Using one measurement to reproduce another

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Outline

- Main question & motivating example
- Single-use & many-use approaches
- Main results & sketch of idea
- Further results & Conclusions

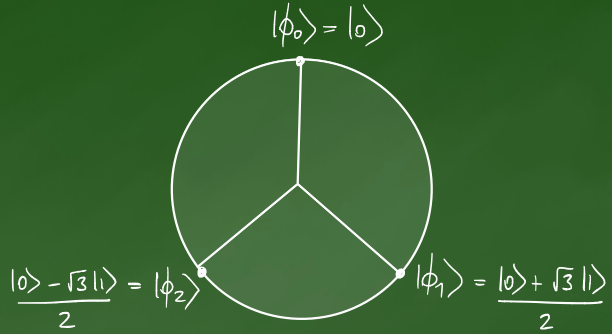
Main question

To what extent can we use one measuring device to reproduce (simulate) another?

Motivating example

- Consider the trine measurement T
  - 3 outcome qubit measurement

$$M_a = \frac{2}{3} |\phi_a\rangle\langle\phi_a| \quad \text{for } a = 0,1,2$$

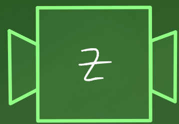


- Q: How well can this be used to reproduce ideal von Neumann measurement

  i.e. simulate $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ measurement

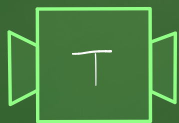
The goal

Reproduce as closely as possible statistics of ideal measurement

ideal $\quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow$ 

$\Longrightarrow \; a$

$$\mathrm{Prob}(+1) = |\alpha|^2$$
$$\mathrm{Prob}(-1) = |\beta|^2$$

trine $\quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow$ 

$\Longrightarrow \; a$

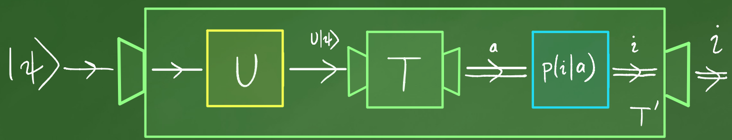
$$\mathrm{Prob}(0) = \frac{2}{3}|\alpha|^2$$
$$\mathrm{Prob}(1) = \frac{1}{6}|\alpha|^2 + \frac{1}{2}|\beta|^2 + \frac{\sqrt{3}}{6}(\alpha\beta^* + \alpha^*\beta)$$
$$\mathrm{Prob}(2) = \frac{1}{6}|\alpha|^2 + \frac{1}{2}|\beta|^2 - \frac{\sqrt{3}}{6}(\alpha\beta^* + \alpha^*\beta)$$

\+ correct post-measurement state too?

Single-copy approach

- Concept of measurement simulation introduced recently considered what can be achieved making single use of available measurement

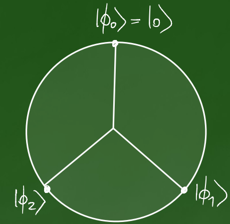
Protocol



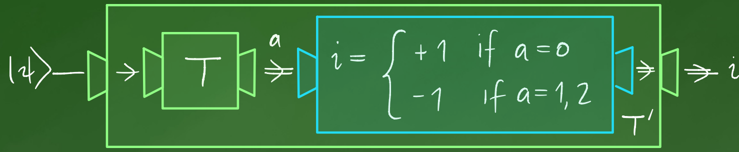
- Simulate $T' = \{M'_i\}$ using measurement $T = \{M_a\}$

$$M'_i = \sum_a \underbrace{p(i|a)}_{\text{classical post-processing of measurement result}} \underbrace{U^\dagger M_a U}_{\text{quantum pre-processing}}$$

Optimal single copy use of trine to reproduce Z



- Optimal protocol is simple:



$$M'_1 = \tfrac{2}{3}|0\rangle\langle 0| \qquad M'_{-1} = \tfrac{1}{3}|0\rangle\langle 0| + |1\rangle\langle 1|$$

– Simulated measurement is a noisy-Z measurement

RMS error

$$\mathcal{E}_{(1)} = \sqrt{\frac{1}{2}\int d\psi \left[\left(|\alpha|^2 - \langle\psi|M'_1|\psi\rangle\right)^2 + \left(|\beta|^2 - \langle\psi|M'_{-1}|\psi\rangle\right)^2\right]}$$

av. over outcomes

av. over all pure states

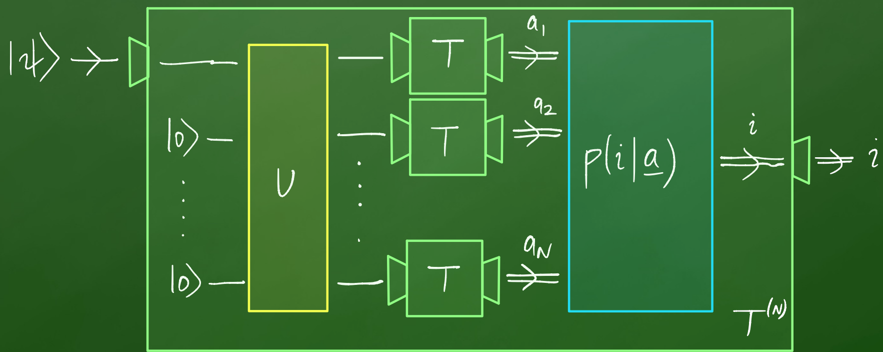
ideal probs in $|\psi\rangle$

reproduced probs in $|\psi\rangle$

$$= \frac{1}{3\sqrt{3}} \approx 0.192$$

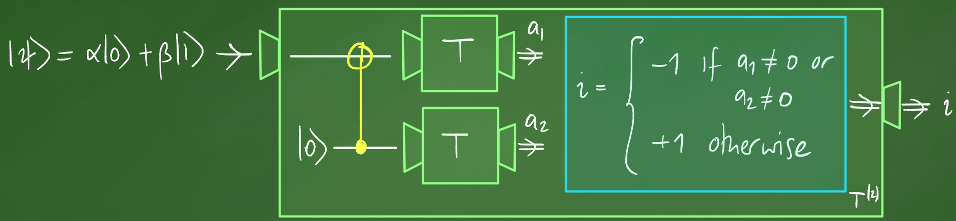
Multi-copy approach

- Our approach is to allow ourselves multiple uses of the available measurement
  (to reproduce single use of target measurement)
  - Allow for preparation of arbitrary ancilliary systems
    + entangling pre-processing
  - Measure system + ancillary systems & collectively post-process string of outcomes



Two use of trine protocol

- Apply CNOT w/ ancillary qubit to prepare $|\Psi\rangle = \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$
- Measure both qubits w/ trine measurement
- output -1 if either measurement result is 1 or 2 ; output $+1$ otherwise



$$M^{(2)}_{+1} = \frac{8}{9}|0\rangle\langle 0| \qquad M^{(2)}_{-1} = \frac{1}{9}|0\rangle\langle 0| + |1\rangle\langle 1|$$

Improved noisy Z measurement

RMS error $\varepsilon_{(2)} = \dfrac{1}{9\sqrt{3}} = \dfrac{1}{3}\varepsilon_{(1)} \approx 0.064$

N use of trine protocol

- Obvious generalisation to N measurements performs very well
- CNOTs between $|\psi\rangle$ & $|0\rangle^{\otimes(N-1)}$ creates $|\Psi\rangle = \alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}$

  interpret this as a form of classical cloning (cloning in basis of target measurement)

- classical post-processing: – output −1 if any measurement result is 1 or 2

  – output +1 if all measurement results are 0

$$M_{+1}^{(N)} = \left(1 - \frac{1}{3^N}\right)|0\rangle\langle 0| \qquad M_{-1}^{(N)} = \frac{1}{3^N}|0\rangle\langle 0| + |1\rangle\langle 1|$$

approaches ideal measurement exponentially fast in N

RMS error $\quad \mathcal{E}_{(N)} = \frac{1}{\sqrt{3}}\frac{1}{3^N} = \frac{\mathcal{E}_{(1)}}{3^N}$

exponentially vanishing error

- Interestingly: this is not optimal protocol!

Any* measurement can reproduce an ideal measurement

* non-trivial

POVM elements not all proportional to identity $M_a \neq q(a)\mathbb{1}$

- Above result is not special to trine.
- In fact, this result holds in almost full generality:

> Using any non-trivial measurement a sufficient number of times can reproduce arbitrarily well ideal projective measurements

Outline of proof

- Only need a few simple observations:

1. Ability to perform ideal projective measurement is equivalent to being able to perfectly distinguish a basis of states.

2. Viewing measurement outcomes as classical random variables, N measurements on a basis state $|i\rangle$ is equivalent to obtaining N samples of random variable A_i

3. All distinct random variables can be discriminated given sufficiently many samples. & error drops off exponentially in no. of samples N.

An ideal measurement can reproduce any other measurement

- Can also easily see that an ideal measurement can be used to perform any measurement
  - including non-trivial post-measurement states

- Most general measurement: instrument $\mathcal{I} = \{\Lambda_a(\cdot)\}$

  CP maps s.t. $\Lambda(\cdot) = \sum_a \Lambda_a(\cdot)$ is CPTP

  $$\mathrm{Prob}(a|\rho) = \mathrm{tr}[\Lambda_a(\rho)]$$

  Post-measurement state $\rho'_a = \dfrac{\Lambda_a(\rho)}{\mathrm{tr}[\Lambda_a(\rho)]}$

- Can define associated channel $\Gamma(\rho) = \sum_a \Lambda_a(\rho) \otimes |a\rangle\langle a|$

  stores measurement result in 2nd register

- Measuring 2nd register with ideal measurement 'reads' result & leaves system in correct post-measurement state.

Any* measurement can reproduce any other measurement

* non-trivial

- Combining above insight w/ previous result implies that

> Using any non-trivial measurement a sufficient number of times can reproduce arbitrarily well any other measurement (instrument)

- Implement channel above (or its unitary dilation) as a pre-processing
- use previous protocol to simulate ideal measurement of 2nd register.

Further results

- In above protocols are zero rate
  N uses of available measurement reproduce 1 use of target measurement. $R = \frac{1}{N} \to 0$
  - Using coding theory can obtain finite rate reproduction
    Reproduce k uses of target measurement in parallel using N measurements. $R = \frac{k}{N} \to \text{constant} > 0$
- Above protocols used deterministic post-processing i.e. $i = f(\underline{a})$
  - Find that probabilistic post-processing can outperform det. post-processings
- (Preliminary) If pre-processing unitaries are imperfect we can still overcome this.
  + imperfect ancillary systems (?)
  ↳ experimental feasibility?

Conclusions

- Main result: Any non-trivial measurement can reproduce arbitrarily well any other measurement given sufficient uses

- Simple protocols & few uses improve measurements

  → lightweight measurement-error mitigation?

- Average errors drop exponentially & coding theory techniques allow for finite rate reproduction

- Preliminary results indicate tolerance to imperfections (noisy ancillary systems, imperfect unitaries)