Metaplectic Structures and Correction of Quantization

After the choice of a polarization, we still do not yet have the structure of a Hilbert space on the space of states; this issue is resolved by placing a *metaplectic structure* on the symplectic manifold, in a process known as *metaplectic correction* (for historical reasons), so that sections are square-integrable. In this chapter, we present a discussion of the insufficiency of polarization to motivate the definition and purpose of metaplectic structures, before building on top of the metaplectic structure to conclude the traditional geometric quantization programme on a given symplectic manifold.

Integrability of States

Consider a prequantizable symplectic manifold (M, ω) equipped with P a choice of polarization. In general, the space $\Gamma_P(L) = \Gamma_P(L \to M)$ of polarized sections does not form a Hilbert space[1], for the following reasons:

- there may be sections $\phi \in \Gamma_P(L)$ that are not square-integrable; and even if this were not the case,
- the inner product may not be defined for all pairs $\phi, \psi \in \Gamma_P(L)$.

In the following sections we will sketch out the following amelioration[2] to the above problem: consider the metaplectic frame bundle on

In addition, we also provide the necessary background on frame bundles, double covers, equivariant lifts, and characteristic classes needed to understand aspects of the correction process.

[1] A notable exception is in the case of a Kähler manifold.

[2] In a process known as "*metaplectic correction*".

Frame Bundles

Suppose we have a vector bundle $E \to B$; then for each point $b \in B$, the fiber E_b is a vector space. For each such E_b, we can consider the ordered bases (or *frame*) for E_b; the bundle over B with fiber given by the collection of all bases over E_b is then called the *frame bundle* of $E \to B$. More formally,

Definition 1. *Let $E \to B$ be a (real) smooth vector bundle of rank k; denote by F_b the collection of all frames of E_b at a point $b \in B$. The* frame bundle $F(E) \to B$ *is defined as the disjoint union*

$$F(E) = \coprod_{b \in B} F_b.$$

The points of the frame bundle are of the form (b, ϕ), for $b \in B$ and ϕ an ordered basis for E_b; the fiber of $F(E)$ at b is F_b.

We have an obvious example of a frame bundle:

Example 1. *Consider the tangent bundle TM of a smooth manifold M; this is a vector bundle with the fiber equal to the tangent space at each point M. The associated frame bundle $F(TM) \to M$ is known as the* tangent frame bundle *of M.*

An important point to note is the close relationship between the general linear group $GL(n, \mathbb{R})$ and the frame bundle $F(E)$ of a rank n vector bundle $E \to B$: we have an action

$$\cdot : GL(n, \mathbb{R}) \times F(E) \to F(E), \; g \cdot (b, \phi) \mapsto (b, g\phi),$$

which is free[3] and has orbits $GL(n, \mathbb{R})_b = F_b$.

Finally, frame bundles are indeed bundles, with bundle structures inherited from the underlying vector bundle:

[3] That is, if $g \in GL(n, \mathbb{R})$ has a fixed point, then $g = Id_n$ the identity matrix.

Theorem 1. *Let $E \to B$ be a rank n real vector bundle, with local trivialization (U_i, ψ_i);*

Double Coverings of Groups

Consider the group of unit complex numbers

$$\mathbb{T} = \{e^{i\theta} | \theta \in [0, 2\pi)\}$$

with the group operation given by multiplication. Consider also $(\mathbb{R}, +)$, the additive group of real numbers, along with the exponential map $\exp : \mathbb{R} \to \mathbb{T}$ given by

$$x \mapsto e^{ix}.$$

The preimage of any element in $\mathbb{T}$ is identically $\mathbb{Z}$; one can also note that this map is a homeomorphism[4] as well as a group homomorphism, since

$$(x + y) \mapsto e^{i(x+y)} = e^{ix}e^{iy}.$$

Hence one can envision this mapping causing the real line "wrapping around" the circle group $\mathbb{T}$ an infinite number of times.

[4] When $\mathbb{R}$ and $\mathbb{T}$ are viewed as topological spaces.

There is a generalization of this phenomenon, known as a *covering* group. The following definition comes from Munkres:

Definition 2. *Let $p : E \to B$ be a continuous, surjective map of topological spaces; an open set $U \subset B$ is* evenly covered *by p if $p^{-1}(U)$ can be written as the union of disjoint open slices $V_\alpha \subset E$, such that $V_\alpha|_p \to U$ is a homeomorphism.*

If every point $b \in B$ has a neighborhood that is evenly covered by p, then p is called a covering map *and E is called a* covering space *of B.*

A covering group *of a topological group G is a covering space $H \to G$ such that H is a topological group and $\pi : H \to G$ is a continuous group homomorphism.*

In other words, it is a covering space within the category of topological groups.

We have the following examples of covering groups:

- The circle group $\mathbb{T}$ can cover itself through the map $e^{i\theta} \mapsto e^{ni\theta}$ for some fixed natural number n.
- $SU(2)$, the special unitary group[5] of 2×2 matrices, is the covering group of $SO(3)$, the special orthogonal group[6] of 3×3 matrices.
- (*Double coverings.*) Let H be covered by G, with $|G|/|H| = 2$.[7] Further suppose that the preimage of each evenly-covered open set in H can be written as the union of two disjoint open slices in G. Then G is said to be a double-covering group of H.

[5] The special unitary group consists of unitary matrices with determinant 1.

[6] The special orthogonal group consists of orthogonal matrices of determinant 1. Orthogonal matrices are those matrices M that satisfy

$$M^T M = Id,$$

and represent Euclidean rotations.

[7] I.e., the index of H in G is 2.

The following theorem applies to covering spaces, and is known as the *lifting lemma*:

Theorem 2. *Suppose $p : E \to B$ is a covering map with E locally path-connected and B path-connected. Let $p(e_0) = b_0$.*

Let $f : Y \to B$ be a continuous map with $f(y_0) = b_0$. Suppose Y is path-connected and locally path-connected. Then f can be lifted to a map $\widetilde{f} : Y \to E$ such that $\widetilde{f}(y_0) = e_0$ if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)),$$

where pi_1 is the fundamental group at a given base point.

Intuitively, the above result states that we can understand all the covering spaces of a given "nice enough" topological space by examining the subgroups of its fundamental group.

Equivariant Liftings of Frame Bundles

In the following, recall that an *equivariant* map (with respect to a group action) is a map that commutes with the group action. More precisely, suppose we have a map $f : X \to Y$ and a group G acting on both X and Y under the same group action

$$(\cdot) : G \times X \to X, \ G \times Y \to Y.$$

Then we say that f is equivariant if for any point x and any choice of group element g, we have that

$$f(g \cdot x) = g \cdot f(x).$$

In our context, we have a certain frame bundle[8] $SF(M) \to M$ with a group G acting on $SF(M)$ with M fixed. Additionally, we have a double-covering group

$$\rho : \hat{G} \to G,$$

such that $\hat{G}$ acts on another frame bundle $MF(M) \to M$.

[8] We will be concerned with the *symplectic frame bundle*, which is a sub-bundle of the tangent frame bundle, corresponding to the symplectic group

$$Sp(n, \mathbb{R}) \subset GL(n, \mathbb{R}).$$

However, we specialize later so that we may present the more general idea here.

We are concerned with the possibility of lifting the double-covering map ρ to a map $MF \to SF$ between the frame bundles. Additionally, we would want such a map between frame bundles to "respect" the group actions in some sense[9].

[9] (In other words, we want a map that is equivariant with respect to the G and $\hat{G}$ actions.)

The main result we are concerned with is the following, which states existence:

Theorem 3. ***FILL THIS IN.***

We now get more specific and give two brief examples which we will elaborate upon below:

- (**Metaplectic Structures.**)
- (**Spin Structures.**)

Stiefel-Whitney Classes and Chern Classes

The *Stiefel-Whitney* classes

$$w_0(E), w_1(E), \ldots, w_k(E)$$

of a real vector bundle $E \to B$ of rank k tell us about the obstructions to finding independent*Here, the independence is used in the linear-algebraic sense; think of independent vectors for each $b \in B$.* sections

$$\sigma : B \to E,$$

and are defined cohomologically.

On the other hand, the Chern classes

$$c_1(K), c_2(K), \ldots, c_k(K)$$

of a *complex* vector bundle $K \to M$ of rank k are defined homotopically and allow us to confirm that two vector bundles over the same base manifold are indeed different[10]. **Work in progress. To be completed.**

[10] However, two different vector bundles can have the same Chern class.

Metaplectic and Spin Structures

In this section, we examine two very special double covering groups: the *metaplectic group*, which is the double covering group of the symplectic group $Sp(2n)$; and the *spin group*, which is the double covering group of the special orthogonal group $SO(n)$.

Definition

A metaplectic structure on a symplectic manifold may be thought of as analogous to a spin structure placed on a Riemannian manifold[11],

[11] Or rather, an *oriented* Riemannian manifold.

in that the definitions of each share the same pattern of being defined as *equivariant lifts* of a certain frame bundle, with respect to a certain double-cover of structure groups. For the sake of illustration and comparison, we present both definitions.

First, the Riemannian case:

Definition 3. *Let (M, g) be an orientable Riemannian manifold. Then there is a double-covering of Lie groups*

$$\rho : \mathrm{Spin}(n) \to SO(n)$$

from the spin group to the special orthogonal group. Consider the oriented orthonormal frame bundle

And now the symplectic case:

Definition 4. *Let (M, ω) be a symplectic manifold. Then there is a double-covering of Lie groups*

Remark. For a discussion of spin and metaplectic groups, see the appendix. We will remark here that the spin group $\mathrm{Spin}(n)$ is the unique double cover of $SO(n)$ such that the sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to SO(n) \to 1$$

is exact, where 1 here refers to the trivial group; and the metaplectic group $Mp(2n)$ is the unique double cover of $Sp(2n)$ the order n symplectic group.

Properties

Metaplectic structures, much like spin structures, may be classified cohomologically:

Theorem 4. *Let (M, ω) be a symplectic manifold. Then (M, ω) admits a metaplectic structure if and only if the second Stiefel-Whitney class vanishes; equivalently, if and only if the first Chern class is even.*

When this is the case, the class of admissible metaplectic structures is classified by

Induced Hilbert Space from Metaplectic Structure

Recalling the obstruction from the first section of this chapter, we now give a presentation of the way in which the existence of a metaplectic structure enables the construction of the "proper" Hilbert space of states.

Examples

We present some examples of metaplectic structures.

Phase Spaces

Let Q be some n-dimensional orientable manifold. Then the cotangent bundle T^*Q is a symplectic manifold; further, it has a natural metaplectic structure.

To introduce the topic, note that we have the following short exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{C}\times$$

which in turn gives us the exact sequence

this latter group is the metaplectic group and is a *double cover* of the symplectic group.

History and References

- **Frame bundles.** Kobayashi & Nomizu, Foundations of Differential Geometry vol 1.
- **Covering groups.** Munkres, Topology.
- **Stiefel-Whitney and Chern classes.** ???
-