Recap of Quantum Physics (Year 1)

- Quantum state: $|\psi\rangle$ complete description of state of quantum particle

  'ket (vector)' in Dirac Notation

- Superposition: $$|\psi\rangle = \sum_{n=1}^{d} \alpha_n |n\rangle$$ distinguishable basis states

  complex numbers

  Normalisation condition: $\sum_n |\alpha_n|^2 = 1$

- Bra vector: $$\langle\psi| = \sum_{n=1}^{d} \alpha_n^* \langle n|$$ "dual vector" to $|\psi\rangle$

- Scalar product: $$\langle\psi|\phi\rangle = \sum_n \alpha_n^* \beta_n$$ $|\phi\rangle = \sum_{n=1}^{d} \beta_n |n\rangle$

  – Orthonormality: $$\langle n|m\rangle = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

- Norm: $$\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$$ "length" of vector. normalisation condition: $\| |\psi\rangle \| = 1$

- Operators: $\hat{A}$ Act on vectors to give a vector: $\hat{A}|\psi\rangle = |\psi'\rangle$
  - Represent dynamics & physical properties
  - linear: $\hat{A}(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha\hat{A}|\psi_1\rangle + \beta\hat{A}|\psi_2\rangle = \alpha|\psi_1'\rangle + \beta|\psi_2'\rangle$
  - Dirac form: $\hat{A} = \sum_{n,m} a_{n,m}|n\rangle\langle m| = \sum_{n,m} \langle n|\hat{A}|m\rangle |n\rangle\langle m|$

    complex numbers "matrix elements"
  - Identity operator: $\hat{I}$ $\quad \hat{I}|\psi\rangle = |\psi\rangle$

    $\hat{I} = \sum_n |n\rangle\langle n|$

- Hermitian conjugate (adjoint): $\hat{A}^\dagger = \sum_{n,m} a^*_{n,m}|m\rangle\langle n|$

  complex conjugate + transpose (ket $\longleftrightarrow$ bra)

  $|\psi\rangle^\dagger = \langle\psi|$
  - $(\hat{A}|\psi\rangle)^\dagger = \langle\psi|\hat{A}^\dagger$

- Hermitian operators: $\hat{A}^\dagger = \hat{A}$
  - Represent observable quantities physical properties that can be measured.

- Eigenvalues & Eigenvectors: $\hat{A}|v_n\rangle = \lambda_n |v_n\rangle$

  eigenvector eigenvalue

  - For Hermitian $\hat{A}$:
    1. λ_n is real
    2. if $\lambda_m \neq \lambda_n$ then $\langle v_n | v_m \rangle = 0$ orthogonal
    3. Eigenvectors $|v_n\rangle$ form an othonormal basis

  - Physically: eigenstates of Hermitian operators represent states with definite value of associated (observable) physical property, given by corresponding eigenvalue.

- Equation of Motion: Schrödinger Eq^{n}: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$ instantaneous rate of change

  'h-bar' $\frac{h}{2\pi}$ 'Hamiltonian' total energy of system

- Time evolution operator: $\hat{U}(t)$: $\hat{U}(t)|\psi(0)\rangle = |\psi(t)\rangle$ finite change

  - Unitary: $\hat{U}^{\dagger}(t)\,\hat{U}(t) = \hat{I}$ $\quad [\,\hat{U}^{\dagger}(t) = \hat{U}(-t)\,]$ reversibility

- $\hat{H} = \sum_n E_n |n\rangle\langle n|$ $\longrightarrow$ $\hat{U}(t) = \sum_n e^{-iE_n t/\hbar} |n\rangle\langle n|$

energy eigenstates

energy eigenvalues

- $|\psi(0)\rangle = \sum_n \alpha_n(0) |n\rangle$ $\longrightarrow$ $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \sum_n \alpha_n(0) e^{-iE_n t/\hbar} |n\rangle$

• Measurements of Observables: $\hat{A} = \sum_{n=1}^{d} \lambda_n |v_n\rangle\langle v_n|$ $\qquad |\psi\rangle = \sum_{n=1}^{d} \alpha_n |v_n\rangle$

$\alpha_n = \langle v_n|\psi\rangle$

- Possible results: eigenvalues $\lambda_1, \dots, \lambda_d$

$$\mathrm{Prob}(\lambda_k) = |\alpha_k|^2$$

- State after measurement: corresponding eigenstate $|\psi'\rangle = |v_k\rangle$

[For degenerate eigenvalues, $\mathrm{Prob}(\lambda) = \sum_{\substack{k \text{ s.t.} \\ \lambda_k = \lambda}} |\alpha_k|^2$, $|\psi'\rangle = \frac{1}{\sqrt{\mathrm{Prob}(\lambda)}} \sum_{\substack{k \text{ s.t.} \\ \lambda_k = \lambda}} \alpha_k |v_k\rangle$]

• Expectation values: $\langle A\rangle = \sum_n \lambda_n \mathrm{prob}(\lambda_n) = \langle\psi|\hat{A}|\psi\rangle$