Compressing states & aligning reference frames

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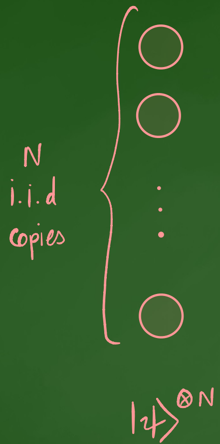
CIFAR

Outline

- Compression of identical qubits
- Aligning refoence frames
- Observations on the links between these directions
  - novel way to quantify quality of frame alignment
  - novel 'effective dimension' of compression
- Conclusions

Compressing identical qubits

Q: How much info$^{\underline{n}}$ is contained in N identical copies of a state $|\psi\rangle$?

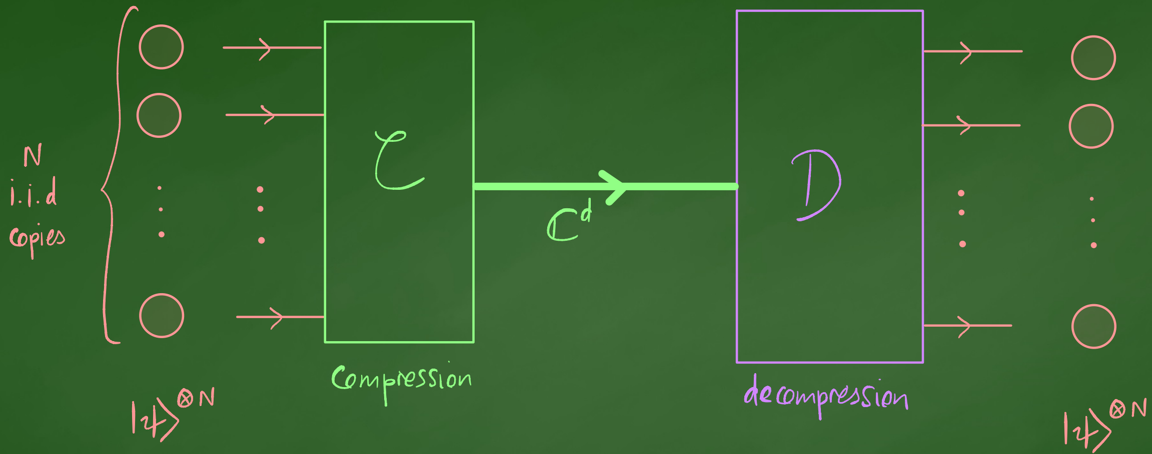


e.g. when doing tomography exactly how much info$^{\underline{n}}$ are you being given?

– what size memory do you need to store state?

Compressing identical qubits

Q: How much info^{n} is contained in N identical copies of a state $|\psi\rangle$?

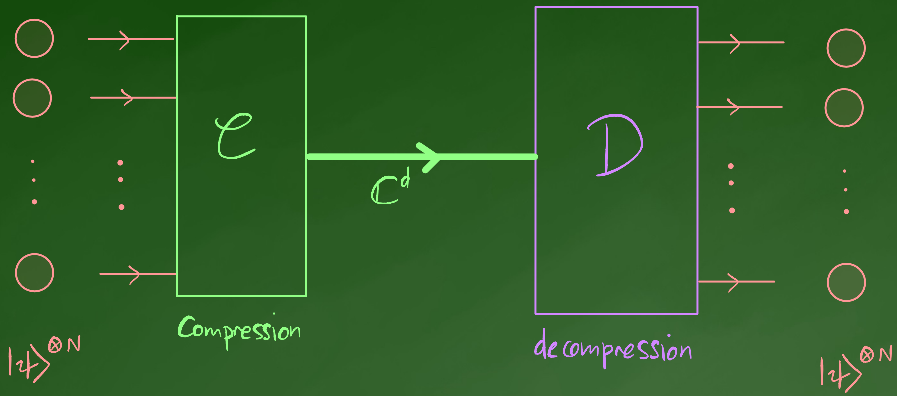


- Can answer this by considering size of smallest memory needed to faithfully store $|\psi\rangle^{\otimes N}$

Can perfectly recover state

Bužek & Hillery PRA (1996)

Compressing identical qubits

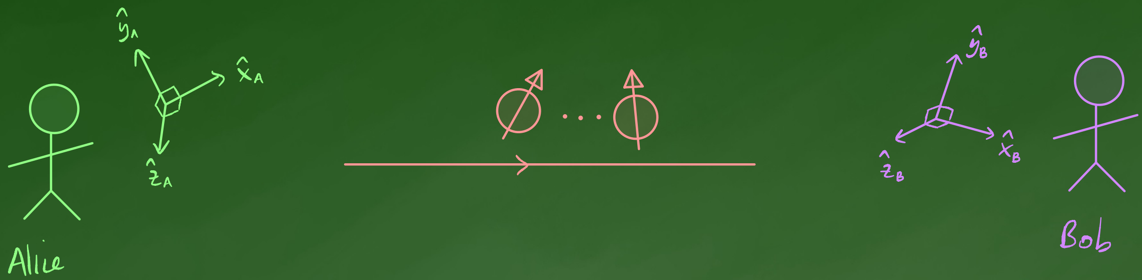


- Can compress state into $d = (N+1)$ dimensions exponentially smaller than 2^N
  - $|\psi\rangle^{\otimes N}$ lives in fully symmetric subspace
  - Compression achieved using quantum Schur-Weyl transform

Bužek & Hillery PRA (1996)

Aligning reference frames

- Consider spatially separated Alice & Bob whose reference frames are unaligned.



- Alice can send spin-½ particles to Bob to communicate relative orientation of her frame
  - quantum reference frame.

- Simplest task: Align single axis ($\hat{z}$ w.l.o.g.)
  - Generally interested in best state of N spins for conveying a direction.

Anti-parallel spins are better than parallel spins

- $|\uparrow_z\rangle|\downarrow_z\rangle$ is better for aligning $\hat{z}$ than $|\uparrow_z\rangle|\uparrow_z\rangle$

  - Figure of merit: Average fidelity between sent direction $\hat{z}_A$ & optimal guess of Bob $\hat{g}_B$

$$F = \int d\hat{z}_A \; p(\hat{g}_B | \hat{z}_A) \, \frac{1 + \hat{z}_A \cdot \hat{g}_B}{2}$$

- For $|\uparrow_z\rangle|\uparrow_z\rangle$ $\qquad F_{\uparrow\uparrow} = \frac{3}{4}$

- For $|\uparrow_z\rangle|\downarrow_z\rangle$ $\qquad F_{\uparrow\downarrow} = \frac{1+\sqrt{3}}{2\sqrt{3}} \approx 0.789$

Gisin & Popescu (1999)

Intuition behind result 1 :

- For uniformly random orientation between A & B, $|\uparrow_z\rangle|\uparrow_z\rangle$ from Bob's perspective is $U|\uparrow_z\rangle U|\uparrow_z\rangle$. Such a state is supported on symmetric subspace of dimension $d=3$.

  $\uparrow$ $\uparrow$ specifies relative rotation between frames

- On other hand $U|\uparrow_z\rangle U|\downarrow_z\rangle$ is supported on (full) Hilbert space of dimension $d=4$.

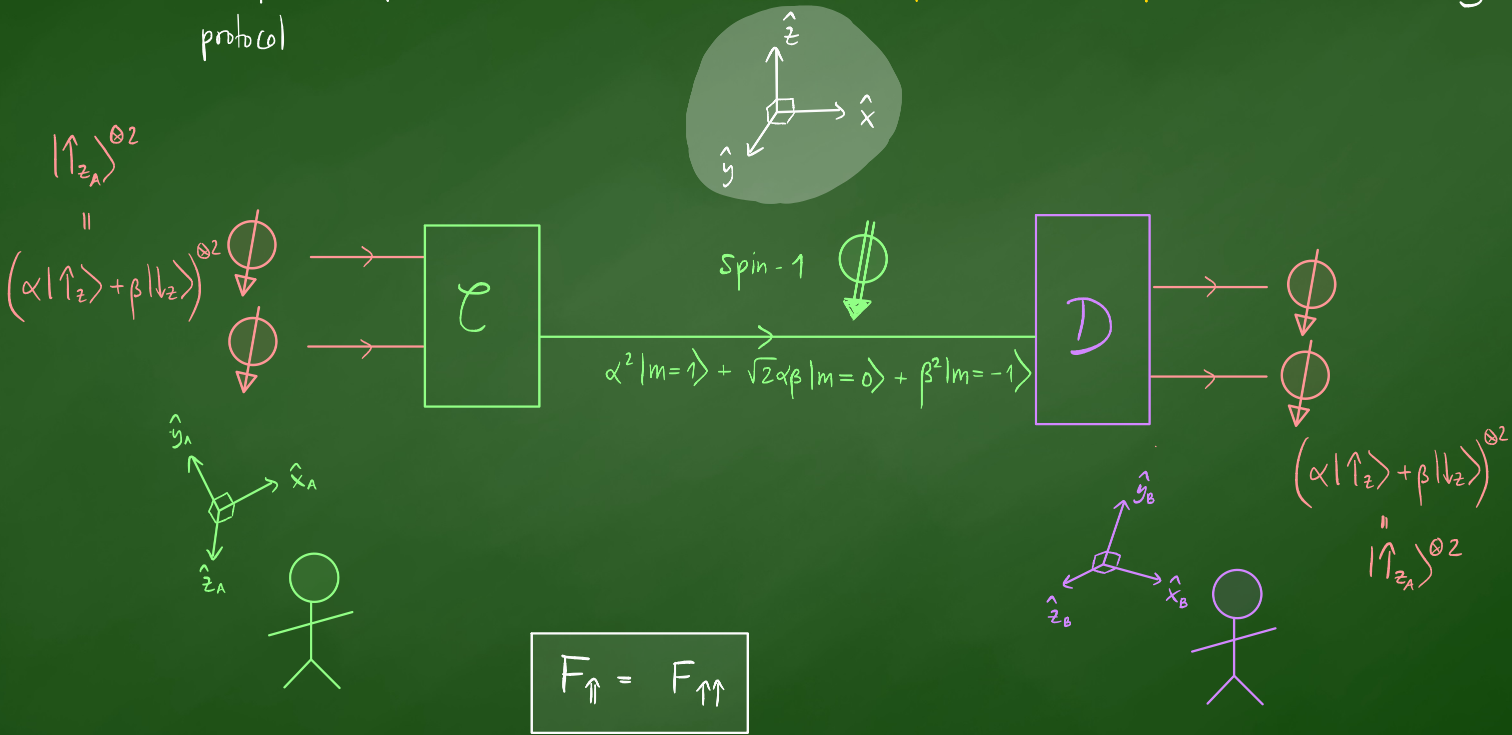
  $\hookrightarrow$ "more orthogonal" & easier to discriminate

$\longrightarrow$ $|\uparrow_z\rangle|\downarrow_z\rangle$ conveys more info$^{\underline{n}}$ about direction than $|\uparrow_z\rangle|\uparrow_z\rangle$

Q: Is there a (formal?) link between these results?

Observation 1

- We can use compression to strength intuition
  - For parallel spins Alice & Bob could add in compression & decompression without affecting protocol

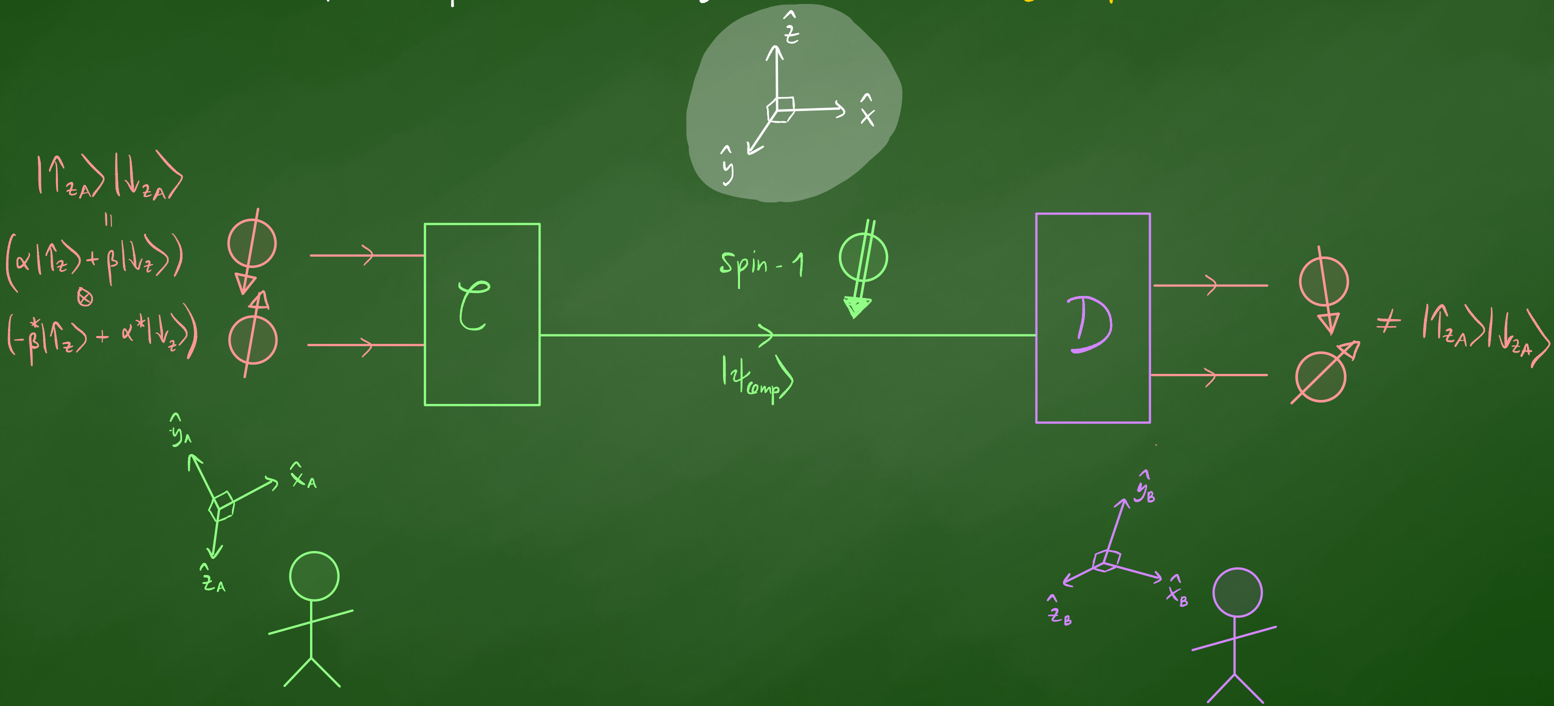


- Same average fidelity achieved using compressed state.

Observation 1

- We can use Compression to strength intuition

  – For anti-parallel spins it is no longer possible to faithfully compress state



  – If we forced Alice & Bob to compress, average fidelity would decrease.

Observation 2

- For $N \geq 3$ it is always possible to compress optimal N spin-$1/2$ direction indicator state.

- Representation theory: $- \mathcal{H}_{1/2}^{\otimes N} = \bigoplus_{j=0}^{N/2} \mathcal{N}_j \otimes \mathcal{M}_j$

Spin-j representation

dim: $d_{\mathcal{N}_j} = 2j+1$

multiplicity of spin-j repr.

dim: $d_{\mathcal{M}_j} = \binom{N}{\frac{N}{2}-j} \frac{2j+1}{\frac{N}{2}+j+1}$

$$- U^{\otimes N} = \bigoplus_{j=0}^{N/2} U_j \otimes \mathbb{1}$$

$$\hookrightarrow \quad |\psi\rangle = \sum_{j=0}^{N/2} \beta_j |\psi_j\rangle \qquad |\psi_j\rangle = \sum_{m=1}^{d_j} \alpha_m |\phi_m^{(j)}\rangle |r_m^{(j)}\rangle$$

d_j: $\min(d_{\mathcal{N}_j}, d_{\mathcal{M}_j})$

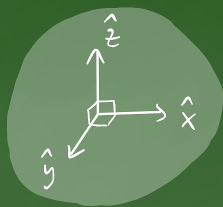
Schmidt decomposition in each spin sector

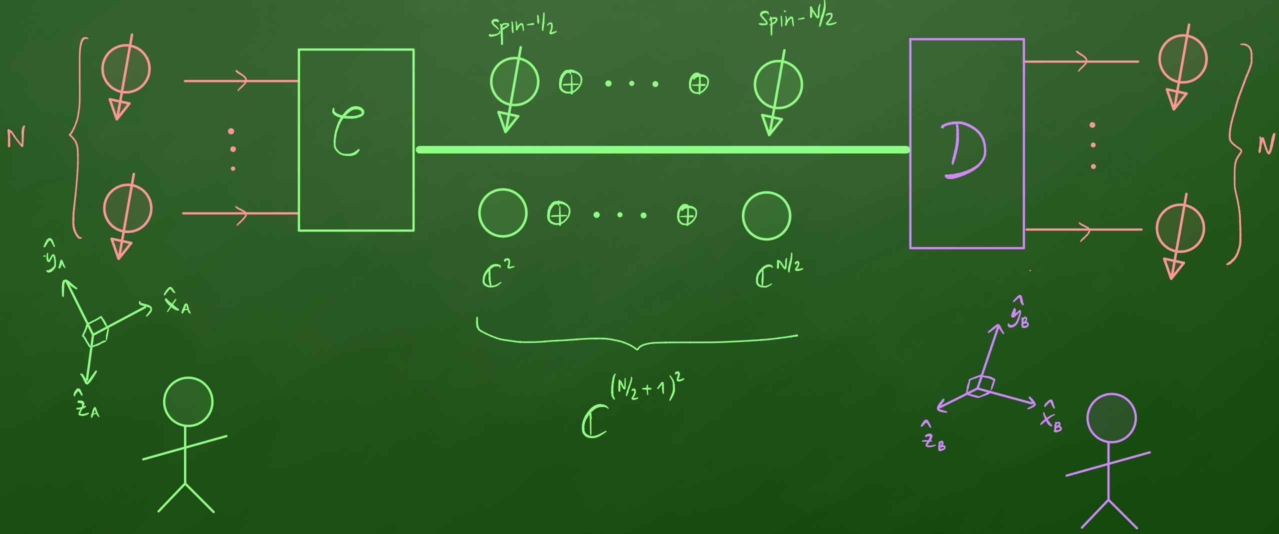
- $$U^{\otimes n} |\psi\rangle = \sum_{j=0}^{N/2} \beta_j \sum_{m=1}^{d_j} \alpha_m \underbrace{U_j |\phi_m^{(j)}\rangle}_{\text{always in spin-}j\text{ subspace}} \underbrace{|r_m^{(j)}\rangle}_{\text{invariant}}$$

- $U^{\otimes n}|\psi\rangle = \sum_{j=0}^{N/2} \beta_j \sum_{m=1}^{d_j} \alpha_m \underbrace{U_j|\phi_m^{(j)}\rangle}_{\text{always in spin-}j\text{ subspace}} \underbrace{|r_m^{(j)}\rangle}_{\hookrightarrow \text{ invariant}}$

$\hookrightarrow$ $U^{\otimes n}|\psi\rangle$ lives in subspace of dimension $\sum_{j=0}^{N/2} d_j^2$

$$= \left(\frac{N}{2}+1\right)^2$$





Observation 3

- Can use compression as a means to quantify performance of state as directional indicator
  - Alternative to average fidelity

- "Good" state is one that isn't very compressible

(Return to this soon)

Observation 4 (largely an aside ...)

- Speakable information helps to transmit unspeakable information

  Speakable information:
  - Something that can be written down or "spoken"
  - bit strings

  Unspeakable information:
  - Something that cannot be written down or "spoken"
  - direction in space, handedness

  (information that requires a shared reference frame)

- Sending purely speakable information has no capacity to send unspeakable information
  - Q: Can sending speakable info$^{\underline{n}}$ alongside unspeakable info$^{\underline{n}}$ help?

    Intuitively: sending some 0's & 1's won't help me indicate a direction.

    But the above shows it does help!

- $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_{z_A}\rangle|0\rangle + |\downarrow_{z_A}\rangle|1\rangle\right)$ is a better direction indicator than $|\uparrow_{z_A}\rangle$.

Observation 4

- $|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow_{zA}\rangle|0\rangle + |\downarrow_{zA}\rangle|1\rangle\Big)$

Average state:
(twirl)

$$\rho = \int dU \, (U\otimes \mathbb{1})|\psi\rangle\langle\psi|(U^\dagger\otimes\mathbb{1})$$

$$= \frac{1}{2}|\phi^+\rangle\langle\phi^+| + \frac{1}{4}|\psi^+\rangle\langle\psi^+| + \frac{1}{4}|\psi^-\rangle\langle\psi^-|$$

$\frac{|\uparrow_z\rangle|0\rangle + |\downarrow_z\rangle|1\rangle}{\sqrt{2}}$ $\quad$ $\frac{|\uparrow_z\rangle|1\rangle + |\downarrow_z\rangle|0\rangle}{\sqrt{2}}$ $\quad$ $\frac{|\uparrow_z\rangle|1\rangle - |\downarrow_z\rangle|0\rangle}{\sqrt{2}}$

- Support on 3-dimensional subspace

  ↳ NOT compressible to a single spin-1/2 particle.

- Although qubit conveys no directional information, the entanglement with spin-1/2 does encode additional directional information!

  – Same reason why superdense coding works.

Observation 5

- Average state is not maximally mixed state on support

$$\rho_{av} = \frac{1}{2}|\phi^+\rangle\langle\phi^+| + \frac{1}{4}|\psi^+\rangle\langle\psi^+| + \frac{1}{4}|\psi^-\rangle\langle\psi^-|$$

↳ Does not have maximum entropy given its support:

$$S(\rho_{av}) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{4}\log\frac{1}{4}$$
$$= \frac{3}{2}\log 2$$

$$d_{eff} = 2^{3/2} \approx 2.8$$

- Entropy of average state is a more refined notion of compression than just dimension of support.

$$d_{eff} = 2^{S(\rho_{av})}$$

- For N uniformly random i.i.d. states, average is max. mixed on symmetric subspace

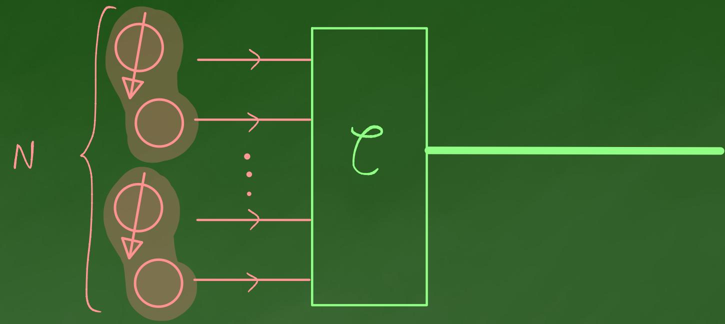
↳ $S(\rho_{av}) = \log d_{supp}$ $\qquad d_{eff} = d_{supp}$

- More generally this isn't case & $S(\rho_{av})$ gives a better measure of how compressible state is

  - by combining with Schumacher compression

N Spin-½ - qubit pairs

$$|\Phi_N\rangle = \left(\frac{1}{\sqrt{2}}\left(|\uparrow_{z_A}\rangle|0\rangle + |\downarrow_{z_A}\rangle|1\rangle\right)\right)^{\otimes N}$$



Q: How compressible is this?

- $$S(\rho_{av}) \underset{N \text{ large}}{\approx} \frac{3}{2}\log N + \frac{3 - \ln(\pi/8)}{\ln 4}$$

i.e. $d_{eff} \approx N^{3/2}$

	dim	compressed dim
$\|\psi\rangle^{\otimes N}$	2^N	$(N+1)$
$\|\Phi_N\rangle$	4^N	$\approx N^{3/2}$

Conclusions

- Compression brings insight field of reference frame alignment
- Entropy of average state gives effective compressed dimension.