In this chapter we will introduce and justify the notion of a *polarization* on a symplectic manifold, as well as give some examples and showing how it solves some of the issues with pre-quantization by adjusting the prequantum Hilbert space into a polarized prequantum Hilbert space and adjusting the prequantized operators into polarized prequantum operators; note that there is one final modification to be made after the polarization step, however, to remedy a final issue with the integrability of states.

Definition and Basic Results

We will hit the ground running with the definitions and proceed to justify it shortly after:

Definition 1. *Let (M^{2n}, ω) be a symplectic manifold. A (complex) distribution P on M, i.e. a sub-bundle*

$$P \hookrightarrow T^{\mathbf{C}}M \equiv TM \otimes_{\mathbf{R}} \mathbf{C}$$

of the (complexified) tangent bundle, is said to be a polarization *of M if all of the following are hold:*

- *P is* Lagrangian*, i.e. it is an* isotropic *distribution of dimension n; recall that a distribution is said to be isotropic if the symplectic form ω_x on each tangent space T_xM vanishes on the subspace $P_x \hookrightarrow T_xM$ defined by the distribution, and that the maximal dimension of an isotropic distribution is half the dimension of the manifold.*
- *P is* involutive*, i.e. for X, Y vector fields in P, the commutator $[X, Y]$ is a vector field in P. In the literature, this is usually written in shorthand as $[P, P] \subset P$.*
- *For all points $m \in M$, we have that $\dim(P_m \cap \overline{P}_m \cap TM)$ is constant, where the overline refers to complex conjugation.*

Having defined the meaning of a polarization, we can now define a way of polarizing the prequantum Hilbert space and polarizing the operators obtained through prequantization.

Definition 2. *Given a symplectic manifold (M^{2n}, Ω) with prequantum line bundle $(L \to M, \nabla)$ and prequantum Hilbert space*[1]

$$\mathcal{H}^{preq} := \{\psi \in \Gamma(L) | \langle \psi, \psi \rangle < \infty\},$$

we define the polarized Hilbert space $\mathcal{H}^P$ for P a choice of polarization on M as the Hilbert space of sections in $\mathcal{H}^{preq}$ that respect the polarization*, i.e.*

$$\nabla_X \psi = 0 \qquad \forall X \in \mathcal{X}(P),$$

[1] (Or more accurately, the metric completion of this space.)

where X is a vector field generated by P, i.e. the space of sections that remain fixed by parallel transport along any X in the polarization.

Further, we say that an observable $f \in C^\infty(M)$ is polarizable, *or* "respects the polarization", *under the polarization P iff X_f the associated Hamiltonian vector field satisfies*[2]

$$[X_f, P] = P.$$

When f is polarizable, its corresponding polarized operator $O_f : \mathcal{H}^P \to \mathcal{H}^P$ *is the same*[3] *as the prequantum operator O_f^{preq}.*

[2] Here, this is shorthand for the fact that X_f keeps P fixed.

[3] In other words, the primary purpose of a choice of polarization is to restrict the class of possible classical observables that may be meaningfully quantized — it does not change the definition of the quantization of a classical observable.

We remark that the condition placed upon the smooth functions on M is necessary as not all observables respect the polarization. For instance, given a polarization P on M such that $\overline{P} = P$, we can find coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that

$$P = span(\{\frac{\partial}{\partial x_i}_{i=1}^{n}\});$$

then one can verify that the oberservables $y_1(p), \ldots, y_n(p) \in C^\infty(M)$ fail to respect the polarization.

Justification and Intuition.

We now justify the reasons for defining polarizations and the polarized Hilbert space and operators as such.

First off, we analyze the intuition behind the three conditions needed to consider a distribution $P \hookrightarrow T^{\mathbb{C}}M$ to be a polarization:

- The requirement that P be *Lagrangian* comes from the need to find a "representation" or "choice" of configuration space inside of the phase space given by the symplectic manifold. Since the quantization procedure acts on symplectic manifolds, the question of a choice of configuration space is implicitly ignored; Lagrangian distributions, which are necessarily rank n when $dim(M) = 2n$, select a configuration space. The choice, however, is usually not unique[4].
- The *involutivity* of P comes frome the Frobenius theorem, which states that involutivity is equivalent to integrability (see [5], pp. 494-505). Recall that an integrable distribution is sort of like an index (or *foliation*) of immersed manifolds:

[4] (Note that $\binom{2n}{n} > 1$ for $n \geq 1$.)

[5]

Definition 3 (Integral Manifolds)**.** *Let $D \subset TM$ be a distribution. We call an immersed submanifold*

$$N \hookrightarrow M$$

an integral manifold *of D if*

$$\forall p \in N \; T_pN = D_p.$$

Definition 4 (Integrable Distribution). *A distribution $D \subset TM$ is an* integrable *distribution if every point $m \in M$ lies in some integral manifold N of D.*

Intuitively, an integrable distribution gives us a way of characterizing a collection of "parallel" immersed manifolds. As an illustrative example, consider $M = \mathbf{R}^4$. We have a distribution spanned by the vector fields $\partial/\partial x^1$ and $\partial/\partial x^2$; the 2-dimensional affine subspaces of $\mathbf{R}^4$ parallel to the plane spanned by these vectors are the integral manifolds of this distribution.

- Define the distribution $D = P \cap \overline{P} \cap TM$; we require that $dim(D)$ be constant in our definition, though it is an excluded condition in other treatments. Perhaps the least obvious condition, the constant rank of D guarantees that D is a valid tangent distribution.

  This is important, for example, when it comes to real polarizations. Suppose P is a real polarization, so that $P = \overline{P}$. Then As long as $P \cap \overline{P} \cap TM$ has constant dimension, it also forms a Lagrangian distribution on M. Conversely, the complexification $D^{\mathbf{C}}$ of any LaGrangian distribution D on M gives us a real polarization.

Finally, we remark that polarizations may be thought of as "representations" (in a physical sense) of a given physical system; just as there aren't always unique choices of polarizations for a given symplectic manifold, there are also often many ways of representing a physical system. In the next section, we illustrate this with a few examples.

Examples: Real and Kähler Polarizations.

In this section we provide a few examples to see what typical polarizations might look like, with specific emphasis on two very special classes of polarizations: *real* polarizations and *Kähler* polarizations. First, some definitions:

Definition 5 (Real and Kähler and Polarizations). *A polarization P on a symplectic manifold (M, ω) is said to be:*

- real *if $P = \overline{P}$;*
- pseudo-Kähler *if $P \cap \overline{P} = 0$.*

Suppose we define a Hermitian form

$$h^P(u, v) = i\omega(u, \overline{v})$$

on P. If P is pseudo-Kähler, then $ker(h^P)$ is non-degenerate; if h^P is positive-definite on P then we say that P is a Kähler polarization.

Recall that a Kähler manifold (M, ω, J) is a manifold M with a symplectic form ω compatible with the complex structure J; the following theorem states that Kähler polarizations and Kähler manifolds give rise to each other.

Theorem 1. *Let (M, ω, J) be a Kähler manifold, and let $T^{\mathbb{C}}M$ be its complexified tangent bundle. Then the two complex distributions*

$$T_{(1,0)} = \{v \in T^{\mathbb{C}_p}M | J(v) = iv\}, \ \ T_{(0,1)} = \{v \in T^{\mathbb{C}_p}M | J(v) = -iv\}$$

are Kähler polarizations.

Conversely, if a Kähler polarization exists on a symplectic manifold (M, ω), then there exists a complex structure J compatible with ω such that (M, ω, J) has the structure of a Kähler manifold.

We now present two different polarizations of the same manifold, and show how the choice of a real polarization versus a Kähler polarization can change the representation of the system. In our case, we have a cotangent bundle of a smooth manifold as our phase space; choosing certain real polarizations gives us the Schrödinger and momentum representations, whereas a particular Kähler polarization gives us the holomorphic, or Bargmann-Fock, representation. The following examples are taken from Shaw, and the symplectic manifold M is set as T^*Q some cotangent bundle in both cases.

Schrödinger and Momentum Representations.

- *Manifold* — Choose a cotangent bundle $M = T^*Q$, where Q is some smooth manifold.
- *Basis* — Choose the *canonical basis* $\{q_i, p_j\}$.
- *Standard Symplectic Form* — With the canonical basis, the standard symplectic form is written as

$$\omega = \sum (dq_i \wedge dp_i).$$

- *Polarization* — Choose the polarization spanned by the momenta $\{\frac{\partial}{partial p_j}\}_j$. Then a section $\phi : M \to \mathbb{C}$ is polarized if

$$\frac{\partial \phi}{\partial p_j} = 0 \; \forall j,$$

  or equivalently, is constant in the p_j dimensions. This is the *Schrödinger* representation, matching up with physical intuition from basic quantum mechanics.

  (If we had chosen the $\frac{\partial}{\partial q_i}{}_i$ to span our polarization, we would have had the *momentum* representation.)

Bargmann-Fock Representation.

- *Manifold* — Choose again the cotangent bundle $M = T^*Q$, where Q is some smooth manifold.
- *Basis* — This time, choose the basis $z_j, \bar{z}_j$, where
$$z_j = p_j + iq_j.$$
- *Standard Symplectic Form* — With the above basis, the standard symplectic form (as in the previous example) becomes
$$\omega = \frac{1}{2} d\bar{z}_j \wedge dz_j,$$
where summation notation is in effect.
- *Complex Structure* — We also have a complex structure this time, viewing M as a Kähler manifold, given by
$$Jz_j = iz_j,\ J\bar{z}_j = -i\bar{z}_j.$$
- *Polarization* — The Kähler polarization corresponding to J is the polarization P spanned by $\{\frac{\partial}{\partial \bar{z}_j}\}_j$. This means that polarized sections ψ are those such that
$$\frac{\partial \psi}{\partial \bar{z}_j} = 0,\ \forall j.$$
This is the *holomorphic*, or *Bargmann-Fock* representation of (M, ω, J).

Remarks on Polarizations and the Polarized Hilbert Space.

We now make some remarks on interesting features of polarizations on a symplectic manifold.

Canonical Polarizations

A natural question to ask is whether there are situations where a choice of polarizations is either canonical in some sense or otherwise unique; a further question might be whether there are categories where we have a class of functors indexed by choices of polarization.

To be completed.

BKS Kernels

Another natural question is whether there are algebraic ways of comparing two different representations corresponding to two different polarizations on the same manifold.

To be completed.

History and References

For an introduction to the theory of polarizations in the context of geometric quantization, see