Convergence of the Gutt star product

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August 22, 2015

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As we have already seen, the answer a mathematician would give to the question "What is (classical) mechanics?" is very closely linked to symplectic and, more generally, Poisson geometry. It is for this reason, that we will have a look at some basics of Poisson geometry. Of course, a systematic way to get a strict deformation quantization for every Poisson manifold would rather be a life's work and is by far beyond the reach of a master thesis. We have to restrict it to a special class of Poisson systems in order to get a viable challenge. Constant Poisson structures were already tackled by Stefan Waldmann and investigated more closely by Matthias Schötz in and respectively. Therefore, the next logical step is to attack linear Poisson structures.

Chapter 3

Algebraic Preliminaries

3.1 Linear Poisson structures in infinite dimensions

As we have seen before, there has already been done some work on how to strictly quantize Poisson structures on vector spaces. The Weyl-Moyal-product on locally convex vector spaces was topologized by Stefan Waldmann in [?] and then investigated more closely by Matthias Schötz in [?]. It is thus clear that in the next step linear Poisson structures on locally convex vector spaces should be done. This will give a new big class of Poisson structures, which will be deformable in a strict way. Before we do so in the rest of this master thesis, we recall briefly some basics on linear Poisson structures.

We will always take a vector space V and look at Poisson structures on the coordinates which are elements of the dual space V^*. In order to cover most of the physically interesting examples by our reflections, our aim will be locally convex vector spaces. Here we have to think about what coordinates should be and how a Poisson structure on them could look like. A priori, it is not even clear which dual we should consider: the algebraic dual V^* of all linear forms on V, or the topological dual V' which contains just the continuous linear forms. Here, one could argue that only V' is of real interest, since otherwise we would encounter the very strange effect of having discontinuous polynomials, and constructing a continuous star product on this space seems somehow pointless. But even if we stick to V', it still remains the question if we want to consider the weak or the strong topology there and why one of them should be better. Once this is done, we have to think about a generalization of Poisson tensors to this context. These are just the very first problems which come up and they won't be the last. For this reason, it is worth looking at some equivalent formulations of $Pol(V^*)$, since they may allow for better generalizations. To do so, we have to go back to the finite-dimensional case.

Let V be a finite dimensional vector space. There is now no questions about the dual or its topology, since $V^* = V'$ is finite-dimensional, too, and we deal with polynomials on it. But a linear Poisson structure on V^* is something very familiar: It is equivalent to a Lie algebra structure on V.

Proposition 3.1.1 *Let V be a vector-space of dimension $n \in \mathbb{N}$ and $\pi \in \Gamma^\infty(\Lambda^2(TV^*))$. Then the two following things are equivalent:*

i.) π is a linear Poisson tensor.

ii.) V has a uniquely determined Lie algebra structure.

PROOF: We choose a basis $e_1, \ldots, e_n \in V$ and denote its dual basis $e^1, \ldots, e^n \in V^*$. Then we call the linear coordinates in these bases $x_1, \ldots, x_n \in \mathscr{C}^\infty(V^*)$ and $\xi^1, \ldots, \xi^n \in \mathscr{C}^\infty(V)$, such that for all $\xi \in V, x \in V^*$

$$\xi = \xi^i(\xi) e_i \quad \text{and} \quad x = x_i(x) e^i.$$

In these coordinates, the Poisson tensor reads

$$\pi = \frac{1}{2}\pi_{ij}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where π is linear in the coordinates and we have

$$\pi_{ij}(x) = c_{ij}^k x_k.$$

This equivalent to a tensor

$$c = \frac{1}{2}c_{ij}^k e_k \otimes e^i \wedge e^j$$

which gives for $f, g \in \mathscr{C}^\infty(V^*)$

$$\{f,g\}(x) = \pi(df,dg)(x) = x_k c_{ij}^k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}, \tag{3.1.1}$$

using the identification $T^*V^* \cong V^{**} \cong V$. But now, the statement is obvious, since antisymmetry of π means antisymmetry of the c_{ij}^k in the indices i and j and the Jacobi identity for the Poisson tensor gives

$$c_{ij}^\ell c_{\ell k}^m + c_{jk}^\ell c_{\ell i}^m + c_{ki}^\ell c_{\ell j}^m = 0 \tag{3.1.2}$$

for all i, j, k, m, since it must be fulfilled for all smooth functions. Vicely versa, (3.1.2) ensures the Jacobi identity of π in (3.1.1). Hence the map

$$[\cdot,\cdot]\colon V \times V \longrightarrow V \quad (e_i, e_j) \longmapsto c_{ij}^k e_k \tag{3.1.3}$$

defines a Lie bracket, since the c_{ij}^k are antisymmetric and fulfil the Jacobi identity and are therefore structures constants. Conversely, the structure constants of a Lie algebra on V define a Poisson tensor on V^* via (3.1.1). □

Since we know now, that V is actually a Lie algebra, we will call it $\mathfrak{g}$ from now on. Since they carry additional structure, these Poisson systems have a special name.

Definition 3.1.2 (Kirillov-Kostant-Souriau bracket) *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the Poisson bracket $\{\cdot,\cdot\}_{KKS}$, which is given by Proposition 3.1.1 on $\mathfrak{g}^*$ is called the Kirillov-Kostant-Souriau bracket.*

The correspondence from Proposittion 3.1.1 is a first hint how we could extend our ideas to infinite dimensionsal systems. Unfortunately, we will not be able to find this nice correspondence of a Lie algebra structure on $\mathfrak{g}$ and the linear polynomials on $\mathfrak{g}'$, since the procedure we used involves the double-dual of $\mathfrak{g}$. In general, this will be really bigger than $\mathfrak{g}$ itself, and starting from some analogon of a linear Poisson structure on $\mathfrak{g}'$, we will find a Lie algebra structure on $\mathfrak{g}''$. Of course, we could just use the canonical embedding $\mathfrak{g} \subseteq \mathfrak{g}''$, but it could (and, in general, it will) happen, that the Lie bracket of $x, y \in \mathfrak{g}$ will not be in $\mathfrak{g}$ any more, but just in its double-dual. In most of the physical cases, we are actually not interested in the double-dual, but in the original vector space. Therefore, it seems to be a good choice to translate "linear Poisson structure on $\mathfrak{g}^*$" as "$\mathfrak{g}$ is a Lie algebra" in infinite dimensions. Remark however that this is a choice and not a necessity, and other choices would have been possible.

The next task are the polynomials on $\mathfrak{g}'$. As already mentioned, it is not easy to find a good generalization for them, since for a locally convex Lie algebra $\mathfrak{g}$, even $\mathfrak{g}'$ will be a rather huge vector space. Again, it is helpful to go back to the finite-dimensional case, where we have the following result:

Proposition 3.1.3 *Let $\mathfrak{g}$ be a vector space of dimension $n \in \mathbb{N}$. Then the algebras $\mathrm{S}^\bullet(\mathfrak{g})$ and $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ are canonically isomorphic.*

PROOF: since this is a very well-known result, we just want to sketch the proof briefly: Take a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ and its linear coordinates $x_1, \ldots, x_n \in \mathscr{C}^\infty(\mathfrak{g}^*)$ with $x_i(\xi) = e_i(\xi)$ for $\xi \in \mathfrak{g}^*$. On homogeneous symmetric tensors this yields the map

$$\mathscr{J} \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{Pol}^\bullet(\mathfrak{g}^*), \quad e_1^{\mu_1} \ldots e_n^{\mu_n} \longmapsto \xi_1^{\mu_1} \ldots \xi_n^{\mu_n}.$$

From the construction, we see that this is an isomorphism, but note, that we have used the identification $\mathfrak{g}^{**} \cong \mathfrak{g}$ via

$$e_i(\xi) = \langle \xi, e_i \rangle. \qquad \square$$

This gives a new idea for generalizing $\mathrm{Pol}(\mathfrak{g}^*)$. One could argue that this isomorphism also uses the double dual which we wanted to avoid, since we could "drop out" of our original algebra and end up in $\mathrm{S}^\bullet(\mathfrak{g}^{**})$. Luckily, things are different now, and this is not possible. We can extend the Poisson bracket on $\mathfrak{g}$ as a bi-derivation to $\mathrm{S}^\bullet(\mathfrak{g})$ and get a closed Poisson algebra now. This way, we never even need to talk about $\mathrm{S}^\bullet(\mathfrak{g}^{**})$. Of course, $\mathrm{S}^\bullet(\mathfrak{g})$ only injects in $\mathrm{Pol}(\mathfrak{g}')$ and we don't have an isomorphism any more, but we have good reason to think that this is enough: we get a closed and reasonably big subalgebra of of the polynomials. Moreover, the symmetric tensor algebra is defined on infinite-dimensional spaces exactly in the syme way as on finite-dimensional ones, there is no question about how to generalize, the construction is canonical.

Finally, we found a suitable way of speaking about object of interest: We replace linear Poisson structures on $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ by $\mathrm{S}^\bullet(\mathfrak{g})$. In finite dimension, this will not make a difference, but for the infinite-dimensional case, this is a choice and it is not mandatory to do it like this. Anyway, a decision had to be made and we have good reasons to believe that we are not completely misguided.

3.2 The Gutt star product

Our final aim is to endow the symmetric algebra, and hence the polynomial algebra, with a new, noncommutative product. This is possible in a very natural way, due to the Poincaré-Birkhoff-Wit theorem. It links the symmetric tensor algebra $\mathrm{S}^\bullet(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $\mathscr{U}(\mathfrak{g})$.

3.2.1 The universal enveloping algebra

If $\mathscr{A}$ is an associative algebra, one can construct a Lie algebra out of it by using the commutator

$$[a, b] = a \cdot b - b \cdot a, \quad a, b \in \mathscr{A}.$$

This construction is in fact functorial, since it doesn't only map associative algebras to Lie algebras, but also morphisms of the former to those of the latter. While getting a Lie algebra out of an associative algebra is easy, the reversed process is more complicated, but also possible. It is a well-known fact that every Lie algebra $\mathfrak{g}$ can be embedded into an associative algebra, known as the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$. It is uniquely determined (up to isomorphism) by the universal property: for every unital associative algebra $\mathscr{A}$ and every homomorphism for Lie algebras $\phi \colon \mathfrak{g} \longrightarrow \mathscr{A}$ using the commutator on $\mathscr{A}$, one gets a unital homomorphism of associative algebras $\Phi \colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{A}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g}) \arrow[dr, "\Phi"] & \\
& \mathscr{A} \\
\mathfrak{g} \arrow[uu, "\iota"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

The proof of existence and uniqueness of the universal enveloping algebra can be found in every standard textbook on Lie theory like [?] or [?], and we won't do it here in detail. Just recall that existence is proven by an explicit construction: one takes the tensor algebra $\mathrm{T}^\bullet(\mathfrak{g})$ and considers the two-sided ideal

$$\mathrm{I} = < \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] >, \quad \forall_{\xi,\eta\in\mathfrak{g}}$$

inside of it. Then one gets the universal enveloping algebra by the quotient

$$\mathscr{U} = \frac{\mathrm{T}^\bullet(\mathfrak{g})}{\mathrm{I}}. \tag{3.2.1}$$

It follows from this construction, that $\mathscr{U}(\mathfrak{g})$ is a filtered algebra

$$\mathscr{U}(\mathfrak{g}) = \bigcup_{k\in\mathbb{N}} \mathscr{U}^k(\mathfrak{g}), \quad \mathscr{U}^k(\mathfrak{g}) = \Big\{ x = \sum_i \xi_1^i \cdot \ldots \cdot \xi_n^i \;\Big|\; \xi_j^i \in \mathfrak{g},\ 1 \le j \le n, i \in \mathbb{N} \Big\}.$$

We just get a filtration, not a graded structure, however, since the ideal I is not homogeneous. Moreover, $\mathscr{U}(\mathfrak{g})$ is commutative (and graded) if and only if $\mathfrak{g}$ was commutative. But $\mathscr{U}(\mathfrak{g})$ is much more than an associative algebra: it is also a Hopf algebra, since one can define a coassociative, cocommutative coproduct on it

$$\Delta\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}), \quad \xi \longmapsto \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi, \quad \forall_{\xi\in\mathfrak{g}}$$

which extends to $\mathscr{U}(\mathfrak{g})$ via algebra homomorphism, as well as an antipode

$$S\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}), \quad \xi \longmapsto -\xi, \quad \forall_{\xi\in\mathfrak{g}}$$

which extends to $\mathscr{U}(\mathfrak{g})$ via algebra antihomomorphism.

3.2.2 The Poincaré-Birkhoff-Witt theorem

It is a well-known fact that one has a basis in $\mathscr{U}(\mathfrak{g})$. This result is due to the already mentioned theorem of Poincaré, Birkhoff and Witt:

Theorem 3.2.1 (Poincaré-Birkhoff-Witt theorem) *Let $\mathfrak{g}$ be a Lie algebra with a basis $\mathcal{B}_\mathfrak{g} = \{\beta_i\}_{i\in I}$. Then the set*

$$\mathcal{B}_{\mathscr{U}(\mathfrak{g})} = \big\{ \beta_{i_1}^{\mu_{i_1}} \cdot \ldots \cdot \beta_{i_n}^{\mu_{i_n}} \;\big|\; n \in \mathbb{N},\ i_k \in I \text{ with } i_1 \preccurlyeq \ldots \preccurlyeq i_n \text{ and } \beta_{i_k} \in \mathcal{B}_\mathfrak{g},\ \mu_{i_1}, \ldots, \mu_{i_n} \in \mathbb{N} \big\}$$

defines a basis of $\mathscr{U}(\mathfrak{g})$.

there are different proofs for this statement. While a geometrical proof is very convenient in the finite-dimensional case, a combinatorial argument must be used in infinite dimensions. Most textbooks give the latter one, but restrict to finite-dimensional Lie algebras in order to avoid speaking of basis in infinite dimensions (except, of course, [?]). Once one has accepted working with the Lemma of Zorn, the proof will work the same way for any Lie algebra, since the idea relies on ordered index sets which can be defined in any dimension. The PBW theorem allows us to set up an isomorphism between $\mathrm{S}^\bullet(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g})$ immediately, since a basis of the former can be given by almost the same expression

$$\mathcal{B}_{\mathrm{S}^\bullet(\mathfrak{g})} = \left\{\beta_{i_1}^{\mu_{i_1}} \dots \beta_{i_n}^{\mu_{i_n}} \;\middle|\; n \in \mathbb{N},\ i_k \in I, 1 \leq k \leq n, i_1 \preccurlyeq \dots \preccurlyeq i_n \text{ and } \beta_{i_k} \in \mathcal{B}_{\mathfrak{g}},\ \mu_{i_1}, \dots, \mu_{i_n} \in \mathbb{N}\right\}$$

where we just have replaced the noncommutative product in $\mathscr{U}(\mathfrak{g})$ by the symmetric tensor product $\vee$ (which we will usually denote without a symbol, if possible) in $\mathrm{S}^\bullet(\mathfrak{g})$. This allows us to write down an isomorphsim between the symmetric tensor algebra and the universal enveloping algebra, just by mapping the basis vectors to each other in a naive way. Of course, this can never be an isomorphism in the sense of algebras, but only of (filtered) vector spaces, because one of the algebras is commutative and the other isn't. Moreover, the symmetric algebra has a grading in the sense that

$$\mathrm{S}^\bullet(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathrm{S}^n(\mathfrak{g}), \quad \mathrm{S}^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \vee \dots \vee \mathfrak{g}}_{n \text{ times}},$$

which induces a filtration by $\mathrm{S}^{(k)}(\mathfrak{g}) = \sum_{j=0}^{k} \mathrm{S}^j(\mathfrak{g})$. Our simple isomorphism will respect the filtration, but it isn't the only isomorphism which one can write down. In [?], Berezin proposed another isomorphism which is more helpful to use:

$$\mathfrak{q}_n \colon \mathrm{S}^n(\mathfrak{g}) \longrightarrow \mathscr{U}^n(\mathfrak{g}), \quad \beta_{i_1} \dots \beta_{i_n} \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \beta_{i_{\sigma(1)}} \cdot \dots \cdot \beta_{i_{\sigma(n)}}, \quad \mathfrak{q} = \sum_{n=0}^{\infty} \mathfrak{q}_n. \tag{3.2.2}$$

We will refer to it as the quantization map, for reasons that will soon become clear. It also respects the filtration and transfers the symmetric product to another symmetric expression. In this sense, we can now switch between both algebras and use the setting, which is more convenient in the current situation: the graded structure of $\mathrm{S}^\bullet(\mathfrak{g})$, or the Hopf algebra structure of $\mathscr{U}(\mathfrak{g})$.

3.2.3 The Gutt star product

Since we know, that the universal enveloping and the symmetric tensor algebra are isomorphic as vector spaces, we have a good tool at hand to endow the symmetric tensor algebra, and hence the polynomials, with a noncommutative product. This is exactly what Gutt did in [?]. She constructed a star product on $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ by pulling back the one from $\mathscr{U}(\mathfrak{g})$ while encoding the noncommutativity in a formal parameter $z \in \mathbb{C}$ in a convenient way.

Definition 3.2.2 (Gutt star product) *Let $\mathfrak{g}$ be a Lie algebra, $z \in \mathbb{C}$, and $f, g \in \mathrm{S}^\bullet(\mathfrak{g})$ of degree k and ℓ respectively. Then we define the Gutt star product:*

$$\star_z \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad (f, g) \longmapsto \sum_{n=0}^{e+\ell-1} z^n \pi_{k+\ell-n}\big(\mathfrak{q}^{-1}(\mathfrak{q}(f) \cdot \mathfrak{q}(g))\big), \tag{3.2.3}$$

where the $\pi_n \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^n(\mathfrak{g})$ are the projections on the homogeneous components of degree n.

This is the original way in which Gutt defined her star product in [?], but there are two more ways to do it. Define

$$\mathrm{I}_z =< \xi \otimes \eta - \eta \otimes \xi - z[\xi, \eta] >$$

for $z \in \mathbb{C}$. Then we set

$$\mathscr{U}(\mathfrak{g}_z) = \frac{\mathrm{T}^\bullet(\mathfrak{g})}{\mathrm{I}_z}, \tag{3.2.4}$$

and get the map

$$\mathfrak{q}_{z,n} \colon \mathrm{S}^n(\mathfrak{g}) \longrightarrow \mathscr{U}^n(\mathfrak{g}_z), \quad \beta_{i_1} \dots \beta_{i_n} \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \beta_{i_{\sigma(1)}} \cdot \dots \cdot \beta_{i_{\sigma(n)}}, \quad \mathfrak{q}_z = \sum_{n=0}^{\infty} \mathfrak{q}_{z,n}. \tag{3.2.5}$$

This way, we also get a star product:

$$\widehat{\star}_z \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad (f, g) \longmapsto \mathfrak{q}_z^{-1}(\mathfrak{q}_z(f) \cdot \mathfrak{q}_z(g)). \tag{3.2.6}$$

In [?], Drinfeld also constructed a star product using the Baker-Campbell-Hausdorff series: take $\xi, \eta \in \mathfrak{g}$ and set

$$\exp(\xi) *_z \exp(\eta) = \exp\left(\frac{1}{z} \mathrm{BCH}(z\xi, z\eta)\right), \tag{3.2.7}$$

where the exponential series is understood a formal power series in ξ and η. By differentiating, on gets the star product on all polynomials.

Of course, our aim is to show that these three maps are in fact identical and that they define a star product. Since this is a long way to go, we postpone the proof to the end of this chapter. It will be useful to learn something about the Baker-Campbell-Hausdorff series and the Bernoulli number first.

3.3 The Baker-Campbell-Hausdorff series

Since we have a formula for $\star_z$ which involves the Baker-Campbell-Hausdorff series, we want to give a short overview about it and introduce some results, that will be helpful later on. Note however, that there is not *the* BCH formula, since one can always rearrange terms using antisymmetry, Jacobi and higher identities, but for $\xi, \eta \in \mathfrak{g}$ for some Lie algebra $\mathfrak{g}$, we can always write it as

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \mathrm{BCH}_n(\xi, \eta) = \sum_{a,b=0}^{\infty} \mathrm{BCH}_{a,b}(\xi, \eta), \tag{3.3.1}$$

where $\mathrm{BCH}_n(\xi, \eta)$ denotes all expressions having n letters and $\mathrm{BCH}_{a,b}(\xi, \eta)$ denotes all expressions with a ξ's and b η's. Clearly we have

$$\mathrm{BCH}_n(\xi, \eta) = \sum_{b+b=n} \mathrm{BCH}_{a,b}(\xi, \eta).$$

This only postpones the problem of non-uniqueness to the partial expressions, but it will be helpful, since we don't have to choose a special form for writing BCH every time.

3.3.1 Some general and historical remarks

Let a noncommutative algebra $\mathscr{A}$ be given. For $\xi, \eta \in \mathscr{A}$ we look at the identity

$$\exp(\chi) = \exp(\xi) \exp(\eta).$$

Then, algebraically speaking, the BCH series is the formal power series for χ as a commutator expression in ξ and η. We can take the formal power series for the exponentials

$$\exp(\xi) \exp(\eta) = \sum_{n,m=0}^{\infty} \frac{\xi^n \eta^m}{n!m!}$$

and use the formal power series for the logarithm

$$\log(\chi) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\chi - 1)^k.$$

This yields

$$\chi = \log(\exp(\xi)\exp(\eta)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{i\in\{1,\dots,k\} \\ n_i, m_i \geq 0 \\ n_i + m_i \geq 1}} \frac{\xi^{n_1}\eta^{m_1}\dots\xi^{n_k}\eta^{m_k}}{n_1!m_1!\dots n_k!m_k!}. \tag{3.3.2}$$

It is obviously far from trivial, if and how this can be expressed using commutators, and the first one to find a general way for this was Dynkin in the 1950's.

Now let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a corresponding Lie group G. From the geometric point of view, the BCH formula is the infinitesimal counterpart of the multiplication law in G. One can express it as a series of Lie brackets, which converges just in a small neighbourhood around the unit element $e \in G$. Since the exponential function $\exp\colon \mathfrak{g} \longrightarrow G$ is locally diffeomorphic there, one gets for $\xi, \eta \in \mathfrak{g}$

$$\begin{aligned} \mathrm{BCH}(\xi,\eta) &= \log(\exp(X)\exp(Y)) \\ &= \xi + \eta + \frac{1}{2}[\xi,\eta] + \frac{1}{12}([\xi,[\xi,\eta]] + [\eta,[\eta,\xi]]) + \dots \end{aligned}$$

The formula was recovered by Campbell [?, ?], Baker [?] and Hausdorff [?] independently, founding on a work by Schur [?]. A lot of work has been done on it since, and one can find a nice summary about it in the introductory part of [?].

3.3.2 Forms of the BCH

As already mentioned, there are different forms of stating the BCH formula and depending on the problem one wants to solve, not each one is well suited. One can classify them roughly into four groups.

i.) There are recursive formulas, which calculate each term from the previous one. The first expression due to Baker, Campbell and Hausdorff were of this kind. Though the idea is old, this approach is still much used and allows powerful applications: Casas and Murua found an efficient algorithms [?] for calculating the BCH series up to high orders, based on a recursive formula given by Varadarajan in his textbook [?].

ii.) Most textbooks prove an integral form of the series, like [?] and [?]. since we will use it, too, we want to introduce it here briefly. Take the meromorphic function

$$g\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{z\log(z)}{z-1} \tag{3.3.3}$$

and denote for $\xi \in \mathfrak{g}$ by

$$\mathrm{ad}_\xi\colon \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \eta \longmapsto [\xi,\eta]$$

the usual ad-operator. Then one has for $\xi, \eta \in \mathfrak{g}$

$$\mathrm{BCH}(\xi,\eta) = \xi + \int_0^1 g(\exp(\mathrm{ad}_\xi)\exp(t\,\mathrm{ad}_\eta))(\eta)dt. \tag{3.3.4}$$

iii.) As already mentioned, Dynkin found a closed form for (3.3.2), which is the only one of this kind known so far. One can its proof for example in [**?**]. It reads

$$\mathrm{BCH}(\xi,\eta)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{\substack{i\in\{1,\dots,k\}\\ n_i,m_i\geq 0\\ n_i+m_i\geq 1}}\frac{1}{\sum_{i=1}^{k}(n_i+m_i)}\frac{[\xi^{n_1}\eta^{m_1}\dots\xi^{n_k}\eta^{m_k}]}{n_1!m_1!\dots n_k!m_k!}, \tag{3.3.5}$$

where the expression $[\dots]$ denotes Lie brackets nested to the left:

$$[\xi\eta\eta\xi]=.[[[\xi,\eta],\eta],\xi]$$

Unfortunately, the combinatorics get extremely complicated for higher degrees, since it contains a lot of redundancies.

iv.) Goldberg gave a form of the series which is based on words in two letters:

$$\mathrm{BCH}(\xi,\eta)=\sum_{n=1}^{\infty}\sum_{|w|=n}g_w w. \tag{3.3.6}$$

The g_w are coefficients, which can be calculated using the recursively defined Goldberg polynomials (see [**?**]). It was put into commutator form by Thompson in [**?**]:

$$\mathrm{BCH}(\xi,\eta)=\sum_{n=1}^{\infty}\sum_{|w|=n}\frac{g_w}{n}[w]. \tag{3.3.7}$$

Again, the $[w]$ are Lie brackets nested to the left. Of course, this formula will also have redundancies, but its combinatorial aspect is much easier than the one of (3.3.5). Since there are estimates for the coefficients g_w, we will use this form for our Main Theorem.

3.3.3 The Goldberg-Thompson formula and some results

Goldberg's theorems

We now introduce the results of Goldberg: he noted a word in the letters ξ and η as

$$w_\xi(s_1,s_2,\dots,s_m)=\xi^{s_1}\eta^{s_2}\dots(\xi\vee\eta)^{s_m},$$

with $m\in\mathbb{N}$ and the last letter will be ξ if m is odd and η if m is even. The index ξ of w_ξ means, that the word starts with a ξ. Now we can assign to each word $w_{\xi\vee\eta}(s_1,\dots,s_m)$ a coefficient $c_{\xi\vee\eta}(s_1,\dots,s_m)$. This is done by the following formula:

$$c_\xi(s_1,\dots,s_m)=\int_0^1 t^{m'}(t-1)^{m''}G_{s_1}(t)\dots G_{s_m}(t)dt, \tag{3.3.8}$$

where we have $m'=\lfloor\frac{m}{2}\rfloor$, $m''=\lfloor\frac{m-1}{2}\rfloor$ with $\lfloor\cdot\rfloor$ denoting the entire part of a real number and we have $n=\sum_{i=1}^{m}s_i$. For c_η we have

$$c_\eta(s_1,\dots,s_m)=\begin{cases}c_\xi(s_1,\dots,s_m) & \text{if } m \text{ is odd}\\ (-1)^{n-1}c_\xi(s_1,\dots,s_m) & \text{if } m \text{ is even.}\end{cases} \tag{3.3.9}$$

The G_s are the recursively defined Goldberg polynomials

$$G_s(t) = \frac{1}{s}\frac{d}{dt}t(t-1)G_{s-1}(t), \tag{3.3.10}$$

for $s > 1$ and $G_1(t) = 1$. From (3.3.9) we get immediately

$$c_\xi(s_1, \ldots, s_m) = c_\eta(s_1, \ldots, s_m) = 0 \quad \text{if } m \text{ is odd and } n \text{ is even.}$$

Of course, Goldberg found interesting identities which are fulfilled by the coefficients. A very remarkable one is that for all permutations $\sigma \in S_m$ one has

$$c_\xi(s_1, \ldots, s_m) = c_\xi\big(s_{\sigma(1)}, \ldots, s_{\sigma(m)}\big),$$

since (3.3.8) obviously doesn't see the ordering of the s_i and m', m'' and n are not affected by reordering. For words with $m = 2$, an easier formula can be found:

$$c_\xi(s_1, s_2) = \frac{(-1)^{s_1}}{s_1!s_2!}\sum_{n=1}^{s_2}\binom{s_2}{n}B_{s_1+s_2-n},$$

where the B_s denote the Bernoulli numbers, which will be explained more precisely in the next paragraph. First, we note that the only case which matters to us is of course $s_1 = 1$, since for $s_1, s_2 > 1$ we will find something like $[[\xi, \xi], \ldots] = 0$. For simplicity, let's set $s_2 = 1$ and to permute $s_1 \leftrightarrow s_2$:

$$c_\xi(1, n-1) = \frac{(-1)^{n-1}}{s!}B_s. \tag{3.3.11}$$

Bernoulli numbers

We have seen the Bernoulli numbers B_n showing up and we will encounter them very often in the following. Hence it is useful to learn a few important things about them. They are defined by the series expansion of

$$g\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n. \tag{3.3.12}$$

Clearly, g has poles at $z = 2k\pi i$, $k \in \mathbb{Z}\backslash\{0\}$. Moreover, one can easily show that all odd Bernoulli numbers are zero, except for $B_1 = -\frac{1}{2}$ and since in some applications one needs B_1 to be positive, there is a different convention for naming them: one often encounters $B_n^* = (-1)^n B_n$ (which only differs for $n = 1$). The nonzero Bernoulli number alternate in sign and for there absolute value, one can show the asymptotic behaviour (see [**?**])

$$B_{2n} \sim (-1)^{n-1}4\sqrt{\pi n}\Big(\frac{n}{\pi e}\Big)^{2n}.$$

This is not surprising, since we now that the generating function g had poles at $\pm 2\pi i$. The Bernoulli numbers can also be calculated by the recursion formula

$$B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k, \tag{3.3.13}$$

which is well-known in the literature. Since we will deal a lot with them, we want to give the first numbers of this series here.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$\frac{691}{2730}$	0	$\frac{7}{6}$	0	$\frac{3617}{510}$

BCH up to first order

Proposition 3.3.1 *Let $\mathfrak{g}$ be a Lie algebra and the Bernoulli numbers as defined before. Then we have for $\xi, \eta \in \mathfrak{g}$*

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=0}^{\infty} \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2) \tag{3.3.14}$$

$$= \sum_{n=0}^{\infty} \frac{B_n}{n!} (\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\xi^2) \tag{3.3.15}$$

PROOF: This is something we can perfectly calculate using the Goldberg coefficients. Remind that we will put words to Lie brackets, and for computing the coefficients we will need the words $\eta\xi^n$ *and* $\xi\eta\xi^{n-1}$ because of antisymmetry. Now let $n \in \mathbb{N}$. We have

$$c\eta(1, n) = (-1)^n c_\xi(1, n) = (-1)^n \frac{(-1)^n}{n!} B_n = \frac{B_n}{n!}.$$

By $n - fold$ skew-symmetry and (3.3.7), we get the contribution

$$\frac{(-1)^n}{(n+1)!} B_n (\mathrm{ad}_\xi)^n(\eta) = \frac{1}{(n+1)!} B_n^* (\mathrm{ad}_\xi)^n(\eta)$$

Now we need $c_\xi(1, 1, n-1)$: let $n > 1$, then

$$\begin{aligned} c_\xi(1, 1, n-1) &= \int_0^1 t(t-1) G_{n-1}(t) dt \\ &= -\int_0^1 t \frac{d}{dt} (t(t-1) G_{n-1}(t)) dt \\ &= -\int_0^1 n t G_n(t) dt \\ &= -n c_\xi(1, n) \\ &= -n \frac{(-1)^n}{n!} B_n \\ &= (-1)^{n+1} \frac{1}{(n-1)!} B_n \end{aligned}$$

So by using $n - 1$ times the skew-symmetry of the Lie bracket, we get

$$\frac{1}{n+1} \cdot (-1)^{n+1} \frac{1}{(n-1)!} B_n [\ldots[\xi, \eta], \xi] \ldots], \xi] = \frac{n}{(n+1)!} B_n (\mathrm{ad}_\xi)^n(\eta)$$

For $n > 1$, we add up those two and use the fact that $B_n = B_n^*$ and find the result we want. For $n = 1$, there is just the first contribution and $c_\xi(1, 1) = -B_1$, which gives

$$B_1^* \,\mathrm{ad}_\xi(\eta)$$

in total. For $n = 0$, we get $c_\xi(1) = c_\eta(1) = 1$ and finally get (3.3.14). For (3.3.15), note that we need $c_\xi(1, n)$ and $c_\eta(1, 1, n-1)$. We have $c_\xi(1, n) = (-1)^n c_\eta(1, n)$ and $c_\eta(1, 1, n-1) = (-1)^n c_\xi(1, 1, n-1$. this gives a global $(-1)^n$ and makes hence B_n out of B_n^*. $\square$

Remark 3.3.2 Note that we could also have used the integral formula (3.3.4) to prove this. We have two exponential series, but the second one can be cut after the constant term, since we are looking for contributions which are linear in η. The function in the integral is nothing but $(g \circ \log)(z)$. Since we insert $\exp(\mathrm{ad}_\xi)$, we get

$$\mathrm{BCH}(\xi, \eta) = \xi + \int_0^1 g(\mathrm{ad}_\xi)(\eta) dt + \mathcal{O}(\eta^2) = \xi + \sum_{n=1}^{\infty} \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2)$$

since there is no dependence on t any more.

3.4 The Equality of the Star Products

The first step is associativity, since it will simplify the proof later.

Proposition 3.4.1 *The three maps $\star_z$, $\widehat{\star}_z$ and $*_z$ define associative multiplications.*

PROOF:

i.) For $\widehat{\star}_z$, associativity is clear from the very construction, since the multiplication in $\mathscr{U}(\mathfrak{g}_z)$ is associative. Bilinearity is also immediate.

ii.) For $\star_z$, we have to move around sums and shift projections. Recall that $\pi_n(f \star_z g) = 0$, if $n > \deg(f) + \deg(g)$. Take homogeneous tensors $f, g, h \in \mathrm{S}^\bullet(\mathfrak{g})$ of degree $k, \ell, m \in \mathbb{N}$ respectively. The we have

$$\begin{aligned}
(f \star_z g) \star_z h &= \sum_{j=0}^{k+\ell-1} \sum_{i=0}^{k+\ell+m-j-1} z^i \big(\pi_{k+\ell+m-j-i} \circ \mathfrak{q}^{-1}\big)\big(\mathfrak{q}\big(z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(a)\mathfrak{q}(b))\big)\mathfrak{q}(h)\big) \\
&= \sum_{j=0}^{k+\ell-1} \sum_{i=0}^{k+\ell+m-1} z^{i-j} \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)\big(\mathfrak{q}\big(z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(a)\mathfrak{q}(b))\big)\mathfrak{q}(h)\big) \\
&= \sum_{i=0}^{k+\ell+m-1} z^i \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)\left(\mathfrak{q}\left(\sum_{j=0}^{k+\ell-1} z^{-j} z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(a)\mathfrak{q}(b))\right)\mathfrak{q}(h)\right) \\
&= \sum_{i=0}^{k+\ell+m-1} z^i \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}a)\mathfrak{q}(b)\mathfrak{q}(h)),
\end{aligned}$$

and we just need to reverse this procedure on the right hand side in order to get the wanted result. All we did was using the linearity of the involved maps. Bilinearity is also clear for $\star_z$, since all the maps involved are (bi-)linear.

iii.) Also for $*_z$, associativity is not complicated, since one can write the star product, using the formal power series of the exponential and the logarithm. Since in this setting, we have

$$\exp(\log(z)) = z, \quad \forall_{z \in \mathfrak{g}},$$

one has

$$\begin{aligned}
(\exp(\xi) *_z \exp(\eta)) *_z \exp(\chi) &= \exp\left(\frac{1}{z}\mathrm{BCH}\left(\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right), z\chi\right)\right) \\
&= \exp\left(\frac{1}{z}\mathrm{BCH}\left(z\xi, \left(\frac{1}{z}\mathrm{BCH}(z\eta, z\chi)\right)\right)\right)
\end{aligned}$$

$$= \exp(\xi) *_z (\exp(\eta) *_z \exp(\chi)),$$

since

$$\begin{aligned}
\mathrm{BCH}\Big(\Big(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\Big), z\chi\Big) &= \log\Big(\exp\Big(\log\Big(\frac{1}{z}\exp(z\xi)\exp(z\eta)\Big)\Big)\exp(z\chi)\Big) \\
&= \log\Big(\frac{1}{z}\exp(z\xi)\exp(z\eta)\exp(z\chi)\Big) \\
&= \log\Big(\exp(z\xi)\log\Big(\Big(\frac{1}{z}\exp(z\eta)\exp(z\chi)\Big)\Big)\Big) \\
&= \mathrm{BCH}\Big(z\xi, \Big(\frac{1}{z}\mathrm{BCH}(z\eta, z\chi)\Big)\Big).
\end{aligned}$$

Bilinearity follows from differentiating the formula and is a simple computation. □

Note that for a star product, the maps must fulfil the classical and the semi-classical limit. This will be done in Corollary 4.2.1. It is left to show, that the star product are in fact equal. It is enough to show that they coincide for terms of the form $\xi^k \star \eta$ with $\xi, \eta \in \mathfrak{g}$ and $k \in \mathbb{N}$, since we get them on homogeneous polynomials of the form $\xi_1 \dots \xi_k \star \eta$ by polarization and for two polynomials like $\xi_1 \dots \xi_k \star \eta_1 \dots \eta_\ell$ by iteration, which is possible due to associativity. The next lemma will be a first big step:

Lemma 3.4.2 *Let $\mathfrak{g}$ be a Lie algebra, then we have*

$$\xi^k \widehat{\star}_z \eta = \sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta), \tag{3.4.1}$$

where the B_n^* are the Bernoulli numbers defined as above.

Proof: This proof is divided into the two following lemmata:

Lemma 3.4.3 *Let $\xi, \eta \in \mathfrak{g}$ and $k \in \mathbb{N}$. Then we have*

$$q_z\left(\sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta)\right) = \sum_{s=0}^{k} \mathcal{K}(k,s) \xi^{k-s} \cdot \eta \cdot \xi^s$$

with

$$\mathcal{K}(k,s) = \frac{1}{k+1} \sum_{n=0}^{k} \binom{k+1}{n} B_n^* \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{\ell=0}^{k-n} \delta_{s,\ell+j}$$

Proof: Since the map $\mathfrak{q}_z$ is linear, we can pull out the constants and get

$$\mathfrak{q}_z\left(\sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^k(\eta)\right) = \sum_{n=0}^{k} \binom{k}{n} B_n^* \mathfrak{q}_z\Big(z^n \xi^{k-n} (\mathrm{ad}_\xi)^k(\eta)\Big)$$

Now we need the two equalities

$$\mathfrak{q}_z\Big(\xi^k \eta\Big) = \frac{1}{k+1} \sum_{j=0}^{k} \xi^{k-l} \cdot \eta \cdot \xi^k$$

and

$$\mathfrak{q}_z\Big((z^n \operatorname{ad}_\xi)^k(\eta)\Big) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\xi^{k-j}\cdot\eta\cdot\xi^j$$

which can easily be shown by induction and plug them in:

$$\begin{aligned}
\sum_{n=0}^{k}\binom{k}{n}B_n^*\mathfrak{q}_z(\ldots) &= \sum_{n=0}^{k}\binom{k}{n}\frac{B_n^*}{k-n+1}\sum_{\ell=0}^{k-n}\xi^{k-n-\ell}\cdot\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}\xi^{n-j}\cdot\eta\cdot\xi^j\right)\cdot\xi^\ell \\
&= \sum_{n=0}^{k}\binom{k}{n}\frac{B_n^*}{k-n+1}\sum_{\ell=0}^{k-n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\xi^{k-\ell-j}\cdot\eta\cdot\xi^{\ell+j} \\
&= \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\xi^{k-\ell-j}\cdot\eta\cdot\xi^{\ell+j}
\end{aligned}$$

We just need to collect those terms for which we have $\ell+j=s$ for all $s=0,\ldots,k$. If we do this with a Kronecker-delta, we will get exactly the $\mathcal{K}(k,s)$. $\triangledown$

For the second lemma, we need some statement on Bernoulli numbers and binomial coefficients: Let $k,m,n\in\mathbb{N}$. Then we have the following identities for Bernoulli numbers and binomial coefficients:

$$\sum_{j=0}^{k}\binom{k+1}{n}B_j = 0 \tag{3.4.2}$$

$$\sum_{j=0}^{k}\binom{k+1}{n}B_j^* = k+1 \tag{3.4.3}$$

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{m+j} = (-1)^m\sum_{i=0}^{m}\binom{m}{i}B_{k+i} \tag{3.4.4}$$

$$\sum_{j=0}^{m}(-1)^j\binom{n}{j} = (-1)^m\binom{n-1}{m} \tag{3.4.5}$$

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}. \tag{3.4.6}$$

The identity (3.4.3) is the standard recursion for the B_n^* (see e.g. [**?**]), (3.4.5) and (3.4.6) can be found in good textbooks on combinatorics such as [**?**]. Equation (3.4.4) is a theorem due to Carlitz, proven in [**?**]. With them, we can show the next lemma which will finish this subproof.

Lemma 3.4.4 *Let $\mathcal{K}(k,s)$ be defined as in Lemma* **??**, *then we have for all $k\in\mathbb{N}$*

$$\mathcal{K}(k,s) = \begin{cases} 1 & s=0 \\ 0 & \textit{else.}\end{cases}$$

PROOF:

(i) $s=0$: The Kronecker-delta will always be zero unless $l=j=0$. So we get

$$\mathcal{K}(k,0) = \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^* = \frac{k+1}{k+1} = 1$$

where we have used (**??**).

(ii) $s = 1$: To get a contribution, we must have $(j, \ell) = (1, 0)$ or $(0, 1)$. Except for $n = 0$ and $n = k$, both cases are possible. We split them off:

$$
\begin{aligned}
\mathcal{K}(k,1) &= \underbrace{\frac{1}{k+1}}_{n=0} - \underbrace{kB_k^*}_{n=k} + \frac{1}{k+1}\sum_{n=1}^{k-1}\binom{k+1}{n}B_n^*\left(1+(-1)\binom{n}{1}\right) \\
&= \frac{1}{k+1} + \frac{1}{k+1}\sum_{n=1}^{k-1}\binom{k+1}{n}B_n^* - \frac{1}{k+1}\sum_{n=1}^{k-1}\binom{k+1}{n}nB_n^* - kB_k^* \\
&= \underbrace{\frac{1}{k+1}\sum_{n=0}^{k-1}}_{=1-B_k^*} - \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}nB_n^* \\
&= 1 - B_k^* - \underbrace{\frac{k+1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*}_{=k+1} + \sum_{n=0}^{k}\underbrace{\frac{k+1-n}{k+1}\binom{k+1}{n}}_{\binom{k}{n}}B_n^* \\
&= 1 - B_k^* - k - 1 + \sum_{n=0}^{k-1}\binom{k}{n}B_n^* - B_k^* \\
&= -k + \sum_{n}^{k-1}\binom{k}{n}B_n^* \\
&= 0
\end{aligned}
$$

(iii) $s \mapsto s+1$: It is sufficient to prove $\mathcal{K}(k, s+1) - \mathcal{K}(k, s) = 0$. In order to do that, we look for a better (i.e. well-suited, not necessarily nicely looking) form of $\mathcal{K}(k, s)$.

$$
\begin{aligned}
\mathcal{K}(k,s) &= \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\delta_{s,\ell+j} \\
&= \frac{1}{k+1}\sum_{n=0}^{s}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\delta_{s,\ell+j} \\
&\quad + \frac{1}{k+1}\sum_{n=s+1}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{s}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\delta_{s,\ell+j} \\
&= \frac{1}{k+1}\sum_{n=0}^{s}\binom{k+1}{n}B_n^*\sum_{j=\max\{0,s+n-k\}}^{n}(-1)^j\binom{n}{j} \\
&\quad + \frac{1}{k+1}\sum_{n=s+1}^{k}\binom{k+1}{n}B_n^*\sum_{j=\max\{0,s+n-k\}}^{s}(-1)^j\binom{n}{j}
\end{aligned}
$$

As long as $\max\{0, s+n-k\} = 0$, the second sum in the first summand will be zero as it is just the binomial expansion of $(1-1)^n$. We may not forget that for $n = 0$ we cannot plug this in. Hence we get a special case and a shorter first sum. In the sums over j we use again the binomial expansion of $(1-1)^n$ and get

$$
\mathcal{K}(k,s) = \frac{1}{k+1}\left[1 + \sum_{k+1-s}^{s}\binom{k+1}{n}B_n^*\left(-\sum_{j=0}^{s+n-k-1}(-1)^j\binom{n}{j}\right)\right.
$$

$$+\sum_{n=s+1}^{k}\binom{k+1}{n}B_n^*\left(-\sum_{j=0}^{s+n-k-1}(-1)^j\binom{n}{j}-\sum_{j=s+1}^{n}(-1)^j\binom{n}{j}\right)\Bigg]$$

Now it is helpful to use (**??**) and $\binom{k}{n-k}=\binom{k}{n}$. We also get some -1-terms which we can put together with the B_n^*:

$$\mathcal{K}(k,s)=\frac{1}{k+1}\Bigg[1+\sum_{n=k+1-s}^{s}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s}$$
$$+\sum_{n=s+1}^{k}\binom{k+1}{n}B_n\left((-1)^{k-s}\binom{n-1}{k-s}+(-1)^{n+s}\binom{n-1}{s}\right)\Bigg]$$

We finally found the "better" form. We hence must compute $\mathcal{K}(k,s+1)-\mathcal{K}(k,s)$. Since we want to show that it is 0, we can multiply it with $k+1$ in order to get rid of the factor in front:

$$(k+1)(\mathcal{K}(k,s+1)-\mathcal{K}(k,s))$$
$$=\sum_{n=k-s}^{s+1}\binom{k+1}{n}B_n(-1)^{k-s-1}\binom{n-1}{k-s-1}-\sum_{n=k+1-s}^{s}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s}$$
$$+\sum_{n=s+2}^{k}\binom{k+1}{n}B_n\left((-1)^{k-s-1}\binom{n-1}{k-s-1}+(-1)^{n+s+1}\binom{n-1}{s+1}\right)$$
$$-\sum_{n=s+1}^{k}\binom{k+1}{n}B_n\left((-1)^{k-s}\binom{n-1}{k-s}+(-1)^{n+s}\binom{n-1}{s}\right)$$
$$=-\sum_{n=k-s}^{k}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s-1}-\sum_{n=k-s+1}^{k}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s}$$
$$-\sum_{n=s+2}^{k}\binom{k+1}{n}B_n(-1)^{n+s}\binom{n-1}{s+1}-\sum_{n=s+1}^{k}\binom{k+1}{n}B_n(-1)^{n+s}\binom{n-1}{s}$$
$$=-\sum_{n=k-s}^{k}\binom{k+1}{n}B_n(-1)^{k-s}\left(\binom{n-1}{k-s-1}+\binom{n-1}{k-s}\right)$$
$$-\sum_{n=s+1}^{k}\binom{k+1}{n}B_n(-1)^{n+s}\left(\binom{n-1}{s+1}+\binom{n-1}{s}\right)$$

We have just rearranged the sums, added some 0-terms and put them together to get it nicer. Now we will use the recursion formula for the binomial coefficients

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}$$

and our binomial multiplication equality (**??**):

$$=-\sum_{n=k-s}^{k}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n}{k-s}-\sum_{n=s+1}^{k}\binom{k+1}{n}B_n(-1)^{n+s}\binom{n}{s+1}$$
$$=-\sum_{n=k-s}^{k}\binom{k+1}{s+1}\binom{s+1}{n+s-k}B_n(-1)^{k-s}-\sum_{n=s+1}^{k}\binom{k+1}{s+1}\binom{k-s}{n-s-1}B_n(-1)^{n+s}$$

Since we want to show that this is 0, we can divide by $\binom{k+1}{s+1}$, because this will not be zero as long as $s \leq k$. After doing so, can use the fact that $n > 1$ in the second sum of this induction (since $s \geq 1$) and hence only even n will come up, as for odd n the Bernoulli numbers are zero. For this reason we get $(-1)^n = 1$. Then we rewrite these sums by shifting the indices and add two zeros:

$$
\begin{aligned}
&-\sum_{n=k-s}^{k} \binom{s+1}{n+s-k} B_n (-1)^{k-s} + \sum_{n=s+1}^{k} \binom{k-s}{n-s-1} B_n (-1)^{s+1} \\
&= (-1)^{s+1} \sum_{\ell=0}^{k-s-1} \binom{k-s}{\ell} B_{\ell+s+1} - (-1)^{k-s} \sum_{\ell=0}^{s} \binom{s+1}{\ell} B_{\ell+k-s} \\
&= (-1)^{s+1} \sum_{\ell=0}^{k-s} \binom{k-s}{\ell} B_{\ell+s+1} - (-1)^{k-s} \sum_{\ell=0}^{s+1} \binom{s+1}{\ell} B_{\ell+k-s} \\
&\quad - (-1)^{s+1} \binom{k-s}{k-s} B_{k+1} + (-1)^{k-s} \binom{s+1}{s+1} B_{k+1}
\end{aligned}
$$

The first two terms are exactly the Carlitz-identity (**??**), they vanish for this reason. So we are left with the last two terms. We get

$$
-(-1)^{s+1} B_{k+1} + (-1)^{k-s} B_{k+1} = (-1)^s B_{k+1} \Big(1 + (-1)^k\Big) = 0
$$

since the bracket will be zero if k is odd and $B_{k+1} = 0$ if k is even. $\triangledown$

In Lemma 4.1.1, we will see that also $*_z$ fulfils this identity. Hence $*_z = \widehat{\star}_z$. We only need to show $\widehat{\star}_z = \star_z$. For $z = 1$, the two maps are clearly identical and therefore we find

$$
\xi^k \star_r \eta = \sum_{n=0}^{k} \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^k(\eta).
$$

But knowing this and inserting it into the definition of $\star_z$, we get $\widehat{\star}_z = \star_z$. So with the proof in chapter 4, we have finally proven the following theorem:

Theorem 3.4.5 *The three maps $\star_z$, $\widehat{\star}_z$ and $*_z$ coincide on $\mathrm{S}^\bullet(\mathfrak{g})$ and define star products.*

Chapter 4

Formulas for the Gutt star product

We have seen some results on the Baker-Campbell-Hausdorff series and an identity for the Gutt star product. The latter one, stated in Theorem 3.4.5, will be a very useful tool in the following, since we want to get explicit formulas for $\star_z$. There is still a part of the proof missing, but this will be caught up at the beginning of the first section of this chapter. From there, we will come to a first easy formula for $\star_z$. Afterwards, we will use the same procedure to find two more formulas for it: a rather involved one for the n-fold star product of vectors, which will not necessarily be helpful for algebraic computations, but will turn out very useful for estimates, and a more explicit one for the product of two monomials.

From those formulas, we will be able to draw some easy, but nice consequences in the next section and we will prove the classical and the semi- calssical limit. Then, we will show how to compute the Gutt star product explicitly by calculating two easy examples.

At the end of this chapter, we will give an easy Mathematica code, which can be used to verify the correctness of our formulas for polynomials of low orders.

4.1 Formulas for the Gutt Star Product

4.1.1 A Monomial with a Linear Term

The easiest case for which we will to develop a formula is surely the following one: For a given Lie algebra $\mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}$ we would like to compute

$$\xi^k \star_{zG} \eta = \sum_{n=0}^{k} z^n C_n(\xi^k, \eta)$$

We have already done this for $\star_z$ and $\widehat{\star}_t$, now we want to do the same for $*_z$. This will finish the proof of Theorem 3.4.5. For this purpose, we will use that

$$\xi^k = \frac{\partial^k}{\partial t^k}\Big|_{t=0} \exp(t\xi). \tag{4.1.1}$$

Now we have all the ingredients to prove the following lemma:

Lemma 4.1.1 *Let $\mathfrak{g}$ be a Lie algebra and $\xi, \eta \in \mathfrak{g}$. We have the following identity for $*_z$:*

$$\xi^k *_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j(\eta). \tag{4.1.2}$$

Proof: We start from the simplified form for the Baker-Campbell-Hausdorff series from Equation (**??**):

$$\mathrm{BCH}(\xi,\eta) = \xi + \sum_{n=0}^{\infty} \frac{B_n^*}{n!}(\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2).$$

Putting things together with the definition of the Drinfel'ds tar product and Equation (4.1.1) we get

$$\begin{aligned}
\xi^k *_z \eta &= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(\frac{1}{z}\mathrm{BCH}(zt\xi, zs\eta)\right) \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(t\xi + \sum_{j=0}^{\infty} z^j \frac{B_j^*}{j!}(\mathrm{ad}_{t\xi})^j(s\eta)\right).
\end{aligned}$$

From this, we see that only terms which have exactly k of the ξ's in them and which are linear in η will contribute. This means we can cut off the sum at $j = k$. If we now write out the exponential series which we can also cut for the same reason. We have

$$\begin{aligned}
\xi^k *_z \eta &= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k}\frac{1}{n!}\left(t\xi + \sum_{j=0}^{k}(zt)^j \frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^n \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k}\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}(t\xi)^{n-m}\left(\sum_{j=0}^{k}(zt)^j \frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^m \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \left(\sum_{n=0}^{k}\frac{1}{n!}(t\xi)^n + \sum_{n=0}^{k}\sum_{j=0}^{k}\frac{1}{(n-1)!}t^{n+j-1}z^j\frac{B_j^*}{j!}\xi^{n-1}(\mathrm{ad}_\xi)^j(s\eta)\right).
\end{aligned}$$

In the last step we just cut off the sum over m since the terms for $m > 1$ will vanish because of the differentiation with respect to s. We can finally differentiate to get the formula

$$\begin{aligned}
\xi^k *_z \eta &= \sum_{n=0}^{k}\sum_{j=0}^{k}\delta_{k,n+j-1}\frac{k!}{j!(n-1)!}z^j B_j^* \xi^{n-1}(\mathrm{ad}_\xi)^j(\eta) \\
&= \sum_{j=0}^{k}\binom{k}{j} z^j B_j^* \xi^{k-j}(\mathrm{ad}_\xi)^j(\eta),
\end{aligned}$$

which is the wanted result. □

Remark 4.1.2 We have now proven the equality of the star products $\widehat{\star}_z$ $*_z$ by deriving an easy formula for both of them. From now on, we will derive all the other formulas from $\star_{zH}$, since this is the one which is easier to compute.

Once this is done, it is actually easy to get the formula for monomials of the form $\xi_1 \ldots \xi_k$ with $\eta \in \mathfrak{g}$:

Proposition 4.1.3 *Let $\mathfrak{g}$ be a Lie algebra and $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$. We have*

$$\xi_1 \ldots \xi_k \star_{zG} \eta = \sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j} z^j B_j^* \sum_{\sigma \in S_k}[\xi_{\sigma(1)}, [\ldots[\xi_{\sigma(j)}, \eta]\ldots]]\xi_{\sigma(j+1)}\ldots\xi_{\sigma(k)} \text{ and} \tag{4.1.3}$$

$$\eta \star_{zG} \xi_1 \ldots \xi_k = \sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j} z^j B_j \sum_{\sigma \in S_k}[\xi_{\sigma(1)}, [\ldots[\xi_{\sigma(j)}, \eta]\ldots]]\xi_{\sigma(j+1)}\ldots\xi_{\sigma(k)}. \tag{4.1.4}$$

PROOF: We get the result by just polarizing the formula from Lemma **??**. Let $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ be given, then we introduce the parameters t_i for $i = 1, \ldots, k$ and set

$$\Xi = \Xi(t_1, \ldots, t_k) = \sum_{i=1}^{k} t_i \xi^i.$$

Then it is immediate to see that

$$\xi_1 \ldots \xi_k = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \ldots \partial t_k}\Big|_{t_1,\ldots,t_k=0} \Xi^k$$

since for every $i = 1, \ldots, k$ we have

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} \Xi = \xi_i. \tag{4.1.5}$$

We also find for every $\eta \in \mathfrak{g}$

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} \operatorname{ad}_\Xi(\eta) = \operatorname{ad}_{\xi_i}(\eta). \tag{4.1.6}$$

Now we just need to calculate $\Xi^k \star_{zG} \eta$ and differentiate with respect to the t_i. In order to do this properly, we define

$$\gamma_n^k(\xi_1, \ldots, \xi_k; \eta) = z^n \binom{k}{n} B_n^*(\operatorname{ad}_{\xi_1} \circ \ldots \circ \operatorname{ad}_{\xi_n})(\eta) \xi_{n+1} \ldots \xi_k$$

and

$$\gamma^k(\xi_1, \ldots, \xi_k; \eta) = \sum_{n=0}^{k} \gamma_n^k(\xi_1, \ldots, \xi_k; \eta).$$

We see that

$$\Xi^k \star_{zG} \eta = \gamma^k(\Xi, \ldots, \Xi; \eta)$$

and can now differentiate this expression, which is linear in the every argument, with respect to the t_i. From the Equations (4.1.5) and (4.1.6) we get with the Leibniz rule

$$\frac{\partial}{\partial t_1} \gamma^k(\Xi, \ldots, \Xi; \eta) = \sum_{j=1}^{k} \gamma^k(\underbrace{\Xi, \ldots, \Xi}_{j-1 \text{ times}}, \xi_1, \underbrace{\Xi, \ldots, \Xi}_{k-j-1 \text{ times}}; \eta)$$

Differentiating now with respect to t_2, we get a second sum, where ξ_2 will be put once in every "free" position, and so on. One by one, all the slots will be taken by ξ_i's. We just need to divide by $k!$, and we finally find the formula from Equation (4.1.3). The proof of Equation 4.1.4 is analogue. □

Remark 4.1.4 This formula, or slightly different version of it, have already been known before. Gutt found it in her paper [**?**] and Neumaier mentioned it in his diploma thesis [**?**]. A version of it, as a multiplication formula in the universal enveloping algebra, can also be found in [**?**], part 2.8.12 (c).

4.1.2 An Iterated Formula for the General Case

Proposition 4.1.3 allows us basically to get a formula for the case of $\xi_1, \ldots, \xi_k \in \mathfrak{g}$

$$\xi_1 \star_z \ldots \star_z \xi_k = \sum_{j=0}^{k} C_{z,j}(\xi_1, \ldots, \xi_k)$$

which we will need to prove the continuity of the coproduct, but which can also help to prove the continuity of the product in a different way.

Unluckily, this approach has a problem: iterating this formula, we get strangely nested Lie brackets, which would be very difficult to bring into a nice form with Jacobi and higher identities. So this is not a good way to find a handy formula for the usual star product of two monomials. Nevertheless, we want to pursue it for a moment, since we will get an equality which will be, although rather unfriendly looking, very useful in the following: for analytic observations, it will be enough to put (even brutal) estimates on it and the exact nature of the combinatorics in the formula will not be important. Hence we rewrite Equation (4.1.3) in order to cook up such a formula.

Let's take $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$, then we have

$$\xi_1 \ldots \xi_k \star_z \eta = \sum_{n=0}^{k} C_n(xi_1 \ldots \xi_k, \eta)$$

with the C_n being given explicitly as

$$C_n^k \colon \mathrm{S}^k(\mathfrak{g}) \times \mathfrak{g} \longrightarrow \mathrm{S}^{k-n+1}(\mathfrak{g}) \tag{4.1.7}$$

$$(\xi_1 \ldots \xi_k, \eta) \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{j} B_j^* z^j [\xi_{\sigma(1)}, [\ldots, [\xi_{\sigma(j)}, \eta]]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)} \tag{4.1.8}$$

with

$$C_n = \sum_{k=0}^{\infty} C_n^k$$

and extended via bilinearity. But this gives us a good way of writing the n- fold star product of vectors:

Proposition 4.1.5 *Let $\mathfrak{g}$, $2 \leq k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. Then we have*

$$\xi_1 \star_z \ldots \star_z \xi_k = \sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \ldots + i_{k-1}} C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k). \tag{4.1.9}$$

PROOF: This is an easy prof by induction over k. For $k = 2$ the statement is clearly true. For the step $k \to k+1$ we can directly apply Equation (**??**):

$$\begin{aligned}
\xi_1 \star_z \ldots \star_z \xi_{k+1} &= \left(\sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \ldots + i_{k-1}} C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k) \right) \star_z \xi_{k+1} \\
&= \sum_{i_k=0}^{k} z^{i_k} C_{i_k} \left(\sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \ldots + i_{k-1}} C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k), \xi_{k+1} \right) \\
&= \sum_{\substack{1 \leq j \leq k \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \ldots + i_k} \big(C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k), \xi_{k+1} \big) \qquad \square
\end{aligned}$$

Remark 4.1.6

i.) Of course, it is almost trivial to get Proposition 4.1.5 from Proposition 4.1.3 and its value doesn't lie in the idea of iterating the C_n's, but in the fact that we know from (4.1.8) how the C_n's look like and in knowing the summation range in (4.1.9). This will allow us to put estimates on terms like $\xi_1 \star_z \ldots \star_z \xi_k$.

ii.) One important goal in this chapter is actually a nice identity for the case of two monomials $\xi_1 \ldots \xi_k \star_z \eta_1 \ldots \eta_\ell$ with $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. Theoretically, we could use Equation (4.1.9) for it, since

$$\xi_1 \ldots \xi_k \star_z \eta_1 \ldots \eta_\ell = \frac{1}{k!\ell!} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \xi_{\sigma(1)} \star_z \ldots \star_z \xi_{\sigma(k)} \star_z \eta_{\tau(1)} \star_z \ldots \star_z \eta_{\tau(\ell)}. \tag{4.1.10}$$

This equality can easily been proven from the definition of the map $\mathfrak{q}_z$. The only flaw in the plan is, however, that we're looking for something *nice*. So we have to go for something different.

4.1.3 A Formula for two Monomials

If we want to get an identity for the star product of two monomials, we have to get back to Equation (3.2.7). The result will not be of overwhelming beauty either, but still a lot better than Equation (4.1.10). We will at least be able to do some computations with concrete examples. As a first step, we must introduce a bit of notation:

Definition 4.1.7 (G-Index) *Let $k, \ell, n \in \mathbb{N}$ and $r = k + \ell - n$. Then we call an r-tuple J*

$$J = (J_1, \ldots, J_r) = ((a_1, b_1), \ldots, (a_r, b_r))$$

a G-index if it fulfils the following properties:

(i) $J_i \in \{0, 1, \ldots, k\} \times \{0, 1, \ldots, \ell\}$

(ii) $|J_i| = a_i + b_i \geq 1 \quad \forall_{i=1,\ldots,r}$

(iii) $\sum\limits_{i=1}^{r} a_i = k$ *and* $\sum\limits_{i=1}^{r} b_i = \ell$

(iv) The tuple is ordered in the following sense: $i > j \Rightarrow |J_i| \geq |J_j| \quad \forall_{i,j=1,\ldots,r}$ *and* $|a_i| \geq |a_j|$ *if* $|J_i| = |J_j|$

(v) If $a_i = 0$ [or $b_i = 0$] for some i, then $b_i = 1$ [or $a_i = 1$].

We call the set of all such G-indices $\mathcal{G}_r(k, \ell)$.

Definition 4.1.8 (G-Factorial) *Let $J = ((a_1, b_1), \ldots, (a_r, b_r)) \in \mathcal{G}_r(k, \ell)$ be a G-Index. We set for a given tuple $(a, b) \in \{0, 1, \ldots, k\} \times \{0, 1, \ldots, \ell\}$*

$$\#_J(a, b) = \text{ number of times that } (a, b) \text{ appears in } J.$$

Then we define the G-factorial of $J \in \{0, 1, \ldots, k\} \times \{0, 1, \ldots, \ell\}$ as

$$J! = \prod_{(a,b) \in \{0,1,\ldots,k\} \times \{0,1,\ldots,\ell\}} (\#_J(a, b))!$$

Each pair (a, b) will later correspond to $\mathrm{BCH}_{a,b}(\xi, \eta)$. Now we can state a good formula for the Gutt star product:

Lemma 4.1.9 *Let $\mathfrak{g}$ be a Lie algebra, $\xi, \eta \in \mathfrak{g}$ and $k, \ell \in \mathbb{N}$. Then we have the following identity for the Gutt star product:*

$$\xi^k \star_z \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n\Big(\xi^k, \eta^\ell\Big),$$

where the C_n are given by

$$C_n\Big(\xi^k, \eta^\ell\Big) = \frac{k!\ell!}{(k+\ell-n)!} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \mathrm{BCH}_{a_i,b_i}(\xi,\eta)\ldots\mathrm{BCH}_{a_r,b_r}(\xi,\eta) \tag{4.1.11}$$

$$= \sum_{J\in\mathcal{G}_{k+\ell-n}(k,\ell)} \frac{k!\ell!}{J!} \prod_{i=1}^{k+\ell-n} \mathrm{BCH}_{a_i,b_i}(\xi,\eta) \tag{4.1.12}$$

and the product is taken in the symmetric tensor algebra.

Proof: We want to calculate what the C_n look like. Let's denote $r = k+\ell-n$ for brevity. Then we have

$$C_n\Big(\xi^k, \eta^\ell\Big) \in \mathrm{S}^r(\mathfrak{g}).$$

Of course, the only part of the series

$$\exp\bigg(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\bigg) = \sum_{n=0}^{k+\ell}\bigg(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\bigg)^n + \mathcal{O}(\xi^{k+1}, \eta^{\ell+1})$$

which lies in $\mathrm{S}^r(\mathfrak{g})$ is the summand for $n = r$. Since we introduce the formal parameters t and s, we don't need to care about terms of higher orders in ξ and η than k and ℓ respectively.

$$\begin{aligned}
z^n C_n\Big(\xi^k, \eta^\ell\Big) &= \frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \frac{1}{z^r}\frac{\mathrm{BCH}(zt\xi, zs\eta)^r}{r!} \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \left(\sum_{j=1}^{k+\ell}\mathrm{BCH}_j(zt\xi, zs\eta)\right)^r \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0} \sum_{\substack{j_1,\ldots,j_r\geq 1\\ j_1+\ldots+j_r=k+\ell}} \mathrm{BCH}_{j_1}(zt\xi, zs\eta)\cdots\mathrm{BCH}_{j_r}(zt\xi, zs\eta) \\
&= z^n\frac{k!\ell!}{r!} \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} \mathrm{BCH}_{a_i,b_i}(\xi,\eta)\ldots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)
\end{aligned}$$

We sum over all possible arrangements of the (a_i, b_i). In order to find a nicer form of the sum, we put the ordering from definition 4.1.7 on these multi-indices and avoid therefore double counting. We loose the freedom of arranging the (a_i, b_i) and need to count the number of multi-indices $((a_1, b_1), \ldots, (a_r, b_r))$ which belong to the same G-index J. This number will be $\frac{r!}{J!}$, since we can't interchange the (a_i, b_i) any more (therefore $r!$), unless they are equal (therefore $J!^{-1}$). Since the ranges of the (a_i, b_i) in Equation (4.1.11) and of the elements in $\mathcal{G}_r(k,\ell)$ are the same. we can change the summation there to $J \in \mathcal{G}_r(k,\ell)$ and multiply by $\frac{r!}{J!}$. We find

$$z^n C_n\Big(\xi^k, \eta^\ell\Big) = z^n\frac{k!\ell!}{J!}\sum_{J\in\mathcal{G}_r(k,\ell)} \mathrm{BCH}_{a_i,b_i}(\xi,\eta)\ldots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)$$

which is precisely Equation (4.1.12). □

Now we just need to generalize this to factorizing tensors. To do so, we need a last definition:

Definition 4.1.10 *Let $k, \ell, n \in \mathbb{N}$ and $J \in \mathcal{G}_{k+\ell-n}(k, \ell)$. Then for $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell$ from a Lie algebra $\mathfrak{g}$ we set*

$$\Gamma_J(\xi_1, \ldots, \xi_k; \eta_1, \ldots, \eta_\ell) = \frac{1}{J!} \prod_{i=1}^{k+\ell-n} \mathrm{BCH}_{a_i,b_i}\left(\xi^{(a_i)}, \eta^{(b_i)}\right) \tag{4.1.13}$$

where the notation $\mathrm{BCH}_{a_i,b_i}\left(\xi^{(a_i)}, \eta^{(b_i)}\right)$ means that we have taken $\prod_{i=1}^{k+\ell-n} \mathrm{BCH}_{a_i,b_i}\left(\xi^{(a_i)}, \eta^{(b_i)}\right)$ and replaced the j-th ξ appearing in it with ξ_j for $j = 1, \ldots, k$ and analogously with the η's.

Proposition 4.1.11 *Let $\mathfrak{g}$ be a Lie algebra, $k, \ell \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. Then we have the following identity for the Gutt star product:*

$$\xi_1 \ldots \xi_k \star_z \eta_1 \ldots \eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1 \ldots \xi_k, \eta_1 \ldots \eta_\ell),$$

where the C_n are given by

$$C_n(\xi_1 \ldots \xi_k, \eta_1 \ldots \eta_\ell) = \sum_{J \in \mathcal{G}_{k+\ell-n}(k,\ell)} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \Gamma_J\left(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}; \eta_{\tau(1)}, \ldots, \eta_{\tau(\ell)}\right) \tag{4.1.14}$$

and the product is taken in the symmetric tensor algebra.

PROOF: The proof relies on polarization again and is completely analogous to the one of Proposition 4.1.3. We set

$$\Xi = \sum_{i=1}^{k} t_i \xi^i \quad \text{and} \quad \mathrm{H} = \sum_{i=1}^{\ell} t_j \eta^j.$$

Then it is easy to see that we will get rid of the factorials in Equation (4.1.12) since

$$\xi_1 \ldots \xi_k \star_z \eta_1 \ldots \eta_\ell = \frac{1}{k!\ell!} \frac{\partial^{k+\ell}}{\partial_{t_1} \ldots \partial_{s_\ell}}\bigg|_{t_1,\ldots,s_\ell=0} \Xi^k \star_z \mathrm{H}^\ell .$$

Instead of the factorials, we get symmetrizations over the ξ_i and the η_j as we did in Proposition 4.1.3, which gives the wanted result. □

4.2 Consequences and examples

Some consequences

Proposition 4.1.11 allows us to get some easy algebraic results. For example, we would like to see that the Gutt star product fulfils the classical and the semi-classical limit from Definition **??**. Now we will prove it using our new formula, and we will have shown finally that $\star_z$ is really a star product.

Corollary 4.2.1 *Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{S}^\bullet(\mathfrak{g})$ edowed with the Gutt star product*

$$x \star_z y = \sum_{n=0}^{\infty} z^n C_n(x, y).$$

i.) On factorizing tensors $\xi_1 \dots \xi_k$ and $\eta_1 \dots \eta_\ell$, C_0 and C_1 give

$$C_0(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) = \xi_1 \dots \xi_k \eta_1 \dots \eta_\ell \tag{4.2.1}$$

$$C_1(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) = \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \xi_1 \dots \widehat{\xi_i} \dots \xi_k \eta_1 \dots \widehat{\eta_j} \dots \eta_\ell [\xi_i \eta_j], \tag{4.2.2}$$

where the hat denotes elements which are left out.

ii.) If $\mathfrak{g}$ is finite-dimensional and we use the canonical isomorphism $\mathcal{J} \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{Pol}^\bullet(\mathfrak{g}^*)$, we have for $f, g \in \mathrm{Pol}^\bullet(\mathfrak{g}^*)$

$$C_1(\mathcal{J}(f), \mathcal{J}(g)) - C_1(\mathcal{J}(f), \mathcal{J}(g)) = \mathcal{J}^{-1}(\{f, g\}_{KKS})$$

where $\{\cdot, \cdot\}_{KKS}$ is the Kirillov-Kostant-Souriau bracket.

iii.) The map $\star_z$ fulfils the classical and the semi-classical limit and is therefore a star product.

PROOF: We take $\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell \in \mathrm{S}^\bullet(\mathfrak{g})$ and consider the G-indices in $\mathcal{G}_{k+\ell}(k, \ell)$ first. This is easy, since there is just one element inside:

$$\mathcal{G}_{k+\ell}(k, \ell) = \Big\{ (\underbrace{(0,1), \dots, (0,1)}_{\ell \text{ times}}, \underbrace{(1,0), \dots, (1,0)}_{k \text{ times}}) \Big\}.$$

So we find

$$\begin{aligned} C_0(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) &= \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{J!} \mathrm{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(1)}\big) \dots \mathrm{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(k)}\big) \\ &\quad \cdot \mathrm{BCH}_{1,0}\big(\eta_{\tau(1)}, \varnothing\big) \dots \mathrm{BCH}_{1,0}\big(\eta_{\tau(\ell)}, \varnothing\big) \\ &= \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{k!\ell!} \xi_{\sigma(1)} \dots \xi_{\sigma(k)} \eta_{\tau(1)} \dots \eta_{\tau(\ell)} \\ &= \xi_{\sigma(1)} \dots \xi_{\sigma(k)} \eta_{\tau(1)} \dots \eta_{\tau(\ell)} \end{aligned}$$

where we used $J! = k!\ell!$ according to Definition 4.1.8. We do the same for $C_1(\dots)$. Also here, we have just one element in $\mathcal{G}_{k+\ell-1}(k, \ell)$:

$$\mathcal{G}_{k+\ell}(k, \ell) = \Big\{ (\underbrace{(0,1), \dots, (0,1)}_{\ell-1 \text{ times}}, \underbrace{(1,0), \dots, (1,0)}_{k-1 \text{ times}}, (1,1)) \Big\}.$$

Using

$$\mathrm{BCH}_{1,1}(\xi, \eta) = \frac{1}{2}[\xi, \eta]$$

and $J! = (k-1)!(\ell-1)!$, we find

$$\begin{aligned} C_1(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) &= \frac{1}{2} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \frac{1}{(k-1)!(\ell-1)!} \xi_{\sigma(1)} \dots \xi_{\sigma(k-1)} \eta_{\tau(1)} \dots \eta_{\tau(\ell-1)} [\xi_{\sigma(k)}, \eta_{\tau(\ell)}] \\ &= \frac{1}{2} \sum_{i=0}^{k} \sum_{j=0}^{\ell} \xi_1 \dots \widehat{\xi_i} \dots \xi_k \eta_1 \dots \widehat{\eta_j} \dots \eta_\ell [\xi_i, \eta_j]. \end{aligned}$$

This finishes part one. From this, the anti-symmetry of the Lie bracket yields

$$C_1(\xi_1 \dots \xi_k, \eta_1 \dots \eta_\ell) - C_1(\eta_1 \dots \eta_\ell, \xi_1 \dots \xi_k) = \sum_{i=0}^{k} \sum_{j=0}^{\ell} \xi_1 \dots \widehat{\xi_i} \dots \xi_k \eta_1 \dots \widehat{\eta_j} \dots \eta_\ell [\xi_i, \eta_j].$$

We now need to compute the KKS brackets on polynomials. Because of the linearity in both arguments, it is sufficient to check it on monomials of coordinates. Let $e_1, \ldots, e_n$ be a basis with linear coordinates $x_1, \ldots, x_n$. Now take $\mu_1, \ldots, \mu_n, \nu_1, \ldots \nu_n \in \mathbb{N}$ and consider the monomials $f = x_1^{\mu_1} \ldots x_n^{\mu_n}$ and $g = x_1^{\nu_1} \ldots x_n^{\nu_n}$. We use the notation fronm Proposition 3.1.1 and find for $x \in \mathfrak{g}^*$

$$\begin{aligned}\{f, g\}_{KKS}(x) &= x_k c_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \\ &= \mu_i \nu_j c_{ij}^k x_k x_1^{\mu_1} \ldots x_i^{\mu_i - 1} \ldots x_n^{\mu_n} x_1^{\nu_1} \ldots x_j^{\nu_j - 1} \ldots x_n^{\nu_n}.\end{aligned}$$

Applying $\mathcal{J}^{-1}$ to it gives

$$\mathcal{J}^{-1}(\{f, g\}_{KKS}) = \sum_{i=0}^{n} \sum_{j=0}^{n} \mu_i \nu_j e_1^{\mu_1} \ldots e_i^{\mu_i - 1} \ldots e_n^{\mu_n} e_1^{\nu_1} \ldots e_j^{\nu_j - 1} \ldots e_n^{\nu_n} [e_i, e_j]. \tag{4.2.3}$$

On the other hand, we have

$$\mathcal{J}(f) = e_1^{\mu_1} \ldots e_n^{\mu_n} \quad \text{and} \quad \mathcal{J}(g) = e_1^{\nu_1} \ldots e_n^{\nu_n}.$$

Together with (4.2.2) this is exactly (4.2.3) and proves part two. From this, the third part is immediate, due to the bilinearity of the C_n. □

It is clear, that the formulas from Proposition 4.1.11 and Proposition 4.1.3 should coincide. However, we want to check it, to have the evidence that everything works as we wanted.

Corollary 4.2.2 *Given $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$, the results of the Equations* (4.1.14) *and* (4.1.3) *are compatible.*

PROOF: We have to compute sets of G-indices for $\xi_1, \ldots, \xi_k, \eta_\ell \in \mathfrak{g}$. Again, they only have one element:

$$\mathcal{G}_{k+1-n}(k, 1) = \Big\{ (\underbrace{(1,0), \ldots, (1,0)}_{k-n \text{ times}}, (n, 1)) \Big\}.$$

So we have with $J! = (k-n)!$

$$z^n C_n(\xi_1 \ldots \xi_k, \eta_\ell) = z^n \sum_{\sigma \in S_k} \frac{1}{(k-n)!} \frac{B_n^*}{n!} \xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} [\xi_{\sigma(k-n+1)}, [\ldots, [\xi_{\sigma(k), \eta}] \ldots]]$$

which gives, after a light reordering

$$z^n C_n(\xi_1 \ldots \xi_k, \eta_\ell) = z^n \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{n} B_n^* [\xi_{\sigma(1)}, [\ldots, [\xi_{\sigma(n), \eta}] \ldots]] \xi_{\sigma(n+1)} \cdots \xi_{\sigma(k)}.$$

Summing up over all n now gives Equation (4.1.3). □

Two examples

Equation (4.1.14) is useful if one wants to do real computations with the star product, but it is maybe not intuitive to apply. This is why we will give two examples here. The easiest one which is not covered by the simpler formula (4.1.3) will be the star product of two quadratic terms. The second one should be the a bit more complex case of a cubic term with a quadratic term.

Two quadratic terms

Let's take $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathfrak{g}$. We want to compute

$$\xi_1\xi_2 \star_z \eta_1\eta_2 = C_0(\xi_1, \xi_2, \eta_1, \eta_2) + zC_1(\xi_1, \xi_2, \eta_1, \eta_2) + z^2C_2(\xi_1, \xi_2, \eta_1, \eta_2) + z^3C_3(\xi_1, \xi_2, \eta_1, \eta_2).$$

The very first thing we have to do is computing the set of G-indices. Then we calculate the G-factorial and finally go through the permutations.

C_0: We already did this in Corollary 4.2.1, and know that the zeroth order in z is just the symmetric product. Therefore we have

$$C_0(\xi_1\xi_2, \eta_1\eta_2) = \xi_1\xi_2\eta_1\eta_2$$

C_1: We also did this one in Corollary 4.2.1: There is just one G-index and we finally get

$$C_1(\xi_1\xi_2, \eta_1\eta_2) = \frac{1}{2}(\xi_2\eta_2[\xi_1, \eta_1] + \xi_2\eta_1[\xi_1, \eta_2] + \xi_1\eta_2[\xi_2, \eta_1] + \xi_1\eta_1[\xi_2, \eta_2]).$$

C_2: This is the first time, something interesting happens. We have three G-indices:

$$\mathcal{G}_2(2, 2) = \{J^1, J^2, J^3\} = \{\big((0, 1), (2, 1)\big), \big((1, 0), (1, 2)\big), \big((1, 1), (1, 1)\big)\}.$$

The G-factorials give $J^1! = J^2! = 1$ and $J^3! = 2$, since the index $(1, 1)$ appears twice in J_3. We take $\mathrm{BCH}_{1,2}(X, Y)$ and $\mathrm{BCH}_{2,1}(X, Y)$ for two variables X and Y from Equation (**??**):

$$\mathrm{BCH}_{1,2}(X, Y) = \frac{1}{12}[Y, [Y, X]] \quad \text{and} \quad \mathrm{BCH}_{2,1}(X, Y) = \frac{1}{12}[X, [X, Y]].$$

So we have to insert the ξ_i and the η_j into $\frac{1}{12}X[Y, [Y, X]]$ and $\frac{1}{12}Y[X, [X, Y]]$ respectively and then we go on with the last one, which is

$$\frac{1}{2}\mathrm{BCH}_{1,1}(X, Y)\mathrm{BCH}_{1,1}(X, Y) = \frac{1}{8}[X, Y][X, Y].$$

We hence get

$$\begin{aligned} C_2(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{1}{12}\big(&\eta_1[[\eta_2, \xi_1], \xi_2] + \eta_1[[\eta_2, \xi_2], \xi_1] + \eta_2[[\eta_1, \xi_1], \xi_2] + \eta_2[[\eta_1, \xi_2], \xi_1] + \\ &\xi_1[[\xi_2, \eta_1], \eta_2] + \xi_1[[\xi_2, \eta_2], \eta_1] + \xi_2[[\xi_1, \eta_1], \eta_2] + \xi_2[[\xi_1, \eta_2], \eta_1]\big) + \\ &\frac{1}{4}\big([\xi_1, \eta_1][\xi_2, \eta_2] + [\xi_1, \eta_2][\xi_2, \eta_1]\big) \end{aligned}$$

C_3: Here, we only have one G-index:

$$\mathcal{G}_1(2, 2) = \{\big((2, 2)\big)\}$$

The G-factorial is 1. We take again Equation (**??**) and see

$$\mathrm{BCH}_{2,2}(X, Y) = \frac{1}{24}[Y, [X, [Y, X]]].$$

This gives

$$C_3(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{1}{24}\big([[[\eta_1, \xi_1], \xi_2], \eta_2] + [[[\eta_1, \xi_2], \xi_1], \eta_2] + [[[\eta_2, \xi_1], \xi_2], \eta_1] + [[[\eta_2, \xi_2], \xi_1], \eta_1]\big)$$

We just have to put all the four terms together and have the star product.

A cubic and a quadratic term

Let $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2 \in \mathfrak{g}$. We compute

$$\xi_1\xi_2\xi_3 \star_G \eta_1\eta_2 = \sum_{n=0}^{4} z^n C_n(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2)$$

C_0: The first part is again just the symmetric product:

$$C_0(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \xi_1\xi_2\xi_3\eta_1\eta_2.$$

C_1: Here we have again the term from Corollary **??**:

$$\begin{aligned} C_1(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \frac{1}{2}\big(&\xi_2\xi_3\eta_2[\xi_1, \eta_1] + \xi_2\xi_3\eta_1[\xi_1, \eta_2] + \xi_1\xi_3\eta_2[\xi_2, \eta_1] + \\ &\xi_1\xi_3\eta_1[\xi_2, \eta_2] + \xi_1\xi_2\eta_2[\xi_3, \eta_1] + \xi_1\xi_2\eta_1[\xi_3, \eta_2]\big) \end{aligned}$$

C_2: Here the calculation is very similar to the one of C_2 in the example before. We have three G-indices:

$$\mathcal{G}_3(3,2) = \{J^1, J^2, J^3\} = \{\big((0,1),(1,0),(2,1)\big), \big((1,0),(1,0),(1,2)\big), \big((1,0),(1,1),(1,1)\big)\}.$$

The G-factorials are now $J^1! = 1$ and $J^2! = J^3! = 2$. Again, we take the BCH terms from Equation (**??**) and see, that we must insert the ξ_i and the η_j into

$$\frac{1}{12}XY[X,[X,Y]] + \frac{1}{24}XX[Y,[Y,X]] + \frac{1}{8}X[X,Y][X,Y].$$

Now we go through all the possible permutations and get

$$\begin{aligned} C_2(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \frac{1}{12}\big(&\xi_1\xi_2[[\xi_3, \eta_1], \eta_2] + \xi_1\xi_2[[\xi_3, \eta_2], \eta_1] + \xi_1\xi_3[[\xi_2, \eta_1], \eta_2] + \\ &\xi_1\xi_3[[\xi_2, \eta_2], \eta_1] + \xi_2\xi_3[[\xi_1, \eta_1], \eta_2] + \xi_2\xi_3[[\xi_1, \eta_2], \eta_1]\big) + \\ &\frac{1}{12}\big(\xi_1\eta_1[[\eta_2, \xi_2], \xi_3] + \xi_1\eta_2[[\eta_1, \xi_2], \xi_3] + \xi_1\eta_1[[\eta_2, \xi_3], \xi_2] + \\ &\xi_1\eta_2[[\eta_1, \xi_3], \xi_2] + \xi_2\eta_1[[\eta_2, \xi_1], \xi_3] + \xi_2\eta_2[[\eta_1, \xi_1], \xi_3] + \\ &\xi_2\eta_1[[\eta_2, \xi_3], \xi_1] + \xi_2\eta_2[[\eta_1, \xi_3], \xi_1] + \xi_3\eta_1[[\eta_2, \xi_2], \xi_1] + \\ &\xi_3\eta_2[[\eta_1, \xi_2], \xi_1] + \xi_3\eta_1[[\eta_2, \xi_1], \xi_2] + \xi_3\eta_2[[\eta_1, \xi_1], \xi_2]\big) + \\ &\frac{1}{4}\big(\xi_1[\xi_2, \eta_1][\xi_3, \eta_2] + \xi_1[\xi_3, \eta_1][\xi_2, \eta_2] + \xi_2[\xi_1, \eta_1][\xi_3, \eta_2] + \\ &\xi_2[\xi_3, \eta_1][\xi_1, \eta_2] + \xi_3[\xi_1, \eta_1][\xi_2, \eta_2] + \xi_3[\xi_2, \eta_1][\xi_1, \eta_2]\big). \end{aligned}$$

C_3: We first calculate the G-indices:

$$\mathcal{G}_2(3,2) = \{J^1, J^2, J^3\} = \{\big((0,1),(3,1)\big), \big((1,0),(2,2)\big), \big((1,1),(2,1)\big)\}.$$

We don't have to care about J^1, since $\mathrm{BCH}_{3,1}(X,Y) = 0$. The G-factorials for the other two indices are 1. The BCH terms have been computed before. So we have to fill in the expression

$$\frac{1}{24}X[Y,[X,[Y,X]]] + \frac{1}{2 \cdot 12}[X,Y][X,[X,Y]].$$

Doing the permutations, we get

$$\begin{aligned}
C_3(\xi_1\xi_2\xi_3,\eta_1\eta_2) = \frac{1}{24}\big(&\xi_1[[[\eta_1,\xi_2],\xi_3],\eta_2] + \xi_1[[[\eta_2,\xi_2],\xi_3],\eta_1] + \xi_1[[[\eta_1,\xi_3],\xi_2],\eta_2]+\\
&\xi_1[[[\eta_2,\xi_3],\xi_2],\eta_1] + \xi_2[[[\eta_1,\xi_1],\xi_3],\eta_2] + \xi_2[[[\eta_2,\xi_1],\xi_3],\eta_1]+\\
&\xi_2[[[\eta_1,\xi_3],\xi_1],\eta_2] + \xi_2[[[\eta_2,\xi_3],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_2],\xi_1],\eta_2]+\\
&\xi_3[[[\eta_2,\xi_2],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_1],\xi_2],\eta_2] + \xi_3[[[\eta_2,\xi_1],\xi_2],\eta_1]\big)+\\
&\frac{1}{24}\big([\xi_1,\eta_1][[\eta_2,\xi_2],\xi_3] + [\xi_1,\eta_2][[\eta_1,\xi_2],\xi_3] + [\xi_1,\eta_1][[\eta_2,\xi_3],\xi_2]+\\
&[\xi_1,\eta_2][[\eta_1,\xi_3],\xi_2] + [\xi_2,\eta_1][[\eta_2,\xi_1],\xi_3] + [\xi_2,\eta_2][[\eta_1,\xi_1],\xi_3]+\\
&[\xi_2,\eta_1][[\eta_2,\xi_3],\xi_1] + [\xi_2,\eta_2][[\eta_1,\xi_3],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_2],\xi_1]+\\
&[\xi_3,\eta_2][[\eta_1,\xi_2],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_1],\xi_2] + [\xi_3,\eta_2][[\eta_1,\xi_1],\xi_2]\big).
\end{aligned}$$

C_4: Now there's only C_4 left. We only have one G-index:

$$\mathcal{G}_1(3,2) = \big\{\big((3,2)\big)\big\},$$

but there are more terms which belong to it. We have to go through

$$\mathrm{BCH}_{3,2}(X,Y) = \frac{1}{120}[[[[X,Y],X],Y],X] + \frac{1}{360}[[[[Y,X],X],X],Y].$$

So we permute and get

$$\begin{aligned}
C_4(\xi_1\xi_2\xi_3,\eta_1\eta_2) = \frac{1}{120}\big(&[[[[\xi_1,\eta_1],\xi_2],\eta_2],\xi_3] + [[[[\xi_1,\eta_2],\xi_2],\eta_1],\xi_3] + [[[[\xi_1,\eta_1],\xi_3],\eta_2],\xi_2]+\\
&[[[[\xi_1,\eta_2],\xi_3],\eta_1],\xi_2] + [[[[\xi_2,\eta_1],\xi_1],\eta_2],\xi_3] + [[[[\xi_2,\eta_2],\xi_1],\eta_1],\xi_3]+\\
&[[[[\xi_2,\eta_1],\xi_3],\eta_2],\xi_1] + [[[[\xi_2,\eta_2],\xi_3],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_2],\eta_2],\xi_1]+\\
&[[[[\xi_3,\eta_2],\xi_2],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_1],\eta_2],\xi_2] + [[[[\xi_3,\eta_2],\xi_1],\eta_1],\xi_2]\big)+\\
&\frac{1}{360}\big([[[[\eta_1,\xi_1],\xi_2],\xi_3],\eta_2] + [[[[\eta_2,\xi_1],\xi_2],\xi_3],\eta_1] + [[[[\eta_1,\xi_1],\xi_3],\xi_2],\eta_2]+\\
&[[[[\eta_2,\xi_1],\xi_3],\xi_2],\eta_1] + [[[[\eta_1,\xi_2],\xi_1],\xi_3],\eta_2] + [[[[\eta_2,\xi_2],\xi_1],\xi_3],\eta_1]+\\
&[[[[\eta_1,\xi_2],\xi_3],\xi_1],\eta_2] + [[[[\eta_2,\xi_2],\xi_3],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_2],\xi_1],\eta_2]+\\
&[[[[\eta_2,\xi_3],\xi_2],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_1],\xi_2],\eta_2] + [[[[\eta_2,\xi_3],\xi_1],\xi_2],\eta_1]\big).
\end{aligned}$$

Now we only have to add up all those terms and we have finally computed the star product.

Chapter 5

A locally convex topology for the Gutt star product

We have finished the algebraic part of this work, except for some little lemmas concerning the Hopf theoretic chapter. Our next goal is setting up a locally convex topology on the symmetric tensor algebra, in which the Gutt star product will converge. At the beginning of this chapter, we will first give a motivation why the setting of locally convex algebras is convenient and necessary. In the second part, we will briefly recall the most important things on locally convex algebras and introduce the topology which we will work with. In the third section, the core of this chapter, the continuity of the star product and the dependence on the formal parameter are proven. Part four treats the case when the formal parameter $z = 1$ and hence talks about a locally convex topology on the universal enveloping algebra of a Lie algebra. We will also show, that our topology is "optimal" in a specific sense.

5.1 Why locally convex?

The first question one could ask is why we want the observable algebra to be a *locally convex* one. There are a lot of different choices and most of them would even make things simpler: we could think of locally multiplicatively convex algebras, Banach algebras, C^*- or even von Neumann algebras. All of them have much more structure than just locally convex algebras. We would have an entire holomorphic calculus within our algebra if we assumed it to be locally m-convex, or even a continuous one if we wanted it to be C^*.

The reason is that, in general, all these nice features are simply not there. Quantum mechanics tells us that the algebra made up by the space and momentum operators $\hat{q}$ and $\hat{p}$ can not be locally m-convex.

Proposition 5.1.1 *Let $\mathcal{A}$ be a unital associative algebra which contains the quantum mechanical observables $\hat{q}$ and $\hat{p}$ and in which the canonical commutation relation*

$$[\hat{q}, \hat{p}] = i\hbar\mathbb{1}$$

is fulfilled. Then the only submultiplicative semi-norm on it is $p = 0$.

Proof: First, we need to show a little lemma:

Lemma 5.1.2 *In the given algebra, we have for $n \in \mathbb{N}$*

$$(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n) = (i\hbar)^n n!\mathbb{1}. \tag{5.1.1}$$

PROOF: To show it, we use the fact that for $a \in \mathcal{A}$ the operator ad_a is a derivation, which is always true for a Lie algebra which comes from an associative algebra with the commutator, since for $a, b, c \in \mathcal{A}$ we have

$$[a, bc] = abc - bca = abc - bac + bac - bca = [a, b]c + b[a, c].$$

For $n = 1$, Equation (5.1.1) is certainly true. So let's look at the step $n \to n + 1$. We make use of the derivation property and have

$$\begin{aligned}
(\mathrm{ad}_{\hat{q}})^{n+1}\left(\hat{p}^{n+1}\right) &= (\mathrm{ad}_{\hat{q}})^n(i\hbar\hat{p}^n + \hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^n(\hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^{n-1}\Big([\hat{q}, \hat{p}]\,\mathrm{ad}_{\hat{q}}(\hat{p}^n) + \hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n)\Big) \\
&= (i\hbar)^{n+1}n! + i\hbar(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n) + (\mathrm{ad}_{\hat{q}})^{n-1}\Big(\hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n)\Big) \\
&= 2(i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^{n-1}\Big(\hat{p}(\mathrm{ad}_{\hat{q}})^2(\hat{p}^n)\Big) \\
&\overset{(*)}{=} \quad \vdots \\
&= n(i\hbar)^{n+1}n! + \mathrm{ad}_{\hat{q}}(\hat{p}(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)) \\
&= n(i\hbar)^{n+1}n! + i\hbar(i\hbar)^n n! \\
&= (i\hbar)^{n+1}(n + 1)!.
\end{aligned}$$

At $(*)$, we actually used another statement which is to be proven by induction over k and says

$$(\mathrm{ad}_{\hat{q}})^{n+1}\left(\hat{p}^{n+1}\right) = k(i\hbar)^{n+1}n! + (\mathrm{ad}_{\hat{q}})^{n+1-k}\Big(\hat{p}(\mathrm{ad}_{\hat{q}})^k(\hat{p}^n)\Big).$$

Since this proof is analogous to the first lines of the computation before, we omit it here and the lemma is proven. $\triangledown$

Now we can go on with the actual proof. Let $\|\cdot\|$ be a submultiplicative semi-norm. Then we see from Equation (5.1.1) that

$$\|(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)\| = |\hbar|^n n! \|\mathbb{1}\|.$$

On the other hand, we have

$$\begin{aligned}
\|(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n)\| &= \|\hat{q}(\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n) - (\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n)\hat{q}\| \\
&\leq 2\|\hat{q}\|\|(\mathrm{ad}_{\hat{q}})^{n-1}(\hat{p}^n)\| \\
&\leq \quad \vdots \\
&\leq 2^n\|\hat{q}\|^n\|\hat{p}^n\| \\
&\leq 2^n\|\hat{q}\|^n\|\hat{p}\|^n
\end{aligned}$$

So in the end we get

$$|\hbar|^n n! \|\mathbb{1}\| \leq c^n$$

for some $c \in \mathbb{R}$. This cannot be fulfilled for all $n \in \mathbb{N}$ unless $\|\mathbb{1}\| = 0$. But then, by submultiplicativity, the semi-norm itself must be equal to 0. $\square$

Remark 5.1.3 The so called Weyl algebra, which fulfils the properties of the foregoing proposition, can be constructed from a Poisson algebra with constant Poisson tensor. On one hand, it is a fair to ask the question, why this restriction of not being locally m-convex should also be put on linear Poisson systems. On the other hand, there is no reason to expect that things become easier when we make the Poisson system more complex. Moreover, the Weyl algebra is actually nothing but a quotient of the universal enveloping algebra of the so called Heisenberg algebra, which is a particular Lie algebra. There is no reason why the original algebra should have a "better" analytical structure than its quotient, since the ideal, which is divided out by this procedure, is a closed one.

There's a second good reason why we should avoid our topology to be locally m-convex. The topology we set up on $\mathrm{S}^\bullet(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ will also give a topology on $\mathcal{U}(\mathfrak{g})$. In Proposition 5.4.1 ,we will show that, under weak (but for our purpose necessary) additional assumptions, there can be no topology on $\mathcal{U}(\mathfrak{g})$ which allows an entire holomorphic calculus. This underlines the results from Proposition 5.1.1, since locally m-convex algebras always have such a calculus.

In this sense, we have good reasons to think that $\mathrm{S}^\bullet(\mathfrak{g})$ will not allow a better setting than the one of a locally convex algebra if we want the Gutt star product to be continuous. Before we attack this task, we have to recall some technology from locally convex analysis.

5.2 Locally convex algebras

5.2.1 Locally convex spaces and algebras

Every locally convex algebra is of course also a locally convex space which is, of course, a topological vector space. To make clear what we talk about, we first give a definition which is taken from [**?**].

Definition 5.2.1 (Topological vector space) *Let V be a vector space endowed with a topology τ. Then we call (V, τ) (or just V, if there is no confusion possible) a topological vector space, if the two following things hold:*

i.) for every point in $x \in V$ the set $\{x\}$ is a closed and

ii.) the vector space operations (addition, scalar multiplication) are continuous.

Not all books require axiom (*i*) for a topological vector space. It is, however, useful, since it assures that the topology in a topological vector space is Hausdorff – a feature which we will always want to have. The proof for this is not difficult, but since we don't want to go too much into detail here, we refer to [**?**] again, where it can be found as Theorem 1.12.

The most important class of topological vector spaces are, at least, but not only, from a physical point of view, locally convex ones. Almost all interesting physical examples belong to this class: Finite-dimensional spaces, inner product (or pre-Hilbert) spaces, Banach spaces, Fréchet spaces, nuclear spaces and many more. There are at least two equivalent definitions of what is a locally convex space. While the first is more geometrical, the second is better suited for our analytic purpose.

Theorem 5.2.2 *For a topological vector space V, the following things are equivalent.*

i.) V has a local base $\mathscr{B}$ of the topology whose members are convex.

ii.) The topology on V is generated by a separating family of semi-norms $\mathcal{P}$.

Proof: This theorem is a very well-known result and can be found in standard literature, such as [**?**] again, where it is divided into two Theorems (namely 1.36 and 1.37). □

Definition 5.2.3 (Locally convex space) *A locally convex space is a topological vector space in which one (and thus all) of the properties from Theorem 5.2.2 are fulfilled.*

The first property explains the term "locally convex". For our intention, the second property is more helpful, since in this setting proving continuity just means putting estimates on semi-norms. For this purpose, one often extends the set of semi-norms $\mathcal{P}$ to the set of all continuous semi-norms $\mathscr{P}$ which contains all semi-norms that are compatible with the topology (e.g. sums, multiples and maxima of (finitely many) semi-norms from $\mathcal{P}$). From here, we can start looking at locally convex algebras.

Definition 5.2.4 (Locally convex algebra) *A locally convex algebra is a locally convex vector space with an additional algebra structure which is continuous.*

More precisely, let $\mathcal{A}$ be a locally convex algebra and $\mathscr{P}$ the set of all continuous semi-norms, then for all $p \in \mathscr{P}$ there exists a $q \in \mathscr{P}$ such that for all $x, y \in \mathcal{A}$ one has

$$p(ab) \leq q(a)q(b). \tag{5.2.1}$$

Remind that we didn't require our algebras to be associative. The product in this equation could also be a Lie bracket. If we talk about associative algebras, we will always say it explicitly.

5.2.2 A special class of locally convex algebras

For our study of the Gutt star product, the usual continuity estimate (5.2.1) will not be enough, since there will be an arbitrarily high number of nested brackets to control. We will need an estimate which does not depend on the number of Lie brackets involved. Since Lie algebras are just one type of algebras, we can define the property we need also for other locally convex algebras.

Definition 5.2.5 (Asymptotic estimate algebra) *Let $\mathcal{A}$ be a locally convex algebra (not necessarily associative) with the set of all continuous semi-norms $\mathscr{P}$. For a given $p \in \mathscr{P}$ we call $q \in \mathscr{P}$ an asymptotic estimate for p, if there exists an $m \in \mathbb{N}$ such that for all $n \geq m$ we have*

$$p(x_1 \cdot \ldots \cdot x_n) \leq q(x_1) \ldots q(x_n) \quad \forall_{x_1,\ldots,x_n \in \mathscr{A}}. \tag{5.2.2}$$

For non-associative algebras, we want this estimate to be fulfilled for all ways of setting brackets on the left hand side. We call a locally convex algebra an AE algebra, if every $p \in \mathscr{P}$ has an asymptotic estimate.

Remark 5.2.6 Without further restrictions, we can set $m = 1$ in the upper definition, since this just means taking the maximum over a finite number of continuous semi-norms. If q satisfies the upper definition for some $m \in \mathbb{N}$ and for all $i = 2, \ldots, m-1$ we have

$$p(x_1 \cdot \ldots \cdot x_i) \leq q^{(i)}(x_1) \ldots q^{(i)}(x_i)$$

for all $x_1, \ldots, x_i \in \mathcal{A}$, then we just set

$$q' = \max\{p, q^{(2)}, \ldots, q^{(m-1)}, q\}.$$

Clearly, q' will again be a continuous semi-norm and an asymptotic estimate for p.

Remark 5.2.7 (The notion "asymptotic estimate")

i.) The term asymptotic estimate has, to the best of our knowledge, first been used by Boseck, Czichowski and Rudolph in [?]. They defined asymptotic estimates in the same way we did, but their idea of an AE algebra was different from ours: for them, in an AE algebra every continuous semi-norm admits a series of asymptotic estimates. This series must fulfil two additional properties, which actually make the algebra locally m-convex. Clearly, our definition is weaker, since it does not imply, a priori, the existence of an topologically equivalent set of submultiplicative semi-norms.

ii.) In [?], Glöckner and Neeb used a property to which they referred as $(*)$ for associative algebras. It was then used in [] by ... and ..., who called it the GN-property. It is easy to see that it is equivalent to our AE condition.

There are, of course, a lot of examples of AE (Lie) algebras. All finite dimensional and Banach (Lie) algebras fulfil (5.2.2), just as locally m-convex (Lie) algebras do. The same is true for nilpotent locally convex Lie algebras, since here again one just has to take the maximum of a finite number of semi-norms, analogously to the procedure in Remark 5.2.6.

It is far from clear what is exactly implied by the AE property. Are there examples for associative algebras which are AE but not locally m-convex, for example? Are there Lie algebras which are truly AE and not locally m-convex or nilpotent? We don't have an answer to this questions, but we can make some simple observations, which allow us to give an answer for special cases.

Proposition 5.2.8 (Entire calculus) *Let $\mathcal{A}$ be an associative AE algebra. Then it has an entire holomorphic calculus.*

Proof: The proof is the same as for locally m-convex algebras: let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function with $f(z) = \sum_n a_n z^n$ and p a continuous semi-norm with an asymptotic estimate q. Then one has $\forall_{x\in\mathcal{A}}$

$$p(f(x)) = p\left(\sum_{n=0}^{\infty} a_n x^n\right) \leq \sum_{n=0}^{\infty} |a_n| p(x^n) \leq \sum_{n=0}^{\infty} |a_n| q(x)^n < \infty. \qquad \square$$

Remark 5.2.9 (Entire Calculus, AE and LMC algebras) The fact that AE algebras have an entire calculus makes them very similar to locally m-convex ones. Now there is something we can say about associative algebras which have an entire calculus: if such an algebra is additionally commutative and Fréchet, then must be even locally m-convex. This statement was proved in [?] by Mitiagin, Rolewicz and Zelazko. Oudadess and El kinani extended this result to commutative, associative algebras, in which the Baire category theorem holds. For non-commutative algebras, the situation is different. There are associative "Baire algebras" having an entire calculus, which are not locally m-convex. Zelazko gave an example for such an algebra in [?]. Unfortunately, his example is also not AE. It seems to be is an interesting (and non-trivial) question, if a non locally m-convex but AE algebra exists at all and if yes, how an example could look like.

5.2.3 The projective tensor product

We want to set up a topology on $\mathrm{S}^\bullet(\mathfrak{g})$. Therefore, we will first construct a topology on the tensor algebra $\mathrm{T}^\bullet(\mathfrak{g})$. As all the following constructions in this section don't use any algebra structure, we will do them on a locally convex vector space V where $\mathscr{P}$ is the set of continuous semi-norms. Then we can use the projective tensor product $\otimes_\pi$ in order to get a locally convex topology on each tensor power $V^{\otimes_\pi n}$. The precise construction can be found in standard textbooks on locally convex analysis like [?] or in the lecture notes [?]. Recall that for $p_1, \ldots, p_n \in \mathscr{P}$ we have a continuous semi-norm on $V^{\otimes_\pi n}$ via

$$(p_1 \otimes_\pi \ldots \otimes_\pi p_n)(x) = \inf\left\{ \sum_i p_1\left(x_i^{(1)}\right) \ldots p_n\left(x_i^{(n)}\right) \,\middle|\, x = \sum_i x_i^{(1)} \otimes \ldots \otimes x_i^{(n)} \right\}.$$

On factorizing tensors, we moreover have the property

$$(p_1 \otimes_\pi \ldots \otimes_\pi p_n)(x_1 \otimes_\pi \ldots \otimes_\pi x_n) = p_1(x_1) \ldots p_n(x_n) \tag{5.2.3}$$

which will be extremely useful in the following and which can be proven by the Hahn-Banach theorem. We also have

$$(p_1 \otimes \ldots \otimes p_n) \otimes (q_1 \otimes \ldots \otimes q_m) = p_1 \otimes \ldots \otimes p_n \otimes q_1 \otimes \ldots \otimes q_m.$$

For a given $p \in \mathscr{P}$ we will denote $p^n = p^{\otimes_\pi n}$ and p^0 is just the absolute value on the field $\mathbb{K}$. The π-topology on $V^{\otimes_\pi n}$ is set up by all the projective tensor products of continuous semi-norms, or, equivalently, by all the p^n for $p \in \mathscr{P}$.

The projective tensor product has a very nice feature: if we want to show a (continuity) estimate on the tensor algebra, it is enough to do it on factorizing tensors. We will use this very often and just refer to it as the "infimum argument".

Lemma 5.2.10 (Infimum argument for the projective tensor product) *Let $V_1, \ldots, V_n, W$ be locally convex vector spaces and*

$$\phi\colon V_1 \times \ldots \times V_n \longrightarrow W$$

a n-linear map, from which we get the linear map $\Phi\colon V_1 \otimes_\pi \ldots \otimes_\pi V_n \longrightarrow W$. Then Φ is continuous if and only if this is true for ϕ and if for $p, q \in \mathscr{P}$ the estimate

$$p(\Phi(x_1 \otimes \ldots \otimes x_n)) \leq q(v_1) \ldots q(v_n)$$

is fulfilled for all $x_i \in V_i, i = 1, \ldots, n$, then we have

$$p(\Phi(x)) \leq q(x)$$

for all $x \in V_1 \otimes \ldots \otimes V_n$.

Proof: If Φ is continuous, the continuity of ϕ is clear. The other implication is more interesting. Continuity for ϕ means, that for every continuous semi-norm q on W we have continuous semi-norms p_i on V_i with $i = 1, \ldots, n$ such that for all $x^{(i)} \in V_i$ the estimate

$$q\Big(\phi\Big(x^{(1)}, \ldots, x^{(n)}\Big)\Big) \leq p_1\Big(x^{(1)}\Big) \ldots p_n\Big(x^{(n)}\Big) \tag{5.2.4}$$

holds. Let $x \in V_1 \otimes_\pi \ldots \otimes_\pi V_n$, then it has a representation in terms of factorizing tensors like

$$x = \sum_j x_j^{(1)} \otimes_\pi \ldots \otimes_\pi x_j^{(n)}.$$

We thus have

$$\begin{aligned} q(\Phi(x)) &= q\left(\sum_j \Phi\Big(x_j^{(1)} \otimes_\pi \ldots \otimes_\pi x_j^{(n)}\Big)\right) \\ &\leq \sum_j q\Big(\phi\Big(x_j^{(1)}, \ldots, x_j^{(n)}\Big)\Big) \\ &\leq \sum_j p_1\Big(x_j^{(1)}\Big) \ldots p_n\Big(x_j^{(n)}\Big). \end{aligned}$$

Now we take the infimum over all possibilities of writing x as a sum of factorizing tensors on both sides. While nothing will happen on the left hand side, on the right hand side we will find $(p_1 \otimes_\pi \ldots \otimes_\pi p_n)(x)$. This gives exactly the estimate we wanted. $\square$

Most of the time, we will deal with the symmetric tensor algebra. Therefore, we want to recall some basic facts about $\mathrm{S}^n(V)$, when it inherits the π-topology from the $V^{\otimes_\pi n}$. We will call it $\mathrm{S}^n_\pi(V)$ when we endow it with this topology.

Lemma 5.2.11 *Let V be a locally convex vector space, p a continuous semi-norm and $n, m \in \mathbb{N}$.*

i.) The symmetrization map

$$\mathscr{S}_n \colon V^{\otimes_\pi n} \longrightarrow V^{\otimes_\pi n}, \quad (x_1 \otimes \ldots \otimes x_n) \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$$

is continuous and we have for all $x \in V^{\otimes_\pi n}$ the estimate

$$p^n(\mathscr{S}_n(x)) \leq p^n(x). \tag{5.2.5}$$

ii.) Each symmetric tensor power $\mathrm{S}^n_\pi(V) \subseteq V^{\otimes_\pi n}$ is a closed subspace.

iii.) For $x \in \mathrm{S}^n_\pi(V)$ and $y \in \mathrm{S}^m_\pi(V)$ we have

$$p^{n+m}(xy) \leq p^n(x) p^m(y).$$

PROOF: The first part is very easy to see and uses most of the tools which are typical for the projective tensor product. We have the estimate for factorizing tensors $x_1 \otimes \ldots \otimes x_n$

$$\begin{aligned}
p^n(\mathscr{S}(x_1 \otimes \ldots \otimes x_n)) &= p^n\left(\frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}\right) \\
&\leq \frac{1}{n!} \sum_{\sigma \in S_n} p^n\big(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}\big) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} p\big(x_{\sigma(1)}\big) \ldots p\big(x_{\sigma(n)}\big) \\
&= p(x_1) \ldots p(x_n) \\
&= p^n(x_1 \otimes \ldots \otimes x_n).
\end{aligned}$$

Then we use the infimum argument from Lemma 5.2.10 and we are done. The second part is also easy since the kernel of a continuous map is always a closed subspace of the initial space and we have

$$\mathrm{S}^n_\pi = \ker(\mathsf{id} - \mathscr{S}_n).$$

The third part is a consequence from the first and also immediate. □

One could maybe think that the inequality in the first part of this lemma is just an artefact which is due to the infimum argument and should actually be an equality, if one looked to it more closely. It is very interesting to see, that this is *not* the case, since it may happen that this inequality is strict. The following example illustrates this.

Example 5.2.12 We take $V = \mathbb{R}^2$ with the standard basis e_1, e_2 and V is endowed with the maximum norm. Now look at $e_1 \otimes e_2$, which has the norm

$$\|e_1 \otimes e_2\| = \|e_1\| \otimes \|e_2\| = 1$$

We now evaluate the symmetrization map on $V \otimes_\pi V$:

$$\mathscr{S}(e_1 \otimes e_2) = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1).$$

Our aim is to show, that the projective tensor product of the norm of this symmetrized vector is not 1. Therefore we need to find another way of writing it which has a norm of less than 1. Observe that

$$\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) = \frac{1}{4}((e_1 + e_2) \otimes (e_1 + e_2) + (-e_1 + e_2) \otimes (e_1 - e_2))$$

and we have

$$\begin{aligned}
&\frac{1}{4}\|(e_1 + e_2) \otimes (e_1 + e_2) + (-e_1 + e_2) \otimes (e_1 - e_2)\| \\
&\leq \frac{1}{4}(\|(e_1 + e_2) \otimes (e_1 + e_2)\| + \|(-e_1 + e_2) \otimes (e_1 - e_2)\|) \\
&= \frac{1}{4}(1 + 1) \\
&= \frac{1}{2}.
\end{aligned}$$

So we have $\|\mathscr{S}(e_1 \otimes e_2)\| \leq \frac{1}{2} < 1$.

5.2.4 A topology for the Gutt star product

The next step is to set up a topology on $\mathrm{T}^\bullet(V)$ which has the π-topology on each component. A priori, there are a lot of such topologies and at least two natural ones: the direct sum topology which is very fine and has a very small closure, and the cartesian product topology which is very coarse and therefore has a very big closure. We need something in between, which we can adjust in a convenient way.

Definition 5.2.13 (R-topology) *Let p be an continuous semi-norm on a locally convex vector space V and $R \in \mathbb{R}$. We define the semi-norm*

$$p_R = \sum_{n=0}^{\infty} n!^R p^n$$

on the Tensor algebra $\mathrm{T}^\bullet(V)$. We write for the tensor or the symmetric algebra endowed with all such semi-norms $\mathrm{T}^\bullet_R(V)$ or $\mathrm{S}^\bullet_R(V)$ respectively.

We now want to collect the most important results on the locally convex algebras $(\mathrm{T}^\bullet_R(V), \otimes)$ and $(\mathrm{S}^\bullet_R(V), \vee)$.

Lemma 5.2.14 *Let $R' \geq R \geq 0$ and q, p are continuous semi-norms on V.*

i.) If $q \geq p$ then $q_R \geq p_R$ and $p_{R'} \geq p_R$.

ii.) The tensor product is continuous and satisfies the following inequality:

$$p_R(x \otimes y) \leq \left(2^R p\right)_R(x)\left(2^R p\right)_R(y)$$

iii.) For all $n \in \mathbb{N}$ the induced topology on $\mathrm{T}^n(V) \subset \mathrm{T}^\bullet_R(V)$ and on $\mathrm{S}^n(V) \subset \mathrm{S}^\bullet_R(V)$ is the π-topology.

iv.) For all $n \in \mathbb{N}$ the projection and the inclusion maps

$$\begin{array}{ccccc}
\mathrm{T}^\bullet_R(V) & \longrightarrow & V^{\otimes_\pi n} & \longrightarrow & \mathrm{T}^\bullet(V) \\
\mathrm{S}^\bullet_R(V) & \longrightarrow & \mathrm{S}^n_\pi(V) & \longrightarrow & \mathrm{S}^\bullet_R(V)
\end{array}$$

are continuous.

v.) The completions $\widehat{\mathrm{T}}^{\bullet}_R(V)$ *of* $\mathrm{T}^{\bullet}_R(V)$ *and* $\widehat{\mathrm{S}}^{\bullet}_R(V)$ *of* $\mathrm{S}^{\bullet}_R(V)$ *can be described explicitly as*

$$\begin{aligned}\widehat{\mathrm{T}}^{\bullet}_R(V) &= \left\{ x = \sum_{n=0}^{\infty} x_n \;\middle|\; p_R(x) < \infty, \text{ for all } p \right\} \subseteq \prod_{n=0}^{\infty} V^{\hat{\otimes}_\pi n} \\ \widehat{\mathrm{S}}^{\bullet}_R(V) &= \left\{ x = \sum_{n=0}^{\infty} x_n \;\middle|\; p_R(x) < \infty, \text{ for all } p \right\} \subseteq \prod_{n=0}^{\infty} \mathrm{S}^n_{\hat{\otimes}_\pi}\end{aligned}$$

with p *running through all continuous semi-norms on* V *and the* p_R *are extended to the Cartesian product allowing the value* $+\infty$*.*

vi.) If $R' > R$*, then the topology on* $\mathrm{T}^{\bullet}_{R'}(V)$ *is strictly finer than the one on* $\mathrm{T}^{\bullet}_R(V)$*, the same holds for* $\mathrm{S}^{\bullet}_{R'}(V)$ *and* $\mathrm{S}^{\bullet}_R(V)$*. Therefore the completions get smaller for bigger* R*.*

vii.) The inclusion maps $\widehat{\mathrm{T}}^{\bullet}_{R'}(V) \longrightarrow \widehat{\mathrm{T}}^{\bullet}_R(V)$ *and* $\widehat{\mathrm{S}}^{\bullet}_{R'}(\mathfrak{g}) \longrightarrow \widehat{\mathrm{S}}^{\bullet}_R(\mathfrak{g})$ *are continuous.*

viii.) The topology on $\mathrm{T}^{\bullet}_R(V)$ *with the tensor product and on* $\mathrm{S}^{\bullet}_R(V)$ *with the symmetric product is locally m-convex if and only if* $R = 0$*.*

ix.) The algebras $\mathrm{T}^{\bullet}_R(V)$ *and* $\mathrm{S}^{\bullet}_R(V)$ *are first countable if and only if this is true for* V*.*

PROOF: The first part is clear on factorizing tensors and extends to the whole tensor algebra via the infimum argument. For part (*ii*), take two factorizing tensors

$$x = x^{(1)} \otimes \ldots \otimes x^{(n)} \quad \text{ and } y = y^{(1)} \otimes \ldots \otimes y^{(m)}$$

and compute:

$$\begin{aligned} p_R(x \otimes y) &= (n+m)!^R p^{n+m}\Big(x^{(1)} \otimes \ldots x^{(n)} \otimes y^{(1)} \otimes \ldots y^{(m)}\Big) \\ &= (n+m)!^R p^n \Big(x^{(1)} \otimes \ldots x^{(n)}\Big) p^m \Big(y^{(1)} \otimes \ldots y^{(m)}\Big) \\ &= \binom{n+m}{n}^R n!^R m!^R p^n \Big(x^{(1)} \otimes \ldots x^{(n)}\Big) p^m \Big(y^{(1)} \otimes \ldots y^{(m)}\Big) \\ &\leq 2^{(n+m)R} p_R \Big(x^{(1)} \otimes \ldots x^{(n)}\Big) p_R \Big(y^{(1)} \otimes \ldots y^{(m)}\Big) \\ &= \big(2^R p\big)_R \Big(x^{(1)} \otimes \ldots x^{(n)}\Big) \big(2^R p\big)_R \Big(y^{(1)} \otimes \ldots y^{(m)}\Big). \end{aligned}$$

The parts (*iii*) and (*iv*) are clear from the construction of the R- topology. In part (*v*) we used the completion of the tensor product $\hat{\otimes}$, the statement itself is clear and implies (*vi*) directly, since we have really more elements in the completion for $R < R'$, like the series over $x^n \frac{1}{n!^t}$ for $t \in (R, R')$ and $0 \neq x \in V$. Statement (*vii*) follows from the first. For (*viii*), it is easy to see that $\mathrm{T}^{\bullet}_0(V)$ and $\mathrm{S}^{\bullet}_0(V)$ are locally m-convex. For every $R > 0$ we have

$$p_R(x^n) = n!^R p(x)^n$$

for all $n \in \mathbb{N}$ and all $x \in V$. If we had a submultiplicative semi-norm $\|\cdot\|$ from an equivalent topology, then we would have some $x \in V$, and a continuous semi-norm p with $p(x) \neq 0$ such that $p_R \leq \|\cdot\|$, and hence

$$n!^R p(x)^R \leq \|x^n\| \leq \|x\|^n.$$

Since this is valid for all $n \in \mathbb{N}$, we get a contradiction. For the last part, the tensor algebras cannot be first countable if V itself isn't. On the other hand, if V has a finite base of the topology, then $\mathrm{T}^{\bullet}_R(V)$ and $\mathrm{S}^{\bullet}_R(V)$ are just a countable multiple of V and stay therefore first countable. □

The projective tensor product obviously keeps a lot of important and strong properties of the original vector space V. But Proposition **??** still leaves some important things. We will not make use of them in the following, but it is worth naming them for completeness. To do this in full generality, we need one more definition, which will be also very important in chapter 5.

Definition 5.2.15 *For a locally convex vector space V and $R \geq 0$ we set*

$$\mathrm{S}^{\bullet}_{R^-}(V) = \operatorname*{proj\,lim}_{\epsilon \longrightarrow 0} \mathrm{S}^{\bullet}_{1-\epsilon}(V)$$

and call its completion $\widehat{\mathrm{S}}^{\bullet}_{R^-}(V)$.

Now we can state two more propositions. Since we won't use them, we omit the proofs here. They can be found in [?].

Proposition 5.2.16 *Let $R \geq 0$ and V a locally convex vector space. If $\{e_i\}_{i \in I}$ is an absolute Schauder basis of V with coefficient functionals $\{\varphi^i\}_{i \in I}$, i.e. for every $x \in V$ we have*

$$x = \sum_{i \in I} \varphi^i(x) e_i$$

such that for every $p \in \mathscr{P}$ there is a $q \in \mathscr{P}$ such that

$$\sum_{i \in I} |\varphi^i(x)| p(e_i) \leq q(x), \tag{5.2.6}$$

then the set $\{e_{i_1} \otimes \ldots \otimes e_{i_n}\}_{i_1, \ldots, i_n \in I}$ defines an absolute Schauder basis of $\mathrm{T}^{\bullet}_R(V)$ together with the linear functionals $\{\varphi^{i_1} \otimes \ldots \otimes \varphi^{i_n}\}_{i_1, \ldots, i_n \in I}$ which satisfy

$$\sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n \in I} \big|(\varphi^{i_1} \otimes \ldots \otimes \varphi^{i_n})(x)\big| p_R(e_{i_1} \otimes \ldots \otimes e_{i_n}) \leq q_R(x)$$

for every $x \in \mathrm{T}^{\bullet}_R(V)$ whenever p and q satisfy (5.2.6). The same statement is true for $\mathrm{S}^{\bullet}_R(V)$ and for $\mathrm{S}^{\bullet}_{R^-}(V)$ (for $R > 0$) when we choose a maximal linearly independent subset out of the set $\{e_{i_1} \ldots e_{i_n}\}_{i_1, \ldots, i_n \in I}$.

Proposition 5.2.17 *Let V be a locally convex space. For $R \geq 0$ the following statements are equivalent:*

i.) *V is nuclear.*

ii.) *$\mathrm{T}^{\bullet}_R(V)$ is nuclear.*

iii.) *$\mathrm{S}^{\bullet}_R(V)$ is nuclear.*

If moreover $R > 0$, then the following statements are equivalent:

i.) *V is strongly nuclear.*

ii.) *$\mathrm{T}^{\bullet}_R(V)$ is strongly nuclear.*

iii.) *$\mathrm{S}^{\bullet}_R(V)$ is strongly nuclear.*

5.3 Continuity results for the Gutt star product

From now on, we start with an AE Lie algebra $\mathfrak{g}$ rather than with a general locally convex space. We have all the tools by the hand to show the continuity of the Gutt star product. We can do it either via the bigger formula (4.1.14) for two monomials or via the smaller one (4.1.3) for a monomial with a vector and iterate it. The results are very similar, but a bit better for the first approach. Nevertheless, both approaches give strong results, and depending on the precise situation, each one has its advantages. This is why we want to give both proofs here.

There will be a very general way how most of the proofs will work, and which tools will be used in the following. If we want to show the continuity of a map $f \colon \mathrm{S}^{\bullet}_R(\mathfrak{g}) \longrightarrow \mathrm{S}^{\bullet}_R(\mathfrak{g})$, we will proceed most of the time like this:

i.) First, we extend a map to the whole tensor algebra by putting the symmetrizer in front: $f = f \circ \mathscr{S}$. This doesn't lead to problems since the symmetrization does not affect symmetric tensors.

ii.) Then, we start with an estimate, which we do only on factorizing tensors in order to use the infimum argument (Lemma 5.2.10).

iii.) During the estimation process, we find symmetric products of Lie brackets. Those will be split up by the continuity of the symmetric product (5.2.5) from Lemma 5.2.11 the AE property (5.2.2).

iv.) Finally, we rearrange the split up semi-norms to the semi-norm of a factorizing tensor by (5.2.3).

5.3.1 Continuity of the product

In the first proof, we want to approach the estimate via the formula

$$\xi_1 \cdots \xi_k \star_{zG} \eta_1 \cdots \eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell)$$

Since this comes from polarizing the formula

$$\xi^k \star_{zG} \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n\Big(\xi^k, \eta^\ell\Big), \tag{5.3.1}$$

we will just give an explicit proof for the latter one. One gets the estimate for the first one easily in the same way, since in the end, all Lie brackets are broken up in step (*iii*). One will get sums over permutations weighted with the inverse of their quantity and, as their semi-norms are just numbers which commute, one ends up with the same estimate as for (5.3.1). First, we want to extend the Gutt star product to the whole tensor algebra: we define

$$\star_{zG} \colon \mathrm{T}^\bullet(\mathfrak{g}) \times \mathrm{T}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{T}^\bullet(\mathfrak{g}), \quad \star_{zG} = \star_{zG} \circ \mathscr{S}\,.$$

Theorem 5.3.1 *Let $\mathfrak{g}$ be an AE-Lie algebra and $R \geq 1$, then for $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$, $z \in \mathbb{C}$ and each continuous semi-norm p on $\mathfrak{g}$ there exists a constant c such that we have for an asymptotic estimate q of p*

$$p_R(x \star_{zG} y) \leq (cq)_R(x)(cq)_R(y). \tag{5.3.2}$$

Hence the Gutt star product is continuous on $\mathrm{S}^\bullet_R(\mathfrak{g})$ for all $z \in \mathbb{C}$.

Proof: We need to give estimates on the $z^n C_n$ in order to show their convergence. Let us use $r = k + \ell - n$ for brevity and recall that the products are taken in the symmetric algebra. Then we can use Equation (4.1.11) in the proof of Lemma 4.1.9 and put estimates on it. Let p be a continuous semi-norm and let q be an asymptotic estimate for it. By using the continuity estimate for the symmetric tensor product in (a), the AE property of the Lie bracket (5.2.2) in (b) and then simplifying the summation by adding more terms in (c), we get

$$p_R\Big(z^n C_n\Big(\xi^k, \eta^\ell\Big)\Big) = p_R\Bigg(z^n \frac{k!\ell!}{r!} \sum_{\substack{a_1, b_1, \ldots, a_r, b_r \geq 0 \\ a_i + b_i \geq 1 \\ a_1 + \ldots + a_r = k \\ b_1 + \ldots + b_r = \ell}} \mathrm{BCH}_{a_1, b_1}(\xi, \eta) \cdots \mathrm{BCH}_{a_r, b_r}(\xi, \eta)\Bigg)$$

$$
\begin{aligned}
&\overset{\text{(a)}}{\leq} |z|^n \frac{k!\ell!}{r!} r!^R \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} p^{a_1+b_1}(\mathrm{BCH}_{a_1,b_1}(\xi,\eta))\cdots p^{a_r+b_r}(\mathrm{BCH}_{a_r,b_r}(\xi,\eta))\\
&\overset{\text{(b)}}{\leq} |z|^n \frac{k!\ell!}{r!^{1-R}} q(\xi)^k q(\eta)^\ell \sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}} |\vartheta_{a_1,b_1}|\ldots|\vartheta_{a_r,b_r}|\\
&\overset{\text{(c)}}{\leq} |z|^n \frac{k!\ell!}{r!^{1-R}} q(\xi)^k q(\eta)^\ell \sum_{\substack{j_1,\ldots,j_r\geq 1\\ j_1+\ldots+j_r=k+\ell}} |\vartheta_{j_1}|\ldots|\vartheta_{j_r}| \qquad (5.3.3)
\end{aligned}
$$

Now we will use the fact that Thompson gave estimate for the growth of the Baker-Campbell-Hausdorff coefficientsin in [?]. More precisely he showed $|\vartheta_j| \leq \frac{2}{j}$. For us, it will be sufficient that for all $j \in \mathbb{N}$ we have $|\vartheta_j| \leq 2$. Knowing this and using some easy combinatoric estimates we find

$$
\sum_{\substack{j_1,\ldots,j_r\geq 1\\ j_1+\ldots+j_r=k+\ell}} |\vartheta_{j_1}|\cdots|\vartheta_{j_r}| \leq 2^r \sum_{\substack{j_1,\ldots,j_r\geq 1\\ j_1+\cdots+j_r=k+\ell}} 1 \leq 2^r \binom{k+\ell+r-1}{k+\ell} \leq 2^{3(k+\ell)-2n-1}.
$$

We put this together with (5.3.3) into the estimate

$$
\begin{aligned}
p_R\Big(\xi^{\otimes k} \star_{zG} \eta^{\otimes \ell}\Big) &= p_R\left(\sum_{n=0}^{k+\ell-1} z^n C_n\Big(\xi^k,\eta^\ell\Big)\right)\\
&\leq \sum_{n=0}^{k+\ell-1} |z|^n \frac{k!\ell!}{(k+\ell-n)!^{1-R}} q(\xi)^k q(\eta)^\ell 2^{3(k+\ell)-2n-1}\\
&= \frac{1}{2}\sum_{n=0}^{k+\ell-1} \frac{|z|^n}{4^n}\left(\frac{k!\ell!n!}{(k+\ell-n)!n!}\right)^{1-R} (8q)_R\Big(\xi^{\otimes k}\Big)(8q)_R\Big(\eta^{\otimes \ell}\Big)\\
&\leq \frac{1}{2}\sum_{n=0}^{k+\ell-1} \frac{|z|^n}{4^n}\frac{1}{n!^{R-1}}\binom{k+\ell}{k}^{R-1}\binom{k+\ell}{n}^{1-R} (8q)_R\Big(\xi^{\otimes k}\Big)(8q)_R\Big(\eta^{\otimes \ell}\Big) \qquad (5.3.4)\\
&\leq \frac{1}{2}\sum_{n=0}^{k+\ell-1} \frac{|z|^n}{4^n n!^{R-1}} 2^{(1-R)(k+\ell)} (8q)_R\Big(\xi^{\otimes k}\Big)(8q)_R\Big(\eta^{\otimes \ell}\Big)\\
&= \frac{1}{2}\sum_{n=0}^{k+\ell-1} \frac{|z|^n}{4^n n!^{R-1}} (8q)_R\Big(\xi^{\otimes k}\Big)(8q)_R\Big(\eta^{\otimes \ell}\Big).
\end{aligned}
$$

Remind that $R \geq 1$, so $2^{1-R} \leq 1$. For $|z| \leq 2$ we get

$$
\begin{aligned}
p_R\Big(\xi^{\otimes k} \star_{zG} \eta^{\otimes \ell}\Big) &\leq \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}(8q)_R(\xi^{\otimes k})(8q)_R(\eta^{\otimes \ell})\\
&= (8q)_R\Big(\xi^{\otimes k}\Big)(8q)_R\Big(\eta^{\otimes \ell}\Big).
\end{aligned}
$$

For $|z| > 1$ we have on the other hand

$$
p_R\Big(\xi^{\otimes k} \star_{zG} \eta^{\otimes \ell}\Big) \leq \frac{1}{2}\sum_{n=0}^{k+\ell-1}\frac{|z|^{k+\ell}}{4^n}(8q)_R(\xi^{\otimes k})(8q)_R(\eta^{\otimes \ell})
$$

$$\leq (8zq)_R\left(\xi^{\otimes k}\right)(8zq)_R\left(\eta^{\otimes \ell}\right).$$

Thus we can find in both cases a continuous semi-norm which fulfils (5.3.2). The proof for factorizing tensors is the same and one finally gets

$$p_R(\xi_1 \otimes \ldots \otimes \xi_k \star_{zG} \eta_1 \otimes \ldots \otimes \eta_\ell) \leq (cq)_R(\xi_1 \otimes \ldots \otimes \xi_k)(cq)_R(\eta_1 \otimes \ldots \otimes \eta_\ell)$$

with the same c's as before. Having established this, we can use the infimum argument to get the statement on arbitrary tensors and we are done. □

Remark 5.3.2 We have actually proven more than just the continuity of the star product. We will come back to this proof later in order to show also the entire holomorphic dependence of the star product on the formal parameter. This will be possible because we put bounds on the series $\sum_n z^n C_n(\cdot,\cdot)$. This again means, that we have proven the continuity of the C_n operators, too: for arbitrary tensors $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ and a continuous semi-norm p, there is a constant c independent of z (and even of R) such that

$$p_R(C_n(x,y)) \leq (cq)_R(x)(cq)_R(y) \tag{5.3.5}$$

for all $n \in \mathbb{N}$, where q is an asymptotic estimate for p. But we even have more than that:

Corollary 5.3.3 *Let $R \geq 0$ and $n \in \mathbb{N}$. Then, for every continuous semi- norm p there exists a constant $c > 0$ such that for an asymptotic estimate q and every $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ we get the estimate*

$$p_R(C_n(x,y)) \leq n!^{1-R}(cq)_R(x)(cq)_R(y). \tag{5.3.6}$$

PROOF: We go back to (5.3.3) in the proof and use the estimate for the sum over the $|\vartheta_i|$:

$$\begin{aligned}
p_R\left(C_n\left(\xi^{\otimes k}, \eta^{\otimes \ell}\right)\right) &\leq |z|^n \frac{k!\ell!}{(k+\ell-n)!^{1-R}} 2^{3(k+\ell)} q(\xi)^k q(\eta)^\ell \\
&= |z|^n \left(\frac{k!\ell!(k+\ell)!n!}{(k+\ell-n)!(k+\ell)!n!}\right)^{1-R} 2^{3(k+\ell)} q_R\left(\xi^{\otimes k}\right) q_R\left(\eta^{\otimes \ell}\right) \\
&\leq |z|^n 2^{(1-R)(k+\ell)} n!^{1-R} 2^{3(k+\ell)} q_R\left(\xi^{\otimes k}\right) q_R\left(\eta^{\otimes \ell}\right) \\
&\leq n!^{1-R}(16(|z|+1))^{k+\ell} q_R\left(\xi^{\otimes k}\right) q_R\left(\eta^{\otimes \ell}\right).
\end{aligned}$$

We just have to absorb the constant in front into the semi-norms and the proof is done. □

Remark 5.3.4 This makes things a bit clearer: The estimate 5.3.6 is exactly the one from (5.3.5) for $R \geq 1$. All the C_n are indeed continuous for any $R \geq 0$, but only for $R \geq 1$ there is something like a uniform continuity. When R decreases, the continuity of the C_n's "gets worse" and the uniform continuity finally breaks down when the threshold $R = 1$ is trespassed. But we need this uniform estimate, since we have to control the operators up to an arbitrarily high order if we want to guarantee the continuity of the star product. Continuity up to a formerly chosen order n does not suffice.

Now, we want to give the second proof, which relies on (4.1.3). Approaching like this, we don't account for the fact that we will encounter terms like $[\eta, \eta]$ which will vanish, but we estimate more brutally. During this procedure, we will also count the formal parameter z more often than it is actually there. This is why we will have to make assumptions on R and z which are a bit stronger than before. Moreover, we will split up tensor products and put them together again various times, which is the reason why an AE Lie algebra will not suffice any more: we will need $\mathfrak{g}$ to be locally m-convex. But if we make these assumptions, we get the following lemma which will finally make the proof easier.

Lemma 5.3.5 *Let $\mathfrak{g}$ be a locally m-convex Lie algebra and $R \geq 1$. Then if $|z| < 2\pi$ or $R > 1$ there exists for $x \in \mathrm{T}^\bullet(\mathfrak{g})$ of degree at most k, $\eta \in \mathfrak{g}$ and each continuous submultiplicative semi-norm p a constant $c_{z,R}$ only depending on z and R such that the following estimate holds:*

$$p_R(x \star_{zG} \eta) \leq c_{z,R}(k+1)^R p_R(x) q(\eta) \tag{5.3.7}$$

PROOF: We start again with factorizing tensors. Since we get the same estimate for monomials and for powers of some $\xi \in \mathfrak{g}$ via polarization, it is enough to consider $\xi^{\otimes k} \star_{zG} \eta$. This gives

$$\begin{aligned} p_R\Big(\xi^{\otimes k} \star_{zG} \eta\Big) &= p_R\left(\sum_{n=0}^{k} \binom{k}{n} B_n^* z^n \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta)\right) \\ &= \sum_{n=0}^{k} \binom{k}{n} |B_n^*||z|^n (k+1-n)!^R p^{k+1-n}\Big(\xi^{k-n}(\mathrm{ad}_\xi)^n(\eta)\Big) \\ &\leq (k+1)^R \sum_{n=0}^{k} |B_n^*||z|^n \frac{k!(k-n)!^R}{(k-n)!n!} p(\xi)^k p(\eta) \\ &= (k+1)^R \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R} \left(\frac{(k-n)!n!}{k!}\right)^{R-1} p_R\Big(\xi^{\otimes k}\Big) p(\eta) \\ &\leq (k+1)^R p_R\Big(\xi^{\otimes k}\Big) p(\eta) \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R}. \end{aligned}$$

Now if $|z| < 2\pi$ the sum can be estimated by extending it to a series which converges. We end up with a constant depending on R and on z such that

$$p_R\Big(\xi^{\otimes k} \star_{zG} \eta\Big) \leq (k+1)^R c_{z,R} p_R\Big(\xi^{\otimes k}\Big) p(\eta).$$

If on the other hand $|z| \geq 2\pi$ and $R > 1$ we can estimate

$$\begin{aligned} p_R\Big(\xi^{\otimes k} \star_{zG} \eta\Big) &\leq (k+1)^R p_R\Big(\xi^{\otimes k}\Big) p(\eta) \left(\sum_{n=0}^{k} \frac{|B_n^*|}{n!}\right)\left(\sum_{n=0}^{k} \frac{|z|^n}{n!^{R-1}}\right) \\ &\leq (k+1)^R \underbrace{2\tilde{c}_{z,R}}_{=c_{z,R}} p_R\Big(\xi^{\otimes k}\Big) p(\eta). \end{aligned}$$

We hence have the estimate on factorizing tensors and can extend this to generic tensors of degree at most k by the infimum argument. $\square$

In the following, we assume again that either $R > 1$ or $R \geq 1$ and $|z| < 2\pi$ in order the use Lemma 5.3.5. Now we can give a simpler proof of Theorem 5.3.1 for the case of a locally m-convex Lie algebra:

PROOF (ALTERNATIVE PROOF OF THEOREM 5.3.1): Assume that $\mathfrak{g}$ is now even locally m-convex. We want to replace η in the foregoing lemma by an arbitrary tensor y of degree at most ℓ. Again, we do that on factorizing tensors first and get

$$\begin{aligned} p_R\Big(\xi^{\otimes k} \star_{zG} \eta^{\otimes \ell}\Big) &= p_R\big(\xi^{\otimes k} \underbrace{\star_{zG} \eta \star \cdots \star_{zG} \eta}_{\ell\text{-times}}\big) \\ &\leq c_{z,R}(k+\ell)^R p_R\big(\xi^{\otimes k} \underbrace{\star_{zG} \eta \star \cdots \star_{zG} \eta}_{\ell-1\text{-times}}\big) p(\eta) \end{aligned}$$

$$
\begin{aligned}
&\leq \quad \vdots \\
&\leq c_{z,R}^{\ell}((k+\ell)\cdots(k+1))^R p_R\left(\xi^{\otimes k}\right)p(\eta)^{\ell} \\
&= c_{z,R}^{\ell}\left(\frac{(k+\ell)!}{k!\ell!}\right)^R p_R\left(\xi^{\otimes k}\right)p_R\left(\eta^{\otimes \ell}\right) \\
&\leq (2^R p)_R\left(\xi^{\otimes k}\right)(2^R c_{z,R} p)_R\left(\eta^{\otimes \ell}\right).
\end{aligned}
$$

Once again, we have the estimate on factorizing tensors via polarization and extend it via the infimum argument to the whole tensor algebra, since the estimate depends no longer on the degree of the tensors. □

Using this approach for continuity, it is easy to see that nilpotency of the Lie algebra changes the estimate substantially: If we knew that we will have at most N brackets because $N+1$ brackets vanish, then the sum in the proof of Lemma 5.3.5 would end at N instead of k and would therefore be independent of the degree of x.

In both proofs, it is easy to see that we need at least $R \geq 1$ to get rid of the factorials which come up because of the combinatorics of the star product. It is nevertheless interesting to see that this result is sharp, that means the Gutt star product really fails continuity, if $R < 1$:

Example 5.3.6 Let $0 \leq R < 1$ and $\mathfrak{g}$ be the Heisenberg algebra in three dimensions, i.e. the Lie algebra generated by the elements P, Q and E with the bracket $[P,Q] = E$ and all other brackets vanishing. This is a very simple example for a non-abelian Lie algebra and if continuity of the star product fails for this one, then we can not expect it to hold for more complex ones. We impose on $\mathfrak{g}$ the ℓ^1-topology with the norm n and $n(P) = n(Q) = n(E) = 1$. This will be helpful, since here we really have the equality

$$
p^{n+m}(X^n Y^m) = p^n(X^n)p^m(Y^m)
$$

for the symmetric product. Then we consider

$$
a_k = \frac{P^k}{k!^R} \quad \text{and} \quad b_k = \frac{Q^k}{k!^R}.
$$

It is easy to see that

$$
n_R(a_k) = n_R(b_k) = 1
$$

We want to show that there is no $c > 0$ such that

$$
n_R(a_k \star_{zG} b_k) \leq (cn)_R(a_k)(cn)_R(b_k)
$$

With other words, $n_R(a_k \star_{zG} b_k)$ grows faster than exponentially. But this is the case, since with our combinatorial formula (4.1.14) we see

$$
\begin{aligned}
n_R(a_k \star_{zG} b_k) &= n_R\left(\sum_{j=0}^{k}\binom{k}{j}\binom{k}{j}j!\frac{1}{k!^{2R}}P^{k-j}Q^{k-j}E^j\right) \\
&= \sum_{j=0}^{k}\frac{k!^2 j!(2k-j)!^R}{(k-j)!^2 j!^2 k!^{2R}}\underbrace{n^{2k-j}(P^{k-j}Q^{k-j}E^j)}_{=1} \\
&= \sum_{j=0}^{k}\underbrace{\binom{k}{j}^2\binom{2k}{k}\binom{2k}{j}^{-1}}_{\geq 1} j!^{1-R}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{j=0}^{k} j!^{1-R} \\
&\geq k!^{1-R},
\end{aligned}
$$

which is exactly what we wanted to show.

5.3.2 Dependence on the formal parameter

We now look at the completion $\widehat{\mathrm{S}}^{\bullet}_R(\mathfrak{g})$ of the symmetric algebra with the Gutt star product $\star_{zG}$ and get the following negative result:

Proposition 5.3.7 *Let $\xi \in \mathfrak{g}$ and $R \geq 1$, then $\exp(\xi) \notin \widehat{\mathrm{S}}^{\bullet}_R(\mathfrak{g})$, where $\exp(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!}$.*

PROOF: Take a semi-norm p such that $p(\xi) \neq 0$. Then set $c = p(\xi)^{-1}$. For ξ^n the powers in the sense of either the usual tensor product, or the symmetric product or the star product are the same. So we have for $N \in \mathbb{N}$

$$
p_R\left(\sum_{n=0}^{N} \frac{c^n}{n!} \xi^n\right) = \sum_{n=0}^{N} \frac{n!^R}{n!} c^n p_R(\xi^n) = \sum_{n=0}^{N} n!^{R-1} \geq N,
$$

and clearly $\exp(\xi)$ does not converge for the semi-norm p_R. □

As already mentioned, the proof of Theorem 5.3.1 gives more than stated before. We know, that for $R \geq 1$ the star product is continuous on $\mathrm{S}^{\bullet}_R(\mathfrak{g})$ and therefore has a continuous extension to $\widehat{\mathrm{S}}^{\bullet}_R(\mathfrak{g})$, but this extension is a priori abstract. It does not need to be the series of the C_n operators again. Yet, this is the case.

Corollary 5.3.8 *Let $\mathfrak{g}$ be an AE Lie algebra and $R \geq 1$. Then, for every $z \in \mathbb{C}$, the Gutt star product converges absolutely, i.e. for every continuous semi-norm p, there is another continuous semi-norm q such that for all $f, g \in \widehat{\mathrm{S}}^{\bullet}_R(\mathfrak{g})$*

$$
p_R(f \star_{zG} g) \leq \sum_{n=0}^{\infty} p_R(z^n C_n(f,g)) \leq q_R(f) q_R(g). \tag{5.3.8}
$$

PROOF: The first inequality is clear. We know that for all $z \in \mathbb{C}$ and for all $p \in \mathscr{P}$ there is a $q \in \mathscr{P}$ such that

$$
p_R(z^n C_n(f,g)) \leq q_R(f) q_R(g)
$$

holds for all $n \in \mathbb{N}$. So there is also a $q' \in \mathscr{P}$ such that this holds for $2z$. We hence get

$$
p_R(z^n C_n(f,g)) \leq 2^{-n} q'_R(f) q'_R(g).
$$

Now the conclusion follows since

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_R(z^n C_n(f,g)) &\leq \sum_{n=0}^{\infty} 2^{-n} q'_R(f) q'_R(g) \\
&\leq 2 q'_R(f) q'_R(g)
\end{aligned}
$$

and the 2 in front can be absorbed in the semi-norms. □

The consequence of Corollary 5.3.8 is that $\star_{zG}$ really converges to the formal series, and not just in an abstract sense. So the formal series

$$x \star_{zG} y = \sum_{n=0}^{\infty} z^n C_n(x, y)$$

remains valid for elements x, y in the completion. Knowing this and using the fact that all the projections on the homogeneous components are continuous from Lemma 5.2.14 (iv), we can reinterpret the continuity result we found in Theorem 5.3.1.

Proposition 5.3.9 *Let $R \geq 1$, then for all $f, g \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ the map*

$$\mathbb{K} \ni z \longmapsto f \star_{zG} g \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}) \tag{5.3.9}$$

is real-analytic if $\mathbb{K} = \mathbb{R}$ and entire-holomorphic if $\mathbb{K} = \mathbb{C}$ with Taylor expansion at $z = 0$ given by Equation (4.1.14). The collection of the algebras $\left\{\left(\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}), \star_{zG}\right)\right\}_{z\in\mathbb{C}}$ is a holomorphic deformation of the completed symmetric tensor algebra $\left(\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}), \vee\right)$.

PROOF: The important point is that for $f, g \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ and every continuous semi-norm p we have another continuous semi-norm q such that

$$\begin{aligned} p_R(f \star_{zG} g) &= p_R\left(\sum_{n=0}^{\infty} z^n C_n(f, g)\right) \\ &= \sum_{n=0}^{\infty} |z|^n p_R(C_n(f, g)) \\ &\leq \sum_{n=0}^{\infty} |z|^n q_R(f) q_R(g). \end{aligned}$$

We already showed that for every $M > 0$, there exists a $c \geq 1$, such that in the open disc $|z| < M \subset \mathbb{C}$ the inequality

$$p_R(f \star_{zG} g) \leq \sum_{n=0}^{\infty} |z|^n q_R(f) q_R(g) \leq (cq)_R(f)(cq)_R(g) < \infty$$

holds. Thus the map (5.3.9) is holomorphic in z on every open disc around 0. This means that the map is actually entire. □

Remark 5.3.10 If $R > 1$, we even have the result

$$p_R(C_n(f, g)) \leq \frac{1}{n!^{R-1}} q_R(f) q_R(g)$$

from (5.3.5) for every $p \in \mathscr{P}$ and a suitable $q \in \mathscr{P}$. In this case, we get

$$p_R(f \star_{zG} g) \leq \sum_{n=0}^{\infty} p_R(z^n C_n(f, g)) \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!^{R-1}} q_R(f) q_R(g),$$

and we see the entire dependence easier.

5.4 Functorialty, Representations and an optimal result

5.4.1 An optimal result

Let's set the formal parameter $z = 1$ for a moment and make some observations. So far, we found a topology on $\mathrm{S}^\bullet_R(\mathfrak{g})$ which gives a continuous star product and which has a reasonably large completion, but it is always fair to ask if we can do better than that: we've seen that our completed algebra will not contain exponential series, which would be a very nice feature to have. So is it possible to put another locally convex topology on $\mathrm{S}^\bullet_R(\mathfrak{g})$ which gives a completion with exponentials? The answer is no, at least under mild additional assumptions.

Proposition 5.4.1 (Optimality the R-topology) *Let $\mathfrak{g}$ be an AE Lie algebra in which one has elements ξ, η for which the Baker-Campbell-Hausdorff series does not converge. Then there is no locally convex topology on $\mathrm{S}^\bullet(\mathfrak{g})$ such that all of the following things are fulfilled:*

i.) The Gutt star product $\star_G$ is continuous.

ii.) For every $\xi \in \mathfrak{g}$ the series $\exp(\xi)$ converges absolutely in the completion of $\mathrm{S}^\bullet(\mathfrak{g})$.

iii.) For all $n \in \mathbb{N}$ the projection and inclusion maps with respect to the graded structure

$$\mathrm{S}^\bullet(\mathfrak{g}) \xrightarrow{\pi_n} \mathrm{S}^n(\mathfrak{g}) \xrightarrow{\iota_n} \mathrm{S}^\bullet(\mathfrak{g})$$

are continuous.

First of all, we should make clear what "the Baker-Campbell-Hausdorff series does not converge" actually means. This may be clear for a finite-dimensional Lie algebra, but in the locally convex setting, it is not that obvious. First, we note here that a sequence in a locally convex space is convergent [or Cauchy], if it is convergent [or Cauchy] with respect to all $p \in \mathscr{P}$. Quite similar to a normed space, we can make the following definition.

Definition 5.4.2 *Let V be a locally convex vector space, $\mathscr{P}$ the set of continuous semi-norms and $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence in V. We set*

$$\rho_p(\alpha) = \left(\limsup_{n\longrightarrow\infty} \sqrt[n]{p(\alpha_n)} \right)^{-1}$$

where $\rho_p(\alpha) = \infty$ if $\limsup_{n\longrightarrow\infty} \sqrt[n]{p(\alpha_n)} = 0$ as usual.

From this, we immediately get the two following lemmas.

Lemma 5.4.3 (Root test in locally convex spaces) *Let V be a complete locally convex vector space, $p \in \mathscr{P}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence. Then, if $\rho_p(\alpha) > 1$, the series*

$$\mathcal{S}_n(\alpha) = \sum_{j=0}^{n} \alpha_j$$

converges absolutely with respect to p. If, conversely, this series convegres with respect to p, then we have $\rho_p(\alpha) \geq 1$.

PROOF: The proof is completely analogous to the one in finite dimensions. □

Lemma 5.4.4 *Let V be a complete locally convex vector space, $p \in \mathscr{P}$, $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence and $M > 0$. Then, if the power series*

$$\lim_{n\longrightarrow\infty} \sum_{j=0}^{n} \alpha_j z^j$$

converges for all $z \in \mathbb{C}$ with $|z| \leq M$, it converges absolutely with respect to p for all $z \in \mathbb{C}$ with $|z| < M$.

PROOF: Like in the finite-dimensional setting, we use the root test: Convergence for $|z| \leq M$ means $\rho_p(\alpha_z) \geq 1$, where we have set $\alpha_z = (\alpha_n z^n)_{n\in\mathbb{N}}$. Hence, for every $z' < z$ we get $\rho_p(\alpha_{z'}) > 1$ and absolute convergence by Lemma 5.4.3. □

Now we can prove a Lemma, from which Proposition 5.4.1 will follow immediately.

Lemma 5.4.5 *Let $\mathfrak{g}$ be an AE Lie algebra and $\mathrm{S}^\bullet(\mathfrak{g})$ is endowed with a locally convex topology, such that the conditions $(i)-(iii)$ from Proposition 5.4.1 are fulfilled. Then the Baker-Campbell-Hausdorff series converges absolutely for all $\xi, \eta \in \mathfrak{g}$.*

PROOF: We ... □

PROOF (PROOF OF PROPOSITION 5.4.1): We will just need the projection π_1 to the Lie algebra itself. Let $\xi, \eta \in \mathfrak{g}$ such that $\mathrm{BCH}(\xi, \eta)$ does not exist, i.e. there is a continuous semi-norm p such that the limit

$$p\left(\lim_{N\to\infty}\lim_{M\to\infty}\sum_{n,m=0}^{N,M}\mathrm{BCH}_{\xi,\eta}(n,m)\right) \tag{5.4.1}$$

does not exist. But if we assume that the Gutt star product is continuous and that the exponential series is absolutely convergent, then we have

$$\begin{aligned}
\pi_1(\exp(\xi)\star_G\exp(\eta)) &= \pi_1\left(\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{\xi^n}{n!}\right)\star_G\lim_{M\to\infty}\left(\sum_{m=0}^{M}\frac{\eta^m}{m!}\right)\right)\\
&\overset{(a)}{=} \pi_1\left(\lim_{N\to\infty}\lim_{M\to\infty}\left(\sum_{n=0}^{N}\frac{\xi^n}{n!}\right)\star_G\left(\sum_{m=0}^{M}\frac{\eta^m}{m!}\right)\right)\\
&\overset{(b)}{=} \lim_{N\to\infty}\lim_{M\to\infty}\pi_1\left(\left(\sum_{n=0}^{N}\frac{\xi^n}{n!}\right)\star_G\left(\sum_{m=0}^{M}\frac{\eta^m}{m!}\right)\right)\\
&\overset{(c)}{=} \lim_{N\to\infty}\lim_{M\to\infty}\sum_{n,m=0}^{N,M}\mathrm{BCH}_{\xi,\eta}(n,m)
\end{aligned}$$

where we used the continuity of the star product in (a), the continuity of the projection in (b) and evaluated the projection in (c). Since $\exp(\xi)$ and $\exp(\eta)$ are elements in the completion, their star product exists and hence each continuous semi-norm has a well-defined value for it. But if we take now the semi-norm from (5.4.1) of it, we get a contradiction. □

Of course, there are many ways of arranging the Baker-Campbell-Hausdorff series. There is no need to write it like in (5.4.1), but this is enough to illustrate the meaning of Proposition 5.4.1. For finite-dimensional Lie algebras, the statement is even better: In a finite-dimensional Lie algebra, the Baker-Campbell-Hausdorff series converges if and only if it converges absolutely, since we can write it as a power series.

5.4.2 Functoriality

For $z = 1$, the algebra $\mathrm{S}^\bullet_1(\mathfrak{g})$ with the Gutt star product $\star_G$ is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Since $\mathcal{U}(\mathfrak{g})$ and $\mathrm{S}^\bullet(\mathfrak{g})$ have universal properties and we endowed them with a topology, we can ask whether we get some functorial properties with our construction. In other words: Let $\mathcal{A}$ be an associative, locally convex algebra and $\phi\colon \mathfrak{g} \longrightarrow \mathcal{A}$ a continuous Lie algebra homomorphism. We have the commuting diagram

$$
\begin{tikzcd}
\mathcal{U}_R(\mathfrak{g}) \arrow[r, "\mathfrak{q}^{-1}"] \arrow[dr, "\Phi"] & \mathrm{S}^\bullet_R(\mathfrak{g}) \arrow[d, "\widetilde{\Phi}"] \\
\mathfrak{g} \arrow[u, "\iota"] \arrow[r, "\phi"'] & \mathcal{A}
\end{tikzcd}
$$

from the algebraic theory. The important question is now whether the algebra homomorphisms Φ and $\widetilde{\Phi}$ are continuous. This question is partly answered by the following result:

Proposition 5.4.6 *Let $\mathfrak{g}$ be an AE-Lie algebra, $\mathcal{A}$ an associative AE-algebra and $\phi\colon \mathfrak{g} \longrightarrow \mathcal{A}$ is a continuous Lie algebra homomorphism. If $R \geq 0$, then the induced algebra homomorphisms Φ and $\widetilde{\Phi}$ are continuous.*

PROOF: We define an extension of Φ on the whole tensor algebra again:

$$\Psi\colon \mathrm{T}^\bullet_R(\mathfrak{g}) \longrightarrow \mathcal{A}, \quad \Psi = \widetilde{\Phi} \circ \mathscr{S}$$

It is clear that if Ψ is continuous on factorizing tensors, we get the continuity of $\widetilde{\Phi}$ and of Φ via the infimum argument. So let p be a continuous semi-norm on $\mathcal{A}$ with its asymptotic estimate q and $\xi_1, \ldots, \xi_n \in \mathfrak{g}$. Since ϕ is continuous, we find a continuous semi-norm r on $\mathfrak{g}$ such that for all $\xi \in \mathfrak{g}$ we have $q(\phi(\xi)) \leq r(\xi)$. Then we have

$$
\begin{aligned}
p(\Psi(\xi_1 \otimes \cdots \otimes \xi_n)) &= p\Big(\widetilde{\Phi}(\xi_1 \star_{zG} \cdots \star_{zG} \xi_n)\Big) \\
&= p(\phi(\xi_1) \cdots \phi(\xi_n)) \\
&\leq q(\phi(\xi_1)) \cdots q(\phi(\xi_n)) \\
&\leq r(\xi_1) \cdots r(\xi_n) \\
&\leq r_R(\xi_1 \otimes \cdots \otimes \xi_n),
\end{aligned}
$$

where the last inequality is true for all $R \geq 0$. $\square$

Although this is a nice result, our construction fails to be universal, since the universal enveloping algebra endowed with our topology is *not* AE in general. This is even very easy to see:

Example 5.4.7 Take $\xi \in \mathfrak{g}$, then we know that $\xi^{\otimes n} = \xi^{\star_G n} = \xi^n$ for $n \in \mathbb{N}$ where the formal parameter is $z = 1$. Let $R > 0$ and p a continuous semi-norm in $\mathfrak{g}$ then we find

$$p_R(\xi^n) = n!^R p(\xi)^n = \frac{n!^R}{c^n} q(\xi)^n \tag{5.4.2}$$

for $c = \frac{p(\xi)}{q(\xi)}$ for a different semi-norm q with $q(\xi) \neq 0$. But since the $\frac{n!^R}{c^n}$ will always diverge for $n \to \infty$ we will never get an asymptotic estimate for p_R.

5.4.3 Representations

Although the construction is not functiorial, we can draw a nice conclusion from Proposition 5.4.6:

Proposition 5.4.8 *Let $R \geq 1$ and $\mathcal{U}_R(\mathfrak{g})$ the universal enveloping algebra of an AE-Lie algebra $\mathfrak{g}$, then for every continuous representation ϕ of $\mathfrak{g}$ into the bounded linear operators $\mathfrak{B}(V)$ on a Banach space V the induced homomorphism of associative algebras $\Phi\colon \mathcal{U}(\mathfrak{g}) \longrightarrow \mathfrak{B}(V)$ is continuous.*

PROOF: This follows directly from Proposition 5.4.6 and $\mathfrak{B}(V)$ being a Banach algebra. □

Remark 5.4.9

i.) From this, it follows in particular that for all finite-dimensional Lie algebras all finite-dimensional representations on some vector space V extend to continuous algebra homomorphisms $\mathcal{U}_R(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$. For representations on infinite-dimensional Banach or Hilbert spaces, the statement is typically rather irrelevant, since there one rarely has norm-continuous representations, but merely strongly continuous ones.

ii.) In [**?**] Schottenloher and Pflaum mention an alternative topology on $\mathcal{U}(\mathfrak{g})$ for finite-dimensional Lie algebras: They took the coarsest locally convex topology, such that all finite-dimensional representations of $\mathfrak{g}$ extend to continuous algebra homomorphisms. This topology is in fact even locally m-convex. Our topology which uses the grading on $\mathrm{S}^\bullet_R(\mathfrak{g})$ is different from that: As we have seen in Proposition **??**, it is finer for $R \geq 0$ and even strictly finer for $R > 0$. For the interesting case $R \geq 1$ it is "just" locally convex, but its advantage (for our purpose) is that it respects the grading, which is helpful for the holomorphic dependence on the formal parameter.

Chapter 6

Nilpotent Lie algebras

6.1 An overview

- Reference to the counter-example before, no big change - Yet: Projective Limit - Module structure - Generalizations to nilpotency

6.2 The Heisenberg and the Weyl algebra

6.3 The projective limit

6.4 A module structure

6.4.1 Generic case and a counter-example

6.4.2 Nilpotent case and good news

6.5 Banach Lie algebras and the finite-dimensional case

6.5.1 Generalizations of nilpotency

6.5.2 A new projective Limit

6.5.3 A result for the finite-dimensional case

Chapter 7

The Hopf algebra structure

7.1 The co-product

7.1.1 A formula for the co-product

7.1.2 Continuity for the co-product

7.2 The whole Hopf algebra structure

Chapter 8

Examples and remarks

8.1 Some classical Lie algebra

8.2 Some new ideas

- The Weyl algebra is not AE. - It is damn difficult to find something which is AE and not LMC.

8.2.1 A subalgebra of the Weyl algebra

8.2.2 Holomorphic vector fields