# Convergence of the Gutt star product

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# Chapter 1

# Introduction

Here will be things like thanks to and so on...

# Chapter 2

# Deformation quantization

Here should already be some stuff...

## 2.1 Mechanics, the classical and the quantum view

## 2.2 Why Deformation quantization

## 2.3 The long way from formal to strict

As we have already seen, the answer a mathematician would give to the question "What is (classical) mechanics?" is very closely linked to symplectic and, more generally, Poisson geometry. It is for this reason, that we will have a look at some basics of Poisson geometry. Of course, a systematic way to get a strict deformation quantization for every Poisson manifold would rather be a life's work and is by far beyond the reach of a master thesis. We have to restrict it to a special class of Poisson systems in order to get a viable challenge. Constant Poisson structures were already tackled by Stefan Waldmann and investigated more closely by Matthias Schötz in and respectively. Therefore, the next logical step is to attack linear Poisson structures.

# Chapter 3

# Algebraic Preliminaries

## 3.1 Linear Poisson structures

As we have seen before, there has already been done some work on how to strictly quantize local Poisson structures. The Weyl-Moyal-product on locally convex vector spaces was topologized by Stefan Waldmann in [?] and then investigated more closely by Matthias Schötz in [?]. It is thus clear that in the next step linear Poisson structures on locally convex vector spaces should be done. This will give a new big class of Poisson structures, which will be deformable in a strict way. Before we do so in the rest of this master thesis, we recall briefly some basics on linear Poisson structures.

**Remark 3.1.1 (The axiom of choice)** Our final goal is to do some locally convex functional analysis. Since in this game it is mandatory for us to sell our souls to the devil for the sake of the axiom of choice (e.g. in form of the Hahn-Banach theorem and the projective tensor product), there is no point in not doing it right from the beginning.

First of all, linear Poisson structures are actually something familiar: We will always take a vector space $V$ and look at Poisson structures on its dual space $V^*$. But a linear Poisson structure on $V^*$ is nothing but the structure of a Lie algebra on $V$ itself, at least for finite-dimensional vector spaces.

**Proposition 3.1.2** *Let $V$ be a vector-space of dimension $n \in \mathbb{N}$ and $\pi \in \Gamma^\infty(\Lambda^2(TV^*))$. Then the two following things are equivalent:*

*i.) $\pi$ is a Poisson tensor.*

*ii.) $V$ has a uniquely determined Lie algebra structure.*

Proof: First, we want to think of what it means, that $\pi$ is a Poisson tensor. It is already a antisymmetric bi-vectorfield. The only thing which it must satisfy is the Jacobi-identity. □

We are interested in the polynomial algebra on the dual of the original vector space, since the construction is inspired by the formal deformation quantization on the cotangent bundle $T^*G$ of a Lie group $G$, which was investigated in [?]. Now we know, that this original vector space really is a Lie algebra and for this reason we will call it in the following $\mathfrak{g}$.

The polynomial algebra $\mathrm{Pol}(\mathfrak{g}^*)$ has many appearances and it is not an easy question how the find a good generalization for infinite-dimensional vector spaces. One could of course think of a good definition of the polynomial functions on the dual of an infinite-dimensional Lie algebra $g$. In order to keep the very most of the physically interesting cases in there, we will assume it to be locally convex. But we know, that even the topological dual $\mathfrak{g}'$ will be a huge vector space and whatever the polynomials there should be, there will be a lot of them. We will furthermore not be able to find the nice correspondence of the Lie algebra structure on $\mathfrak{g}$ and the linear polynomials on $\mathfrak{g}'$, since the procedure we used involved the double-dual of $\mathfrak{g}$. In general, this will be really bigger than $\mathfrak{g}$ and starting from a linear Poisson structure on $\mathfrak{g}'$, we will find a Lie algebra structure on $\mathfrak{g}''$. Of course, we could just use the canonical embedding $\mathfrak{g} \subseteq \mathfrak{g}''$, but it could (and, in general, it will) happen, that the Lie bracket of $x, y \in \mathfrak{g}$ will not be in $\mathfrak{g}$ any more, but just in its double-dual. In most of the physical cases, we are actually not interested in the double-dual, but in the original vector space.

The little reflection shows, that we will have to think of a different generalization of linear Poisson structure for the infinite-dimensional case. Luckily, in the finite-dimensional case, there's a different way of seeing the polynomials on $\mathfrak{g}^* = \mathfrak{g}'$, which allows a much easier generalization to infinite dimensions: It is the symmetric tensor algebra over $\mathfrak{g}$ itself.

**Proposition 3.1.3** *Let $\mathfrak{g}$ be a vector space of dimension $n \in \mathbb{N}$. Then the algebras $\mathrm{S}^\bullet(\mathfrak{g})$ and $\mathrm{Pol}(\mathfrak{g}^*)$ are canonically isomorphic.*

PROOF: Again via basis and double-dual... □

Here again, we used the double dual, so one could ask why this situation should differ from the foregoing one. But there is a difference: instead of looking at $\mathrm{S}^\bullet(\mathfrak{g}^{**})$ we can directly restrict to $\mathrm{S}^\bullet(\mathfrak{g})$. This is a setting in which closedness of the Poisson bracket is automatically fulfilled, since we don't to to characterize the object in $\mathrm{S}^\bullet(\mathfrak{g}) \subseteq \mathrm{S}^\bullet(\mathfrak{g}^{**})$. We can take the interesting linear Poisson structure on $\mathrm{Pol}(\mathfrak{g}^*)$ directly to be bilinear maps

$$\mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g})$$

which satisfy certain conditions on the degree. This is the way we want to go. We generalize the finite-dimensional situation, where we have $\mathrm{Pol}(\mathfrak{g}^*) \cong \mathrm{S}^\bullet(\mathfrak{g})$ in this sense, that we look directly at $\mathrm{S}^\bullet(\mathfrak{g})$ for infinite-dimensional Lie algebras, since those (and their tensor products) are much better known and much easier to control.

## 3.2 The universal enveloping algebra

## 3.3 The Baker-Campbell-Hausdorff formula

# Chapter 4

# Formulas for the Gutt star product

We have seen some results on the Baker-Campbell-Hausdorff series and an identity for the Gutt star product. The latter one, stated in Theorem **??**, will be a very useful tool in the following, since we want to get explicit formulas for $\star_{zG}$. There is still one part of the proof missing, but this will be caught up at the beginning of the first section of this chapter. From there, we will come to a first easy formula for $\star_{zG}$. Afterwards, we will use the same procedure to find two more formulas for it: a rather involved one for the $n$-fold star product of vectors, which will not necessarily be helpful for algebraic computations, but will turn out very useful for estimates, and a more explicit one for the product of two monomials.

From those formulas, we will be able to draw some easy, but nice consequences in the next section and we will show how to compute the Gutt star product explicitly by calculating two easy examples.

At the end of this chapter, we will give an easy Mathematica code, which can be used to verify the correctness of our formulas for polynomials of low orders.

## 4.1 Formulas for the Gutt star product

### 4.1.1 An Iterative Approach from Linear Terms

The easiest case for which we will to develop a formula is surely the following one: For a given Lie algebra $\mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}$ we would like to compute

$$\xi^k \star_{zG} \eta = \sum_{n=0}^{k} z^n C_n(\xi^k, \eta)$$

We have already done this for the Gutt star product, now we want to do the same for the BCH star product. This will finish the proof of Theorem **??**. For this purpose, we will use that

$$\xi^k = \frac{\partial^k}{\partial t^k}\Big|_{t=0} \exp(t\xi). \tag{4.1.1}$$

We now have all the ingredients to prove the following proposition:

**Lemma 4.1.1** *Let $\mathfrak{g}$ be a Lie algebra and $\xi, \eta \in \mathfrak{g}$. We have the following identity for the BCH star product $\star_{zG}$*

$$\xi^k \star_{zH} \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j(\eta). \tag{4.1.2}$$

PROOF: We start from the simplified form for the Baker-Campbell-Hausdorff series from Equation (**??**):

$$\mathtt{BCH}(\xi,\eta)=\xi+\sum_{n=0}^{\infty}\frac{B_n^*}{n!}(\mathrm{ad}_\xi)^n(\eta)+\mathcal{O}(\eta^2).$$

Putting things together with the definition of the BCH star product and Equation (4.1.1) we get

$$\begin{aligned}\xi^k\star_{zH}\eta&=\frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0}\exp\left(\frac{1}{z}\mathtt{BCH}(zt\xi,zs\eta)\right)\\&=\frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0}\exp\left(t\xi+\sum_{j=0}^{\infty}z^j\frac{B_j^*}{j!}(\mathrm{ad}_{t\xi})^j(s\eta)\right).\end{aligned}$$

From this, we see that only terms which have exactly $k$ of the $\xi$'s in them and which are linear in $\eta$ will contribute. This means we can cut off the sum at $j=k$. If we now write out the exponential series which we can also cut for the same reason. We have

$$\begin{aligned}\xi^k\star_{zH}\eta&=\frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0}\sum_{n=0}^{k}\frac{1}{n!}\left(t\xi+\sum_{j=0}^{k}(zt)^j\frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^n\\&=\frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0}\sum_{n=0}^{k}\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}(t\xi)^{n-m}\left(\sum_{j=0}^{k}(zt)^j\frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^m\\&=\frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0}\left(\sum_{n=0}^{k}\frac{1}{n!}(t\xi)^n+\sum_{n=0}^{k}\sum_{j=0}^{k}\frac{1}{(n-1)!}t^{n+j-1}z^j\frac{B_j^*}{j!}\xi^{n-1}(\mathrm{ad}_\xi)^j(s\eta)\right).\end{aligned}$$

In the last step we just cut off the sum over $m$ since the terms for $m>1$ will vanish because of the differentiation with respect to $s$. We can finally differentiate to get the formula

$$\begin{aligned}\xi^k\star_{zH}\eta&=\sum_{n=0}^{k}\sum_{j=0}^{k}\delta_{k,n+j-1}\frac{k!}{j!(n-1)!}z^jB_j^*\xi^{n-1}(\mathrm{ad}_\xi)^j(\eta)\\&=\sum_{j=0}^{k}\binom{k}{j}z^jB_j^*\xi^{k-j}(\mathrm{ad}_\xi)^j(\eta),\end{aligned}$$

which is the wanted result. □

**Remark 4.1.2** We have now proven the equality of the two star products $\star_{zG}$ and $\star_{zH}$ by deriving an easy formula from both of them. From now on, we will derive all the other formulas from $\star_{zH}$, since this is the one which is easier to compute.

Once this is done, it is actually easy to get the formula for monomials of the form $\xi_1\ldots\xi_k$ with $\eta\in\mathfrak{g}$:

**Proposition 4.1.3** *Let $\mathfrak{g}$ be a Lie algebra and $\xi_1,\ldots,\xi_k,\eta\in\mathfrak{g}$. We have*

$$\xi_1\ldots\xi_k\star_{zG}\eta=\sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}z^jB_j^*\sum_{\sigma\in S_k}[\xi_{\sigma(1)},[\ldots[\xi_{\sigma(j)},\eta]\ldots]]\xi_{\sigma(j+1)}\ldots\xi_{\sigma(k)}.\tag{4.1.3}$$

PROOF: We get the result by just polarizing the formula from Lemma 4.1.1. Let $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ be given, then we introduce the parameters $t_i$ for $i = 1, \ldots, k$ and set

$$\Xi = \Xi(t_1, \ldots, t_k) = \sum_{i=1}^{k} t_i \xi^i.$$

Then it is immediate to see that

$$\xi_1 \ldots \xi_k = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big|_{t_1,\ldots,t_k=0} \Xi^k$$

since for every $i = 1, \ldots, k$ we have

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} \Xi = \xi_i. \tag{4.1.4}$$

We also find for every $\eta \in \mathfrak{g}$

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} \mathrm{ad}_\Xi(\eta) = \mathrm{ad}_{\xi_i}(\eta). \tag{4.1.5}$$

Now we just need to calculate $\Xi^k \star_{zG} \eta$ and differentiate with respect to the $t_i$. In order to do this properly, we define

$$\gamma_n^k(\xi_1, \ldots, \xi_k; \eta) = z^n \binom{k}{n} B_n^*(\mathrm{ad}_{\xi_1} \circ \ldots \circ \mathrm{ad}_{\xi_n})(\eta) \xi_{n+1} \ldots \xi_k$$

and

$$\gamma^k(\xi_1, \ldots, \xi_k; \eta) = \sum_{n=0}^{k} \gamma_n^k(\xi_1, \ldots, \xi_k; \eta).$$

We see that

$$\Xi^k \star_{zG} \eta = \gamma^k(\Xi, \ldots, \Xi; \eta)$$

and can now differentiate this expression, which is linear in the every argument, with respect to the $t_i$. From the Equations (4.1.4) and (4.1.5) we get with the Leibniz rule

$$\frac{\partial}{\partial t_1} \gamma^k(\Xi, \ldots, \Xi; \eta) = \sum_{j=1}^{k} \gamma^k(\underbrace{\Xi, \ldots, \Xi}_{j-1 \text{ times}}, \xi_1, \underbrace{\Xi, \ldots, \Xi}_{k-j-1 \text{ times}}; \eta)$$

Differentiating now with respect to $t_2$, we get a second sum, where $\xi_2$ will be put once in every "free" position, and so on. One by one, all the slots will be taken by $\xi_i$'s. We just need to divide by $k!$, and we finally find the formula from Equation (4.1.3). $\square$

### 4.1.2 A first general Formula

Proposition 4.1.3 allows us basically to get a formula for the case of $\xi_1, \ldots, \xi_k \in \mathfrak{g}$

$$\xi_1 \star_{zG} \ldots \star_{zG} \xi_k = \sum_{j=0}^{k} C_{z,j}(\xi_1, \ldots, \xi_k)$$

which we will need to prove the continuity of the coproduct, but which can also help to prove the continuity of the product in a different way.

Unluckily, this approach has a problem: iterating this formula, we get strangely nested Lie brackets, which would be very difficult to bring into a nice form with Jacobi and higher identities. So this is not a good way to find an handy formula for the usual star product of two monomials.

Nevertheless, we want to pursue it for a moment, since we will get an equality which will be, although rather unfriendly looking, very useful in the following: for analytic observations, it will be enough to put (even brutal) estimates on it and the exact nature of the combinatorics in the formula will not be important. Hence we rewrite Equation (4.1.3) in order to cook up such a formula.

**Definition 4.1.4** *Let $j, k \in \mathbb{N}_0$, $j \leq k$ and $B_j^*$ as usual, then we define bilinear maps via*

$$\begin{aligned} B_z^{k,j} \colon \mathrm{S}^k(\mathfrak{g}) \times \mathfrak{g} &\longrightarrow \mathrm{S}^{k-j+1}(\mathfrak{g}) \\ (\xi_1 \dots \xi_k, \eta) &\longmapsto \tfrac{1}{k!} \sum_{\sigma \in S_k} \tbinom{k}{j} B_j^* z^j [\xi_{\sigma(1)}, [\dots, [\xi_{\sigma(j)}, \eta]]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)} \end{aligned}$$

*and*

$$B_z^j \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathfrak{g} \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad B_z^j = \sum_{k=0}^{\infty} B_z^{k,j}$$

*where we set $B_z^j(x) = 0$ if $\deg(x) < j$ .*

We immediately get an easier identity for Equation (4.1.3):

$$\xi_1 \dots \xi_k \star_{zG} \eta = \sum_{j=0}^{k} B_z^j(\xi_1 \dots \xi_k, \eta). \tag{4.1.6}$$

More than that: We can extend it to arbitrary symmetric tensors:

**Lemma 4.1.5** *Let $\mathfrak{g}$ be a Lie-algebra an $x \in \mathrm{S}^\bullet(\mathfrak{g})$. Then we have the formula*

$$x \star_{zG} \eta = \sum_{j=0}^{\infty} B_z^j(x, \eta). \tag{4.1.7}$$

PROOF: First it is clear that the sum over $j$ in Equation (4.1.7) is actually finite, since for $j > \deg(x)$ there is no further contribution. Using the grading we can write

$$x = \sum_{k=0}^{\deg(x)} \sum_i x_i^{(k)}.$$

The $B_z^i$-maps are linear in the first argument and the $x_i^{(k)}$ can be chosen to be factorizing tensors. But on factorizing tensors, this is just Equation (4.1.6). We hence have by the linearity of $\star_{zG}$

$$\begin{aligned} x \star_{zG} \eta &= \sum_{k=0}^{\deg(x)} \sum_i x_i^{(k)} \star_{zG} \eta \\ &= \sum_{k=0}^{\deg(x)} \sum_i \sum_{j=0}^{\infty} B_z^j \left( x_i^{(k)}, \eta \right) \\ &= \sum_{j=0}^{\infty} B_z^j \left( \sum_{k=0}^{\deg(x)} \sum_i x_i^{(k)}, \eta \right) \\ &= \sum_{j=0}^{\infty} B_z^j(x, \eta). \end{aligned}$$

□

We can use this approach to go on:

**Proposition 4.1.6** *Let* $\mathfrak{g}$, $2 \leq k \in \mathbb{N}$ *and* $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. *Then we have*

$$\xi_1 \star_{zG} \ldots \star_{zG} \xi_k = \sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0,\ldots,j\}}} B_z^{i_{k-1}}\big(\ldots B_z^{i_2}\big(B_z^{i_1}(\xi_1, \xi_2), \xi_3\big) \ldots, \xi_k\big). \tag{4.1.8}$$

Proof: We will prove this by induction over $k$. For $k = 2$ we get

$$\xi_1 \star_{zG} \xi_2 = B_z^0(\xi_1, \xi_2) + B_z^1(\xi_1, \xi_2) = \xi_1\xi_2 + \frac{1}{2}[\xi_1, \xi_2]$$

Which is clearly true. For the step $k \to k+1$ we can directly apply Equation (4.1.7):

$$\begin{aligned} \xi_1 \star_{zG} \ldots \star_{zG} \xi_{k+1} &= \left( \sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0,\ldots,j\}}} B_z^{i_{k-1}}\big(\ldots B_z^{i_2}\big(B_z^{i_1}(\xi_1, \xi_2), \xi_3\big) \ldots, \xi_k\big) \right) \star_{zG} \xi_{k+1} \\ &= \sum_{i_k=0}^{k} B_z^{i_k} \left( \sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0,\ldots,j\}}} B_z^{i_{k-1}}\big(\ldots B_z^{i_2}\big(B_z^{i_1}(\xi_1, \xi_2), \xi_3\big) \ldots, \xi_k\big), \xi_{k+1} \right) \\ &= \sum_{\substack{1 \leq j \leq k \\ i_j \in \{0,\ldots,j\}}} B_z^{i_k}\Big(B_z^{i_{k-1}}\big(\ldots B_z^{i_2}\big(B_z^{i_1}(\xi_1, \xi_2), \xi_3\big) \ldots, \xi_k\big), \xi_{k+1}\Big) \end{aligned}$$

□

**Remark 4.1.7** Our final goal in this chapter is actually a nice identity for the case of two monomials $\xi_1 \ldots \xi_k \star_{zG} \eta_1 \ldots \eta_\ell$ with $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. Theoretically, we could use Equation (4.1.8) for it, since

$$\xi_1 \ldots \xi_k \star_{zG} \eta_1 \ldots \eta_\ell = \frac{1}{k!\ell!} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \xi_{\sigma(1)} \star_{zG} \ldots \star_{zG} \xi_{\sigma(k)} \star_{zG} \eta_{\tau(1)} \star_{zG} \ldots \star_{zG} \eta_{\tau(\ell)}. \tag{4.1.9}$$

This equality can easily been proven from the definition of the map $\mathfrak{q}$. The only flaw in the plan is, however, that we're looking for something *nice*. So we have to go for something different.

### 4.1.3 A Formula for two Monomials

If we want to get an identity for the star product of two monomials, we have to get back to Equation (**??**). The result will not be very explicit either, but still by far better than Equation (4.1.9). We will at least be able to do some computations with concrete examples. As a first step, we must introduce a bit of notation:

**Definition 4.1.8 (G-Index)** *Let* $k, \ell, n \in \mathbb{N}$ *and* $r = k + \ell - n$. *Then we call an* $r$*-tuple* $J$

$$J = (J_1, \ldots, J_r) = ((a_1, b_1), \ldots, (a_r, b_r))$$

*a G-index if it fulfils the following properties:*

*(i)* $J_i \in \{0, 1, \ldots, k\} \times \{0, 1, \ldots, \ell\}$

*(ii)* $|J_i| = a_i + b_i \geq 1 \quad \forall_{i=1,\ldots,r}$

*(iii)* $\sum_{i=1}^{r} a_i = k$ *and* $\sum_{i=1}^{r} b_i = \ell$

*(iv) The tuple is ordered in the following sense:* $i > j \Rightarrow |J_i| \geq |J_j| \quad \forall_{i,j=1,\ldots,r}$ *and* $|a_i| \geq |a_j|$ *if* $|J_i| = |J_j|$

*(v) If* $a_i = 0$ *[or* $b_i = 0$*] for some* $i$*, then* $b_i = 1$ *[or* $a_i = 1$*].*

*We call the set of all such G-indices* $\mathcal{G}_r(k,\ell)$*.*

**Definition 4.1.9 (G-Factorial)** *Let* $J = ((a_1,b_1),\ldots,(a_r,b_r)) \in \mathcal{G}_r(k,\ell)$ *be a G-Index. We set for a given tuple* $(a,b) \in \{0,1,\ldots,k\} \times \{0,1,\ldots,\ell\}$

$$\#_J(a,b) = \text{ number of times that } (a,b) \text{ appears in } J.$$

*Then we define the G-factorial of* $J \in \{0,1,\ldots,k\} \times \{0,1,\ldots,\ell\}$ *as*

$$J! = \prod_{(a,b)\in\{0,1,\ldots,k\}\times\{0,1,\ldots,\ell\}} (\#_J(a,b))!$$

This allows us to state an explicit formula for the Gutt star product:

**Lemma 4.1.10** *Let* $\mathfrak{g}$ *be a Lie algebra,* $\xi,\eta \in \mathfrak{g}$ *and* $k,\ell \in \mathbb{N}$*. Then we have the following identity for the Gutt star product:*

$$\xi^k \star_{zG} \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n\Big(\xi^k,\eta^\ell\Big),$$

*where the* $C_n$ *are given by*

$$C_n\Big(\xi^k,\eta^\ell\Big) = \sum_{J\in\mathcal{G}_{k+\ell-n}(k,\ell)} \frac{k!\ell!}{J!} \prod_{i=1}^{k+\ell-n} \mathtt{BCH}_{a_i,b_i}(\xi,\eta) \tag{4.1.10}$$

*and the product is taken in the symmetric tensor algebra.*

Proof: We want to calculate what the $C_n$ look like. Let's denote $r = k+\ell-n$ for brevity. Then we have

$$C_n\Big(\xi^k,\eta^\ell\Big) \in \mathrm{S}^r(\mathfrak{g}).$$

Of course, the only part of the series

$$\exp\bigg(\frac{1}{z}\mathtt{BCH}(z\xi,z\eta)\bigg) = \sum_{n=0}^{k+\ell}\bigg(\frac{1}{z}\mathtt{BCH}(z\xi,z\eta)\bigg)^n + \mathcal{O}(\xi^{k+1},\eta^{\ell+1})$$

which lies in $\mathrm{S}^r(\mathfrak{g})$ is the summand for $n=r$. Since we introduce the formal parameters $t$ and $s$, we don't need to care about terms of higher orders in $\xi$ and $\eta$ than $k$ and $\ell$ respectively.

$$\begin{aligned}
z^n C_n\Big(\xi^k,\eta^\ell\Big) &= \frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \frac{1}{z^r}\frac{\mathtt{BCH}(zt\xi,zs\eta)^r}{r!} \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \left(\sum_{j=1}^{k+\ell}\mathtt{BCH}_j(zt\xi,zs\eta)\right)^r \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \sum_{\substack{j_1,\ldots,j_r\geq 1\\ j_1+\ldots+j_r=k+\ell}} \mathtt{BCH}_{j_1}(zt\xi,zs\eta)\cdots\mathtt{BCH}_{j_r}(zt\xi,zs\eta)
\end{aligned}$$

$$= z^n \frac{k!\ell!}{r!} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \mathtt{BCH}_{a_i,b_i}(\xi,\eta)\ldots\mathtt{BCH}_{a_r,b_r}(\xi,\eta) \tag{4.1.11}$$

We sum over all possible arrangements of the $(a_i, b_i)$. In order to find a nicer form of the sum, we put the ordering from definition 4.1.8 on these multi-indices and avoid therefore double counting. We loose the freedom of arranging the $(a_i, b_i)$ and need to count the number of multi-indices $((a_1, b_1), \ldots, (a_r, b_r))$ which belong to the same G-index $J$. This number will be $\frac{r!}{J!}$, since we can't interchange the $(a_i, b_i)$ any more (therefore $r!$), unless they are equal (therefore $J!^{-1}$). Since the ranges of the $(a_i, b_i)$ in Equation (4.1.11) and of the elements in $\mathcal{G}_r(k, \ell)$ are the same. we can change the summation there to $J \in \mathcal{G}_r(k, \ell)$ and multiply by $\frac{r!}{J!}$. We find

$$z^n C_n\left(\xi^k, \eta^\ell\right) = z^n \frac{k!\ell!}{J!} \sum_{J\in\mathcal{G}_r(k,\ell)} \mathtt{BCH}_{a_i,b_i}(\xi,\eta)\ldots\mathtt{BCH}_{a_r,b_r}(\xi,\eta)$$

which is precisely Equation (4.1.10). □

Now we just need to generalize this to factorizing tensors. To do so, we need a last definition:

**Definition 4.1.11** *Let $k, \ell, n \in \mathbb{N}$ and $J \in \mathcal{G}_{k+\ell-n}(k, \ell)$. Then for $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell$ from a Lie algebra $\mathfrak{g}$ we set*

$$\Gamma_J(\xi_1, \ldots, \xi_k; \eta_1, \ldots, \eta_\ell) = \frac{1}{J!} \prod_{i=1}^{k+\ell-n} \mathtt{BCH}_{a_i,b_i}\left(\xi^{(a_i)}, \eta^{(b_i)}\right) \tag{4.1.12}$$

*where the notation $\mathtt{BCH}_{a_i,b_i}\big(\xi^{(a_i)}, \eta^{(b_i)}\big)$ means that we have taken $\prod_{i=1}^{k+\ell-n} \mathtt{BCH}_{a_i,b_i}\big(\xi^{(a_i)}, \eta^{(b_i)}\big)$ and replaced the $j$-th $\xi$ appearing in it with $\xi_j$ for $j = 1, \ldots, k$ and analogously with the $\eta$'s.*

**Proposition 4.1.12** *Let $\mathfrak{g}$ be a Lie algebra, $k, \ell \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. Then we have the following identity for the Gutt star product:*

$$\xi_1 \ldots \xi_k \star_{zG} \eta_1 \ldots \eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1 \ldots \xi_k, \eta_1 \ldots \eta_\ell),$$

*where the $C_n$ are given by*

$$C_n(\xi_1 \ldots \xi_k, \eta_1 \ldots \eta_\ell) = \sum_{J\in\mathcal{G}_{k+\ell-n}(k,\ell)} \sum_{\sigma\in S_k} \sum_{\tau\in S_\ell} \Gamma_J\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}; \eta_{\tau(1)}, \ldots, \eta_{\tau(\ell)}\big) \tag{4.1.13}$$

*and the product is taken in the symmetric tensor algebra.*

Proof: The proof relies on polarization again and is completely analogous to the one of proposition 4.1.3. We set

$$\Xi = \sum_{i=1}^k t_i \xi^i \quad \text{and} \quad \mathrm{H} = \sum_{i=1}^\ell t_j \eta^j.$$

Then it is easy to see that we will get rid of the factorials in Equation (4.1.10) since

$$\xi_1 \ldots \xi_k \star_{zG} \eta_1 \ldots \eta_\ell = \frac{1}{k!\ell!} \frac{\partial^{k+\ell}}{\partial_{t_1} \ldots \partial_{s_\ell}}\Big|_{t_1,\ldots,s_\ell=0} \Xi^k \star_{zG} \mathrm{H}^\ell .$$

Instead of the factorials, we get symmetrizations over the $\xi_i$ and the $\eta_j$ as we did in Proposition 4.1.3, which gives the wanted result. □

## 4.2 Consequences and examples

### Some consequences

Proposition 4.1.12 allows us to get some easy algebraic results. For example, we know that the Gutt star product should fulfil the classical and the semi-classical limit from Definition **??** and this was also proven by Simone Gutt in the paper [?] where she discovered it, but it is good to see that the formula we set up really gives the same result.

**Corollary 4.2.1** *Given two arbitrary tensors $x, y \in \mathrm{S}^\bullet(\mathfrak{g})$, we find*

$$x \star_{zG} y = C_0(x,y) + zC_1(x,y) + \sum_{n=2}^{\deg x + \deg y - 1} z^n C_n(x,y).$$

*The $C_n$ satisfy the identities*

$$C_0(x,y) = xy$$

*and*

$$C_1(x,y) - C_1(y,x) = \{x,y\}_{KKS}$$

*where $\{\cdot,\cdot\}_{KKS}$ denotes the Kirillov-Kostant-Souriau bracket.*

PROOF: Since the $C_n$ are bilinear, it is sufficient to check those identities on factorizing tensors. Again we take $\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell \in \mathrm{S}^\bullet(\mathfrak{g})$. We have to look at the G-indices in $\mathcal{G}_{k+\ell}(k,\ell)$. This is easy, since there is just one element inside:

$$\mathcal{G}_{k+\ell}(k,\ell) = \Big\{ (\underbrace{(0,1),\dots,(0,1)}_{\ell \text{ times}}, \underbrace{(1,0),\dots,(1,0)}_{k \text{ times}}) \Big\}.$$

So we find

$$\begin{aligned} C_0(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) &= \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{J!} \mathtt{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(1)}\big) \dots \mathtt{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(k)}\big) \\ &\quad \cdot \mathtt{BCH}_{1,0}\big(\eta_{\tau(1)}, \varnothing\big) \dots \mathtt{BCH}_{1,0}\big(\eta_{\tau(\ell)}, \varnothing\big) \\ &\overset{(a)}{=} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{k!\ell!} \xi_{\sigma(1)} \dots \xi_{\sigma(k)} \eta_{\tau(1)} \dots \eta_{\tau(\ell)} \\ &= \xi_{\sigma(1)} \dots \xi_{\sigma(k)} \eta_{\tau(1)} \dots \eta_{\tau(\ell)} \end{aligned}$$

where we used $J! = k!\ell!$ in (a) according to Definition 4.1.9. We do the same for $C_1(\dots)$. Also here, there is just one element in $\mathcal{G}_{k+\ell-1}(k,\ell)$:

$$\mathcal{G}_{k+\ell}(k,\ell) = \Big\{ (\underbrace{(0,1),\dots,(0,1)}_{\ell-1 \text{ times}}, \underbrace{(1,0),\dots,(1,0)}_{k-1 \text{ times}}, (1,1)) \Big\}.$$

Of course we have

$$\mathtt{BCH}_{1,1}(\xi,\eta) = \frac{1}{2}[\xi,\eta]$$

and find with $J! = (k-1)!(\ell-1)!$

$$C_1(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) = \frac{1}{2} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{(k-1)!(\ell-1)!} \xi_{\sigma(1)} \dots \xi_{\sigma(k-1)} \eta_{\tau(1)} \dots \eta_{\tau(\ell-1)} [\xi_{\sigma(k)}, \eta_{\tau(\ell)}]$$

$$= \frac{1}{2} \sum_{i=0}^{k} \sum_{j=0}^{\ell} \xi_1 \dots \widehat{\xi_i} \dots \xi_k \eta_1 \dots \widehat{\eta_j} \dots \eta_\ell [\xi_i, \eta_j]$$

where the hat means that the $\xi_i$ and the $\eta_j$ are left out. From this, the anti-symmetry of the Lie bracket yields

$$C_1(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) - C_1(\eta_1 \dots \eta_\ell, \xi_1 \dots \xi_k) = \sum_{i=0}^{k} \sum_{j=0}^{\ell} \xi_1 \dots \widehat{\xi_i} \dots \xi_k \eta_1 \dots \widehat{\eta_j} \dots \eta_\ell [\xi_i, \eta_j].$$

We just need to verify, that this is really the KKS-Poisson bracket for given polynomials $\xi_1 \dots \xi_k$ and $\eta_1 \dots \eta_\ell$. □

Moreover, we have compatibility of the bigger formula from Proposition 4.1.12 with the smaller one from Proposition 4.1.3.

**Corollary 4.2.2** *Given $\xi_1, \dots, \xi_k, \eta \in \mathfrak{g}$, the results of the two Equations* (4.1.13) *and* (4.1.3) *coincide.*

PROOF: We have to compute sets of G-indices for $\xi_1, \dots, \xi_k, \eta_\ell \in \mathfrak{g}$. Again, they only have one element:

$$\mathcal{G}_{k+1-n}(k, 1) = \Big\{ (\underbrace{(1,0), \dots, (1,0)}_{k-n \text{ times}}, (n, 1)) \Big\}.$$

So we have with $J! = (k-n)!$

$$z^n C_n(\xi_1 \dots \xi_k, \eta_\ell) = z^n \sum_{\sigma \in S_k} \frac{1}{(k-n)!} \frac{B_n^*}{n!} \xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} [\xi_{\sigma(k-n+1)}, [\dots, [\xi_{\sigma(k),\eta}] \dots]]$$

which gives, after a light reordering

$$z^n C_n(\xi_1 \dots \xi_k, \eta_\ell) = z^n \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{n} B_n^* [\xi_{\sigma(1)}, [\dots, [\xi_{\sigma(n),\eta}] \dots]] \xi_{\sigma(n+1)} \cdots \xi_{\sigma(k)}.$$

Summing up over all $n$ now gives Equation (4.1.3). □

Just to make it it complete, we also want to state what it looks like when we change the left and the right hand side.

**Proposition 4.2.3** *Let $\mathfrak{g}$ be a Lie algebra and $\xi_1, \dots, \xi_k, \eta \in \mathfrak{g}$. We have*

$$\eta \star_{zG} \xi_1 \dots \xi_k = \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} z^j B_j \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\dots [\xi_{\sigma(j)}, \eta] \dots]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)}. \tag{4.2.1}$$

PROOF: The proof is completely analogue to the one of Proposition 4.1.3. The only difference is that we take Equation (**??**) which gives the BCH series up to first order in the first and not in the second argument. □

### Two examples

Equation (4.1.13) is useful if one wants to do real computations with the star product, but it does not look very easy to apply on the first sight. This is why we will give two examples here. The easiest one which is not covered by the simpler formula (4.1.3) will be the star product of two quadratic terms. The second one should be the a bit more complex case of a cubic term with a quadratic term.

**Two quadratic terms**

Let's take $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathfrak{g}$. We want to compute

$$\xi_1\xi_2 \star_{zG} \eta_1\eta_2 = C_0(\xi_1, \xi_2, \eta_1, \eta_2) + zC_1(\xi_1, \xi_2, \eta_1, \eta_2) + z^2C_2(\xi_1, \xi_2, \eta_1, \eta_2) + z^3C_3(\xi_1, \xi_2, \eta_1, \eta_2).$$

The very first thing we have to do is computing the set of G-indices. Then we calculate the G-factorial and finally go through the permutations.

$C_0$: We already did this in Corollary 4.2.1 and know, that the zeroth order in $z$ is just the symmetric product. Therefore we have

$$C_0(\xi_1\xi_2, \eta_1\eta_2) = \xi_1\xi_2\eta_1\eta_2$$

$C_1$: We also did this one in Corollary 4.2.1: There is just one G-index and we finally get

$$C_1(\xi_1\xi_2, \eta_1\eta_2) = \frac{1}{2}(\xi_2\eta_2[\xi_1, \eta_1] + \xi_2\eta_1[\xi_1, \eta_2] + \xi_1\eta_2[\xi_2, \eta_1] + \xi_1\eta_1[\xi_2, \eta_2]).$$

$C_2$: This is the first time, something interesting happens. We have three G-indices:

$$\mathcal{G}_2(2,2) = \{J^1, J^2, J^3\} = \{\big((0,1),(2,1)\big), \big((1,0),(1,2)\big), \big((1,1),(1,1)\big)\}.$$

The G-factorials give $J^1! = J^2! = 1$ and $J^3! = 2$, since the index $(1,1)$ appears twice in $J_3$. We take $\mathtt{BCH}_{1,2}(X,Y)$ and $\mathtt{BCH}_{2,1}(X,Y)$ for two variables $X$ and $Y$ from Equation (**??**):

$$\mathtt{BCH}_{1,2}(X,Y) = \frac{1}{12}[Y,[Y,X]] \quad \text{and} \quad \mathtt{BCH}_{2,1}(X,Y) = \frac{1}{12}[X,[X,Y]].$$

So we have to insert the $\xi_i$ and the $\eta_j$ into $\frac{1}{12}X[Y,[Y,X]]$ and $\frac{1}{12}Y[X,[X,Y]]$ respectively and then we go on with the last one, which is

$$\frac{1}{2}\mathtt{BCH}_{1,1}(X,Y)\mathtt{BCH}_{1,1}(X,Y) = \frac{1}{8}[X,Y][X,Y].$$

We hence get

$$\begin{aligned} C_2(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{1}{12}\big(&\eta_1[[\eta_2, \xi_1], \xi_2] + \eta_1[[\eta_2, \xi_2], \xi_1] + \eta_2[[\eta_1, \xi_1], \xi_2] + \eta_2[[\eta_1, \xi_2], \xi_1] + \\ &\xi_1[[\xi_2, \eta_1], \eta_2] + \xi_1[[\xi_2, \eta_2], \eta_1] + \xi_2[[\xi_1, \eta_1], \eta_2] + \xi_2[[\xi_1, \eta_2], \eta_1]\big) + \\ &\frac{1}{4}\big([\xi_1, \eta_1][\xi_2, \eta_2] + [\xi_1, \eta_2][\xi_2, \eta_1]\big) \end{aligned}$$

$C_3$: Here, we only have one G-index:

$$\mathcal{G}_1(2,2) = \{\big((2,2)\big)\}$$

The G-factorial is 1. We take again Equation (**??**) and see

$$\mathtt{BCH}_{2,2}(X,Y) = \frac{1}{24}[Y,[X,[Y,X]]].$$

This gives

$$C_3(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{1}{24}\big([[[\eta_1, \xi_1], \xi_2], \eta_2] + [[[\eta_1, \xi_2], \xi_1], \eta_2] + [[[\eta_2, \xi_1], \xi_2], \eta_1] + [[[\eta_2, \xi_2], \xi_1], \eta_1]\big)$$

We just have to put all the four terms together and have the star product.

### A cubic and a quadratic term

Let $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2 \in \mathfrak{g}$. We compute

$$\xi_1\xi_2\xi_3 \star_G \eta_1\eta_2 = \sum_{n=0}^{4} z^n C_n(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2)$$

$C_0$: The first part is again just the symmetric product:

$$C_0(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \xi_1\xi_2\xi_3\eta_1\eta_2.$$

$C_1$: Here we have again the term from Corollary 4.2.1:

$$\begin{aligned} C_1(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \frac{1}{2}\big(&\xi_2\xi_3\eta_2[\xi_1, \eta_1] + \xi_2\xi_3\eta_1[\xi_1, \eta_2] + \xi_1\xi_3\eta_2[\xi_2, \eta_1] + \\ &\xi_1\xi_3\eta_1[\xi_2, \eta_2] + \xi_1\xi_2\eta_2[\xi_3, \eta_1] + \xi_1\xi_2\eta_1[\xi_3, \eta_2]\big) \end{aligned}$$

$C_2$: Here the calculation is very similar to the one of $C_2$ in the example before. We have three G-indices:

$$\mathcal{G}_3(3, 2) = \left\{J^1, J^2, J^3\right\} = \left\{\big((0, 1), (1, 0), (2, 1)\big), \big((1, 0), (1, 0), (1, 2)\big), \big((1, 0), (1, 1), (1, 1)\big)\right\}.$$

The G-factorials are now $J^1! = 1$ and $J^2! = J^3! = 2$. Again, we take the BCH terms from Equation (**??**) and see, that we must insert the $\xi_i$ and the $\eta_j$ into

$$\frac{1}{12}XY[X, [X, Y]] + \frac{1}{24}XX[Y, [Y, X]] + \frac{1}{8}X[X, Y][X, Y].$$

Now we go through all the possible permutations and get

$$\begin{aligned} C_2(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \frac{1}{12}\big(&\xi_1\xi_2[[\xi_3, \eta_1], \eta_2] + \xi_1\xi_2[[\xi_3, \eta_2], \eta_1] + \xi_1\xi_3[[\xi_2, \eta_1], \eta_2] + \\ &\xi_1\xi_3[[\xi_2, \eta_2], \eta_1] + \xi_2\xi_3[[\xi_1, \eta_1], \eta_2] + \xi_2\xi_3[[\xi_1, \eta_2], \eta_1]\big) + \\ \frac{1}{12}\big(&\xi_1\eta_1[[\eta_2, \xi_2], \xi_3] + \xi_1\eta_2[[\eta_1, \xi_2], \xi_3] + \xi_1\eta_1[[\eta_2, \xi_3], \xi_2] + \\ &\xi_1\eta_2[[\eta_1, \xi_3], \xi_2] + \xi_2\eta_1[[\eta_2, \xi_1], \xi_3] + \xi_2\eta_2[[\eta_1, \xi_1], \xi_3] + \\ &\xi_2\eta_1[[\eta_2, \xi_3], \xi_1] + \xi_2\eta_2[[\eta_1, \xi_3], \xi_1] + \xi_3\eta_1[[\eta_2, \xi_2], \xi_1] + \\ &\xi_3\eta_2[[\eta_1, \xi_2], \xi_1] + \xi_3\eta_1[[\eta_2, \xi_1], \xi_2] + \xi_3\eta_2[[\eta_1, \xi_1], \xi_2]\big) + \\ \frac{1}{4}\big(&\xi_1[\xi_2, \eta_1][\xi_3, \eta_2] + \xi_1[\xi_3, \eta_1][\xi_2, \eta_2] + \xi_2[\xi_1, \eta_1][\xi_3, \eta_2] + \\ &\xi_2[\xi_3, \eta_1][\xi_1, \eta_2] + \xi_3[\xi_1, \eta_1][\xi_2, \eta_2] + \xi_3[\xi_2, \eta_1][\xi_1, \eta_2]\big). \end{aligned}$$

$C_3$: We first calculate the G-indices:

$$\mathcal{G}_2(3, 2) = \left\{J^1, J^2, J^3\right\} = \left\{\big((0, 1), (3, 1)\big), \big((1, 0), (2, 2)\big), \big((1, 1), (2, 1)\big)\right\}.$$

We don't have to care about $J^1$, since $\mathtt{BCH}_{3,1}(X, Y) = 0$. The G-factorials for the other two indices are 1. The BCH terms have been computed before. So we have to fill in the expression

$$\frac{1}{24}X[Y, [X, [Y, X]]] + \frac{1}{2 \cdot 12}[X, Y][X, [X, Y]].$$

Doing the permutations, we get

$$\begin{aligned}C_3(\xi_1\xi_2\xi_3,\eta_1\eta_2) = \frac{1}{24}\big(&\xi_1[[[\eta_1,\xi_2],\xi_3],\eta_2] + \xi_1[[[\eta_2,\xi_2],\xi_3],\eta_1] + \xi_1[[[\eta_1,\xi_3],\xi_2],\eta_2]+\\
&\xi_1[[[\eta_2,\xi_3],\xi_2],\eta_1] + \xi_2[[[\eta_1,\xi_1],\xi_3],\eta_2] + \xi_2[[[\eta_2,\xi_1],\xi_3],\eta_1]+\\
&\xi_2[[[\eta_1,\xi_3],\xi_1],\eta_2] + \xi_2[[[\eta_2,\xi_3],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_2],\xi_1],\eta_2]+\\
&\xi_3[[[\eta_2,\xi_2],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_1],\xi_2],\eta_2] + \xi_3[[[\eta_2,\xi_1],\xi_2],\eta_1]\big)+\\
&\frac{1}{24}\big([\xi_1,\eta_1][[\eta_2,\xi_2],\xi_3] + [\xi_1,\eta_2][[\eta_1,\xi_2],\xi_3] + [\xi_1,\eta_1][[\eta_2,\xi_3],\xi_2]+\\
&[\xi_1,\eta_2][[\eta_1,\xi_3],\xi_2] + [\xi_2,\eta_1][[\eta_2,\xi_1],\xi_3] + [\xi_2,\eta_2][[\eta_1,\xi_1],\xi_3]+\\
&[\xi_2,\eta_1][[\eta_2,\xi_3],\xi_1] + [\xi_2,\eta_2][[\eta_1,\xi_3],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_2],\xi_1]+\\
&[\xi_3,\eta_2][[\eta_1,\xi_2],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_1],\xi_2] + [\xi_3,\eta_2][[\eta_1,\xi_1],\xi_2]\big).\end{aligned}$$

$C_4$: Now there's only $C_4$ left. We only have one G-index:

$$\mathcal{G}_1(3,2) = \big\{\big((3,2)\big)\big\},$$

but there are more terms which belong to it. We have to go through

$$\mathtt{BCH}_{3,2}(X,Y) = \frac{1}{120}[[[[X,Y],X],Y],X] + \frac{1}{360}[[[[Y,X],X],X],Y].$$

So we permute and get

$$\begin{aligned}C_4(\xi_1\xi_2\xi_3,\eta_1\eta_2) = \frac{1}{120}\big(&[[[[\xi_1,\eta_1],\xi_2],\eta_2],\xi_3] + [[[[\xi_1,\eta_2],\xi_2],\eta_1],\xi_3] + [[[[\xi_1,\eta_1],\xi_3],\eta_2],\xi_2]+\\
&[[[[\xi_1,\eta_2],\xi_3],\eta_1],\xi_2] + [[[[\xi_2,\eta_1],\xi_1],\eta_2],\xi_3] + [[[[\xi_2,\eta_2],\xi_1],\eta_1],\xi_3]+\\
&[[[[\xi_2,\eta_1],\xi_3],\eta_2],\xi_1] + [[[[\xi_2,\eta_2],\xi_3],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_2],\eta_2],\xi_1]+\\
&[[[[\xi_3,\eta_2],\xi_2],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_1],\eta_2],\xi_2] + [[[[\xi_3,\eta_2],\xi_1],\eta_1],\xi_2]\big)+\\
&\frac{1}{360}\big([[[[\eta_1,\xi_1],\xi_2],\xi_3],\eta_2] + [[[[\eta_2,\xi_1],\xi_2],\xi_3],\eta_1] + [[[[\eta_1,\xi_1],\xi_3],\xi_2],\eta_2]+\\
&[[[[\eta_2,\xi_1],\xi_3],\xi_2],\eta_1] + [[[[\eta_1,\xi_2],\xi_1],\xi_3],\eta_2] + [[[[\eta_2,\xi_2],\xi_1],\xi_3],\eta_1]+\\
&[[[[\eta_1,\xi_2],\xi_3],\xi_1],\eta_2] + [[[[\eta_2,\xi_2],\xi_3],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_2],\xi_1],\eta_2]+\\
&[[[[\eta_2,\xi_3],\xi_2],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_1],\xi_2],\eta_2] + [[[[\eta_2,\xi_3],\xi_1],\xi_2],\eta_1]\big).\end{aligned}$$

Now we only have to add up all those terms and we have finally computed the star product.

## 4.3 Low-Verifications of the formulas

### 4.3.1 First verifications with Mathematica

### 4.3.2 Ideas for an algorithm beyond

# Chapter 5

# A locally convex topology for the Gutt star product

We have finished the algebraic part of this work, except for some little lemmas concerning the Hopf theoretic chapter. Our next goal is setting up a locally convex topology on the symmetric tensor algebra, in which the Gutt star product will be continuous. At the beginning of this chapter, we will first give a motivation why the setting of locally convex algebras is convenient and necessary. In the second part, we will briefly recall the most important things on locally convex algebras and introduce the topology which we will work with. In the third section, the core of this chapter, the continuity of the star product and the dependence on the formal parameter are proven. Part four treats the case when the formal parameter $z = 1$ and hence talks about a locally convex topology on the universal enveloping algebra of a Lie algebra. We will also show, that our topology is "optimal" in a specific sense.

## 5.1 Why locally convex?

The first question one could ask is, why we want the observable algebra to be a *locally convex* one. There are a lot of different choices and most of them would even make things simpler: we could think of locally multiplicatively convex algebras, Banach algebras, $C^*$- or even von Neumann algebras. All of them have much more structure than just locally convex algebras. We would have an entire holomorphic calculus within our algebra if we assumed it to be locally m-convex or even a continuous one if we wanted it to be $C^*$.

The reason is, that all these nice features are simply not there, in general. Quantum mechanics shows us, that the algebra made up by $\hat{q}$ and $\hat{p}$ can not be locally m-convex.

**Proposition 5.1.1** *Let $\mathscr{A}$ be a unital associative algebra which contains the quantum mechanical observables $\hat{q}$ and $\hat{p}$ and in which the canonical commutation relation*

$$[\hat{q}, \hat{p}] = i\hbar$$

*is fulfilled. Then the only submultiplicative semi-norm on it is $p = 0$.*

PROOF: First, we need to show a little lemma:

**Lemma 5.1.2** *In the given algebra, we have for $n \in \mathbb{N}$*

$$(\mathrm{ad}_{\hat{q}})^n (\hat{p}^n) = (i\hbar)^n n! \mathbb{1}. \tag{5.1.1}$$

PROOF: To show it, we use the fact that for $a \in \mathscr{A}$ the operator $\mathrm{ad}_a$ is a derivation, which is always true for a Lie algebra which comes from an associative algebra with the commutator, since for $a, b, c \in \mathscr{A}$ we have

$$[a, bc] = abc - bca = abc - bac + bac - bca = [a, b]c + b[a, c].$$

Now for $n = 1$ Equation (5.1.2) is certainly true. So let's look at the step $n \to n + 1$. We make use of the derivation property and have

$$\begin{aligned}
(\mathrm{ad}_{\hat{q}})^{n+1}(\hat{p}^{n+1}) &= (\mathrm{ad}_{\hat{q}})^n (i\hbar\hat{p}^n + \hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^n (\hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n-1} \Big( [\hat{q}, \hat{p}]\,\mathrm{ad}_{\hat{q}}(\hat{p}^n) + \hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n) \Big) \\
&= (i\hbar)^{n+1} n! + i\hbar(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n) + (\mathrm{ad}_{\hat{q}})^{n-1} \Big( \hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n) \Big) \\
&= 2(i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n-1} \Big( \hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n) \Big) \\
&\overset{(*)}{=} \ \vdots \\
&= n(i\hbar)^{n+1} n! + \mathrm{ad}_{\hat{q}}(\hat{p}(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)) \\
&= n(i\hbar)^{n+1} n! + i\hbar(i\hbar)^n n! \\
&= (i\hbar)^{n+1}(n+1)!.
\end{aligned}$$

At $(*)$ we actually used another statement which is to be proven by induction over $k$ and says

$$(\mathrm{ad}_{\hat{q}})^{n+1}(\hat{p}^{n+1}) = k(i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n+1-k} \Big( \hat{p}(\mathrm{ad}_{\hat{q}})^k(\hat{p}^n) \Big).$$

Since this proof is analogous to the first lines of the computation before, we omit it here and the lemma is proven. ▽

Now we can go on with the actual proof. Let $\|\cdot\|$ be a submultiplicative semi-norm. Then we see from Equation (5.1.2) that

$$\|(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)\| = |\hbar|^n n! \|\mathbb{1}\|.$$

On the other hand, we have

$$\begin{aligned}
\|(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)\| &= \|\hat{q}(\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n) - (\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n)\hat{q}\| \\
&\leq 2\|\hat{q}\| \|(\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n)\| \\
&\leq \ \vdots \\
&\leq 2^n \|\hat{q}\|^n \|\hat{p}^n\| \\
&\leq 2^n \|\hat{q}\|^n \|\hat{p}\|^n
\end{aligned}$$

So in the end we get

$$|\hbar|^n n! \|\mathbb{1}\| \leq c^n$$

for some $c \in \mathbb{R}$. This cannot be fulfilled for all $n \in \mathbb{N}$ unless $\|\mathbb{1}\| = 0$. But then, by submultiplicativity, the semi-norm itself must be equal to 0. □

**Remark 5.1.3** The so called Weyl algebra, which fulfils the properties of the foregoing proposition, can be constructed from a Poisson algebra with constant Poisson tensor. On one hand, it is a fair question, why this restriction of not being locally m-convex should also be put on linear

Poisson systems. On the other hand, there is no reason to expect that things become easier when we make the Poisson system more complex. Moreover, the Weyl algebra is actually nothing but a quotient of the Universal enveloping algebra of the so called Heisenberg algebra, a particular Lie algebra. So there is also no reason why the original algebra should have a "better" analytical structure than the quotient, since the ideal, which is divided out by this procedure, is a closed one.

There is a second good reason why we should avoid our topology to be locally m-convex. The topology we set up on $\mathrm{S}^\bullet(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ will also give a topology on $\mathcal{U}(\mathfrak{g})$. In Proposition **??** we will show, that, under weak (but for our purpose necessary) additional assumptions, there can be no topology on $\mathcal{U}(\mathfrak{g})$ which allows an entire holomorphic calculus. This underlines the results from Proposition 5.1.1, since locally m-convex algebras always have such a calculus.

In this sense, we have good reasons to think that $\mathrm{S}^\bullet(\mathfrak{g})$ will not allow a better setting than the one of a locally convex algebra if we want the Gutt star product to be continuous. Before we attack this task, we have to recall some technology from locally convex analysis.

## 5.2 Locally convex algebras

### 5.2.1 Locally convex spaces and algebras

Every locally convex algebra is of course also a locally convex space which is, of course, a topological vector space. To make clear, what we talk about, we first give a definition, which is taken from [**?**].

**Definition 5.2.1 (Topological vector space)** *Let $V$ be a vector space endowed with a topology $\tau$. Then we call $(V, \tau)$ (or for short just $V$, if there is no confusion about the topology possible) a topological vector space, if the two following things hold:*

*i.) for every point in $x \in V$ the set $\{x\}$ is a closed and*

*ii.) the vector space operations (addition, scalar multiplication) are continuous.*

Not all books require axiom (*i*) for a topological vector space. It is, however useful, since it assures that the topology in a topological vector space is Hausdorff – a feature, which we will always want to have. The proof for this is not difficult, but since we don't want to go too much into detail here, we refer to [**?**] again, where it can be found as Theorem 1.12.

The most important class of topological vector spaces are, at least, but not only, from a physical point of view, locally convex ones. Almost all interesting physical examples belong to this class: Finite-dimensional spaces, inner product (or pre-Hilbert) spaces, Banach spaces, Fréchet spaces, nuclear spaces and many more. Now, there are at least two equivalent definitions of what is a locally convex space. While the first is more geometrical, the second is better suited for our analytic purpose.

**Theorem 5.2.2** *For a topological vector space $V$, the following things are equivalent.*

*i.) $V$ has a local base $\mathscr{B}$ of the topology whose members are convex.*

*ii.) The topology on $V$ is generated by a separating family of semi-norms $\mathscr{P}$.*

PROOF: This theorem is a very well-known result and can be found in standard literature, such as [**?**], where it is divided into the Theorems 1.36 and 1.37. □

**Definition 5.2.3 (Locally convex space)** *A locally convex space is a topological vector space in which one (and thus all) of the properties from Theorem 5.2.2 are fulfilled.*

The first property explains the term "locally convex". For our intention, the second property is more helpful, since in this setting proving continuity just means putting estimates on semi-norms. For this purpose, one often extends the set of semi-norms $\mathscr{P}$ to the set of all continuous semi-norms $\mathcal{P}$, which contains all semi-norms that are compatible with the topology (e.g. sums, multiples and maxima of (finitely many) semi-norms from $\mathscr{P}$).

The next step are locally convex algebras.

**Definition 5.2.4 (Locally convex algebra)** *A locally convex algebra is a locally convex vector space with an additional algebra structure, which is continuous.*

More precisely, let $\mathscr{A}$ be a locally convex algebra and $\mathcal{P}$ the set of all continuous semi-norms, then for all $p \in \mathcal{P}$ there exists a $q \in \mathcal{P}$ such that for all $x, y \in \mathscr{A}$ one has

$$p(ab) \leq q(a)q(b). \tag{5.2.1}$$

Remind that we didn't require our algebras to be associative. The product in this equation could also be a Lie bracket. If we talk about associative algebras, we will always say it explicitly.

### 5.2.2 A special class of locally convex algebras

For our study of the Gutt star product, the usual continuity estimate (5.2.1) will not be enough, since there will be an arbitrarily high number of nested brackets to control. We will need an estimate which does not depend on the number of Lie brackets implied. But Lie also algebras are just a special type of algebras and the property we need makes sense for other types of locally convex algebra, too. This motivates the following definition.

**Definition 5.2.5 (Asymptotic estimate algebra)** *Let $\mathscr{A}$ be a locally convex algebra (not necessarily associative) with $\cdot$ denoting the multiplication and $\mathscr{P}$ the set of all continuous semi-norms. For a given $p \in \mathscr{P}$ we call $q \in \mathscr{P}$ an asymptotic estimate for $p$, if there exists an $m \in \mathbb{N}$ such that for all $n \geq m$ $q$ satisfies the following condition: $\forall_{x_1,\ldots,x_n \in \mathscr{A}}$*

$$p(x_1 \cdot \ldots \cdot x_n) \leq q(x_1) \ldots q(x_n). \tag{5.2.2}$$

*We call a locally convex algebra an AE-algebra, if every continuous semi-norm admits an asymptotic estimate.*

**Remark 5.2.6** Without further restrictions, we can set $m = 1$ in the upper definition, since this just means taking the maximum over a finite number of continuous semi-norms. If $q$ satisfies the upper definition for some $m \in \mathbb{N}$ and for all $i = 2, \ldots, m-1$ we have

$$p(x_1 \cdot \ldots \cdot x_i) \leq q^{(i)}(x_1) \ldots q^{(i)}(x_i)$$

for all $x_1, \ldots, x_i \in \mathcal{A}$, then we just set

$$q' = \max\{p, q^{(2)}, \ldots, q^{(m-1)}, q\}.$$

Clearly, $q'$ will again be a continuous semi-norm and an asymptotic estimate for $p$.

**Remark 5.2.7**

*i.)* The term asymptotic estimate has, to the best of our knowledge, first been used by Czichowski at all in [REFERENCE]. They defined asymptotic estimates in the same way we did, but their notion of AE-algebra was different from ours: in their definition of an AE algebra, not just one but a series of asymptotic estimates has to exist which fulfils two more properties. This is not the case in our definition, which is, in general, weaker.

*ii.)* In [], Neeb and Glöckner used a property to which they referred as $(*)$ for associative algebras. It was then used in [] by ... and ..., who called it the $GN$-property. It is easy to see that it is equivalent to being AE.

*iii.)* There are, of course, a lot of example for AE (Lie) algebras. All finite dimensional and Banach (Lie) algebras fulfil (5.2.2), just as locally m-convex (Lie) algebras do. The same is true for nilpotent locally convex Lie algebras, since here again one just has to take the maximum of a finite number of semi-norms. It is, however, far from clear what is exactly implied by (**??**). Are there examples for associative algebras which are AE but not locally m-convex, for example? Are there Lie algebras which are truly and not nilpotent?

It is at least possible to make some easy observations: an associative AE-algebra $\mathscr{A}$ will admit an entire holomorphic calculus: let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function and $a_n$ such that $f(z) = \sum_n a_n z^n$. Then one has $\forall_{x\in\mathscr{A}}$

$$p(f(x)) = p\left(\sum_{n=0}^{\infty} a_n x^n\right) \leq \sum_{n=0}^{\infty} |a_n| p(x^n) \leq \sum_{n=0}^{\infty} |a_n| q(x)^n < \infty$$

where $p \in \mathscr{P}$ and $q$ its asymptotic estimate. So in some sense, at least in the associative case, AE algebras are very close to locally m-convex ones. If the algebra is even commutative and Fréchet, then the two notions coincide: in [**?**] Mitiagin, Rolewicz and Zelazko proved that a commutative Fréchet algebra admitting an entire calculus is in fact locally m-convex. For non-commutative algebras, the situation is different. It is a very interesting (and non-trivial) question, if a non locally m-convex but AE algebra exists at all and if yes, how it could look like.

### 5.2.3 The projective tensor product

- Inequality of the symmetric tensor product

## 5.3 A topology for the Gutt star product

### 5.3.1 Continuity of the product

- a counter-example

### 5.3.2 Dependence on the formal parameter

### 5.3.3 Completion

### 5.3.4 Nuclearity

## 5.4 Alternative topologies and an optimal result

# Chapter 6

# Nilpotent Lie algebras

## 6.1 An overview

- Reference to the counter-example before, no big change - Yet: Projective Limit - Module structure - Generalizations to nilpotency

## 6.2 The Heisenberg and the Weyl algebra

## 6.3 The projective limit

## 6.4 A module structure

### 6.4.1 Generic case and a counter-example

### 6.4.2 Nilpotent case and good news

## 6.5 Banach Lie algebras and the finite-dimensional case

### 6.5.1 Generalizations of nilpotency

### 6.5.2 A new projective Limit

### 6.5.3 A result for the finite-dimensional case

# Chapter 7

# The Hopf algebra structure

## 7.1 The co-product

### 7.1.1 A formula for the co-product

### 7.1.2 Continuity for the co-product

## 7.2 The whole Hopf algebra structure

# Chapter 8

# Examples and remarks

## 8.1 Some classical Lie algebra

## 8.2 Some new ideas

### 8.2.1 A subalgebra of the Weyl algebra

### 8.2.2 Holomorphic vector fields