Using one measurement to reproduce another

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joint work with Noah Linden

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Outline

- Main question & motivating example
- Main result & sketch of proof
  - two key subroutines
- outline briefly other results
- Current & future directions
- Conclusions

Main question

To what extent can we use one measuring device to simulate (reproduce) another?

- Simulate ideal measurement from noisy one
- Completely change the measurement

Motivating example

- Consider the noisy-Z measurement

$$M_1 = (1-p)|0\rangle\langle 0| + q|1\rangle\langle 1|$$

$$M_{-1} = p|0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$$

$p = q = 0$: ideal Z meas.

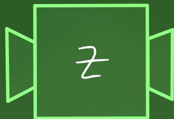
$p = q$: sym. noisy Z

- Q: How well can this be used to simulate ideal von Neumann measurement

i.e. simulate $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ measurement

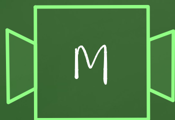
The goal

Simulate as closely as possible statistics of ideal measurement

ideal $\quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow$ 

$\Longrightarrow a$

$$\mathrm{Prob}(+1) = |\alpha|^2$$
$$\mathrm{Prob}(-1) = |\beta|^2$$

noisy Z $\quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow$ 

$\Longrightarrow a$

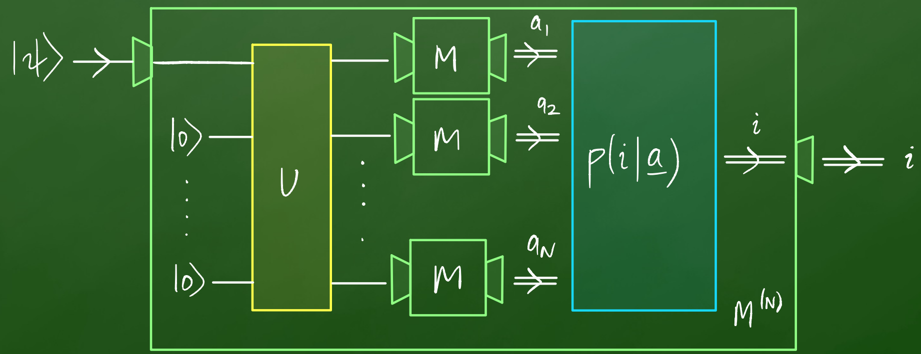
$$\mathrm{Prob}(+1) = (1-p)|\alpha|^2 + q|\beta|^2$$
$$\mathrm{Prob}(-1) = p|\alpha|^2 + (1-q)|\beta|^2$$

\+ correct post-measurement state too

Multi-copy approach

- Our approach is to allow ourselves multiple uses of the available measurement (to simulate single use of target measurement)
  - Allow for preparation of arbitrary ancillary systems + entangling pre-processing
  - Measure system + ancillary systems & collectively post-process string of outcomes

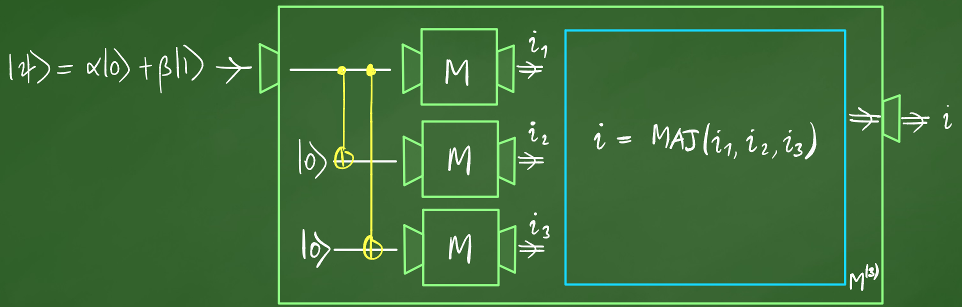
e.g.



3-use CNOT protocol

- Apply CNOT w/ ancillary qubits to prepare $|\Psi\rangle = \alpha|0\rangle|0\rangle|0\rangle + \beta|1\rangle|1\rangle|1\rangle$
- Measure all qubits w/ noisy-Z
- output majority of results

"repetition code"



$$M_1^{(3)} = (1-p^{(3)})|0\rangle\langle 0| + q^{(3)}|1\rangle\langle 1|$$

$$M_{-1}^{(3)} = p^{(3)}|0\rangle\langle 0| + (1-q^{(3)})|1\rangle\langle 1|$$

$$p^{(3)} = 3p^2 - 2p^3$$

$$q^{(3)} = 3q^2 - 2q^3$$

i.e. $p \rightarrow O(p^2)$

$q \rightarrow O(q^2)$

Any* measurement can simulate any other measurement

* non-trivial

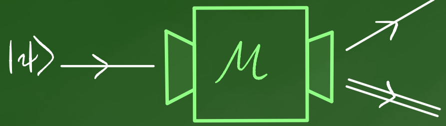
PDVM elements not all proportional to identity $M_a \neq q(a)\mathbb{1}$

- Above result is not special to noisy-Z & Z
- In fact, this result holds in almost full generality:

> Using any non-trivial measurement a sufficient number of times can simulate arbitrarily well any other measurement (instrument)

Quantum Instruments

- Most general type of non-destructive measurement possible.



$\rho'_a = \Lambda_a(|\psi\rangle\langle\psi|)$

$a \qquad \mathrm{Prob}(a) = \mathrm{tr}\left[\Lambda_a(|\psi\rangle\langle\psi|)\right]$

- $\Lambda_a(\cdot)$ completely positive trace non-increasing map.
- $\Lambda_a(\cdot) = \sum_i K_i^a (\cdot) K_i^{a\dagger}$ Kraus representation
- $\sum_a \Lambda_a(\cdot) = \Lambda(\cdot)$ q. channel.

- Previously: trivial to get post-measurement state correct
  - depends only upon meas. outcome & no memory of pre-meas. state $|\psi\rangle$
- Here: non-trivial dependence between pre-meas. & post-meas. state.

Two key subroutines

- Main result is proven by combining two key simulation sub routines:

Subroutine 1: post-measurement sub routine

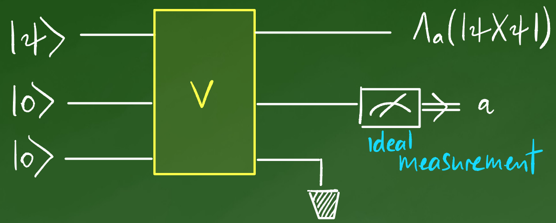
Given ability to perform (destructive) projective measurement it is possible to simulate an arbitrary instrument

Subroutine 2: Generalised classical cloning sub routine

Given ability to perform any (non-trivial) instrument it is possible to simulate arbitrarily well an ideal projective measurement

Post-measurement subroutine

- Essentially Naimark's theorem



$$V|\psi\rangle|0\rangle|0\rangle = \sum_{a,j} K_j^a|\psi\rangle|a\rangle|j\rangle$$

- Projective measurement is destructive but measurement steers first system in desired post-measurement state.

- Routine shows that in order to simulate arbitrary instrument $\mathcal{M}$ it suffices to be able to simulate ideal projective measurement

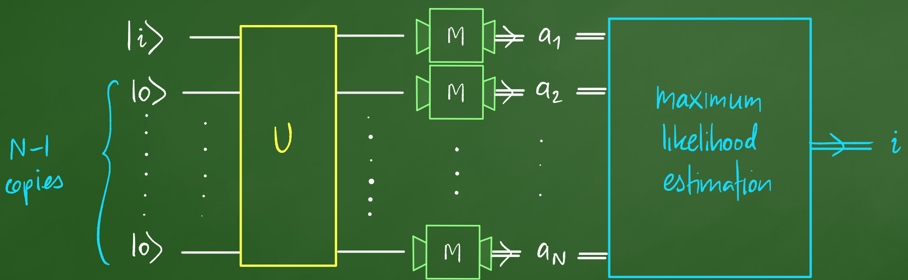
  this simplifies problem dramatically!

Generalised classical cloning sub routine

key observation 1: Ability to perform ideal projective measurement is equivalent* being able to distinguish (perfectly) between a basis of states.

↳ we will focus on distinguishing a basis of states.

Generalised classical cloning sub-routine:



$$U|i\rangle|0\rangle^{\otimes(N-1)} = |e_i\rangle^{\otimes N}$$

where $|e_i\rangle$ forms some O.N.B.

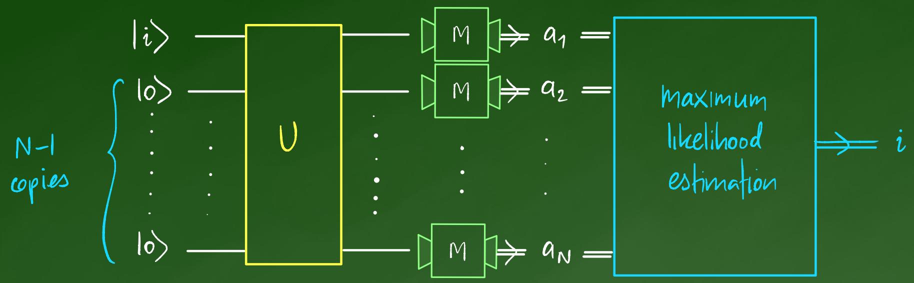
- For large N this protocol will correctly identify the basis state $|i\rangle$
- Why?
  - $P(a|i) = \mathrm{tr}(M_a |e_i\rangle\langle e_i|)$

    ↳ $P^{(N)}(\underline{a}|i) = \prod_{i=1}^{N} P(a_k|i)$
  - i.e. N i.i.d. samples from one of the distributions $\{P(a|i)\}_i$.
  - d-ary hypothesis testing

    ↳ all RVs can be distinguished asymptotically as $N \to \infty$, with error vanishing exponentially fast.

*up to trivial post-processing of measurement results.

Generalised classical cloning sub-routine:



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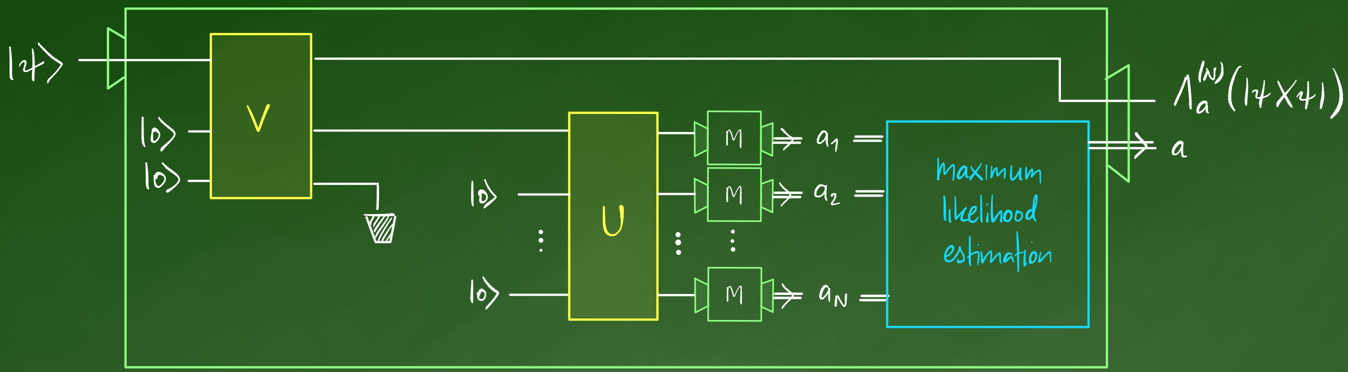
    $\hookrightarrow$ all RVs can be distinguished asymptotically as $N \to \infty$, with error vanishing exponentially fast.

Subtlety: How do we know that distributions $P(a|i)$ are distinct for all i?

i.e. what if $P(a|0) = P(a|1)\ \forall a$? e.g. $M_0 = |0\rangle\langle 0|$ $M_1 = |1\rangle\langle 1| + |2\rangle\langle 2|$

we can deal with this by considering running different U on different subsets. (leave details for offline).

Putting it all together



$$V|\psi\rangle|0\rangle|0\rangle = \sum_{a,j} k_j^a |\psi\rangle|a\rangle|j\rangle \qquad U|i\rangle|0\rangle^{\otimes(N-1)} = |e_i\rangle^{\otimes N}$$

- Uses N+1 ancillary systems & N uses of available measurement.
- $\Lambda_a^{(N)}$ approaches target instrument as $N \to \infty$ (becomes physically indistinguishable)
- Not optimal but universal protocol

Other Results

- In general optimal protocol minimising average TVD error $\langle \mathcal{E}_{(N)} \rangle$ for fixed N will not be Classical Cloning (CNOT) protocol.
  - Optimal to apply unitaries that take $|0\rangle|0\rangle^{\otimes(N-1)}$ & $|1\rangle|0\rangle^{\otimes(N-1)}$ to non-orthogonal states.
  - Intuition: this performs worse for basis states but better for equatorial states $\frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}$

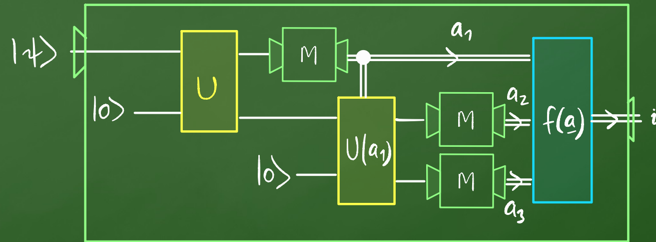
- Similarly optimal protocols sometimes involve probabilistic post-processing of measurement results.
  - Intuition: injecting randomness into simulation can again be useful (especially for equatorial states).

- Above protocol is zero rate: N uses of available measurement to simulate 1 use of target
  - Using ideas of (classical) block coding we can achieve finite rate protocols.
  - Given k-particle state & want to simulate k measurement results in parallel.
  - Rate lower bounded by classical capacity of classical channel $p(a|x) = tr(M_a |x\rangle\langle x|)$.

Current & Future directions

- Practical question: what happens if unitaries & ancillary systems are noisy?
  - More general error correction ideas can be applied.
    Not all noise is relevant e.g. dephasing irrelevant for z meas.
  - Architecture specific analysis ions, Rydberg atoms, SC qubits etc.

- what is optimal protocol for a fixed number of uses?
  - more general meas. protocols e.g.



  - Don't have example of measurement where classical feed forward provably useful yet.

- what is the optimal rate of measurement simulation?
  - Analogue of coding theorem for measurements.

Conclusions

- Main result: Any non-trivial measurement can simulate arbitrarily well any other measurement given sufficient uses
  - Ability to perform projective meas. equivalent to ability to perform arbitrary instrument
  - Any measurement can distinguish a basis of states $\equiv$ simulate a projective measurement

- Simple protocols & few uses improve measurements
  → lightweight measurement-error mitigation

- Lots of interesting future directions to explore.