Convergence of the Gutt star product

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Here will be things like thanks to and so on...

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Here should already be some stuff...

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As we have already seen, the answer a mathematician would give to the question "What is (classical) mechanics?" is very closely linked to symplectic and, more generally, Poisson geometry. It is for this reason, that we will have a look at some basics of Poisson geometry. Of course, a systematic way to get a strict deformation quantization for every Poisson manifold would rather be a life's work and is by far beyond the reach of a master thesis. We have to restrict it to a special class of Poisson systems in order to get a viable challenge. Constant Poisson structures were already tackled by Stefan Waldmann and investigated more closely by Matthias Schötz in and respectively. Therefore, the next logical step is to attack linear Poisson structures.

Chapter 3

Algebraic Preliminaries

3.1 Linear Poisson structures

As we have seen before, there has already been done some work on how to strictly quantize local Poisson structures. The Weyl-Moyal-product on locally convex vector spaces was topologized by Stefan Waldmann in [**?**] and then investigated more closely by Matthias Schötz in [**?**]. It is thus clear that in the next step linear poisson structures on locally convex vector spaces must be done. Before we do so in the rest of this master thesis, we recall briefly some basics on linear Poisson structures.

Remark 3.1.1 (The axiom of choice) Our final goal is to do some locally convex functional analysis. Since in this game it is mandatory for us to sell our souls to the devil for the sake of the axiom of choice (e.g. in form of the Hahn-Banach theorem and the projective tensor product), there is no point in not doing it right from the beginning.

First of all, linear Poisson structures are actually something familiar: They are nothing but Lie algebras: Let $\{e^i\}_{i\in I}$ be a basis of a vector space $V*$ equipped with a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$.

3.2 The universal enveloping algebra

3.3 The Baker-Campbell-Hausdorff formula

Chapter 4

Formulas for the Gutt star product

We have seen some results on the Baker-Campbell-Hausdorff series and one on an identity for the Gutt star product. In Theorem **??** we found a very useful tool to get explicit formulas for the Gutt star product. We moreover may not forget that we still have to prove one part of it. We will do at the beginning of the first section of this chapter. This will then lead to a first easy formula for $\star_{zG}$. Afterwards, we will use the same procedure to find two more formulas for it: a rather involved one for the n-fold star product of vectors, which will not necessarily be helpful for algebraic computations, but will turn out very useful for estimates, and a more explicit one for the product of two monomials.

From those formulas, we will be able to draw some easy, but nice consequences in the next section and we will show how to compute the Gutt star product explicitly by doing two easy examples.

At the end of this chapter, we will give an easy Mathematica code, which can be used to verify the correctness of our formulas for low orders.

4.1 Formulas for the Gutt star product

4.1.1 An Iterative Approach from Linear Terms

The easiest case for which we will to develop a formula is surely the following one: For a given Lie algebra $\mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}$ we would like to compute

$$\xi^k \star_{zG} \eta = \sum_{n=0}^{k} z^n C_n(\xi^k, \eta)$$

We have already done this already for the Gutt star product, now we want to do the same for the BCH star product. This will finish the proof of Theorem **??**. For this purpose, we will use

$$\xi^k = \frac{\partial^k}{\partial t^k}\Big|_{t=0} \exp(t\xi).$$

We now have all the ingredients to prove the following proposition:

Lemma 4.1.1 *Let $\mathfrak{g}$ be a Lie algebra and $\xi, \eta \in \mathfrak{g}$. We have the following identity for the BCH star product $\star_{zG}$*

$$\xi^k \star_{zH} \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j (\eta). \tag{4.1.1}$$

Proof: We start from the simplified form for the Baker-Campbell-Hausdorff series from Equation (**??**). Putting things together we get

$$
\begin{aligned}
\xi^k \star_{zH} \eta &= \frac{\partial^k}{\partial t^k} \frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(\frac{1}{z}\mathtt{BCH}(zt\xi, zs\eta)\right) \\
&= \frac{\partial^k}{\partial t^k} \frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(t\xi + \sum_{j=0}^{\infty} z^j \frac{B_j^*}{j!} (\mathrm{ad}_{t\xi})^j (s\eta)\right).
\end{aligned}
$$

From this, we see that only terms which have exactly k of the ξ's in them and which are linear in η will contribute. This means we can cut off the sum at $j = k$. If we now write out the exponential series, we can cut it, too, for the same reason. We have

$$
\begin{aligned}
\xi^k \star_{zH} \eta &= \frac{\partial^k}{\partial t^k} \frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k} \frac{1}{n!} \left(t\xi + \sum_{j=0}^{k} (zt)^j \frac{B_j^*}{j!} (\mathrm{ad}_{\xi})^j (s\eta)\right)^n \\
&= \frac{\partial^k}{\partial t^k} \frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (t\xi)^{n-m} \left(\sum_{j=0}^{k} (zt)^j \frac{B_j^*}{j!} (\mathrm{ad}_{\xi})^j (s\eta)\right)^m \\
&= \frac{\partial^k}{\partial t^k} \frac{\partial}{\partial s}\Big|_{t=0,s=0} \left(\sum_{n=0}^{k} \frac{1}{n!}(t\xi)^n + \sum_{n=0}^{k}\sum_{j=0}^{k} \frac{1}{(n-1)!} t^{n+j-1} z^j \frac{B_j^*}{j!} \xi^{n-1} (\mathrm{ad}_{\xi})^j (s\eta)\right).
\end{aligned}
$$

In the last step we just cut off the sum over m since the terms for $m > 1$ will vanish because of the differentiation with respect to s. We can finally differentiate to get the formula

$$
\begin{aligned}
\xi^k \star_{zH} \eta &= \sum_{n=0}^{k}\sum_{j=0}^{k} \delta_{k,n+j-1} \frac{k!}{j!(n-1)!} z^j B_j^* \xi^{n-1} (\mathrm{ad}_{\xi})^j (\eta) \\
&= \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_{\xi})^j (\eta),
\end{aligned}
$$

which is the wanted result. □

Remark 4.1.2 We have now proven the equality of the two star products $\star_{zG}$ and $\star_{zH}$, by deriving an easy formula from both of them. From now on, we will derive all the other formulas from $\star_{zH}$, since this is the one which is easier to compute.

Once this is done, it is actually easy to get the formula for monomials of the form $\xi_1 \ldots \xi_k$ with $\eta \in \mathfrak{g}$:

Proposition 4.1.3 *Let $\mathfrak{g}$ be a Lie algebra and $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$. We have*

$$
\xi_1 \ldots \xi_k \star_{zG} \eta = \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} z^j B_j^* \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\ldots [\xi_{\sigma(j)}, \eta] \ldots]] \xi_{\sigma(j+1)} \ldots \xi_{\sigma(k)}. \tag{4.1.2}
$$

Proof: We get the result by just polarizing the formula from Lemma **??**. Let $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ be given, then we introduce the parameters t_i for $i = 1, \ldots, k$ and set

$$
\Xi = \Xi(t_1, \ldots, t_k) = \sum_{i=1}^{k} t_i \xi^k.
$$

Then it is immediate to see that

$$\xi_1 \dots \xi_k = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \dots \partial t_k}\Big|_{t_1,\dots,t_k=0} \Xi^k$$

since for every $i = 1, \dots, k$ we have

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} \Xi = \xi_i. \tag{4.1.3}$$

We also find for every $\eta \in \mathfrak{g}$

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} \mathrm{ad}_\Xi(\eta) = \mathrm{ad}_{\xi_i}(\eta). \tag{4.1.4}$$

Now we just need to calculate $\Xi^k \star_{zG} \eta$ and differentiate with respect to the t_i. In order to do this properly, we define

$$\gamma_n^k(\xi_1, \dots, \xi_k; \eta) = z^n \binom{k}{n} B_n^*(\mathrm{ad}_{\xi_1} \circ \dots \circ \mathrm{ad}_{\xi_n})(\eta)\xi_{n+1} \dots \xi_k$$

and

$$\gamma^k(\xi_1, \dots, \xi_k; \eta) = \sum_{n=0}^{k} \gamma_n^k(\xi_1, \dots, \xi_k; \eta).$$

We see that

$$\Xi^k \star_{zG} \eta = \gamma^k(\Xi, \dots, \Xi; \eta)$$

and can now differentiate this expression, which is linear in the every argument, with respect to the t_i. From the Equations (4.1.3) and (4.1.4) we get with the Leibniz rule

$$\frac{partial}{\partial t_1} \gamma^k(\Xi, \dots, \Xi; \eta) = \sum_{j=1}^{k} \gamma^k(\underbrace{\Xi, \dots, \Xi}_{j-1 \text{ times}}, \xi_1, \underbrace{\Xi, \dots, \Xi}_{k-j-1 \text{ times}}; \eta)$$

Differentiating now with respect to t_2, we get a second sum, where ξ_2 will be put once in every "free" position, and so on. One by one, all the slots will be taken by ξ_i's. We just need to divide by $k!$, and we finally find the formula from Equation (4.1.2). □

4.1.2 A first general Formula

Proposition 4.1.3 allows us basically to get a formula for the case of $\xi_1, \dots, \xi_k \in \mathfrak{g}$

$$\xi_1 \star_{zG} \dots \star_{zG} \xi_k = \sum_{j=0}^{k} C_{z,j}(\xi_1, \dots, \xi_k)$$

which we will need to prove the continuity of the coproduct, but which can also help to prove the continuity of the product in a different way.

Unluckily, this approach has a problem: iterating this formula, we get strangely nested Lie brackets, which would be very difficult to bring into a nice form with Jacobi and higher identities. So this is not a good way to find an handy formula for the usual star product of two monomials. Nevertheless, we want to pursue it for a moment, since we will get an equality which will be, although rather unfriendly looking, very useful in the following: for analytic observations, it will be enough to put (even brutal) estimates on it and the exact nature of the combinatorics in the formula is just not important. Hence we rewrite Equation (4.1.2) in order to cook up such a formula.

Definition 4.1.4 *Let $j, k \in \mathbb{N}_0$, $j \leq k$ and B_j^* as usual, then we define bilinear maps via*

$$\begin{aligned} B_z^{k,j} \colon \mathrm{S}^k(\mathfrak{g}) \times \mathfrak{g} &\longrightarrow \mathrm{S}^{k-j+1}(\mathfrak{g}) \\ (\xi_1 \ldots \xi_k, \eta) &\longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{j} B_j^* z^j [\xi_{\sigma(1)}, [\ldots, [\xi_{\sigma(j)}, \eta]]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)} \end{aligned}$$

and

$$B_z^j \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathfrak{g} \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad B_z^j = \sum_{k=0}^{\infty} B_z^{k,j}$$

where we set $B_z^j(x) = 0$ if $\deg(x) < j$.

We immediately get an easier identity for Equation (4.1.2). More than that: We can extend it to arbirtry symmetric tensors:

Lemma 4.1.5 *Let $\mathfrak{g}$ be a Lie-algebra an $x \in \mathrm{S}^\bullet(\mathfrak{g})$. Then we have the formula*

$$x \star_{zG} \eta = \sum_{j=0}^{\infty} B_z^j(x, \eta). \tag{4.1.5}$$

PROOF: First it is clear that the sum over j in Equation (4.1.5) is actually finite, since for $j > \deg(x)$ there is no further contribution. Using the grading we can write

$$x = \sum_{k=0}^{\deg(x)} \sum_i x_i^{(k)}.$$

The B_z^i-maps are linear in the first argument and the $x_i^{(k)}$ can be chosen to be factorizing tensors. But on factorizing tensors, Equation (4.1.5) is clearly true. We hence have by the linearity of $\star_{zG}$

$$\begin{aligned} x \star_{zG} \eta &= \sum_{k=0}^{\deg(x)} \sum_i x_i^{(k)} \star_{zG} \eta \\ &= \sum_{k=0}^{\deg(x)} \sum_i \sum_{j=0}^{\infty} B_z^j(x_i^{(k)}, \eta) \\ &= \sum_{j=0}^{\infty} B_z^j \left(\sum_{k=0}^{\deg(x)} \sum_i x_i^{(k)}, \eta \right) \\ &= \sum_{j=0}^{\infty} B_z^j(x, \eta). \end{aligned}$$

□

Now we want to use this approach to go on:

Proposition 4.1.6 *Let $\mathfrak{g}$, $2 \leq k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. Then we have*

$$\xi_1 \star_{zG} \ldots \star_{zG} \xi_k = \sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \ldots, j\}}} B_z^{i_{k-1}} \big(\ldots B_z^{i_2} \big(B_z^{i_1}(\xi_1, \xi_2), \xi_3 \big) \ldots, \xi_k \big).$$

PROOF: We will prove this by induction over k. For $k = 2$ we get

$$\xi_1 \star_{zG} \xi_2 = B_z^0(\xi_1, \xi_2) + B_z^1(\xi_1, \xi_2) = \xi_1\xi_2 + \frac{1}{2}[\xi_1, \xi_2].$$

For the step $k \to k+1$ we can directly apply Equation (4.1.5):

$$\begin{aligned}
\xi_1 \star_{zG} \dots \star_{zG} \xi_{k+1} &= \Bigg(\sum_{\substack{1 \le j \le k-1 \\ i_j \in \{0,\dots,j\}}} B_z^{i_{k-1}}\big(\dots B_z^{i_2}\big(B_z^{i_1}(\xi_1, \xi_2), \xi_3\big) \dots, \xi_k\big) \Bigg) \star_{zG} \xi_{k+1} \\
&= \sum_{i_k=0}^{k} B_z^{i_k} \Bigg(\sum_{\substack{1 \le j \le k-1 \\ i_j \in \{0,\dots,j\}}} B_z^{i_{k-1}}\big(\dots B_z^{i_2}\big(B_z^{i_1}(\xi_1, \xi_2), \xi_3\big) \dots, \xi_k\big), \xi_{k+1} \Bigg) \\
&= \sum_{\substack{1 \le j \le k \\ i_j \in \{0,\dots,j\}}} B_z^{i_k}\Big(B_z^{i_{k-1}}\big(\dots B_z^{i_2}\big(B_z^{i_1}(\xi_1, \xi_2), \xi_3\big) \dots, \xi_k\big), \xi_{k+1}\Big) \qquad \square
\end{aligned}$$

Theoretically, we could now use this formula to get another one for the case of two monomials with $\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_\ell \in \mathfrak{g}$ since

$$\xi_1 \dots \xi_k \star_{zG} \eta_1 \dots \eta_\ell = \frac{1}{k!\ell!} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \xi_{\sigma(1)} \star_{zG} \dots \star_{zG} \xi_{\sigma(k)} \star_{zG} \eta_{\tau(1)} \star_{zG} \dots \star_{zG} \eta_{\tau(\ell)}$$

This equality can easily be gotten from the definition of the map $\mathfrak{q}$.

4.1.3 A Formula for two Monomials

Now we need a formula for two monomials, which will be useful to prove the continuity of $\star_{zG}$. It will not be very explicit, but it will good enough to do computations with concrete examples. As a first step, we must introduce a bit of notation:

Definition 4.1.7 (G-Index) *Let $k, \ell, n \in \mathbb{N}$ and $r = k + \ell - n$. Then we call an r-tuple J*

$$J = (J_1, \dots, J_r) = ((a_1, b_1), \dots, (a_r, b_r))$$

a G-index if it fulfils the following properties:

(i) $J_i \in \{0, 1, \dots, k\} \times \{0, 1, \dots, \ell\}$

(ii) $|J_i| = a_i + b_i \ge 1 \quad \forall_{i=1,\dots,r}$

(iii) $\sum_{i=1}^{r} a_i = k$ *and* $\sum_{i=1}^{r} b_i = \ell$

(iv) The tuple is ordered in the following sense: $i > j \Rightarrow |J_i| \ge |J_j| \quad \forall_{i,j=1,\dots,r}$ *and* $|a_i| \ge |a_j|$ *if* $|J_i| = |J_j|$

(v) If $a_i = 0$ [or $b_i = 0$] for some i, then $b_i = 1$ [or $a_i = 1$].

We call the set of all such G-indices $\mathcal{G}_r(k, \ell)$.

Definition 4.1.8 (G-Factorial) *Let $J = ((a_1, b_1), \ldots, (a_r, b_r)) \in \mathcal{G}_r(k, \ell)$ be a G-Index. We set for a given tuple $(a, b) \in \{0, 1, \ldots, k\} \times \{0, 1, \ldots, \ell\}$*

$$\#_J(a, b) = \text{ number of times that } (a, b) \text{ appears in } J.$$

Then we define the G-factorial of $J \in \{0, 1, \ldots, k\} \times \{0, 1, \ldots, \ell\}$ as

$$J! = \prod_{(a,b)\in\{0,1,\ldots,k\}\times\{0,1,\ldots,\ell\}} (\#_J(a, b))!$$

This allows us to state an explicit formula for the Gutt star product:

Lemma 4.1.9 *Let $\mathfrak{g}$ be a Lie algebra, $\xi, \eta \in \mathfrak{g}$ and $k, \ell \in \mathbb{N}$. Then we have the following identity for the Gutt star product:*

$$\xi^k \star_{zG} \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n\Big(\xi^k, \eta^\ell\Big),$$

where the C_n are given by

$$C_n\Big(\xi^k, \eta^\ell\Big) = \sum_{J\in\mathcal{G}_{k+\ell-n}(k,\ell)} \frac{k!\ell!}{J!} \prod_{i=1}^{k+\ell-n} \mathtt{BCH}_{a_i,b_i}(\xi, \eta) \tag{4.1.6}$$

and the product is taken in the symmetric tensor algebra.

Proof: We want to calculate what the C_n look like. Let's denote $r = k + \ell - n$ for brevity. Then we have

$$\begin{aligned}
z^n C_n(\xi^k, \eta^\ell) &= \frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \frac{1}{z^r}\frac{\mathtt{BCH}(zt\xi, zs\eta)^r}{r!} \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \left(\sum_{j=1}^{k+\ell} \mathtt{BCH}_j(zt\xi, zs\eta)\right)^r \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \sum_{\substack{j_1,\ldots,j_r\geq 1\\ j_1+\ldots+j_r=k+\ell}} \mathtt{BCH}_{j_1}(zt\xi, zs\eta)\cdots\mathtt{BCH}_{j_r}(zt\xi, zs\eta) \\
&= z^n\frac{k!\ell!}{r!} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \mathtt{BCH}_{a_i,b_i}(\xi, \eta)\cdots\mathtt{BCH}_{a_r,b_r}(\xi, \eta) \qquad (4.1.7)
\end{aligned}$$

We sum over all possible arrangements of the (a_i, b_i). In order to get rid of redundancies, we put an ordering on these multi- indices: We use the one from definition 4.1.7. Obviously, we loose the freedom of arranging the (a_i, b_i) a we need to count the number of multi-indices $((a_1, b_1), \ldots, (a_r, b_r))$ which belong to the same G-index J. This number will be $\frac{r!}{J!}$, since we can't interchange the (a_i, b_i) any more (therefore $r!$), unless they are equal (therefore $J!^{-1}$). Changing the summation to $J \in \mathcal{G}_r(k, \ell)$ and multiplying by $\frac{r!}{J!}$ we get the identity (4.1.6). □

Now we just need to generalize this to factorizing tensors. To do so, we need a last definition:

Definition 4.1.10 *Let $\ell, n \in \mathbb{N}$ and $J \in \mathcal{G}_{k+\ell-n}(k, \ell)$. Then for $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell$ from a Lie algebra $\mathfrak{g}$ we set*

$$\Gamma_J(\xi_1, \ldots, \xi_k; \eta_1, \ldots, \eta_\ell) = \frac{1}{J!}\prod_{i=1}^{k+\ell-n} \mathtt{BCH}_{a_i,b_i}\Big(\xi^{(a_i)}, \eta^{(b_i)}\Big) \tag{4.1.8}$$

where the notation $\mathtt{BCH}_{a_i,b_i}\big(\xi^{(a_i)},\eta^{(b_i)}\big)$ *means that we have taken* $\prod_{i=1}^{k+\ell-n}\mathtt{BCH}_{a_i,b_i}\big(\xi^{(a_i)},\eta^{(b_i)}\big)$ *and replaced the* j*-th* ξ *appearing in it with* ξ_j *for* $j=1,\dots,k$ *and analogously with the* η*'s.*

Proposition 4.1.11 *Let* $\mathfrak{g}$ *be a Lie algebra,* $k,\ell\in\mathbb{N}$ *and* $\xi_1,\dots,\xi_k,\eta_1,\dots,\eta_\ell\in\mathfrak{g}$*. Then we have the following identity for the Gutt star product:*

$$\xi_1\dots\xi_k\star_{zG}\eta_1\dots\eta_\ell=\sum_{n=0}^{k+\ell-1}z^nC_n(\xi_1\dots\xi_k,\eta_1\dots\eta_\ell),$$

where the C_n *are given by*

$$C_n(\xi_1\dots\xi_k,\eta_1\dots\eta_\ell)=\sum_{J\in\mathcal{G}_{k+\ell-n}(k,\ell)}\sum_{\sigma\in S_k}\sum_{\tau\in S_\ell}\Gamma_J\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(k)};\eta_{\tau(1)},\dots,\eta_{\tau(\ell)}\big) \tag{4.1.9}$$

and the product is taken in the symmetric tensor algebra.

PROOF: The proof relies on polarization again and is completely analogous to the one of proposition **??**. We get rid of the factorials in Equation (4.1.6) and get symmetrizations over the ξ_i and the η_j instead, which gives the wanted result. □

Remark 4.1.12 Equation (4.1.9) allows us to do some explicit computations. One can check for example the following things:

i.) The classical limit holds:

$$C_0(\xi_1\dots\xi_k,\eta_1\dots\eta_\ell)=\xi_1\dots\xi_k\eta_1\dots\eta_\ell$$

ii.) The semi-classical limit holds:

$$\xi_1\dots\xi_k\star_{zG}\eta_1\dots\eta_\ell-\eta_1\dots\eta_\ell\star_{zG}\xi_1\dots\xi_k=z\{\xi_1\dots\xi_k,\eta_1\dots\eta_\ell\}_{KKS}$$

with $\{\cdot,\cdot\}_{KKS}$ being the Kirillov-Konstant- Souriau bracket.

iii.) For $\xi_1\dots\xi_k\star_{zG}\eta$ we get the smaller formula in Equation (4.1.2) again.

We don't want to check this here, since these are easy calculations, but it is interesting to see that the formulas really give the wanted identities for the Gutt star product.

4.2 Consequences and examples

4.3 Low-Verifications of the formulas

4.3.1 First verifications with Mathematica

4.3.2 Ideas for an algorithm beyond

Chapter 5

A locally convex topology for the Gutt star product

Here should already be some stuff...

5.1 Why locally convex?

The first question one could ask is, why we want the observable algebra to be a *locally convex* one. There are a lot of different choices and most of them would even make things simpler: we could think of locally multiplicatively convex algebras, Banach algebras, C^*- or even von Neumann algebras. All of them have much more structure than just locally convex algebras. We would have an entire holomorphic calculus within our algebra if assumed it to be locally m-convex or even a continuous one if we wanted it to be C^*.

The reason is, that all these nice features are simply not there, in general. Quantum mechanics shows us, that the algebra made up by $\hat{q}$ and $\hat{p}$ can not be locally m-convex.

Proposition 5.1.1 *Let $\mathscr{A}$ be a unital associative algebra which contains the quantum mechanical observables $\hat{q}$ and $\hat{p}$ and in which the canonical commutation relation*

$$[\hat{q}, \hat{p}] = i\hbar$$

is fulfilled. Then the only submultiplicative semi-norm on it is $p = 0$.

PROOF: First, we need to show a little lemma:

Lemma 5.1.2 *In the given algebra, we have for $n \in \mathbb{N}$*

$$(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n) = (i\hbar)^n n! \mathbb{1}. \tag{5.1.1}$$

PROOF: To show it, we use the fact that for $a \in \mathscr{A}$ the operator ad_a is a derivation, which is always true for a Lie algebra which comes from an associative algebra with the commutator, since for $a, b, c \in \mathscr{A}$ we have

$$\begin{aligned} [a, bc] &= abc - bca \\ &= abc - bac + bac - bca \\ &= [a, b]c + b[a, c]. \end{aligned}$$

Now for $n = 1$ Equation (5.1.2) is certainly true. So let's look at the step $n \to n + 1$. We make use of the derivation property and have

$$(\mathrm{ad}_{\hat{q}})^{n+1}\big(\hat{p}^{n+1}\big) = (\mathrm{ad}_{\hat{q}})^n(i\hbar\hat{p}^n + \hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n))$$

$$
\begin{aligned}
&= (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^n (\hat{p}\, \mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n-1} \Big([\hat{q}, \hat{p}]\, \mathrm{ad}_{\hat{q}}(\hat{p}^n) + \hat{p} (\mathrm{ad}_{\hat{q}})^2 (\hat{p}^n) \Big) \\
&= (i\hbar)^{n+1} n! + i\hbar (\mathrm{ad}_{\hat{q}})^n (\hat{p}^n) + (\mathrm{ad}_{\hat{q}})^{n-1} \Big(\hat{p} (\mathrm{ad}_{\hat{q}})^2 (\hat{p}^n) \Big) \\
&= 2 (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n-1} \Big(\hat{p} (\mathrm{ad}_{\hat{q}})^2 (\hat{p}^n) \Big) \\
&\overset{(*)}{=} \quad \vdots \\
&= n (i\hbar)^{n+1} n! + \mathrm{ad}_{\hat{q}} (\hat{p} (\mathrm{ad}_{\hat{q}})^n (\hat{p}^n)) \\
&= n (i\hbar)^{n+1} n! + i\hbar (i\hbar)^n n! \\
&= (i\hbar)^{n+1} (n+1)!.
\end{aligned}
$$

At $(*)$ we actually used another statement which is to be proven by induction over k and says

$$
(\mathrm{ad}_{\hat{q}})^{n+1} (\hat{p}^{n+1}) = k (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n+1-k} \Big(\hat{p} (\mathrm{ad}_{\hat{q}})^k (\hat{p}^n) \Big).
$$

Since this proof is analogous to the first lines of the computation before, we omit it here and the lemma is proven. ▽

Now we can go on with the actual proof. Let $\|\cdot\|$ be a submultiplicative semi-norm. Then we see from Equation (5.1.2) that

$$
\|(\mathrm{ad}_{\hat{q}})^n (\hat{p}^n)\| = |\hbar|^n n! \|\mathbb{1}\|.
$$

On the other hand, we have

$$
\begin{aligned}
\|(\mathrm{ad}_{\hat{q}})^n (\hat{p}^n)\| &= \|\hat{q} (\mathrm{ad}_{\hat{q}})^{n-1} (\hat{p}^n) - (\mathrm{ad}_{\hat{q}})^{n-1} (\hat{p}^n) \hat{q}\| \\
&\leq 2 \|\hat{q}\| \|(\mathrm{ad}_{\hat{q}})^{n-1} (\hat{p}^n)\| \\
&\leq \quad \vdots \\
&\leq 2^n \|\hat{q}\|^n \|\hat{p}^n\| \\
&\leq 2^n \|\hat{q}\|^n \|\hat{p}\|^n
\end{aligned}
$$

So in the end we get

$$
|\hbar|^n n! \|\mathbb{1}\| \leq c^n
$$

for some $c \in \mathbb{R}$. This cannot be fulfilled for all $n \in \mathbb{N}$ unless $\|\mathbb{1}\| = 0$. But then, by submultiplicativity, the semi-norm itself must be equal to 0. □

Remark 5.1.3 The so called Weyl algebra, which fulfils the properties of the foregoing proposition, can be constructed from a Poisson algebra with constant Poisson tensor. It is a fair question, why this restriction of not being locally m-convex should also be put on linear Poisson systems. On the other hand, there is no reason to expect that things become easier when we make the Poisson system more complex. Moreover, the Weyl algebra is actually nothing but a quotient of the Universal enveloping algebra of the so called Heisenberg algebra, a particular Lie algebra. So there is also no reason why the original algebra should have a "better" analytical structure than the quotient, since the ideal, which is divided out by this procedure, is closed.

5.2 Locally convex algebras

5.2.1 Locally convex spaces and algebras

5.2.2 A special class of algebras

For a locally convex Lie algebra $\mathfrak{g}$, the Lie bracket is usually assumed to be continuous. This means that for every continuous semi-norms p there exists another continuous semi-norms q such that for all $\xi, \eta \in \mathfrak{g}$ one has

$$p([\xi, \eta]) \leq q(\xi)q(\eta).$$

For our study of the Gutt star product, this will not be enough, since we will have to estimate an arbitrary high number of nested brackets. We will need an estimate which does not depend on the number of lie brackets implied. But Lie algebras are just a special case and the property makes sense for any kind of locally convex algebra. This motivates the following definition.

Definition 5.2.1 *Let $\mathscr{A}$ be a Hausdorff, locally convex algebra (not necessarily associative) with $\cdot$ denoting the multiplication and $\{p_i\}_{i\in I}$ the set of all continuous semi-norms which defines the topology. For a given continuous semi-norm p we call another continuous semi-norm q an asymptotic estimate for p, if there exists an $m \in \mathbb{N}$ such that for all $n \geq m$ the following holds:*

$$p(x_1 \cdot \ldots \cdot x_n) \leq q(x_1) \ldots q(x_n) \quad \forall_{x_1,\ldots,x_n \in \mathscr{A}}. \tag{5.2.1}$$

We call a locally convex algebra an AE-algebra, if every continuous semi-norm has an asymptotic estimate.

Remark 5.2.2 We can without restrictions define $m = 1$ in the upper definition, since this just means taking the maximum over a finite number of continuous semi-norms. Clearly the result will be a continuous semi- norm again.

Remark 5.2.3

i.) The term asymptotic estimate has, to the best of our knowledge, first been used by Czichowski at all in [REFERENCE]. They defined asymptotic estimates in the same way we did, but their notion of AE-algebra was different from ours: in their definition of an AE algebra, not just one but a series of asymptotic estimates has to exist which fulfils two more properties. This is not the case in our definition, which is, in general, weaker.

ii.) In [], Neeb and Glöckner used a property to which they referred as $(*)$ for associative algebras. It was then used in [] by ... and ..., who called it the GN-property. It is easy to see that it is equivalent to being AE.

iii.) There are, of course, a lot of example for AE (Lie) algebras. All finite dimensional and Banach (Lie) algebras fulfil (5.2.1), just as locally m-convex (Lie) algebras do. The same is true for nilpotent locally convex Lie algebras, since here again one just has to take the maximum of a finite number of semi-norms. We are, however, not sure what is exactly implied by (5.2.1). Are there examples for associative or Lie algebras which are AE but not locally m-convex, for example?

It is at least possible to make some easy observations: an associative AE-algebra $\mathscr{A}$ will admit an entire holomorphic calculus: let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function and a_n such that $f(z) = \sum_n a_n z^n$. Then one has $\forall_{x\in\mathscr{A}}$

$$p(f(x)) = p\left(\sum_{n=0}^{\infty} a_n x^n\right) \leq \sum_{n=0}^{\infty} |a_n| p(x^n) \leq \sum_{n=0}^{\infty} |a_n| q(x)^n < \infty$$

where p is a continuous semi-norm and q its asymptotic estimate. So in some sense, at least in the associative case, AE-algebras are very close to locally m-convex ones. If the algebra is even commutative, then it seems like they are the same. In [] ... claims that an associative, commutative locally convex algebra admitting an entire calculus must already be locally m-convex.

For non-commutative algebras, the situation is different. It is a very interesting (and non-trivial) question, how an example of an associative, non locally m-convex but AE-algebra could look like.

5.2.3 The projective tensor product

- Inequality of the symmetric tensor product

5.3 A topology for the Gutt star product

5.3.1 Continuity of the product

- a counter-example

5.3.2 Dependence on the formal parameter

5.3.3 Completion

5.3.4 Nuclearity

5.4 Alternative topologies and an optimal result

Chapter 6

Nilpotent Lie algebras

6.1 An overview

- Reference to the counter-example before, no big change - Yet: Projective Limit - Module structure - Generalizations to nilpotency

6.2 The Heisenberg and the Weyl algebra

6.3 The projective limit

6.4 A module structure

6.4.1 Generic case and a counter-example

6.4.2 Nilpotent case and good news

6.5 Banach Lie algebras and the finite-dimensional case

6.5.1 Generalizations of nilpotency

6.5.2 A new projective Limit

6.5.3 A result for the finite-dimensional case

Chapter 7

The Hopf algebra structure

7.1 The co-product

7.1.1 A formula for the co-product

7.1.2 Continuity for the co-product

7.2 The whole Hopf algebra structure

Chapter 8

Examples and remarks

8.1 Some classical Lie algebra

8.2 Some new ideas

8.2.1 A subalgebra of the Weyl algebra

8.2.2 Holomorphic vector fields