GEOMETRY

Through Algebra

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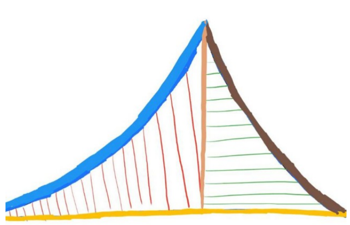

Contents

Introduction

This book shows how to solve problems in geometry using trigonometry and coordinate geometry.

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1}$$

1.1. Altitude

1.1.1. $\mathbf{D}_1$ is a point on $\mathbf{BC}$ such that

$$\mathbf{AD_1} \perp \mathbf{BC} \tag{1.1.1.1}$$

and $\mathbf{AD_1}$ is defined to be the altitude. Find the normal vector of $\mathbf{AD_1}$.

Solution:

Given,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \tag{1.1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \tag{1.1.1.3}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.1.4}$$

Normal vector of $\mathbf{AD_1}$ is orthogonal to $\mathbf{AD_1}$ and hence parallel to $\mathbf{BC}$. Direction vector $\mathbf{m_{BC}}$

$$= \mathbf{C} - \mathbf{B} \tag{1.1.1.5}$$

$$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.1.6}$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.1.7}$$

$$\text{Normal vector of } \mathbf{AD_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.1.8}$$

1.1.2. Find the equation of $\mathbf{AD_1}$.

Solution:

The normal vector of $\mathbf{AD_1}$ is

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.2.1}$$

The equation of $\mathbf{AD_1}$ is

$$\mathbf{n}^\top(\mathbf{x}-\mathbf{A}) = 0 \tag{1.1.2.2}$$

$$\mathbf{n}^\top(\mathbf{x}) = \mathbf{n}^\top(\mathbf{A}) \tag{1.1.2.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.2.4}$$

$$\begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = -17 \tag{1.1.2.5}$$

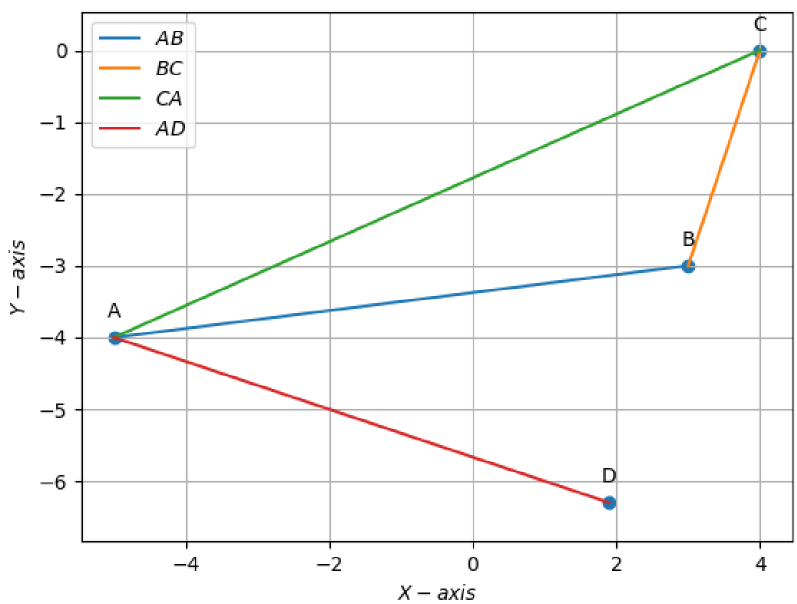

Figure 1.1: Altitude $\mathbf{AD_1}$

1.1.3. Find the equations of the altitudes $\mathbf{BE_1}$ and $\mathbf{CF_1}$ to the sides $\mathbf{AC}$ and $\mathbf{AB}$ respectively.

Solution:

The normal equation of $\mathbf{BE_1}$ is

$$\mathbf{n} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \tag{1.1.3.1}$$

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.1.3.2}$$

$$\mathbf{n}^\top (\mathbf{x}) = \mathbf{n}^\top (\mathbf{B}) \tag{1.1.3.3}$$

$$\implies \begin{pmatrix} 9 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 9 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{1.1.3.4}$$

$$\implies \begin{pmatrix} 9 & 4 \end{pmatrix} \mathbf{x} = 15 \tag{1.1.3.5}$$

The normal equation of $\mathbf{CF_1}$ is

$$\mathbf{n} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.1.3.6}$$

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.1.3.7}$$

$$\mathbf{n}^\top (\mathbf{x}) = \mathbf{n}^\top (\mathbf{C}) \tag{1.1.3.8}$$

$$\implies \begin{pmatrix} 8 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.3.9}$$

$$\implies \begin{pmatrix} 8 & 1 \end{pmatrix} \mathbf{x} = 32 \tag{1.1.3.10}$$

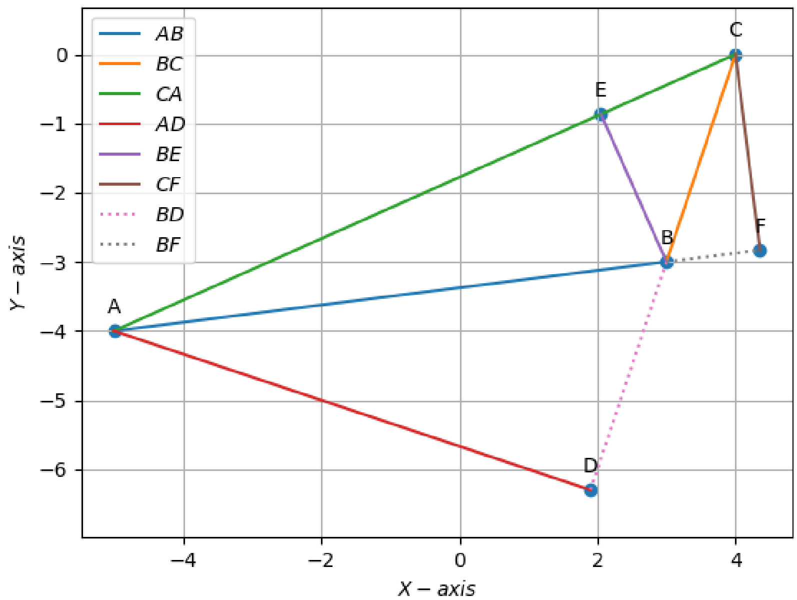

Figure 1.2: Altitudes $\mathbf{BE_1}$ and $\mathbf{CF_1}$

1.1.4. Find the intersection $\mathbf{H}$ of $\mathbf{BE_1}$ and $\mathbf{CF_1}$.

Solution:

Equation of $\mathbf{BE_1}$:

$$\begin{pmatrix} 9 & 4 \end{pmatrix} \mathbf{x} = 15 \tag{1.1.4.1}$$

Equation of $\mathbf{CF_1}$:

$$\begin{pmatrix} 8 & 1 \end{pmatrix} \mathbf{x} = 32 \tag{1.1.4.2}$$

Therefore, we need to solve the following equation to get $\mathbf{H}$:

$$\begin{pmatrix} 9 & 4 \\ 8 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 15 \\ 32 \end{pmatrix} \tag{1.1.4.3}$$

Solving the above equation by Gauss-Jordan method

$$\begin{pmatrix} 9 & 4 & 15 \\ 8 & 1 & 32 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{9}} \begin{pmatrix} 1 & \frac{4}{9} & \frac{5}{3} \\ 8 & 1 & 32 \end{pmatrix} \tag{1.1.4.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 8R_1} \begin{pmatrix} 1 & \frac{4}{9} & \frac{5}{3} \\ 0 & \frac{-23}{9} & \frac{56}{3} \end{pmatrix} \tag{1.1.4.5}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-9R_2}{23}} \begin{pmatrix} 1 & \frac{4}{9} & \frac{5}{3} \\ 0 & 1 & \frac{-168}{23} \end{pmatrix} \tag{1.1.4.6}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{4R_2}{9}} \begin{pmatrix} 1 & 0 & \frac{113}{23} \\ 0 & 1 & \frac{-168}{23} \end{pmatrix} \tag{1.1.4.7}$$

Therefore point of intersection $\mathbf{H}$ is

$$= \frac{1}{23} \begin{pmatrix} 113 \\ -168 \end{pmatrix} \tag{1.1.4.8}$$

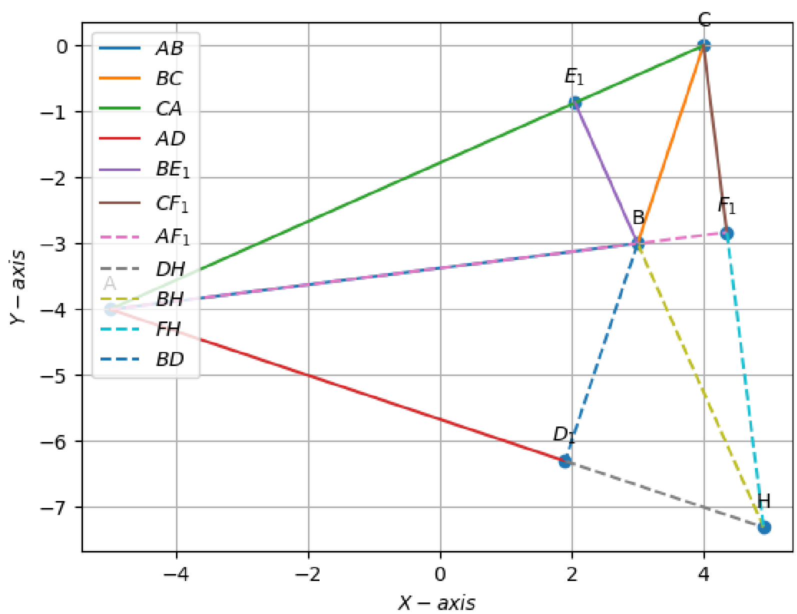

Figure 1.3: Intersection point $\mathbf{H}$ of altitudes BE_1 and CF_1

1.1.5. Verify that

$$\left(\mathbf{A}-\mathbf{H}\right)^{\top}\left(\mathbf{B}-\mathbf{C}\right)=0 \tag{1.1.5.1}$$

Solution:

Given,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \tag{1.1.5.2}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}. \tag{1.1.5.3}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{1.1.5.4}$$

$$\mathbf{H} = \frac{1}{23} \begin{pmatrix} 113 \\ -168 \end{pmatrix} \tag{1.1.5.5}$$

To solve the equation

$$\mathbf{A} - \mathbf{H} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \frac{1}{23} \begin{pmatrix} 113 \\ -168 \end{pmatrix} \tag{1.1.5.6}$$

$$= \frac{1}{23} \begin{pmatrix} -228 \\ 76 \end{pmatrix} \tag{1.1.5.7}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.5.8}$$

$$= \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.1.5.9}$$

$$\implies (\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = \frac{1}{23} \begin{pmatrix} -228 & 76 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{1.1.5.10}$$

$$= 0 \tag{1.1.5.11}$$

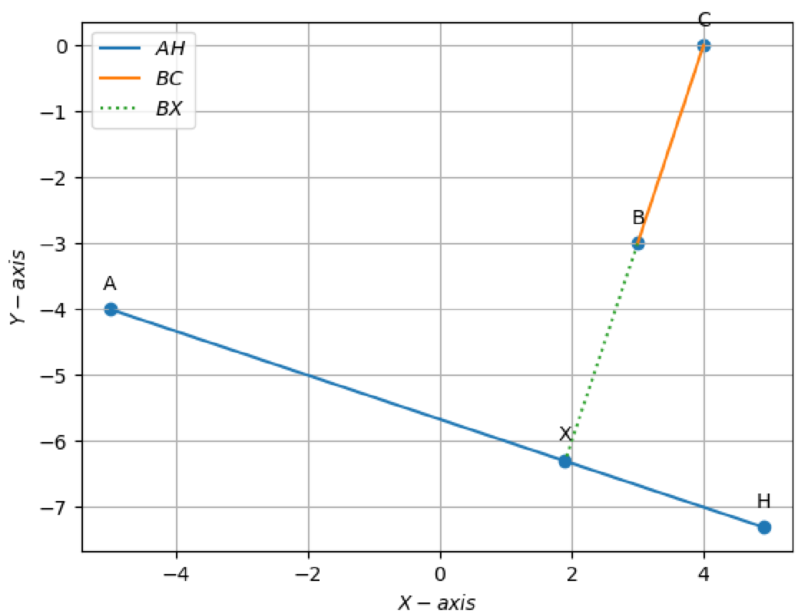

Figure 1.4: Plot of points A, B, C and H

All cosdes for this section are available at

geometry/Triangle/Altitude/codes/All_Altitudes.py