The standard geometric quantization procedure works on symplectic manifolds, a significant subclass of which — including all unconstrained physical phase spaces — is comprised of cotangent bundles. But when do we obtain a symplectic manifold that is *not* equivalent to a cotangent bundle?

In practice, we obtain a symplectic manifold that is inequivalent to a cotangent bundle when we have a symmetry acting on phase space (viewed as a Lie group action) and proceed to obtain the "reduction" of the space via the action of the symmetry. In general, the process is known as *symplectic reduction*; this section will be devoted to the explanation of this reduction procedure.

Generally speaking, symplectic reduction assumes a symplectic manifold M equipped with a Lie group action GM on M, producing a symplectic manifold $M//G$ — known as the *reduced space* — in the following manner:

- Let g be the Lie algebra associated to the Lie group G, i.e. the exponential map takes g to G. If the action GM preserves the symplectic form on M, then we obtain the following homomorphisms of Lie algebras:

  $$(\cdot)^M : g \to \Gamma(M, TM), X \mapsto X^M$$

  constructed by composing the group action and the exponential map, and

  $$J_{(\cdot)} : g \to C^\infty(M), X \mapsto J_X \text{ s.t. } X^M \cong \xi_{J_X},$$

  where ξ_Y is the Hamiltonian vector field of Y.

- The functions J_X may be assembled into a *moment map*[1] $J : M \to g^*$ for the action:

  $$\langle J(p), X \rangle = J_X(p).$$

  Here, g^* refers to the vector space dual of the Lie algebra g.

- The reduced space M^0 is then defined via $M^0 = J^{-1}(0)/G$, the preimage of $0 \in g^0$ through J, modulo G Hence M^0 is a subspace of M/G, the orbit space of the action GM. Generally, we cannot guarantee that M^0 is a smooth manifold unless certain topological assumptions are made on the Lie group and its action, which we review below; however, if those conditions hold and M^0 is a smooth manifold, it also inherits a natural symplectic form.

[1] Also written as "momentum mapping".

In this chapter, we define and discuss the notion of symplectic reduction, with a particular focus on the symplectic reduction process of Marsden and Weinstein developed in [2]. We also discuss the prerequisites necessary to understand the Marsden-Weinstein reduction

[2]

procedure; we begin with a quick review of Lie groups and Lie algebras (as well as the correspondence between the two categories) and duals of Lie algebras, before discussing Lie groups acting on symplectic manifolds via symplectic and Poisson actions; we present the full reduction procedure in the final part.

Lie Groups and Lie Algebras

Lie Groups

Lie groups are mathematical objects that are at once both smooth manifolds and groups[3], with the group structure and the manifold structure interacting in a canonical way.

[3] In the group theory sense.

More formally,

Definition 1. *Let $(G, \cdot)$ be a group such that the set G also has the structure of a smooth manifold. Then G is said to be a* Lie group *if the group multiplication*

$$(\cdot) : G \times G \to G$$

is a smooth map from the product manifold $G \times G$ to G and the inversion map

$$(-)^{-1} : G \to G$$

is a smooth automorphism.

Equivalently, G is a Lie group if the inverse product

$$(g_1, g_2) \to g_1^{-1} \cdot g_2$$

is a smooth map.

Lie groups often appear as matrix groups and manifolds with a large amount of symmetry:

- $GL(n, \mathbb{R})$, the general linear group, is a Lie group under matrix multiplication.
- $Sp(2n, \mathbb{R})$, the symplectic group, is a Lie group under matrix multiplication.
- $\mathbb{T}^1$, the dimension 1 torus, is a Lie group when viewed as the complex numbers of norm 1 under multiplication.

Lie Algebras and their Duals

A Lie algebra is a vector space with a special kind of product, called a *Lie bracket*.

Definition 2. *Let g be a vector space over some field. Let*

$$[\cdot,\cdot] : g \times g \to g$$

be an operation satisfying

- *bilinearity:*
- *alternativity:*
- *anticommutativity:*
- Jacobi's identity*:*

The the pair $(g, [\cdot,\cdot])$ *is called a* Lie algebra*.*

As with all vector spaces, there is the notion of a *dual* vector space associated to a Lie algebra g:

Lie Algebra Associated to a Lie Group

Lie algebras and Lie groups share an important association, reflecting a *global* versus *local* dichotomy.

Theorem 1.

Lie Group Actions on Symplectic Manifolds

Lie groups acting on symplectic manifolds often have a physical interpretation as the group of symmetries acting on a physical system;

Lie Group Actions on Symplectic and Poisson Manifolds

In general, actions of Lie groups on smooth manifolds allow us a secondary perspective towards understanding complicated manifold geometries. Lie group actions form "layerings" in the sense that we may study $g \cdot M$ for each $ginG$; at the same time, we can also look at the *orbit-types* of G

More formally, suppose we had an action of a Lie group on a smooth manifold given by

$$\sigma : G \times M \to M, (g, p) \mapsto g \cdot p;$$

for a point $p \in M$, consider the orbit map $\sigma_p : G \to M$ given by $\sigma_p(g) = g \cdot p$. Then σ_p is a differentiable map, and its differential at $e \in G$ is

On Symplectic Manifolds

Fix a connected Lie group G with an action GM on a symplectic manifold (M, ω). Suppose that the action GM leaves ω fixed, i.e. G

acts on M through symplectomorphisms $M \to M$. Let g be the Lie algebra associated to G, i.e. $\exp G = g$. Then for $X \in g$, we have that

$$0 = X \cdot \omega = \mathrm{d}\iota(\xi_X)\omega + \iota(\xi_X)\mathrm{d}\omega = \mathrm{d}\iota(\xi_X)\omega \text{ (closed one-form)},$$

where ι is the map that [...FIGURE THIS PHRASING OUT], d is the exterior derivative, and ξ_X is the vector field on M associated to X. The first equality follows from the definition of a symplectomorphism; the second from the definition of the Lie algebra action; and the third from the fact that ω is closed.

The above chain of equalities leads to two useful definitions. Call a vector field $\xi \in (M)$ a *symplectic vector field* if

$$\mathrm{d}\iota(\xi)\omega \text{ is closed,}$$

and call it a *Hamiltonian vector field* if

$$\xi^\flat \text{ is exact,}$$

i.e. if there exists a function $\phi_\xi \in C^\infty(M)$ such that $\xi^\flat + \mathrm{d}\phi_\xi = 0$[4].

[4] Observe that ϕ_ξ is in general *not* unique.

Define also the corresponding spaces $sym(M)$ and $ham(M)$ consisting of the above vector fields:

- Define $sym(M)$ to be the space of symplectic vector fields on M; we also note that

$$sym(M) = \sharp(\Omega^1_{closed}(M))$$

  where $\sharp$ is the *musical isomorphism* taking a differential 1-form to its corresponding vector field.

- Define $ham(M)$ to be the space of hamiltonian vector fields on M; we note that

$$ham(M) = \sharp(\Omega^1_{exact}(M)).$$

With the above definitions in hand, we can now define the notions of *Hamiltonian* and *Poisson* actions of Lie groups:

Definition 3. *Let GM be a Lie group action, and let $g = (G)$ be the Lie algebra associated to G. We say that GM is* Hamiltonian *if, for all $X \in g$, we can find a function $\phi_X \in C^\infty(M)$ such that*

$$\xi_X^\flat + \mathrm{d}\phi_X = 0,$$

in which case we obtain an associated map $g \to C^\infty(M)$ given by

Further, consider the Poisson algebra on $C^\infty(M)$ given by the Poisson bracket

$$\{f, g\} := \omega(\xi_f, \xi_g),$$

where ξ_f (resp., ξ_g) is the hamiltonian vector field asociated to f (resp., g), i.e. $\xi_f^\flat + \mathrm{d}f = 0$ ($\xi_g^\flat + \mathrm{d}g = 0$). This gives us a Lie algebra structure on $C^\infty(M)$, and a Hamiltonian action GM is said to be Poisson *if there exists a homomorphism of Lie algebras $g \to C^\infty(M), X \mapsto \phi_X$ such that*

$$\xi_X^\flat + \mathrm{d}\phi_X = 0 \text{ and } \phi_{[X,Y]} = \{\phi_X, \phi_Y\},$$

for all $X, Y \in g$.

Intuitively,

On Poisson Manifolds

Fix a Poisson manifold P, i.e. a smooth manifold with a Lie bracket $\{\cdot,\cdot\}$ on its space of smooth functions $C^\infty(P)$ such that the *Leibniz rule* holds:

$$\forall f, g, h \in C^\infty(M), \{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Equivalently, we may say that Poisson manifolds are smooth manifolds such that $C^\infty(M)$ is equipped with a Lie algebra structure such that the

Example 1. *As an example of a Poisson bracket, consider the* Hamiltonian vector field ξ_f *of a function f, i.e. the unique vector field satisfying*

The Lie bracket of two vector fields is defined via $[X, Y] = XY - YX$, viewing the vector fields as derivations. It is not difficult to verify that the following map is a Poisson bracket on $C^\infty(M)$:

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \{f, g\} := [\xi_f, \xi_g].$$

For a Lie group G, a *Lie group action* on a Poisson manifold P

Closing Remarks on Symplectic and Poisson Manifolds

There are a few remarks to make here regarding the connection between symplectic and Poisson manifolds. The most obvious is that every symplectic manifold possesses a Poisson manifold if one takes the Poisson bracket to be given by

$$\{f, g\} = \omega(\xi_f, \xi_g),$$

where as above ξ_f (resp., ξ_g) is the Hamiltonian vector field of f (resp., g).

Something more can be said here — symplectic manifolds are those special Poisson manifolds in which the Hamiltonian vector fields *exhaust* the tangent bundle:

Theorem 2.

Marsden-Weinstein Reduction and Co-Isotropic Reduction

To define the Marsden-Weinstein reduction process, one must first introduce the moment map taking a manifold to the dual of the Lie algebra acting on it.

Definition

Properties

$$\iota^*\omega = \pi^*\omega^0,$$

where the maps ι and π are inclusion and projection maps, i.e.

$$M\iota \hookleftarrow J^{-1}(0)\pi \to M^0.$$

Coisotropic Reduction

Aside from the notion of Marsden-Weinstein reduction, there is also *coisotropic* reduction, in which

De Rham and Lie Algebra Cohomologies

We would like a way of measuring the difference between symplectic actions and Poisson actions of Lie groups, i.e. a way to quantify the obstructions that would prevent a symplectic action from becoming a Poisson action. This is possible through a combination of the *de Rham* and *Chevalley-Eilenberg*[5] cohomology theories — and more generally, via *equivariant cohomology*.

[5] Also known as *Lie algebra* cohomology.

Historical Notes and References

The notion of a Lie group action on a manifold has existed since Sophus Lie's investigations into

In general, the orbit space M/G of a G-manifold, when G is a general Lie group, fails to be a manifold unless we impose requirements on G. For a study of how to deal with orbifold orbit spaces, see [6].

[6]

The genesis of Lie algebra cohomology is in the 1948 paper of Chevalley and Eilenberg, "Cohomology Theory of Lie Groups and Lie Algebras" [7]; a modern introduction may be found in the seventh chapter of Hilton & Stammbach's "A Course in Homological Algebra" [8].

[7]

[8]

For a cursory introduction to equivariant cohomology, there is the short expository by L. Tu [9]; deeper discussions may be found in the first chapter of Guillemin and Sternberg's "Supersymmetry and Equivariant de Rham Theory" and

[9]

For a reference to the theory of Lie algebras, see http://www.tufts.edu/ fgonzale/lie.algebras.book.pdf