Convergence of the Gutt star product

Paul Stapor

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Chapter 1

Introduction

HThroughout the history, the fields of mathematics and physics have always been closely linked to each other. The great physicists of the past have always been great mathematicians and vice versa: Carl Friedrich Gauss, for some persons the most brilliant mathematician of all, did not only find innumerable mathematical relations, prove myriads of theorems and develop countless new ideas, which should become rich and fruitful new fields in later mathematics, he also has a large number of credits in physics: the recovery of the dwarf planet Ceres ion astronomy, new results in electromagnetism (like the *Gauss's law* or a representation for the unit of magnetism, which was named after him) and the Gaussian lens formula in geometric optics are just some of his best known merits. Isaac Newton on the other hand, may have been rather a physicist, but it was his so called second law of motion that was the first foundation of differential calculus and therefore opened up the door for a completely new branch of mathematics. Even if one wants to claim that differential calculus was actually invented by Gottfried Wilhelm Leibniz, this does not change anything, since Leibniz wrote many essays on physics and can be considered also as the inventor of the concept of kinetic energy (or, as he called it, the *vis viva*) and its conservation in certain mechanical systems. Of course one has to name Joseph-Louis Lagrange, who was an ingenious mathematician with rich contributions to number theory and algebra, but also to the fields of analytical mechanics and astronomy. Still today, for every second years physics student about half of the mandatory lecture on theoretical mechanics is devoted to the Lagrangian formalism and way one can derive the laws of motion for various mechanical systems from it. A last name we want to mention here is Paul Dirac, who is certainly one of the founding fathers of quantum mechanics. He provided the ideas for a lot of structures and relations in differential geometry, functional analysis and distribution theory. Many of the concept he introduced using his physicist's intuition were later proven to be right or used as starting points for new theories by mathematicians.

Many developments in mathematics can be seen as triggered by physics: they were necessary to describe the physical behaviour of the world and therefore pushed forward by scientists. We already mentioned differential calculus, without whom modern analysis, the theory of ordinary or partial differential equations or differential geometry would not be possible. Besides the also named field of functional analysis, also Lie theory and many parts of geometry provide examples for physics inspired mathematics. Of course, this correspondence is not a one way street, since the understanding of nature made great progress due to a better knowledge of the mathematical laws of her formation. A good example therefor is Lebesgue's theory of integration and its application to quantum mechanics: the space $L^2(\mathbb{R}^{3n})$ is the state space of standard n-particle system in quantum mechanics.

There are good reasons to say that this very tight binding of mathematics and physics persisted until the 20th century. Without any doubt, those two areas are still closely linked, but one could say that at a certain point in history they started walking away from each other. Of course, there have always been mathematicians who did not take their motivation from physics and physicists who did not use elaborate or even invent new math to describe aspects of the world around them, but for a long time, the vast majority of both groups showed at least an interest for the other domain. This definitely changed during the 20th century. The main reason for this can surely be found in the extremely fast development which of both domains experienced in this time. It is already impossible for one person to overview the whole field of mathematics or the one of physics, since there are too many new things coming up every day. Another reason is surely the fact that modern mathematics is strongly influenced by the desire to formulate things as clean as possible, without using handwaving "physical arguments". This is a principle which surely allowed many new and powerful evolutions in the last decades and which is mostly due to the Bourbaki movement in the middle of the last century, but it also forgets about the fact that physical intuition was often a powerful tool for new ideas or also for heuristics which led to proofs of important theorems. Another reason, which is more situated in the domain of physics, is certainly the incredibly fast development of the knowledge about semiconductors. This became possible due to quantum mechanics which forms the foundation of this theory, but for the very most of modern applications, a basic understanding of the quantum theory behind is enough, or one can even get new results with so called semi-classical approaches. Here, a lot of new and fruitful results can be established without going deep into mathematics and hence without giving a new stimulus to it. In this sense, it is enough for many modern physicists to acquire a certain amount of mathematical knowledge and then, they never have to care about mathematical theories again.

Certainly, the situation is not bad, neither for mathematics nor for physics and it would be by far too much to say that those fields are falling apart. There are still a lot of intersections of the two sciences and there is no reason for pessimism, since these contact areas provide rich and fruitful domains of research. The big number of fields, where either mathematics takes its motivation from physics, or where theoretical physics needs very elaborate mathematical methods, is usually grouped under the name *mathematical physics*. One of its younger areas is the theory of quantization and therein the theory deformation quantization can be settled. It belongs to the area of pure mathematics, but takes its inspiration from physics and is therefore a part of mathematical physics. The idea is, roughly spoken, to find a correspondence between the quantum and the classical world in physics. The mathematical description of their laws are different but yet they show a lot of similarities. It is more or less clear, how the classical world is created out of a huge number of quantum objects and the mathematics of classical mechanics can be understood as a limit case the behaviour of n quantum objects where $n \longrightarrow \infty$. The other way round, it is not clear how one can create the mathematical description of a quantum system out of the one of a classical system. This reversed process is usually called quantization and its understanding is a mathematical task, not a physical one. Deformation quantization tries to "deform" the idealized algebra of classical physical observables by making it noncommutative and to get an idealized algebra of quantum mechanical observables this way. This is done by replacing the pointwise product of functions (since the classical algebra of obervables is usually modelled as the smooth function on a Poisson manifold) with a noncommutative product, which takes into account certain derivatives of the functions and plugs in the formal parameter $\hbar$. This new product is called a star product and becomes a formal power series in $\hbar$. The zeroth order in the formal parameter represents classical mechanics and the first order quantum mechanics. Different mechanical systems allow different star products and their classification has been one of the main tasks of the theory for a long time. Besides the purely algebraic aspects of this theory, one also wants that this deformation is continuous or smooth in a certain sense and that a suitable subalgebra can be found, for which the formal power series is also convergent, since $\hbar$

is not a formal parameter in physics, but a nature constant with a specific value.

This work focusses on a particular star product, the so called Gutt star product, which can be established on a certain class of Poisson manifolds. Its goal is to find a large subalgebra of the smooth function and a locally convex topology on them, such that the Gutt star product is convergent and that the commutative classical algebra can be deformed smoothly into the non-commutative quantum algebra. Of course, one has to give a proper definition for this smoothness. Moreover, we will try to find as many good properties of this construction as possible and relate it to other fields of mathematics, such as Lie theory or functional analysis. The work is organized as follows:

Chapter 2

Deformation quantization

2.1 Mechanics: The Classical and the Quantum World

2.1.1 Classical Mechanical Systems

2.1.2 Quantum Mechanics

2.1.3 The Correspondence Principle

2.2 Making Things Noncommutative: Quantization

2.2.1 Noncommutative Geometry

2.2.2 Deformation Quantization

2.3 Formal vs. Strict Deformation Quantization

2.3.1 A Mathematical Theory

Example how Physics gives rise to new Math, Including a short history, Things got stuck somewhere

2.3.2 From Formal to Strict

Three steps: Formal -> Strict -> Representations Concepts of strict DQ (Rieffel vs. Waldmann, meaning C* vs. Locally convex) So far everything is Math, Physics would start after the last step, Maybe one day...

Chapter 3

Algebraic Preliminaries

3.1 Linear Poisson structures in infinite dimensions

As we have seen before, there has already been done some work on how to strictly quantize Poisson structures on vector spaces. Star products of exponential type on locally convex vector spaces were topologized by Stefan Waldmann in [34] and then investigated more closely by Matthias Schötz in [27]. Hence, as a the next step, we want to do linear Poisson structures on locally convex vector spaces. This will give a new big class of Poisson structures, which will be deformable in a strict way. Before we do so in the rest of this master thesis, we recall briefly some basics on linear Poisson structures.

We will always take a vector space V and look at Poisson structures on the coordinates which are elements of the dual space V^*. In order to cover most of the physically interesting examples by our reflections, we will assume that V is a locally convex vector space. Every finite- dimensional vector space is normable and complete and hence locally convex, so it fits in this framework. It is clear what V^* should be and there is just one interesting topology on it. For infinite-dimensional spaces, the situation is more delicate: we have to think about what coordinates should be and how a Poisson structure on them could look like. A priori, it is not clear which dual we should consider: the algebraic dual V^* of all linear forms on V, or the topological dual V' which contains just the continuous linear forms? Here, one could argue that only V' is of real interest, since otherwise we would encounter the very strange effect of having discontinuous polynomials, and the aim of constructing a continuous star product on them seems somehow pointless. But even if we stick to V', the question of the topology still remains: do we want to consider the weak or the strong topology there and why one of them should be more interesting. In any case, we have to choose a topology on this space. Once this is done, we have to think about a good notion of Poisson tensors in this context. However, we encounter quite a number of question, which have no trivial answer. For this reason, it is worth looking at some equivalent formulations of $Pol(V^*)$ in the finite- dimensional case, since they may allow better generalizations.

Let V be a finite dimensional vector space. Now, there is now question about the dual or its topology, since $V^* = V'$ is finite-dimensional, too, and we deal with polynomials on it. A linear Poisson structure on V^* is something very familiar: it is equivalent to a Lie algebra structure on V.

Proposition 3.1.1 *Let V be a vector-space of dimension $n \in \mathbb{N}$ and $\pi \in \Gamma^\infty(\Lambda^2(TV^*))$. Then the two following things are equivalent:*

i.) π is a linear Poisson tensor.

ii.) V has a uniquely determined Lie algebra structure.

PROOF: We choose a basis $e_1, \ldots, e_n \in V$ and denote its dual basis $e^1, \ldots, e^n \in V^*$. Then we call the linear coordinates in these bases $x_1, \ldots, x_n \in \mathscr{C}^\infty(V^*)$ and $\xi^1, \ldots, \xi^n \in \mathscr{C}^\infty(V)$, such that for all $\xi \in V, x \in V^*$

$$\xi = \xi^i(\xi)e_i \quad \text{and} \quad x = x_i(x)e^i.$$

In these coordinates, the Poisson tensor reads

$$\pi = \frac{1}{2}\pi_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where π is linear in the coordinates and we have

$$\pi_{ij}(x) = c_{ij}^k x_k.$$

This equivalent to a tensor

$$c = \frac{1}{2}c_{ij}^k e_k \otimes e^i \wedge e^j$$

which gives for $f, g \in \mathscr{C}^\infty(V^*)$

$$\{f, g\}(x) = \pi(df, dg)(x) = x_k c_{ij}^k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}, \tag{3.1.1}$$

using the identification $T^*V^* \cong V^{**} \cong V$. But now, the statement is obvious, since antisymmetry of π means antisymmetry of the c_{ij}^k in the indices i and j and the Jacobi identity for the Poisson tensor gives

$$c_{ij}^\ell c_{\ell k}^m + c_{jk}^\ell c_{\ell i}^m + c_{ki}^\ell c_{\ell j}^m = 0 \tag{3.1.2}$$

for all i, j, k, m, since it must be fulfilled for all smooth functions. Vicely versa, (3.1.2) ensures the Jacobi identity of π in (3.1.1). Hence the map

$$[\cdot, \cdot] \colon V \times V \longrightarrow V \quad (e_i, e_j) \longmapsto c_{ij}^k e_k \tag{3.1.3}$$

defines a Lie bracket, since the c_{ij}^k are antisymmetric and fulfil the Jacobi identity and are therefore structures constants. Conversely, the structure constants of a Lie algebra on V define a Poisson tensor on V^* via (3.1.1). □

From now on, we will call the original vector space $\mathfrak{g}$ instead of V which is more intuitive for a Lie algebra. Since this kind of Poisson systems has a particular structure, there is a proper name for them.

Definition 3.1.2 (Kirillov-Kostant-Souriau bracket) *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the Poisson bracket $\{\cdot, \cdot\}_{KKS}$, which is given by Proposition 3.1.1 on $\mathfrak{g}^*$ is called the Kirillov-Kostant-Souriau bracket.*

Proposittion 3.1.1 gives us a hint how we could think of "infinite-dimensionsal vector spaces with linear Poisson tensor":We take $\mathfrak{g}$ to be an infinite-dimensional Lie algebra, which gives something like a linear Poisson structure on $\mathfrak{g}'$. If, on the other hand, we chose directly $\mathfrak{g}'$ to have a linear Poisson tensor, we would get a Lie algebra structure on $\mathfrak{g}''$. Of course, we could think of "just using this structure on $\mathfrak{g}$", but in general, this will *not* be closed: taking the Lie bracket of $\xi, \eta \in \mathfrak{g}$, we might drop out of $\mathfrak{g}$ and have $[\xi, \eta] \subseteq \mathfrak{g}'' \backslash \mathfrak{g}$. Usually, such a behaviour will not be of physical interest, since the algebras of physical systems are usually closed objects and the double-dual is not object of interest. This is why we will translate the term "linear Poisson structure on $\mathfrak{g}^*$" by "$\mathfrak{g}$ is a Lie algebra" in infinite dimensions and from now on,we will restrict our observations to systems which can be described as such. Remark however, that, from a mathematical point of view, this is a choice and not a necessity and other choices would be possible.

The next task are the polynomials on $\mathfrak{g}'$. As already mentioned, it is not easy to find a good generalization for them, since for a locally convex Lie algebra $\mathfrak{g}$, even $\mathfrak{g}'$ will be a huge vector space. Again, it is helpful to go back to the finite-dimensional case, where we have the following result:

Proposition 3.1.3 *Let $\mathfrak{g}$ be a vector space of dimension $n \in \mathbb{N}$. Then the algebras $\mathrm{S}^\bullet(\mathfrak{g})$ and $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ are canonically isomorphic.*

PROOF: Since this is a very well-known result, we just want to sketch the proof briefly: Take a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ and its linear coordinates $x_1, \ldots, x_n \in \mathscr{C}^\infty(\mathfrak{g}^*)$ with $x_i(x) = e_i(x)$ for $x \in \mathfrak{g}^*$. On homogeneous symmetric tensors this yields the map

$$\mathscr{J}\colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{Pol}^\bullet(\mathfrak{g}^*), \quad e_1^{\mu_1} \ldots e_n^{\mu_n} \longmapsto \xi_1^{\mu_1} \ldots \xi_n^{\mu_n}.$$

From the construction, we see that this is an isomorphism, but note, that we have used the identification $\mathfrak{g}^{**} \cong \mathfrak{g}$ via

$$e_i(x) = \langle x, e_i \rangle. \qquad \square$$

The last identification we used in the last step will not work in both directions any more, but just in one: we have a canonical injection $\mathrm{S}^\bullet(\mathfrak{g}) \subseteq \mathrm{Pol}^\bullet(\mathfrak{g}')$, so every symmetric tensor still gives a polynomial. Anyway, this gives an idea how to avoid speaking about $\mathrm{Pol}(\mathfrak{g}^*)$ and its topology: we restrict right from the beginning to $\mathrm{S}^\bullet(\mathfrak{g})$. For finite-dimensional systems, both points of view are equivalent, but in infinite dimensions, this becomes a choice. However, we have good reasons to think that this is enough: we get a closed and reasonably big subalgebra of of the polynomials. Moreover, the symmetric tensor algebra is defined on infinite-dimensional spaces exactly in the same way as on finite-dimensional ones, and the construction is identical.

So finally, we found a suitable way of speaking about our object of interest: We replace linear Poisson structures on $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ by $\mathrm{S}^\bullet(\mathfrak{g})$.

3.2 The Gutt star product

The aim of this chapter is to endow the symmetric algebra, and hence the polynomial algebra, with a new, noncommutative product. This is possible in a very natural way, due to the Poincaré-Birkhoff-Witt theorem. It links the symmetric tensor algebra $\mathrm{S}^\bullet(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $\mathscr{U}(\mathfrak{g})$.

3.2.1 The universal enveloping algebra

If $\mathscr{A}$ is an associative algebra, one can construct a Lie algebra out of it by using the commutator

$$[a, b] = a \cdot b - b \cdot a, \quad a, b \in \mathscr{A}.$$

This construction is functorial, since it doesn't only map associative algebras to Lie algebras, but also morphisms of the former to those of the latter. While constructing a Lie algebra out of an associative algebra is easy, the reversed process is more complicated, but also possible. Every Lie algebra $\mathfrak{g}$ can be embedded into a particular associative algebra, known as the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$, whichis uniquely determined (up to isomorphism) by the universal property: for every unital associative algebra $\mathscr{A}$ and every homomorphism for Lie algebras $\phi\colon \mathfrak{g} \longrightarrow \mathscr{A}$ using the commutator on $\mathscr{A}$, one gets a unital homomorphism of associative algebras $\Phi\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{A}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \arrow[dr, "\Phi"] & \\ & \mathscr{A} \\ \mathfrak{g} \arrow[uu, "\iota"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

The proof of existence and uniqueness of the universal enveloping algebra can be found in every standard textbook on Lie theory like [20] or [33], and we won't do it here in detail. Just recall that existence is proven by an explicit construction: one takes the tensor algebra $\mathrm{T}^\bullet(\mathfrak{g})$ and considers the two-sided ideal

$$\mathfrak{I} = \langle \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \rangle, \quad \forall_{\xi,\eta\in\mathfrak{g}}$$

inside of it. Then one gets the universal enveloping algebra by the quotient

$$\mathscr{U}(\mathfrak{g}) = \frac{\mathrm{T}^\bullet(\mathfrak{g})}{\mathfrak{I}}. \tag{3.2.1}$$

To avoid confusion, we will always denote the multiplication in $\mathscr{U}(\mathfrak{g})$ by $\odot$, whereas the commutative product in $\mathrm{S}^\bullet(\mathfrak{g})$ will be denoted without a sign. It follows from this construction, that $\mathscr{U}(\mathfrak{g})$ is a filtered algebra

$$\mathscr{U}(\mathfrak{g}) = \bigcup_{k\in\mathbb{N}} \mathscr{U}^k(\mathfrak{g}), \quad \mathscr{U}^k(\mathfrak{g}) = \Big\{ x = \sum_i \xi_1^i \odot \ldots \odot \xi_n^i \;\Big|\; \xi_j^i \in \mathfrak{g},\ 1 \le j \le n, i \in I \Big\}.$$

Generally, we just get a filtration, not a graded structure, since the ideal $\mathfrak{I}$ is not homogeneous in the symmetric degree. We only get a graded structure on $\mathscr{U}(\mathfrak{g})$, if and only if $\mathfrak{g}$ was commutative. Then $\mathscr{U}(\mathfrak{g})$ is isomorphic to the symmetric tensor algebra and hence commutative, too. But $\mathscr{U}(\mathfrak{g})$ is much more than an associative algebra: it is also a Hopf algebra, since one can define a coassociative, cocommutative coproduct on it

$$\Delta\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}), \quad \xi \longmapsto \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi, \quad \forall_{\xi\in\mathfrak{g}}$$

which extends to $\mathscr{U}(\mathfrak{g})$ via algebra homomorphism, as well as an antipode

$$S\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}), \quad \xi \longmapsto -\xi, \quad \forall_{\xi\in\mathfrak{g}}$$

which extends to $\mathscr{U}(\mathfrak{g})$ via algebra antihomomorphism. We will come back to those two maps and to the Hopf structure in chapter 7, when we talk about their continuity. More details on the algebraic aspect of deformation theory using Hopf algebras can be found in [11] and [23], for example.

3.2.2 The Poincaré-Birkhoff-Witt theorem

The algebra $\mathscr{U}(\mathfrak{g})$ always admits a basis, which must be infinite. This result is due to the already mentioned theorem of Poincaré, Birkhoff and Witt:

Theorem 3.2.1 (Poincaré-Birkhoff-Witt theorem) *Let $\mathfrak{g}$ be a Lie algebra with a basis $\mathcal{B}_\mathfrak{g} = \{\beta_i\}_{i\in I}$. Then the set*

$$\mathcal{B}_{\mathscr{U}(\mathfrak{g})} = \big\{ \beta_{i_1}^{\mu_{i_1}} \odot \ldots \odot \beta_{i_n}^{\mu_{i_n}} \;\big|\; n \in \mathbb{N},\ i_k \in I \text{ with } i_1 \preccurlyeq \ldots \preccurlyeq i_n \text{ and } \beta_{i_k} \in \mathcal{B}_\mathfrak{g},\ \mu_{i_1}, \ldots, \mu_{i_n} \in \mathbb{N} \big\}$$

defines a basis of $\mathscr{U}(\mathfrak{g})$.

There are different proofs for this statement. While a geometrical proof (like it can be found in [35]) is very convenient in the finite-dimensional case, a combinatorial argument must be used for infinite-dimensional Lie algebras. Most textbooks restrict to finite-dimensional Lie algebras and give a version of the latter one, except [6], which does it ni full generality. The idea of most of the combinatoric proof works with minor changes also for any Lie algebra, since in relies on ordered index sets which can be defined in any dimension. The PBW theorem allows us to set up an isomorphism between $\mathrm{S}^\bullet(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g})$ immediately, since a basis of the former can be given by almost the same expression

$$\mathcal{B}_{\mathrm{S}^\bullet(\mathfrak{g})} = \left\{\beta_{i_1}^{\mu_{i_1}} \cdots \beta_{i_n}^{\mu_{i_n}} \;\middle|\; n \in \mathbb{N},\ i_k \in I, 1 \leq k \leq n, i_1 \preccurlyeq \cdots \preccurlyeq i_n \text{ and } \beta_{i_k} \in \mathcal{B}_{\mathfrak{g}},\ \mu_{i_1}, \ldots, \mu_{i_n} \in \mathbb{N}\right\}$$

where we just have replaced the noncommutative product in $\mathscr{U}(\mathfrak{g})$ by the symmetric tensor product $\cdot$ (which we will usually denote without a symbol, if possible) in $\mathrm{S}^\bullet(\mathfrak{g})$. This allows us to write down an isomorphism between the symmetric tensor algebra and the universal enveloping algebra, just by mapping the basis vectors to each other in a naive way. Of course, this can never be an isomorphism in the sense of algebras, but only of (filtered) vector spaces, because one of the algebras is commutative and the other isn't. Moreover, the symmetric algebra has a graded structure from the one on the tensor algebra which the universal enveloping algebra does not have:

$$\mathrm{S}^\bullet(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathrm{S}^n(\mathfrak{g}), \quad \mathrm{S}^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \vee \ldots \vee \mathfrak{g}}_{n \text{ times}}.$$

We will denote by $\pi_n \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^n(\mathfrak{g})$ the canonical projections of this grading. This induces a filtration by $\mathrm{S}^{(k)}(\mathfrak{g}) = \sum_{j=0}^{k} \mathrm{S}^j(\mathfrak{g})$. Our simple isomorphism will respect the filtration, but not the grading. However, it isn't the only isomorphism which one can write down. In [4], Berezin proposed another isomorphism which is more helpful to use:

$$\mathfrak{q}_n \colon \mathrm{S}^n(\mathfrak{g}) \longrightarrow \mathscr{U}^n(\mathfrak{g}), \quad \beta_{i_1} \ldots \beta_{i_n} \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \beta_{i_{\sigma(1)}} \cdot \ldots \cdot \beta_{i_{\sigma(n)}}, \quad \mathfrak{q} = \sum_{n=0}^{\infty} \mathfrak{q}_n. \tag{3.2.2}$$

We will refer to it as the quantization map, for reasons that will soon become clear. It also respects the filtration and transfers the symmetric product to another symmetric expression. In this sense, we can now switch between both algebras and use the setting, which is more convenient in the current situation: the graded structure of $\mathrm{S}^\bullet(\mathfrak{g})$, or the Hopf algebra structure of $\mathscr{U}(\mathfrak{g})$.

3.2.3 The Gutt star product

Since we know, that the universal enveloping and the symmetric tensor algebra are isomorphic as vector spaces, we have a good tool at hand to endow the symmetric tensor algebra, and hence the polynomials, with a noncommutative product by pulling back the product from $\mathscr{U}(\mathfrak{g})$ to $\mathrm{S}^\bullet(\mathfrak{g})$ via $\mathfrak{q}$. This is exactly what Gutt did in [17]. She constructed a star product on $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ from $\mathscr{U}(\mathfrak{g})$ while encoding the noncommutativity in a formal parameter $z \in \mathbb{C}$ in a convenient way.

Definition 3.2.2 (Gutt star product) *Let $\mathfrak{g}$ be a Lie algebra, $z \in \mathbb{C}$, and $f, g \in \mathrm{S}^\bullet(\mathfrak{g})$ of degree k and ℓ respectively. Then we define the Gutt star product by:*

$$\star_z \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad (f, g) \longmapsto \sum_{n=0}^{k+\ell-1} z^n \pi_{k+\ell-n}\big(\mathfrak{q}^{-1}(\mathfrak{q}(f) \cdot \mathfrak{q}(g))\big). \tag{3.2.3}$$

This is the original way in which Gutt defined her star product in [17], but there are two more ways to do it. Define

$$\mathfrak{I}_z =< \xi \otimes \eta - \eta \otimes \xi - z[\xi, \eta] >$$

for $z \in \mathbb{C}$. Then we set

$$\mathscr{U}(\mathfrak{g}_z) = \frac{\mathrm{T}^\bullet(\mathfrak{g})}{\mathfrak{I}_z}, \tag{3.2.4}$$

and get the map

$$\mathfrak{q}_{z,n} \colon \mathrm{S}^n(\mathfrak{g}) \longrightarrow \mathscr{U}^n(\mathfrak{g}_z), \quad \beta_{i_1} \ldots \beta_{i_n} \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \beta_{i_{\sigma(1)}} \cdot \ldots \cdot \beta_{i_{\sigma(n)}}, \quad \mathfrak{q}_z = \sum_{n=0}^{\infty} \mathfrak{q}_{z,n}. \tag{3.2.5}$$

This way, we also get a star product:

$$\widehat{\star}_z \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad (f, g) \longmapsto \mathfrak{q}_z^{-1}(\mathfrak{q}_z(f) \cdot \mathfrak{q}_z(g)). \tag{3.2.6}$$

In [13], Drinfeld also constructed a star product using the Baker-Campbell-Hausdorff series: take $\xi, \eta \in \mathfrak{g}$ and set

$$\exp(\xi) *_z \exp(\eta) = \exp\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right), \tag{3.2.7}$$

where the exponential series is understood a formal power series in ξ and η. By formally differentiating, on gets the star product on all polynomials.

Of course, our aim is to show that these three maps are in fact identical and that they define a star product. Since this is a long way to go, we postpone the proof to the end of this chapter. It will be useful to learn something about the Baker-Campbell-Hausdorff series and the Bernoulli number first.

3.3 The Baker-Campbell-Hausdorff series

Since we have a formula for $\star_z$ which involves the Baker-Campbell-Hausdorff series, we want to give a short overview about it and introduce some results that will be helpful later on. Note however, that there is not *the* BCH *formula*, since one can always rearrange terms using anti-symmetry or Jacobi identity, but for $\xi, \eta \in \mathfrak{g}$, we can always write it as

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \mathrm{BCH}_n(\xi, \eta) = \sum_{a,b=0}^{\infty} \mathrm{BCH}_{a,b}(\xi, \eta), \tag{3.3.1}$$

where $\mathrm{BCH}_n(\xi, \eta)$ denotes all expressions having n letters and $\mathrm{BCH}_{a,b}(\xi, \eta)$ denotes all expressions with a ξ's and b η's. We have $\mathrm{BCH}_{0,0}(\xi, \eta) = 0$, $\mathrm{BCH}_{1,0}(\xi, \eta) = \xi$ and $\mathrm{BCH}_{0,1}(\xi, \eta) = \eta$. Clearly this gives

$$\mathrm{BCH}_n(\xi, \eta) = \sum_{a+b=n} \mathrm{BCH}_{a,b}(\xi, \eta).$$

Of course, this only moves the problem of non-uniqueness to a later point when we will have to discuss the partial expressions. Yet, in the beginning, this will be helpful.

3.3.1 Some general and historical remarks

Assume $\mathfrak{g}$ to be the Lie algebra of a finite-dimensional Lie group G. From the geometric point of view, the BCH formula is the infinitesimal counterpart of the multiplication law in G. Since the multiplication is smooth and and the exponential function locally diffeomorphic around the unit element e, we would expect that there is a Lie algebraic analogon to the group multiplication, at least near the origin, which depends somehow "smoothly on the arguments". Finding this expression is, however, a different task.

One approach to this would be the following: consider an algebra $\mathscr{A}$ with the noncommuting elements ξ, η. We want to study the identity of formal power series

$$\exp(\chi) = \exp(\xi)\exp(\eta).$$

There should be a $\chi \in \mathscr{A}$, which fulfils this relation. We can rewrite the right hand side as

$$\exp(\xi)\exp(\eta) = \sum_{n,m=0}^{\infty} \frac{\xi^n \eta^m}{n!m!}$$

and use the formal power series for the logarithm

$$\log(\chi) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(\chi - 1)^k$$

in order to get an expression for χ. This yields

$$\chi = \log(\exp(\xi)\exp(\eta)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{i\in\{1,\dots,k\}\\ n_i,m_i\geq 0\\ n_i+m_i\geq 1}} \frac{\xi^{n_1}\eta^{m_1}\dots\xi^{n_k}\eta^{m_k}}{n_1!m_1!\dots n_k!m_k!}. \tag{3.3.2}$$

It is far from trivial, if and how this can be expressed using Lie brackets. The first one who found a general way for this was Dynkin in the 1950's [14,15]. Of course, the question of convergence still remains, although we would expect the expression to converge at least in a neighbourhood of 0.

A different approach works via differential equations. We can consider flows on the Lie group. This gives also an expression of the group multiplication in logarithmic coordinates just using Lie brackets. One gets recursive relations for the $\mathrm{BCH}_n(\xi,\eta)$ and the first formulas due to Baker [3], Campbell [7,8] and Hausdorff [19] were of this kind. For the first terms, one finds

$$\begin{aligned}
\mathrm{BCH}(\xi,\eta) &= \log(\exp(X)\exp(Y))\\
&= \xi + \eta + \frac{1}{2}[\xi,\eta] + \frac{1}{12}([[\eta,\xi],\xi] + [[\xi,\eta],\eta]) + \frac{1}{24}[[[\eta,\xi],\xi],\eta]\\
&\quad + \frac{1}{120}([[[[\eta,\xi],\eta],\xi],\eta] + [[[[\xi,\eta],\xi],\eta],\xi]) + \frac{1}{360}([[[[\eta,\xi],\xi],\xi],\eta] + [[[[\xi,\eta],\eta],\eta],\xi])\\
&\quad - \frac{1}{720}([[[[\eta,\xi],\xi],\xi],\xi] + [[[[\xi,\eta],\eta],\eta],\eta]) + \cdots
\end{aligned} \tag{3.3.3}$$

which coincides of course with the result from Dynkin.

3.3.2 Forms of the BCH

As already mentioned, there are different forms of stating the BCH formula and depending on the problem one wants to solve, not every one is equally well suited. One can classify them roughly into four groups.

i.) There are recursive formulas, which calculate each term from the previous one. The first expressions due to Baker, Campbell and Hausdorff were of this kind. Though the idea is old, this approach is still much in use and allows powerful applications: Casas and Murua found an efficient algorithm [10] for calculating a form of BCH series without redundancies based on a recursive formula, which was given by Varadarajan in his textbook [33]. For such a non-redundant formula one needs a notion of basis of the free Lie algebra. There are approaches to such (Hall or Hall-Viennot) basis, which can e.g. be found in [28].

ii.) Most textbooks prove an integral form of the series, like [18] and [20]. Since we will use it, too, we want to introduce it briefly. Take the function

$$g\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{z\log(z)}{z-1} \tag{3.3.4}$$

and denote for $\xi \in \mathfrak{g}$ by

$$\mathrm{ad}_\xi\colon \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \eta \longmapsto [\xi, \eta]$$

the usual ad-operator. Then one has for $\xi, \eta \in \mathfrak{g}$

$$\mathrm{BCH}(\xi, \eta) = \xi + \int_0^1 g(\exp(\mathrm{ad}_\xi)\exp(t\,\mathrm{ad}_\eta))(\eta)dt. \tag{3.3.5}$$

iii.) As already mentioned, Dynkin found a closed form for (3.3.2), which is the only one of this kind known so far. A proof can be found in [21], for example. It reads

$$\mathrm{BCH}(\xi, \eta) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{i\in\{1,\dots,k\}\\ n_i, m_i \geq 0\\ n_i+m_i\geq 1}} \frac{1}{\sum_{i=1}^{k}(n_i+m_i)} \frac{[\xi^{n_1}\eta^{m_1}\dots\xi^{n_k}\eta^{m_k}]}{n_1!m_1!\dots n_k!m_k!}, \tag{3.3.6}$$

where the expression $[\dots]$ denotes Lie brackets nested to the left:

$$[\xi\eta\eta\xi] = [[[\xi, \eta], \eta], \xi].$$

Unfortunately, the combinatorics get extremely complicated for higher degrees and increasingly many terms belong to the same Lie bracket expression.

iv.) Goldberg gave a form of the series which is based on words in two letters:

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{|w|=n} g_w w. \tag{3.3.7}$$

The g_w are coefficients, which can be calculated using the recursively defined Goldberg polynomials (see [16]). It was put into commutator form by Thompson in [31]:

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{|w|=n} \frac{g_w}{n}[w]. \tag{3.3.8}$$

Again, the $[w]$ are Lie brackets nested to the left. Of course, this formula will also have redundancies, but its combinatorial aspect is much easier than the one of (3.3.6). Since there are estimates for the coefficients g_w, we will use this form for our Main Theorem.

3.3.3 The Goldberg-Thompson formula and some results

Goldberg's theorems

We now introduce the results of Goldberg: he noted a word in the letters ξ and η as

$$w_\xi(s_1, s_2, \ldots, s_m) = \xi^{s_1}\eta^{s_2}\ldots(\xi \vee \eta)^{s_m},$$

with $m \in \mathbb{N}$ and the last letter will be ξ if m is odd and η if m is even. The index ξ of w_ξ means that the word starts with a ξ. Now we can assign to each word $w_{\xi\vee\eta}(s_1, \ldots, s_m)$ a coefficient $c_{\xi\vee\eta}(s_1, \ldots, s_m)$. This is done by the following formula:

$$c_\xi(s_1, \ldots, s_m) = \int\limits_0^1 t^{m'}(t-1)^{m''}G_{s_1}(t)\ldots G_{s_m}(t)dt, \tag{3.3.9}$$

where we have $m' = \lfloor \frac{m}{2} \rfloor$, $m'' = \lfloor \frac{m-1}{2} \rfloor$ with $\lfloor\cdot\rfloor$ denoting the entire part of a real number and we have $n = \sum_{i=1}^m s_i$. The G_s are the recursively defined Goldberg polynomials

$$G_s(t) = \frac{1}{s}\frac{d}{dt}t(t-1)G_{s-1}(t), \tag{3.3.10}$$

for $s > 1$ and $G_1(t) = 1$. For c_η we have

$$c_\eta(s_1, \ldots, s_m) = (-1)^{n-1}c_\xi(s_1, \ldots, s_m) \tag{3.3.11}$$

and furthermore

$$c_\eta(s_1, \ldots, s_m) = c_\xi(s_1, \ldots, s_m)$$

if m is odd. This yields immediately

$$c_\xi(s_1, \ldots, s_m) = c_\eta(s_1, \ldots, s_m) = 0$$

if m is odd and n is even. Of course, Goldberg found interesting identities which are fulfilled by the coefficients. A very remarkable one is that for all permutations $\sigma \in S_m$ one has

$$c_\xi(s_1, \ldots, s_m) = c_\xi\big(s_{\sigma(1)}, \ldots, s_{\sigma(m)}\big),$$

since (3.3.9) obviously doesn't see the ordering of the s_i and m', m'' and n are not affected by reordering. For words with $m = 2$, an easier formula can be found:

$$c_\xi(s_1, s_2) = \frac{(-1)^{s_1}}{s_1!s_2!}\sum_{n=1}^{s_2}\binom{s_2}{n}B_{s_1+s_2-n},$$

where the B_s denote the Bernoulli numbers, which will be explained more precisely in the next paragraph. First, we note that the only case which matters to us is of course $s_1 = 1$, since for $s_1, s_2 > 1$ we will find something like $[[\xi, \xi], \ldots] = 0$. For simplicity, let's set $s_2 = 1$ and to permute $s_1 \leftrightarrow s_2$:

$$c_\xi(1, s) = \frac{(-1)^s}{s!}B_s. \tag{3.3.12}$$

Bernoulli numbers

We have seen the Bernoulli numbers B_n showing up and we will encounter them very often in the following. Hence it is useful to learn a few important things about them. They are defined by the series expansion of

$$g\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \tag{3.3.13}$$

Clearly, g has poles at $z = 2k\pi i$, $k \in \mathbb{Z}\backslash\{0\}$. Moreover, one can easily show that all odd Bernoulli numbers are zero, except for $B_1 = -\frac{1}{2}$ and since in some applications one wants B_1 to be positive, there is a different convention for naming them: one often encounters $B_n^* = (-1)^n B_n$ (which only differs for $n = 1$). The nonzero Bernoulli numbers alternate in sign. For their absolute value, one can show the asymptotic behaviour (see [29, 30])

$$|B_{2n}| \sim (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2*n}}.$$

This is not surprising, since we know that the generating function g had poles at $\pm 2\pi i$. The Bernoulli numbers can also be calculated by the recursion formula

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \tag{3.3.14}$$

which is well-known in the literature (e.g. [2]). Since we will deal with them, we want to give the first numbers of this series here.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$\frac{691}{2730}$	0	$\frac{7}{6}$	0	$\frac{3617}{510}$

BCH up to first order

Proposition 3.3.1 *Let $\mathfrak{g}$ be a Lie algebra and the Bernoulli numbers as defined before. Then we have for $\xi, \eta \in \mathfrak{g}$*

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=0}^{\infty} \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2) \tag{3.3.15}$$

$$= \sum_{n=0}^{\infty} \frac{B_n}{n!} (\mathrm{ad}_\eta)^n(\xi) + \mathcal{O}(\xi^2) \tag{3.3.16}$$

PROOF: We want to can calculate this using the Goldberg coefficients. Remind that we will put words to Lie brackets, and for computing the coefficients we will need the words $\eta\xi^n$ *and* $\xi\eta\xi^{n-1}$ because of antisymmetry and words of the form $\xi^k\eta\xi^{n-k}$ with $k > 1$ will give vanishing expressions. Now let $n \in \mathbb{N}$. We have

$$c_\eta(1, n) = (-1)^n c_\xi(1, n) = (-1)^n \frac{(-1)^n}{n!} B_n = \frac{B_n}{n!}.$$

By n-fold skew-symmetry and (3.3.8), we get the contribution

$$\frac{(-1)^n}{(n+1)!} B_n (\mathrm{ad}_\xi)^n(\eta) = \frac{1}{(n+1)!} B_n^* (\mathrm{ad}_\xi)^n(\eta)$$

Now we need $c_\xi(1,1,n-1)$: let $n>1$, then

$$
\begin{aligned}
c_\xi(1,1,n-1) &= \int\limits_0^1 t(t-1)G_{n-1}(t)dt \\
&= -\int\limits_0^1 t\frac{d}{dt}(t(t-1)G_{n-1}(t))dt \\
&= -\int\limits_0^1 ntG_n(t)dt \\
&= -nc_\xi(1,n) \\
&= -n\frac{(-1)^n}{n!}B_n \\
&= (-1)^{n+1}\frac{1}{(n-1)!}B_n,
\end{aligned}
$$

where we have done an integration by parts in the third step. So by using $n-1$ times the skew-symmetry of the Lie bracket, we get

$$\frac{1}{n+1}\cdot(-1)^{n+1}\frac{1}{(n-1)!}B_n[\ldots[\xi,\eta],\xi]\ldots],\xi] = \frac{n}{(n+1)!}B_n(\mathrm{ad}_\xi)^n(\eta)$$

For $n>1$ we add up those two and use the fact that $B_n = B_n^*$ and find the result we want. For $n=1$, there is just the first contribution and $c_\xi(1,1) = -B_1$, which gives

$$B_1^*\,\mathrm{ad}_\xi(\eta)$$

in total. For $n=0$, we get $c_\xi(1) = c_\eta(1) = 1$ and finally get (3.3.15). For (3.3.16), note that we need $c_\xi(1,n)$ and $c_\eta(1,1,n-1)$. We have $c_\xi(1,n) = (-1)^n c_\eta(1,n)$ and $c_\eta(1,1,n-1) = (-1)^n c_\xi(1,1,n-1$. This gives $(-1)^n$ and switches B_n to B_n^*. □

Remark 3.3.2 (Alternative Proof) Note that we could also have used the integral formula (3.3.5) to prove this. We want to sketch an alternative proof here: if we write the second of the two exponential functions as a series, we see that it can be cut after the constant term, since we are looking for contributions which are linear in η. The function left to integrate is then just $(g\circ\log)(z)$. Since we insert $\exp(\mathrm{ad}_\xi)$, we get

$$\mathrm{BCH}(\xi,\eta) = \xi + \int_0^1 g(\mathrm{ad}_\xi)(\eta)dt + \mathcal{O}(\eta^2) = \xi + \sum_{n=1}^{\infty}\frac{B_n^*}{n!}(\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2),$$

since there is no dependence on t left and we get the same result.

3.4 The Equality of the Star Products

We want to prove the equality of the three star products. For a general and possibly infinite-dimensional Lie algebra, this is quite tedious. As a first step, it will be helpful to show their associativity.

Remark 3.4.1 In the finite-dimensional case, there are different proofs for Theorem 3.4.6, the main theorem of this section, which mostly rely on geometric arguments, like the one in [5].

Unluckily, these techniques are not at hand in infinite dimensions and one has to find an algebraic proof instead. Since in the community of deformation quantization, this statement is somehow folklore and believed for any Lie algebra, it strongly seems like such a proof already exists. However, the author was not able to trace it down in literature and therefore gives an own proof.

Proposition 3.4.2 *The three maps $\star_z$, $\widehat{\star}_z$ and $*_z$ define associative multiplications.*

PROOF: All maps are defined as $\mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g})$, so we have to show bilinearity and associativity.

i.) For $\widehat{\star}_z$, associativity and bilinearity are clear from the construction, since we just pull-back the multiplication in $\mathscr{U}(\mathfrak{g}_z)$.

ii.) For $\star_z$, bilinearity follows from the fact that all maps, that are used in its definition, are (bi-)linear. For associativity, we have to interchange sums and shift projections. Recall that $\pi_n(f \star_z g) = 0$, if $n > \deg(f) + \deg(g)$. Take homogeneous tensors $f, g, h \in \mathrm{S}^\bullet(\mathfrak{g})$ of degree $k, \ell, m \in \mathbb{N}$ respectively. The we have

$$
\begin{aligned}
&\big(f \star_z g\big) \star_z h \\
&\quad= \sum_{j=0}^{k+\ell-1} \sum_{i=0}^{k+\ell+m-j-1} z^i \big(\pi_{k+\ell+m-j-i} \circ \mathfrak{q}^{-1}\big)\big(\mathfrak{q}\big(z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g))\big) \odot \mathfrak{q}(h)\big) \\
&\quad= \sum_{j=0}^{k+\ell-1} \sum_{i=0}^{k+\ell+m-1} z^{i-j} \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)\big(\mathfrak{q}\big(z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g))\big) \odot \mathfrak{q}(h)\big) \\
&\quad= \sum_{i=0}^{k+\ell+m-1} z^i \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big) \left(\mathfrak{q} \left(\sum_{j=0}^{k+\ell-1} z^{-j} z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g)) \right) \odot \mathfrak{q}(h) \right) \\
&\quad= \sum_{i=0}^{k+\ell+m-1} z^i \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g) \odot \mathfrak{q}(h)),
\end{aligned}
$$

and we just need to do the reversed process on the right hand side to get the wanted result.

iii.) For $*_z$, we get associativity using the exponential function and the logarithm. We have

$$
\begin{aligned}
(\exp(\xi) *_z \exp(\eta)) *_z \exp(\chi) &= \exp\left(\frac{1}{z}\mathrm{BCH}\left(\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right), z\chi\right)\right) \\
&= \exp\left(\frac{1}{z}\mathrm{BCH}\left(z\xi, \left(\frac{1}{z}\mathrm{BCH}(z\eta, z\chi)\right)\right)\right) \\
&= \exp(\xi) *_z (\exp(\eta) *_z \exp(\chi)),
\end{aligned}
$$

since

$$
\begin{aligned}
\mathrm{BCH}\left(\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right), z\chi\right) &= \log\left(\exp\left(\log\left(\frac{1}{z}\exp(z\xi)\exp(z\eta)\right)\right)\exp(z\chi)\right) \\
&= \log\left(\frac{1}{z}\exp(z\xi)\exp(z\eta)\exp(z\chi)\right) \\
&= \log\left(\exp(z\xi)\log\left(\left(\frac{1}{z}\exp(z\eta)\exp(z\chi)\right)\right)\right) \\
&= \mathrm{BCH}\left(z\xi, \left(\frac{1}{z}\mathrm{BCH}(z\eta, z\chi)\right)\right).
\end{aligned}
$$

Bilinearity follows from differentiating the formula and is a simple computation. $\square$

Note star products must fulfil the classical and the semi-classical limit. We will do this in Corollary 4.2.1 and so just the equality is left to show. It is enough to prove the coincidence for terms of the form $\xi^k \star \eta$ with $\xi, \eta \in \mathfrak{g}$ and $k \in \mathbb{N}$, because $\mathrm{S}^\bullet(\mathfrak{g})$ is a commutative algebra and hence we get them on arbitrary monomials by polarization. The equality for the product of two monomials then follows by iteration, which is possible due to associativity. The next lemma will be a first big step.

Lemma 3.4.3 *Let $\xi, \eta\mathfrak{g}$, then we have*

$$\xi^k \widehat{\star}_z \eta = \sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta). \tag{3.4.1}$$

PROOF: This proof is divided into the two following lemmata:

Lemma 3.4.4 *Let $\xi, \eta \in \mathfrak{g}$ and $k \in \mathbb{N}$. Then we have*

$$\mathfrak{q}_z \left(\sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta) \right) = \sum_{s=0}^{k} \mathcal{K}(k,s) \xi^{k-s} \odot \eta \odot \xi^s$$

with

$$\mathcal{K}(k,s) = \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} B_n^* \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{\ell=0}^{k-n} \delta_{s,\ell+j}.$$

PROOF: Since the map $\mathfrak{q}_z$ is linear, we can pull out the constants and get

$$\mathfrak{q}_z \left(\sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta) \right) = \sum_{n=0}^{k} \binom{k}{n} B_n^* \mathfrak{q}_z \Big(z^n \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta) \Big).$$

Now we need the two equalities

$$\mathfrak{q}_z(\xi^n \eta) = \frac{1}{n+1} \sum_{\ell=0}^{n} \xi^{k-\ell} \odot \eta \odot \xi^\ell$$

and

$$\mathfrak{q}_z((z^n \,\mathrm{ad}_\xi)^n(\eta)) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \xi^{n-j} \odot \eta \odot \xi^j$$

which can easily be shown by induction. They give

$$\begin{aligned}
&\sum_{n=0}^{k} \binom{k}{n} B_n^* \mathfrak{q}_z \Big(z^n \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta) \Big) \\
&\qquad = \sum_{n=0}^{k} \binom{k}{n} \frac{B_n^*}{k-n+1} \sum_{\ell=0}^{k-n} \xi^{k-n-\ell} \odot \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} \xi^{n-j} \odot \eta \odot \xi^j \right) \odot \xi^\ell \\
&\qquad = \sum_{n=0}^{k} \binom{k}{n} \frac{B_n^*}{k-n+1} \sum_{\ell=0}^{k-n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \xi^{k-\ell-j} \odot \eta \odot \xi^{\ell+j} \\
&\qquad = \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} B_n^* \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{\ell=0}^{k-n} \xi^{k-\ell-j} \odot \eta \odot \xi^{\ell+j}
\end{aligned}$$

We just need to collect those terms for which we have $\ell + j = s$ for all $s = 0, \ldots, k$. If we do this with a Kronecker-delta, we will get exactly the $\mathcal{K}(k,s)$. $\triangledown$

For the second lemma, we need some statements on Bernoulli numbers and binomial coefficients. Let $k, m, n \in \mathbb{N}$. Then we have the following identities:

$$\sum_{j=0}^{k} \binom{k+1}{n} B_j^* = k+1 \tag{3.4.2}$$

$$(-1)^k \sum_{j=0}^{k} \binom{k}{j} B_{m+j} = (-1)^m \sum_{i=0}^{m} \binom{m}{i} B_{k+i} \tag{3.4.3}$$

$$\sum_{j=0}^{m} (-1)^j \binom{n}{j} = (-1)^m \binom{n-1}{m} \tag{3.4.4}$$

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}. \tag{3.4.5}$$

The first one can easily be proven using the recursive definition of the Bernoulli numbers (3.3.14). Equation (3.4.4) and (3.4.5) are standard identities in combinatorics and can be found in the textbook of Aigner [1]. Finally, Equation (3.4.3) is a theorem due to Carlitz [9]. With them, we can show the next lemma which will finish this proof.

Lemma 3.4.5 *Let $\mathcal{K}(k,s)$ be defined as in Lemma 3.4.4, then we have for all $k \in \mathbb{N}$*

$$\mathcal{K}(k,s) = \begin{cases} 1 & s = 0 \\ 0 & \textit{else.} \end{cases}$$

PROOF: This is divided into three parts. First, we show the statement for $s = 0$, then we show it for $s = 1$ and then proceed by induction.

(i) $s = 0$: The Kronecker-delta will always be zero unless $l = j = 0$. So we get

$$\mathcal{K}(k,0) = \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} B_n^* = \frac{k+1}{k+1} = 1,$$

where we have used (3.4.2).

(ii) $s = 1$: To get a contribution from the δ, we must have $(j,\ell) = (1,0)$ or $(0,1)$. Except for $n = 0$ and $n = k$, both cases are possible. We split them off:

$$\begin{aligned}
\mathcal{K}(k,1) &= \underbrace{\frac{1}{k+1}}_{n=0} - \underbrace{kB_k^*}_{n=k} + \frac{1}{k+1} \sum_{n=1}^{k-1} \binom{k+1}{n} B_n^* \left(1 + (-1)\binom{n}{1}\right) \\
&= \frac{1}{k+1} + \frac{1}{k+1} \sum_{n=1}^{k-1} \binom{k+1}{n} B_n^* - \frac{1}{k+1} \sum_{n=1}^{k-1} \binom{k+1}{n} nB_n^* - kB_k^* \\
&= \underbrace{\frac{1}{k+1} \sum_{n=0}^{k-1} \binom{k+1}{n} B_n^*}_{=1-B_k^*} - \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} nB_n^* \\
&= 1 - B_k^* - \underbrace{\frac{k+1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} B_n^*}_{=k+1} + \sum_{n=0}^{k} \underbrace{\frac{k+1-n}{k+1} \binom{k+1}{n}}_{\binom{k}{n}} B_n^*
\end{aligned}$$

$$
\begin{aligned}
&= 1 - B_k^* - k - 1 + \sum_{n=0}^{k-1} \binom{k}{n} B_n^* + B_k^* \\
&= -k + \sum_{n}^{k-1} \binom{k}{n} B_n^* \\
&= 0.
\end{aligned}
$$

(iii) $s \mapsto s+1$: Due to the induction, it is sufficient to prove $\mathcal{K}(k, s+1) - \mathcal{K}(k, s) = 0$. In order to do that, we must get rid of the δ's and therefore rewrite $\mathcal{K}(k, s)$:

$$
\begin{aligned}
\mathcal{K}(k, s) &= \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} B_n^* \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{\ell=0}^{k-n} \delta_{s,\ell+j} \\
&= \frac{1}{k+1} \sum_{n=0}^{s} \binom{k+1}{n} B_n^* \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{\ell=0}^{k-n} \delta_{s,\ell+j} \\
&\quad + \frac{1}{k+1} \sum_{n=s+1}^{k} \binom{k+1}{n} B_n^* \sum_{j=0}^{s} (-1)^j \binom{n}{j} \sum_{\ell=0}^{k-n} \delta_{s,\ell+j} \\
&= \frac{1}{k+1} \sum_{n=0}^{s} \binom{k+1}{n} B_n^* \sum_{j=\max\{0,s+n-k\}}^{n} (-1)^j \binom{n}{j} \\
&\quad + \frac{1}{k+1} \sum_{n=s+1}^{k} \binom{k+1}{n} B_n^* \sum_{j=\max\{0,s+n-k\}}^{s} (-1)^j \binom{n}{j}.
\end{aligned}
$$

As long as $\max\{0, s+n-k\} = 0$, the first sum over j will be zero as it is just the binomial expansion of $(1-1)^n$, except for $n = 0$. Hence we get a special case and a shorter first sum over n. In the sums over j we use again the binomial expansion of $(1-1)^n$ and get

$$
\begin{aligned}
\mathcal{K}(k, s) = \frac{1}{k+1} \Bigg[1 &+ \sum_{k+1-s}^{s} \binom{k+1}{n} B_n^* \left(- \sum_{j=0}^{s+n-k-1} (-1)^j \binom{n}{j} \right) \\
&+ \sum_{n=s+1}^{k} \binom{k+1}{n} B_n^* \left(- \sum_{j=0}^{s+n-k-1} (-1)^j \binom{n}{j} - \sum_{j=s+1}^{n} (-1)^j \binom{n}{j} \right) \Bigg].
\end{aligned}
$$

Now it is helpful to use (3.4.4) and $\binom{k}{n-k} = \binom{k}{n}$. We also get $(-1)^n$-terms which we can put together with the B_n^* to get B_n:

$$
\begin{aligned}
\mathcal{K}(k, s) = \frac{1}{k+1} \Bigg[1 &+ \sum_{n=k+1-s}^{s} \binom{k+1}{n} B_n (-1)^{k-s} \binom{n-1}{k-s} \\
&+ \sum_{n=s+1}^{k} \binom{k+1}{n} B_n \left((-1)^{k-s} \binom{n-1}{k-s} + (-1)^{n+s} \binom{n-1}{s} \right) \Bigg].
\end{aligned}
$$

We finally made the δ disappear. Hence we must compute $\mathcal{K}(k, s+1) - \mathcal{K}(k, s)$. Since we want to show that it is 0, we can multiply it with $k+1$ in order to get rid of the factor in front:

$$
(k+1)(\mathcal{K}(k, s+1) - \mathcal{K}(k, s))
$$

$$
\begin{aligned}
&= \sum_{n=k-s}^{s+1} \binom{k+1}{n} B_n (-1)^{k-s-1} \binom{n-1}{k-s-1} - \sum_{n=k+1-s}^{s} \binom{k+1}{n} B_n (-1)^{k-s} \binom{n-1}{k-s} \\
&\quad + \sum_{n=s+2}^{k} \binom{k+1}{n} B_n \left((-1)^{k-s-1} \binom{n-1}{k-s-1} + (-1)^{n+s+1} \binom{n-1}{s+1} \right) \\
&\quad - \sum_{n=s+1}^{k} \binom{k+1}{n} B_n \left((-1)^{k-s} \binom{n-1}{k-s} + (-1)^{n+s} \binom{n-1}{s} \right) \\
&= - \sum_{n=k-s}^{k} \binom{k+1}{n} B_n (-1)^{k-s} \binom{n-1}{k-s-1} - \sum_{n=k-s+1}^{k} \binom{k+1}{n} B_n (-1)^{k-s} \binom{n-1}{k-s} \\
&\quad - \sum_{n=s+2}^{k} \binom{k+1}{n} B_n (-1)^{n+s} \binom{n-1}{s+1} - \sum_{n=s+1}^{k} \binom{k+1}{n} B_n (-1)^{n+s} \binom{n-1}{s} \\
&= - \sum_{n=k-s}^{k} \binom{k+1}{n} B_n (-1)^{k-s} \left(\binom{n-1}{k-s-1} + \binom{n-1}{k-s} \right) \\
&\quad - \sum_{n=s+1}^{k} \binom{k+1}{n} B_n (-1)^{n+s} \left(\binom{n-1}{s+1} + \binom{n-1}{s} \right).
\end{aligned}
$$

We have rearranged the sums, added some zeros and shortened the expression. Now we will use the recursion formula for the binomial coefficients

$$
\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
$$

and our binomial multiplication equality (3.4.5):

$$
\begin{aligned}
&= - \sum_{n=k-s}^{k} \binom{k+1}{n} B_n (-1)^{k-s} \binom{n}{k-s} - \sum_{n=s+1}^{k} \binom{k+1}{n} B_n (-1)^{n+s} \binom{n}{s+1} \\
&= - \sum_{n=k-s}^{k} \binom{k+1}{s+1} \binom{s+1}{n+s-k} B_n (-1)^{k-s} - \sum_{n=s+1}^{k} \binom{k+1}{s+1} \binom{k-s}{n-s-1} B_n (-1)^{n+s}.
\end{aligned}
$$

Since we want to show that this is 0, we can divide by $\binom{k+1}{s+1}$ which will never be zero because $s \in \{0, 1, \ldots, k\}$. After doing so, can use $n > 1$ in the second sum and thus only even n will show up, because for odd n the Bernoulli numbers are zero. For this reason we have $(-1)^n = 1$. Then we rewrite these sums by shifting the indices and we add two zeros:

$$
\begin{aligned}
&- \sum_{n=k-s}^{k} \binom{s+1}{n+s-k} B_n (-1)^{k-s} + \sum_{n=s+1}^{k} \binom{k-s}{n-s-1} B_n (-1)^{s+1} \\
&= (-1)^{s+1} \sum_{\ell=0}^{k-s-1} \binom{k-s}{\ell} B_{\ell+s+1} - (-1)^{k-s} \sum_{\ell=0}^{s} \binom{s+1}{\ell} B_{\ell+k-s} \\
&= (-1)^{s+1} \sum_{\ell=0}^{k-s} \binom{k-s}{\ell} B_{\ell+s+1} - (-1)^{k-s} \sum_{\ell=0}^{s+1} \binom{s+1}{\ell} B_{\ell+k-s} \\
&\quad - (-1)^{s+1} \binom{k-s}{k-s} B_{k+1} + (-1)^{k-s} \binom{s+1}{s+1} B_{k+1}.
\end{aligned}
$$

The first two terms give the Carlitz-identity (3.4.3) and vanish. So we are left with the last two terms and get

$$-(-1)^{s+1}B_{k+1} + (-1)^{k-s}B_{k+1} = (-1)^s B_{k+1}\Big(1 + (-1)^k\Big) = 0,$$

since the bracket will be zero if k is odd and $B_{k+1} = 0$ if k is even. $\triangledown$

In Lemma 4.1.1, we will see that also $*_z$ fulfils this identity. Hence $*_z = \widehat{\star}_z$. We only need to show $\widehat{\star}_z = \star_z$. For $z = 1$, the two maps are clearly identical and therefore we find

$$\xi^k \star_1 \eta = \sum_{n=0}^{k} \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^k(\eta).$$

But now $\widehat{\star}_z = \star_z$ follows from the definition of $\star_z$: we just have to plug in powers of z and find (3.4.1). So with the proofs in Chapter 4, we will have proven the following theorem:

Theorem 3.4.6 *The three maps $\star_z$, $\widehat{\star}_z$ and $*_z$ coincide on $\mathrm{S}^\bullet(\mathfrak{g})$ and define star products.*

Chapter 4

Formulas for the Gutt star product

We have seen some results on the Baker-Campbell-Hausdorff series and an identity for the Gutt star product. The latter one, stated in Theorem 3.4.6, will be a very useful tool in the following, since we want to get explicit formulas for $\star_z$. There is still a part of the proof missing, but this will be caught up at the beginning of the first section of this chapter. From there, we will come to a first easy formula for $\star_z$. Afterwards, we will use the same procedure to find two more formulas for it: the first is a rather involved one for the n-fold star product of vectors. It will not be helpful for algebraic computations, but very useful for estimates. The second one is a more explicit formula for the product of two monomials.

From those formulas, we will be able to draw some easy, but nice conclusion in the second section and we will prove the classical and the semi-classical limit. Then, we will show how to calculate the Gutt star product explicitly by computing two easy examples.

4.1 Formulas for the Gutt Star Product

4.1.1 A Monomial with a Linear Term

The easiest case for which we will develop a formula is surely the following one: For a given Lie algebra $\mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}$ we would like to compute

$$\xi^k \star_z \eta = \sum_{n=0}^{k} z^n C_n(\xi^k, \eta)$$

We have already done this for $\star_z$ and $\widehat{\star}_z$, now we want to do the same for $*_z$. This will finish the proof of the equality of the star products from Theorem 3.4.6. We will use that

$$\xi^k = \frac{\partial^k}{\partial t^k}\Big|_{t=0} \exp(t\xi). \tag{4.1.1}$$

Now we have all the ingredients to prove the following lemma:

Lemma 4.1.1 *Let $\mathfrak{g}$ be a Lie algebra and $\xi, \eta \in \mathfrak{g}$. We have the following identity for $*_z$:*

$$\xi^k *_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j(\eta). \tag{4.1.2}$$

PROOF: We start from the simplified form for the Baker-Campbell-Hausdorff series from Equation (3.3.15) in Proposition 3.3.1:

$$\mathrm{BCH}(\xi, \eta) = \xi + \sum_{n=0}^{\infty} \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2).$$

If we insert this into the definition of the Drinfel'd star product and use Equation (4.1.1) we get

$$\begin{aligned}
\xi^k *_z \eta &= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(\frac{1}{z}\mathrm{BCH}(zt\xi, zs\eta)\right) \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(t\xi + \sum_{j=0}^{\infty} z^j \frac{B_j^*}{j!}(\mathrm{ad}_{t\xi})^j(s\eta) + \mathcal{O}(\eta^2)\right).
\end{aligned}$$

We see that only terms which have exactly k of the ξ's in them and which are linear in η will contribute. This means we can cut off the sum at $j = k$ and omit higher orders in η. We now use the exponential series, cut it at k for the same reason and get

$$\begin{aligned}
\xi^k *_z \eta &= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k}\frac{1}{n!}\left(t\xi + \sum_{j=0}^{k}(zt)^j\frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^n \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k}\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}(t\xi)^{n-m}\left(\sum_{j=0}^{k}(zt)^j\frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^m \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \left(\sum_{n=0}^{k}\frac{1}{n!}(t\xi)^n + \sum_{n=0}^{k}\sum_{j=0}^{k}\frac{1}{(n-1)!}t^{n+j-1}z^j\frac{B_j^*}{j!}\xi^{n-1}(\mathrm{ad}_\xi)^j(s\eta)\right).
\end{aligned}$$

In the last step we set $m = 1$ since the other term have either too many or not enough η's and will vanish because of the differentiation with respect to s. We can finally differentiate to get the formula

$$\begin{aligned}
\xi^k *_z \eta &= \sum_{n=0}^{k}\sum_{j=0}^{k}\delta_{k,n+j-1}\frac{k!}{j!(n-1)!}z^j B_j^*\xi^{n-1}(\mathrm{ad}_\xi)^j(\eta) \\
&= \sum_{j=0}^{k}\binom{k}{j}z^j B_j^*\xi^{k-j}(\mathrm{ad}_\xi)^j(\eta),
\end{aligned}$$

which is the wanted result. □

Remark 4.1.2 We have proven the equality of the star products $\widehat{\star}_z$ $*_z$ by deriving an easy formula for both of them. From now on, we will get all other formulas from $*_z$, since this is the one which is easier to compute.

Now it is actually easy to get the formula for monomials of the form $\xi_1 \dots \xi_k$ with $\eta \in \mathfrak{g}$:

Proposition 4.1.3 *Let $\mathfrak{g}$ be a Lie algebra and $\xi_1, \dots, \xi_k, \eta \in \mathfrak{g}$. We have*

$$\xi_1 \cdots \xi_k \star_z \eta = \sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}z^j B_j^* \sum_{\sigma\in S_k}[\xi_{\sigma(1)}, [\dots[\xi_{\sigma(j)}, \eta]\dots]]\xi_{\sigma(j+1)}\cdots\xi_{\sigma(k)} \text{ and} \tag{4.1.3}$$

$$\eta \star_z \xi_1 \cdots \xi_k = \sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}z^j B_j \sum_{\sigma\in S_k}[\xi_{\sigma(1)}, [\dots[\xi_{\sigma(j)}, \eta]\dots]]\xi_{\sigma(j+1)}\cdots\xi_{\sigma(k)}. \tag{4.1.4}$$

PROOF: We get the result by just polarizing the formula from Lemma 4.1.1. Let $\xi_1, \dots, \xi_k \in \mathfrak{g}$, then we introduce the parameters t_i for $i = 1, \dots, k$ and set

$$\Xi = \Xi(t_1, \dots, t_k) = \sum_{i=1}^{k} t_i \xi^i.$$

Then we see that

$$\xi_1 \cdots \xi_k = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1,\cdots,t_k=0} \Xi^k$$

since for every $i = 1, \ldots, k$ we have

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} \Xi = \xi_i. \tag{4.1.5}$$

By writing out the Ξ's and using multilinearity, we find

$$\begin{aligned}
\xi_1 \cdots \xi_k \star_z \eta &= \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1,\cdots,t_k=0} \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \Xi^{k-j} (\mathrm{ad}_\Xi)^j(\eta) \\
&= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \sum_{\{i_1,\ldots,i_k\} \in \{1,\ldots,k\}^k} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1,\cdots,t_k=0} t_{i_1} \cdots t_{i_k} \\
&\quad \cdot \xi_{i_1} \cdots \xi_{i_{k-j}} \mathrm{ad}_{\xi_{i_{k-j+1}}} \circ \cdots \circ \mathrm{ad}_{\xi_{i_k}}(\eta) \\
&= \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} z^j B_j^* \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\ldots [\xi_{\sigma(j)}, \eta] \ldots]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)}.
\end{aligned}$$

In the last step, all expression which did not contain each ξ_i exactly once disappeared due to the differentiation. The proof of Equation 4.1.4 works analogously. □

Remark 4.1.4 This formula is actually not a new result: Gutt already gave it in her paper [17, Prop. 1] and referred to Dixmier [12, part 2.8.12 (c)], who already gave it in his textbook. It can also be found in the diploma thesis of Neumaier [25, Rem. 5.2.8] and a work due to Kathotia [22, Eq. 2.23]. Probably the first one to mention it was Berezin in [4, Eq. 30].

4.1.2 An Iterated Formula for the General Case

Proposition 4.1.3 allows theoretically to get a formula for the case of $\xi_1, \ldots, \xi_k \in \mathfrak{g}$

$$\xi_1 \star_z \ldots \star_z \xi_k = \sum_{j=0}^{k} C_{z,j}(\xi_1, \ldots, \xi_k)$$

which we will need to prove the functoriality of our later construction. This could also be used to give an alternative proof for our main theorem. Unluckily, this approach has a problem: iterating this formula, we get strangely nested Lie brackets, which would be very difficult to bring into a nice form with Jacobi identity. So this is not a good way to find a handy formula for the usual star product of two monomials. Nevertheless, we want to pursue it for a moment, since we will get an equality which will be, although rather involved looking, very useful in the following: for analytic observations, it will be enough to put (even rough) estimates on it and the exact nature of the combinatorics in the formula will not be important. Hence we rewrite Equation (4.1.3) in order to cook up such a formula.

Let's take $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$, then we have

$$\xi_1 \cdots \xi_k \star_z \eta = \sum_{n=0}^{k} C_n(\xi_1 \ldots \xi_k, \eta)$$

with the C_n being as bilinear operators which are given explicitly on monomials by

$$C_n^k \colon \mathrm{S}^k(\mathfrak{g}) \times \mathfrak{g} \longrightarrow \mathrm{S}^{k-n+1}(\mathfrak{g}) \tag{4.1.6}$$

$$(\xi_1 \cdots \xi_k, \eta) \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{j} B_j^* z^j \xi_{\sigma(1)} \cdots \xi_{\sigma(k-j)} [\xi_{\sigma(k-j+1)}, [\dots, [\xi_{\sigma(k)}, \eta]]] \tag{4.1.7}$$

with

$$C_n = \sum_{k=0}^{\infty} C_n^k.$$

This gives us a good way of writing the n-fold star product of vectors:

Proposition 4.1.5 *Let $\mathfrak{g}$, $2 \leq k \in \mathbb{N}$ and $\xi_1, \dots, \xi_k \in \mathfrak{g}$. Then we have*

$$\xi_1 \star_z \dots \star_z \xi_k = \sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \dots, j\}}} z^{i_1 + \dots + i_{k-1}} C_{i_{k-1}}(\dots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \dots, \xi_k). \tag{4.1.8}$$

Proof: This is an easy prof by induction over k. For $k = 2$ the statement is clearly true. For the step $k \to k+1$ we get

$$\begin{aligned}
\xi_1 \star_z \dots \star_z \xi_{k+1} &= \left(\sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \dots, j\}}} z^{i_1 + \dots + i_{k-1}} C_{i_{k-1}}(\dots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \dots, \xi_k) \right) \star_z \xi_{k+1} \\
&= \sum_{i_k=0}^{k} z^{i_k} C_{i_k} \left(\sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \dots, j\}}} z^{i_1 + \dots + i_{k-1}} C_{i_{k-1}}(\dots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \dots, \xi_k), \xi_{k+1} \right) \\
&= \sum_{\substack{1 \leq j \leq k \\ i_j \in \{0, \dots, j\}}} z^{i_1 + \dots + i_k} \big(C_{i_{k-1}}(\dots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \dots, \xi_k), \xi_{k+1} \big) \qquad \square
\end{aligned}$$

Remark 4.1.6

i.) Of course, Proposition 4.1.5 is an easy consequence from Proposition 4.1.3. It's value, however, is that we know how the C_n's look like and what the summation range in (4.1.8) is. This will allow us to put estimates on things like iterated star products.

ii.) As already mentioned, we would get an identity for the star product of two monomials via

$$\xi_1 \cdots \xi_k \star_z \eta_1 \cdots \eta_\ell = \frac{1}{k!\ell!} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(k)} \star_z \eta_{\tau(1)} \star_z \cdots \star_z \eta_{\tau(\ell)}. \tag{4.1.9}$$

This can be proven from the definition of the map $\mathfrak{q}_z$. Unfortunately, this would give a very clumsy formula to deal with.

4.1.3 A Formula for two Monomials

If we want to get an identity for the star product of two monomials, we have to go back to Equation (3.2.7). This will not give a simple looking formula either, but we will at least be able to do some computations with concrete examples. As a first step, we must introduce a bit of notation:

Definition 4.1.7 (G-Index) *Let $k, \ell, n \in \mathbb{N}$ and $r = k + \ell - n$. Then we call an r-tuple J*

$$J = (J_1, \dots, J_r) = ((a_1, b_1), \dots, (a_r, b_r))$$

a G-index if it fulfils the following properties:

(i) $J_i \in \{0, 1, \dots, k\} \times \{0, 1, \dots, \ell\}$

(ii) $|J_i| = a_i + b_i \geq 1 \quad \forall_{i=1,\dots,r}$

(iii) $\sum_{i=1}^{r} a_i = k$ *and* $\sum_{i=1}^{r} b_i = \ell$

(iv) The tuple is ordered in the following sense: $i > j \Rightarrow |J_i| \geq |J_j| \quad \forall_{i,j=1,\dots,r}$ *and* $|a_i| \geq |a_j|$ *if* $|J_i| = |J_j|$

(v) If $a_i = 0$ *[or* $b_i = 0$*] for some* i*, then* $b_i = 1$ *[or* $a_i = 1$*].*

We call the set of all such G-indices $\mathcal{G}_r(k, \ell)$*.*

Definition 4.1.8 (G-Factorial) *Let* $J = ((a_1, b_1), \dots, (a_r, b_r)) \in \mathcal{G}_r(k, \ell)$ *be a G-Index. We set for a given tuple* $(a, b) \in \{0, 1, \dots, k\} \times \{0, 1, \dots, \ell\}$

$$\#_J(a, b) = \text{ number of times that } (a, b) \text{ appears in } J.$$

Then we define the G-factorial of $J \in \{0, 1, \dots, k\} \times \{0, 1, \dots, \ell\}$ *as*

$$J! = \prod_{(a,b) \in \{0,1,\dots,k\} \times \{0,1,\dots,\ell\}} (\#_J(a, b))!$$

Each pair (a, b) will later correspond to $\mathrm{BCH}_{a,b}(\xi, \eta)$. Now we can state a good formula for the Gutt star product:

Lemma 4.1.9 *Let* $\mathfrak{g}$ *be a Lie algebra,* $\xi, \eta \in \mathfrak{g}$ *and* $k, \ell \in \mathbb{N}$*. Then we have the following identity for the Gutt star product:*

$$\xi^k \star_z \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n\left(\xi^k, \eta^\ell\right),$$

where the C_n *are given by*

$$C_n\left(\xi^k, \eta^\ell\right) = \frac{k!\ell!}{(k+\ell-n)!} \sum_{\substack{a_1,b_1,\dots,a_r,b_r \geq 0 \\ a_i+b_i \geq 1 \\ a_1+\dots+a_r=k \\ b_1+\dots+b_r=\ell}} \mathrm{BCH}_{a_i,b_i}(\xi, \eta) \cdots \mathrm{BCH}_{a_r,b_r}(\xi, \eta) \tag{4.1.10}$$

$$= \sum_{J \in \mathcal{G}_{k+\ell-n}(k,\ell)} \frac{k!\ell!}{J!} \prod_{i=1}^{k+\ell-n} \mathrm{BCH}_{a_i,b_i}(\xi, \eta) \tag{4.1.11}$$

and the product is taken in the symmetric tensor algebra.

PROOF: We want to calculate what the C_n look like. Let's denote $r = k + \ell - n$ for brevity. Then we have

$$C_n\left(\xi^k, \eta^\ell\right) \in \mathrm{S}^r(\mathfrak{g}).$$

Of course, the only part of the series

$$\exp\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right) = \sum_{n=0}^{k+\ell}\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right)^n + \mathcal{O}(\xi^{k+1}, \eta^{\ell+1})$$

which lies in $S^r(\mathfrak{g})$ is the summand for $n = r$. We introduce the formal parameters t and s. Since we differentiate with respect to them, we can omit terms of higher orders in ξ and η than k and ℓ respectively.

$$\begin{aligned}
z^n C_n\Big(\xi^k, \eta^\ell\Big) &= \frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \frac{1}{z^r}\frac{\mathrm{BCH}(zt\xi, zs\eta)^r}{r!} \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \left(\sum_{j=1}^{k+\ell} \mathrm{BCH}_j(zt\xi, zs\eta)\right)^r \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \sum_{\substack{j_1,\dots,j_r\geq 1\\ j_1+\dots+j_r=k+\ell}} \mathrm{BCH}_{j_1}(zt\xi, zs\eta)\cdots\mathrm{BCH}_{j_r}(zt\xi, zs\eta) \\
&= z^n\frac{k!\ell!}{r!} \sum_{\substack{a_1,b_1,\dots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\dots+a_r=k\\ b_1+\dots+b_r=\ell}} \mathrm{BCH}_{a_i,b_i}(\xi,\eta)\cdots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)
\end{aligned}$$

We sum over all possible arrangements of the (a_i, b_i). In order to find n easier summation range, we put the ordering from Definition 4.1.7 on these multi-indices and avoid therefore double counting. We loose the freedom of arranging the (a_i, b_i) and need to count the number of multi-indices $((a_1, b_1), \dots, (a_r, b_r))$ which belong to the same G-index J. This number will be $\frac{r!}{J!}$, since we can not interchange the (a_i, b_i) any more (therefore $r!$), unless they are equal (therefore $J!^{-1}$). Since the ranges of the (a_i, b_i) in Equation (4.1.10) and of the elements in $\mathcal{G}_r(k,\ell)$ are the same, we can change the summation there to $J \in \mathcal{G}_r(k,\ell)$ and need to multiply by $\frac{r!}{J!}$. This gives

$$z^n C_n\Big(\xi^k, \eta^\ell\Big) = z^n\frac{k!\ell!}{J!} \sum_{J\in\mathcal{G}_r(k,\ell)} \mathrm{BCH}_{a_i,b_i}(\xi,\eta)\cdots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)$$

which is equivalent to Equation (4.1.11). □

Now we need to generalize this to factorizing tensors. To do so, we need a last definition:

Definition 4.1.10 *Let $a, b \in \mathbb{N}$ and $\xi_1, \dots, \xi_a, \eta_1, \dots, \eta_b \in \mathfrak{g}$. Then we define by*

$$\widetilde{\mathrm{BCH}}_{a,b}\colon \mathfrak{g}^{a+b} \longrightarrow \mathfrak{g}$$

the map which we get when we replace in$\mathrm{BCH}_{a,b}(\xi,\eta)$ the i-th ξ by ξ_i and the j-th η by η_j for $i = 1, \dots, a$ and $j = 1, \dots, b$.

Proposition 4.1.11 *Let $\mathfrak{g}$ be a Lie algebra, $k, \ell \in \mathbb{N}$ and $\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_\ell \in \mathfrak{g}$. Then we have the following identity for the Gutt star product:*

$$\xi_1 \dots \xi_k \star_z \eta_1 \dots \eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1\cdots\xi_k, \eta_1\cdots\eta_\ell),$$

where the C_n are given by

$$\begin{aligned}
C_n(\xi_1\cdots\xi_k, \eta_1\cdots\eta_\ell) = \sum_{J\in\mathcal{G}_{k+\ell-n}(k,\ell)} \sum_{\sigma\in S_k}\sum_{\tau\in S_\ell} \prod_{i=1}^{l+\ell-n} \widetilde{\mathrm{BCH}}_{a_i,b_i}\big(\xi_{\sigma(a_1+\dots+a_{i-1}+1)}, \dots, \\
\dots, \xi_{\sigma(a_1+\dots+a_i)}\big)\big(\eta_{\tau(b_1+\dots+b_{i-1}+1)}, \dots, \eta_{\tau(b_1+\dots+b_i)}\big). \qquad (4.1.12)
\end{aligned}$$

PROOF: The proof relies on polarization again and is completely analogous to the one of Proposition 4.1.3. We set

$$\Xi = \sum_{i=1}^{k} t_i \xi^i \quad \text{and} \quad \mathrm{H} = \sum_{i=1}^{\ell} t_j \eta^j .$$

Then it is easy to see that we will get rid of the factorials in Equation (4.1.11) since

$$\xi_1 \cdots \xi_k \star_z \eta_1 \cdots \eta_\ell = \frac{1}{k!\ell!} \frac{\partial^{k+\ell}}{\partial_{t_1} \cdots \partial_{s_\ell}}\Big|_{t_1,\ldots,s_\ell=0} \Xi^k \star_z \mathrm{H}^\ell .$$

Instead of the factorials, we get symmetrizations over the ξ_i and the η_j as we did in Proposition 4.1.3, which gives the wanted result. □

4.2 Consequences and examples

Some consequences

Proposition 4.1.11 allows us to get some algebraic results. For example, we would like to see that the Gutt star product fulfils the classical and the semi-classical limit from Definition **??**. We can prove this using Proposition 4.1.3. This will finish the proof of Theorem 3.4.6.

Corollary 4.2.1 *Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{S}^\bullet(\mathfrak{g})$ edowed with the Gutt star product*

$$x \star_z y = \sum_{n=0}^{\infty} z^n C_n(x, y).$$

i.) On factorizing tensors $\xi_1 \ldots \xi_k$ and $\eta_1 \ldots \eta_\ell$, C_0 and C_1 give

$$C_0(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell) = \xi_1 \cdots \xi_k \eta_1 \cdots \eta_\ell \tag{4.2.1}$$

$$C_1(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell) = \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \xi_1 \cdots \widehat{\xi_i} \cdots \xi_k \eta_1 \cdots \widehat{\eta_j} \cdots \eta_\ell [\xi_i \eta_j], \tag{4.2.2}$$

where the hat denotes elements which are left out.

ii.) For $\mathfrak{g}$ finite-dimensional and the canonical isomorphism $\mathcal{J} \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{Pol}^\bullet(\mathfrak{g}^*)$ from Proposition 3.1.3, we have for $f, g \in \mathrm{Pol}^\bullet(\mathfrak{g}^*)$

$$C_1\big(\mathcal{J}^{-1}(f), \mathcal{J}^{-1}(g)\big) - C_1\big(\mathcal{J}^{-1}(f), \mathcal{J}^{-1}(g)\big) = \mathcal{J}^{-1}(\{f, g\}_{KKS})$$

where $\{\cdot,\cdot\}_{KKS}$ is the Kirillov-Kostant-Souriau bracket.

iii.) The map $\star_z$ fulfils the classical and the semi-classical limit and is therefore a star product.

PROOF: We take $\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell \in \mathrm{S}^\bullet(\mathfrak{g})$ and consider the G-indices in $\mathcal{G}_{k+\ell}(k, \ell)$ first. This is easy, since there is just one element inside:

$$\mathcal{G}_{k+\ell}(k, \ell) = \Big\{\big(\underbrace{(0,1), \ldots, (0,1)}_{\ell \text{ times}}, \underbrace{(1,0), \ldots, (1,0)}_{k \text{ times}}\big)\Big\}.$$

So we find

$$C_0(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell) = \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{J!} \mathrm{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(1)}\big) \cdots \mathrm{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(k)}\big)$$

$$
\begin{aligned}
&\cdot \mathrm{BCH}_{1,0}\big(\eta_{\tau(1)}, \varnothing\big) \cdots \mathrm{BCH}_{1,0}\big(\eta_{\tau(\ell)}, \varnothing\big) \\
&= \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{k!\ell!} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)} \eta_{\tau(1)} \cdots \eta_{\tau(\ell)} \\
&= \xi_1 \cdots \xi_k \eta_1 \cdots \eta_\ell
\end{aligned}
$$

where we used $J! = k!\ell!$ according to Definition 4.1.8. We do the same for C_1. Also here, we have just one element in $\mathcal{G}_{k+\ell-1}(k,\ell)$:

$$
\mathcal{G}_{k+\ell}(k,\ell) = \Big\{ (\underbrace{(0,1),\ldots,(0,1)}_{\ell-1 \text{ times}}, \underbrace{(1,0),\ldots,(1,0)}_{k-1 \text{ times}}, (1,1)) \Big\}.
$$

Using

$$
\mathrm{BCH}_{1,1}(\xi,\eta) = \frac{1}{2}[\xi,\eta]
$$

and $J! = (k-1)!(\ell-1)!$, we find

$$
\begin{aligned}
C_1(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell) &= \frac{1}{2} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{(k-1)!(\ell-1)!} \xi_{\sigma(1)} \cdots \xi_{\sigma(k-1)} \eta_{\tau(1)} \cdots \eta_{\tau(\ell-1)} [\xi_{\sigma(k)}, \eta_{\tau(\ell)}] \\
&= \frac{1}{2} \sum_{i=0}^{k} \sum_{j=0}^{\ell} \xi_1 \cdots \widehat{\xi_i} \cdots \xi_k \eta_1 \cdots \widehat{\eta_j} \cdots \eta_\ell [\xi_i, \eta_j].
\end{aligned}
$$

This finishes part one. From this, the anti-symmetry of the Lie bracket yields

$$
C_1(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell) - C_1(\eta_1 \cdots \eta_\ell, \xi_1 \cdots \xi_k) = \sum_{i=0}^{k} \sum_{j=0}^{\ell} \xi_1 \cdots \widehat{\xi_i} \cdots \xi_k \eta_1 \cdots \widehat{\eta_j} \cdots \eta_\ell [\xi_i, \eta_j].
$$

We now need to compute the KKS brackets on polynomials. Because of the linearity in both arguments, it is sufficient to check it on monomials of coordinates. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ with linear coordinates $x_1, \ldots, x_n$ on $\mathfrak{g}^*$. Now take $\mu_1, \ldots, \mu_n, \nu_1, \ldots \nu_n \in \mathbb{N}$ and consider the monomials $f = x_1^{\mu_1} \cdots x_n^{\mu_n}$ and $g = x_1^{\nu_1} \cdots x_n^{\nu_n}$. We use the notation from Proposition 3.1.1 and find for $x \in \mathfrak{g}^*$

$$
\begin{aligned}
\{f,g\}_{KKS}(x) &= x_k c_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \\
&= \mu_i \nu_j c_{ij}^k x_k x_1^{\mu_1} \cdots x_i^{\mu_i - 1} \cdots x_n^{\mu_n} x_1^{\nu_1} \cdots x_j^{\nu_j - 1} \cdots x_n^{\nu_n}.
\end{aligned}
$$

Applying $\mathcal{J}^{-1}$ to it gives

$$
\mathcal{J}^{-1}(\{f,g\}_{KKS}) = \sum_{i=0}^{n} \sum_{j=0}^{n} \mu_i \nu_j e_1^{\mu_1} \cdots e_i^{\mu_i - 1} \cdots e_n^{\mu_n} e_1^{\nu_1} \cdots e_j^{\nu_j - 1} \cdots e_n^{\nu_n} [e_i, e_j]. \tag{4.2.3}
$$

On the other hand, we have

$$
\mathcal{J}^{-1}(f) = e_1^{\mu_1} \cdots e_n^{\mu_n} \quad \text{and} \quad \mathcal{J}^{-1}(g) = e_1^{\nu_1} \cdots e_n^{\nu_n}.
$$

Together with (4.2.2) this gives (4.2.3) and proves part two. Due to the bilinearity of the C_n, the third part follows. $\square$

It is clear, that the formulas from Proposition 4.1.11 and Proposition 4.1.3 should coincide. However, we want to check it, to have the evidence that everything works as we wanted.

Corollary 4.2.2 *Given $\xi_1, \dots, \xi_k, \eta \in \mathfrak{g}$, the results of the Equations* (4.1.12) *and* (4.1.3) *are compatible.*

PROOF: We have to compute sets of G-indices for $\xi_1, \dots, \xi_k, \eta_\ell \in \mathfrak{g}$. Again, they only have one element:

$$\mathcal{G}_{k+1-n}(k,1) = \Big\{\big(\underbrace{(1,0), \dots, (1,0)}_{k-n \text{ times}}, (n,1)\big)\Big\}.$$

So we have with $J! = (k-n)!$ and $\mathrm{BCH}_{n,1}(\xi,\eta) = \frac{B_n^*}{n!}(\mathrm{ad}_\xi)^n(\eta)$

$$\begin{aligned} z^n C_n(\xi_1 \dots \xi_k, \eta_\ell) &= z^n \sum_{\sigma \in S_k} \frac{1}{(k-n)!} \frac{B_n^*}{n!} \xi_{\sigma(1)} \dots \xi_{\sigma(k-n)} [\xi_{\sigma(k-n+1)}, [\dots, [\xi_{\sigma(k),\eta}] \dots]] \\ &= z^n \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{n} B_n^* \xi_{\sigma(1)} \dots \xi_{\sigma(k-n)} [\xi_{\sigma(k-n+1)}, [\dots, [\xi_{\sigma(k),\eta}] \dots]] \end{aligned}$$

Summing up over all n gives Equation (4.1.3). □

Two examples

Equation (4.1.12) is useful if one wants to do real computations with the star product, but it is maybe not intuitive to apply. This is why we will give two examples here. The easiest one which is not covered by the simpler formula (4.1.3) will be the star product of two quadratic terms. The second one should be the a bit more complex case of a cubic term with a quadratic term.

Two quadratic terms

Let $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathfrak{g}$. We want to compute

$$\xi_1\xi_2 \star_z \eta_1\eta_2 = C_0(\xi_1\xi_2, \eta_1\eta_2) + zC_1(\xi_1\xi_2, \eta_1\eta_2) + z^2 C_2(\xi_1\xi_2, \eta_1\eta_2) + z^3 C_3(\xi_1\xi_2, \eta_1\eta_2).$$

The very first thing we have to do is computing the set of G-indices. Then we calculate the G-factorial and finally go through the permutations.

C_0: We already did this in Corollary 4.2.1, and know that the zeroth order in z is just the symmetric product. Therefore we have

$$C_0(\xi_1\xi_2, \eta_1\eta_2) = \xi_1\xi_2\eta_1\eta_2$$

C_1: We also did this one in Corollary 4.2.1: There is just one G-index and we finally get

$$C_1(\xi_1\xi_2, \eta_1\eta_2) = \frac{1}{2}(\xi_2\eta_2[\xi_1, \eta_1] + \xi_2\eta_1[\xi_1, \eta_2] + \xi_1\eta_2[\xi_2, \eta_1] + \xi_1\eta_1[\xi_2, \eta_2]).$$

C_2: This is the first time, something interesting happens. We have three G-indices:

$$\mathcal{G}_2(2,2) = \{J_1, J_2, J_3\} = \big\{\big((0,1),(2,1)\big), \big((1,0),(1,2)\big), \big((1,1),(1,1)\big)\big\}.$$

The G-factorials give $J_1! = J_2! = 1$ and $J^3! = 2$, since the index $(1,1)$ appears twice in J_3. We take $\mathrm{BCH}_{a,b}(\xi,\eta)$ from Equation (3.3.3) for $(a,b) \in \{(1,2),(2,1)\}$:

$$\mathrm{BCH}_{1,2}(\xi,\eta) = \frac{1}{12}[[\xi,\eta],\eta] \quad \text{and} \quad \mathrm{BCH}_{2,1}(\xi,\eta) = \frac{1}{12}[[\eta,\xi],\xi].$$

So we have to insert the ξ_i and the η_j into $\frac{1}{12}\xi[[\xi,\eta],\eta]$ and $\frac{1}{12}\eta[[\eta,\xi],\xi]$ respectively and then we go on with the last one, which is

$$\frac{1}{2}\mathrm{BCH}_{1,1}(\xi,\eta)\mathrm{BCH}_{1,1}(\xi,\eta)=\frac{1}{8}[\xi,\eta][\xi,\eta].$$

We hence get

$$\begin{aligned}C_2(\xi_1,\xi_2,\eta_1,\eta_2)=\frac{1}{12}\big(&\eta_1[[\eta_2,\xi_1],\xi_2]+\eta_1[[\eta_2,\xi_2],\xi_1]+\eta_2[[\eta_1,\xi_1],\xi_2]+\eta_2[[\eta_1,\xi_2],\xi_1]+\\&\xi_1[[\xi_2,\eta_1],\eta_2]+\xi_1[[\xi_2,\eta_2],\eta_1]+\xi_2[[\xi_1,\eta_1],\eta_2]+\xi_2[[\xi_1,\eta_2],\eta_1]\big)+\\&\frac{1}{4}\big([\xi_1,\eta_1][\xi_2,\eta_2]+[\xi_1,\eta_2][\xi_2,\eta_1]\big)\end{aligned}$$

C_3: Here, we only have one G-index:

$$\mathcal{G}_1(2,2)=\big\{\big((2,2)\big)\big\}$$

The G-factorial is 1. We take again Equation (3.3.3) and see

$$\mathrm{BCH}_{2,2}(\xi,\eta)=\frac{1}{24}[[[\eta,\xi],\xi],\eta].$$

This gives

$$C_3(\xi_1,\xi_2,\eta_1,\eta_2)=\frac{1}{24}\big([[[\eta_1,\xi_1],\xi_2],\eta_2]+[[[\eta_1,\xi_2],\xi_1],\eta_2]+[[[\eta_2,\xi_1],\xi_2],\eta_1]+[[[\eta_2,\xi_2],\xi_1],\eta_1]\big)$$

We just have to put all the four terms together and have the star product.

A cubic and a quadratic term

Let $\xi_1,\xi_2,\xi_3,\eta_1,\eta_2\in\mathfrak{g}$. We compute

$$\xi_1\xi_2\xi_3\star_G\eta_1\eta_2=\sum_{n=0}^{4}z^nC_n(\xi_1\xi_2\xi_3,\eta_1\eta_2)$$

C_0: The first part is again just the symmetric product:

$$C_0(\xi_1\xi_2\xi_3,\eta_1\eta_2)=\xi_1\xi_2\xi_3\eta_1\eta_2.$$

C_1: Here we have again the term from Corollary 4.2.1:

$$\begin{aligned}C_1(\xi_1\xi_2\xi_3,\eta_1\eta_2)=\frac{1}{2}\big(&\xi_2\xi_3\eta_2[\xi_1,\eta_1]+\xi_2\xi_3\eta_1[\xi_1,\eta_2]+\xi_1\xi_3\eta_2[\xi_2,\eta_1]+\\&\xi_1\xi_3\eta_1[\xi_2,\eta_2]+\xi_1\xi_2\eta_2[\xi_3,\eta_1]+\xi_1\xi_2\eta_1[\xi_3,\eta_2]\big)\end{aligned}$$

C_2: Here the calculation is very similar to the one of C_2 in the example before. We have three G-indices:

$$\mathcal{G}_3(3,2)=\{J_1,J_2,J_3\}=\big\{\big((0,1),(1,0),(2,1)\big),\big((1,0),(1,0),(1,2)\big),\big((1,0),(1,1),(1,1)\big)\big\}.$$

The G-factorials are now $J_1!=1$ and $J_2!=J_3!=2$. Again, we take the BCH terms from Equation (3.3.3) and see, that we must insert the ξ_i and the η_j into

$$\frac{1}{12}\xi\eta[[\eta,\xi],\xi]+\frac{1}{24}\xi\xi[[\xi,\eta],\eta]+\frac{1}{8}\xi[\xi,\eta][\xi,\eta].$$

Now we go through all the possible permutations and get

$$\begin{aligned}C_2(\xi_1\xi_2\xi_3,\eta_1\eta_2) = &\frac{1}{12}\big(\xi_1\xi_2[[\xi_3,\eta_1],\eta_2] + \xi_1\xi_2[[\xi_3,\eta_2],\eta_1] + \xi_1\xi_3[[\xi_2,\eta_1],\eta_2]+\\ &\xi_1\xi_3[[\xi_2,\eta_2],\eta_1] + \xi_2\xi_3[[\xi_1,\eta_1],\eta_2] + \xi_2\xi_3[[\xi_1,\eta_2],\eta_1]\big)+\\ &\frac{1}{12}\big(\xi_1\eta_1[[\eta_2,\xi_2],\xi_3] + \xi_1\eta_2[[\eta_1,\xi_2],\xi_3] + \xi_1\eta_1[[\eta_2,\xi_3],\xi_2]+\\ &\xi_1\eta_2[[\eta_1,\xi_3],\xi_2] + \xi_2\eta_1[[\eta_2,\xi_1],\xi_3] + \xi_2\eta_2[[\eta_1,\xi_1],\xi_3]+\\ &\xi_2\eta_1[[\eta_2,\xi_3],\xi_1] + \xi_2\eta_2[[\eta_1,\xi_3],\xi_1] + \xi_3\eta_1[[\eta_2,\xi_2],\xi_1]+\\ &\xi_3\eta_2[[\eta_1,\xi_2],\xi_1] + \xi_3\eta_1[[\eta_2,\xi_1],\xi_2] + \xi_3\eta_2[[\eta_1,\xi_1],\xi_2]\big)+\\ &\frac{1}{4}\big(\xi_1[\xi_2,\eta_1][\xi_3,\eta_2] + \xi_1[\xi_3,\eta_1][\xi_2,\eta_2] + \xi_2[\xi_1,\eta_1][\xi_3,\eta_2]+\\ &\xi_2[\xi_3,\eta_1][\xi_1,\eta_2] + \xi_3[\xi_1,\eta_1][\xi_2,\eta_2] + \xi_3[\xi_2,\eta_1][\xi_1,\eta_2]\big).\end{aligned}$$

C_3: We first calculate the G-indices:

$$\mathcal{G}_2(3,2) = \{J_1,J_2,J_3\} = \big\{\big((0,1),(3,1)\big),\big((1,0),(2,2)\big),\big((1,1),(2,1)\big)\big\}.$$

We can omit J_1, since $\mathrm{BCH}_{3,1}(\xi,\eta)=0$. The G-factorials for the other two indices are 1. The BCH terms have been computed before. So we have to fill in the expression

$$\frac{1}{24}\xi[[[\eta,\xi],\xi],\eta] + \frac{1}{2\cdot 12}[\xi,\eta][[\eta,\xi],\xi].$$

Going through the permutations we get

$$\begin{aligned}C_3(\xi_1\xi_2\xi_3,\eta_1\eta_2) = &\frac{1}{24}\big(\xi_1[[[\eta_1,\xi_2],\xi_3],\eta_2] + \xi_1[[[\eta_2,\xi_2],\xi_3],\eta_1] + \xi_1[[[\eta_1,\xi_3],\xi_2],\eta_2]+\\ &\xi_1[[[\eta_2,\xi_3],\xi_2],\eta_1] + \xi_2[[[\eta_1,\xi_1],\xi_3],\eta_2] + \xi_2[[[\eta_2,\xi_1],\xi_3],\eta_1]+\\ &\xi_2[[[\eta_1,\xi_3],\xi_1],\eta_2] + \xi_2[[[\eta_2,\xi_3],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_2],\xi_1],\eta_2]+\\ &\xi_3[[[\eta_2,\xi_2],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_1],\xi_2],\eta_2] + \xi_3[[[\eta_2,\xi_1],\xi_2],\eta_1]\big)+\\ &\frac{1}{24}\big([\xi_1,\eta_1][[\eta_2,\xi_2],\xi_3] + [\xi_1,\eta_2][[\eta_1,\xi_2],\xi_3] + [\xi_1,\eta_1][[\eta_2,\xi_3],\xi_2]+\\ &[\xi_1,\eta_2][[\eta_1,\xi_3],\xi_2] + [\xi_2,\eta_1][[\eta_2,\xi_1],\xi_3] + [\xi_2,\eta_2][[\eta_1,\xi_1],\xi_3]+\\ &[\xi_2,\eta_1][[\eta_2,\xi_3],\xi_1] + [\xi_2,\eta_2][[\eta_1,\xi_3],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_2],\xi_1]+\\ &[\xi_3,\eta_2][[\eta_1,\xi_2],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_1],\xi_2] + [\xi_3,\eta_2][[\eta_1,\xi_1],\xi_2]\big).\end{aligned}$$

C_4: Now there is only C_4 left. We have one G-index:

$$\mathcal{G}_1(3,2) = \big\{\big((3,2)\big)\big\},$$

but there are more terms which belong to it. We have to go through

$$\mathrm{BCH}_{3,2}(\xi,\eta) = \frac{1}{120}[[[[\xi,\eta],\xi],\eta],\xi] + \frac{1}{360}[[[[\eta,\xi],\xi],\xi],\eta].$$

So we permute and get

$$\begin{aligned}C_4(\xi_1\xi_2\xi_3,\eta_1\eta_2) = &\frac{1}{120}\big([[[[\xi_1,\eta_1],\xi_2],\eta_2],\xi_3] + [[[[\xi_1,\eta_2],\xi_2],\eta_1],\xi_3] + [[[[\xi_1,\eta_1],\xi_3],\eta_2],\xi_2]+\\ &[[[[\xi_1,\eta_2],\xi_3],\eta_1],\xi_2] + [[[[\xi_2,\eta_1],\xi_1],\eta_2],\xi_3] + [[[[\xi_2,\eta_2],\xi_1],\eta_1],\xi_3]+\\ &[[[[\xi_2,\eta_1],\xi_3],\eta_2],\xi_1] + [[[[\xi_2,\eta_2],\xi_3],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_2],\eta_2],\xi_1]+\end{aligned}$$

$$
\begin{aligned}
&[[[[\xi_3,\eta_2],\xi_2],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_1],\eta_2],\xi_2] + [[[[\xi_3,\eta_2],\xi_1],\eta_1],\xi_2]\big)+\\
&\frac{1}{360}\big([[[[\eta_1,\xi_1],\xi_2],\xi_3],\eta_2] + [[[[\eta_2,\xi_1],\xi_2],\xi_3],\eta_1] + [[[[\eta_1,\xi_1],\xi_3],\xi_2],\eta_2]+\\
&[[[[\eta_2,\xi_1],\xi_3],\xi_2],\eta_1] + [[[[\eta_1,\xi_2],\xi_1],\xi_3],\eta_2] + [[[[\eta_2,\xi_2],\xi_1],\xi_3],\eta_1]+\\
&[[[[\eta_1,\xi_2],\xi_3],\xi_1],\eta_2] + [[[[\eta_2,\xi_2],\xi_3],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_2],\xi_1],\eta_2]+\\
&[[[[\eta_2,\xi_3],\xi_2],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_1],\xi_2],\eta_2] + [[[[\eta_2,\xi_3],\xi_1],\xi_2],\eta_1]\big).
\end{aligned}
$$

Now we only have to add up all those terms and we have finally computed the star product.

Chapter 5

A locally convex topology for the Gutt star product

We have finished the algebraic part of this work, except for some little lemmas concerning the Hopf theoretic chapter. Our next goal is setting up a locally convex topology on the symmetric tensor algebra, in which the Gutt star product will converge. At the beginning of this chapter, we will first give a motivation why the setting of locally convex algebras is convenient and necessary. In the second part, we will briefly recall the most important things on locally convex algebras and introduce the topology which we will work with. In the third section, the core of this chapter, the continuity of the star product and the dependence on the formal parameter are proven. We also show that representations of Lie algebras lift in a good way to those of the deformed symmetric algebra and that our construction is in fact functorial. Part four treats the case when the formal parameter $z = 1$ and hence talks about a locally convex topology on the universal enveloping algebra of a Lie algebra. We will also show, that our topology is "optimal" in a specific sense.

5.1 Why locally convex?

The first question one could ask is why we want the observable algebra to be a *locally convex* one. There are a lot of different choices and most of them would even make things simpler: we could think of locally multiplicatively convex algebras, Banach algebras, C^*- or even von Neumann algebras. All of them have much more structure than just locally convex algebras. We would have an entire holomorphic calculus within our algebra if we assumed it to be locally m-convex, or even a continuous one if we wanted it to be C^*.

The reason is that, in general, all these nice features are simply not there. Quantum mechanics tells us that the algebra made up by the space and momentum operators $\hat{q}$ and $\hat{p}$ can not be locally m-convex.

Proposition 5.1.1 *Let $\mathcal{A}$ be a unital associative algebra which contains the quantum mechanical observables $\hat{q}$ and $\hat{p}$ and in which the canonical commutation relation*

$$[\hat{q}, \hat{p}] = i\hbar\mathbb{1}$$

is fulfilled. Then the only submultiplicative semi-norm on it is $p = 0$.

PROOF: First, we need to show a little lemma:

Lemma 5.1.2 *In the given algebra, we have for $n \in \mathbb{N}$*

$$(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n) = (i\hbar)^n n!\mathbb{1}. \tag{5.1.1}$$

PROOF: To show it, we use the fact that for $a \in \mathcal{A}$ the operator ad_a is a derivation, which is always true for a Lie algebra which comes from an associative algebra with the commutator, since for $a, b, c \in \mathcal{A}$ we have

$$[a, bc] = abc - bca = abc - bac + bac - bca = [a, b]c + b[a, c].$$

For $n = 1$, Equation (5.1.1) is certainly true. So let's look at the step $n \to n + 1$. We make use of the derivation property and have

$$\begin{aligned}
(\mathrm{ad}_{\hat{q}})^{n+1}\big(\hat{p}^{n+1}\big) &= (\mathrm{ad}_{\hat{q}})^n(i\hbar\hat{p}^n + \hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^n(\hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^{n-1}\Big([\hat{q}, \hat{p}]\,\mathrm{ad}_{\hat{q}}(\hat{p}^n) + \hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n)\Big) \\
&= (i\hbar)^{n+1}n! + i\hbar(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n) + (\mathrm{ad}_{\hat{q}})^{n-1}\Big(\hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n)\Big) \\
&= 2(i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^{n-1}\Big(\hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n)\Big) \\
&\overset{(*)}{=} \quad \vdots \\
&= n(i\hbar)^{n+1}n! + \mathrm{ad}_{\hat{q}}(\hat{p}(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)) \\
&= n(i\hbar)^{n+1}n! + i\hbar(i\hbar)^n n! \\
&= (i\hbar)^{n+1}(n+1)!.
\end{aligned}$$

At $(*)$, we actually used another statement which is to be proven by induction over k and says

$$(\mathrm{ad}_{\hat{q}})^{n+1}\big(\hat{p}^{n+1}\big) = k(i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^{n+1-k}\Big(\hat{p}(\mathrm{ad}_{\hat{q}})^k(\hat{p}^n)\Big).$$

Since this proof is analogous to the first lines of the computation before, we omit it here and the lemma is proven. $\triangledown$

Now we can go on with the actual proof. Let $\|\cdot\|$ be a submultiplicative semi-norm. Then we see from Equation (5.1.1) that

$$\|(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)\| = |\hbar|^n n! \|\mathbb{1}\|.$$

On the other hand, we have

$$\begin{aligned}
\|(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)\| &= \|\hat{q}(\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n) - (\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n)\hat{q}\| \\
&\leq 2\|\hat{q}\|\|(\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n)\| \\
&\leq \quad \vdots \\
&\leq 2^n\|\hat{q}\|^n\|\hat{p}^n\| \\
&\leq 2^n\|\hat{q}\|^n\|\hat{p}\|^n
\end{aligned}$$

So in the end we get

$$|\hbar|^n n! \|\mathbb{1}\| \leq c^n$$

for some $c \in \mathbb{R}$. This cannot be fulfilled for all $n \in \mathbb{N}$ unless $\|\mathbb{1}\| = 0$. But then, by submultiplicativity, the semi-norm itself must be equal to 0. $\square$

Remark 5.1.3 The so called Weyl algebra, which fulfils the properties of the foregoing proposition, can be constructed from a Poisson algebra with constant Poisson tensor. On one hand, it is a fair to ask the question, why this restriction of not being locally m-convex should also be put on linear Poisson systems. On the other hand, there is no reason to expect that things become easier when we make the Poisson system more complex. Moreover, the Weyl algebra is actually nothing but a quotient of the universal enveloping algebra of the so called Heisenberg algebra, which is a particular Lie algebra. There is no reason why the original algebra should have a "better" analytical structure than its quotient, since the ideal, which is divided out by this procedure, is a closed one.

There's a second good reason why we should avoid our topology to be locally m-convex. The topology we set up on $\mathrm{S}^\bullet(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ will also give a topology on $\mathcal{U}(\mathfrak{g})$. In Proposition **??** ,we will show that, under weak (but for our purpose necessary) additional assumptions, there can be no topology on $\mathcal{U}(\mathfrak{g})$ which allows an entire holomorphic calculus. This underlines the results from Proposition 5.1.1, since locally m-convex algebras always have such a calculus.

In this sense, we have good reasons to think that $\mathrm{S}^\bullet(\mathfrak{g})$ will not allow a better setting than the one of a locally convex algebra if we want the Gutt star product to be continuous. Before we attack this task, we have to recall some technology from locally convex analysis.

5.2 Locally convex algebras

5.2.1 Locally convex spaces and algebras

Every locally convex algebra is of course also a locally convex space which is, of course, a topological vector space. To make clear what we talk about, we first give a definition which is taken from [**?**].

Definition 5.2.1 (Topological vector space) *Let V be a vector space endowed with a topology τ. Then we call (V, τ) (or just V, if there is no confusion possible) a topological vector space, if the two following things hold:*

i.) for every point in $x \in V$ the set $\{x\}$ is a closed and

ii.) the vector space operations (addition, scalar multiplication) are continuous.

Not all books require axiom (i) for a topological vector space. It is, however, useful, since it assures that the topology in a topological vector space is Hausdorff – a feature which we will always want to have. The proof for this is not difficult, but since we don't want to go too much into detail here, we refer to [**?**] again, where it can be found as Theorem 1.12.

The most important class of topological vector spaces are, at least, but not only, from a physical point of view, locally convex ones. Almost all interesting physical examples belong to this class: Finite-dimensional spaces, inner product (or pre-Hilbert) spaces, Banach spaces, Fréchet spaces, nuclear spaces and many more. There are at least two equivalent definitions of what is a locally convex space. While the first is more geometrical, the second is better suited for our analytic purpose.

Theorem 5.2.2 *For a topological vector space V, the following things are equivalent.*

i.) V has a local base $\mathscr{B}$ of the topology whose members are convex.

ii.) The topology on V is generated by a separating family of semi-norms $\mathcal{P}$.

Proof: This theorem is a very well-known result and can be found in standard literature, such as [**?**] again, where it is divided into two Theorems (namely 1.36 and 1.37). $\square$

Definition 5.2.3 (Locally convex space) *A locally convex space is a topological vector space in which one (and thus all) of the properties from Theorem 5.2.2 are fulfilled.*

The first property explains the term "locally convex". For our intention, the second property is more helpful, since in this setting proving continuity just means putting estimates on semi-norms. For this purpose, one often extends the set of semi-norms $\mathcal{P}$ to the set of all continuous semi-norms $\mathscr{P}$ which contains all semi-norms that are compatible with the topology (e.g. sums, multiples and maxima of (finitely many) semi-norms from $\mathcal{P}$). From here, we can start looking at locally convex algebras.

Definition 5.2.4 (Locally convex algebra) *A locally convex algebra is a locally convex vector space with an additional algebra structure which is continuous.*

More precisely, let $\mathcal{A}$ be a locally convex algebra and $\mathscr{P}$ the set of all continuous semi-norms, then for all $p \in \mathscr{P}$ there exists a $q \in \mathscr{P}$ such that for all $x, y \in \mathcal{A}$ one has

$$p(ab) \leq q(a)q(b). \tag{5.2.1}$$

Remind that we didn't require our algebras to be associative. The product in this equation could also be a Lie bracket. If we talk about associative algebras, we will always say it explicitly.

5.2.2 A special class of locally convex algebras

For our study of the Gutt star product, the usual continuity estimate (5.2.1) will not be enough, since there will be an arbitrarily high number of nested brackets to control. We will need an estimate which does not depend on the number of Lie brackets involved. Since Lie algebras are just one type of algebras, we can define the property we need also for other locally convex algebras.

Definition 5.2.5 (Asymptotic estimate algebra) *Let $\mathcal{A}$ be a locally convex algebra (not necessarily associative) with the set of all continuous semi-norms $\mathscr{P}$. For a given $p \in \mathscr{P}$ we call $q \in \mathscr{P}$ an asymptotic estimate for p, if there exists an $m \in \mathbb{N}$ such that for all $n \geq m$ we have*

$$p(x_1 \cdot \ldots \cdot x_n) \leq q(x_1) \ldots q(x_n) \quad \forall_{x_1,\ldots,x_n \in \mathscr{A}}. \tag{5.2.2}$$

For non-associative algebras, we want this estimate to be fulfilled for all ways of setting brackets on the left hand side. We call a locally convex algebra an AE algebra, if every $p \in \mathscr{P}$ has an asymptotic estimate.

Remark 5.2.6 Without further restrictions, we can set $m = 1$ in the upper definition, since this just means taking the maximum over a finite number of continuous semi-norms. If q satisfies the upper definition for some $m \in \mathbb{N}$ and for all $i = 2, \ldots, m-1$ we have

$$p(x_1 \cdot \ldots \cdot x_i) \leq q^{(i)}(x_1) \ldots q^{(i)}(x_i)$$

for all $x_1, \ldots, x_i \in \mathcal{A}$, then we just set

$$q' = \max\{p, q^{(2)}, \ldots, q^{(m-1)}, q\}.$$

Clearly, q' will again be a continuous semi-norm and an asymptotic estimate for p.

Remark 5.2.7 (The notion "asymptotic estimate")

i.) The term asymptotic estimate has, to the best of our knowledge, first been used by Boseck, Czichowski and Rudolph in [?]. They defined asymptotic estimates in the same way we did, but their idea of an AE algebra was different from ours: for them, in an AE algebra every continuous semi-norm admits a series of asymptotic estimates. This series must fulfil two additional properties, which actually make the algebra locally m-convex. Clearly, our definition is weaker, since it does not imply, a priori, the existence of an topologically equivalent set of submultiplicative semi-norms.

ii.) In [**?**], Glöckner and Neeb used a property to which they referred as $(*)$ for associative algebras. It was then used in [] by ... and ..., who called it the GN-property. It is easy to see that it is equivalent to our AE condition.

There are, of course, a lot of examples of AE (Lie) algebras. All finite dimensional and Banach (Lie) algebras fulfil (5.2.2), just as locally m-convex (Lie) algebras do. The same is true for nilpotent locally convex Lie algebras, since here again one just has to take the maximum of a finite number of semi-norms, analogously to the procedure in Remark 5.2.6.

It is far from clear what is exactly implied by the AE property. Are there examples for associative algebras which are AE but not locally m-convex, for example? Are there Lie algebras which are truly AE and not locally m-convex or nilpotent? We don't have an answer to this questions, but we can make some simple observations, which allow us to give an answer for special cases.

Proposition 5.2.8 (Entire calculus) *Let $\mathcal{A}$ be an associative AE algebra. Then it has an entire holomorphic calculus.*

PROOF: The proof is the same as for locally m-convex algebras: let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function with $f(z) = \sum_n a_n z^n$ and p a continuous semi-norm with an asymptotic estimate q. Then one has $\forall_{x\in\mathcal{A}}$

$$p(f(x)) = p\left(\sum_{n=0}^{\infty} a_n x^n\right) \leq \sum_{n=0}^{\infty} |a_n| p(x^n) \leq \sum_{n=0}^{\infty} |a_n| q(x)^n < \infty. \qquad \square$$

Remark 5.2.9 (Entire Calculus, AE and LMC algebras) The fact that AE algebras have an entire calculus makes them very similar to locally m-convex ones. Now there is something we can say about associative algebras which have an entire calculus: if such an algebra is additionally commutative and Fréchet, then must be even locally m-convex. This statement was proved in [**?**] by Mitiagin, Rolewicz and Zelazko. Oudadess and El kinani extended this result to commutative, associative algebras, in which the Baire category theorem holds. For non-commutative algebras, the situation is different. There are associative "Baire algebras" having an entire calculus, which are not locally m-convex. Zelazko gave an example for such an algebra in [**?**]. Unfortunately, his example is also not AE. It seems to be is an interesting (and non-trivial) question, if a non locally m-convex but AE algebra exists at all and if yes, how an example could look like.

5.2.3 The projective tensor product

We want to set up a topology on $\mathrm{S}^\bullet(\mathfrak{g})$. Therefore, we will first construct a topology on the tensor algebra $\mathrm{T}^\bullet(\mathfrak{g})$. As all the following constructions in this section don't use any algebra structure, we will do them on a locally convex vector space V where $\mathscr{P}$ is the set of continuous semi-norms. Then we can use the projective tensor product $\otimes_\pi$ in order to get a locally convex topology on each tensor power $V^{\otimes_\pi n}$. The precise construction can be found in standard textbooks on locally convex analysis like [**?**] or in the lecture notes [**?**]. Recall that for $p_1, \ldots, p_n \in \mathscr{P}$ we have a continuous semi-norm on $V^{\otimes_\pi n}$ via

$$(p_1 \otimes_\pi \ldots \otimes_\pi p_n)(x) = \inf\left\{ \sum_i p_1\left(x_i^{(1)}\right) \ldots p_n\left(x_i^{(n)}\right) \,\middle|\, x = \sum_i x_i^{(1)} \otimes \ldots \otimes x_i^{(n)} \right\}.$$

On factorizing tensors, we moreover have the property

$$(p_1 \otimes_\pi \ldots \otimes_\pi p_n)(x_1 \otimes_\pi \ldots \otimes_\pi x_n) = p_1(x_1) \ldots p_n(x_n) \tag{5.2.3}$$

which will be extremely useful in the following and which can be proven by the Hahn-Banach theorem. We also have

$$(p_1 \otimes \ldots \otimes p_n) \otimes (q_1 \otimes \ldots \otimes q_m) = p_1 \otimes \ldots \otimes p_n \otimes q_1 \otimes \ldots \otimes q_m.$$

For a given $p \in \mathscr{P}$ we will denote $p^n = p^{\otimes_\pi n}$ and p^0 is just the absolute value on the field $\mathbb{K}$. The π-topology on $V^{\otimes_\pi n}$ is set up by all the projective tensor products of continuous semi-norms, or, equivalently, by all the p^n for $p \in \mathscr{P}$.

The projective tensor product has a very nice feature: if we want to show a (continuity) estimate on the tensor algebra, it is enough to do it on factorizing tensors. We will use this very often and just refer to it as the "infimum argument".

Lemma 5.2.10 (Infimum argument for the projective tensor product) *Let $V_1, \ldots, V_n, W$ be locally convex vector spaces and*

$$\phi\colon V_1 \times \ldots \times V_n \longrightarrow W$$

a n-linear map, from which we get the linear map $\Phi\colon V_1 \otimes_\pi \ldots \otimes_\pi V_n \longrightarrow W$. Then Φ is continuous if and only if this is true for ϕ and if for $p, q \in \mathscr{P}$ the estimate

$$p(\Phi(x_1 \otimes \ldots \otimes x_n)) \leq q(v_1) \ldots q(v_n)$$

is fulfilled for all $x_i \in V_i, i = 1, \ldots, n$, then we have

$$p(\Phi(x)) \leq q(x)$$

for all $x \in V_1 \otimes \ldots \otimes V_n$.

Proof: If Φ is continuous, the continuity of ϕ is clear. The other implication is more interesting. Continuity for ϕ means, that for every continuous semi-norm q on W we have continuous semi-norms p_i on V_i with $i = 1, \ldots, n$ such that for all $x^{(i)} \in V_i$ the estimate

$$q\left(\phi\left(x^{(1)}, \ldots, x^{(n)}\right)\right) \leq p_1\left(x^{(1)}\right) \ldots p_n\left(x^{(n)}\right) \tag{5.2.4}$$

holds. Let $x \in V_1 \otimes_\pi \ldots \otimes_\pi V_n$, then it has a representation in terms of factorizing tensors like

$$x = \sum_j x_j^{(1)} \otimes_\pi \ldots \otimes_\pi x_j^{(n)}.$$

We thus have

$$\begin{aligned} q(\Phi(x)) &= q\left(\sum_j \Phi\left(x_j^{(1)} \otimes_\pi \ldots \otimes_\pi x_j^{(n)}\right)\right) \\ &\leq \sum_j q\left(\phi\left(x_j^{(1)}, \ldots, x_j^{(n)}\right)\right) \\ &\leq \sum_j p_1\left(x_j^{(1)}\right) \ldots p_n\left(x_j^{(n)}\right). \end{aligned}$$

Now we take the infimum over all possibilities of writing x as a sum of factorizing tensors on both sides. While nothing will happen on the left hand side, on the right hand side we will find $(p_1 \otimes_\pi \ldots \otimes_\pi p_n)(x)$. This gives exactly the estimate we wanted. □

Most of the time, we will deal with the symmetric tensor algebra. Therefore, we want to recall some basic facts about $\mathrm{S}^n(V)$, when it inherits the π-topology from the $V^{\otimes_\pi n}$. We will call it $\mathrm{S}^n_\pi(V)$ when we endow it with this topology.

Lemma 5.2.11 *Let V be a locally convex vector space, p a continuous semi-norm and $n, m \in \mathbb{N}$.*

i.) The symmetrization map

$$\mathscr{S}_n \colon V^{\otimes_\pi n} \longrightarrow V^{\otimes_\pi n}, \quad (x_1 \otimes \ldots \otimes x_n) \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$$

is continuous and we have for all $x \in V^{\otimes_\pi n}$ the estimate

$$p^n(\mathscr{S}_n(x)) \leq p^n(x). \tag{5.2.5}$$

ii.) Each symmetric tensor power $\mathrm{S}^n_\pi(V) \subseteq V^{\otimes_\pi n}$ is a closed subspace.

iii.) For $x \in \mathrm{S}^n_\pi(V)$ and $y \in \mathrm{S}^m_\pi(V)$ we have

$$p^{n+m}(xy) \leq p^n(x) p^m(y).$$

Proof: The first part is very easy to see and uses most of the tools which are typical for the projective tensor product. We have the estimate for factorizing tensors $x_1 \otimes \ldots \otimes x_n$

$$\begin{aligned} p^n(\mathscr{S}(x_1 \otimes \ldots \otimes x_n)) &= p^n\left(\frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)} \right) \\ &\leq \frac{1}{n!} \sum_{\sigma \in S_n} p^n\big(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}\big) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} p\big(x_{\sigma(1)}\big) \ldots p\big(x_{\sigma(n)}\big) \\ &= p(x_1) \ldots p(x_n) \\ &= p^n(x_1 \otimes \ldots \otimes x_n). \end{aligned}$$

Then we use the infimum argument from Lemma 5.2.10 and we are done. The second part is also easy since the kernel of a continuous map is always a closed subspace of the initial space and we have

$$\mathrm{S}^n_\pi = \ker(\mathsf{id} - \mathscr{S}_n).$$

The third part is a consequence from the first and also immediate. □

One could maybe think that the inequality in the first part of this lemma is just an artefact which is due to the infimum argument and should actually be an equality, if one looked to it more closely. It is very interesting to see, that this is *not* the case, since it may happen that this inequality is strict. The following example illustrates this.

Example 5.2.12 We take $V = \mathbb{R}^2$ with the standard basis e_1, e_2 and V is endowed with the maximum norm. Now look at $e_1 \otimes e_2$, which has the norm

$$\|e_1 \otimes e_2\| = \|e_1\| \otimes \|e_2\| = 1$$

We now evaluate the symmetrization map on $V \otimes_\pi V$:

$$\mathscr{S}(e_1 \otimes e_2) = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1).$$

Our aim is to show, that the projective tensor product of the norm of this symmetrized vector is not 1. Therefore we need to find another way of writing it which has a norm of less than 1. Observe that

$$\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) = \frac{1}{4}((e_1 + e_2) \otimes (e_1 + e_2) + (-e_1 + e_2) \otimes (e_1 - e_2))$$

and we have

$$\begin{aligned}
&\frac{1}{4}\|(e_1 + e_2) \otimes (e_1 + e_2) + (-e_1 + e_2) \otimes (e_1 - e_2)\| \\
&\leq \frac{1}{4}(\|(e_1 + e_2) \otimes (e_1 + e_2)\| + \|(-e_1 + e_2) \otimes (e_1 - e_2)\|) \\
&= \frac{1}{4}(1 + 1) \\
&= \frac{1}{2}.
\end{aligned}$$

So we have $\|\mathscr{S}(e_1 \otimes e_2)\| \leq \frac{1}{2} < 1$.

5.2.4 A topology for the Gutt star product

The next step is to set up a topology on $\mathrm{T}^\bullet(V)$ which has the π-topology on each component. A priori, there are a lot of such topologies and at least two natural ones: the direct sum topology which is very fine and has a very small closure, and the cartesian product topology which is very coarse and therefore has a very big closure. We need something in between, which we can adjust in a convenient way.

Definition 5.2.13 (R-topology) *Let p be an continuous semi-norm on a locally convex vector space V and $R \in \mathbb{R}$. We define the semi-norm*

$$p_R = \sum_{n=0}^{\infty} n!^R p^n$$

on the Tensor algebra $\mathrm{T}^\bullet(V)$. We write for the tensor or the symmetric algebra endowed with all such semi-norms $\mathrm{T}^\bullet_R(V)$ or $\mathrm{S}^\bullet_R(V)$ respectively.

We now want to collect the most important results on the locally convex algebras $(\mathrm{T}^\bullet_R(V), \otimes)$ and $(\mathrm{S}^\bullet_R(V), \vee)$.

Lemma 5.2.14 (The R-topology) *Let $R' \geq R \geq 0$ and q, p are continuous semi-norms on V.*

i.) If $q \geq p$ then $q_R \geq p_R$ and $p_{R'} \geq p_R$.

ii.) The tensor product is continuous and satisfies the following inequality:

$$p_R(x \otimes y) \leq \left(2^R p\right)_R(x) \left(2^R p\right)_R(y)$$

iii.) For all $n \in \mathbb{N}$ the induced topology on $\mathrm{T}^n(V) \subset \mathrm{T}^\bullet_R(V)$ and on $\mathrm{S}^n(V) \subset \mathrm{S}^\bullet_R(V)$ is the π-topology.

iv.) For all $n \in \mathbb{N}$ the projection and the inclusion maps

$$\begin{array}{ccccc}
\mathrm{T}^\bullet_R(V) & \longrightarrow & V^{\otimes_\pi n} & \longrightarrow & \mathrm{T}^\bullet(V) \\
\mathrm{S}^\bullet_R(V) & \longrightarrow & \mathrm{S}^n_\pi(V) & \longrightarrow & \mathrm{S}^\bullet_R(V)
\end{array}$$

are continuous.

v.) The completions $\widehat{\mathrm{T}}^\bullet_R(V)$ *of* $\mathrm{T}^\bullet_R(V)$ *and* $\widehat{\mathrm{S}}^\bullet_R(V)$ *of* $\mathrm{S}^\bullet_R(V)$ *can be described explicitly as*

$$\begin{aligned}\widehat{\mathrm{T}}^\bullet_R(V) &= \left\{ x = \sum_{n=0}^{\infty} x_n \;\middle|\; p_R(x) < \infty, \text{ for all } p \right\} \subseteq \prod_{n=0}^{\infty} V^{\hat{\otimes}_\pi n} \\ \widehat{\mathrm{S}}^\bullet_R(V) &= \left\{ x = \sum_{n=0}^{\infty} x_n \;\middle|\; p_R(x) < \infty, \text{ for all } p \right\} \subseteq \prod_{n=0}^{\infty} \mathrm{S}^n_{\hat{\otimes}_\pi}\end{aligned}$$

with p *running through all continuous semi-norms on* V *and the* p_R *are extended to the Cartesian product allowing the value* $+\infty$*.*

vi.) If $R' > R$*, then the topology on* $\mathrm{T}^\bullet_{R'}(V)$ *is strictly finer than the one on* $\mathrm{T}^\bullet_R(V)$*, the same holds for* $\mathrm{S}^\bullet_{R'}(V)$ *and* $\mathrm{S}^\bullet_R(V)$*. Therefore the completions get smaller for bigger* R*.*

vii.) The inclusion maps $\widehat{\mathrm{T}}^\bullet_{R'}(V) \longrightarrow \widehat{\mathrm{T}}^\bullet_R(V)$ *and* $\widehat{\mathrm{S}}^\bullet_{R'}(\mathfrak{g}) \longrightarrow \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ *are continuous.*

viii.) The topology on $\mathrm{T}^\bullet_R(V)$ *with the tensor product and on* $\mathrm{S}^\bullet_R(V)$ *with the symmetric product is locally m-convex if and only if* $R = 0$*.*

ix.) The algebras $\mathrm{T}^\bullet_R(V)$ *and* $\mathrm{S}^\bullet_R(V)$ *are first countable if and only if this is true for* V*.*

Proof: The first part is clear on factorizing tensors and extends to the whole tensor algebra via the infimum argument. For part (*ii*), take two factorizing tensors

$$x = x^{(1)} \otimes \ldots \otimes x^{(n)} \quad \text{and } y = y^{(1)} \otimes \ldots \otimes y^{(m)}$$

and compute:

$$\begin{aligned}p_R(x \otimes y) &= (n+m)!^R p^{n+m}\left(x^{(1)} \otimes \ldots x^{(n)} \otimes y^{(1)} \otimes \ldots y^{(m)}\right) \\ &= (n+m)!^R p^n\left(x^{(1)} \otimes \ldots x^{(n)}\right) p^m\left(y^{(1)} \otimes \ldots y^{(m)}\right) \\ &= \binom{n+m}{n}^R n!^R m!^R p^n\left(x^{(1)} \otimes \ldots x^{(n)}\right) p^m\left(y^{(1)} \otimes \ldots y^{(m)}\right) \\ &\leq 2^{(n+m)R} p_R\left(x^{(1)} \otimes \ldots x^{(n)}\right) p_R\left(y^{(1)} \otimes \ldots y^{(m)}\right) \\ &= \left(2^R p\right)_R\left(x^{(1)} \otimes \ldots x^{(n)}\right) \left(2^R p\right)_R\left(y^{(1)} \otimes \ldots y^{(m)}\right).\end{aligned}$$

The parts (*iii*) and (*iv*) are clear from the construction of the R- topology. In part (*v*) we used the completion of the tensor product $\hat{\otimes}$, the statement itself is clear and implies (*vi*) directly, since we have really more elements in the completion for $R < R'$, like the series over $x^n \frac{1}{n!^t}$ for $t \in (R, R')$ and $0 \neq x \in V$. Statement (*vii*) follows from the first. For (*viii*), it is easy to see that $\mathrm{T}^\bullet_0(V)$ and $\mathrm{S}^\bullet_0(V)$ are locally m-convex. For every $R > 0$ we have

$$p_R(x^n) = n!^R p(x)^n$$

for all $n \in \mathbb{N}$ and all $x \in V$. If we had a submultiplicative semi-norm $\|\cdot\|$ from an equivalent topology, then we would have some $x \in V$, and a continuous semi-norm p with $p(x) \neq 0$ such that $p_R \leq \|\cdot\|$, and hence

$$n!^R p(x)^R \leq \|x^n\| \leq \|x\|^n.$$

Since this is valid for all $n \in \mathbb{N}$, we get a contradiction. For the last part, the tensor algebras cannot be first countable if V itself isn't. On the other hand, if V has a finite base of the topology, then $\mathrm{T}^\bullet_R(V)$ and $\mathrm{S}^\bullet_R(V)$ are just a countable multiple of V and stay therefore first countable. □

The projective tensor product obviously keeps a lot of important and strong properties of the original vector space V. But Proposition **??** still leaves some important things. We will not make use of them in the following, but it is worth naming them for completeness. To do this in full generality, we need one more definition, which will be also very important in chapter 5.

Definition 5.2.15 *For a locally convex vector space V and $R \geq 0$ we set*

$$\mathrm{S}^\bullet_{R^-}(V) = \operatorname*{proj\,lim}_{\epsilon \longrightarrow 0} \mathrm{S}^\bullet_{1-\epsilon}(V)$$

and call its completion $\widehat{\mathrm{S}}^\bullet_{R^-}(V)$.

Now we can state two more propositions. Since we won't use them, we omit the proofs here. They can be found in [**?**].

Proposition 5.2.16 (Schauder bases) *Let $R \geq 0$ and V a locally convex vector space. If $\{e_i\}_{i\in I}$ is an absolute Schauder basis of V with coefficient functionals $\{\varphi^i\}_{i\in I}$, i.e. for every $x \in V$ we have*

$$x = \sum_{i\in I} \varphi^i(x) e_i$$

such that for every $p \in \mathscr{P}$ there is a $q \in \mathscr{P}$ such that

$$\sum_{i\in I} |\varphi^i(x)| p(e_i) \leq q(x), \tag{5.2.6}$$

then the set $\{e_{i_1} \otimes \ldots \otimes e_{i_n}\}_{i_1,\ldots,i_n \in I}$ defines an absolute Schauder basis of $\mathrm{T}^\bullet_R(V)$ together with the linear functionals $\{\varphi^{i_1} \otimes \ldots \otimes \varphi^{i_n}\}_{i_1,\ldots,i_n\in I}$ which satisfy

$$\sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n\in I} \big|\big(\varphi^{i_1} \otimes \ldots \otimes \varphi^{i_n}\big)(x)\big| p_R(e_{i_1} \otimes \ldots \otimes e_{i_n}) \leq q_R(x)$$

for every $x \in \mathrm{T}^\bullet_R(V)$ whenever p and q satisfy (5.2.6). The same statement is true for $\mathrm{S}^\bullet_R(V)$ and for $\mathrm{S}^\bullet_{R^-}(V)$ (for $R > 0$) when we choose a maximal linearly independent subset out of the set $\{e_{i_1} \ldots e_{i_n}\}_{i_1,\ldots,i_n\in I}$.

Proposition 5.2.17 (Nuclearity) *Let V be a locally convex space. For $R \geq 0$ the following statements are equivalent:*

i.) *V is nuclear.*

ii.) *$\mathrm{T}^\bullet_R(V)$ is nuclear.*

iii.) *$\mathrm{S}^\bullet_R(V)$ is nuclear.*

If moreover $R > 0$, then the following statements are equivalent:

i.) *V is strongly nuclear.*

ii.) *$\mathrm{T}^\bullet_R(V)$ is strongly nuclear.*

iii.) *$\mathrm{S}^\bullet_R(V)$ is strongly nuclear.*

5.3 Continuity results for the Gutt star product

From now on, we start with an AE Lie algebra $\mathfrak{g}$ rather than with a general locally convex space V. We have all the tools by the hand to show the continuity of the Gutt star product. We can do it either via the bigger formula (4.1.12) for two monomials or via the smaller one (4.1.3) for a monomial with a vector and iterate it. The results are very similar, but a bit better for the first approach. Nevertheless, both approaches give strong results, and depending on the precise situation, each one has its advantages. This is why we want to give both proofs here.

There will be a very general way how most of the proofs will work, and which tools will be used in the following. If we want to show the continuity of a map $f \colon \mathrm{S}^\bullet_R(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet_R(\mathfrak{g})$, we will proceed most of the time like this:

i.) First, we extend a map to the whole tensor algebra by putting the symmetrizer in front: $f = f \circ \mathscr{S}$. This doesn't lead to problems since the symmetrization does not affect symmetric tensors.

ii.) Then, we start with an estimate, which we do only on factorizing tensors in order to use the infimum argument (Lemma 5.2.10).

iii.) During the estimation process, we find symmetric products of Lie brackets. Those will be split up by the continuity of the symmetric product (5.2.5) from Lemma 5.2.11 the AE property (5.2.2).

iv.) Finally, we rearrange the split up semi-norms to the semi-norm of a factorizing tensor by (5.2.3).

5.3.1 Continuity of the product

In the first proof, we want to approach the estimate via the formula

$$\xi_1 \cdots \xi_k \star_z \eta_1 \cdots \eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell)$$

Since this comes from polarizing the formula

$$\xi^k \star_z \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n\Big(\xi^k, \eta^\ell\Big), \tag{5.3.1}$$

we will just give an explicit proof for the latter one. One gets the estimate for the first one easily in the same way, since in the end, all Lie brackets are broken up in step (*iii*). One will get sums over permutations weighted with the inverse of their quantity and, as their semi-norms are just numbers which commute, one ends up with the same estimate as for (5.3.1).

Before we star, we must fix two things: first, we want to extend the Gutt star product to the whole tensor algebra: define

$$\star_z \colon \mathrm{T}^\bullet(\mathfrak{g}) \times \mathrm{T}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{T}^\bullet(\mathfrak{g}), \quad \star_z = \star_z \circ \mathscr{S}\,.$$

Second, we will need the next lemma:

Lemma 5.3.1 *Let $\mathfrak{g}$ be a AE-Lie algebra, $\xi, \eta \in \mathfrak{g}$, p a continuous seminorm, q an asymptotic estimate for it and $a, b, n \in \mathbb{N}$ with $a + b = n$. Then, using the Goldberg-Thompson form of the Baker-Campbell-Hausdorff series, we have the following estimates:*

i.) The coefficients g_w from (3.3.8) *fulfil the estimate*

$$\sum_{|w|=n} \Big|\frac{g_w}{n}\Big| \leq \frac{2}{n}. \tag{5.3.2}$$

Recall that $|w|$ denotes the length of a word w and $[w]$ is the word put in Lie brackets nested to the left.

ii.) For every word w which has consists of a ξ's and b η's, we have

$$p([w]) \leq q(\xi)^a q(\eta)^b. \tag{5.3.3}$$

iii.) We have the estimate

$$p(\mathrm{BCH}_{a,b}(\xi, \eta)) \leq \frac{2}{a+b} q(\xi)^a q(\eta)^b. \tag{5.3.4}$$

PROOF: As already mentioned, Thompson showed in [31] how the Goldberg expression could be rearranged in Lie bracket form. In [32], he put estimates on this coefficients and proved (5.3.2). The next estimate (5.3.3) is due to the AE property, which does not see the way how brackets are set but just counts the number of ξ's and η's in the whole expression. Let's now use the notation $|w|_\xi$ for the number of ξ's appearing in a word w and $|w|_\eta$ for the number of η's. Clearly, $|w| = |w|_\xi + |w|_\eta$. With (5.3.2) and the AE property of $\mathfrak{g}$, we get

$$\begin{aligned} p(\mathrm{BCH}_{a,b}(\xi,\eta)) &\leq \sum_{\substack{|w|_\xi=a \\ |w|_\eta=b}} p^{a+b}\left(\frac{g_w}{a+b}[w]\right) \\ &\leq \sum_{\substack{|w|_\xi=a \\ |w|_\eta=b}} \left|\frac{g_w}{a+b}\right| p([w]) \\ &\leq \sum_{\substack{|w|_\xi=a \\ |w|_\eta=b}} \left|\frac{g_w}{a+b}\right| q(\xi)^a q(\eta)^b \\ &\leq \frac{2}{a+b} q(\xi)^a q(\eta)^b. \end{aligned}$$

□

Theorem 5.3.2 (Continuity of the star product) *Let $\mathfrak{g}$ be an AE-Lie algebra, $R \geq 0$, p a continuous seminorm with an asymptotic estimate q and $z \in \mathbb{C}$.*

i.) For $n \in \mathbb{N}$, the operator C_n is continuous and for all $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ we have the estimate:

$$p_R(C_n(x,y)) \leq \frac{n!^{1-R}}{2 \cdot 8^n}(16q)_R(x)(16q)_R(y) \tag{5.3.5}$$

ii.) For $R \geq 1$, the Gutt star product is continuous and for all $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ we have the estimate:

$$p_R(x \star_z y) \leq (cq)_R(x)(cq)_R(y) \tag{5.3.6}$$

with $c = 16(|z|+1)$. Hence, the Gutt star product is continuous and the estimate (5.3.6) holds on $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ for all $z \in \mathbb{C}$

PROOF: We need to give an estimate on the C_n in order to show their convergence. Let us use $r = k + \ell - n$ for brevity and recall that the products are taken in the symmetric algebra. We use Equation (4.1.10) from Lemma 4.1.9 and put estimates on it using Lemma 5.3.1 *(iii)*. So let p be a continuous seminorm and let q be an asymptotic estimate for it:

$$\begin{aligned} p_R\Big(C_n\Big(\xi^k, \eta^\ell\Big)\Big) &= p_R\Bigg(\frac{k!\ell!}{r!} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r \geq 0 \\ a_i+b_i \geq 1 \\ a_1+\ldots+a_r=k \\ b_1+\ldots+b_r=\ell}} \mathrm{BCH}_{a_1,b_1}(\xi,\eta)\cdots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)\Bigg) \\ &\leq \frac{k!\ell!}{r!} r!^R \sum_{\substack{a_1,b_1,\ldots,a_r,b_r \geq 0 \\ a_i+b_i \geq 1 \\ a_1+\ldots+a_r=k \\ b_1+\ldots+b_r=\ell}} p^{a_1+b_1}(\mathrm{BCH}_{a_1,b_1}(\xi,\eta))\cdots p^{a_r+b_r}(\mathrm{BCH}_{a_r,b_r}(\xi,\eta)) \\ &\leq \frac{k!\ell!}{r!^{1-R}} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r \geq 0 \\ a_i+b_i \geq 1 \\ a_1+\ldots+a_r=k \\ b_1+\ldots+b_r=\ell}} \frac{2q(\xi)^{a_1}q(\eta)^{b_1}}{a_1+b_1}\cdots\frac{2q(\xi)^{a_r}q(\eta)^{b_r}}{a_r+b_r} \end{aligned}$$

$$\leq q(\xi)^k q(\eta)^\ell 2^r \frac{k!\ell!}{r!^{1-R}} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} 1,$$

where we just used $\frac{2}{a_i+b_i} \leq 2$ in the last step. We need to count the number of terms in the sum:

$$\sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} 1 \leq \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_1+b_1+\ldots+a_r+b_r=k+\ell}} 1 = \binom{k+\ell+2r-1}{k+\ell} \leq 2^{3(k+\ell)-2n-1},$$

where we gave up some restrictions in the first estimate and got therefore more terms. Now we have

$$\begin{aligned}
p_R\Big(C_n\big(\xi^k,\eta^\ell\big)\Big) &\leq q(\xi)^k q(\eta)^\ell 2^{k+\ell-n} \frac{k!\ell!}{(k+\ell-n)!^{1-R}} 2^{3(k+\ell)-2n-1}\\
&= q_R\Big(\xi^{\otimes k}\Big) q_R\Big(\eta^{\otimes \ell}\Big) 2^{4(k+\ell)-3n-1} \left(\frac{k!\ell!n!}{(k+\ell-n)!n!}\right)^{1-R}\\
&\leq q_R\Big(\xi^{\otimes k}\Big) q_R\Big(\eta^{\otimes \ell}\Big) 2^{4(k+\ell)-3n-1} 2^{(1-R)(k+\ell)} n!^{1-R}\\
&= \frac{n!^{1-R}}{2\cdot 8^n} (16q)_R\Big(\xi^{\otimes k}\Big)(16q)_R\Big(\eta^{\otimes \ell}\Big)
\end{aligned}$$

and the first part is proven on factorizing tensors by polarization:

$$p_R(C_n(\xi_1\otimes\ldots\otimes\xi_k,\eta_1\otimes\ldots\otimes\eta_\ell)) \leq \frac{n!^{1-R}}{2\cdot 8^n}(16q)_R(\xi_1\otimes\ldots\otimes\xi_k)(16q)_R(\eta_1\otimes\ldots\otimes\eta_\ell).$$

For general tensors $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$, we use the infimum argument from Lemma **??** and the first part is done. For the second part, let x and y be tensors of degree at most k and ℓ respectively. We use (5.3.5) in (a), the fact that $R \geq 1$ in (b) and have

$$\begin{aligned}
p_R(x \star_z y) &= p_R\left(\sum_{n=0}^{l+\ell-1} z^n C_n(x,y)\right)\\
&\leq \sum_{n=0}^{k+\ell-1} p_R(z^n C_n(x,y))\\
&\overset{(a)}{\leq} \sum_{n=0}^{k+\ell-1} \frac{|z|^n}{2\cdot 8^n} n!^{1-R}(16q)_R(x)(16q)_R(y)\\
&\overset{(b)}{\leq} \frac{(|z|+1)^{k+\ell}}{2}(16q)_R(x)(16q)_R(y)\sum_{n=0}^{\infty}\frac{1}{8^n}\\
&\leq (16(|z|+1)q)_R(x)(16(|z|+1)q)_R(y).
\end{aligned} \tag{5.3.7}$$

Since estimates on $\mathrm{S}^\bullet_R(\mathfrak{g})$ also hold for the completion, the Theorem is proven. □

Remark 5.3.3 (Uniform continuity) Part *(i)* of the theorem makes clear, why exactly continuity will only hold if $R \geq 1$: the estimate in (5.3.5) shows, that all the C_n are indeed continuous for any $R \geq 0$, but only for $R \geq 1$ there is something like a uniform continuity. When R decreases, the continuity of the C_n's "gets worse" and the uniform continuity finally breaks down when the threshold $R = 1$ is trespassed. But we need this uniform estimate, since we have to control the operators up to an arbitrarily high order if we want to guarantee the continuity of the star product. Continuity up to a formerly chosen order n does not suffice.

Now, we want to give the second proof, which relies on (4.1.3). Approaching like this, we don't account for the fact that we will encounter terms like $[\eta,\eta]$ which will vanish, but we estimate more brutally. During this procedure, we will also count the formal parameter z more often than it is actually there. This is why we will have to make assumptions on R and z which are a bit stronger than before. Moreover, we will split up tensor products and put them together again various times, which is the reason why an AE Lie algebra will not suffice any more: we will need $\mathfrak{g}$ to be locally m-convex. But if we make these assumptions, we get the following lemma which will finally make the proof easier.

Lemma 5.3.4 *Let $\mathfrak{g}$ be a locally m-convex Lie algebra and $R \geq 1$. Then if $|z| < 2\pi$ or $R > 1$ there exists for $x \in \mathrm{T}^\bullet(\mathfrak{g})$ of degree at most k, $\eta \in \mathfrak{g}$ and each continuous submultiplicative semi-norm p a constant $c_{z,R}$ only depending on z and R such that the following estimate holds:*

$$p_R(x \star_z \eta) \leq c_{z,R}(k+1)^R p_R(x) q(\eta) \tag{5.3.8}$$

PROOF: We start again with factorizing tensors. Since we get the same estimate for monomials and for powers of some $\xi \in \mathfrak{g}$ via polarization, it is enough to consider $\xi^{\otimes k} \star_z \eta$. This gives

$$\begin{aligned}
p_R\left(\xi^{\otimes k} \star_z \eta\right) &= p_R\left(\sum_{n=0}^{k} \binom{k}{n} B_n^* z^n \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta)\right) \\
&= \sum_{n=0}^{k} \binom{k}{n} |B_n^*||z|^n (k+1-n)!^R p^{k+1-n}\left(\xi^{k-n}(\mathrm{ad}_\xi)^n(\eta)\right) \\
&\leq (k+1)^R \sum_{n=0}^{k} |B_n^*||z|^n \frac{k!(k-n)!^R}{(k-n)!n!} p(\xi)^k p(\eta) \\
&= (k+1)^R \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R} \left(\frac{(k-n)!n!}{k!}\right)^{R-1} p_R\left(\xi^{\otimes k}\right) p(\eta) \\
&\leq (k+1)^R p_R\left(\xi^{\otimes k}\right) p(\eta) \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R}.
\end{aligned}$$

Now if $|z| < 2\pi$ the sum can be estimated by extending it to a series which converges. We end up with a constant depending on R and on z such that

$$p_R\left(\xi^{\otimes k} \star_z \eta\right) \leq (k+1)^R c_{z,R} p_R\left(\xi^{\otimes k}\right) p(\eta).$$

If on the other hand $|z| \geq 2\pi$ and $R > 1$ we can estimate

$$\begin{aligned}
p_R\left(\xi^{\otimes k} \star_z \eta\right) &\leq (k+1)^R p_R\left(\xi^{\otimes k}\right) p(\eta) \left(\sum_{n=0}^{k} \frac{|B_n^*|}{n!}\right) \left(\sum_{n=0}^{k} \frac{|z|^n}{n!^{R-1}}\right) \\
&\leq (k+1)^R \underbrace{2\tilde{c}_{z,R}}_{=c_{z,R}} p_R\left(\xi^{\otimes k}\right) p(\eta).
\end{aligned}$$

We hence have the estimate on factorizing tensors and can extend this to generic tensors of degree at most k by the infimum argument. □

In the following, we assume again that either $R > 1$ or $R \geq 1$ and $|z| < 2\pi$ in order the use Lemma **??**. Now we can give a simpler proof of Theorem 5.3.2 for the case of a locally m-convex Lie algebra:

Proof (Alternative Proof of Theorem 5.3.2): Assume that $\mathfrak{g}$ is now even locally m-convex. We want to replace η in the foregoing lemma by an arbitrary tensor y of degree at most ℓ. Again, we do that on factorizing tensors first and get

$$
\begin{aligned}
p_R\Big(\xi^{\otimes k} \star_z \eta^{\otimes \ell}\Big) &= p_R\big(\xi^{\otimes k} \underbrace{\star_z \eta \star \cdots \star_z \eta}_{\ell\text{-times}}\big) \\
&\leq c_{z,R}(k+\ell)^R p_R\big(\xi^{\otimes k} \underbrace{\star_z \eta \star \cdots \star_z \eta}_{\ell-1\text{-times}}\big) p(\eta) \\
&\leq \ \vdots \\
&\leq c_{z,R}^{\ell}((k+\ell)\cdots(k+1))^R p_R\Big(\xi^{\otimes k}\Big) p(\eta)^{\ell} \\
&= c_{z,R}^{\ell}\left(\frac{(k+\ell)!}{k!\ell!}\right)^R p_R\Big(\xi^{\otimes k}\Big) p_R\Big(\eta^{\otimes \ell}\Big) \\
&\leq (2^R p)_R\Big(\xi^{\otimes k}\Big)(2^R c_{z,R} p)_R\Big(\eta^{\otimes \ell}\Big).
\end{aligned}
$$

Once again, we have the estimate on factorizing tensors via polarization and extend it via the infimum argument to the whole tensor algebra, since the estimate depends no longer on the degree of the tensors. □

Using this approach for continuity, it is easy to see that nilpotency of the Lie algebra changes the estimate substantially: If we knew that we will have at most N brackets because $N+1$ brackets vanish, then the sum in the proof of Lemma **??** would end at N instead of k and would therefore be independent of the degree of x.

In both proofs, it is easy to see that we need at least $R \geq 1$ to get rid of the factorials which come up because of the combinatorics of the star product. It is nevertheless interesting to see that this result is sharp, that means the Gutt star product really fails continuity, if $R < 1$:

Example 5.3.5 (A counter-example) Let $0 \leq R < 1$ and $\mathfrak{g}$ be the Heisenberg algebra in three dimensions, i.e. the Lie algebra generated by the elements P, Q and E with the bracket $[P, Q] = E$ and all other brackets vanishing. This is a very simple example for a non-abelian Lie algebra and if continuity of the star product fails for this one, then we can not expect it to hold for more complex ones. We impose on $\mathfrak{g}$ the ℓ^1-topology with the norm n and $n(P) = n(Q) = n(E) = 1$. This will be helpful, since here we really have the equality

$$p^{n+m}(X^n Y^m) = p^n(X^n) p^m(Y^m)$$

for the symmetric product. Then we consider

$$a_k = \frac{P^k}{k!^R} \quad \text{and} \quad b_k = \frac{Q^k}{k!^R}.$$

It is easy to see that

$$n_R(a_k) = n_R(b_k) = 1$$

We want to show that there is no $c > 0$ such that

$$n_R(a_k \star_z b_k) \leq (cn)_R(a_k)(cn)_R(b_k)$$

With other words, $n_R(a_k \star_z b_k)$ grows faster than exponentially. But this is the case, since with our combinatorial formula (4.1.12) we see

$$n_R(a_k \star_z b_k) = n_R\left(\sum_{j=0}^{k} \binom{k}{j}\binom{k}{j} j! \frac{1}{k!^{2R}} P^{k-j} Q^{k-j} E^j\right)$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \frac{k!^2 j!(2k-j)!^R}{(k-j)!^2 j!^2 k!^{2R}} \underbrace{n^{2k-j}(P^{k-j}Q^{k-j}E^j)}_{=1} \\
&= \sum_{j=0}^{k} \underbrace{\binom{k}{j}^2 \binom{2k}{k} \binom{2k}{j}^{-1}}_{\geq 1} j!^{1-R} \\
&\geq \sum_{j=0}^{k} j!^{1-R} \\
&\geq k!^{1-R},
\end{aligned}
$$

which is exactly what we wanted to show.

5.3.2 Dependence on the formal parameter

We now look at the completion $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ of the symmetric algebra with the Gutt star product $\star_z$ and get the following negative result:

Proposition 5.3.6 *Let $\xi \in \mathfrak{g}$ and $R \geq 1$, then $\exp(\xi) \notin \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$, where $\exp(\xi) = \sum_{n=0}^\infty \frac{\xi^n}{n!}$.*

PROOF: Take a semi-norm p such that $p(\xi) \neq 0$. Then set $c = p(\xi)^{-1}$. For ξ^n the powers in the sense of either the usual tensor product, or the symmetric product or the star product are the same. So we have for $N \in \mathbb{N}$

$$
p_R\left(\sum_{n=0}^{N} \frac{c^n}{n!}\xi^n\right) = \sum_{n=0}^{N} \frac{n!^R}{n!} c^n p_R(\xi^n) = \sum_{n=0}^{N} n!^{R-1} \geq N,
$$

and clearly $\exp(\xi)$ does not converge for the semi-norm p_R. □

When we go back to Theorem 5.3.2, we see that we have actually proven slightly more than we stated: We showed that the star product converges absolutely and locally uniform in z. This means, that the star product does is not only continuous, but also that the formal series converges to the star product in the completion. We can take a closer look at this proof in order to get a new result for the dependence on the formal parameter z:

Proposition 5.3.7 (Dependence on z) *Let $R \geq 1$, then for all $x, y \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ the map*

$$
\mathbb{K} \ni z \longmapsto x \star_z y \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}) \tag{5.3.9}
$$

is analytic with (absolutely convergent) Taylor expansion at $z = 0$ given by Equation (4.1.11). For $\mathbb{K} = \mathbb{C}$, the collection of algebras $\left\{\left(\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}), \star_z\right)\right\}_{z\in\mathbb{C}}$ is an entire holomorphic deformation of the completed symmetric tensor algebra $\left(\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}), \vee\right)$.

PROOF: The crucial point is that for $x, y \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ and every continuous seminorm p we have an asymptotic estimate q such that

$$
\begin{aligned}
p_R(x \star_z y) &= p_R\left(\sum_{n=0}^{\infty} z^n C_n(x, y)\right) \\
&= \sum_{n=0}^{\infty} |z|^n p_R(C_n(x, y))
\end{aligned}
$$

$$\leq (16q)_R(x)(16q)_R(y)\sum_{n=0}^{\infty}\frac{|z|^n n!^{1-R}}{2\cdot 8^n},$$

where we used the fact that the estimate (5.3.5) extends to the completion. For $R > 1$, this map is clearly analytic and absolutely convergent for all $z \in \mathbb{K}$. If $R = 1$, then for every $M \geq 1$ we go back to homogeneous, factorizing tensors $x^{(k)}$ and $y^{(\ell)}$ of degree k and ℓ respectively, and have

$$\begin{aligned} M^n p_R\Big(C_n\Big(x^{(k)}, y^{(\ell)}\Big)\Big) &\leq \frac{M^n}{2\cdot 8^n}(16q)_R\Big(x^{(k)}\Big)(16q)_R\Big(y^{(\ell)}\Big) \\ &\leq M^{k+\ell}(16q)_R\Big(x^{(k)}\Big)(16q)_R\Big(y^{(\ell)}\Big) \\ &= (16Mq)_R\Big(x^{(k)}\Big)(16Mq)_R\Big(y^{(\ell)}\Big), \end{aligned}$$

where we used that $0 \leq n \leq k+\ell-1$. The infimum argument gives the estimate on all Tensors $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ and it extends to the completion such that

$$p_R(z^n C_n(x,y)) \leq (16Mq)_R(x)(16Mq)_R(y)\frac{|z|^n}{2\cdot(8M)^n}$$

and hence

$$p_R(x \star_z y) \leq (16Mq)_R(x)(16Mq)_R(y)\sum_{n=0}^{\infty}\frac{|z|^n}{2\cdot(8M)^n}.$$

So the power series converges for all $z \in \mathbb{C}$ with $|z| < 8M$ and converges uniformly if $|z| \leq cM$ for $c < 8$. But then, the map (5.3.9) converges on all open discs centered around $z = 0$, and it must therefore be entire. □

Remark 5.3.8 ((Weakly) holomorphic maps with values in locally convex spaces) One could argue, that the term "holomorphic" in a locally convex space V does not necessarily mean that a map has a absolutely convergent Taylor expansion, but one should rather see the map (5.3.9) as a collection of paths $\mathbb{C} \longrightarrow \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ and ask for their complex differentiability in the sense of a differential quotient. Of course, this would be is a valid formulation of the word "holomorphic". In this sense, one can call a map "weakly holomorphic", if every continuous linear form $\lambda\colon V \longrightarrow \mathbb{C}$ applied to it gives a holomorphic map in $\mathbb{C}$. Then, we would just have proven the map (5.3.9) to be weakly holomorphic. Yet, we have proven real holomorphicity, since in [**?**] Rudin gave a prove the two notions of holomorphicity coincide in locally convex spaces.

5.3.3 Representations

Let's see the symmetric algebra with the Gutt star product as the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_z)$. We know, that this algebra has the universal property that Lie algebra homomorphisms into associative algebras can be lifted to unital homomorphisms of associative algebras. As a commutative diagram, this reads

$$\begin{tikzcd} \mathcal{U}_R(\mathfrak{g}) \arrow[r, "\mathfrak{q}^{-1}"] \arrow[dr, "\Phi"] & \mathrm{S}^\bullet_R(\mathfrak{g}) \arrow[d, "\widetilde{\Phi}"] \\ \mathfrak{g} \arrow[u, "\iota"] \arrow[r, "\phi"'] & \mathscr{A} \end{tikzcd}$$

where $\mathscr{A}$ is an associative algebra as mentioned. Since we endowed $\mathscr{U}_R(\mathfrak{g}_z)$ and $\mathrm{S}^\bullet_R(\mathfrak{g})$ with a topology, we can now ask if the homomorphisms Φ and $\widetilde{\Phi}$ are continuous. This question is partly answered by the following result:

Proposition 5.3.9 (Universal property) *Let $\mathfrak{g}$ be an AE-Lie algebra, $\mathcal{A}$ an associative AE-algebra and $\phi\colon \mathfrak{g} \longrightarrow \mathscr{A}$ is a continuous Lie algebra homomorphism. If $R \geq 1$, then the induced algebra homomorphisms Φ and $\widetilde{\Phi}$ are continuous.*

PROOF: We define an extension of Φ on the whole tensor algebra again:

$$\Psi\colon \mathrm{T}^\bullet_R(\mathfrak{g}) \longrightarrow \mathscr{A}, \quad \Psi = \widetilde{\Phi} \circ \mathscr{S}$$

It is clear that if Ψ is continuous on factorizing tensors, we get the continuity of $\widetilde{\Phi}$ and of Φ via the infimum argument. So let p be a continuous semi-norm on $\mathcal{A}$ with its asymptotic estimate q and $\xi_1, \dots, \xi_n \in \mathfrak{g}$. Since ϕ is continuous, we find a continuous semi-norm r on $\mathfrak{g}$ such that for all $\xi \in \mathfrak{g}$ we have $q(\phi(\xi)) \leq r(\xi)$. Then we have

$$\begin{aligned} p(\Psi(\xi_1 \otimes \cdots \otimes \xi_n)) &= p\Big(\widetilde{\Phi}(\xi_1 \star_z \cdots \star_z \xi_n)\Big) \\ &= p(\phi(\xi_1) \cdots \phi(\xi_n)) \\ &\leq q(\phi(\xi_1)) \cdots q(\phi(\xi_n)) \\ &\leq r(\xi_1) \cdots r(\xi_n) \\ &\leq r_R(\xi_1 \otimes \cdots \otimes \xi_n), \end{aligned}$$

where the last inequality is true for all $R \geq 0$ and hence for all $R \geq 1$. □

Although this is a nice result, our construction fails to be universal, since the universal enveloping algebra endowed with our topology is *not* AE in general. This is even very easy to see:

Example 5.3.10 Take $\xi \in \mathfrak{g}$, then we know that $\xi^{\otimes n} = \xi^{\star_z n} = \xi^n$ for $n \in \mathbb{N}$ where the formal parameter is $z = 1$. Let $R > 0$ and p a continuous semi-norm in $\mathfrak{g}$ then we find

$$p_R(\xi^n) = n!^R p(\xi)^n = \frac{n!^R}{c^n} q(\xi)^n \tag{5.3.10}$$

for $c = \frac{p(\xi)}{q(\xi)}$ for a different semi-norm q with $q(\xi) \neq 0$. But since the $\frac{n!^R}{c^n}$ will always diverge for $n \to \infty$ we will never get an asymptotic estimate for p_R.

Although the construction is not universal, we can draw a nice conclusion from Proposition 5.3.9:

Corollary 5.3.11 (Continuous Representations) *Let $R \geq 1$ and $\mathcal{U}_R(\mathfrak{g})$ the universal enveloping algebra of an AE-Lie algebra $\mathfrak{g}$, then for every continuous representation ϕ of $\mathfrak{g}$ into the bounded linear operators $\mathfrak{B}(V)$ on a Banach space V the induced homomorphism of associative algebras $\Phi\colon \mathcal{U}(\mathfrak{g}) \longrightarrow \mathfrak{B}(V)$ is continuous.*

PROOF: This follows directly from Proposition 5.3.9 and $\mathfrak{B}(V)$ being a Banach algebra. □

Remark 5.3.12

i.) From this, we get the special case that all finite-dimensional representations of an AE-Lie algebra can be lifted to continuous representations of $\mathscr{U}(\mathfrak{g}_z)$. We will come back to this in the next section and see another result of it.

ii.) For infinite-dimensional Lie algebras, this statement will not be very important, since there, one rather has strongly continuous representations and no norm-continuous ones. Nevertheless, our topology may help to think about continuous linear functionals on $\mathscr{U}(\mathfrak{g}_z)$ and to do finally some GNS-representation theory of it. This would finally give us a representation of $\mathscr{U}(\mathfrak{g}_z)$ on a (Pre-)Hilbert space and we could talk about, what strongly continuous representations of the universal enveloping algebra should be. In this sense, the results of this chapter may also open a door towards some new approaches in this field.

5.3.4 Functoriality

Now let $\mathfrak{g}, \mathfrak{h}$ be two AE-Lie algebras. We know that a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathscr{U}(\mathfrak{h}_z)$ would lift to a homomorphism $\mathscr{U}(\mathfrak{g}_z) \longrightarrow \mathscr{U}(\mathfrak{h}_z)$, if the latter one was AE, which is not the case. Yet, we would like to have this result and get the functoriality of our construction, but it won't be that easy. Let's draw the commutative diagram to make things clear:

$$
\begin{tikzcd}
\mathrm{S}^\bullet_R(\mathfrak{g}) \arrow[r, "\widetilde{\Phi}_z"] & \mathrm{S}^\bullet_R(\mathfrak{h}) \\
\mathscr{U}_R(\mathfrak{g}_z) \arrow[u, "\mathfrak{q}_z^{-1}"] \arrow[r, "\Phi_z"] & \mathscr{U}_R(\mathfrak{h}_z) \arrow[u, "\mathfrak{q}_z^{-1}"'] \\
\mathfrak{g} \arrow[u, "\iota_z"] \arrow[r, "\phi"'] & \mathfrak{h} \arrow[u, "\iota_z"']
\end{tikzcd}
$$

If ϕ is a continuous Lie algebra homomorphism, we want to know if Φ_z and $\widetilde{\Phi}_z$ will be continuous, too. Luckily, the answer is yes and our construction is functorial. For the proof, we will need the next lemma.

Lemma 5.3.13 *Let $\mathfrak{g}$ be an AE-Lie algebra, $R \geq 1$ and $z \in \mathbb{C}$. Then for p a continuous seminorm, q an asymptotic estimate, $n \in \mathbb{N}$ and all $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ the following estimate*

$$p_R(\xi_1 \star_z \cdots \star_z \xi_n) \leq c^n n!^R q^n(\xi_1 \otimes \cdots \otimes \xi_n) \tag{5.3.11}$$

holds with $c = 8\mathrm{e}(|z|+1)$.

PROOF: We start with a continuous seminorm p:

$$
\begin{aligned}
p_R(\xi_1 \star_z \cdots \star_z \xi_n) &= p_R\left(\sum_{\ell=0}^{n-1} \sum_{\substack{1\leq j\leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} z^{i_{n-1}} C_{i_{n-1}}\big(\ldots z^{i_2} C_{i_2}\big(z^{i_1} C_{i_1}(\xi_1, \xi_2), \xi_3\big) \ldots, \xi_n\big)\right) \\
&\leq \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\leq j\leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} p^{n-\ell}\Big(z^{i_{n-1}} C_{i_{n-1}}\big(\ldots z^{i_2} C_{i_2}\big(z^{i_1} C_{i_1}(\xi_1, \xi_2), \xi_3\big) \ldots, \xi_n\big)\Big) \\
&\leq \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\leq j\leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |B^*_{i_{n-1}}| \cdots |B^*_{i_1}| |z|^{i_{n-1}} \cdots |z|^{i_1}
\end{aligned}
$$

$$\cdot \binom{1}{i_1}\binom{2-i_1}{i_2}\cdots\binom{n-1-i_1-\cdots i_{n-2}}{i_{n-1}} q(\xi_1)\cdots q(\xi_n) \tag{5.3.12}$$

By using the fact that $|B_m^*| \le m!$ for all $m \in \mathbb{N}$ and grouping together the powers of $|z|$, we find

$$\begin{aligned}
&p_R(\xi_1 \star_z \cdots \star_z \xi_n) \\
&\le \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\le j\le n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^\ell \frac{1!(2-i_1)!\cdots(n-1-i_1-\cdots-i_{n-2})!}{(1-i_1)!\cdots(n-1-i_1-\cdots i_{n-1})!} q(\xi_1)\cdots q(\xi_n) \\
&\le \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\le j\le n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^\ell 1^{i_1} 2^{i_2}\cdots(n-1)^{i_{n-1}} q(\xi_1)\cdots q(\xi_n) \\
&\le \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\le j\le n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^\ell n^\ell q(\xi_1)\cdots q(\xi_n) \\
&\le \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\le j\le n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^\ell (2\mathrm{e})^n \ell! q(\xi_1)\cdots q(\xi_n),
\end{aligned}$$

where in the last step we used $n^\ell \le \mathrm{e}^n \frac{n!}{(n-\ell)!} = \mathrm{e}^n \binom{n}{\ell}\ell! \le \mathrm{e}^n 2^n \ell!$. But now we can simply estimate $|z|^\ell \le (|z|+1)^n$ and $(n-\ell)!^R \ell! \le n!^R$ for $R \ge 1$. We just need to count the number of summands and get

$$\begin{aligned}
p_R(\xi_1 \star_z \cdots \star_z \xi_n) &\le n!^R (2\mathrm{e})^n (|z|+1)^n q(\xi_1)\cdots q(\xi_n) \sum_{\ell=0}^{n-1} \sum_{\substack{1\le j\le n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} 1 \\
&\overset{(a)}{\le} n!^R (2\mathrm{e})^n (|z|+1)^n q(\xi_1)\cdots q(\xi_n) \sum_{\ell=0}^{n-1} \binom{n-1+\ell-1}{\ell-1} \\
&\overset{(b)}{\le} n!^R (2\mathrm{e})^n (|z|+1)^n q(\xi_1)\cdots q(\xi_n) 2^{2n} \\
&\le c^n n!^R q^n(\xi_1 \otimes \cdots \otimes \xi_n),
\end{aligned}$$

with $c = 8\mathrm{e}(|z|+1)$. In (a) the estimate for the big sum is the following: for every $j = 1, \ldots, n-1$ we have surely $i_j \in \{0, 1, \ldots, n-1\}$ and the sum of all the i_j is ℓ. If we forget about all other restrictions, we will get even more terms. But then the number of summands is same as there are ways to distribute ℓ items on $n-1$ places, which is given by $\binom{n-1+\ell-1}{\ell-1}$. Then in (b) we use

$$\binom{n-1+\ell-1}{\ell-1} \le \binom{2n}{\ell-1}$$

with the binomial coefficient being zero for $\ell = 0$. Then it is just the standard estimate for binomial coefficients via the sum over all ℓ. $\square$

Proposition 5.3.14 (Functoriality) *Let $R \geq 1$, $\mathfrak{g}, \mathfrak{h}$ be AE-Lie algebras and $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ a continuous homomorphism between them. Then it lifts to a continuous unital homomorphism of locally convex algebras $\Phi_z\colon \mathscr{U}_R(\mathfrak{g}_z) \longrightarrow \mathscr{U}_R(\mathfrak{h}_z)$.*

PROOF: First, if $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is continuous, then for every continuous seminorm q on $\mathfrak{h}$, we have a continuous seminorm r on $\mathfrak{g}$ such that for all $\xi \in \mathfrak{g}$

$$q(\phi(\xi)) \leq r(\xi).$$

Second, we define Ψ_z on factorizing tensors via

$$\Psi_z\colon \mathrm{T}^\bullet_R(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet_R(\mathfrak{h}), \quad \Psi_z = \widetilde{\Phi}_z \circ \mathscr{S}$$

and extend it linearly to $\mathrm{T}^\bullet_R(\mathfrak{g})$. Clearly, Φ_z and $\widetilde{\Phi}_z$ will be continuous if Ψ_z is continuous. From this, we get for a seminorm p on $\mathfrak{h}$, an asymptotic estimate q and $\xi_1, \ldots, \xi_n$

$$\begin{aligned} p_R(\Psi_z(\xi_1 \otimes \ldots \otimes \xi_n)) &= p_R(\phi(\xi_1) \star_z \ldots \star_z \phi(\xi_1)) \\ &\overset{\text{(a)}}{\leq} c^n n!^R q(\phi(\xi_1)) \ldots q(\phi(\xi_n)) \\ &\overset{\text{(b)}}{\leq} c^n n!^R r(\xi_1) \ldots r(\xi_n) \\ &= (cr)_R(\xi_1 \otimes \ldots \otimes \xi_n). \end{aligned}$$

Again, we use the infimum argument and we have the estimate on all tensor in $\mathrm{T}^\bullet_R(\mathfrak{g})$. It extends to the completion and the statement is proven. □

5.4 Alternative topologies and an optimal result

Let's set the formal parameter $z = 1$ for a moment and make some observations. So far, we found a topology on $\mathrm{S}^\bullet_R(\mathfrak{g})$ which gives a continuous star product and which has a reasonably large completion, but it is always fair to ask if we can do better than that: we've seen that our completed algebra will not contain exponential series, which would be a very nice feature to have. So is it possible to put another locally convex topology on $\mathrm{S}^\bullet_R(\mathfrak{g})$ which gives a completion with exponentials? The answer is no, at least under mild additional assumptions.

Proposition 5.4.1 (Optimality of the R-topology) *Let $\mathfrak{g}$ be an AE Lie algebra in which one has elements ξ, η for which the Baker-Campbell-Hausdorff series does not converge. Then there is no locally convex topology on $\mathrm{S}^\bullet(\mathfrak{g})$ such that all of the following things are fulfilled:*

i.) *The Gutt star product $\star_G$ is continuous.*

ii.) *For every $\xi \in \mathfrak{g}$ the series $\exp(\xi)$ converges absolutely in the completion of $\mathrm{S}^\bullet(\mathfrak{g})$.*

iii.) *For all $n \in \mathbb{N}$ the projection and inclusion maps with respect to the graded structure*

$$\mathrm{S}^\bullet(\mathfrak{g}) \xrightarrow{\pi_n} \mathrm{S}^n(\mathfrak{g}) \xrightarrow{\iota_n} \mathrm{S}^\bullet(\mathfrak{g})$$

are continuous.

First of all, we should make clear what "the Baker-Campbell-Hausdorff series does not converge" actually means. This may be clear for a finite-dimensional Lie algebra, but in the locally convex setting, it is not that obvious. For simplicity, let's assume our local convex space to be complete in the following. First, we note here that a net or a sequence in a locally convex space is convergent [or Cauchy], if it is convergent [or Cauchy] with respect to all $p \in \mathscr{P}$. Quite similar to a normed space, we can make the following definition.

Definition 5.4.2 *Let V be a locally convex vector space, $\mathscr{P}$ the set of continuous semi-norms, $p \in \mathscr{P}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence in V. We set*

$$\rho_p(\alpha) = \left(\limsup_{n\longrightarrow\infty} \sqrt[n]{p(\alpha_n)} \right)^{-1}$$

where $\rho_p(\alpha) = \infty$ if $\limsup_{n\longrightarrow\infty} \sqrt[n]{p(\alpha_n)} = 0$ as usual.

From this, we immediately get the two following lemmas.

Lemma 5.4.3 (Root test in locally convex spaces) *Let V be a complete locally convex vector space, $p \in \mathscr{P}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence. Then, if $\rho_p(\alpha) > 1$, the series*

$$\mathcal{S}_n(\alpha) = \sum_{j=0}^{n} \alpha_j$$

converges absolutely with respect to p. If, conversely, this series convegres with respect to p, then we have $\rho_p(\alpha) \geq 1$.

PROOF: The proof is completely analogous to the one in finite dimensions. □

Lemma 5.4.4 *Let V be a complete locally convex vector space, $p \in \mathscr{P}$, $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence and $M > 0$. Then, if the power series*

$$\lim_{n\longrightarrow\infty} \sum_{j=0}^{n} \alpha_j z^j$$

converges for all $z \in \mathbb{C}$ with $|z| \leq M$, it converges absolutely with respect to p for all $z \in \mathbb{C}$ with $|z| < M$.

PROOF: Like in the finite-dimensional setting, we use the root test: Convergence for $|z| \leq M$ means $\rho_p(\alpha_z) \geq 1$, where we have set $\alpha_z = (\alpha_n z^n)_{n\in\mathbb{N}}$. Hence, for every $z' < z$ we get $\rho_p(\alpha_{z'}) > 1$ and absolute convergence by Lemma **??**. □

This helps us to make clear, what "BCH does not converge" can be interpreted. BCH can be read as a power series in two variables:

$$\mathrm{BCH}(t\xi, s\eta) = \sum_{a,b=0}^{\infty} t^a s^b \mathrm{BCH}_{a,b}(\xi, \eta).$$

So if it converges for ξ and η, then it converges absolutely for all $t\xi$ and $s\eta$ with $t, s \in [0, 1)$. If it converges for all $t\xi$ and $s\eta$ with $t, s \in \mathbb{K}$, then it converges absolutely for all t, s. If this is the case, we can reorder the sums as we like to, and the series

$$\sum_{n=0}^{N} \mathrm{BCH}_n(\xi, \eta)$$

converges if and only if the series

$$\mathrm{BCH}(t\xi, s\eta) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \mathrm{BCH}_{a,b}(\xi, \eta)$$

converges. Now we can prove a Lemma, from which Proposition 5.4.1 will follow immediately.

Lemma 5.4.5 *Let $\mathfrak{g}$ be an AE Lie algebra and $\mathrm{S}^\bullet(\mathfrak{g})$ is endowed with a locally convex topology, such that the conditions $(i) - (iii)$ from Proposition* **??** *are fulfilled. Then the Baker-Campbell-Hausdorff series converges absolutely for all $\xi, \eta \in \mathfrak{g}$.*

PROOF: First, we complete the algebra to $\widehat{\mathrm{S}}^\bullet(\mathfrak{g})$. We will need the projection π_1 to the Lie algebra. Take $\xi, \eta \in \mathfrak{g}$. Now, since the the Gutt star product is continuous and that the exponential series is absolutely convergent for all $\xi, \eta \in \mathfrak{g}$, we get for $t, s \in \mathbb{K}$

$$\begin{aligned}
\pi_1(\exp(t\xi) \star_G \exp(s\eta)) &= \pi_1\left(\lim_{N\to\infty}\left(\sum_{n=0}^{N} \frac{t^n\xi^n}{n!}\right) \star_G \lim_{M\to\infty}\left(\sum_{m=0}^{M} \frac{s^m\eta^m}{m!}\right)\right) \\
&\overset{(a)}{=} \pi_1\left(\lim_{N\to\infty}\lim_{M\to\infty}\left(\sum_{n=0}^{N} \frac{t^n\xi^n}{n!}\right) \star_G \left(\sum_{m=0}^{M} \frac{s^m\eta^m}{m!}\right)\right) \\
&\overset{(b)}{=} \lim_{N\to\infty}\lim_{M\to\infty} \pi_1\left(\left(\sum_{n=0}^{N} \frac{t^n\xi^n}{n!}\right) \star_G \left(\sum_{m=0}^{M} \frac{s^m\eta^m}{m!}\right)\right) \\
&\overset{(c)}{=} \lim_{N\to\infty}\lim_{M\to\infty} \sum_{n,m=0}^{N,M} \mathrm{BCH}_{t\xi,s\eta}(n,m)
\end{aligned}$$

where we used the continuity of the star product in (a), the continuity of the projection in (b) and evaluated the projection in (c). Since $\exp(t\xi)$ and $\exp(s\eta)$ are elements in the completion, their star product exists and hence the double series at the end of this equation converges for any two elements $\xi, \eta \in \mathfrak{g}$. but now we can use the result, that in this setting, the BCH series converges absolutely. We can rearrange the terms and get the convergence of

$$\sum_{n=1}^{N} \mathrm{BCH}_n(t\xi, s\eta).$$

Therefore, the BCH series must converge globally. □

Obviously, this proves Proposition 5.4.1.

Now we want to use a result due to Wojtyński [?], who showed that for Banach-Lie algebras, global convergence of the BCH series is equivalent to the fact that for any ξ we have

$$\|(\mathrm{ad}_\xi)^n\|^{\frac{1}{n}} \xrightarrow{n\to\infty} 0.$$

A Banach-Lie algebra with this property is sometimes called quasi-nilpotent, radical or nil, see for example [?] for various generalizations of nilpotency in the case of Banach algebras. For finite-dimensional Lie algebras, quasi-nilpotency implies nilpotency. Hence for a finite-dimensional Lie algebra $\mathfrak{g}$, BCH is globally convergent if and only if $\mathfrak{g}$ is nilpotent.

From this, we see that at least for "non-quasi-nilpotent" Banach-Lie algebras, our result is in some sense optimal, at least if we want the grading structure to compatible with the topology.

Remark 5.4.6 (Another topology in $\mathscr{U}(\mathfrak{g})$) In [26] Schottenloher and Pflaum mention an alternative topology on the universal enveloping algebra for finite- dimensional Lie algebras: They took the coarsest locally convex topology, such that all finite-dimensional representations of $\mathfrak{g}$ extend to continuous algebra homomorphisms. This topology is in fact even locally m- convex and has therefore an entire holomorphic calculus. In particular, the completion will contain exponential functions for all Lie algebra elements. Therefore, as we have seen in Proposition 5.4.1, it can not respect the grading structure, as our topology does. The R-topology must hence be different from that. As we have seen in Proposition **??**, it is finer for $R \geq 0$, ans since the topologies are different, it is strictly finer.One could argue that the R-topology is "just" locally convex, but its advantage (for our purpose) is that the grading is necessary for the holomorphic dependence on the formal parameter, which is a wanted feature.

Chapter 6

Nilpotent Lie algebras

At the end of the last chapter, we have seen that the Baker-Campbell-Hausdorff series and its convergence plays an important role for a topology on the universal enveloping algebra. It is thus natural to ask whether things will change, if we look at Lie algebras from which have a globally convergent BCH series. To make things not too complicated from the beginning, we focus on locally convex and truly nilpotent Lie algebras. Recall that a Lie algebra $\mathfrak{g}$ is nilpotent, if there exists a $N \in \mathbb{N}$, such that for all $n \geq N$ and all $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ we have

$$\mathrm{ad}_{\xi_1} \circ \ldots \circ \mathrm{ad}_{\xi_n} = 0. \tag{6.0.1}$$

In the infinite-dimensional case, this is *not* the same as

$$(\mathrm{ad}_\xi)^n = 0$$

for all $\xi \in \mathfrak{g}$ and and $n \geq N$, but (typically strictly) stronger. In the case of finite-dimensional Lie algebras, the notions (6.0.1) and (6) coincide due to the so-called theorem of Engel, which makes use of the existence of a finite series of nilpotent ideals in the Lie algebra. Such a terminating series doesn't need to exist in infinite dimensions and there are known counter-examples to it.

Before we look at this case more closely, let's first make a list of things, that we expect to change or not when we go to this more particular setting.

i.) In Example 5.3.5 we have seen that we can not expect to get a continuous algebra structure for $R < 1$, even for very simple nilpotent, but non-abelian Lie algebras. Therefore, we should not expect to get much larger completions now.

ii.) In [34], Waldmann showed that the Weyl-Moyal star product converges in the R-topology for $R \geq \frac{1}{2}$. This so-called Weyl algebra is, however, nothing but a quotient of the Heisenberg algebra. It would be interesting to understand this a bit better, since we know that we need $R \geq 1$ for the latter. The quotient procedure must therefore have some strong influence on this construction. Can we reproduce the value $R \geq \frac{1}{2}$ somehow by dividing out an ideal?

iii.) The argument we used in Propositipon 5.4.1, namely the non-global convergence of BCH, is not given any more. Now, we don't have a reason any longer to expect that exponentials won't be part of the completion. In this sense, it would be at least nice to have something more than "just" $R = 1$. Can we do that?

iv.) As already mentioned, there are generalizations or weaker forms of nilpotency in infinite-dimensions, especially for Banach-Lie algebras, which are equivalent to the usual notion of nilpotency in finite dimensions. If we get a stronger result for nilpotent Lie algebras, can we extend it to some of these generalizations?

The very fascinating and highly interesting answer to the three questions we just posed is: yes, we can. The first section of this chapter will be devoted to the question from point (*iii*): we get a bigger completion by using a projective limit. We will also see how to get again the nice functorial properties we had before. In second section, we will reproduce the result by Waldmann, at least for the finite-dimensional case. The third part will take care of some generalizations of nilpotentcy for Banach-Lie algebras and will extend the result of the projective limit to a particular subcase there.

6.1 The projective limit

6.1.1 Continuity of the Product

As already mentioned, it is possible to extend the continuity result. Therefore, we take a locally convex, nilpotent Lie algebra $\mathfrak{g}$ and look at

$$\mathrm{S}^\bullet_{1^-}(\mathfrak{g}) = \operatorname*{proj\,lim}_{\epsilon \longrightarrow 0} \mathrm{S}^\bullet_{1-\epsilon}(\mathfrak{g}).$$

A tensor is in the completion $\widehat{Sym}^\bullet_{1^-}(\mathfrak{g})$, when it lies for every $\epsilon > 0$ in the completion or $\mathrm{S}^\bullet_{1-\epsilon}(\mathfrak{g})$. Otherwise stated: Let $\mathscr{P}$ be the set of all continuous seminorms of the Lie algebra $\mathfrak{g}$, then

$$f \in \widehat{Sym}^\bullet_{1^-}(\mathfrak{g}) \quad \Longleftrightarrow \quad p_{1-\epsilon}(x) < \infty \quad \forall_{p\in\mathscr{P}} \forall_{\epsilon>0}.$$

So, if we want to show, that the Gutt star product is continuous in $\widehat{Sym}^\bullet_{1^-}(\mathfrak{g})$, we need to show that for every $p \in \mathscr{P}$ and $R < 1$, there exists a $q \in \mathscr{P}$ and a $R' < 1$, such that for all $x, y \in \mathrm{S}^\bullet(\mathfrak{g})$ we have

$$p_R(x \star_z y) \leq q_{R'}(x) q_{R'}(y).$$

Before we prove the next proposition, we want to remind that locally convex, nilpotent Lie algebras are always AE Lie algebras. So the results we have found so far, are valid in this case.

Theorem 6.1.1 *Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra with continuous Lie bracket and $N \in \mathbb{N}$ such that $N + 1$ Lie brackets vanish.*

i.) If $0 \leq R < 1$, the C_n-operators are continuous and fulfil the estimate

$$p_R(C_n(x, y)) \leq \frac{1}{2 \cdot 8^n} (32\mathrm{e}q)_{R+\epsilon}(x) (32\mathrm{e}q)_{R+\epsilon}(y), \tag{6.1.1}$$

for all $x, y \in \mathrm{S}^\bullet_R(\mathfrak{g})$, where p is a continuous seminorm, q an asymptotic estimate and $\epsilon = \frac{N-1}{N}(1 - R)$.

ii.) The Gutt star product $\star_z$ is continuous for the locally convex projective limit $\mathrm{S}^\bullet_{1^-}(\mathfrak{g})$ and we have

$$p_R(x \star_z y) \leq (cq)_{R+\epsilon}(x) (cq)_{R+\epsilon}(y) \tag{6.1.2}$$

with $c = 32\mathrm{e}(|z|+1)$ and the ϵ from the first part. The Gutt star product extends continuously to $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$, where it converges absolutely and coincides with the formal series.

Proof: Again we use $\star_z$ on the whole tensor algebra and compute the estimate for $\xi^{\otimes k}$ and $\eta^{\otimes \ell}$. The important point is that now, we get restrictions for the values of n. Recall that $k + \ell - n$ is the symmetric degree of $C_n\big(\xi^{\otimes k}, \eta^{\otimes \ell}\big)$ and that we must have

$$(k + \ell - n)N \geq k + \ell \quad \Longleftrightarrow \quad n \leq (k + \ell)\frac{N-1}{N}$$

for $C_n(\xi^{\otimes k}, \eta^{\otimes \ell}) \neq 0$. This makes it possible to estimate $n!^{1-R}$ in (5.3.5): set $\delta = \frac{N-1}{N}$ and also denote a factorial where we have non-integers, meaning the gamma function. We get

$$\begin{aligned}
n!^{1-R} &\leq (\delta(k+\ell)!)^{1-R} \\
&\leq (\delta(k+\ell))^{(1-R)\delta(k+\ell)} \\
&\leq (k+\ell)^{(1-R)\delta(k+\ell)} \\
&= \Big((k+\ell)^{(k+\ell)}\Big)^{(1-R)\delta} \\
&\leq \Big(\mathrm{e}^{k+\ell} 2^{k+\ell} k!\ell!\Big)^{(1-R)\delta} \\
&= \Big((2\mathrm{e})^{\delta(1-R)}\Big)^{k+\ell} k!^{\epsilon} \ell!^{\epsilon},
\end{aligned}$$

using $\epsilon = \delta(1-R)$. Hence

$$\begin{aligned}
p_R\Big(C_n\Big(\xi^{\otimes k}, \eta^{\otimes \ell}\Big)\Big) &\leq \frac{\Big((2\mathrm{e})^{\delta(1-R)}\Big)^{k+\ell} k!^{\epsilon}\ell!^{\epsilon}}{2 \cdot 8^n} (16q)_R\Big(\xi^{\otimes k}\Big)(16q)_R\Big(\eta^{\otimes \ell}\Big) \\
&\leq \frac{1}{2 \cdot 8^n} (cq)_{R+\epsilon}\Big(\xi^{\otimes k}\Big)(cq)_{R+\epsilon}\Big(\eta^{\otimes \ell}\Big)
\end{aligned}$$

with $c = 16(2\mathrm{e})^{\delta(1-R)} \leq 32\mathrm{e}$. We then get the estimate on all tensors by the infimum argument and extend it to the completion. Note, that for every $R < 1$ we also have $R + \epsilon < 1$ with the $\epsilon = \delta(1-R)$ from above. Iterating this continuity estimate, we get closer and closer to 1 and it is not possible to repeat this process an arbitrary number of times and stop at some value strictly less than 1. For the second part, we can conclude analogously to the second part of Proposition **??**. □

Again, we can do an easier proof by assuming submultiplicativity of the seminorms, since we get an alternative version of Lemma 5.3.4:

Lemma 6.1.2 *Let $\mathfrak{g}$ be a locally m-convex, nilpotent Lie algebra such that more N nested Lie brackets vanish. Let p be a continuous seminorm, $z \in \mathbb{K}$ and $R \geq 0$. Then, for every tensor $x \in \mathrm{S}^\bullet_R(\mathfrak{g})$ of degree at most $k \in \mathbb{N}$ and $\eta \in \mathfrak{g}$, we have the estimate*

$$p_R(x \star_z \eta) \leq (k+1)^R k^{N(1-R)} c p_R(x) p(\eta) \tag{6.1.3}$$

with the constant $c = \sum_{n=0}^N \frac{|B_n^*|}{n!}|z|^n$.

PROOF: Again, we do the estimate on factorizing tensors and apply the infimum argument later. So let $\xi, \eta \in \mathfrak{g}$, $k \in \mathbb{N}$, p a continuous seminorm on $\mathfrak{g}$ and $z \in \mathbb{K}$. Then, we have for $R \geq 0$

$$\begin{aligned}
p_R\Big(\xi^{\otimes k} \star_z \eta\Big) &= \sum_{n=0}^{k} (k+1-n)!^R \binom{k}{n} |B_n^*||z|^n p^{k+1-n}\Big(\xi^{k-n}(\mathrm{ad}_\xi)^n(\eta)\Big) \\
&\leq (k+1)^R \sum_{n=0}^{N} \frac{k!(k-n)!^R}{(k-n)!n!} |B_n^*||z|^n p(\xi)^k p(\eta) \\
&= (k+1)^R p_R\Big(\xi^{\otimes k}\Big) p(\eta) \sum_{n=0}^{N} \left(\frac{k!}{(k-n)!}\right)^{1-R} \frac{|B_n^*||z|^n}{n!} \\
&\leq (k+1)^R k^{N(1-R)} p_R\Big(\xi^{\otimes k}\Big) p(\eta) \sum_{n=0}^{N} \frac{|B_n^*||z|^n}{n!}.
\end{aligned}$$

□

Now, we can iterate Lemma 6.1.2 in the same way, we did it in Chapter 5:

PROOF (ALTERNATIVE PROOF OF THEOREM **??**): Again, we do the calculation only on factorizing tensors. We need to transform the $k^{N(1-R)}$ into a very small factorial somehow. This is possible, since for given $N \in \mathbb{N}$ and $0 \leq R < 1$, the sequence

$$\left(\frac{k^N}{\sqrt{k!}}\right)^{1-R}$$

converges to 0 for $k \longrightarrow \infty$ and is therefore bounded by some $\kappa_N > 0$. Hence we get

$$k^{N(1-R)} \leq \kappa_N \sqrt{k!}^{1-R},$$

and together with Lemma 6.1.2 we find

$$p_R(x \star_z \eta) \leq (k+1)^R k!^{\frac{1-R}{2}} c\kappa_N p_R(x) p(\eta)$$

for any tensor x of degree at most k. Now, we can iterate this result for $\xi, \eta \in \mathfrak{g}$, $R \geq 0$, $k, \ell \in \mathbb{N}$:

$$\begin{aligned}
p_R\Big(\xi^{\otimes k} star_z \eta^{\otimes \ell}\Big) &= p_R\Big(\xi^{\otimes k} star_z \eta^{\star_z \ell}\Big) \\
&\leq (k+\ell)^R (k+\ell-1)!^{\frac{1-R}{2}} c\kappa_N p_R\Big(\xi^{\otimes k} star_z \eta^{\star_z(\ell-1)}\Big) p(\eta) \\
&\leq \;\vdots \\
&\leq \left(\frac{(k+\ell)!}{k!}\right)^R (k+\ell-1)!^{\frac{1-R}{2}} \dots k!^{\frac{1-R}{2^N}} (c\kappa_N)^\ell p_R\Big(\xi^{\otimes k}\Big) p(\eta)^\ell \\
&\leq \binom{k+\ell}{k}^R \ell!^R (k+\ell)!^{\frac{(2^N-1)(1-R)}{2^N}} (c\kappa_N)^\ell p_R\Big(\xi^{\otimes k}\Big) p(\eta)^\ell \\
&\leq 2^{(k+\ell)R} \ell!^R k!^{\frac{(2^N-1)(1-R)}{2^N}} \ell!^{\frac{(2^N-1)(1-R)}{2^N}} 2^{(k+\ell)\frac{(2^N-1)(1-R)}{2^N}} (c\kappa_N)^\ell p_R\Big(\xi^{\otimes k}\Big) p(\eta)^\ell \\
&\leq (2p)_{R+\epsilon}\Big(\xi^{\otimes k}\Big) (2c\kappa_N p)_{R+\epsilon}\Big(\eta^{\otimes \ell}\Big),
\end{aligned}$$

where we have set $\epsilon = \frac{(2^N-1)(1-R)}{2^N}$. From this, we have clearly $R + \epsilon < 1$, and we get the wanted result for the projective limit. □

Of course, the projective limit case gives us a bigger completion. We immediately end up with the following result:

Corollary 6.1.3 *Let $\mathfrak{g}$ be a nilpotent, locally convex Lie algebra.*

i.) Let $\exp(\xi)$ be the exponential series for $\xi \in \mathfrak{g}$, the we have $\exp(t\xi) \in \widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g})$ for all $t \in \mathbb{K}$.

ii.) For $\xi, \eta \in \mathfrak{g}$ and $z \neq 0$ we have $\exp(\xi) \star_z \exp(\eta) = \exp\big(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\big)$.

iii.) For $s, t \in \mathbb{K}$ and $\xi \in \mathfrak{g}$ we have $\exp(t\xi) \star_z \exp(s\xi) = \exp((t+s)\xi)$.

PROOF: For the first part, recall that the completion of the projective limit 1^- will contain all those series $(a_n)_{n \in \mathbb{N}_0}$ such that

$$\sum_{n=0}^{\infty} a_n n!^{1-\epsilon} c^n < \infty$$

for all $c > 0$. This is the case for the exponential series of $t\xi$ for $t \in \mathbb{K}$ and $\xi \in \mathfrak{g}$. The second part follows from the fact that all the projections π_n onto the homogeneous subspaces S^n_π are continuous. The third part is then a direct consequence of the second. □

6.1.2 Representations and Functoriality

In the general AE case, we had some useful results concerning representations of Lie algebras and the functorialty of our construction. These results can be extended to the projective limit $\mathrm{S}^{\bullet}_{1^-}(\mathfrak{g})$.

Proposition 6.1.4 (universal property) *Let $\mathfrak{g}$ be a locally convex nilpotent Lie algebra, $\mathscr{A}$ an associative AE algebra and $\phi\colon\ :\mathfrak{g}\longrightarrow\mathscr{A}$ is a continuous homomorphism of Lie algebras. Then, the lifted homomorphisms from $\mathrm{S}^{\bullet}_{1^-}(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g}_z)$ to $\mathscr{A}$ are continuous.*

Proof: The proof is exactly the same as in the general AE case, since there, $R \geq 0$ was enough.□

Again, this construction will be not a universal in the categorial sense, since $\mathrm{S}^{\bullet}_{1^-}(\mathfrak{g})$ fails to be AE. But also here, we get the case of continuous representations into a Banach space (and in particular into a finite-dimensional space) as a corollary.

Corollary 6.1.5 (Continuous Representations) *Let $\mathcal{U}_R(\mathfrak{g})$ the universal enveloping algebra of locally convex nilpotent Lie algebra $\mathfrak{g}$, then for every continuous representation ϕ of $\mathfrak{g}$ into the bounded linear operators $\mathfrak{B}(V)$ on a Banach space V, the induced homomorphism of associative algebras $\Phi\colon\mathcal{U}(\mathfrak{g})\longrightarrow\mathfrak{B}(V)$ is continuous.*

We can also extend the functoriality statement to the projective limit, but we need to get another version of Lemma **??** for nilpotent Lie algebras, since thisis the corner stone of the functoriality proof.

Lemma 6.1.6 *Let $\mathfrak{g}$ be locally convex nilpotent Lie algebra and $N\in\mathbb{N}$ such that $N+1$ Lie brackets vanish, $0\leq R<1$ and $z\in\mathbb{C}$. Then for p a continuous seminorm, q an asymptotic estimate, $n\in\mathbb{N}$ and all $\xi_1,\ldots,\xi_n\in\mathfrak{g}$ the following estimate*

$$p_R(\xi_1\star_z\cdots\star_z\xi_n)\leq c^n n!^{R+\epsilon}q^n(\xi_1\otimes\cdots\otimes\xi_n) \tag{6.1.4}$$

holds with $c=16\mathrm{e}^2(|z|+1)$ and $\epsilon=\frac{N-1}{N}(1-R)$ and the estimate is locally uniform in z.

Proof: We take $R<1$ and go directly into the proof of Lemma **??** at (5.3.12). We know that, since we may have at most N brackets, also the values for ℓ are restricted to

$$\ell\leq\frac{N-1}{N}n=\delta n$$

Using that in the proof of Lemma **??** leads to

$$\begin{aligned}
&p_R(\xi_1\star_z\cdots\star_z\xi_n)\\
&\quad\leq\sum_{\ell=0}^{\delta n}(n-\ell)!^R\sum_{\substack{1\leq j\leq n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1}i_j=\ell}}|z|^\ell\frac{1!(2-i_1)!\cdots(n-1-i_1-\cdots-i_{n-2})!}{(1-i_1)!\cdots(n-1-i_1-\cdots-i_{n-1})!}q(\xi_1)\cdots q(\xi_n)\\
&\quad\leq\sum_{\ell=0}^{\delta n}(n-\ell)!^R\sum_{\substack{1\leq j\leq n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1}i_j=\ell}}|z|^\ell(2\mathrm{e})^n\ell!q(\xi_1)\cdots q(\xi_n)\\
&\quad\leq(2\mathrm{e})^n(|z|+1)^n q(\xi_1)\cdots q(\xi_n)\sum_{\ell=0}^{\delta n}(n-\ell)!^R\ell!\binom{n+\ell-2}{\ell-1}
\end{aligned}$$

We have

$$\ell! = \ell!^R \ell!^{1-R} \le \ell!^R \Big((\delta n)^{\delta n} \Big)^{1-R} \le \ell!^R n^{\delta n(1-R)} \le \ell!^R n!^{\delta(1-R)} \mathrm{e}^{\delta n(1-R)}.$$

Together with $\ell!^R (n-\ell)!^R \le n!^R$ this gives

$$\begin{aligned} p_R(\xi_1 \star_z \cdots \star_z \xi_n) &\le (2\mathrm{e})^n (|z|+1)^n n!^R n!^{\delta(1-R)} q(\xi_1) \cdots q(\xi_n) \sum_{\ell=0}^{\delta n} \binom{n+\ell-2}{\ell-1} e^{\delta n(1-R)} \\ &\le (2\mathrm{e})^n (|z|+1)^n \Big(\mathrm{e}^{(1-R)\delta} \Big)^n 4^n n!^{R+\epsilon} q(\xi_1) \cdots q(\xi_n), \end{aligned}$$

with $\epsilon = \delta(1-R)$. It is clear that for all $R < 1$ we have $R + \epsilon < 1$. Set

$$c = 8\mathrm{e}(|z|+1)\mathrm{e}^{(1-R)\delta} \le 16\mathrm{e}^2(|z|+1)$$

and note that the estimate is locally uniform in z, even though it will not be uniform in z. □

Proposition 6.1.7 *Let $R \ge 1$, $\mathfrak{g}, \mathfrak{h}$ be two locally convex nilpotent Lie algebras and $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ a continuous homomorphism between them. Then it lifts to a continuous unital homomorphism of locally convex algebras $\Phi_z\colon \mathscr{U}_R(\mathfrak{g}_z) \longrightarrow \mathscr{U}_R(\mathfrak{h}_z)$.*

PROOF: The proof is mostly analogous to the one of Proposition 5.3.14. □

6.2 Module structures

The projective limit 1^- is not the only additional structure we will get, if our Lie algebra $\mathfrak{g}$ is nilpotent. Lemma 6.1.2 allows the existence of certain module structures, if the seminorms on $\mathfrak{g}$ are in addition submultiplicative. For every $R \in \mathbb{R}$,the symmetric tensor algebra $\mathrm{S}^\bullet_R(\mathfrak{g})$ is a locally convex vector space. For $R \ge 0$, the (symmetric) tensor product is continuous, which is very important for many estimates, and for $R \ge 1^-$, we have an algebra structure. In between however, we have more than "only" vector spaces: The spaces $\mathrm{S}^\bullet_R(\mathfrak{g})$ form locally convex modules over the $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ for certain values of R'. The next proposition will makes this more exact.

Proposition 6.2.1 (Bimodules in $\mathrm{S}^\bullet_R(\mathfrak{g})$) *Let $\mathfrak{g}$ be a nilpotent, locally m-convex Lie algebra such that $N+1$ Lie brackets vanish, $z \in \mathbb{C}$ and $0 \le R < 1$. Then, for all $x, y \in \mathrm{S}^\bullet(\mathfrak{g})$ and every continuous seminorm p, we have the estimates*

$$p_R(x \star_z y) \le \big(2^{N+1}p\big)_R(x) \big(2^{N+1}cp\big)_{R+N(1-R)}(y) \tag{6.2.1}$$

and

$$p_R(x \star_z y) \le \big(2^{N+1}cp\big)_{R+N(1-R)}(x) \big(2^{N+1}p\big)_R(y) \tag{6.2.2}$$

with $c = \sum_{n=0}^N \frac{|B_n^*||z|^n}{n!}$. Hence, the vector space $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ forms a bimodule over the algebra $\widehat{\mathrm{S}}^\bullet_{R+N(1-R)}(\mathfrak{g})$. In particular, if $\mathfrak{g}$ is 2-step nilpotent, the vector space $\widehat{\mathrm{S}}^\bullet_0(\mathfrak{g})$ is a $\widehat{\mathrm{S}}^\bullet_1(\mathfrak{g})$-bimodule.

PROOF: Again, we do the calculation on factorizing tensors: Let $\xi, \eta \in \mathfrak{g}$, $R \ge 0$, $k, \ell \in \mathbb{N}$. Using Lemma 6.1.2, we get

$$\begin{aligned} p_R\Big(\xi^{\otimes k} star_z \eta^{\otimes \ell}\Big) &= p_R\Big(\xi^{\otimes k} star_z \eta^{\star_z \ell}\Big) \\ &\le (k+\ell)^R (k+\ell-1)^{N(1-R)} p_R\Big(\xi^{\otimes k} star_z \eta^{\star_z(\ell-1)}\Big) p(\eta) \end{aligned}$$

$$
\begin{aligned}
&\leq \vdots \\
&\leq \left(\frac{(k+\ell)!}{k!}\right)^{R}\left(\frac{(k+\ell-1)!}{(k-1)!}\right)^{N(1-R)} c^{\ell} p_R\Big(\xi^{\otimes k}\Big) p(\eta)^{\ell} \\
&\leq 2^{k+l} 2^{N(k+\ell)} \ell!^{N(1-R)} c^{\ell} p_R\Big(\xi^{\otimes k}\Big) p(\eta)^{\ell} \\
&= \left(2^{N+1}p\right)_R\Big(\xi^{\otimes k}\Big)\left(2^{N+1}cp\right)_{R+N(1-R)}\Big(\eta^{\otimes \ell}\Big).
\end{aligned}
$$

The proof of the second estimate is analogous. □

Remark 6.2.2 (Possible extensions) This result immediately poses new questions, like the dependence on the formal parameter in this case, possible generalizations to "weaker forms" of nilpotency and so on. They may be issues of some future work, but can't be addressed here, since we rather want to present something like a part of the "big picture" which is opened by the R-topology, instead of getting lost in its details too much. There are, without any doubt, questions that are more significant than extending those estimates to very special cases and finding sharp bounds there, although this is interesting and important, too.

Although it seems clear from the construction, that these bimodules cannot be there for general Lie algebras, we can give a concrete counter-example, which shows that there are Lie algebras, which don't allow them.

Example 6.2.3 Choose $R < 1$ and take $\mathfrak{g} = \mathbb{R}^3$ with the basis e_1, e_2, e_3 and the vector product as Lie bracket:

$$[e_1, e_2] = e_3 \qquad [e_2, e_3] = e_1 \qquad [e_3, e_1] = e_2$$

Again, we take a ℓ^1-norm n such that $n(e_1) = n(e_2) = n(e_3) = 1$. It has the nice property that for $k, \ell, m \in \mathbb{N}$ we get on the projective tensor product

$$n^{k+\ell+m}\Big(e_1^k e_2^\ell e_3^m\Big) = 1.$$

Now we define the sequence $(a_k)_{k\in\mathbb{N}}$

$$a_k = \frac{1}{k!^R} e_1^k,$$

for which we get $n_R(a_k) = 1$. Now, we want to show that $a_k \star_z e_2$ grows faster than exponentially:

$$
\begin{aligned}
n_R(a_k \star_z e_2) &= n_R\left(\sum_{j=0}^{k}\binom{k}{j} B_j^* \frac{1}{k!^R} e_1^{n-j} (\mathrm{ad}_{e_1})^j (e_2)\right) \\
&= \sum_{j=0}^{k}\binom{k}{j} |B_j^*| \frac{1}{k!^R}(k-j+1)!^R \underbrace{n^{k-j}(e_1(e_2 \wedge e_3))}_{=1} \\
&= \sum_{j=0}^{k}(k-j+1)^R \binom{k}{j} \frac{|B_j^*|}{j!} \frac{(k-j)!^R j^R}{k!^R} j!^{1-R} \\
&= \sum_{j=0}^{k}(k-j+1)^R \binom{k}{j}^{1-R} \frac{|B_j^*|}{j!} j!^{1-R} \\
&\geq \sum_{j=0}^{k} \frac{|B_j^*|}{j!} j!^{1-R}
\end{aligned}
$$

$$\geq \frac{|B_k^*|}{k!^R}.$$

We know, that for $R < 1$ and any $c > 0$

$$\limsup_{n \longrightarrow \infty} \frac{|B_n^*|}{c^n n!^R} = \infty,$$

and hence the limes superior of $n_R(a_k \star_z e_2)$ grows faster than any exponential function.

6.3 The Heisenberg and the Weyl algebra

Now we want to see how we get the link to the Weyl algebra from [34], since we have something like a discrepancy for the parameter R concerning the continuity of the product in the Weyl and the Heisenberg algebra. In the following, we will show that this gap actually makes a lot of sense. For simplicity, we consider the easiest case of the Weyl/Heisenberg algebra with two generators Q and P, but the generalization to the Heisenberg [Weyl] algebra in $2n+1$ [$2n$] dimensions is immediate and goes without problems. Recall that the Weyl algebra is a quotient of the enveloping algebra of the Heisenberg algebra $\mathfrak{h}$ which one gets from dividing out its center. So let $\mathsf{h} \in \mathbb{C}$ and we have a projection

$$\pi \colon \widehat{\mathrm{S}}^\bullet_R(\mathfrak{h}) \longrightarrow \widehat{\mathcal{W}}_R(\mathfrak{h}) = \left(\frac{\mathrm{S}^\bullet_R(\mathfrak{h})}{\langle E - \mathsf{h}\mathbb{1} \rangle} \right)^{\widehat{}} \tag{6.3.1}$$

Of course we want to know if this projection is continuous.

Proposition 6.3.1 *The projection π is continuous for $R \geq 0$.*

PROOF: We extend π to the whole tensor algebra by symmetrizing beforehand. Let then p be a continuous seminorm on $\mathfrak{h}$, $k, \ell, m \in \mathbb{N}_0$. We have

$$\begin{aligned} p_R(\pi(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m})) &= p_R(Q^k P^\ell \mathsf{h}^m) \\ &= |\mathsf{h}|^m (k+\ell)!^R p^{k+\ell}(Q^k P^\ell) \\ &\leq (|\mathsf{h}|+1)^{k+\ell+m} (k+\ell+m)!^R p(Q)^k p(P)^\ell p(E)^m \\ &= ((|\mathsf{h}|+1)p)_R(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}). \end{aligned}$$

Then we do the usual infimum argument and have the result on arbitrary tensors again. □

To establish the link to the continuity results of the Weyl algebra, we need more: $\pi \circ \star_z$ should to be continuous for $R \geq \frac{1}{2}$.

Proposition 6.3.2 *Let $R \geq \frac{1}{2}$ and π the projection from* (6.3.1)*. Then the map $\pi \circ \star_z$ is continuous.*

PROOF: We need to get the estimate on factorizing tensors: Let p be a continuous seminorm, q an asymptotic estimate and $k, k', \ell, \ell', m, m' \in \mathbb{N}_0$. Then we have to get an estimate for

$$p_R\Big(\pi\Big(Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'}\Big)\Big).$$

If we calculate the star product explicitly, we see, that we only get Lie brackets where we have P's and Q's. Let $r = k+\ell+m$ and $s = k'+\ell'+m'$, then we can actually simplify the calculations by

$$p_R\Big(\pi(Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'})\Big) = (p_R \circ \pi)\left(\sum_{n=0}^{r+s-1} z^n C_n(Q^k P^\ell E^m, Q^{k'} P^{\ell'} E^{m'}) \right)$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{r+s-1} |z|^n (p_R \circ \pi)\Big(C_n(Q^k P^\ell E^m, Q^{k'} P^{\ell'} E^{m'})\Big) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n (p_R \circ \pi)(C_n(Q^r, P^s)) \\
&= \sum_{n=0}^{r+s-1} |z|^n \frac{r!s!}{(r-n)!(s-n)!n!} (p_R \circ \pi)\big(Q^{r-n} P^{s-n} E^n\big) \\
&= \sum_{n=0}^{r+s-1} |z|^n |\mathsf{h}|^n \frac{r!s!}{(r-n)!(s-n)!n!} p_R\big(Q^{r-n} P^{s-n}\big) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n |\mathsf{h}|^n \frac{r!s!}{(r-n)!(s-n)!n!} \frac{(r+s-2n)!^R}{r!^R s!^R} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n |\mathsf{h}|^n \binom{r}{n}\binom{s}{n} \frac{(r+s-2n)!^R n!}{r!^R s!^R} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n |\mathsf{h}|^n \binom{r}{n}\binom{s}{n} \frac{(r+s-2n)!^R n!}{r!^R s!^R} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\overset{\text{(a)}}{\leq} \sum_{n=0}^{r+s-1} |z|^n |\mathsf{h}|^n \binom{r}{n}\binom{s}{n} \binom{r+s}{s}^R \binom{r+s}{2n}^{-R} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\leq \sum_{n=0}^{r+s-1} (|z|+1)^n (|\mathsf{h}|+1)^n 4^{r+s} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\leq \underbrace{(8(|z|+1)(|c|+1))^{r+s}}_{=c^{r+s}} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&= (cp)_R\big(Q^{\otimes r}\big)(cp)_R\big(P^{\otimes s}\big) \\
&\overset{\text{(b)}}{=} (cp)_R\Big(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}\Big)(cp)_R\Big(Q^{\otimes k'} \otimes P^{\otimes \ell'} \otimes E^{\otimes m'}\Big),
\end{aligned}
$$

where in (a) we expanded the fraction with $(r+s)!^R$ to get the two binomial coefficients. It is clear, that this step just works for $R \geq \frac{1}{2}$. In (b) we used the fact that we can ask for $p(Q) = p(P) = p(E)$. Now we just need to use

$$
\Big(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}\Big) \star_z \Big(Q^{\otimes k'} \otimes P^{\otimes \ell'} \otimes E^{\otimes m'}\Big) = Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'}
$$

in the first line and we are done, since we can use again the infimum argument to expand this estimate to all tensors. $\square$

The previous proposition can be seen as something like the "finite-dimensional version" of Lemma 3.10 in [34], just that we took a large detour for proving it. One could, most probably, redo some more results of this paper using finite-dimensional versions the Heisenberg algebra and the projection onto the Weyl algebra, but this would yield, also most probably, nothing new. It is good to know that this connections exists, but it is not something which is very helpful to pursue, since an evident generalization to infinite dimensions doesn't seem be quite easy.

6.4 Banach-Lie algebras

Now we want to focus a bit on weaker notions than true nilpotency. Since there are many of them, we want to focus on the easier case of Banach-Lie algebras, where a certain classification and quite a lot of results already exist.

6.4.1 Generalizations of nilpotency

In [?], Müller gives a list of weaker forms of nilpotency in associative Banach algebras. We can mostly copy the ideas and use it for Banach-Lie algebras, too

Definition 6.4.1 *Let $\mathfrak{g}$ be a Banach-Lie algebra in which the Lie bracket fulfils the estimate*

$$\|[\xi,\eta]\| \leq \|\xi\|\|\eta\|.$$

Denote by $\mathbb{B}_1(0)$ all elements $\xi \in \mathfrak{g}$ with $\|\xi\| = 1$. We say that

i.) $\mathfrak{g}$ is topologically nil (or radical, or quasi-nilpotent), if every $\xi \in \mathfrak{g}$ is quasi-nilpotent, i.e.

$$\lim_{n\longrightarrow\infty} \|\mathrm{ad}_\xi^n\|^{\frac{1}{n}} = 0.$$

ii.) $\mathfrak{g}$ is uniformly topologically nil, if

$$\lim_{n\longrightarrow\infty} \mathcal{N}_1(n) = 0.$$

for

$$\mathcal{N}_1(n) = \sup\Big\{ \|\mathrm{ad}_\xi^n\|^{\frac{1}{n}} \Big| \xi \in \mathbb{B}_1(0)\Big\}. \tag{6.4.1}$$

iii.) $\mathfrak{g}$ is topologically nilpotent, if for every sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{B}_1(0)$ we have

$$\lim_{n\longrightarrow\infty} \|\mathrm{ad}_{\xi_1} \circ \ldots \circ \mathrm{ad}_{\xi_n}\|^{\frac{1}{n}} = 0.$$

iv.) $\mathfrak{g}$ is uniformly topologically nilpotent, if

$$\lim_{n\longrightarrow\infty} \mathcal{N}(n) = 0.$$

for

$$\mathcal{N}(n) = \sup\Big\{ \|\mathrm{ad}_{\xi_1} \circ \ldots \circ \mathrm{ad}_{\xi_n}\|^{\frac{1}{n}} \Big| \xi_1,\ldots,\xi_n \in \mathbb{B}_1(0)\Big\}. \tag{6.4.2}$$

It is clear that $(ii) \Rightarrow (i)$ and $(iv) \Rightarrow (iii)$. In the associative case, we have $(iii) \Leftrightarrow (iv)$ and hence $(iii) \Rightarrow (ii)$. Of course, it is a good question, if this remains true for Banach-Lie algebras. We have already encountered notion (i): Wojtyński proved it to be equivalent to the fact that the BCH series converges globally in [?]. In the following, we will make use of notion (iv): we will show, that it is possible to generalize the result of Theorem 6.1.1 to this case.

6.4.2 An adapted R-topology

The idea will be to slightly change the topology: Instead of taking $n!^R$ as weights with $0 \leq R < 1$, we take sequence $(\alpha_n)_{n\in\mathbb{N}}$ (or just (α), for short) with a certain asymptotic behaviour for $n \longrightarrow \infty$ and use $\frac{n!}{\alpha_n}$ as weights. This will generalize the idea of $n!^R$ and be the starting point for the estimates.

First, we observe that every uniformly topologically nilpotent Banach-Lie algebra $\mathfrak{g}$ comes with a characteristic, monotonously decreasing sequence

$$\omega_n = \sup_{m\geq n} \mathcal{N}(m). \tag{6.4.3}$$

If there exists a $N \in \mathbb{N}$, such that $\omega_n = 0$ for all $n \geq N$, then $\mathfrak{g}$ is actually nilpotent and we can use the results of the first section in this chapter. We may hence restrict to those Banach-Lie algebras, where we have $\omega_n > 0$ for all $n \in \mathbb{N}$. This allows the next definition.

Definition 6.4.2 (Rapidly increasing sequences) *Let $\mathfrak{g}$ be a uniformly topologically nilpotent Banach-Lie algebra and (ω) the sequence defined in (6.4.3). Then we define the characteristic sequence $\chi(\mathfrak{g})$ of $\mathfrak{g}$ by*

$$\chi(\mathfrak{g})_n = \max\left\{\frac{1}{\omega_n}, 2\right\}. \tag{6.4.4}$$

We moreover say, that a sequence (α) in $(1, \infty)$ is $\mathfrak{g}$- rapidly increasing, if it fulfils the following properties:

i.) It grows fast than exponentially, i.e.

$$\lim_{n\longrightarrow\infty} \frac{\log(\alpha_n)}{n} = \infty.$$

ii.) There exists a constant $c > 0$, such that

$$\alpha_n \leq c^n \chi(\mathfrak{g})_n.$$

We denote by $\mathscr{I}_{\mathfrak{g}}$ the set of all $\mathfrak{g}$-rapidly increasing sequences.

Clearly, $(\chi(\mathfrak{g}))$ is a $\mathfrak{g}$-rapidly increasing series itself.

Remark 6.4.3 The number 2 in (6.4.4) may look a bit confusing at the first sight, since it is somehow arbitrary, but for technical reasons, we will need $\chi(g) > 1$ later. So actually every real number $c > 1$ could have been put there. In this sense, the previous definition is rather a technical tool than a "general idea".

Each $(\alpha) \in \mathscr{I}_{\mathfrak{g}}$ gives a continuous seminorm p_α on the tensor algebra with the projective tensor product $\mathrm{T}^\bullet_\pi(\mathfrak{g})$ by

$$p_\alpha = \sum_{n=0}^{\infty} \frac{n!}{\alpha_n} \|\cdot\|^{\otimes_\pi n},$$

and we denote the set of all those seminorms by $\mathscr{P}_{\mathfrak{g}}$. Furthermore, For every sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{B}_1(0)$ we get

$$\|[\ldots[[\xi_1, \xi_2], \xi_3], \ldots \xi_n]\| \leq \frac{2^n}{\chi(\mathfrak{g})_n}.$$

Every rapidly increasing sequence (α) yields a continuous function f_α by

$$f_\alpha \colon \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+, \quad f_\alpha(0) = 2,\ f_\alpha(n) = \log(\alpha_n),\ \forall_{n\in\mathbb{N}}, \tag{6.4.5}$$

and linear interpolation between the values at the integers. The idea behind is that this will allow us to use a lemma, which we now introduce. It is taken from a work [24] by Mitiagin, Rolewicz and Żelazko, where it is stated in Lemma 2.1 and Lemma 2.2.

Lemma 6.4.4 *Let $f \colon \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+$ be a continuous functions, such that*

$$\lim_{n\longrightarrow\infty} \frac{f(x)}{x} = \infty. \tag{6.4.6}$$

Then there exists a convex, continuous function $g \colon \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+$, fulfilling (6.4.6) and

$$g(t_1 + \ldots + t_n) \leq 8(g(t_1) + \ldots + g(t_n)) + f(n), \quad \forall_{n\in\mathbb{N}} \text{ and all } t_i \in \mathbb{R}^+_0. \tag{6.4.7}$$

It is clear that Lemma 6.4.4 is applicable to our function f_α.

6.4.3 A new continuity result

Now, we have finally prepared our toolbox well enough to prove a new result.

Proposition 6.4.5 *Let $\mathfrak{g}$ be a uniformly topologically nilpotent Banach-Lie algebra, $\alpha \in \mathscr{I}_{\mathfrak{g}}$, p_α the corresponding seminorm according to* (6.4.5) *and $z \in \mathbb{C}$. Then, there exists a series $\beta = (\beta_n)_{n\in\mathbb{N}} \in \mathscr{I}_{\mathfrak{g}}$, such that for all $x, y \in \mathrm{T}^\bullet_\pi(\mathfrak{g})$ we have the estimate*

$$p_\alpha(x \star_z y) \leq (cp)_\beta(x)(cp)_\beta(y) \tag{6.4.8}$$

with a $c > 0$, which only depends on α.

Proof: We can again do the estimate on factorizing tensors and extend it with the infimum argument later. Let hence $k, \ell \in \mathbb{N}$ and $\xi, \eta \in \mathfrak{g}$. we need to estimate the C_n-operators for $n = 0, 1, \ldots, k+\ell-1$ and note therefore $r = k+\ell-n$ again. We get for $p_\alpha \in \mathscr{P}_{\mathfrak{g}}$

$$\begin{aligned}
p_\alpha\Big(C_n\Big(\xi^{\otimes k}, \eta^{\otimes \ell}\Big)\Big) &= p_\alpha\Bigg(\frac{k!\ell!}{r!} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \mathrm{BCH}_{a_1,b_1}(\xi,\eta)\ldots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)\Bigg)\\
&\leq \frac{k!\ell!}{r!}\frac{r!}{\alpha_r} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \|\mathrm{BCH}_{a_1,b_1}(\xi,\eta)\|\ldots\|\mathrm{BCH}_{a_r,b_r}(\xi,\eta)\|\\
&\leq \frac{k!\ell!}{\alpha_r} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} 2^r \frac{1}{\chi(\mathfrak{g})_{a_1+b_1}}\cdots\frac{1}{\chi(\mathfrak{g})_{a_r+b_r}}\|\xi\|^k\|\eta\|^\ell,
\end{aligned}$$

where we have used the estimate from Lemma 5.3.1 *(iii)* in the last step. Rearranging this, we have

$$p_\alpha\Big(C_n\Big(\xi^{\otimes k}, \eta^{\otimes \ell}\Big)\Big) \leq k!\ell! 2^r \|\xi\|^k\|\eta\|^\ell \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \frac{1}{\alpha_r\cdot\chi(\mathfrak{g})_{a_1+b_1}\cdots\chi(\mathfrak{g})_{a_r+b_r}},$$

and we would like to find a $(\beta) \in \mathscr{I}_{\mathfrak{g}}$ such that

$$\sup\left\{\frac{\beta_k\cdot\beta_\ell}{\alpha_r\cdot\chi(\mathfrak{g})_{a_1+b_1}\cdots\chi(\mathfrak{g})_{a_r+b_r}}\;\middle|\; k,\ell\in\mathbb{N},\; a_i+b_i\geq 1,\; \sum_i a_i = k,\; \sum_j b_j = \ell\right\} \leq \kappa^{k+\ell} \tag{6.4.9}$$

for some $\kappa > 0$, just depending on (α). Then we would have

$$\begin{aligned}
p_\alpha\Big(C_n\Big(\xi^{\otimes k}, \eta^{\otimes \ell}\Big)\Big) &\leq \frac{k!\ell!}{\beta_k\beta_\ell} 2^r \|\xi\|^k\|\eta\|^\ell \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \kappa^{k+\ell}\\
&= 2^{-n}(2\kappa)^{k+\ell} p_\beta\Big(\xi^{\otimes k}\Big) p_\beta\Big(\eta^{\otimes \ell}\Big) \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} 1
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\cdot 8^n}(16\kappa)^{k+\ell} p_\beta\left(\xi^{\otimes k}\right) p_\beta\left(\eta^{\otimes \ell}\right)\\
&= \frac{1}{2\cdot 8^n}(16\kappa p)_\beta\left(\xi^{\otimes k}\right)(16\kappa p)_\beta\left(\eta^{\otimes \ell}\right).
\end{aligned}$$

From this we could conclude analogously to the procedure in the proof of Theorem 5.3.2 and the statement would be proven. We now need to show the existence of a $(\beta) \in \mathscr{I}_{\mathfrak{g}}$ and a $\kappa > 0$, such that (6.4.9) holds.

Claim 6.4.6 *For f_α defined as in* (6.4.5)*, we take the function g we get from Lemma 6.4.4. Then the sequence (β) defined by*

$$\beta_n = \exp\left(\frac{g(n)}{8}\right) \tag{6.4.10}$$

and $\kappa = c' \mathrm{e}^{8g(1)}$ have the desired properties, where $c' > 0$ is a constant, such that $\alpha_n \leq c'^n \chi(\mathfrak{g})_n$.

Proof: First note that there is a fixed $c' > 0$

$$\frac{1}{\chi(\mathfrak{g})_n} \leq \frac{c'^n}{\alpha_n}$$

Denote $a_i + b_i = n_i$. Then we have

$$\frac{\beta_k \cdot \beta_\ell}{\alpha_r \cdot \chi(\mathfrak{g})_{n_1} \cdots \chi(\mathfrak{g})_{n_r}} \leq \frac{c'^{k+\ell}\beta_k \cdot \beta_\ell}{\alpha_r \cdot \alpha_{n_1} \cdots \alpha_{n_r}}$$

and hence

$$\begin{aligned}
\log\left(\frac{\beta_k \cdot \beta_\ell}{\alpha_r \cdot \chi(\mathfrak{g})_{a_1+b_1} \cdots \chi(\mathfrak{g})_{a_r+b_r}}\right) &\leq \log\left(\frac{c'^{k+\ell}\beta_k \cdot \beta_\ell}{\alpha_r \cdot \alpha_{n_1} \cdots \alpha_{n_r}}\right)\\
&= (k+\ell)\log(c') + \frac{g(k)}{8} + \frac{g(\ell)}{8} - f_\alpha(r) - f_\alpha(n_1) - \ldots - f_\alpha(n_r)\\
&\overset{(a)}{\leq} (k+\ell)\log(c') + \frac{g(k+\ell)}{8} - f_\alpha(r) - f_\alpha(n_1) - \ldots - f_\alpha(n_r)\\
&\overset{(b)}{\leq} (k+\ell)\log(c') + g(n_1) + \ldots + g(n_r) + \frac{f_\alpha(r)}{8}\\
&\qquad - f_\alpha(r) - f_\alpha(n_1) - \ldots - f_\alpha(n_r)\\
&\overset{(c)}{\leq} (k+\ell)\log(c') + (g(n_1) - f_\alpha(n_1)) + \ldots + (g(n_r) - f_\alpha(n_r))\\
&\overset{(d)}{\leq} (k+\ell)\log(c') + 8g(1)n_1 + \ldots + 8g(1)n_r\\
&= (k+\ell)(\log(c') + 8g(1)),
\end{aligned}$$

where we have used the convexity of g in (a), Lemma 6.4.4 in (b) and $\frac{g(n)}{8} \leq g(n)$ in (c). In (d) we used the lemma again by estimating

$$g(n) = g(1 + \ldots - 1) \leq 8ng(1) + f_\alpha(n).$$

This shows that $\kappa = c'\mathrm{e}^{8g(1)}$ really gives the estimate. ▽

Now, we only need to sum up the $|z|^n C_n(\xi^{\otimes} k, \eta^{\otimes \ell})$. We know this is possible from Chapter 5, and we finally get a continuity estimate which is locally uniform in z. □

At the end of Chapter 5, we showed that a finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if its universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ admitted a locally convex topology, such that the following three things are fulfilled:

i.) The product in $\mathscr{U}(\mathfrak{g})$ is continuous.

ii.) For every $\xi \in \mathfrak{g}$ the series $\exp(\xi)$ converges absolutely in the completion of $\mathscr{U}(\mathfrak{g})$.

iii.) Pulling back the topology to the symmetric tensor algebra, the projection and inclusion maps with respect to the graded structure

$$\mathrm{S}^\bullet(\mathfrak{g}) \xrightarrow{\pi_n} \mathrm{S}^n(\mathfrak{g}) \xrightarrow{\iota_n} \mathrm{S}^\bullet(\mathfrak{g})$$

are continuous for all $n \in \mathbb{N}$.

For Banach-Lie algebras, we came quite close to a similar statement: We know from Proposition 5.4.1 and the result of Wojtyǹski, that a Banach-Lie algebra must at least be *topologically nil* to satisfy the three upper points. We also know, that a uniformly topologically nilpotent Banach-Lie algebra $\mathfrak{g}$ allows us to construct such a locally convex topology on $\mathscr{U}(\mathfrak{g})$ explicitly. Maybe it is possible to find a notion of generalized nilpotency, which characterizes those three points exactly.

Chapter 7

The Hopf algebra structure

7.1 Everything works

In the above sections we discussed a locally convex topology on the universal enveloping algebra considered as an associative algebra, which makes the Gutt star product continuous. In the following we investigate the continuity of the structures of Hopf structure on $\mathscr{U}_R(\mathfrak{g}_z)$. For this purpose, we need an explicit formula for the antipode

$$S_z\colon \mathscr{U}(\mathfrak{g}_z) \longrightarrow \mathscr{U}(\mathfrak{g}_z) \tag{7.1.1}$$

and the coproduct

$$\Delta_z\colon \mathscr{U}(\mathfrak{g}_z) \longrightarrow \mathscr{U}(\mathfrak{g}_z) \otimes \mathscr{U}(\mathfrak{g}_z) \tag{7.1.2}$$

in the universal enveloping algebra. We pull them back on the symmetric algebra and extend them to the whole tensor algebra by symmetrizing beforehand. We set

$$\mathcal{S}_z\colon\ \mathrm{T}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad \mathcal{S}_z = \mathfrak{q}_z^{-1} \circ S_z \circ \mathfrak{q}_z \circ \mathscr{S}\,. \tag{7.1.3}$$

and

$$\Delta_z\colon\ \mathrm{T}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}) \otimes \mathrm{S}^\bullet(\mathfrak{g}), \quad \Delta_z = (\mathfrak{q}_z^{-1} \otimes \mathfrak{q}_z^{-1}) \circ \Delta_z \circ \mathfrak{q}_z \circ \mathscr{S}\,. \tag{7.1.4}$$

Now we can prove two explicit formulas.

Lemma 7.1.1 *For $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ we have the identities*

$$\mathcal{S}_z(\xi_1 \otimes \cdots \otimes \xi_n) = (-1)^n \xi_1 \cdots \xi_n \tag{7.1.5}$$

and

$$\Delta_z(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{I \subseteq \{1,\ldots,n\}} \xi_I \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n \tag{7.1.6}$$

where ξ_I denotes the symmetric tensor product of all ξ_i with $i \in I$ and $\widehat{\xi_I}$ means that the ξ_i with $i \in I$ are left out.

PROOF: First, we derive Formula 7.1.5: the antipode gives $S_z(\xi) = -\xi$ for $\xi \in \mathfrak{g}$ and extends to $\mathscr{U}(\mathfrak{g}_z)$ by algebra antihomomorphism, hence

$$S_z(\xi_1 \odot \cdots \odot \xi_n) = (-1)^n \xi_n \odot \cdots \odot \xi_1$$

in $\mathscr{U}(\mathfrak{g}_z)$ for $\xi_1, \ldots, \xi_n \in \mathfrak{g}$. This means

$$\mathcal{S}_z(\xi_1 \star_z \cdots \star_z \xi_n) = (-1)^n \xi_n \star_z \cdots \star_z \xi_1$$

in $\mathrm{S}^\bullet(\mathfrak{g})$. But now, using the linearity of S_z we get

$$\begin{aligned} S_z(\xi_1 \cdots \xi_n) &= S_z\left(\frac{1}{n!}\sum_{\sigma\in S_n} \xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\right) \\ &= \frac{1}{n!}\sum_{\sigma\in S_n} S_z\big(\xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\big) \\ &= \frac{1}{n!}\sum_{\sigma\in S_n} (-1)^n \xi_{\sigma(n)} \star_z \cdots \star_z \xi_{\sigma(1)} \\ &= (-1)^n \xi_1 \cdots \xi_n. \end{aligned}$$

For the coproduct, we have well-known formula with shuffle permutations:

$$\Delta_z(\xi_1 \odot \cdots \odot \xi_n) = \sum_{k=0}^{n} \sum_{\substack{1\leq i_1<\ldots<i_k\leq n \\ I=\{i_1,\ldots,i_k\}}} \xi_{i_1} \odot \cdots \odot \xi_{i_k} \otimes \xi_1 \odot \cdots \widehat{\xi_I} \cdots \odot \xi_n.$$

Pulling this back to $\mathrm{S}^\bullet(\mathfrak{g}_z)$, we get star products instead of $\odot$. For symmetric tensor we have by linearity

$$\begin{aligned} \Delta_z(\xi_1 \cdots \xi_n) &= \Delta_z\left(\frac{1}{n!}\sum_{\sigma\in S_n} \xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\right) \\ &= \frac{1}{n!}\sum_{\sigma\in S_n} \Delta_z\big(\xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\big) \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{k=0}^{n} \sum_{\substack{1\leq i_1<\ldots<i_k\leq n \\ I=\{i_1,\ldots,i_k\}}} \xi_{i_{\sigma(1)}} \star_z \cdots \star_z \xi_{i_{\sigma(k)}} \otimes \xi_{\sigma(1)} \star_z \cdots \widehat{\xi_{\sigma(I)}} \cdots \star_z \xi_{\sigma(n)} \\ &= \frac{1}{n!}\sum_{k=0}^{n}\sum_{\sigma\in S_k}\sum_{\tau\in S_{n-k}}\sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} \frac{n!}{k!(n-k)!} \\ &\quad \cdot \xi_{i_{\sigma(1)}} \star_z \cdots \star_z \xi_{i_{\sigma(k)}} \otimes \xi_{\tau(1)} \star_z \cdots \widehat{\xi_I} \cdots \star_z \xi_{\tau(n)} \\ &= \sum_{k=0}^{n} \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} \xi_{i_1} \cdots \xi_{i_k} \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n. \end{aligned}$$ □

Remark 7.1.2 From Equations 7.1.5 and 7.1.6, we see that the maps S_z and Δ_z do not depend on z, since only symmetric tensor products are involved. Hence $S_z = S_0$ and $\Delta_z = \Delta_0$. This means that the product is the only map of the Hopf algebra, which is deformed.

We need a topology on the tensor product in (7.1.6), for which we take again the projective tensor product. The continuity of the two maps is now easy to prove.

Proposition 7.1.3 *Let $\mathfrak{g}$ be an AE-Lie algebra and $R \geq 0$. For every continuous seminorm p and all $x \in \widehat{\mathrm{T}}^\bullet_R(\mathfrak{g})$ the following estimates hold:*

$$p_R(S_z(x)) \leq p_R(x) \tag{7.1.7}$$

and

$$(p_R \otimes p_R)(\Delta_z(x)) \leq (2p)_R(x). \tag{7.1.8}$$

PROOF: We use the extension to the whole tensor algebra. Inequality (7.1.7) is clear on factorizing tensors and extends to all tensors by the infimum argument. To get the estimate (7.1.8), we compute it on factorizing tensors:

$$
\begin{aligned}
(p_R \otimes p_R)(\Delta_z(\xi_1 \otimes \cdots \otimes \xi_n)) &= (p_R \otimes p_R)\left(\sum_{I \subseteq \{1,\ldots,n\}} \xi_I \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n \right) \\
&\leq \sum_{I \subseteq \{1,\ldots,n\}} |I|!^R (n-|I|)!^R p^{|I|}(\xi_I) p^{n-|I|}\Big(\xi_1 \cdots \widehat{\xi_I} \cdots \xi_n\Big) \\
&\leq \sum_{I \subseteq \{1,\ldots,n\}} |I|!^R (n-|I|)!^R p(\xi_1) \cdots p(\xi_n) \\
&\leq \sum_{I \subseteq \{1,\ldots,n\}} n!^R p(\xi_1) \cdots p(\xi_n) \\
&= 2^n n!^R p(\xi_1) \cdots p(\xi_n) \\
&= (2p)_R(\xi_1 \otimes \cdots \otimes \xi_n).
\end{aligned}
$$

This extends to all tensors by the infimum argument. □

Since the continuity of the unit and the counit is clear by the definition of our topology, we have the following result.

Proposition 7.1.4 *Let $\mathfrak{g}$ be an AE-Lie algebra and $z \in \mathbb{C}$. Then, if $R \geq 1$, $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ is a topological Hopf algebra. The same holds for $\widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g})$, if $\mathfrak{g}$ is a nilpotent locally convex Lie algebra with continuous Lie bracket.*

Appendices

Appendix A

Important theorems in Lie theory

A.1 The Poincaré-Birkhoff-Witt theorem

A.2 The Integral form of Baker-Campbell-Hausdorff

Appendix B

Locally convex algebras

B.1 Locally convex algebras with entire calculus

B.2 Locally m-convex and AE-algebras

Appendix C

More explicit formulas for the Gutt star product

C.1 Particular Lie algebras

C.2 An Idea for a Mathematica Code

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