GEOMETRY

Through Algebra

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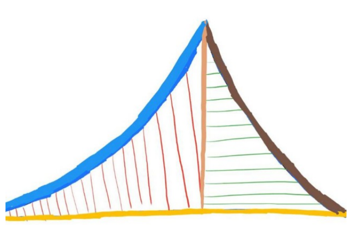

Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1}$$

1.1. Angular Bisector

1.1.1. Let $\mathbf{D_3}, \mathbf{E_3}, \mathbf{F_3}$, be points on $\mathbf{AB}, \mathbf{BC}$ and $\mathbf{CA}$ respectively such that

$$AE_3 = AF_3 = m, \tag{1.1.1.1}$$

$$BD_3 = BF_3 = n, \tag{1.1.1.2}$$

$$CD_3 = CE_3 = p \tag{1.1.1.3}$$

Obtain $\mathbf{m}, \mathbf{n}, \mathbf{p}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Solution:

$$a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{10} \tag{1.1.1.4}$$

$$b = \|\mathbf{C} - \mathbf{A}\| = \sqrt{97} \tag{1.1.1.5}$$

$$c = \|\mathbf{A} - \mathbf{B}\| = \sqrt{65} \tag{1.1.1.6}$$

From the given information,

$$a = m + n, \tag{1.1.1.7}$$

$$b = n + p, \tag{1.1.1.8}$$

$$c = m + p \tag{1.1.1.9}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.10}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.11}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.1.1.12}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.13}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.14}$$

yielding to :

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.1.1.15}$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{65}+\sqrt{97}-\sqrt{10}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{10}+\sqrt{65}-\sqrt{97}}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{10}+\sqrt{97}-\sqrt{65}}{2} \end{aligned} \tag{1.1.1.16}$$

on solving above equations we get

$$p = 7.374418944 \tag{1.1.1.17}$$

$$m = 0.687838803 \tag{1.1.1.18}$$

$$n = 2.474438856 \tag{1.1.1.19}$$

1.1.2. Using section formula, find $\mathbf{D_3}, \mathbf{E_3}, \mathbf{F_3}$.

Solution: Given

$$\mathbf{D_3} = \frac{m\mathbf{C} + n\mathbf{B}}{m+n}, \ \mathbf{E_3} = \frac{n\mathbf{A} + p\mathbf{C}}{n+p}, \ \mathbf{F_3} = \frac{p\mathbf{B} + m\mathbf{A}}{p+m} \tag{1.1.2.1}$$

Here,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{1.1.2.2}$$

$$p = 7.374418944, \ m = 0.687838803, \ n = 2.474438856, \tag{1.1.2.3}$$

On substituting (1.1.2.2) and (1.1.2.3) in (1.1.2.1) We get

$$\mathbf{D_3} = \frac{0.687838803 \begin{pmatrix} 4 \\ 0 \end{pmatrix} + 2.474438856 \begin{pmatrix} 3 \\ -3 \end{pmatrix}}{0.687838803 + 2.474438856} \tag{1.1.2.4}$$

$$\mathbf{E_3} = \frac{2.474438856 \begin{pmatrix} -5 \\ -4 \end{pmatrix} + 7.374418944 \begin{pmatrix} 4 \\ 0 \end{pmatrix}}{2.474438856 + 7.374418944} \tag{1.1.2.5}$$

$$\mathbf{F_3} = \frac{7.374418944 \begin{pmatrix} 3 \\ -3 \end{pmatrix} + 0.687838803 \begin{pmatrix} -5 \\ -4 \end{pmatrix}}{7.374418944 + 0.687838803} \tag{1.1.2.6}$$

On solving above equations We get

$$\mathbf{D_3} = \begin{pmatrix} 3.21751373 \\ -2.34745882 \end{pmatrix} \tag{1.1.2.7}$$

$$\mathbf{E_3} = \begin{pmatrix} 1.7388292 \\ -1.0049648 \end{pmatrix} \tag{1.1.2.8}$$

$$\mathbf{F_3} = \begin{pmatrix} 2.31747277 \\ -3.0853159 \end{pmatrix} \tag{1.1.2.9}$$

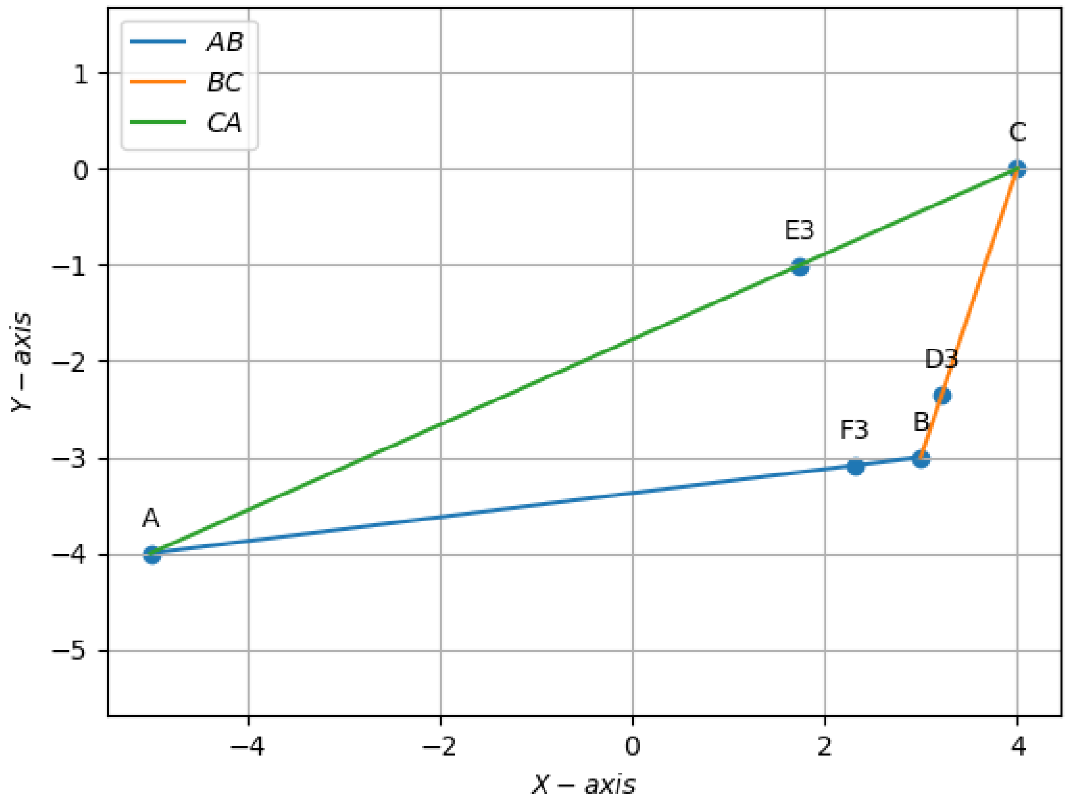

Figure 1.1: Points D3 ,E3 ,F3

1.1.3. Find the circumcentre and circumradius of $\triangle \mathbf{D_3E_3F_3}$. These are the incentre and inradius of $\triangle \mathbf{ABC}$.

Solution: Given

$$\mathbf{D_3} = \begin{pmatrix} 3.21751373 \\ -2.34745882 \end{pmatrix} \tag{1.1.3.1}$$

$$\mathbf{E_3} = \begin{pmatrix} 1.7388292 \\ -1.0049648 \end{pmatrix} \tag{1.1.3.2}$$

$$\mathbf{F_3} = \begin{pmatrix} 2.31747277 \\ -3.0853159 \end{pmatrix} \tag{1.1.3.3}$$

(a) For circumcentre :

Vector equation of $\mathbf{D} - \mathbf{E}$ is

$$(\mathbf{D_3} - \mathbf{E_3})^\top \left(\mathbf{x} - \frac{\mathbf{D_3} + \mathbf{E_3}}{2} \right) = 0 \tag{1.1.3.4}$$

$$(\mathbf{D_3} - \mathbf{F_3})^\top \left(\mathbf{x} - \frac{\mathbf{D_3} + \mathbf{F_3}}{2} \right) = 0 \tag{1.1.3.5}$$

on Substituting the values of $\mathbf{D_3}, \mathbf{E_3}, \mathbf{F_3}$ and solving We get,

$$\begin{pmatrix} 1.47868 & -1.34249 \end{pmatrix} \mathbf{x} = 5.49147 \tag{1.1.3.6}$$

$$\begin{pmatrix} 0.900040 & 0.73785 \end{pmatrix} \mathbf{x} = 0.48655 \tag{1.1.3.7}$$

Thus on solving (1.1.3.6) and (1.1.3.7) using gauss elimination We get

$$\begin{pmatrix}1.47868 & -1.34249 & 5.49147\end{pmatrix} \tag{1.1.3.8}$$

$$\begin{pmatrix}0.900040 & 0.73785 & 0.48655\end{pmatrix} \tag{1.1.3.9}$$

$$\therefore \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2.18209833\\ -2.00232035\end{pmatrix} \tag{1.1.3.10}$$

$$\implies \mathbf{x} = \begin{pmatrix}2.18209833\\ -2.00232035\end{pmatrix} \tag{1.1.3.11}$$

(b) The circiumradius is obtained from $\mathbf{r} = \|\mathbf{I} - \mathbf{D}_3\|$

$$\mathbf{I} = \begin{pmatrix}2.18209833\\ -2.00232035\end{pmatrix} \tag{1.1.3.12}$$

$$\mathbf{D_3} = \begin{pmatrix}3.21751373\\ -2.34745882\end{pmatrix} \tag{1.1.3.13}$$

$$\mathbf{I} - \mathbf{D_3} = \begin{pmatrix}-1.0354154\\ 0.34513547\end{pmatrix} \tag{1.1.3.14}$$

$$\mathbf{r} = \|\mathbf{I} - \mathbf{D_3}\| = \sqrt{(\mathbf{I} - \mathbf{D_3})^\top (\mathbf{I} - \mathbf{D_3})} \tag{1.1.3.15}$$

$$\mathbf{r} = 1.091422715179266 \tag{1.1.3.16}$$

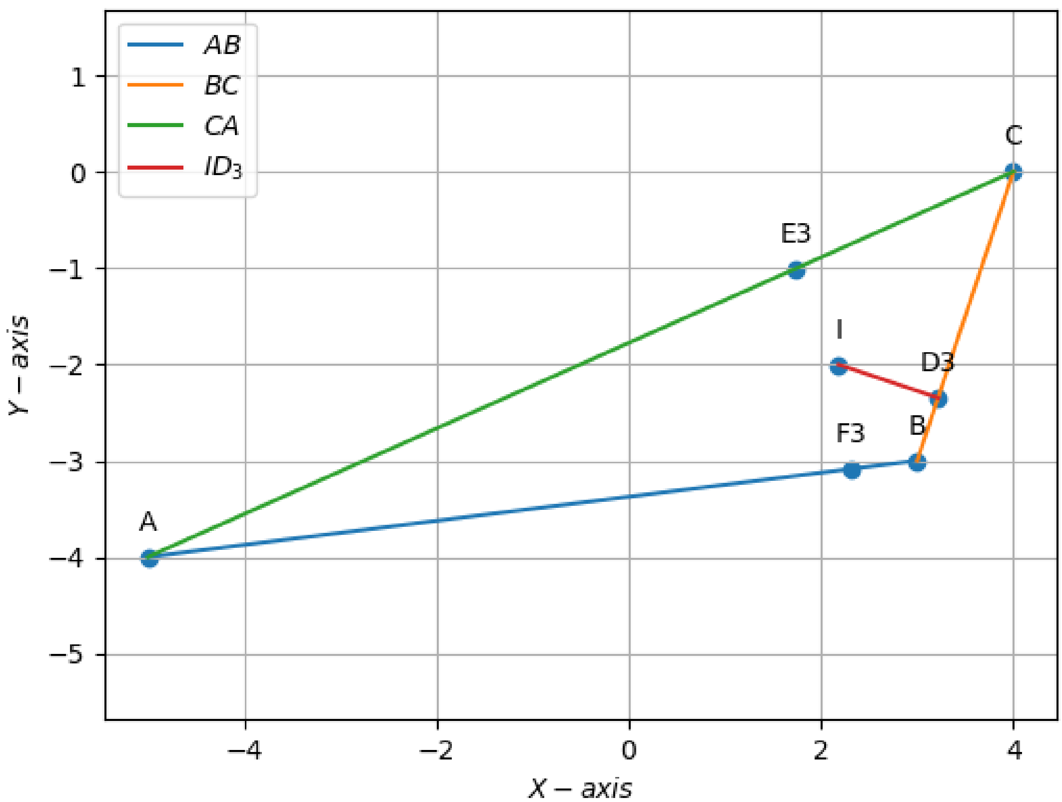

Figure 1.2: incentre and inradius of $\triangle ABC$

1.1.4. Draw the circumcircle of $\triangle\mathbf{D_3E_3F_3}$. This is known as the incircle of $\triangle\mathbf{ABC}$.

Solution:

$$\mathbf{D_3} = \begin{pmatrix} 3.21751373 \\ -2.34745882 \end{pmatrix} \tag{1.1.4.1}$$

$$\mathbf{E_3} = \begin{pmatrix} 1.7388292 \\ -1.0049648 \end{pmatrix} \tag{1.1.4.2}$$

$$\mathbf{F_3} = \begin{pmatrix} 2.31747277 \\ -3.0853159 \end{pmatrix} \tag{1.1.4.3}$$

Incentre

$$\mathbf{I} = \begin{pmatrix} 2.18209833 \\ -2.00232035 \end{pmatrix} \tag{1.1.4.4}$$

Radius

$$\mathbf{r} = 1.091422715179266 \tag{1.1.4.5}$$

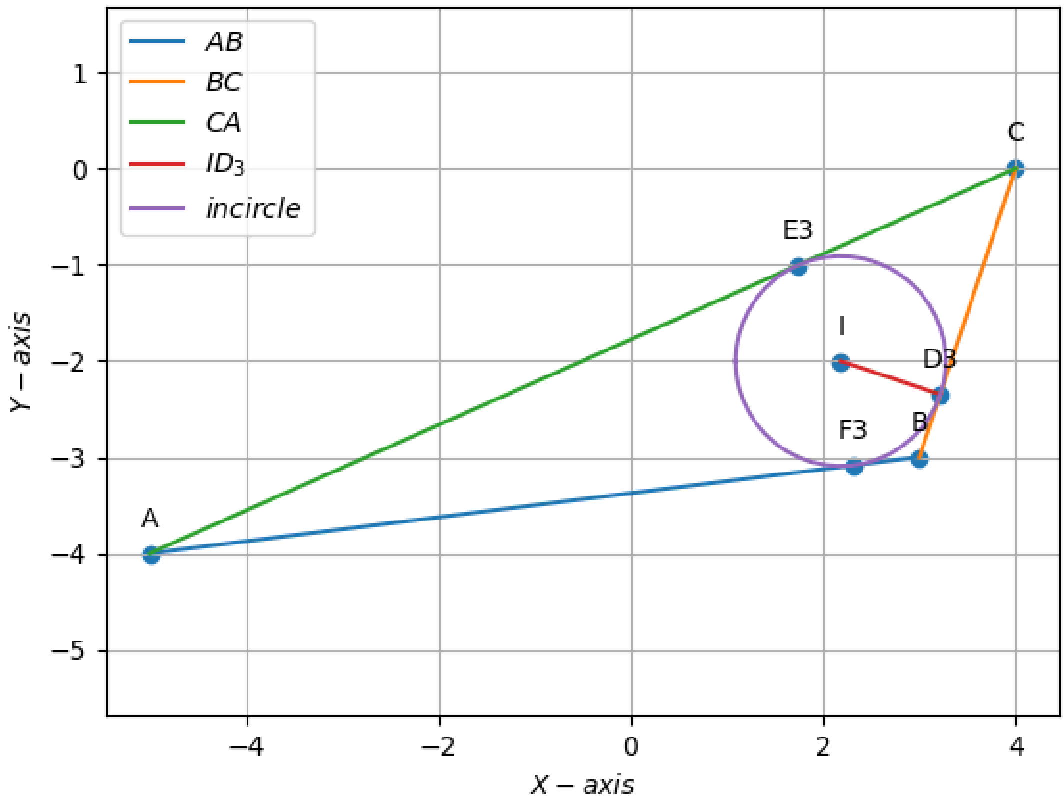

Figure 1.3: incircle of $\triangle ABC$

1.1.5. Verify that

$$\angle BAI = \angle CAI. \tag{1.1.5.1}$$

AI is the bisector of $\angle A$.

Solution:

$$\cos\angle BAI \triangleq \frac{(\mathbf{B}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.2}$$

$$\cos\angle CAI \triangleq \frac{(\mathbf{C}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.3}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{I}$,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{1.1.5.4}$$

$$\mathbf{I} = \begin{pmatrix} 2.18209833 \\ -2.00232035 \end{pmatrix} \tag{1.1.5.5}$$

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.1.5.6}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \tag{1.1.5.7}$$

$$\mathbf{I}-\mathbf{A} = \begin{pmatrix} 7.18209833 \\ 1.99767665 \end{pmatrix} \tag{1.1.5.8}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{A}\| = \sqrt{65} \tag{1.1.5.9}$$

$$\|\mathbf{C}-\mathbf{A}\| = \sqrt{97} \tag{1.1.5.10}$$

$$\|\mathbf{I}-\mathbf{A}\| = 7.454746703 \tag{1.1.5.11}$$

$$\tag{1.1.5.12}$$

(a) calculating for $\angle BAI$:

On substituting the values in (1.1.5.2) ,We get

$$\cos \angle BAI \triangleq \frac{\begin{pmatrix} 8 & 1 \end{pmatrix} \begin{pmatrix} 7.18209833 \\ 1.99767665 \end{pmatrix}}{\sqrt{65} \times 7.4547467039} \tag{1.1.5.13}$$

$$\tag{1.1.5.14}$$

On solving we get,

$$\angle BAI = 8.4187^\circ \tag{1.1.5.15}$$

(b) Calculating for $\angle CAI$:

On substituting the values in (1.1.5.2) ,We get

$$\cos \angle CAI \triangleq \frac{\begin{pmatrix} 9 & 4 \end{pmatrix} \begin{pmatrix} 7.18209833 \\ 1.99767665 \end{pmatrix}}{\sqrt{97} \times 7.4547467039} \tag{1.1.5.16}$$

$$\tag{1.1.5.17}$$

On solving we get,

$$\angle CAI = 8.4187^\circ \tag{1.1.5.18}$$

Therefore $\angle BAI = \angle CAI$. and AI is the bisector of $\angle A$.

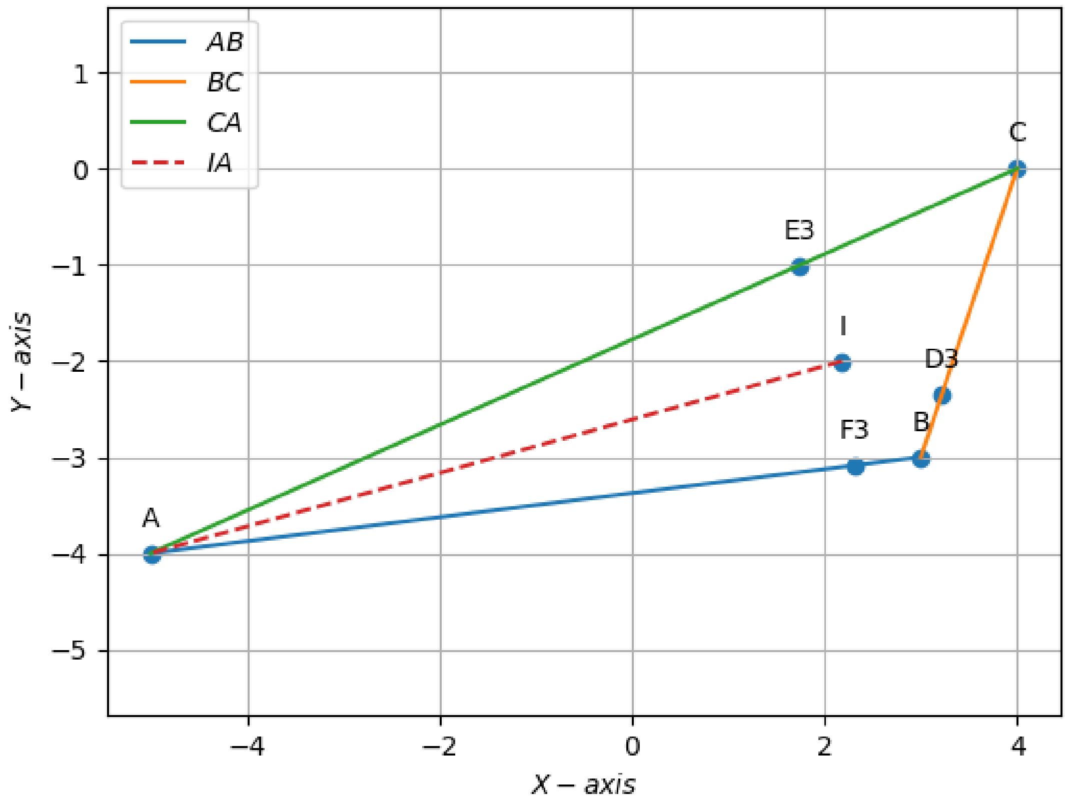

Figure 1.4: Angular bisector AI

1.1.6. Verify that $\mathbf{BI}, \mathbf{CI}$ are also the angle bisectors of $\triangle\mathbf{ABC}$.

Solution:

(a) To prove BI is an angular bisector of $\angle B$

$$\cos\angle ABI \triangleq \frac{(\mathbf{A}-\mathbf{B})\top(\mathbf{I}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{I}-\mathbf{B}\|} \tag{1.1.6.1}$$

$$\cos\angle CBI \triangleq \frac{(\mathbf{C}-\mathbf{B})\top(\mathbf{I}-\mathbf{B})}{\|\mathbf{C}-\mathbf{B}\|\,\|\mathbf{I}-\mathbf{B}\|} \tag{1.1.6.2}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{I}$,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{1.1.6.3}$$

$$\mathbf{I} = \begin{pmatrix} 2.18209833 \\ -2.00232035 \end{pmatrix} \tag{1.1.6.4}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -8 \\ -1 \end{pmatrix} \tag{1.1.6.5}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.6.6}$$

$$\mathbf{I} - \mathbf{B} = \begin{pmatrix} -0.81790167 \\ 0.99767665 \end{pmatrix} \tag{1.1.6.7}$$

also calculating the values of norms

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{65} \tag{1.1.6.8}$$

$$\|\mathbf{C} - \mathbf{B}\| = \sqrt{10} \tag{1.1.6.9}$$

$$\|\mathbf{I} - \mathbf{B}\| = 1.290085982 \tag{1.1.6.10}$$

$$\tag{1.1.6.11}$$

(a) calculating for $\angle ABI$:

On substituting the values in (1.1.6.1) ,We get

$$\cos\angle ABI \triangleq \frac{\begin{pmatrix}-8 & -1\end{pmatrix}\begin{pmatrix}-0.81790167\\ 0.99767665\end{pmatrix}}{\sqrt{65}\times 1.290085982} \tag{1.1.6.12}$$

$$\tag{1.1.6.13}$$

On solving we get ,

$$\angle ABI = 57.7998^\circ \tag{1.1.6.14}$$

(b) calculating for $\angle CBI$:

On substituting the values in (1.1.6.1) ,We get

$$\cos\angle CBI \triangleq \frac{\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}-0.81790167\\ 0.99767665\end{pmatrix}}{\sqrt{10}\times 1.290085982} \tag{1.1.6.15}$$

$$\tag{1.1.6.16}$$

On solving

$$\angle CBI = 57.7998^\circ \tag{1.1.6.17}$$

Therefore $\angle ABI = \angle CBI$. and BI is the bisector of $\angle B$.

(b) To prove CI is an angular bisector of $\angle C$

$$\cos\angle BCI \triangleq \frac{(\mathbf{B}-\mathbf{C})\top(\mathbf{I}-\mathbf{C})}{\|\mathbf{B}-\mathbf{C}\|\,\|\mathbf{I}-\mathbf{C}\|} \tag{1.1.6.18}$$

$$\cos\angle ACI \triangleq \frac{(\mathbf{A}-\mathbf{C})\top(\mathbf{I}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{I}-\mathbf{C}\|} \tag{1.1.6.19}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C} and \mathbf{I}$,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{1.1.6.20}$$

$$\mathbf{I} = \begin{pmatrix} 2.18209833 \\ -2.00232035 \end{pmatrix} \tag{1.1.6.21}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.1.6.22}$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -9 \\ -4 \end{pmatrix} \tag{1.1.6.23}$$

$$\mathbf{I} - \mathbf{C} = \begin{pmatrix} -1.81790167 \\ -2.0023235 \end{pmatrix} \tag{1.1.6.24}$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{10} \tag{1.1.6.25}$$

$$\|\mathbf{A} - \mathbf{C}\| = \sqrt{97} \tag{1.1.6.26}$$

$$\|\mathbf{I} - \mathbf{C}\| = 2.704452972 \tag{1.1.6.27}$$

$$\tag{1.1.6.28}$$

(a) calculating for $\angle BCI$:

On substituting the values in (1.1.6.18) ,We get

$$\cos \angle BCI \triangleq \frac{\begin{pmatrix} -1 & -3 \end{pmatrix} \begin{pmatrix} -1.81790167 \\ -2.0023235 \end{pmatrix}}{\sqrt{10} \times 2.704452972} \tag{1.1.6.29}$$

$$\tag{1.1.6.30}$$

On solving we get ,

$$\angle BCI = 23.8013726^\circ \tag{1.1.6.31}$$

(b) similarly for $\angle ACI$:

On substituting the values in (1.1.6.18) ,We get

$$\cos \angle ACI \triangleq \frac{\begin{pmatrix} -9 & -4 \end{pmatrix} \begin{pmatrix} -1.81790167 \\ -2.0023235 \end{pmatrix}}{\sqrt{97} \times 2.704452972} \tag{1.1.6.32}$$

$$\tag{1.1.6.33}$$

On solving we get ,

$$\angle ACI = 23.8013726^\circ \tag{1.1.6.34}$$

Therefore $\angle BCI = \angle ACI$. and CI is the bisector of $\angle C$.

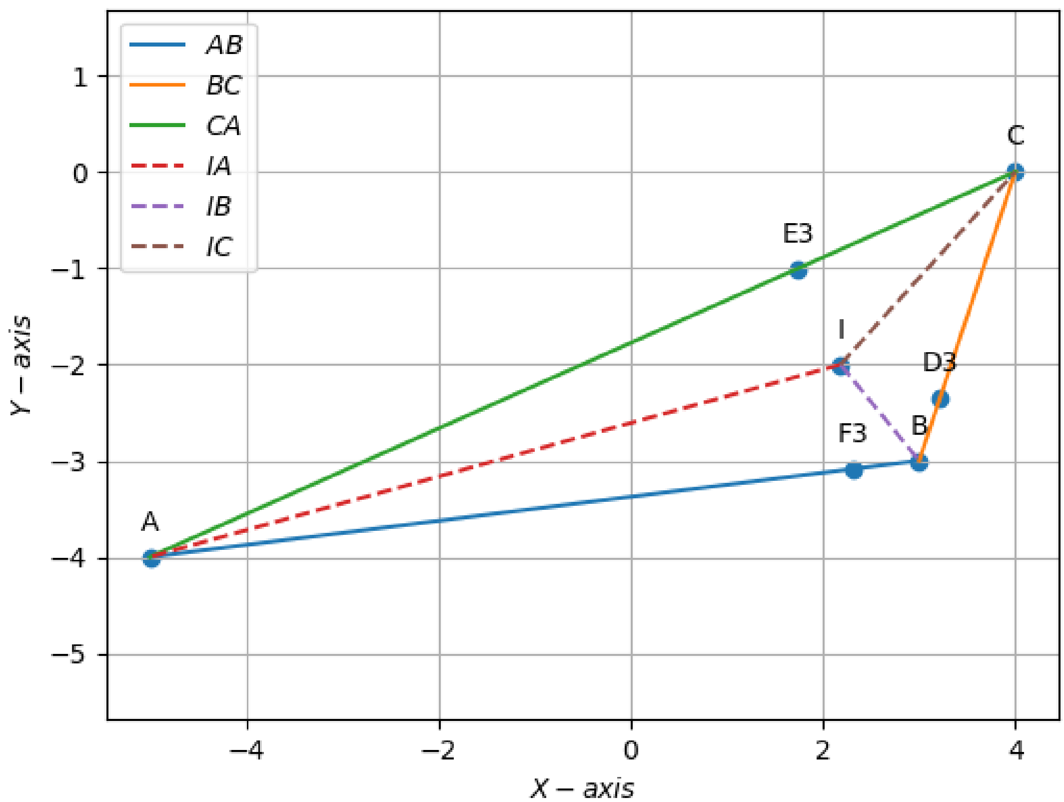

Figure 1.5: Angular bisectors BI and CI

All the codes for this section are available at

geometry/Triangle/Angle_bisector/codes/All_AngleBisectors.py