Definition and Basic Results

The first step of the process is prequantization — in its simplest formulation, a symplectic manifold is prequantized into a Hilbert space given by the sections of a complex line bundle built from the manifold which possesses a connection[1] on the bundle satisfying a curvature condition. We have the following definition:

[1] That is, a *Koszul* connection, as the line bundle is viewed as a rank one vector bundle.

Definition 1. *Let (M^{2n}, ω) be a symplectic manifold. Then a complex line bundle*

$$\pi : L \to M$$

paired with a (Koszul) connection

$$\nabla : \Gamma(L) \to \gamma(L \otimes T^* M)$$

is called a prequantum line bundle *for (M, ω) if the following curvature condition holds:*

$$\omega = \mathrm{curv}(\nabla),$$

i.e. if the curvature 2-form and the symplectic form are equivalent[2].

If such a line bundle and connection exist for (M, ω), the symplectic manifold is said to be prequantizable.

[2] For a review of the curavture 2-form, consult Kobayashi & Nomizu.

In the definition above, $\Gamma(L)$ refers to the space of smooth sections, i.e. smooth functions $M \to L$ that are right inverses of π:

$$\pi \circ \sigma = \iota_M.$$

We will see later that the curvature condition induces a morphism of Lie algebras between the classical and quantum observables. For now, one can think of the curvature 2-form as a way to measure how much the horizontal distribution fails to be integrable; the condition then links this to the symplectic form.

We note that a symplectic manifold is prequantizable if and only if the symplectic form satisfies a cohomological property:

Theorem 1. *A symplectic manifold (M, ω) is prequantizable if and only if*

$$\left[\omega\right] \in H^2_{dR}(M, \mathbb{Z}),$$

i.e. the second de Rham cohomology class of the symplectic form is integral.

Equivalently, (M, ω) is prequantizable iff the period of ω is integral for each integer 2-cocycle, i.e.

$$\int_S \omega \in \mathbb{Z}$$

for each closed[3] 2-cochain S with integer coefficients.

[3] "Closed" here means that $\partial S = 0$.

For historical reasons that we will not get into, this cohomological condition is called the *Bohr-Sommerfeld condition*, and the above theorem is due to A. Weil.

From the existence of a prequantum line bundle, we can define a Hilbert space on the space of sections and define the prequantized observables corresponding to each $f \in C^\infty(M, \mathbb{R})$:

Definition 2. *Let (M, ω) be a prequantizable symplectic manifold with prequantum line bundle given by*

$$(\pi : L \to M, \nabla : \Gamma(L) \to \Gamma(L \otimes T^*M));$$

then the prequantum Hilbert space $\mathcal{H}_{preq}$ *is given by (the completion of) the space of square-integrable sections of L, i.e.*

$$\mathcal{H}_{preq} := \{\phi | \langle \phi, \phi \rangle < \infty\},$$

where the inner product is given by

$$\langle \phi, \psi \rangle := \int_M \langle \phi, \psi \rangle_h \omega^n$$

where $\langle \cdot, \cdot \rangle_h$ is a hermitian inner product compatible with the connection.

For a classical observable $f \in C^\infty(M)$, the associated prequantized observable $O_{preq}(f)$ is defined as

$$O_{preq}(f) = i\hbar \nabla_{X_f} + f,$$

where $X_f \in \mathcal{X}(M)$ is the associated symplectic vector field to f, i.e. the unique $X_f \in \mathcal{X}(M)$ such that

$$\omega(X_f, Y) \equiv \mathrm{d}f(Y) = Yf.$$

Examples

We now give three examples: a basic example of prequantization, and two examples displaying the inherent un-physicality of the prequantized Hilbert space and the prequantized operators.

- *(Cotangent Bundles.)* We first consider the prequantization of a cotangent bundle T^*Q where Q is a smooth manifold of dimension n. The cotangent bundle has a natural global symplectic 2-form given by

  $$\omega = \sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i$$

  where (q_i, p_i) are *canonical coordinates*; note also that $\omega = -\mathrm{d}\theta$ for

  $$\theta = \sum_{i=1}^n p_i \mathrm{d}q_i,$$

the canonical 1-form on T^*Q.

We note that the cohomology class $[\omega] = 0 \in \mathbb{Z}$ since we have that ω is exact ($\mathrm{d}(-\theta) = \omega$).

Hence the cotangent bundle is prequantizable; since ω is equivalently the curvature form, we have a flat connection[4] ∇ and it is thus sufficient to take $T^*Q \times \mathbb{C}$ as the prequantum line bundle.

We obtain a prequantum Hilbert space of $\mathcal{H} = \mathcal{C}(T^*Q)$ with metric given by integration over T^*Q.

The quantized operator corresponding to $f \in C^\infty(T^*Q)$ is then given by

$$Q[f]|\psi\rangle = \nabla_{X_f}|\psi\rangle + f|\psi\rangle.$$

[4] See theorems 9.1 and 9.2 of Kobayashi & Nomizu, for example, for a proof of the claim that a connection is flat iff the bundle is trivial.

- *($\mathbb{R}^3$.)* As a specific example of the above, we consider $Q = \mathbb{R}^3$. According to the above, we have the symplectic manifold $T^*\mathbb{R}^3 \equiv \mathbb{R}^6$, with the hermitian line bundle given by $\mathbb{R}^6 \times \mathbb{C}$; we note that this is unique as Euclidean space is simply connected (i.e. all loops in the fundamental group are contractible), and the associated connection ∇ is given by the standard partial derivatives[5]:

$$\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_6}).$$

The prequantum Hilbert space is then defined as

$$\mathcal{H} = \{\psi : \mathbb{R}^6 \to \mathbb{C} | \psi \text{ smooth } \};$$

note that we do not impose the L^2 condition at this stage of the quantization procedure yet.

Suppose we choose the classical Hamiltonian function H to be the kinetic energy

$$H : \mathbb{R}^6 \to \mathbb{R},\ H(x) = \frac{1}{2}(x_1 + x_2 + x_3)^2.$$

The quantized Hamiltonian operator is then given by

$$\mathcal{O}_H : \mathcal{H} \to \mathcal{H},\ \mathcal{O}_H(\psi) = i\hbar\nabla_{X_H}(\psi) + (H)\cdot(\psi).$$

[5] Notice that if we choose ∇ to be the Levi-Civita connection on Euclidean space, the Christoffel symbols all disappear, leaving us with the familiar gradient from calculus.

- *(Harmonic Oscillator.)* In the case of the n-dimensional harmonic oscillator, we have the phase space/symplectic manifold (M, Ω) given by

$$M = \{(q^j, p_j) \in \mathbb{R}^{2n}\}, \Omega = \sum_j dq^j \wedge dp_j,$$

with the Hamiltonian H given by

$$H(q, p) = \frac{1}{2}\sum_j((q^j)^2 + (p_j)^2).$$

We see that Born-Sommerfeld condition is satisfied, since we have $[\Omega] = 0 \in \mathbb{Z}$, and the line bundle is trivial and unique since M is simply connected, with $L = M \times \mathbb{C}$ and the associated connection ∇ given by

$$\nabla_X \psi = X\psi + (2\pi i)\omega(X)\psi = X(\psi),$$

where ω is the pullback so that

$$\omega(X)\psi = \omega(X, X_\psi).$$

Now let us try to apply the prequantization map to the energy, i.e. H; we have that

$$O_H = -i\hbar \sum_j \left[p_j \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial p_j} \right];$$

further, we have that

$$O_H(\psi) = i\hbar\{H, \psi\},$$

where the section ψ is interpreted as a function $M \to \mathbb{C}$.

There's an issue here, however; the operator O_H and the classical Hamiltonian H have the same spectrum, and hence O_H has a continuous spectrum. But we know from physical experience that quantum operators must have discretized spectra. We will discuss this issue further in the context of polarizations.

Structural Properties

As we mentioned before, the prequantization map satisfies "fairly decent" uniqueness properties, in the sense that the first de Rham cohomology group $H^1(M, \mathbb{T})$ parametrizes the possible prequantum line bundles one may have when given a symplectic manifold.

In particular, we have that the following:

Theorem 2. *Let (M, ω) be a prequantizable symplectic manifold. Then the prequantum line bundle $(L \to M, \nabla)$ is unique if and only if the first de Rham cohomology group $H^1(M, \mathbb{T})$ is trivial, where $\mathbb{T} \subset \mathbb{C}$ is the group of complex numbers with unit modulus.*

In other words, the prequantum line bundle is unique when M is simply connected[6].

[6] Proofs may be found in **[Sniatycki]**, pp. 53-56, and **[Woodhouse]**, p. 160.

History and References

1. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Wiley, 2009.

2. A. Weil, "*Sur quelques questiones...*"

3. J. Sniatycki, Geometric Quantization and Quantum Mechanics, Applied Mathematical Science, Vol. 30, Springer Verlag, New York, 1980.

4. N. M. J. Woodhouse, Geometric Quantization, Oxford Mathematical Monographs, Oxford University Press, 1997.