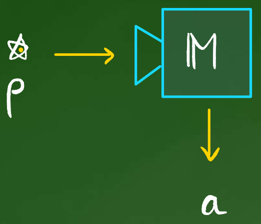
Free object

- Uninformative measurement
  $\mathbb{C} = \{C_a\}_a$
  such that $C_a = q(a)\,\mathbb{1}$

  $q = \{q(a)\}_a$ probabilities.

- For all ρ, $p(a|\rho) = q(a)$
  independent of ρ
  i.e. no measurement performed.

Informativeness of a measurement



$$\mathbb{M} = \{M_a\}$$

$$a = 1, \ldots, O$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

Resourceful object

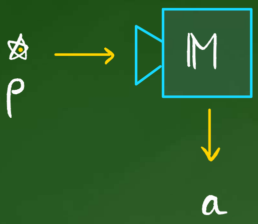
- Any measurement $\mathbb{M}$ that is not completely uninformative.

$$M_a \neq q(a)\mathbb{1} \quad \text{for some } a.$$

Q: How to quantify how informative?

Come back to this!

Informativeness of a measurement



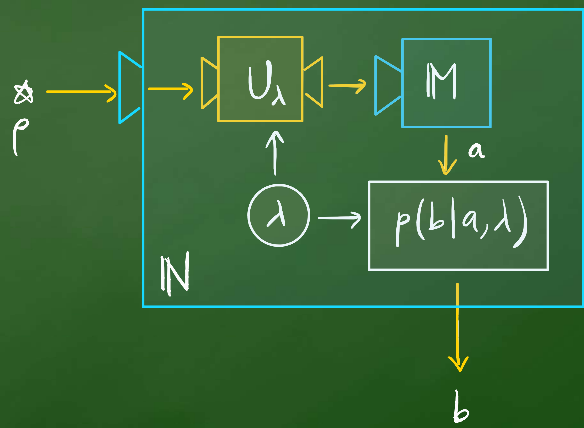
$$\mathbb{M} = \{M_a\}$$

$$a = 1, \dots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

Allowed operations

- Transform one measurement into another: measurement simulation.

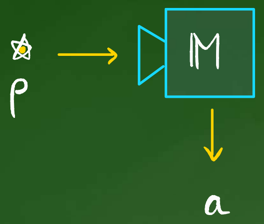


$$N_b = \sum_{a,\lambda} p(\lambda)\, p(b|a,\lambda)\, U_\lambda^\dagger M_a U_\lambda$$

write $\mathbb{M} \succ \mathbb{N}$

Quantifiers of Informativeness

Quantifying measurement informativeness



$\mathbb{M} = \{M_a\}$

$a = 1, \ldots, o$

$p(a|\rho) = \mathrm{tr}(M_a \rho)$

Q: How to quantify the informativeness of one measurement compared to another?

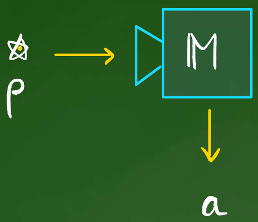
(i) How much noise can be added before informativeness is ruined?

Robustness

(ii) How often does an informative measurement need to be used to reproduce measurement?

Weight

Robustness of Measurement



$\mathbb{M} = \{M_a\}$

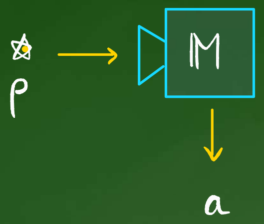
$a = 1, \ldots, o$

$p(a|\rho) = \mathrm{tr}(M_a \rho)$

Instead of performing $\mathbb{M}$ all time, with prob. p perform $\mathbb{M}$ & prob $(1-p)$ perform $\mathbb{N}$. 'noise measurement'

- Robustness of Measurement (informativeness) $R(\mathbb{M})$ quantifies minimal amount of noise that makes measurement uniformative using the worst case noise.

Robustness of Measurement



$$R(\mathbb{M}) = \min_{r,\mathbb{N},q} r$$

$$\text{s.t.} \quad \frac{M_a + r N_a}{1+r} = q(a)\mathbb{1} \quad \forall a$$

$p = \frac{1}{1+r}$ for mathematical reasons.

$$\mathbb{M} = \{M_a\}$$

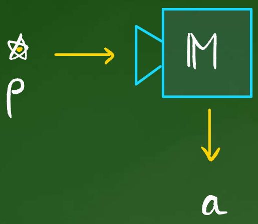
$$a = 1, \ldots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

- If $\mathbb{M}$ is uniformative $M_a = q(a)\mathbb{1}$ & $R(\mathbb{M}) = 0$

- If $\mathbb{M}$ is 'highly informative' then must use lots of noise $\mathbb{N}$ in order to bring it to an uninformative measurement.

Robustness of Measurement



$$R(\mathbb{M}) = \min_{r,\mathbb{N},q} r$$

$$\text{s.t.} \quad \frac{M_a + r N_a}{1+r} = q(a)\mathbb{1} \quad \forall a$$

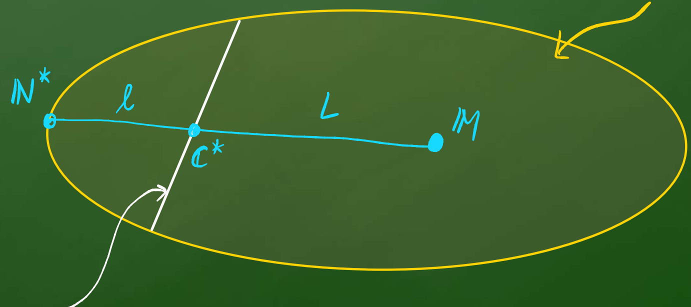
$$\mathbb{M} = \{M_a\}$$

$$a = 1, \dots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

Geometrically:

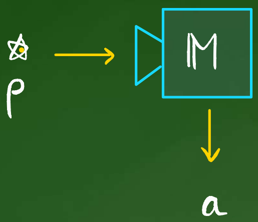
Space of all measurements $\mathbb{M}$



$$\frac{L}{\ell} = R(\mathbb{M})$$

Subset of uninformative measurements $\mathbb{C}$

Weight of Measurement



$$\mathbb{M} = \{M_a\}$$

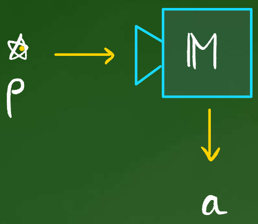
$$a = 1, \dots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

Instead of performing $\mathbb{M}$, with prob. p peform informative measurement $\mathbb{N}$ and with prob. $(1-p)$ peform uninformative measurement $\mathbb{C}$.

- Weight of Measurement (informativeness) $W(\mathbb{M})$ quantifies the minimal amount of resourceful (informative) measurement that must be used in best case

Weight of Measurement



$$W(\mathbb{M}) = \min_{p,\, \mathbb{N},\, q} \; p$$

$$\text{s.t.} \quad p N_a + (1-p)\, q(a)\, \mathbb{1} = M_a \quad \forall a$$

$$\mathbb{M} = \{ M_a \}$$

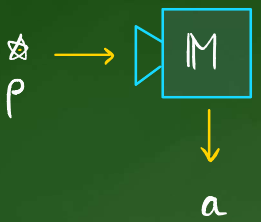
$$a = 1, \dots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

- If $\mathbb{M}$ is uninformative $\quad M_a = q(a)\mathbb{1}$
  & $\quad W(\mathbb{M}) = 0$

- If $\mathbb{M}$ is 'highly informative' then must use lots of resource $\mathbb{N}$ in order to reproduce measurement

Weight of Measurement



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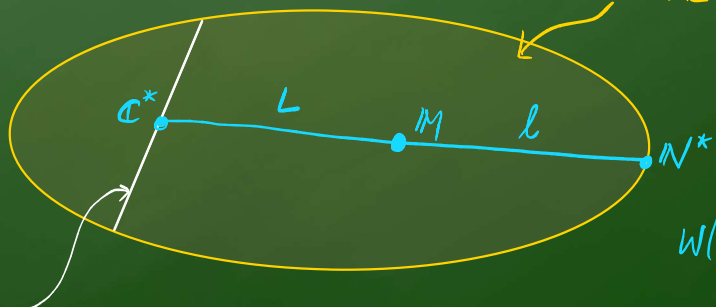
$$\mathbb{M} = \{M_a\}$$

$$a = 1, \dots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

Geometrically:

Space of all measurements $\mathbb{M}$



$$W(\mathbb{M}) = \frac{L}{\ell + L}$$

Subset of uninformative measurements $\mathbb{C}$

Properties of Robustness & Weight

$$R(\mathbb{M}) = \min_{r,\mathbb{N},q} r \quad \text{s.t.} \quad \frac{M_a + r N_a}{1+r} = q(a)\mathbb{1} \;\; \forall a$$

$$W(\mathbb{M}) = \min_{p,\mathbb{N},q} r \quad \text{s.t.} \quad p N_a + (1-p)\, q(a)\mathbb{1} = M_a \;\; \forall a$$

Both quantifiers satisfy:

1. Faithfulness: $R(\mathbb{M}) = W(\mathbb{M}) = 0$ iff $\mathbb{M}$ is uninformative

2. Convexity:
$$R(p\mathbb{M}_1 + (1-p)\mathbb{M}_2) \leq pR(\mathbb{M}_1) + (1-p)R(\mathbb{M}_2)$$
$$W(p\mathbb{M}_1 + (1-p)\mathbb{M}_2) \leq pW(\mathbb{M}_1) + (1-p)W(\mathbb{M}_2)$$

3. Non-increasing under simulation: $\mathbb{M} \succ \mathbb{N} \implies$
$$R(\mathbb{M}) \geq R(\mathbb{N})$$
$$W(\mathbb{M}) \geq W(\mathbb{N})$$

SDP formulations of Robustness & Weight

$$R(\mathbb{M}) = \min_{r, \mathbb{N}, q} \; r \quad \text{s.t.} \quad \frac{M_a + r N_a}{1 + r} = q(a)\mathbb{1} \;\; \forall a$$

$$W(\mathbb{M}) = \min_{p, \mathbb{N}, q} \; r \quad \text{s.t.} \quad p N_a + (1-p)\, q(a)\mathbb{1} = M_a \;\; \forall a$$

Both quantifiers can be recast as semidefinite programs (SDPs)

$$1 + R(\mathbb{M}) = \min_{\tilde{q}} \sum_a \tilde{q}(a) \quad \text{s.t.} \quad \tilde{q}(a)\mathbb{1} \geq M_a \;\; \forall a$$

$$1 - W(\mathbb{M}) = \max_{\tilde{s}} \sum_a \tilde{s}(a) \quad \text{s.t.} \quad \tilde{s}(a)\mathbb{1} \leq M_a \;\; \forall a$$

- Convex optimisation problems

  → "easy" to solve in practice

  → duality theory

Explicit formulations of Robustness & Weight

$$1 + R(\mathbb{M}) = \min_{\tilde{q}} \sum_a \tilde{q}(a) \quad \text{s.t.} \quad \tilde{q}(a)\mathbb{1} \geq M_a \ \forall a$$

$$1 - W(\mathbb{M}) = \max_{\tilde{s}} \sum_a \tilde{s}(a) \quad \text{s.t.} \quad \tilde{s}(a)\mathbb{1} \leq M_a \ \forall a$$

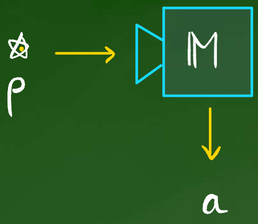
- SDP forms make explicit solutions manifest:

$$1 + R(\mathbb{M}) = \sum_a \underbrace{\| M_a \|_\infty}_{= \lambda_{max}(M_a)}$$

$$1 - W(\mathbb{M}) = \sum_a \underbrace{\| M_a \|_{-\infty}}_{= \lambda_{min}(M_a)}$$

- Not obvious from original definitions just functions of largest & smallest eigenvalues respectively.

Bounds on Robustness & Weight



$$0 \leqslant R(\mathbb{M}) \leqslant \min(o, d) - 1 \leqslant d - 1$$

dimension

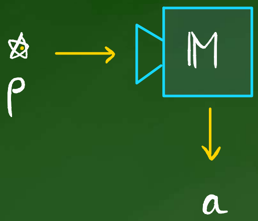
$$0 \leqslant W(\mathbb{M}) \leqslant 1$$

$$\mathbb{M} = \{M_a\}$$

$$a = 1, \dots, o$$

$$p(a|\rho) = \mathrm{tr}(M_a \rho)$$

Maximally resourceful measurements



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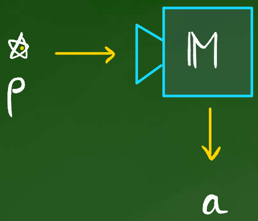
- ideal rank-1 projective measurements

$$M_a = |\phi_a\rangle\langle\phi_a| \quad \text{for } a = 1, \dots, d$$

$$\|M_a\|_\infty = 1 \qquad \|M_a\|_{-\infty} = 0$$

$$R(\mathbb{M}) = d - 1 \qquad W(\mathbb{M}) = 1 \quad \checkmark$$

Maximally resourceful measurements



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$$\|M_a\|_\infty = 1 \qquad \|M_a\|_{-\infty} = 0$$

$$R(\mathbb{M}) = d-1 \qquad W(\mathbb{M}) = 1 \qquad \checkmark$$

- rank-1 measurements

$$M_a = \alpha_a |\psi_a\rangle\langle\psi_a| \quad \text{for } a = 1, \dots, o \qquad o > d$$

$$\sum_a M_a = \mathbb{1} \rightarrow \sum_a \alpha_a = d$$

$$R(\mathbb{M}) = d-1 \qquad W(\mathbb{M}) = 1 \qquad \checkmark$$

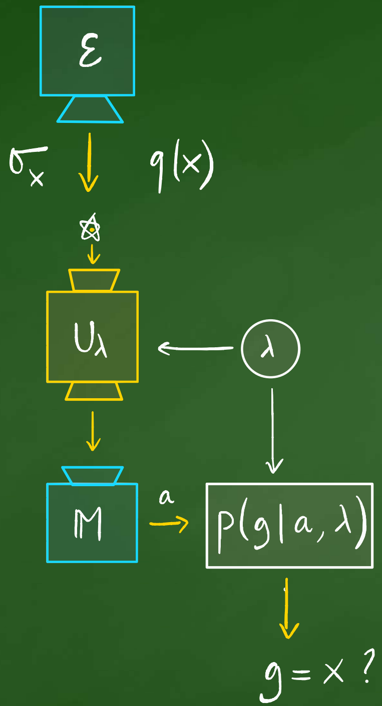
- rank-$(d-1)$ measurements $\quad M_a = \beta_a \Pi_a \quad \mathrm{tr}\,\Pi_a = d-1$

$$R(\mathbb{M}) = \frac{1}{d-1} \qquad W(\mathbb{M}) = 1 \quad \text{boundary}$$

Operational Significance

of Quantifiers

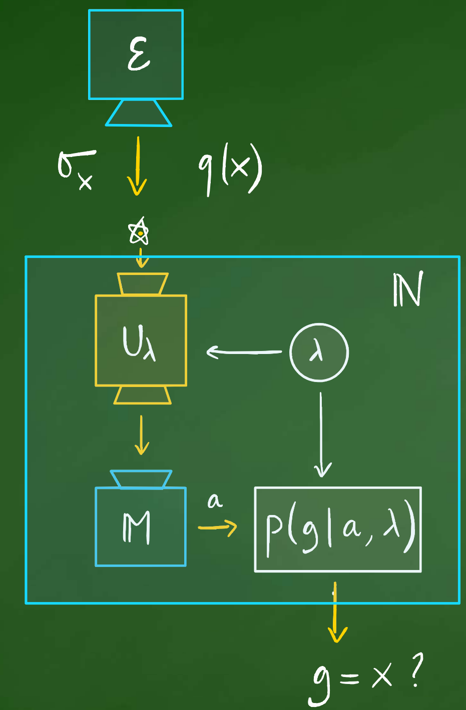
Quantum State Discrimination



- state σ_x is produced with prob. $q(x)$
- Goal: correctly identify state σ_x sent
- Consider a fixed measurement $\mathbb{M}$ and look for best strategy
- $$P^{Q}_{guess}(\mathcal{E}, \mathbb{M}) = \max_{\substack{q(x),\, p(g|a,\lambda) \\ U_\lambda}} \sum_{x,g} q(x) \underbrace{p(g|x)\, \delta_{g,x}}_{p(g=x|x)}$$

$$p(g|x) = \sum_{\lambda} p(\lambda)\, tr\left(U_\lambda \sigma_x U_\lambda^{\dagger} M_a\right) p(g|a,\lambda)$$

Quantum State Discrimination



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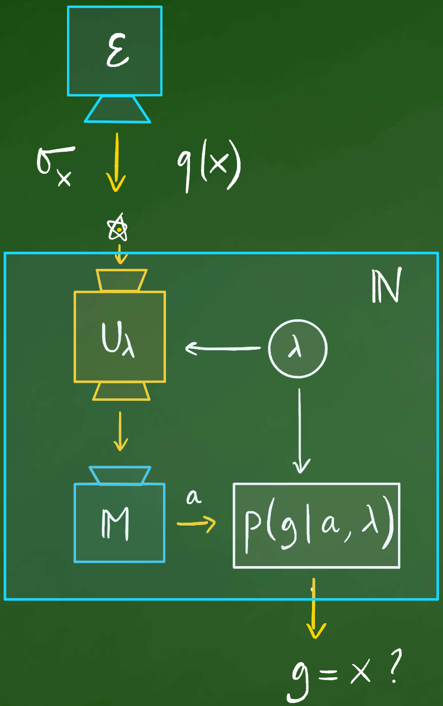
$$p(g|x) = tr(\sigma_x N_g)$$

Quantum State Discrimination



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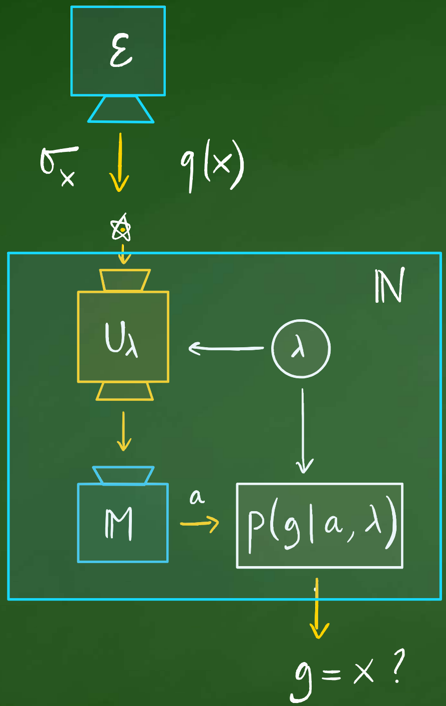
$$p(g|x) = \mathrm{tr}(\sigma_x N_g)$$

- Compare to uninformative measurement
  $=$ no ability to measure q. state

$$P^{c}_{guess}(\mathcal{E}) = \max_{x} q(x)$$

i.e. always guess most likely state.

Operational Significance of Robustness of Measurement



$$\max_{\mathcal{E}} \frac{P^{Q}_{guess}(\mathcal{E}, \mathbb{M})}{P^{c}_{guess}(\mathcal{E})} = 1 + R(\mathbb{M})$$

- Robustness of Measurement determines the biggest advantage that can be gained by using $\mathbb{M}$ in a QSD game.

Proof

(i) upper bound from SDP for $1 + R(\mathbb{M})$ (easy)

(ii) extract optimal QSD game $\mathcal{E}^*$ from dual SDP

Dual SDP for Robustness & Weight

- Every SDP has a dual formulation in terms of dual (lagrange) variables
- Almost always the dual is equivalent to the primal (strong duality)

$$1 + R(\mathbb{M}) = \max_{\rho_a} \sum_a \mathrm{tr}(M_a \rho_a) \quad \text{s.t.} \quad \rho_a \geq 0 \;\; \forall a, \quad \mathrm{tr}\, \rho_a = 1 \;\; \forall a$$

$$1 - W(\mathbb{M}) = \min_{\omega_a} \sum_a \mathrm{tr}(M_a \omega_a) \quad \text{s.t.} \quad \omega_a \geq 0 \;\; \forall a, \quad \mathrm{tr}\, \omega_a = 1 \;\; \forall a$$

- Both problems satisfy strong duality

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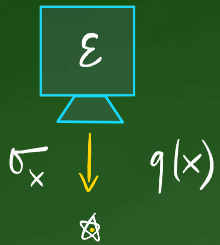
- Both problems satisfy strong duality
- optimal Dual variables ρ_a^* naturally define QSD game: $\mathcal{E}^* = \left\{ \rho_x^*, \frac{1}{O} \right\}$

$p(x)$ i.e. uniform

Quantum State Exclusion



- state σ_x is produced with prob. $q(x)$
- Goal: correctly avoid state σ_x sent

  i.e. guess any $g \neq x$

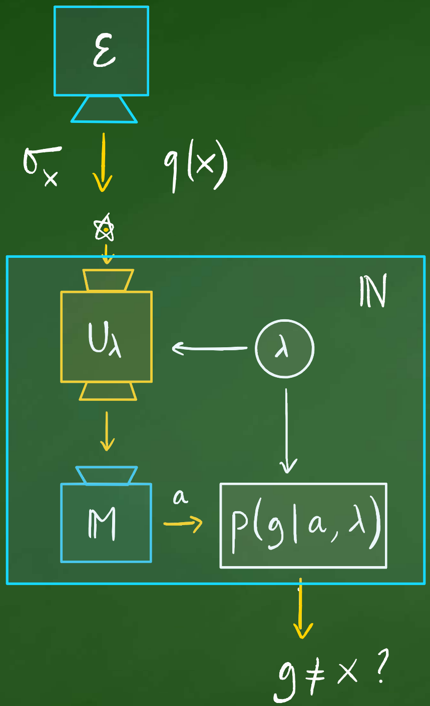
  Silly example

  - Bomb will explode if blue wire is cut, will be deactivated if any other wire is cut
  - σ_x encodes the wire not to be cut.

- Arises in context of PBR theorem

  ("quantum state cannot be interpreted statistically)

Quantum State Exclusion



- state σ_x is produced with prob. $q(x)$
- Goal: correctly avoid state σ_x sent

  i.e. guess any $g \neq x$

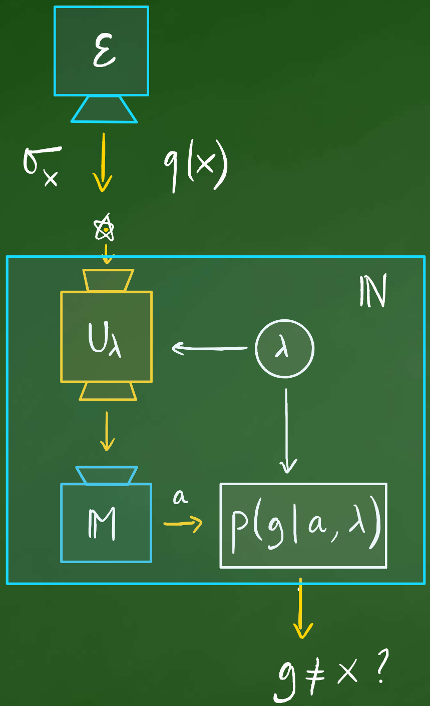
- $$P^{Q}_{err}(\mathcal{E}, M) = \min_{N \prec M} \sum_{x,g} q(x)\, p(g|x)\, \delta_{g,x}$$

  error prob. of incorrectly guessing actual state

- $$P^{c}_{err}(\mathcal{E}) = \min_{x} q(x)$$

  i.e. always guess least likely state

Quantum State Exclusion



$$\min_{\mathcal{E}} \frac{P^{Q}_{err}(\mathcal{E}, \mathbb{M})}{P^{c}_{err}(\mathcal{E})} = 1 - W(\mathbb{M})$$

- Weight of measurement quantifies biggest advantage that can be gained by using $\mathbb{M}$ in a QSE game.

Proof: (i) primal SDP $\rightarrow$ lower bound

Same (ii) Dual SDP $\rightarrow$ optimal QSE game $\mathcal{E}^*$

Examples

- Rank 1 measurements: $M_a = \alpha_a |\phi_a\rangle\langle\phi_a|$ $\quad a = 1, \dots, o \quad o \geq d$

$$1 + R(\mathbb{M}) = d \qquad\qquad 1 - W(\mathbb{M}) = 0$$

$$\mathcal{E}_D^* = \left\{ |\phi_a\rangle\langle\phi_a|, \frac{1}{o} \right\} \qquad\qquad \mathcal{E}_E^* = \left\{ \frac{\mathbb{1} - |\phi_a\rangle\langle\phi_a|}{d-1}, \frac{1}{o} \right\}$$

$$P^Q_{guess}(\mathcal{E}_D^*, \mathbb{M}) = \frac{d}{o} \qquad P^C_{guess}(\mathcal{E}_D^*) = \frac{1}{o} \qquad\qquad P^Q_{err}(\mathcal{E}_E^*, \mathbb{M}) = 0 \qquad P^C_{err}(\mathcal{E}_E^*) = \frac{1}{o}$$

$$\frac{P^Q_{guess}(\mathcal{E}_D^*, \mathbb{M})}{P^C_{guess}(\mathcal{E}_D^*)} = d \qquad\qquad \frac{P^Q_{err}(\mathcal{E}_E^*, \mathbb{M})}{P^C_{guess}(\mathcal{E}_E^*)} = 0$$

Examples

- Rank-(d-1) measurements: $M_a = \alpha_a \Pi_a \qquad a = 1, \dots, 0 \qquad \mathrm{tr}(\Pi_a) = d-1$

$$1 + R(\mathbb{M}) = \frac{d}{d-1} \qquad\qquad 1 - W(\mathbb{M}) = 0$$

$$\mathcal{E}_D^* = \left\{ \frac{\Pi_a}{d-1}, \frac{1}{0} \right\} \qquad\qquad \mathcal{E}_E^* = \left\{ \mathbb{1} - \Pi_a, \frac{1}{0} \right\}$$

$$P^Q_{guess}(\mathcal{E}_D^*, \mathbb{M}) = \frac{1}{0}\frac{d}{d-1} \quad P^C_{guess}(\mathcal{E}_D^*) = \frac{1}{0} \qquad\qquad P^Q_{err}(\mathcal{E}_E^*, \mathbb{M}) = 0 \quad P^C_{err}(\mathcal{E}_E^*) = \frac{1}{0}$$

$$\frac{P^Q_{guess}(\mathcal{E}_D^*, \mathbb{M})}{P^C_{guess}(\mathcal{E}_D^*)} = \frac{d}{d-1} \qquad\qquad \frac{P^Q_{err}(\mathcal{E}_E^*, \mathbb{M})}{P^C_{guess}(\mathcal{E}_E^*)} = 0$$

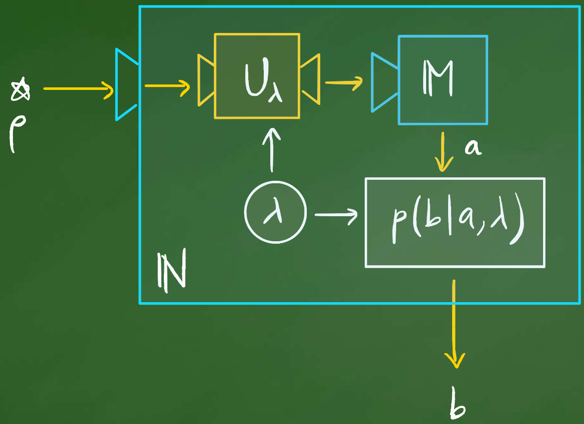
Complete sets of

monotones for measurement

Simulation

Measurement Simulation

- Allowed transformations in resource theory of meas. informativeness



$$N_b = \sum_{a,\lambda} p(\lambda)\, p(b|a,\lambda)\, U_\lambda^\dagger M_a U_\lambda$$

write $\mathbb{M} > \mathbb{N}$

- A complete set of monotones are a set of functions $f_i(\mathbb{M})$ such that $\mathbb{M} > \mathbb{N}$ iff $f_i(\mathbb{M}) \geq f_i(\mathbb{N}) \quad \forall\ i$

i.e. a set of neccessary & sufficient conditions

Monotones in terms of state discrimination & exclusion

- $P^{Q}_{guess}(\mathcal{E}, \mathbb{M})$ & $P^{Q}_{err}(\mathcal{E}, \mathbb{M})$ both form complete sets of monotones

- $\mathbb{M} \succ \mathbb{N}$ iff for all QSD $\mathcal{E}$ $P^{Q}_{guess}(\mathcal{E}, \mathbb{M}) \geq P^{Q}_{guess}(\mathcal{E}, \mathbb{N})$

$$-\log P^{Q}_{guess}(\mathcal{E}, \mathbb{M}) \leq -\log P^{Q}_{guess}(\mathcal{E}, \mathbb{N})$$

$$H_{\infty}(X|A)_{\mathcal{E},\mathbb{M}} \leq H_{\infty}(X|A)_{\mathcal{E},\mathbb{N}}$$

- $\mathbb{M} \succ \mathbb{N}$ iff for all QSE $\mathcal{E}$ $P^{Q}_{err}(\mathcal{E}, \mathbb{M}) \leq P^{Q}_{err}(\mathcal{E}, \mathbb{N})$

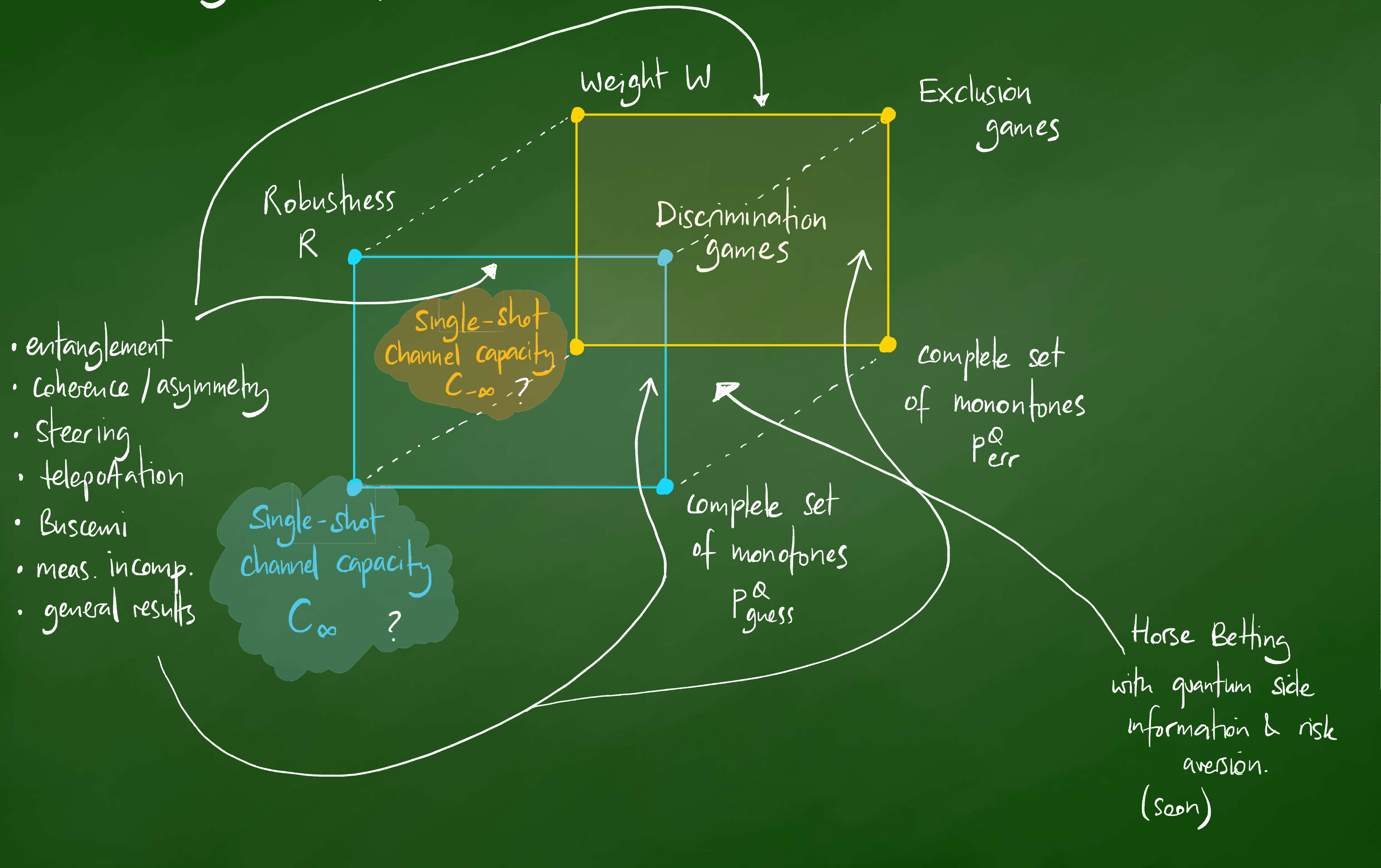
$$-\log P^{Q}_{err}(\mathcal{E}, \mathbb{M}) \geq -\log P^{Q}_{err}(\mathcal{E}, \mathbb{N})$$

$$H_{-\infty}(X|A)_{\mathcal{E},\mathbb{M}} \geq H_{-\infty}(X|A)_{\mathcal{E},\mathbb{N}}$$

- Outperforming in all QSD or QSE games is sufficient to guarantee simulation.

More general resource theories

Four-way correspondence



Conclusions

Conclusions

- Resource-theory approach to measurement informativeness is useful
  - → geometric quantifiers with operational significance
  - → neccessary & sufficient conditions for measurement simulation

- Four-way correspondence seems to hold in many resource theories
  - → deep underlying structure?

- Future work on bridging results using ideas on horse-betting & risk aversion.