GEOMETRY

Through Algebra

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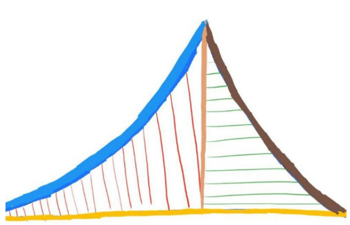

Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1}$$

1.1. Median

1.1.1. If $\mathbf{D}$ divides $\mathbf{BC}$ in the ratio $k\colon 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} \tag{1.1.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides $\mathbf{BC}, \mathbf{CA}$ and $\mathbf{AB}$ respectively.

Solution:

Since $\mathbf{D}$ is the midpoint of $\mathbf{BC}$,

$$k = 1, \tag{1.1.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} 7 \\ -3 \end{pmatrix} \tag{1.1.1.3}$$

Similarly, $\mathbf{E}$ is the midpoint of $\mathbf{AC}$, and $\mathbf{F}$ is the midpoint of $\mathbf{AB}$,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \frac{1}{2}\begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{1.1.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -2 \\ -7 \end{pmatrix} \tag{1.1.1.5}$$

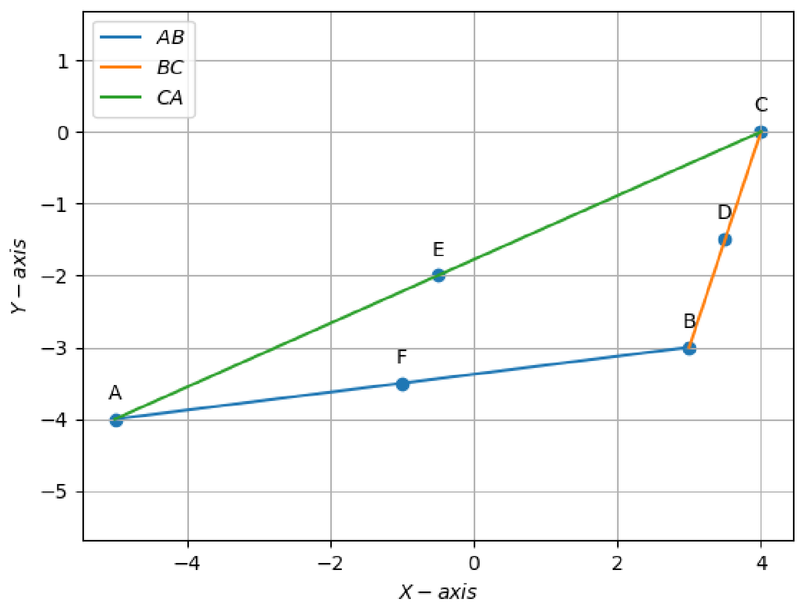

Figure 1.1: Triangle $\mathbf{ABC}$ with midpoints $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$

1.1.2. Find the equations of $\mathbf{AD}, \mathbf{BE}$ and $\mathbf{CF}$.

Solution: :

$\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ are the midpoints of $\mathbf{BC}, \mathbf{CA}$ and $\mathbf{AB}$ respectively, then

$$\mathbf{D} = \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix} \tag{1.1.2.1}$$

$$\mathbf{E} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \tag{1.1.2.2}$$

$$\mathbf{F} = \begin{pmatrix} -1 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.2.3}$$

(a) The normal equation for the median $\mathbf{AD}$ is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.1.2.4}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{1.1.2.5}$$

We have to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2.6}$$

Here $\mathbf{m} = \mathbf{D} - \mathbf{A}$ for median $\mathbf{AD}$

$$\mathbf{m} = \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.2.7}$$

$$= \begin{pmatrix} \frac{17}{2} \\ \frac{5}{2} \end{pmatrix} \tag{1.1.2.8}$$

Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{17}{2} \\ \frac{5}{2} \end{pmatrix} \tag{1.1.2.10}$$

$$= \begin{pmatrix} \frac{5}{2} \\ \frac{-17}{2} \end{pmatrix} \tag{1.1.2.11}$$

$$\mathbf{n}^\top = \begin{pmatrix} \frac{5}{2} & \frac{-17}{2} \end{pmatrix} \tag{1.1.2.12}$$

Hence the normal equation of median $\mathbf{AD}$ is

$$\begin{pmatrix} \frac{5}{2} & \frac{-17}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{5}{2} & \frac{-17}{2} \end{pmatrix} \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.2.13}$$

$$\implies \begin{pmatrix} \frac{5}{2} & \frac{-17}{2} \end{pmatrix} \mathbf{x} = \frac{43}{2} \tag{1.1.2.14}$$

(b) The normal equation for the median $\mathbf{BE}$ is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B} \right) = 0 \tag{1.1.2.15}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{1.1.2.16}$$

Here $\mathbf{m} = \mathbf{E} - \mathbf{B}$ for median $\mathbf{BE}$

$$\mathbf{m} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.2.17}$$

$$= \begin{pmatrix} \frac{-7}{2} \\ 1 \end{pmatrix} \tag{1.1.2.18}$$

Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2.19}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-7}{2} \\ 1 \end{pmatrix} \tag{1.1.2.20}$$

$$= \begin{pmatrix} 1 \\ \frac{7}{2} \end{pmatrix} \tag{1.1.2.21}$$

$$\mathbf{n}^\top = \begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \tag{1.1.2.22}$$

Hence, the normal equation of median **BE** is

$$\begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.2.23}$$

$$\implies \begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \mathbf{x} = \frac{-15}{2} \tag{1.1.2.24}$$

(c) The normal equation for the median **CF** is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C} \right) = 0 \tag{1.1.2.25}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{1.1.2.26}$$

Here $\mathbf{m} = \mathbf{F} - \mathbf{C}$ for median **CF**

$$\mathbf{m} = \begin{pmatrix} -1 \\ \frac{-7}{2} \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.2.27}$$

$$= \begin{pmatrix} -5 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.2.28}$$

Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2.29}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.2.30}$$

$$= \begin{pmatrix} \frac{-7}{2} \\ -5 \end{pmatrix} \tag{1.1.2.31}$$

$$\mathbf{n}^\top = \begin{pmatrix} \frac{-7}{2} & 5 \end{pmatrix} \tag{1.1.2.32}$$

Hence the normal equation of median **CF** is

$$\begin{pmatrix} \frac{-7}{2} & 5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-7}{2} & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.2.33}$$

$$\implies \begin{pmatrix} \frac{-7}{2} & 5 \end{pmatrix} \mathbf{x} = -14 \tag{1.1.2.34}$$

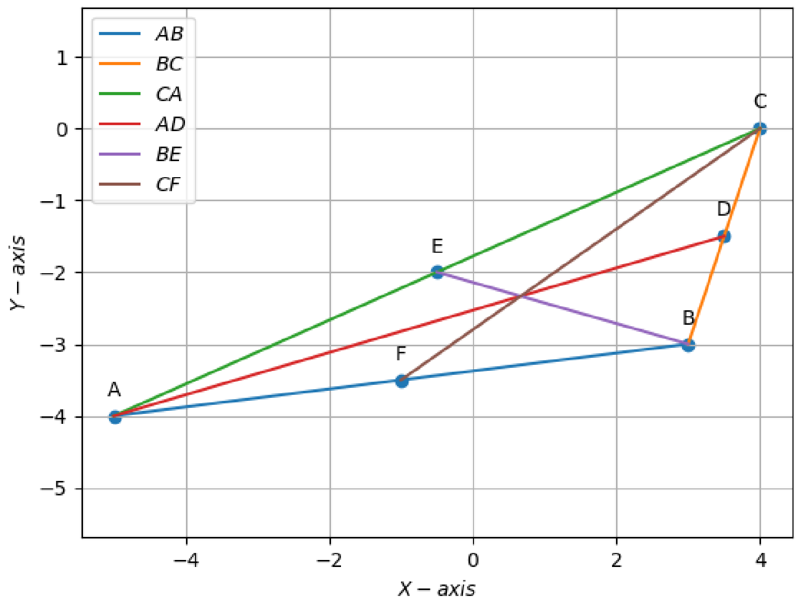

Figure 1.2: Medians AD , BE and CF

1.1.3. Find the intersection $\mathbf{G}$ of $\mathbf{BE}$ and $\mathbf{CF}$

Solution:

$\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are vertices of triangle:

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.3.1}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.3.2}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.3.3}$$

Since $\mathbf{E}$ and $\mathbf{F}$ are midpoints of $\mathbf{CA}$ and $\mathbf{AB}$,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.1.3.4}$$

$$= \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \tag{1.1.3.5}$$

$$\mathbf{F} = \frac{\mathbf{B}+\mathbf{A}}{2} \tag{1.1.3.6}$$

$$= \begin{pmatrix} -1 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.3.7}$$

The line $\mathbf{BE}$ in vector form is given by

$$\begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \mathbf{x} = \frac{-15}{2} \tag{1.1.3.8}$$

The line $\mathbf{CF}$ in vector form is given by

$$\begin{pmatrix} \frac{-7}{2} & 5 \end{pmatrix} \mathbf{x} = -14 \tag{1.1.3.9}$$

From (1.1.3.8) and (1.1.3.9) the augmented matrix is:

$$\begin{pmatrix} 1 & \frac{7}{2} & \frac{-15}{2} \\ \frac{-7}{2} & 5 & -14 \end{pmatrix} \tag{1.1.3.10}$$

Solve for $\mathbf{G}$ using Gauss-Elimination method :

$$\begin{pmatrix} 1 & \frac{7}{2} & \frac{-15}{2} \\ \frac{-7}{2} & 5 & -14 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{7R_1}{2}} \begin{pmatrix} 1 & \frac{7}{2} & \frac{-15}{2} \\ 0 & \frac{69}{4} & \frac{-161}{4} \end{pmatrix} \tag{1.1.3.11}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{4R_2}{69}} \begin{pmatrix} 1 & \frac{7}{2} & \frac{-15}{2} \\ 0 & 1 & \frac{-7}{3} \end{pmatrix} \tag{1.1.3.12}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{7R_2}{2}} \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-7}{3} \end{pmatrix} \tag{1.1.3.13}$$

Therefore,

$$\mathbf{G} = \begin{pmatrix} \frac{2}{3} \\ \frac{-7}{3} \end{pmatrix} \tag{1.1.3.14}$$

From Fig. 1.3, We can see that $\mathbf{G} = \begin{pmatrix} \frac{2}{3} \\ \frac{-7}{3} \end{pmatrix}$ is the intersection of **BE** and **CF**.

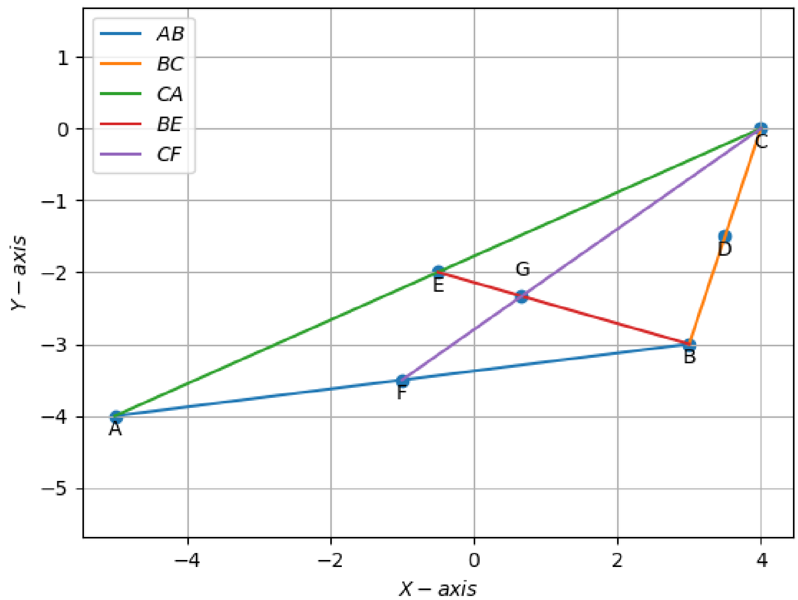

Figure 1.3: G is the centroid of triangle ABC

1.1.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.1.4.1}$$

Solution:

In order to verify the above equation we first need to find **G**.

G is the intersection of **BE** and **CF**, Using the value of **G** from (1.2.3).

$$\mathbf{G} = \begin{pmatrix} \frac{2}{3} \\ \frac{-7}{3} \end{pmatrix} \tag{1.1.4.2}$$

Also, We know that $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ are midpoints of $\mathbf{BC}, \mathbf{CA}$ and $\mathbf{AB}$ respectively from (1.2.1).

$$\mathbf{D} = \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix}, \mathbf{E} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} -1 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.4.3}$$

(a) Calculating the ratio of $\mathbf{BG}$ and $\mathbf{GE}$,

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} \frac{-7}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.1.4.4}$$

$$\mathbf{E} - \mathbf{G} = \begin{pmatrix} \frac{7}{6} \\ \frac{1}{3} \end{pmatrix} \tag{1.1.4.5}$$

$$\|\mathbf{G} - \mathbf{B}\| = \sqrt{\left(\frac{7}{3}\right)^2 + \left(\frac{2}{3}\right)^2} \qquad = \frac{\sqrt{53}}{3} \tag{1.1.4.6}$$

$$\|\mathbf{E} - \mathbf{G}\| = \sqrt{\left(\frac{7}{6}\right)^2 + \left(\frac{1}{3}\right)^2} \qquad = \frac{\sqrt{53}}{6} \tag{1.1.4.7}$$

$$\frac{BG}{GE} = \frac{\|\mathbf{G} - \mathbf{B}\|}{\|\mathbf{E} - \mathbf{G}\|} \qquad = \frac{\frac{\sqrt{53}}{3}}{\frac{\sqrt{53}}{6}} = 2 \tag{1.1.4.8}$$

(b) Calculating the ratio of **CG** and **GF** ,

$$\mathbf{G} - \mathbf{C} = \begin{pmatrix} \frac{-10}{3} \\ \frac{-7}{3} \end{pmatrix} \tag{1.1.4.9}$$

$$\mathbf{F} - \mathbf{G} = \begin{pmatrix} \frac{-5}{3} \\ \frac{7}{6} \end{pmatrix} \tag{1.1.4.10}$$

$$\|\mathbf{G} - \mathbf{C}\| = \sqrt{\left(\frac{-10}{3}\right)^2 + \left(\frac{-7}{3}\right)^2} \qquad = \frac{\sqrt{149}}{3} \tag{1.1.4.11}$$

$$\|\mathbf{F} - \mathbf{G}\| = \sqrt{\left(\frac{-5}{3}\right)^2 + \left(\frac{7}{6}\right)^2} \qquad = \frac{\sqrt{149}}{6} \tag{1.1.4.12}$$

$$\frac{CG}{GF} = \frac{\|\mathbf{G} - \mathbf{C}\|}{\|\mathbf{F} - \mathbf{G}\|} \qquad = \frac{\frac{\sqrt{149}}{3}}{\frac{\sqrt{149}}{6}} = 2 \tag{1.1.4.13}$$

(c) Calculating the ratio of **AG** and **GD** ,

$$\mathbf{G} - \mathbf{A} = \begin{pmatrix} \frac{17}{3} \\ \frac{5}{3} \end{pmatrix} \tag{1.1.4.14}$$

$$\mathbf{D} - \mathbf{G} = \begin{pmatrix} \frac{17}{6} \\ \frac{5}{6} \end{pmatrix} \tag{1.1.4.15}$$

$$\|\mathbf{G} - \mathbf{A}\| = \sqrt{\left(\frac{17}{3}\right)^2 + \left(\frac{5}{3}\right)^2} \qquad = \frac{\sqrt{314}}{3} \tag{1.1.4.16}$$

$$\|\mathbf{D} - \mathbf{G}\| = \sqrt{\left(\frac{17}{6}\right)^2 + \left(\frac{5}{6}\right)^2} \qquad = \frac{\sqrt{314}}{6} \tag{1.1.4.17}$$

$$\frac{AG}{GD} = \frac{\|\mathbf{G} - \mathbf{A}\|}{\|\mathbf{D} - \mathbf{G}\|} \qquad = \frac{\frac{\sqrt{314}}{3}}{\frac{\sqrt{314}}{6}} = 2 \tag{1.1.4.18}$$

From (1.1.4.8), (1.1.4.13), (1.1.4.18)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.1.4.19}$$

Hence verified.

1.1.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.

Solution: Given that,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.5.1}$$

We need to show that points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are collinear. From Problem 1.2.3 We know that, The point $\mathbf{G}$ is

$$\mathbf{G} = \begin{pmatrix} \frac{2}{3} \\ \frac{-7}{3} \end{pmatrix} \tag{1.1.5.2}$$

And from Problem 1.2.1 We know that, The point $\mathbf{D}$ is

$$\mathbf{D} = \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix} \tag{1.1.5.3}$$

In Problem 1.1.3, There is a theorem/law mentioned i.e.,

Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.1.5.4}$$

Using the above law/Theorem Let

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ -5 & \frac{7}{2} & \frac{2}{3} \\ -4 & \frac{-3}{2} & \frac{-7}{3} \end{pmatrix} \tag{1.1.5.5}$$

The matrix $\mathbf{R}$ can be row reduced as follows,

$$\begin{pmatrix} 1 & 1 & 1 \\ -5 & \frac{7}{2} & \frac{2}{3} \\ -4 & \frac{-3}{2} & \frac{-7}{3} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 5R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{17}{2} & \frac{17}{3} \\ -4 & \frac{-3}{2} & \frac{-7}{3} \end{pmatrix} \tag{1.1.5.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 4R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{17}{2} & \frac{17}{3} \\ 0 & \frac{5}{2} & \frac{5}{3} \end{pmatrix} \tag{1.1.5.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{R_2}{17}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{17}{2} & \frac{17}{3} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.5.8}$$

Rank of above matrix is 2.

Hence, we proved that that points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are collinear.

1.1.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.1.6.1}$$

$\mathbf{G}$ is known as the centroid of $\triangle ABC$.

Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.1.6.2}$$

Solution:

$\mathbf{G}$ is known as the centroid of $\triangle ABC$

let us first evaluate the R.H.S of the equation

$$\mathbf{G} = \frac{\begin{pmatrix} -5 \\ -4 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}}{3} = \begin{pmatrix} \frac{-5+3+4}{3} \\ \frac{-4-3+0}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-7}{3} \end{pmatrix} \tag{1.1.6.3}$$

Hence verified.

1.1.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.1.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

Question : Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.1.7.2}$$

The quadrilateral $AFDE$ is defined to be parallelogram.

Solution:

Given that,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.7.3}$$

From Problem 1.2.1 We know that, The point $\mathbf{D}, \mathbf{E}, \mathbf{F}$ is

$$\mathbf{D} = \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} -1 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.7.4}$$

Evaluating the R.H.S of the equation

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -1 \\ \frac{-7}{2} \end{pmatrix} \tag{1.1.7.5}$$

$$= \begin{pmatrix} -4 \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.7.6}$$

Evaluating the L.H.S of the equation

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} - \begin{pmatrix} \frac{7}{2} \\ \frac{-3}{2} \end{pmatrix} \tag{1.1.7.7}$$

$$= \begin{pmatrix} -4 \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.7.8}$$

Hence verified that, R.H.S = L.H.S i.e. ,

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.1.7.9}$$

From the Fig. 1.4, It is verified that $AFDE$ is a parallelogram

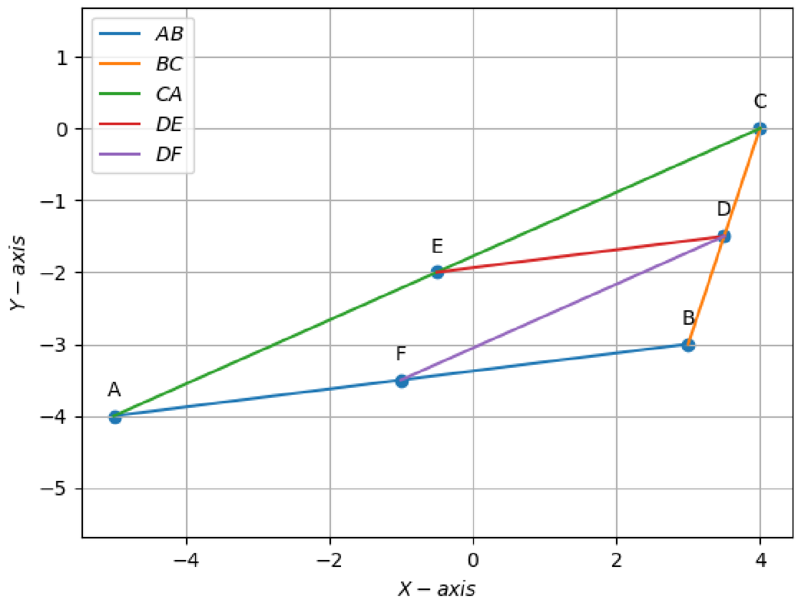

Figure 1.4: $AFDE$ form a parallelogram in triangle ABC

All codes for this section are available at

geometry/Triangle/median/codes/Triangle_medians.py