Expected Utility Theory & its novel application in Quantum Information Science

Paul Skrzypczyk

QIP 2024

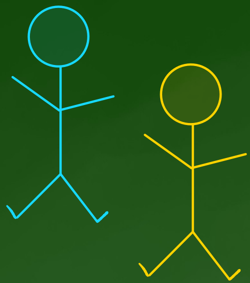
17th January 2024

Based upon:

A. F. Ducuara & PS
PRX Quantum 3, 020366 (2022)
arXiv: 2306.07975

A.F. Ducuara, PS, F. Buscemi, P. Sidajaya & V. Scarani
Phys. Rev. Lett. 131, 197103 (2023)

University of Bristol
The Royal Society
NULLIUS IN VERBA
CIFAR



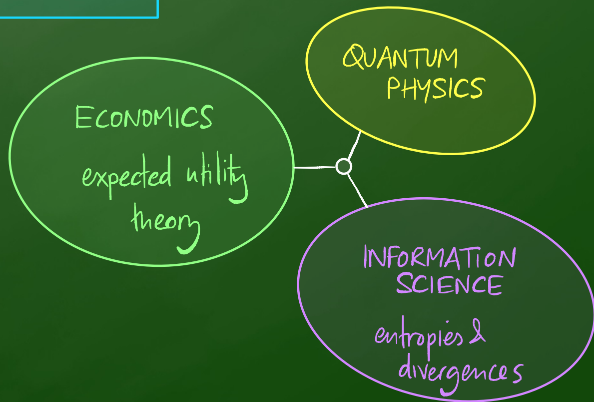
Alice & Bob are ubiquitous in QIS

- genuinely helpful to consider agents who make decisions
- Often interested in optimising some figure of merit / cost function

  e.g.
  - Succ. prob. in q. state discrimination
  - no. of queries to a quantum oracle
  - work extracted from q. system in thermodynamic cycle

- With uncertainty comes risk

<u>Question:</u> Can we account for an agent's attitude towards risk & is this an interesting / useful thing to do?



Outline

- Introduction to expected utility theory and risk aversion
- Application I: quantum state betting
- Application II: stochastic (quantum) thermodynamics
- Outlook & future directions

Introduction to expected utility theory

Choosing between lotteries

- Consider the following lottery setup:
  - Random variable X with pmf $p_X(x)$
  - Bookmaker offers odds $o(x)$–for–1
    i.e. pays out £$o(x)$ on £1 bet if $X = x$
  - Gambler bets proportion $b(x)$ of wealth on x
  - expected wealth after the lottery is

$$\langle W \rangle = \sum_x p_X(x)\, o(x)\, b(x)$$

- Consider being given the choice between two alternative lotteries:

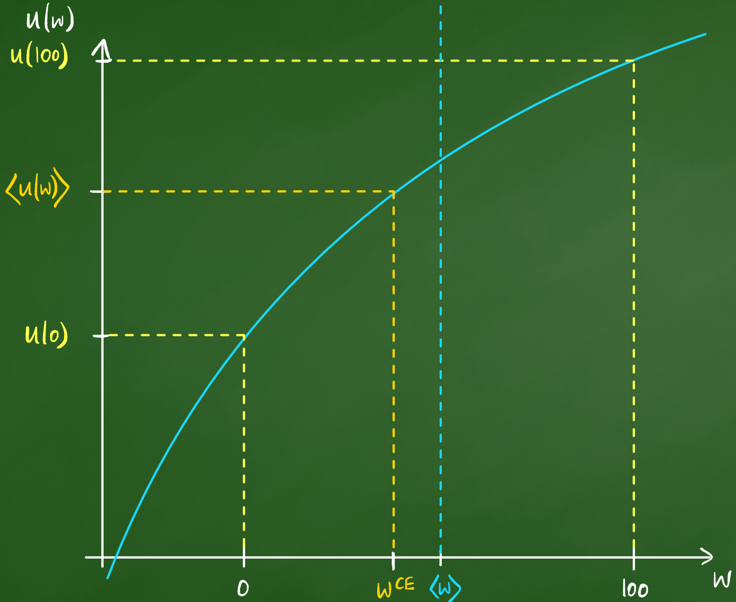
LOTTERY I	LOTTERY II
– X = fair coin; $p(H) = p(T) = \frac{1}{2}$	– X = perfectly biased coin; $p(H) = 1$, $p(T) = 0$
– odds: $o(H) = 100$; $o(T) = 0$	– odds: $o(H) = 30$
	'walk away'

  – In either case, always rational to bet all wealth on $x = H$, i.e. $b(H) = 1$, $b(T) = 0$

- Question: which lottery would you choose?
  - lottery I has higher final (average) wealth but comes with uncertainty
  - lottery II has lower final (deterministic) wealth.

Expected Utility Theory

- Model the behavior of gamblers by introducing concept of utility = happiness / satisfaction
  - Renormalise wealth to account for attitude to risk
- $u(w)$ – utility function



'certainty-equivalent wealth'

= winnings in deterministic lottery that have same utility as lottery 1.

- for risk averse gamblers, utility grows slower than wealth
- certainty-equivalent wealth $w^{CE} = u^{-1}(\langle u(w) \rangle)$ determines boundary between preferences:
  - if $w_{det} < w^{CE}$ is offered in lottery 2 (walk away) then preference is to gamble i.e. lottery 1
  - if $w_{det} > w^{CE}$ is offered, then preference is to walk away with w_{det}.
  - for risk averse gamblers $w^{CE} < \langle w \rangle$ i.e. gambler will walk away if offered below expected return.
- Central tenant of EUT: gamblers optimise expected utility (not average wealth!)

Two canonical attitudes towards risk

- Infinitely many different utility functions to choose from, each characterising different attitude to risk
- Here focus on two simplest / idealised attitudes (= H_2O of economics)

  - Constant relative risk aversion (isoelastic utility)

$$u_R^I(w) = \begin{cases} \dfrac{w^{1-R} - 1}{1-R} & R \neq 1 \\ \ln(w) & R = 1 \end{cases}$$

larger R
more risk averse

$w \longrightarrow \alpha w \qquad w^{CE} \longrightarrow \alpha w^{CE}$

  - Constant absolute risk aversion (exponential utility)

$$u_r^E(w) = \begin{cases} \dfrac{1 - e^{-rw}}{r} & r \neq 0 \\ w & r = 0 \end{cases}$$

larger r
more risk averse

$w \longrightarrow w + \beta \qquad w^{CE} \longrightarrow w^{CE} + \beta$

Application I : Quantum State Betting

Quantum State Betting

- Extension of quantum state discrimination to include betting
- Ensemble of quantum states $\mathcal{E} = \{ p(x), \rho_x \}$ $\qquad$ ρ_x quantum side info$^{\underline{o}}$ about x
- Bookmaker offers odds $o(x)$-for-1 on state ρ_x
- Gambler performs measurement $\mathbb{M} = \{ M_a \}$ and places conditional bet $b(x|a)$

- Figure of Merit: maximised isoelastic certainty-equivalent wealth

$$W_R^{ICE}(\mathcal{E}, \mathcal{O}, \mathbb{M}) = \max_{b(x|a)} \left[\sum_{x a} p(x) \, \mathrm{tr}\left[M_a \rho_x \right] \big(b(x|a) \, o(x) \big)^{1-R} \right]^{\frac{1}{1-R}}$$

$$\boxed{\text{Result 1:} \quad \log W_R^{ICE}(\mathcal{E}, \mathcal{O}, \mathbb{M}) = D_{1/R}\left(P_{X|A} \,\|\, R_X \,|\, P_A \right)}$$

Conditional Renyi divergence of order $\alpha = \frac{1}{R}$

$$\| \quad \frac{\frac{1}{R}}{\frac{1}{R} - 1} \log \sum_a p(a) \left[\sum_x p(x|g)^{1/R} \, r(x)^{1 - 1/R} \right]^R$$

$r(x) \propto \frac{1}{o(x)}$ [fair odds]

- Renyi parameter $\alpha = \frac{1}{R}$ quantifies risk attitude of gambler

- In case of constant odds $O(x) = $ constant

$$\boxed{\underline{\text{Corollary 1}}: \quad \log W_R^{ICE}(\mathcal{E}, O = \text{const.}, \mathbb{M}) = H_{1/R}(P_X | P_A) + \log n}$$

Conditional Renyi entropy of order $\alpha = 1/R$

- Interesting to compare to certainty-equivalent wealth when no quantum side information
  - i.e. when we take away states ρ_x ability to perform measurement $\mathbb{M}$

$$\boxed{\underline{\text{Result 2}}: \quad \log \frac{W_R^{ICE}(\mathcal{E}, O = \text{const.}, \mathbb{M})}{W_R^{ICE}(P_X, O = \text{Const.})} = H_{1/R}(P_X | P_A) - H_{1/R}(P_X) = I_{1/R}(P_X : P_A)}$$

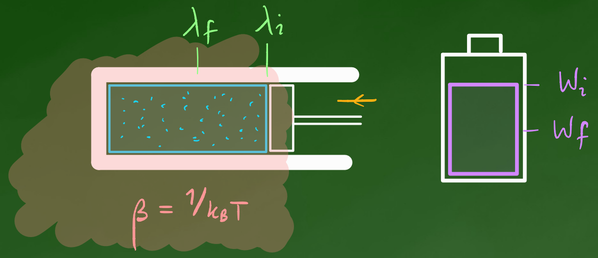
Renyi - Arimoto mutual information of order $\alpha = 1/R$

- Generalisation of recent results relating generalised robustness $(= D_{max})$ & advantages in discrimination tasks beyond $\alpha = \infty$.

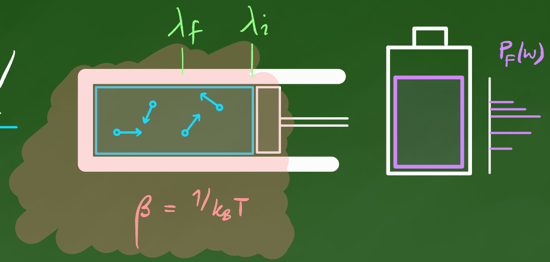
Application II : Stochastic (quantum) thermodynamics

Stochastic (quantum) thermodynamics

traditional



quantum/
stochastic



- process : $\lambda_i \longrightarrow \lambda_f$
- work done : $\Delta W = W_i - W_f \overset{\text{2nd law}}{\geq} \Delta F$
- dissipated work : $W_{diss} = \Delta W - \Delta F \geq 0$

- $P_F(w)$: prob. distribution of work fluctuations
- Jarzynski equality : $\langle e^{-\beta W_{diss}} \rangle = 1$
- Crook's : $w \equiv \beta W_{diss} = \ln \dfrac{P_F(w)}{P_R(-w)}$

"Reverse" process $\lambda_f \longrightarrow \lambda_i$

- Certainty-equivalent dissipated work $W^{ECE}_{diss,r}$ [exponential utility, const. absolute risk aversion r]

  = amount of deterministic dissipated work equally preferable to average dissipated work from process for an agent with risk attitude r

$$\boxed{\underline{\text{Result } 1} : \quad \beta W^{ECE}_{diss,r} = D_{1+r}\left(P_F(w) \,\|\, P_R(-w) \right)}$$

Renyi Divergence of order $\alpha = 1+r$

- Generalisation of 2^{nd} law : $r = 0$ (risk neutral)

$$\beta \langle W_{diss} \rangle = D\left(P_F(w) \,\|\, P_R(-w) \right) = \beta \Delta \langle W \rangle - \beta \Delta F$$

$$\boxed{\underline{\text{Result } 2} : \quad \left\langle e^{r\beta W_{diss}} \right\rangle = e^{r\beta W^{ECE}_{diss,r}} = e^{r D_{1+r}\left(P_F(w) \,\|\, P_R(-w) \right)}}$$

- Generalisation of Jarzynski equality : $r = -1$

Gambler who bets on stochastic violations of 2^{nd} law!

Outlook & Future Directions

Outlook & Future Directions

- Expected utility theory shed new light on old tasks
  - new interpretation of Renyi parameter as quantifying attitude towards risk
- Results so far not very quantum!
  - Actively investigating 'genuinely quantum' extensions of results
- Results so far use only basics of EUT / economic theory
  - Extensions to 'prospect theory' but much more investigation to be done
- General hope: Interesting & worthwhile direction to pursue; results so far tip of iceberg
  - Ideas & tools from economics will give us alternative perspective which might be equally powerful to harness.

See also upcoming work of Arcos, Oppenheim, Renner & Sagawa