Quantization: Examples and Structures

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CHAPTER 1

Preface

1. Comments

This current document is a survey of two popular quantization schemes: geometric quantization and deformation quantization. An emphasis is placed on the structural, category-theoretic properties of these maps rather than their analytic components. Additionally, further generalizations and current work in these fields are discussed.

No claim of originality is made, as the dual purposes of this survey are to function as an exposition of these two techniques, and to summarize some components of contemporary work on the methods involved. The outline of the survey is as follows:

- Part I is an overview of geometric quantization, as well as a discussion of some modern views on the process;
- Part II discusses algebraic quantization;
- Part III discusses the inter-relations and connections between algebraic and geometric quantization, and quantization in general; and finally,
- the final part is a series of appendices on pre-requisites and asides that elaborate on topics mentioned in the previous parts that were not fully explored.

The reader will find this document most helpful if they possess familiarity with some elements of differential geometry (symplectic geometry, fibre bundles, and connections), functional analysis (Hilbert spaces, algebras of operators, basic operator theory), representation theory and group theory (Lie groups and Lie algebras, representations) and category theory (categories, functors, natural transformations, some aspects of the theory of n-categories and higher category theory, and topos theory). There will be some exposition of the physical models involved, but the primary motivation is mathematical; hence, some intuition on the connections between Hamiltonian mechanics and symplectic geometry and on the connections between quantum mechanics and Hilbert spaces will be helpful.

Some initial remarks on the content of this survey are in order. *Quantization* is generally accepted in this context to refer to a structured process that passes from the formalism of classical mechanics to the formalism of quantum mechanics. In general, *there is no exact recipe* for quantization. Some reasons for this involve the existence of "no-go" theorems that prevent certain convenient properties from holding, and the somewhat unavoidable arbitrary choices required during quantization[1]. Consequently, there are many different quantization methods, each with their own motivation — in the current survey, we discuss two of them: geometric quantization (which attempts to preserve geometric structure and hence avoids

[1]Polarizations, for example, are often not unique, and there may not be a preferred choice.

reliance on coordinates), and deformation quantization (which attempts to find a "deformation" of the algebra of observables from the commutative classical context to the non-commutative quantum context). It must be said that these two schemes correspond to quantization with a focus on physical states versus quantization with a focus on observables, not unlike the distinction between Heisenberg's matrix mechanics and Schrödinger's wave function.

Special thanks go to Prakash Panangaden (School of Computer Science, McGill University) for introducing the author to the topic, and to Jesse Madnick (Department of Mathematics, Stanford University) for helpful discussions on Kähler geometry.

2. The Dirac Axioms for Quantization

It is well established that classical mechanics has its mathematical formalisms in the context of smooth manifold theory (namely, symplectic and Poisson manifolds, as is discussed in this survey), and quantum mechanics has its mathematical formalisms in the context of Hilbert spaces.

To elaborate, the Hamiltonian view of classical mechanics postulates that, given a physical system, there is a *configuration space* given by a smooth manifold Q that represents the space of possible states of the physical system; we then obtain its *phase space* by considering its cotangent bundle, T^*Q, which carries a natural symplectic structure. Observables of this physical system are given by smooth functions $T^*Q \to \mathbf{R}$, which together form a Poisson algebra $C^\infty(T^*Q)$; there is a distinguished observable H — called the *Hamiltonian* — that represents the energy of the system; Hamilton's equations then state that the dynamics of the system are governed by the equation

$$\frac{f}{t} = \{f, H\}$$

for $f \in C^\infty(T^*Q)$ an observable quanity, and $\{\cdot,\cdot\}$ the standard Poisson bracket on $C^\infty(T^*Q)$. For further details and a more comprehensive treatment, see the classic text by Arnol'd [**?**]; see also the related appendix.

On the other hand, the mathematical formulation of quantum mechanics calls for a Hilbert space $\mathcal{H}$ to represent the space of states of a physical quantum system [2], with observables given by self-adjoint operators $O : \mathcal{H} \to \mathcal{H}$. Similar to the classical case, the collection of all such observables forms an algebra, albeit a non-commutative one. Dynamics are determined by Schrödinger's equation

$$i\hbar\frac{\psi}{t} = H\psi$$

or equivalently by the Heisenberg equation

$$\frac{O}{t} = \frac{i}{\hbar}[H, O]$$

depending on whether one considers the states to be time-varying or the observables to be time-varying. Here, H is a distinguished operator on the Hilbert space called the Hamiltonian, O is an observable, and $[\cdot,\cdot]$ is the commutator bracket on the algebra of observables. Detailed discussions on the mathematical formulations of

[2] Actually, the projectivization of this Hilbert space $\mathbf{P}\mathcal{H}$ is the space of states, since we have to account for equivalence modulo scalar multiples, i.e. $\lambda \cdot |\psi\rangle$ represents the same state as $|\psi\rangle$, for any complex nonzero λ.

quantum mechanics may be found in von Neumann's classic text [**?**] as well as in the relevant appendix.

It is natural to seek a way of relating these two frameworks. Historically, the development of the theory of quantization began with Dirac, who introduced (what is now called) canonical quantization for the purpose of explaining quantum phenomena by way of a "classical analogy" in his doctoral thesis [**?**]; in particular, the correspondence between the Poisson bracket and the commutator is developed. In Dirac's later text ("*Principle of Quantum Mechanics*", [**?**]), the axiomatic framework for quantization was proposed.

In general, we have what are known as the *Dirac axioms* for quantization: for a symplectic manifold (M, Ω), a quantization $\mathcal{Q}$ of M should give us an associated Hilbert space $\mathcal{H}$ as well as an association between smooth functions $C^\infty(M) \ni f : M \to \mathbb{R}$ and self-adjoint operators $\mathcal{Q}(f) = \mathcal{O}_f : \mathcal{H} \to \mathcal{H}$ such that the following properties hold:

Definition 1.

- *Poisson brackets correspond to commutators, i.e.* $\mathcal{Q}(\{f, g\}) = [\mathcal{Q}(f), \mathcal{Q}(g)]$*;*
- $\mathcal{Q}(1) = i \cdot Id$*, i.e. the constant function* 1 *should be mapped to the identity operator times the scalar* $\sqrt{-1}$*;*
- *for* $\alpha, \beta \in \mathbb{C}$*,* $\mathcal{Q}(\alpha f + \beta g) = \alpha\mathcal{Q}(f) + \beta\mathcal{Q}(g)$ (Linearity)*; and*
- *a* Minimality Condition*: any* complete family *of functions maps to a* complete family *of operators.*

These four ideas correspond to intuitive points regarding the structures we might wish to see preserved through the quantization. The first point states, essentially, that Lie-algebraic structures should be preserved; the second point states that the trivial classical observable should correspond to the trivial quantum observable; the third point states that quantization is a linear map; and the last point, the minimiality condition, states that complete families of functions (i.e., families of functions that separate points on the manifold M) should correspond to families of operators (i.e., operators that act irreducibly on $\mathcal{H}$).

Additionally, physical intuition stipulates that the position and momentum functions $x^i : M \to \mathbf{R}$ and $p_j : M \to \mathbf{R}$ must be quantized to the operators $\psi \mapsto x^i\psi$ and $\psi \mapsto -i\hbar_j\psi$, respectively; we shall see, however, that these conditions are rarely a point of difficulty during the quantization process and serve mainly as a guide. The important structural content of a quantization scheme is codified mainly in the above four axioms.

Unfortunately, it is not possible to quantize the entire Poisson algebra of smooth functions, and there are many obstructions against quantization; see *Obstruction Results in Quantization Theory* [**?**] for a survey. In particular, we have the *Groenewald-van Hove theorem*, which informally states that the algebra of polynomials on the phase space $\mathbb{R}^{2n}$ has no unitary representation ρ such that ρ extends the Schrödinger representation of the Heisenberg algebra. Consequently, this provides a counterexample to the claim that it is possible to quantize the entire algebra of observables on a classical phase space. This obstruction, as well as others, are discussed in the last chapter of Part I.

Some of the more common approaches to quantization are *canonical* (first) and *second* quantization. The former refers to a semi-classical quantization of mechanical systems, where the potentials are treated classically and particles are quantized into elements of a Hilbert space of functions (for example, an L^2 space);

the latter then refers to a functor from the category of Hilbert spaces (i.e., state spaces of single-particle physical systems) to a category of Fock spaces (a direct sum of tensor products of Hilbert spaces, representing state spaces of physical systems that may have multiple particles).

This current survey will not discuss either of those approaches; standard references such as [**?**], [**?**] or [**?**] may be consulted for an overview. Instead, the current survey will present *geometric* quantization and *deformation* quantization, for two main reasons: first, the mathematical sophistication and depth of these techniques often cloud their intuitive structural properties, which form a rich theory that attempts to preserve as much geometric and algebraic insight as possible; and second, much active work is being done on furthering these structural properties — for example, [**?**] posits that the "correct" category of symplectic manifolds to consider may be one with arrows given by canonical relations rather than symplectomorphisms, and [**?**] frames quantization in a higher category-theoretic context.

To motivate the development of the geometric and deformation theories of quantization, we note that the basic intuition behind these theories is to deal with the inability to satisfy all of Dirac's axioms[3] by either compromising on either the correspondence of the Poisson bracket and the commutator, or by limiting the observables that we are "allowed" to quantize. Deformation quantization does the former by treating $\hbar$ as a formal parameter that "adjusts" the bracket, so that we obtain Poisson brackets in the "limit" $\hbar \to 0$. Geometric quantization does the latter by preserving geometric structure through the quantization and choosing a structure (called a *polarization*) on the quantized geometry that limits which operators are able to be meaningfully quantized.

[3]In fact, it is only possible to satisfy two of them at most!

Part 1

Geometric Quantization

CHAPTER 2

Introduction: Geometric Quantization, Symplectic Manifolds, and Hilbert Spaces

In this chapter, we will review some basic and relevant results on the mathematical background behind the geometric quantization process — mainly, the categories **Symp** of symplectic manifolds and symplectomorphisms, and **Hilb** of Hilbert spaces and unitary maps — and outline the three main steps of the process:

- *Pre-quantization*, where we construct an associated *pre-quantized Hilbert space* out of a given symplectic manifold by considering the space of sections of the associated complex line bundle; this sub-process is a natural one, but unfortunately the resulting Hilbert space still contains some rather un-physical properties. This leads to the next step of the process, polarization.
- *Polarization*, where we "cut down" the pre-quantum Hilbert space by half (using a Lagrangian distribution on the symplectic manifold) in order to enforce a physicality condition on the states of the quantum system, i.e. make sure that states rely only on half of the dimensions of the manifold.
- Finally, we present a *metaplectic correction*, where we modify the space of states by tensoring it with half-forms to ensure square-integrability on the states and to make sure that the quantized operators have spectra matching physical experience.

1. Symplectic Manifolds and Classical Mechanics

In this section we define symplectic manifolds and maps between them (symplectomorphisms), along with providing some structural information on **Symp** the category of symplectic manifolds and some motivation for their ubiquity in classical mechanics (via Hamiltonian mechanics).

1.1. Symplectic Vector Spaces. We first consider the flat case, i.e. a symplectic vector space[1], in which we have the following definition:

Definition 2. *A vector space V over a field F along with a bilinear map $\Omega : V \times V \to F$ is said to be* symplectic *if the map Ω is* alternating*, i.e. $\forall v \in V, \Omega(v, v) = 0$ as well as* non-degenerate*, i.e. $\Omega(v, w) = 0 \forall w \in V v = 0$.*

In this scenario, the bilinear map Ω is called the symplectic form *on V.*

[1]A symplectic vector space is analogous to an inner product space, albeit with an important caveat — instead of a symmetric bilinear form, we have an *alternating* form, and the intuition is not that the symplectic form measures volume in some sense but rather that it measures "symplectic" volume. To fully explain this is outside the scope of the current conversation, though an important theorem to keep in mind for the general manifold case is the celebrated *non-squeezing theorem* of Gromov; essentially, there is no symplectomorphism from a sphere to a cylinder with radius smaller than that of the sphere.

For the purpose of illustration, we have the following examples of symplectic vector spaces:

- Let
$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$
where I_n is the $n \times n$ identity matrix; then Ω is called the *standard symplectic form* on $\mathbb{R}^{2n}$.
- Let V be any finite-dimensional vector space over $\mathbb{R}$, with dual space V^*. Then the direct sum $V \oplus V^*$ is a symplectic vector space with the symplectic form given by:
$$\omega(x \oplus \phi, y \oplus \psi) := \psi(x) - \phi(y).$$
In fact, it can be shown that every finite-dimensional symplectic vector space is of this form (up to isomorphism).

We will mainly be concerned with symplectic vector spaces and manifolds over real and complex numbers.

Now there are some special subspaces of symplectic vector spaces; we are mainly concerned with *Lagrangian*, *(co-)isotropic*, and *symplectic* subspaces of symplectic vector spaces. We have the following definitions:

Definition 3. *Let (V, Ω) be a symplectic vector space. For a subspace $F \subset V$, the* symplectic complement $F^\perp$ *of F is defined as the space of vectors "perpendicular" (in analogy to an inner product space) to every vector in F:*
$$F^\perp := \{x \in V | \forall y \in F, \Omega(x, y) = 0\},$$
and F is said to be:

- Lagrangian *if $F = F^\perp$;*
- isotropic *(resp.,* co-isotropic*) if $F \subset F^\perp$ (resp., $F^\perp \subset F$); and*
- symplectic *if $F \cap F^\perp = \{0\}$.*

Note that the Lagrangian subspaces necessarily have half the dimension of the overall space; in fact, an intuitive picture of Lagrangian subspaces are embeddings of the physical configuration space into the phase space[2]. This will gain further relevance in the context of polarizations, in which a Lagrangian distribution is chosen to "filter out" certain sections of the prequantum line bundle.

Before moving on, we make the quick remark that if there exists a symplectic form on a (finite-dimensional) vector space, the dimension of the vector space is necessarily even — for if not, then we would have an odd-dimensional matrix representation of the symplectic form, which is necessarily singular due to the antisymmetry.

1.2. Symplectic Manifolds. We now introduce the geometric analogue of symplectic vector spaces, i.e. symplectic manifolds. We call a smooth manifold and 2-form pair (M, ω) a *symplectic manifold* if it has a symplectic structure:

Definition 4. *A differential 2-form ω on a manifold M^{2n} gives a* symplectic structure *if it is both* closed, *i.e.* $\mathrm{d}\omega = 0$ *where* d *is the exterior derivative, and* non-degenerate, *i.e.* $\omega \wedge \ldots \wedge \omega$ *(n times)* $\neq 0$.

[2] Indeed, it is a theorem that for V symplectic there exists Lagrangian E such that $V = E \oplus E^*$.

We have the following examples:

- *(Kähler Sphere.)* Consider $\mathbb{S}^2 \cong \mathbb{CP}^1$; there are natural *complex*, *symplectic*, and *Riemannian* structures on this manifold, and manifolds that admit these three structures with each being compatible with the others (in some suitable sense) are called *Kähler*.
- *(Cotangent Bundles.)* Let Q be a smooth manifold of dimension n (not necessarily even); then the cotangent bundle T^*Q carries with it a natural symplectic structure, as follows: consider the *tautological one-form* — also known as the *symplectic potential* — given by
$$\theta = \sum_i p_i \mathrm{d}q^i,$$
where the (p_i, q^i) are canonical coordinates; then the exterior derivative of this one-form is a two-form (known as the *canonical*, or *Poincare*, two-form):
$$\omega = -\mathrm{d}\theta = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i.$$
One may verify that this is closed and non-degenerate.

At this point it becomes relevant to introduce the physical importance of symplectic geometry; while we will not introduce all of Hamiltonian mechanics (see the relevant chapter of the appendix for a quick cursory overview), we will make the following definitions:

Definition 5.

Additionally, an important theorem of Darboux provides a way of working locally. *Darboux' theorem* states the following:

Theorem 1. *(Darboux.) Let (M, ω) be a symplectic manifold; for any point $p \in M$, there is a local chart U_p with coordinates x^i, y^i such that*
$$\omega = \sum_i dx^i \wedge dy^i.$$
In other words, there is a local representation of the symplectic form as a wedge product of exterior derivatives of coordinates around each point; symplectic manifolds are locally isomorphic to $\mathbb{R}^{2n}$ with the standard symplectic form.

We also have the following definition of a *symmetry* of a symplectic manifold:

Definition 6. *Let (M^{2n}, ω) be a symplectic manifold and suppose a Lie group G acts on (M, ω) symplectically, i.e. each $g \in G$ respects the symplectic form:*
$$\forall m \in M, \omega_m(v, w) = \omega_{g \cdot m}(v, w)$$
Then G is said to be a group of symmetries*, and each $g \in G$ is said to be a symmetry of the system represented by (M, ω).*

Every symmetry gives rise to a symplectomorphism $\phi_g : M \to M$ such that $\phi_g^* \Omega = \Omega$, where
$$\phi_g^* : \Omega^2 \to \Omega^2, \phi_g^*(F)(v, w) = F(\phi_g(v), \phi_g(w))$$
is the pullback of ϕ_g to a map between differential forms. Hence, each group of symmetries is represented by a subgroup of $Sp(M, \Omega)$ the group of symplectomorphisms on M.

Finally, we conclude by examining some structure of the category **Symp** of symplectic manifolds and symplectomorphisms.

2. Hilbert Spaces and Quantum Mechanics

In this section, we will review the rudiments of the standard Hilbert space formalism behind (non-relativistic) quantum mechanics.

2.1. Hilbert Spaces. In general, the following ingredients are necessary for the mathematical structure behind quantum mechanics: for a given physical system, we have $\mathcal{H}$, a (separable) Hilbert space over $\mathbb{C}$ the complex numbers, along with

- *states* being given by either equivalence classes of elements of $\mathcal{H}$ (up to scalar multiples) — i.e., classes of the form $[\psi] = \{\lambda \cdot \psi : \lambda \in \mathbf{C}, \lambda \neq 0\}$ — or equivalently, elements of the projective Hilbert space $\mathbf{P}\mathcal{H}$, which is given by the map $\pi : \mathcal{H} \to \mathbf{P}\mathcal{H}, \phi \mapsto [\phi]$.
- *observables* being given by self-adjoint (i.e., Hermitian) linear operators $\mathcal{O} : \mathcal{H} \to \mathcal{H}$; since they are self-adjoint, all eigenvalues are real (by the same proof as for Hermitian matrices). Results of physical measurements on observables are given by the eigenvalues of the observable in question.

As for the dynamics, there are two main interpretations stemming from equivalent physical viewpoints: the *Heisenberg* picture, where one views the observables as the dynamical objects, or the *Schrödinger* picture, where one views the states as the dynamic objects.

We have a special observable that defines the dynamics on the system, called the *Hamiltonian*, denoted either H (in Schrödinger picture) or $H(t)$ (in Heisenberg picture), and the following equations determining time-evolution:

- for the Heisenberg picture — assuming states are time-independent and observables are time-varying — we have
$$i\hbar \frac{d}{dt}\mathcal{O}(t) = [\mathcal{O}(t), H(t)];$$
- for the Schrödinger picture — assuming states are time-varying and observables are constant — we have
$$i\hbar \frac{d}{dt}\psi(t) = H\psi(t).$$

We note that dynamics are not often considered in "vanilla" geometric quantization; rather, the process focuses on finding a mapping between symplectic manifolds and Hilbert spaces, i.e. between the phase space and the state space of the classical and quantum systems, respectively.

2.2. Symmetries on a Hilbert Space. We also have the following definition of a *symmetry* on a Hilbert space, due to E. Wigner:

Definition 7. *Let $\mathcal{H}$ be a complex Hilbert space; then a bijective map $U : \mathcal{H} \to \mathcal{H}$ is called a (quantum)* symmetry *iff the "transition probabilities" are preserved under U, i.e.*
$$\forall \psi, \phi \in \mathcal{H}, \frac{|\langle U(\psi), U(\phi)\rangle|^2}{||U(\phi)||^2 \cdot ||U(\psi)||^2} = \frac{|\langle \psi, \phi\rangle|^2}{||\phi||^2 \cdot ||\psi||^2}.$$

Intuitively, symmetries may be thought of as unitary or anti-unitary operators:

Further, we have the following properties regarding the category of Hilbert spaces (as objects) and unitary operators (as arrows):

We will not mention much more about the theory of Hilbert spaces here; for further details, consult the appendix and the references.

3. Overview of the Geometric Quantization Process

We now give an overview of the geometric quantization of symplectic manifolds. The process is done in three steps, each with subtleties in their own right:

- The prequantization step does indeed provide us with a Hilbert space, but this space possesses un-physical properties in many cases[3] (necessitating the polarization step); further, not all arbitrary symplectic manifolds may be prequantized — prequantizability is decided by the *Weil integrality condition*, a cohomological constraint.
- The notion of a polarization "cuts down" the dependency on the extraneous variables that the states have, through the choice of a Lagrangian distribution (i.e. an isotropic distribution of dimension half that of the symplectic manifold); however, we end up with states that are not square-integrable or otherwise do not belong in the quantized Hilbert space. This also leads to observables with spectra that do not match their physics.
- Finally, the metaplectic "correction" — the term "correction" stemming from historical reasons — ameliorates this last issue by tensoring the state space by a space of *half-forms* so as to render the states (square-)integrable and to correct the spectra of the quantized operators, thought it requires the introduction of new structures (the metaplectic and metalinear groups). A detailed discussion of the structures involved is discussion in the section on the metaplectic correction.

The main goal is to construct a functorial correspondence from **Symp** — the category of symplectic manifolds and symplectomorphisms — to **Hilb** — the category of Hilbert spaces and unitary maps. As we will see, this is not necessarily natural in the sense that the geometric quantization construction may not be applicable to all symplectic manifolds, and not all observables $f \in C^\infty(M)$ are quantizable.

We first give an overall summary of the geometric quantization process in this section; we will elaborate with intuition and theorems that justify the definition of each portion of the process in the corresponding section.

DEFINITION 8. *Let (M^n, Ω) be a symplectic manifold with $\frac{\Omega}{2\pi\hbar}$ having integral cohomology class, i.e.*

$$\left[\frac{\Omega}{2\pi\hbar}\right] \in H^2(M, \mathbb{R})$$

Then:

- ***(Prequantization.)*** *Let $\pi : L \to M$ be some* complex line bundle *over M, and let $\Gamma(M, L)$ be the space of smooth sections $s : M \to L$. Then*
- ***(Polarization.)*** *Let $T^{\mathbf{C}}M$ be the complexified tangent bundle of M, i.e. $T^{\mathbf{C}}M = TM \otimes_{\mathbf{R}} \mathbf{C}$. A* polarization *of (M, Ω) is an* involutive, Lagrangian sub-bundle *$\mathcal{P} \hookrightarrow T^{\mathbf{C}}M$.*

[3]...Such as obtaining quantum wavefunctions that depend on more variables than the dimension of the configuration space, for example.

- *(**Metaplectic Correction.**)*

From these definitions we may define the following:

- *The* quantized Hilbert space $\mathcal{H}$ *is defined to be the (metric completion of the) complex vector space generated by elements of the form* $s \otimes \sigma$*, where*

  $$s \in \Gamma(M, L), \sigma \in \Gamma(M, N^{1/2}), \nabla_X s = 0, \mathcal{L}_X \sigma = 0 \forall X$$

  where X *is any locally Hamiltonian vector field in* $\mathcal{P} \cup \bar{\mathcal{P}}$ *for* $\mathcal{P}$ *some choice of polarization.*

  The inner product of this Hilbert space is given by

  $$< s_1 \otimes \sigma_1, s_2 \otimes \sigma_2 >= \int_{\mathcal{D}} < s_1, s_2 > \cdot < \sigma_1, \sigma_2 > .$$

  The space of states is defined to be $\mathbf{P}\mathcal{H}$*, the projectivization of* $\mathcal{H}$*.*
- *We remark that, given a choice of polarization, not all classical observables can be quantized; it is a condition that the observable's associated Hamiltonian vector field must preserve the polarization in question, i.e. for* $f \in C^\infty(M)$ *with associated Hamiltonian vector field* X_f*,* f *is quantizable if and only if* $[X_f, \mathcal{P}] \subset \mathcal{P}$*.*

  Suppose f *is quantizable. Then we define the* quantized observable *associated to* f*,* $O_f^{\mathcal{P}}$*, by its action on the generator states:*

  $$O_f^{\mathcal{P}}(s \otimes \sigma) = (O_f^{preq} s) \otimes \sigma + i \cdot s \otimes (\mathcal{L}_{X_f} \sigma),$$

  where O_f^{preq} *is the prequantized operator associated to* f*, given by*

  $$O_f^{preq} \psi =$$

The historical truth is that the Hilbert space was not defined as such in one attempt; rather, the latter two steps of the process above were successive improvements and corrections to the pre-quantum Hilbert space (what one may think of as a naive first attempt at defining a Hilbert space from which one may canonically recover the geometry of M). This series of improvements will be discussed in the respective sections following this one.

CHAPTER 3

Pre-quantization

1. Definition and Basic Results

The first step of the process is prequantization — in its simplest formulation, a symplectic manifold is prequantized into a Hilbert space given by the sections of a complex line bundle built from the manifold which possesses a connection[1] on the bundle satisfying a curvature condition. We have the following definition:

DEFINITION 9. *Let* (M^{2n}, ω) *be a symplectic manifold. Then a complex line bundle*

$$\pi : L \to M$$

paired with a (Koszul) connection

$$\nabla : \Gamma(L) \to \gamma(L \otimes T^* M)$$

is called a prequantum line bundle *for* (M, ω) *if the following curvature condition holds:*

$$\omega = \mathrm{curv}(\nabla),$$

i.e. if the curvature 2-form and the symplectic form are equivalent.

If such a line bundle and connection exist for (M, ω)*, the symplectic manifold is said to be* prequantizable.

In the definition above, $\Gamma(L)$ refers to the space of smooth sections, i.e. smooth functions $M \to L$ that are right inverses of π:

$$\pi \circ \sigma = \iota_M.$$

We will see later that the curvature condition induces a morphism of Lie algebras between the classical and quantum observables.

For further details on the curvature form, confer the classic text of Kobayashi and Nomizu [**?**].

We note that a symplectic manifold is prequantizable if and only if the symplectic form satisfies a cohomological property:

THEOREM 2. *A symplectic manifold* (M, ω) *is prequantizable if and only if*

$$\Big[\omega\Big] \in H^2_{dR}(M, \mathbb{Z}),$$

i.e. the second de Rham cohomology class of the symplectic form is integral.

Equivalently, (M, ω) *is prequantizable iff the period of* ω *is integral for each integer 2-cocycle, i.e.*

$$\int_S \omega \in \mathbb{Z}$$

[1] That is, a *Koszul* connection, as the line bundle is viewed as a rank one vector bundle.

for each closed 2*-cochain* S *(i.e.* $S = 0$*) with integer coefficients.*

For historical reasons that we will not get into, this cohomological condition is called the *Bohr-Sommerfeld condition*, and the above theorem is due to A. Weil [**?**].

From the existence of a prequantum line bundle, we can define a Hilbert space on the space of sections and define the prequantized observables corresponding to each $f \in C^\infty(M, \mathbf{R})$:

DEFINITION 10. *Let* (M, ω) *be a prequantizable symplectic manifold with prequantum line bundle given by*

$$(\pi : L \to M, \nabla : \Gamma(L) \to \Gamma(L \otimes T^*M));$$

then the prequantum Hilbert space $\mathcal{H}_{preq}$ *is given by (the completion of) the space of square-integrable sections of* L*, i.e.*

$$\mathcal{H}_{preq} := \{\phi | \langle \phi, \phi \rangle < \infty\},$$

where the inner product is given by

$$\langle \phi, \psi \rangle := \int_M \langle \phi, \psi \rangle_h \omega^n$$

where $\langle \cdot, \cdot \rangle_h$ *is a hermitian inner product compatible with the connection, i.e.*

For a classical observable $f \in C^\infty(M)$*, the associated prequantized observable* $O_{preq}(f)$ *is defined as*

$$O_{preq}(f) = i\hbar \nabla_{X_f} + f,$$

where $X_f \in \mathcal{X}(M)$ *is the associated symplectic vector field to* f*, i.e. the unique* $X_f \in \mathcal{X}(M)$ *such that*

$$\omega(X_f, Y) \equiv \mathrm{d}f(Y) = Yf.$$

2. Examples

We now give three examples: a basic example of prequantization, and two examples displaying the inherent un-physicality of the prequantized Hilbert space and the prequantized operators.

- *(Cotangent Bundles.)* We first consider the prequantization of a cotangent bundle T^*Q where Q is a smooth manifold of dimension n. The cotangent bundle has a natural global symplectic 2-form given by

$$\omega = \sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i$$

where (q_i, p_i) are *canonical coordinates*; note also that $\omega = -\mathrm{d}\theta$ for

$$\theta = \sum_{i=1}^n p_i \mathrm{d}q_i,$$

the canonical 1-form on T^*Q.

We note that the cohomology class $[\omega] = 0 \in \mathbb{Z}$ since we have that ω is exact $(\mathrm{d}(-\theta) = \omega)$.

Hence the cotangent bundle is prequantizable; since ω is equivalently the curvature form, we have a flat connection[2] ∇ and it is thus sufficient to take $T^*Q \times \mathbb{C}$ as the prequantum line bundle.

We obtain a prequantum Hilbert space of $\mathcal{H} = \mathcal{C}(T^*Q)$ with metric given by

The quantized operator corresponding to $f \in C^\infty(T^*Q)$ is then given by

- *($\mathbb{R}^3$.)* As a specific example of the above, we consider $Q = \mathbb{R}^3$. According to the above, we have the symplectic manifold $T^*\mathbb{R}^3 \equiv \mathbb{R}^6$, with the hermitian line bundle given by $\mathrm{R}^6 \times \mathbb{C}; wenotethatthisisuniqueasEuclideanspaceissimplyconnected(i.e.all$
is given by

  Choose the Hamiltonian function H to be
- *(Harmonic Oscillator.)* In the case of the n-dimensional harmonic oscillator, we have the phase space/symplectic manifold (M, Ω) given by

$$M = \{(q^j, p_j) \in \mathbf{R}^{2n}\}, \Omega = \sum_j dq^j \wedge dp_j,$$

with the Hamiltonian H given by

$$H(q,p) = \frac{1}{2} \sum_j ((q^j)^2 + (p_j)^2).$$

We see that Born-Sommerfeld condition is satisfied, since we have $[\Omega] = 0 \in \mathbf{Z}$, and the line bundle is trivial and unique since M is simply connected, with $L = M \times \mathbf{C}$ and the associated connection $\nabla given by \nabla_X \psi = X\psi + (2\pi i)\omega(X)\psi = where \omega$ is the pullback given by

Now let us try to apply the prequantization map to the energy, i.e. H; we have that

$$O_H = -i\hbar \sum_j \left[p_j \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial p_j} \right];$$

further, we have that

$$O_H(\psi) = i\hbar\{H, \psi\},$$

where the section ψ is interpreted as a function $M \to \mathbf{C}$.

There's an issue here, however; the operator O_H and the classical Hamiltonian H have the same spectrum, and hence O_H has a continuous spectrum. But we know from physical experience that quantum operators must have discretized spectra.

3. Structural Properties

As we mentioned before, the prequantization map satisfies "fairly decent" uniqueness properties, in the sense that the first de Rham cohomology group $H^1(M, \mathbb{T})$ parametrizes the possible prequantum line bundles one may have when given a symplectic manifold.

In particular, we have that the following:

[2] See theorems 9.1 and 9.2 of [**?**], for example, for a proof of the claim that a connection is flat iff the bundle is trivial.

THEOREM 3. *Let* (M, ω) *be a prequantizable symplectic manifold. Then the prequantum line bundle* $(L \to M, \nabla)$ *is unique if and only if the first de Rham cohomology group* $H^1(M, \mathbb{T})$ *is trivial, where* $\mathbb{T} \subset \mathbb{C}$ *is the group of complex numbers with unit modulus.*

In other words, the prequantum line bundle is unique when M is simply connected.

In the first section of this chapter, we lied by omission — the prequantum Hilbert space is defined not only by a complex bundle $L \to M$ and a connection ∇, but also by a choice of hermitian metric $\langle \cdot, \cdot \rangle_h$ (or just $h : M \times M \to \mathbb{C}$). Towards this, we have the following:

THEOREM 4.

Further, as implied earlier, we have a morphism of Lie algebras between the classical and quantum observables:

THEOREM 5.

CHAPTER 4

Polarizations

In this chapter we will introduce and justify the notion of a *polarization* on a symplectic manifold, as well as give some examples and showing how it solves some of the issues with pre-quantization by adjusting the prequantum Hilbert space into a polarized prequantum Hilbert space and adjusting the prequantized operators into polarized prequantum operators; note that there is one final modification to be made after the polarization step, however, to remedy a final issue with the integrability of states.

1. Definition and Basic Results

We will hit the ground running with the definitions and proceed to justify it shortly after:

DEFINITION 11. *Let (M^{2n}, ω) be a symplectic manifold. A (complex) distribution P on M, i.e. a sub-bundle*

$$P \hookrightarrow T^{\mathbb{C}}M \equiv TM \otimes_{\mathbb{R}} \mathbb{C}$$

of the (complexified) tangent bundle, is said to be a polarization *of M if all of the following are hold:*

- *P is* Lagrangian*, i.e. it is an* isotropic *distribution of dimension n; recall that a distribution is said to be isotropic if the symplectic form ω_x on each tangent space T_xM vanishes on the subspace $P_x \hookrightarrow T_xM$ defined by the distribution, and that the maximal dimension of an isotropic distribution is half the dimension of the manifold.*
- *P is* involutive*, i.e. for X, Y vector fields in P, the commutator $[X, Y]$ is a vector field in P. In the literature, this is usually written in shorthand as $[P, P] \subset P$.*
- *For all points $m \in M$, we have that $\dim(P_m \cap \overline{P}_m \cap TM)$ is constant, where the overline refers to complex conjugation.*

Having defined the meaning of a polarization, we can now define a way of polarizing the prequantum Hilbert space and polarizing the operators obtained through prequantization.

DEFINITION 12. *Given a symplectic manifold (M^{2n}, Ω) with prequantum line bundle $(L \to M, \nabla)$ and prequantum Hilbert space*

$$\mathcal{H}^{preq} := \{\psi \in \Gamma(L) | \langle \psi, \psi \rangle < \infty\},$$

we define the polarized Hilbert space $\mathcal{H}^P$ for P a choice of polarization on M as the Hilbert space of sections in $\mathcal{H}^{preq}$ that respect the polarization*, i.e.*

$$\nabla_X \psi = 0 \forall X \in (P),$$

where X is a vector field generated by P, i.e. the space of sections that remain fixed by parallel transport along any X in the polarization.

Further, we say that an observable $f \in C^\infty(M)$ is polarizable*, or* "respects the polarization"*, under the polarization P iff X_f the associated Hamiltonian vector field satisfies*

$$[X_f, P] = P.$$

When f is polarizable, its corresponding polarized operator $O_f : \mathcal{H}^P \to \mathcal{H}^P$ *is the same*[1] *as the prequantum operator O_f^{preq}.*

We remark that the condition placed upon the smooth functions on M is necessary as not all observables respect the polarization, since

1.1. Justification and Intuition. We now justify the reasons for defining polarizations and the polarized Hilbert space and operators as such.

First off, we analyze the intuition behind the three conditions needed to consider a distribution $P \hookrightarrow T^{\mathbb{C}}M$ to be a polarization:

- The requirement that P be *Lagrangian* comes from the need to find a "representation" or "choice" of configuration space inside of the phase space given by the symplectic manifold. Since the quantization procedure acts on symplectic manifolds, the question of a choice of configuration space is implicitly ignored; Lagrangian distributions, which are necessarily rank n when $dim(M) = 2n$, select a configuration space. The choice, however, is usually not unique.
- The *involutivity* of P comes frome the Frobenius theorem, which states that involutivity is equivalent to integrability (see [**?**], pp. 494-505). Recall that an integrable distribution is
- Define the distribution $D = P \cap \overline{P} \cap TM$; we require that $dim(D)$ be constant in our definition, though it is an excluded condition in other treatments. Perhaps the least obvious condition, the constant rank of D guarantees that

2. Examples: Real and Kähler Polarizations.

In this section we give a few examples to see what average, "run-of-the-mill" polarizations look like, as well as shine a light on two very special classes of polarizations: *real* polarizations and *Kähler* polarizations.

We now present two different polarizations of the same manifold, and show how the choice of a real polarization versus a Kähler polarization changes the representation. In our case, we have a cotangent bundle of a smooth manifold as our phase space; choosing certain real polarizations gives us the Schrödinger and momentum representations, whereas a particular Kähler polarization gives us the holomorphic, or Bargmann-Fock, representation.

2.1. Schrödinger and Momentum Representations. We first present the two real polarizations on the cotangent bundle.

[1] In other words, the primary purpose of a choice of polarization is to restrict the class of possible classical observables that may be meaningfully quantized — it does not change the definition of the quantization of a classical observable.

2.2. Bargmann-Fock Representation. Now we show an example of a Kähler polarization.

2.3. Bargmann Representation of the Harmonic Oscillator. As a bonus, we present an example with the Harmonic Oscillator

3. Remarks on Polarizations and the Polarized Hilbert Space.

We now make some remarks on interesting features of polarizations on a symplectic manifold.

3.1. Canonical Polarizations. A natural question to ask is whether there are situations where a choice of polarizations is either canonical in some sense or otherwise unique; a further question might be whether there are categories where we have a class of functors indexed by choices of polarization.

3.2. BKS Kernels. Another natural question is whether there are algebraic ways of comparing two different representations corresponding to two different polarizations on the same manifold.

CHAPTER 5

Metaplectic Correction

After the choice of a polarization, we still do not yet have the structure of a Hilbert space on the space of states; this issue is resolved by placing a *metaplectic structure* on the symplectic manifold, in a process known as *metaplectic correction* (for historical reasons), so that sections are square-integrable. In this chapter, we present a discussion of the insufficiency of polarization to motivate the definition and purpose of metaplectic structures, before building on top of the metaplectic structure to conclude the traditional geometric quantization programme on a given symplectic manifold.

1. Integrability of States

Consider a prequantizable symplectic manifold (M, ω) equipped with P a choice of polarization. In general, the space $\Gamma_P(L) = \Gamma_P(L \to M)$ of polarized sections does not form a Hilbert space[1], for the following reasons:

- there may be sections $\phi \in \Gamma_P(L)$ that are not square-integrable; and even if this were the case,
- the inner product may not be defined for all pairs $\phi, \psi \in \Gamma_P(L)$.

The reason for the former is

The reason for the latter is

In the following sections we will sketch out the following amelioration[2] to the above problem:

2. Metaplectic Structures

2.1. Definition. A metaplectic structure on a symplectic manifold may be thought of as analogous to a spin structure placed on a Riemannian manifold[3], in that the definitions of each share the same pattern of being defined as *equivariant lifts* of a certain frame bundle, with respect to a certain double-cover of structure groups. For the sake of illustration and comparison, we present both definitions.

First, the Riemannian case:

Definition 13. *Let (M, g) be an orientable Riemannian manifold. Then there is a double-covering of Lie groups*

$$\rho : \mathrm{Spin}(n) \to SO(n)$$

[1]A notable exception is, of course, the case of a Kähler manifold.

[2]Or *correction*, if you will.

[3]Or rather, an *oriented* Riemannian manifold

from the spin group to the special orthogonal group. Consider the oriented orthonormal frame bundle

And now the symplectic case:

Definition 14.

Remark. For a discussion of spin and metaplectic groups, see the appendix. We will remark here that the spin group Spin(n) is the unique double cover of $SO(n)$ such that the sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to SO(n) \to 1$$

is exact, where 1 here refers to the trivial group; and the metaplectic group $Mp(2n)$ is the unique double cover of $Sp(2n)$ the order n symplectic group.

2.2. Properties. Metaplectic structures, much like spin structures, may be classified cohomologically:

Theorem 6. *Let (M, ω) be a symplectic manifold. Then (M, ω) admits a metaplectic structure if and only if the second Stiefel-Whitney class vanishes; equivalently, if and only if the first Chern class is even.*

When this is the case, the class of admissible metaplectic structures is classified by

3. Induced Hilbert Space from Metaplectic Structure

Recalling the obstruction from the first section of this chapter, we now give a presentation of the way in which the existence of a metaplectic structure enables the construction of the "proper" Hilbert space of states.

4. Examples

We present some examples of metaplectic structures.

4.1. Phase Spaces. Let Q be some n-dimensional orientable manifold. Then the cotangent bundle T^*Q is a symplectic manifold; further, it has a natural metaplectic structure.

To introduce the topic, note that we have the following short exact sequence:

$$0 \to \mathbf{Z} \to \mathbf{C}\times$$

which in turn gives us the exact sequence

this latter group is the metaplectic group and is a *double cover* of the symplectic group.

http://en.wikipedia.org/wiki/Metaplectic$_s$*tructure*
http://ncatlab.org/nlab/show/metaplectic+group
http://en.wikipedia.org/wiki/Spin$_s$*tructure*
http://ncatlab.org/nlab/show/metaplectic+structure
http://ncatlab.org/nlab/show/metaplectic+correction+

CHAPTER 6

Worked Examples of Geometric Quantization

In this section we present two fully worked examples of the overall geometric quantization process. The first example is a continuation of the quantization process of the harmonic oscillator, as in previous examples, and the second example is of flat Euclidean space.

1. Harmonic Oscillator

In the case of the harmonic oscillator, we have the phase space given by the manifold

$$M = \{(p, q) \in \mathbf{R}^{2n} :\}$$

with the symplectic form given by

$$\omega = \frac{1}{2} \sum_{i=1}^{n} \mathrm{d}p^i \wedge \mathrm{d}q^i.$$

1.1. Prequantization.

1.2. Polarizations.

1.3. Correction.

2. Cotangent Bundles

As a simpler example, we will quantize a cotangent bundle T^*Q with the standard symplectic form.

2.1. Prequantization.

2.2. Polarizations.

2.3. Correction.

3. $\mathbf{S}^2$

We now present a quantization of the 2-sphere $\mathbb{S}^2$ viewed as a Kähler manifold[1].

3.1. Prequantization.

3.2. Polarizations.

3.3. Correction.

[1]Ordinarily, we would consider the general n-dimensional sphere for $n = 2k$, but it actually turns out that the only sphere for which a symplectic structure exists is $n = 2$! This is a consequence of the fact that the second de Rham cohomology $H^2(S^n)$ is trivial for $n > 2$.

Part 2

Selected Results in the Theory of Quantization

CHAPTER 7

Structural Properties and Further Results

1. Obstructions to Geometric Quantization

THEOREM 7. *Let **Pois** and **Hilb** be the categories of Poisson manifolds and Hilbert spaces, respectively, and let **Diff** be the category of smooth manifolds; there are natural functors T^* and $\Lambda^{1/2}$ out of **Diff** to **Pois** and **Hilb** (respectively), called the cotangent bundle and half-density functors, given by*

Then the Groenewald-van Hove theorem states that there does not exist a functor $\mathcal{Q} : \mathbf{Pois} \to \mathbf{Hilb}$ such that

In particular, this shows that deformation quantization (in the style of Kontsevich) is not functorial.

CHAPTER 8

No-Go Theorems

CHAPTER 9

Symplectic Groupoids

In this chapter we will present the basic details of *symplectic groupoids*, a mutual category-theoretic generalization of symplectic and Poisson manifolds into the groupoid setting; if groups naturally arise out of symmetry, groupoids naturally arise out of "internal" symmetries with respect to a bundle-type structure, i.e. geometric objects with algebro-geometric objects attached at each point. We proceed in a gradual build-up, first defining groupoids and then making our way through Lie groupoids/algebroids and then encountering symplectic groupoids.

The first definition of a symplectic groupoid comes frome Weinstein, "Symplectic Groupoids and Poisson Manifolds" [**?**] as well as Karasev [] and Zakrzewski []; all were motivated by the problem of understanding quantization. The notion of groupoids as categories generalizing groups came earlier [**?**]. A good reference for this topic is [].

1. Groupoids

There are two equivalent definitions of the notion of a groupoid — one is a traditional set-theoretic definition and the other defines them as a certain type of category. First, the traditional definition:

Definition 15. *Let Γ be a set with subset Γ_0 of* identity elements *along with projections $\alpha, \beta : \Gamma \to \Gamma_0$. Let there be a multiplication operation $(x, y) \mapsto x \cdot y$ on the set*

$$\{(x, y) \in \Gamma \times \Gamma : \beta(x) = \alpha(y)\},$$

and let there be an inverse operation $(\cdot)^{-1} : \Gamma \to \Gamma$ such that the product and the inverse satisfy the usual group axioms on their domain.

Essentially, a groupoid is defined as a set with a partially defined product operation — we can only multiply two elements if they map to the same identity. (Note also that we can enforce commutativity when we specify $\alpha = \beta$.)

The category-theoretic definition is much easier to state:

Definition 16. *A* groupoid *is a category in which all morphisms are invertible.*

The proof of equivalence (with the caveat that the category is *small*) is not difficult, but is tiresome and mainly syntactic manipulation. One may consult any exposition of groupoids for the proof (in fact, even Wikipedia suffices).

We have the following examples of groupoids occurring in mathematics:

- Every group may be thought of as a single-object category with invertible morphisms; hence the notion of groupoid is a proper generalization of the notion of a group.

- *Fundamental Groupoids.* Recall that path-connected topological spaces have fundamental groups that are invariant with respect to choices of base points; the fundamental groupoid $\pi_1(X)$ for a general topological space X, then, is defined as the category with points of X as objects and paths (up to homotopy equivalence) as arrows. Concatenation of paths is used as composition of arrows. The standard fundamental group $\pi_1(X, x)$ at a point $x \in X$ can then be seen as the vertex group (at x) of the fundamental groupoid.
- *Action Groupoids.* Suppose we have a group G acting on a set S; then the associated action groupoid $S//G$ is given by the category with elements of S as objects and group elements $g \cdot s_1 = s_2$ as arrows from s_1 to s_2. Compositions are done in the natural way, i.e. by applying one group element after another using the action. We note that any two objects s_1, s_2 in the same orbit are isomorphic (in the category-theoretic sense).

2. Lie Groupoids and Lie Algebroids

Lie groupoids may be thought of as groupoids in the setting of smooth manifolds, or as "many-object" versions of Lie groups (in the same way that groupoids are "many-object" versions of groups). More specifically,

DEFINITION 17. *A* Lie groupoid *is a groupoid in which the inclusion maps $\alpha, \beta : \Gamma \to \Gamma_0$ are submersions, the multiplication and inversion operations are smooth, and Γ, Γ_0 are both smooth manifolds.*

Equivalently, a Lie groupoid is a groupoid internal to the category of smooth manifolds; in other words, a Lie groupoid Γ over a base manifold Γ_0 has objects as elements of Γ_0, hom-sets $hom_\Gamma(p, q)$ a manifold for $p, q \in \Gamma_0$, with all arrows invertible and source/target maps α, β both submersive.

We have the following commutative diagram:

We also have the notion of Lie algebroid:

DEFINITION 18. *A* Lie algebroid *is a vector bundle $\pi : E \to M$ along with a Lie bracket $[\cdot, \cdot]$ on the space of sections of E and an* anchor*, i.e. a vector bundle morphism $E \to TM$ from the bundle to the tangent bundle of the base manifold.*

The functorial association of a Lie algebra to Lie groups extends to a functorial association of a Lie algebroid to a Lie groupoid:

THEOREM 8. *Let $G \rightrightarrows M$ be a Lie groupoid (with overall manifold G, base manifold M, and source/target maps $s, t : G \to M$). Then there exists a unique Lie algebroid $A \to M$ given by the following:*
- *The bundle A is given by*
- *The bracket $[\cdot, \cdot]$ is given by*
- *The anchor map is given by the differential ds of the source map s.*

The association is functorial and the following diagram is commutative:

Further, the three theorems of Sophus Lie (see appendix) also generalize in a straightforward manner to the theory of Lie groupoids and algebroids:

THEOREM 9. –

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We will not proceed further with the discussion of the theory of Lie groupoids. A detailed introduction to Lie groupoids and algebroids may be found in Mackenzie, "General Theory of Lie Groupoids and Lie Algebroids" [**?**].

3. Symplectic Groupoids

Symplectic groupoids are Lie groupoids with a symplectic structure on the manifold of morphisms such that the internal base manifold is Poisson and the groupoid product respects the symplectic form (in a suitable sense).

More formally, we have the following definition:

Definition 19. *A* symplectic groupoid *is a Lie groupoid* $\Gamma \rightrightarrows M$ *with* Γ, M *both smooth manifolds and source/target maps* $s, t : \Gamma \to M$ *submersive; further, there is a symplectic structure on* Γ *given by a form* ω *such that the submanifold*

$$L = \{(x, y, z) | xy = z\} \subset (\Gamma, -\omega) \times (\Gamma, -\omega) \times (\Gamma, \omega)$$

is Lagrangian.

In other words, a Lie groupoid is symplectic *if*

The notion of a symplectic groupoid is remarkable for a few reasons. Recalling that symplectic manifolds may be viewed as Poisson manifolds (by noting that the bracket given by the symplectic form[1] defines a natural Poisson structure), we note that

Theorem 10.

Further, we have a functorial correspondence between symplectic groupoids and Poisson manifolds, in the style of the Lie group-Lie algebra correspondence:

Theorem 11.

The proof of this statement is outside the scope of this writeup; a presentation may be found in [**?**].

Symplectic groupoids were introduced in [**?**]; further references and introductions may be found in [**?**].

[1]

CHAPTER 10

Weinstein Dual Pairs

CHAPTER 11

The Guillemin-Sternberg-Bott Conjecture

CHAPTER 12

Towards a Quantization Functor

Part 3

Deformation Quantization

Part 4

Appendix