GEOMETRY

Through Algebra

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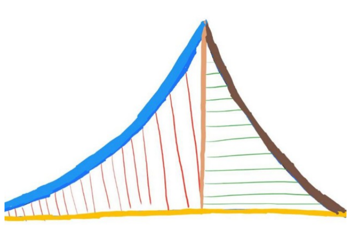

Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1}$$

1.1. Perpendicular Bisector

1.1.1. The equation of the perpendicular bisector of $\mathbf{BC}$ is

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B} - \mathbf{C})^\top = 0 \tag{1.1.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of $\mathbf{AB}, \mathbf{BC}$ and $\mathbf{CA}$.

Solution:

(a) Equation for the perpendicular bisector of $\mathbf{BC}$:

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B} - \mathbf{C})^\top = 0 \tag{1.1.1.2}$$

$$\mathbf{x}(\mathbf{B} - \mathbf{C})^\top = \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B} - \mathbf{C})^\top \tag{1.1.1.3}$$

On substituting the values,

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix} \tag{1.1.1.4}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.1.1.5}$$

$$(\mathbf{B}-\mathbf{C})^\top = \begin{pmatrix} -1 & -3 \end{pmatrix} \tag{1.1.1.6}$$

solving using matrix multiplication

$$(\mathbf{B}-\mathbf{C})^\top \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = 0 \tag{1.1.1.7}$$

$$(\mathbf{B}-\mathbf{C})^\top \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix} -1 & -3 \end{pmatrix} \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix} \tag{1.1.1.8}$$

$$= \frac{-7}{2} + \frac{9}{2} \tag{1.1.1.9}$$

$$= 1 \tag{1.1.1.10}$$

Therefore equation for perpendicular bisector of **BC** is

$$\begin{pmatrix} -1 & -3 \end{pmatrix} \mathbf{x} = 1 \tag{1.1.1.11}$$

(b) Similarly the equation for the perpendicular bisector of **AB** :

$$\left(\mathbf{x} - \frac{\mathbf{A}+\mathbf{B}}{2}\right)(\mathbf{A}-\mathbf{B})^\top = 0 \tag{1.1.1.12}$$

$$\mathbf{x}(\mathbf{A}-\mathbf{B})^\top = \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)(\mathbf{A}-\mathbf{B})^\top \tag{1.1.1.13}$$

On substituting the values,

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \begin{pmatrix} -1 \\ -\frac{-7}{2} \end{pmatrix} \tag{1.1.1.14}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -8 \\ -1 \end{pmatrix} \tag{1.1.1.15}$$

$$(\mathbf{A}-\mathbf{B})^\top = \begin{pmatrix} -8 & -1 \end{pmatrix} \tag{1.1.1.16}$$

solving using matrix multiplication yields

$$(\mathbf{A}-\mathbf{B})^\top \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = 0 \tag{1.1.1.17}$$

$$(\mathbf{A}-\mathbf{B})^\top \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix} -8 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.1.18}$$

$$= \frac{23}{2} \tag{1.1.1.19}$$

Therefore equation for perpendicular bisector of **AB** is

$$\begin{pmatrix} -8 & -1 \end{pmatrix} \mathbf{x} = \frac{23}{2} \tag{1.1.1.20}$$

(c) Similarly the equation for the perpendicular bisector of **CA** :

$$\left(\mathbf{x} - \frac{\mathbf{C}+\mathbf{A}}{2}\right) (\mathbf{C}-\mathbf{A})^\top = 0 \tag{1.1.1.21}$$

$$\mathbf{x}\,(\mathbf{C}-\mathbf{A})^\top = \left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) (\mathbf{C}-\mathbf{A})^\top \tag{1.1.1.22}$$

On substituting the values,

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \tag{1.1.1.23}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \tag{1.1.1.24}$$

$$(\mathbf{C}-\mathbf{A})^\top = \begin{pmatrix} 9 & 4 \end{pmatrix} \tag{1.1.1.25}$$

solving using matrix multiplication

$$(\mathbf{C}-\mathbf{A})^\top \left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = 0 \tag{1.1.1.26}$$

$$(\mathbf{C}-\mathbf{A})^\top \left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix} 9 & 4 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \qquad = \frac{-25}{2} \tag{1.1.1.27}$$

Therefore equation for perpendicular bisector of **CA** is

$$\begin{pmatrix} 9 & 4 \end{pmatrix} \mathbf{x} = \frac{-25}{2} \tag{1.1.1.28}$$

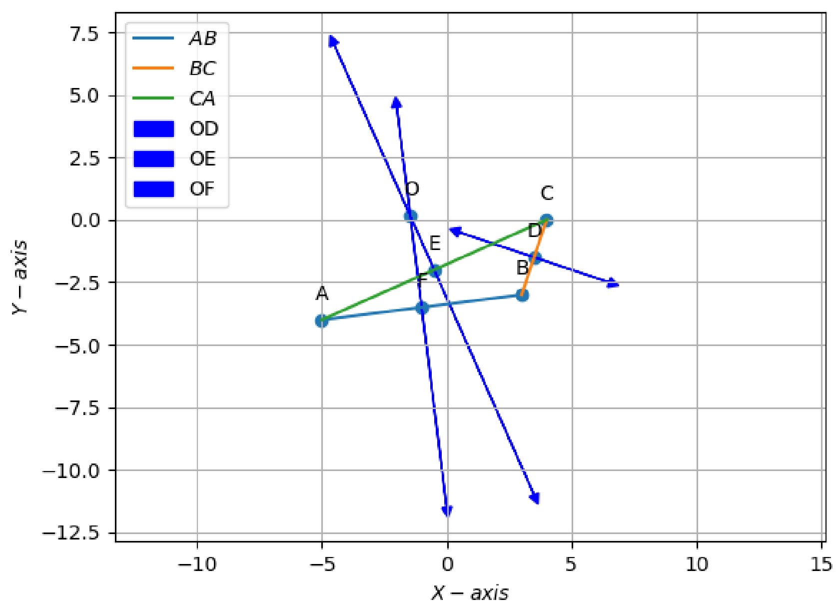

Figure 1.1: Plot of the perpendicular bisectors

1.1.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of $\mathbf{AB}$ and $\mathbf{AC}$.

Solution:

Given,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{1.1.2.1}$$

Vector equation of perpendicular bisector of $\mathbf{A} - \mathbf{B}$ is

$$(\mathbf{A} - \mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{A} + \mathbf{B}}{2} \right) = 0 \tag{1.1.2.2}$$

where,

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} -2 \\ -7 \end{pmatrix} \tag{1.1.2.3}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.2.4}$$

$$= \begin{pmatrix} -8 \\ -1 \end{pmatrix} \tag{1.1.2.5}$$

$$\implies (\mathbf{A}-\mathbf{B})^{\top} = \begin{pmatrix} -8 & -1 \end{pmatrix} \tag{1.1.2.6}$$

$\therefore$ The vector equation of $\mathbf{O}-\mathbf{F}$ is

$$\begin{pmatrix} -8 & -1 \end{pmatrix} \left(\mathbf{x} - \frac{1}{2} \begin{pmatrix} -2 \\ 7 \end{pmatrix} \right) = 0 \tag{1.1.2.7}$$

$$\implies \begin{pmatrix} -8 & -1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \left(\begin{pmatrix} -8 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -7 \end{pmatrix} \right) \tag{1.1.2.8}$$

Performing matrix multiplication yields

$$\begin{pmatrix} -8 & -1 \end{pmatrix} \mathbf{x} = \frac{23}{2} \tag{1.1.2.9}$$

Similarly vector equation of perpendicular bisector of $\mathbf{A}-\mathbf{C}$ is

$$(\mathbf{A}-\mathbf{C})^{\top} \left(\mathbf{x} - \frac{\mathbf{A}+\mathbf{C}}{2} \right) = 0 \tag{1.1.2.10}$$

where,

$$\mathbf{A}+\mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.2.11}$$

$$= \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{1.1.2.12}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.2.13}$$

$$= \begin{pmatrix} -9 \\ -4 \end{pmatrix} \tag{1.1.2.14}$$

$$\implies (\mathbf{A}-\mathbf{C})^\top = \begin{pmatrix} -9 & -4 \end{pmatrix} \tag{1.1.2.15}$$

$\therefore$ The vector equation of $\mathbf{O}-\mathbf{E}$ is

$$\begin{pmatrix} -9 & -4 \end{pmatrix} \left(\mathbf{x} - \frac{1}{2} \begin{pmatrix} -1 \\ -4 \end{pmatrix} \right) = 0 \tag{1.1.2.16}$$

$$\implies \begin{pmatrix} -9 & -4 \end{pmatrix} \mathbf{x} = \frac{1}{2} \left(\begin{pmatrix} -9 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ -4 \end{pmatrix} \right) \tag{1.1.2.17}$$

Performing matrix multiplication yields

$$\begin{pmatrix} -9 & -4 \end{pmatrix} \mathbf{x} = \frac{25}{2} \tag{1.1.2.18}$$

Thus, solving equations (1.1.2.9) and (1.1.2.18) :

$$\begin{pmatrix} -8 & -1 & \frac{23}{2} \\ -9 & -4 & \frac{25}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{-R_1}{8}} \begin{pmatrix} 1 & \frac{1}{8} & -\frac{23}{16} \\ -9 & -4 & \frac{25}{2} \end{pmatrix} \tag{1.1.2.19}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 9R_1} \begin{pmatrix} 1 & \frac{1}{8} & -\frac{23}{16} \\ 0 & -\frac{23}{8} & -\frac{7}{16} \end{pmatrix} \tag{1.1.2.20}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{8R_2}{23}} \begin{pmatrix} 1 & \frac{1}{8} & -\frac{23}{16} \\ 0 & 1 & \frac{7}{46} \end{pmatrix} \tag{1.1.2.21}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{8}} \begin{pmatrix} 1 & 0 & -\frac{67}{46} \\ 0 & 1 & \frac{7}{46} \end{pmatrix} \tag{1.1.2.22}$$

Therefore, the point of intersection of perpendicular bisectors of $\mathbf{A} - \mathbf{B}$ and $\mathbf{A} - \mathbf{C}$ is

$$\mathbf{O} = \begin{pmatrix} -\frac{67}{46} \\ \frac{7}{46} \end{pmatrix}. \tag{1.1.2.23}$$

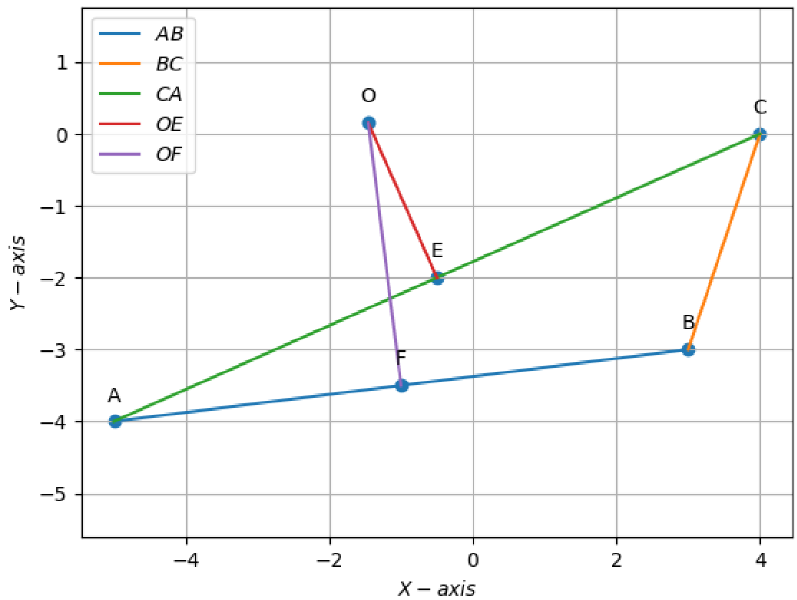

Figure 1.2: $\mathbf{O}-\mathbf{E}$ and $\mathbf{O}-\mathbf{F}$ are perpendicular bisectors of $\mathbf{A}-\mathbf{C}$ and $\mathbf{A}-\mathbf{B}$ respectively

1.1.3. Verify that $\mathbf{O}$ satisfies (1.1.1.1). $\mathbf{O}$ is known as the circumcentre.

Solution:

From the equation (1.1.2.23). we get,

$$\mathbf{O} = \begin{pmatrix} -\frac{67}{46} \\ \frac{7}{46} \end{pmatrix} \tag{1.1.3.1}$$

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C})^{\top} = 0 \tag{1.1.3.2}$$

when substitute value in the above equation we get,

$$= \left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C})^\top \tag{1.1.3.3}$$

$$= \left(\begin{pmatrix} -\frac{67}{46} \\ \frac{7}{46} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 7 \\ -3 \end{pmatrix}\right)\begin{pmatrix} -1 & -3 \end{pmatrix} \tag{1.1.3.4}$$

$$= \begin{pmatrix} -\frac{114}{23} \\ \frac{38}{23} \end{pmatrix}\begin{pmatrix} -1 & -3 \end{pmatrix} \tag{1.1.3.5}$$

$$= 0 \tag{1.1.3.6}$$

It is hence proved that **O** satisfies the equation (1.1.1.1)

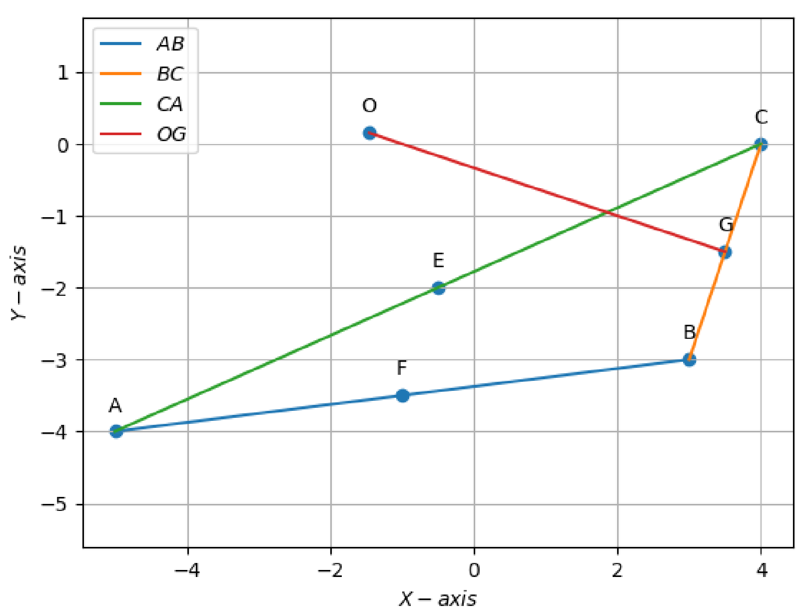

Figure 1.3: Circumcenter plotted using python

1.1.4. Verify that

$$\mathbf{OA} = \mathbf{OB} = \mathbf{OC} \tag{1.1.4.1}$$

Solution:

Given

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.4.3}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.4.4}$$

From equation (1.1.2.23) :

$$\mathbf{O} = \begin{pmatrix} -\frac{67}{46} \\ \frac{7}{46} \end{pmatrix} \tag{1.1.4.5}$$

$$= \begin{pmatrix} -1.45652 \\ 0.15217 \end{pmatrix} \tag{1.1.4.6}$$

(a) Solving of **OA** :

$$\mathbf{OA} = \sqrt{(\mathbf{O}-\mathbf{A})^\top(\mathbf{O}-\mathbf{A})} \tag{1.1.4.7}$$

$$= \sqrt{\begin{pmatrix} \frac{163}{46} & \frac{191}{46} \end{pmatrix} \begin{pmatrix} \frac{163}{46} \\ \frac{191}{46} \end{pmatrix}} \tag{1.1.4.8}$$

$$= \sqrt{\frac{63050}{2116}} \tag{1.1.4.9}$$

$$= \frac{\sqrt{63050}}{46} \tag{1.1.4.10}$$

(b) Solving of **OB** :

$$\mathbf{OB} = \sqrt{(\mathbf{O}-\mathbf{B})^\top(\mathbf{O}-\mathbf{B})} \tag{1.1.4.11}$$

$$= \sqrt{\begin{pmatrix} \frac{-205}{46} & \frac{145}{46} \end{pmatrix} \begin{pmatrix} \frac{-205}{46} \\ \frac{145}{46} \end{pmatrix}} \tag{1.1.4.12}$$

$$= \sqrt{\frac{63050}{2116}} \tag{1.1.4.13}$$

$$= \frac{\sqrt{63050}}{46} \tag{1.1.4.14}$$

(c) Solving of **OC** :

$$\mathbf{OC} = \sqrt{(\mathbf{O}-\mathbf{C})^\top(\mathbf{O}-\mathbf{C})} \tag{1.1.4.15}$$

$$= \sqrt{\begin{pmatrix} \frac{-251}{46} & \frac{7}{46} \end{pmatrix} \begin{pmatrix} \frac{-251}{46} \\ \frac{7}{46} \end{pmatrix}} \tag{1.1.4.16}$$

$$= \sqrt{\frac{63050}{2116}} \tag{1.1.4.17}$$

$$= \frac{\sqrt{63050}}{46} \tag{1.1.4.18}$$

From above equations (1.1.4.10) ,(1.1.4.14) and (1.1.4.18) :

$$\mathbf{OA} = \mathbf{OB} = \mathbf{OC} \tag{1.1.4.19}$$

Hence verified.

1.1.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$\mathbf{R} = \mathbf{OA} \tag{1.1.5.1}$$

This is known as the circumradius.

Solution: Given

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.5.2}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.5.3}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.5.4}$$

From (1.1.2.23), the circumcentre is

$$\mathbf{O} = \begin{pmatrix} \frac{-67}{46} \\ \frac{7}{46} \end{pmatrix} \tag{1.1.5.5}$$

Now we will calculate the radius,

$$\mathbf{R} = \mathbf{OA} \tag{1.1.5.6}$$

$$= \|\mathbf{A} - \mathbf{O}\| \tag{1.1.5.7}$$

$$= \left\| \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} \frac{-67}{46} \\ \frac{7}{46} \end{pmatrix} \right\| \tag{1.1.5.8}$$

$$= \left\| \begin{pmatrix} \frac{-163}{46} \\ \frac{-191}{46} \end{pmatrix} \right\| \tag{1.1.5.9}$$

$$= \sqrt{\begin{pmatrix} \frac{-163}{46} & \frac{-191}{46} \end{pmatrix} \begin{pmatrix} \frac{-163}{46} \\ \frac{-191}{46} \end{pmatrix}} \tag{1.1.5.10}$$

$$= \sqrt{\frac{63050}{2116}} \tag{1.1.5.11}$$

$$= \frac{\sqrt{63050}}{46} \tag{1.1.5.12}$$

see Fig. 1.4

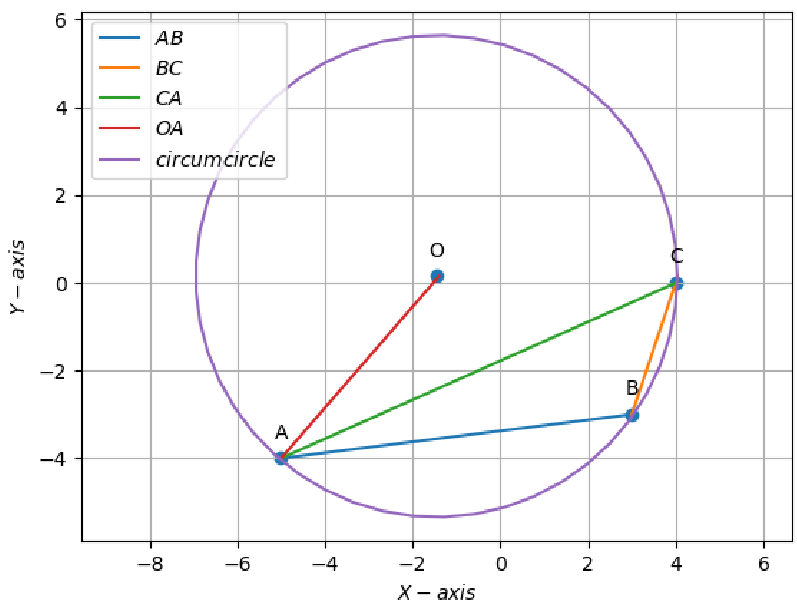

Figure 1.4: circumcircle of Triangle ABC with centre O

1.1.6. Verify that

$$\angle BOC = 2\angle BAC. \tag{1.1.6.1}$$

Solution:

(a) To find the value of $\angle BOC$:

$$\mathbf{B}-\mathbf{O}=\begin{pmatrix}\frac{205}{46}\\ \frac{-145}{46}\end{pmatrix} \tag{1.1.6.2}$$

$$\mathbf{C}-\mathbf{O}=\begin{pmatrix}\frac{251}{46}\\ \frac{-7}{46}\end{pmatrix} \tag{1.1.6.3}$$

$$\implies (\mathbf{B}-\mathbf{O})^{\top}(\mathbf{C}-\mathbf{O})=\frac{26235}{1058} \tag{1.1.6.4}$$

$$=24.79678 \tag{1.1.6.5}$$

$$\implies \|\mathbf{B}-\mathbf{O}\|=\frac{\sqrt{63050}}{46} \tag{1.1.6.6}$$

$$\|\mathbf{C}-\mathbf{O}\|=\frac{\sqrt{63050}}{46} \tag{1.1.6.7}$$

Thus,

$$\cos BOC=\frac{(\mathbf{B}-\mathbf{O})^{\top}(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{O}\|}=0.83219 \tag{1.1.6.8}$$

$$\implies \angle BOC=\cos^{-1}(0.83219) \tag{1.1.6.9}$$

$$=33.6756^{\circ} \tag{1.1.6.10}$$

(b) To find the value of $\angle BAC$:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.1.6.11}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \tag{1.1.6.12}$$

$$\implies (\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = 76 \tag{1.1.6.13}$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{65} \tag{1.1.6.14}$$

$$\|\mathbf{C} - \mathbf{A}\| = \sqrt{97} \tag{1.1.6.15}$$

Thus,

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\|} = 0.95713 \tag{1.1.6.16}$$

$$\implies \angle BAC = \cos^{-1}(0.95713) \tag{1.1.6.17}$$

$$= 16.8375^\circ \tag{1.1.6.18}$$

$$2 \times \angle BAC = 33.675^\circ \tag{1.1.6.19}$$

From (1.1.6.10) and (1.1.6.18),

$$2 \times \angle BAC = \angle BOC \tag{1.1.6.20}$$

Hence Verified.

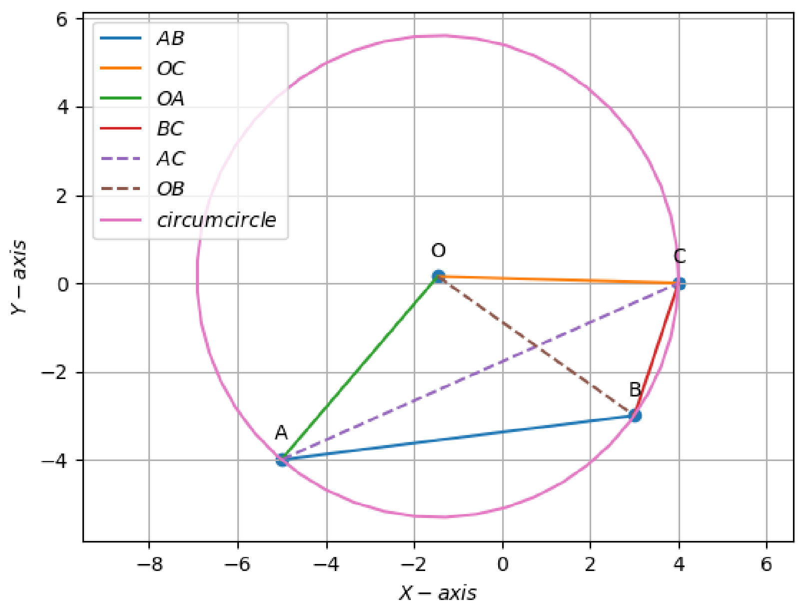

Figure 1.5: $\angle BOC$ and $\angle BAC$

1.1.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.1.7.1}$$

Find θ if

$$\mathbf{C} - \mathbf{O} = \mathbf{P}\left(\mathbf{A} - \mathbf{O}\right) \tag{1.1.7.2}$$

Solution:

$$\mathbf{C}-\mathbf{O}=\begin{pmatrix}-4\\\frac{-3}{4}\end{pmatrix} \tag{1.1.7.3}$$

$$\mathbf{A}-\mathbf{O}=\begin{pmatrix}-3\\\frac{-11}{4}\end{pmatrix} \tag{1.1.7.4}$$

$$\mathbf{P}=\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix} \tag{1.1.7.5}$$

$$\mathbf{C}-\mathbf{O}=\mathbf{P}\left(\mathbf{A}-\mathbf{O}\right) \tag{1.1.7.6}$$

Now from (1.1.7.6)

$$\begin{pmatrix}\frac{251}{46}\\\frac{-7}{46}\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\frac{-163}{46}\\\frac{-191}{46}\end{pmatrix} \tag{1.1.7.7}$$

solving using matrix multiplication,we get

$$\begin{pmatrix}\frac{251}{46}\\\frac{-7}{46}\end{pmatrix}=\begin{pmatrix}\frac{-163}{46}\cos\theta+\frac{191}{46}\sin\theta\\\frac{-163}{46}\sin\theta+\frac{-191}{46}\cos\theta\end{pmatrix} \tag{1.1.7.8}$$

Comparing on Both sides ,we get

$$\frac{-163}{46}\cos\theta+\frac{191}{46}\sin\theta=\frac{251}{46} \tag{1.1.7.9}$$

$$\frac{-163}{46}\sin\theta+\frac{-191}{46}\cos\theta=\frac{-7}{46} \tag{1.1.7.10}$$

On solving equations (1.1.7.9) and (1.1.7.10)

$$\cos\theta = 0.80365 \tag{1.1.7.11}$$

$$\sin\theta = 0.59423 \tag{1.1.7.12}$$

$$\theta = \cos^{-1}(0.80365) \tag{1.1.7.13}$$

$$= 36.55 \tag{1.1.7.14}$$

$$\therefore \theta = 36.55^\circ \tag{1.1.7.15}$$

All codes for this section are available at

$$\boxed{\text{geometry/Triangle/perp\_Bisector/codes/Allperp\_bisectors.py}}$$