GEOMETRY

Through Algebra

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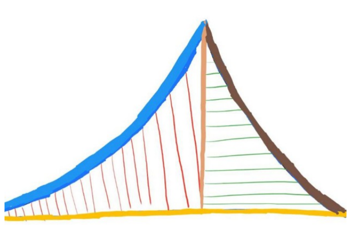

Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1}$$

1.1. Vectors

1.1.1. The direction vector of $\mathbf{AB}$ is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of $\mathbf{AB}, \mathbf{BC}$ and $\mathbf{CA}$.

Solution:

(a) The direction vector of **AB** is

$$= \mathbf{B} - \mathbf{A} \tag{1.1.1.2}$$

$$= \begin{pmatrix} 3-(-5) \\ -3-(-4) \end{pmatrix} \tag{1.1.1.3}$$

$$= \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.1.1.4}$$

(b) The direction vector of **BC** is

$$= \mathbf{C} - \mathbf{B} \tag{1.1.1.5}$$

$$= \begin{pmatrix} 4-(3) \\ 0-(-3) \end{pmatrix} \tag{1.1.1.6}$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.1.7}$$

(c) The direction vector of **CA** is

$$= \mathbf{A} - \mathbf{C} \tag{1.1.1.8}$$

$$= \begin{pmatrix} -5-(4) \\ -4-(0) \end{pmatrix} \tag{1.1.1.9}$$

$$= \begin{pmatrix} -9 \\ -4 \end{pmatrix} \tag{1.1.1.10}$$

1.1.2. The length of side **BC** is

$$\|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^{\top} \mathbf{B} - \mathbf{A}} \tag{1.1.2.1}$$

where,

$$\mathbf{A}^\top \triangleq \begin{pmatrix} -5 & -4 \end{pmatrix} \tag{1.1.2.2}$$

Solution:

Given,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.2.3}$$

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{(\mathbf{B} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C})} \tag{1.1.2.4}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.2.5}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.1.2.6}$$

$$(\mathbf{B} - \mathbf{C})^\top = \begin{pmatrix} -1 \\ -3 \end{pmatrix}^\top = \begin{pmatrix} -1 & -3 \end{pmatrix} \tag{1.1.2.7}$$

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} -1 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.1.2.8}$$

$$= 1 + 9 \tag{1.1.2.9}$$

$$= 10 \tag{1.1.2.10}$$

$$\sqrt{(\mathbf{B} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C})} = \sqrt{10} \tag{1.1.2.11}$$

$$\implies \|\mathbf{B} - \mathbf{C}\| = \sqrt{10} \tag{1.1.2.12}$$

Now solving for **AB**,

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}-8\\-1\end{pmatrix} \tag{1.1.2.13}$$

$$\left\|\mathbf{A}-\mathbf{B}\right\|=\sqrt{\begin{pmatrix}-8 & -1\end{pmatrix}\begin{pmatrix}-8\\-1\end{pmatrix}} \tag{1.1.2.14}$$

$$=\sqrt{(8)^2+(1)^2} \tag{1.1.2.15}$$

$$=\sqrt{65} \tag{1.1.2.16}$$

Now solving for **CA**,

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}9\\4\end{pmatrix} \tag{1.1.2.17}$$

$$\left\|\mathbf{C}-\mathbf{A}\right\|=\sqrt{\begin{pmatrix}9 & 4\end{pmatrix}\begin{pmatrix}9\\4\end{pmatrix}} \tag{1.1.2.18}$$

$$=\sqrt{(9)^2+(4)^2} \tag{1.1.2.19}$$

$$=\sqrt{97} \tag{1.1.2.20}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}=2 \tag{1.1.3.1}$$

Are the given points in (1.1) collinear?

Question : Check the collinearity of $\mathbf{A}, \mathbf{B}, \mathbf{C}$

Solution: Given that,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.3.2}$$

Given that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} < 3 \tag{1.1.3.3}$$

Let

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 4 \\ -4 & -3 & 0 \end{pmatrix} \tag{1.1.3.4}$$

The matrix $\mathbf{R}$ can be row reduced as follows,

$$\begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 4 \\ -4 & -3 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 5R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 8 & 9 \\ -4 & -3 & 0 \end{pmatrix} \tag{1.1.3.5}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 4R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 8 & 9 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.1.3.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 8 & 9 \\ 0 & 0 & -5 \end{pmatrix} \tag{1.1.3.7}$$

There are no zero rows. So,

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.8}$$

Hence, from (1.1.3.3) the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. From Fig. 1.1, We can see that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear .

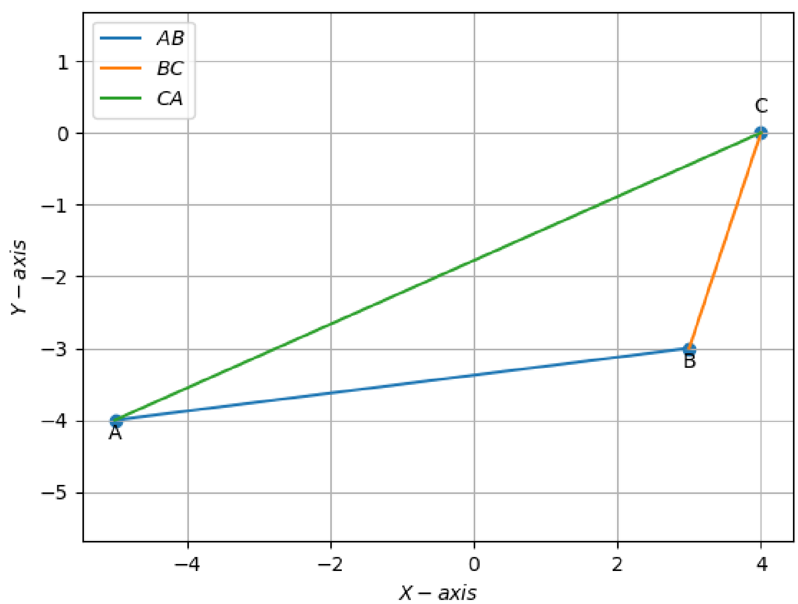

Figure 1.1: $\mathbf{A}, \mathbf{B}, \mathbf{C}$ plot

1.1.4. The parametric form of the equation of $\mathbf{AB}$ is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of $\mathbf{AB}$. Now Find the parameteric equations of $\mathbf{AB}, \mathbf{BC}$ and $\mathbf{CA}$.

Solution:

(a) Parametric form of $\mathbf{AB}$:

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.3}$$

where,

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.4}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.4.5}$$

$$= \begin{pmatrix} 3 - (-5) \\ (-3) - (-4) \end{pmatrix} \tag{1.1.4.6}$$

$$\implies \mathbf{m} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.1.4.7}$$

therefore,

$$\mathbf{AB} : \mathbf{x} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} + k \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.1.4.8}$$

(b) Parametric form of line **BC** :

$$\mathbf{x} = \mathbf{B} + k\mathbf{m} \tag{1.1.4.9}$$

$$\text{BC} : \mathbf{x} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + k \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.4.10}$$

(c) Parametric form of line **CA** :

$$\mathbf{x} = \mathbf{C} + k\mathbf{m} \tag{1.1.4.11}$$

$$\text{CA} : \mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + k \begin{pmatrix} -9 \\ -4 \end{pmatrix} \tag{1.1.4.12}$$

1.1.5. The normal form of the equation of **AB** is

$$\mathbf{n}^{\top} \left(\mathbf{x} - \mathbf{A} \right) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^{\top}\mathbf{m} = \mathbf{n}^{\top} \left(\mathbf{B} - \mathbf{A} \right) = 0 \tag{1.1.5.2}$$

or,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

Find the normal form of the equations of **AB**, **BC** and **CA**

Solution: :

The normal equation for the side **AB** is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.1.5.4}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{1.1.5.5}$$

Now our task is to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. As given.

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.6}$$

Here, $\mathbf{m} = \mathbf{B} - \mathbf{A}$ for side **AB**

$$\implies \mathbf{m} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.5.7}$$

$$= \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.1.5.8}$$

Now as we have obtained vector $\mathbf{m}$.we can use this to obtain vector $\mathbf{n}$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} \tag{1.1.5.9}$$

The transpose of $\mathbf{n}$ is

$$\mathbf{n}^\top = \begin{pmatrix} 1 & -8 \end{pmatrix} \tag{1.1.5.10}$$

Hence the normal equation of side $\mathbf{AB}$ is

$$\begin{pmatrix} 1 & -8 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -8 \end{pmatrix} \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.5.11}$$

$$\implies \begin{pmatrix} 1 & -8 \end{pmatrix} \mathbf{x} = 27 \tag{1.1.5.12}$$

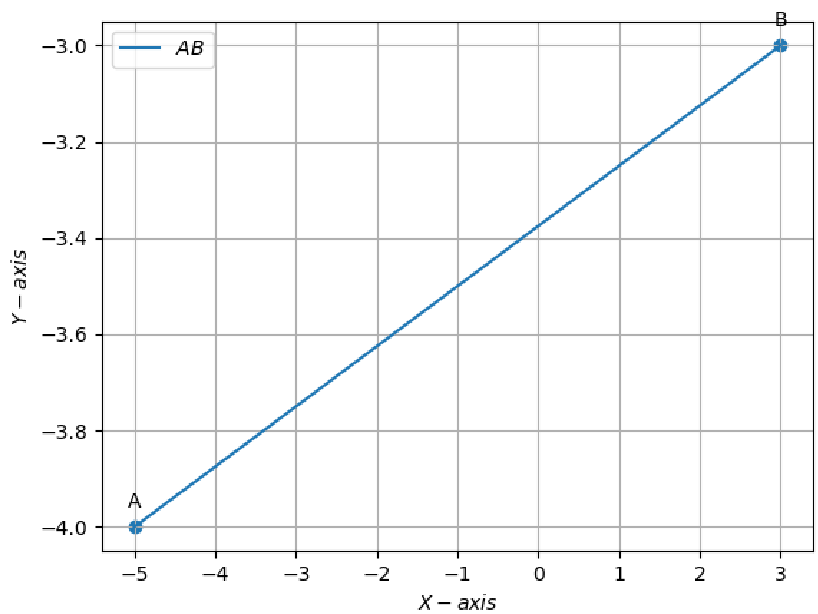

Figure 1.2: The line $\mathbf{AB}$ plotted

Similarly,

$$\implies \mathbf{BC} : \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 12 \tag{1.1.5.13}$$

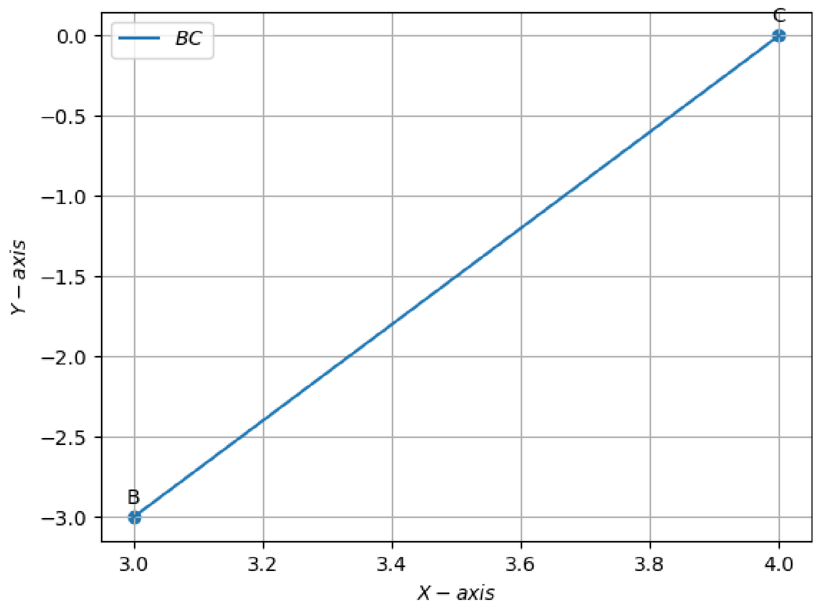

Figure 1.3: The line **BC** plotted

$$\implies \mathbf{CA} : \begin{pmatrix} -4 & 9 \end{pmatrix} \mathbf{x} = -16 \tag{1.1.5.14}$$

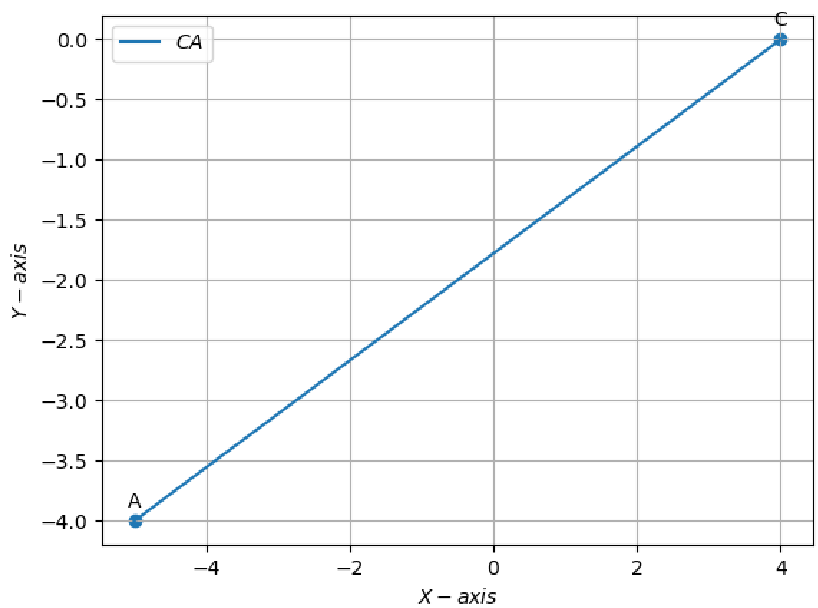

Figure 1.4: The line **CA** plotted

1.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| \tag{1.1.6.1}$$

where,

$$\mathbf{A}\times\mathbf{B} \triangleq \begin{vmatrix} -5 & 3 \\ -4 & -3 \end{vmatrix} \tag{1.1.6.2}$$

Find the area of $\triangle ABC$.

Solution:

Given,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}; \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.6.3}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \end{pmatrix} \tag{1.1.6.4}$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -9 \\ -4 \end{pmatrix} \tag{1.1.6.5}$$

$$\therefore (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} -8 & -9 \\ -1 & -4 \end{vmatrix} \tag{1.1.6.6}$$

$$= (-8 \times -4) - (-9 \times -1) \tag{1.1.6.7}$$

$$= 32 - 9 \tag{1.1.6.8}$$

$$= 23 \tag{1.1.6.9}$$

$$\implies \frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| = \frac{1}{2} \left\| 23 \right\| = \frac{23}{2} \tag{1.1.6.10}$$

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} \tag{1.1.7.1}$$

Solution:

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$,

(a) Finding the value of angle A

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}8\\1\end{pmatrix} \tag{1.1.7.2}$$

and

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}9\\4\end{pmatrix} \tag{1.1.7.3}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{A}\|=\sqrt{65} \tag{1.1.7.4}$$

$$\|\mathbf{C}-\mathbf{A}\|=\sqrt{97} \tag{1.1.7.5}$$

and by doing matrix multiplication we get,

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=\begin{pmatrix}8 & 1\end{pmatrix}\begin{pmatrix}9\\4\end{pmatrix}=76 \tag{1.1.7.6}$$

So, we get

$$\cos A=\frac{76}{\sqrt{65}\sqrt{97}} \tag{1.1.7.7}$$

$$=\frac{76}{\sqrt{6305}} \tag{1.1.7.8}$$

$$\implies A=\cos^{-1}\frac{76}{\sqrt{6305}} \tag{1.1.7.9}$$

(b) Finding the value of angle B

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}1\\3\end{pmatrix} \tag{1.1.7.10}$$

and

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}-8\\-1\end{pmatrix} \tag{1.1.7.11}$$

also calculating the values of norms

$$\|\mathbf{C}-\mathbf{B}\|=\sqrt{10} \tag{1.1.7.12}$$

$$\|\mathbf{A}-\mathbf{B}\|=\sqrt{65} \tag{1.1.7.13}$$

and by doing matrix multiplication we get,

$$(\mathbf{C}-\mathbf{B})^{\top}(\mathbf{A}-\mathbf{B})=\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}-8\\-1\end{pmatrix}=-11 \tag{1.1.7.14}$$

So, we get

$$\cos B=\frac{-11}{\sqrt{10}\sqrt{65}} \tag{1.1.7.15}$$

$$=\frac{-11}{5\sqrt{26}} \tag{1.1.7.16}$$

$$\implies B=\cos^{-1}\frac{-11}{5\sqrt{26}} \tag{1.1.7.17}$$

(c) Finding the value of angle C

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -9 \\ -4 \end{pmatrix} \tag{1.1.7.18}$$

and

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.1.7.19}$$

also calculating the values of norms

$$\|\mathbf{A} - \mathbf{C}\| = \sqrt{97} \tag{1.1.7.20}$$

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{10} \tag{1.1.7.21}$$

and by doing matrix multiplication we get,

$$\begin{aligned}(\mathbf{A} - \mathbf{C})^\top(\mathbf{B} - \mathbf{C}) &= \begin{pmatrix} -9 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} \\ &= 21\end{aligned} \tag{1.1.7.22}$$

so,

$$\cos C = \frac{21}{\sqrt{97}\sqrt{10}} \tag{1.1.7.23}$$

$$= \frac{21}{\sqrt{970}} \tag{1.1.7.24}$$

$$\implies C = \cos^{-1} \frac{21}{\sqrt{970}} \tag{1.1.7.25}$$

All codes for this section are available at

geometry/Triangle/Vectors/codes/Triangle_sides.py