Maxwell's Demon walks into Wall street

Stochastic thermo. — expected utility theory

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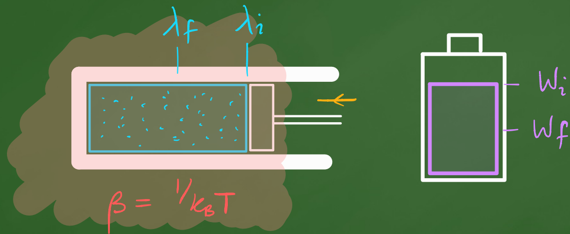
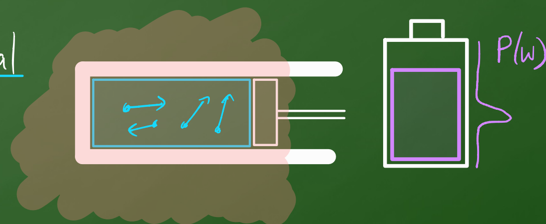
Main result: EUT allows us to quantify fluctuations in ST via divergence (entropic)

Stochastic Thermo

traditional



statistical



process: $\lambda_i \longrightarrow \lambda_f$

work done: $\Delta W = W_i - W_f \overset{\text{2nd law}}{\geqslant} \Delta F = \underbrace{F_f - F_i}_{\text{free energy}}$, $\quad F_f = U_f - TS_f$

$W_{diss} = \Delta W - \Delta F$ dissipated work

- $\lambda_i \to \lambda_f$ has fluctuations
- Associated work distribution $P(W_{diss})$
- $\langle W_{diss} \rangle = \langle \Delta W \rangle - \Delta F \geqslant 0$

Jarzynski & Crooks

1997 : Jarzynski equality : $\boxed{\left\langle e^{-\beta W_{diss}} \right\rangle = 1}$

$\int dW_{diss}\, P(W_{diss})\, e^{-\beta W_{diss}}$

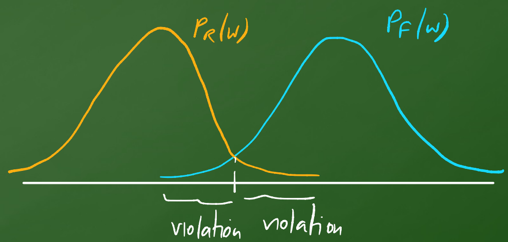
Jensen : $\langle e^{-x} \rangle \geq e^{-\langle x \rangle}$

$\longrightarrow$ Recover 2nd law

1999 : Crooks : $\lambda_i \longrightarrow \lambda_f$ 'forward' process $P_F(W_{diss})$

$\lambda_f \longrightarrow \lambda_i$ 'Reverse' process $P_R(W_{diss})$

$$w \equiv \boxed{\beta W_{diss} = \ln \frac{P_F(w)}{P_R(-w)}}$$



$$\beta \langle W_{diss} \rangle = D\left(P_F(w) \,\|\, P_R(-w) \right)$$

KL Divergence $\int dw\, P_F(w) \ln \left(\frac{P_F(w)}{P_R(-w)} \right)$ distinguishability measure

Expected Utility theory

- Model rational agents decision process in light of uncertainties

Archetypal example: offered two alternatives

option 1: lottery

fair coin: $p = \frac{1}{2}$ H lose £100

$p = \frac{1}{2}$ T win £20

£ = 'loss'

$$\langle £ \rangle = 0.5 \times (£100) + 0.5 \times (-£20)$$
$$= £40$$

option 2: certainty

pay me fixed amount £60

Q: which would you pick?

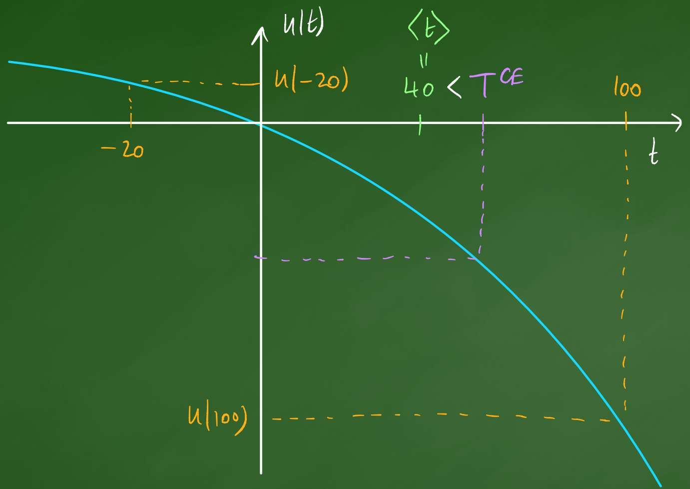
key idea: amount which would make $\underbrace{\text{you pay fixed amount}}_{\text{"certainty equivalent (CE)"}}$ is a measure of risk attitude

lower CE → more risk averse

higher CE → more risk seeking

Utility function:

- utility = 'happiness' or 'satisfaction' with value
- 'Renormalisation' of values, to take into account value of certainty



- Curvature of utility function determines attitude to risk
  - concave u → risk averse
  - linear u → risk neutral
  - convex u → risk seeking
- Measure of risk aversion:

$$ARA_u(t) = \frac{u''(t)}{u'(t)}$$

How happy you are on average from lottery
↓
expected

- certainty equivalent: amount of money that leads to same happiness

$$u(T^{CE}) = \langle u(t) \rangle$$

or

$$\boxed{T^{CE} = u^{-1}\left(\langle u(t) \rangle\right)}$$

- Agent w/ constant risk aversion:

$$u_r(t) = \begin{cases} \frac{1}{r}(1-e^{rt}) & \text{if } r \neq 0 \quad r>0 \text{ averse} \\ -t & \text{if } r=0 \quad \text{neutral} \end{cases}$$

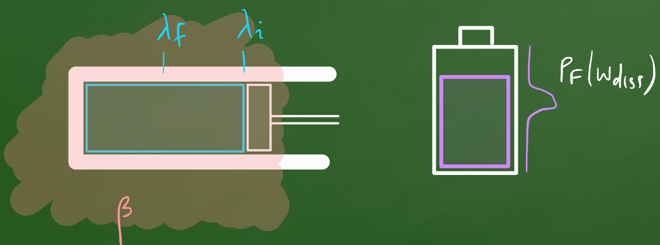
Stat. Thermo & EUT

- Q: Can we quantify thermo. fluctuations using EUT?

  ↳ what is the certainty equivalent dissipated work of a process?

- two alternatives:

1)



2)

- W_{diss}^{CE}: amount of dissipated work happy to spend w/ certainty rather than have prob. $W_{diss} \sim P_F(W_{diss})$

Main result:

$$\boxed{\beta W^{CE}_{diss,r} = D_{1+r}\left(P_F(w) \,\|\, P_R(-w)\right)}$$

$$D_\alpha\left(P(x) \,\|\, Q(x)\right) = \frac{1}{\alpha - 1} \ln \int dx \; P(x)^\alpha \, Q(x)^{1-\alpha}$$

Renyi divergence of order α

- Renyi paramter $\alpha = 1+r$ measure of risk att.
- $r \to 0$: recover Crooks: $\beta W_{diss} = D\left(P_F(w) \,\|\, P_R(-w)\right)$
- $r \to \infty$ extreme risk aversion
$$\begin{aligned} \beta W^{CE}_{diss,\infty} &= D_\infty\left(P_F(w) \,\|\, P_R(-w)\right) \\ &= \ln \min\left\{\lambda : P_F(w) \leq \lambda P_R(-w)\right\} \\ &= \operatorname*{argmax}_{w} \left\{P_F(w) > 0\right\} \\ &= \text{largest possible fluctuation} \end{aligned}$$
- $r = -1$: $W^{CE}_{diss,-1} = 0$ i.e. player willing to bet on fluctuations of 2nd law!

Generalised Jarzynski :

new

$$\left\langle e^{r\beta W_{diss}} \right\rangle_{p_F} = e^{r\beta W^{CE}_{diss,r}}$$

$$= e^{r D_{1+r}\left(p_F(w) \,\|\, p_R(-w)\right)}$$

- $r=-1$: Jarzynski $\left[W^{CE}_{diss,-1} = 0 \right]$
- r.h.s. only independent of p_F for $r=-1$.