Minimum Description Length Principle

With An Application For Credit Risk Models

Paul van Leeuwen (paul.vanleeuwen@devolksbank.nl)

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Introduction

The Speaker

- Paul van Leeuwen, employee of dVB as of September 2022.
  - KRT's RDS 5.0 and Modellen voor Bankbalans.
- Up to 2017 part of the Modelling team at dVB.
- After that as Model Validator at Achmea and Lead Data Scientist at Wageningen University & Research.
- Now self-employed and currently working in the financial sector and providing R workshops.

Why this subject?

- Improve current approaches and modelling techniques.
- Stay in the forefront of statistical innovation.
- Connection with other realms of statistics.

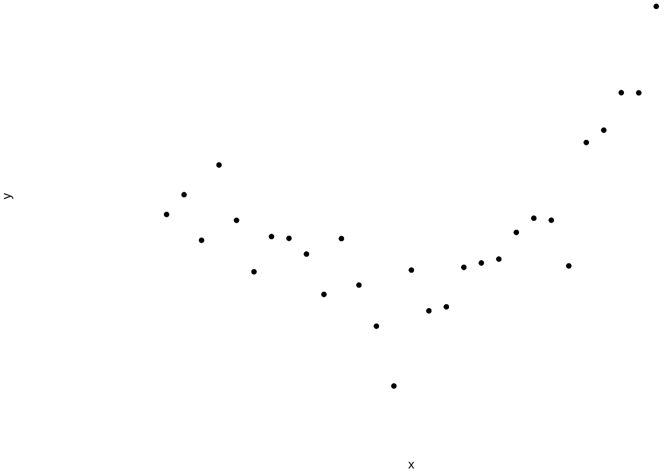
What is MDL?

Introduction

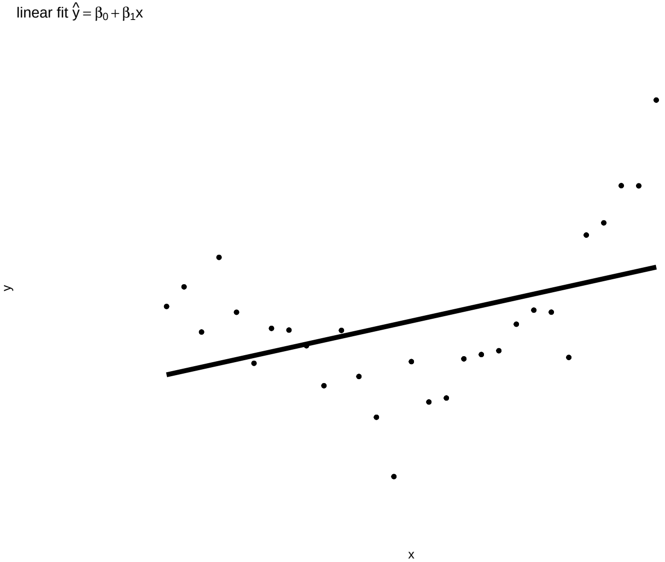
- The Minimum Description Length (MDL) principle aims to describe the data and its description mechanism with the smallest possible information 'length'.
- Applications (among else):
  - model selection (e.g. what order of the Markov model family do we want?);
  - deal with overfitting (e.g. how many explanatory variables to include);
  - exploratory data analysis (what prior knowledge can we confirm?).
- Close ties with frequentist statistics, Bayesian statistics, and machine learning.
- Why is MDL relatively unknown?
  - MDL is the intersection of advanced measure theory, information theory, and statistics.
  - For a decent introduction into MDL, see (Grünwald 2007).

Example of dealing with overfitting

What polynomial generated by this dataset?



Example of dealing with overfitting



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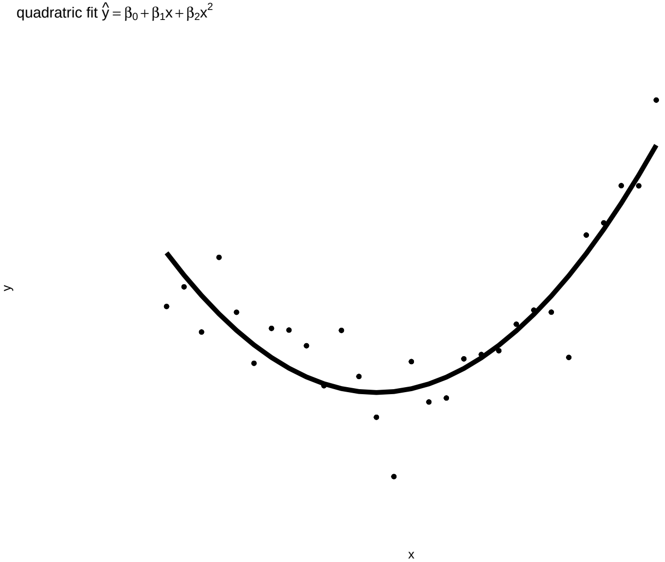

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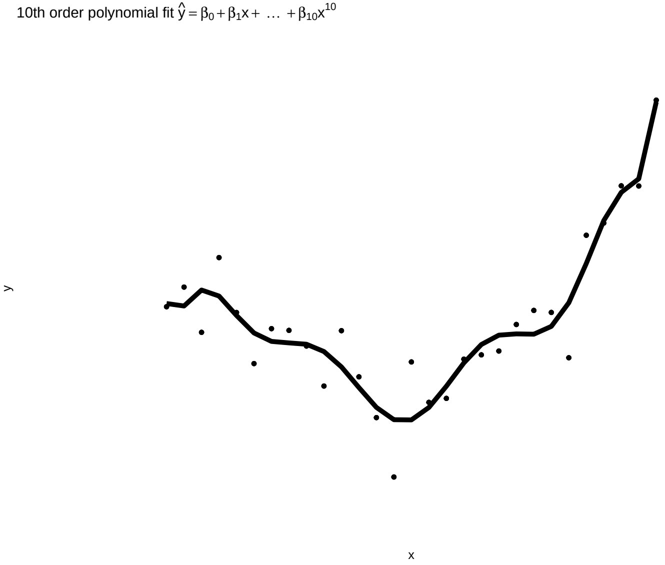

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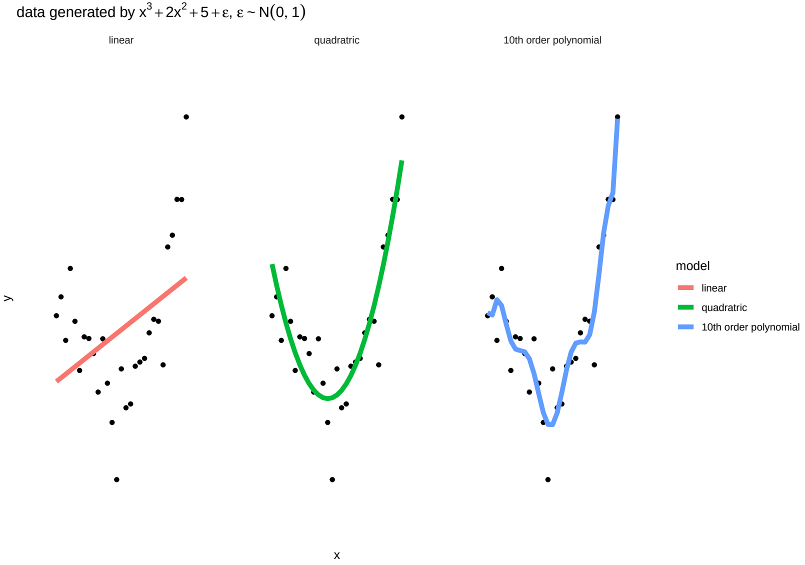

How does MDL work?

- Patterns or regularities in the data can be described with less information 'length' compared to the data alone.
- less information 'length' = compression
- Choose the model that gives the shortest description of the data.
- Note that MDL is an approach, not an algorithm.
    - The modeller has to make choices to implement the MDL principle.

Example of MDL

- Consider three data-generating processes (dgp's) that generate each a binary sequence of length 1000:
  1. 1000100010001000100010001000100010001000100 ... 1000
  2. 1001111110111100011001110001010100110001101 ... 1010
  3. 0100010000000101110010000000001101000110000 ... 0100

Example of MDL

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- All sequences written out take 1000 bits to be reproduced.
- However, because of regularities present in the dgp's, we require less bits to reproduce the same sequences.
- Question: how many bits of Matlab code does each sequence take to be *exactly* reproduced?

Example of MDL

- The sequences as before:
  1. 10001000100010001000100010001000100010001000100 . . . 1000:

Example of MDL

- The sequences as before:
  1. 10001000100010001000100010001000100010001000100 ... 1000: repetition of `[1 0 0 0]` 250 times.
     Matlab-code:
     ```
     textDgp_1 = 'repmat([1 0 0 0],1,250)'
     whos_textDgp_1 = whos('textDgp_1')
     whos_textDgp_1.bytes
     ```
     yields 23 bytes = 184 bits, a compression ratio of $1 - \frac{184}{1000} = 81.6\%$!

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  2. 10011111011110001100111000101010011000110 1 ... 1010:

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  2. 1001111110111100011001110001010100110001101 . . . 1010: a coin toss with heads (0) or tails (1); no compression possible because of the randomness involved.
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  2. 1001111101111000110011100010101001100011 01 ... 1010: a coin toss with heads (0) or tails (1); no compression possible because of the randomness involved.
  3. 0100010000001011100100000000001101000110000 ... 0100: a roll with a four-sided die with outcomes 2, 3, 4 assigned to 0 and outcome 1 assigned to 1; no compression possible because of the randomness involved.

The Principle of MDL

- The more randomness involved, the less data compression is possible.
- When the restriction of exact reproduction is alleviated we may obtain some data compression.
- Dgp 2 (the coin toss) implies no data compression.
  - As the data sequence is completely random.
- Dgp 3 (the 4-sided die) implies some possible data compression.
  - Although exact reproduction is not possible the set of this type of data sequences requires around 821 bits.
- Most datasets are almost incompressible.
  - Only a small fraction can be significantly compressed.

Approach

From Data to Model Selection

- From data to code.
- From code to code length.
- From code length to probabilities.
- From probabilities to model selection.

From Data to Code

- Examples:
  - `Hello, world` could map to 0.
  - `aabbaccdaaadd` could map to 110100.
- In general, a description method maps a sequence of symbols to a binary sequence.
  - The coding alphabet $\mathbb{B}$ can be binary ($\mathbb{B} = \{0, 1\}$), the Western alphabet ($\mathbb{B} = \{\mathrm{a}, \mathrm{b}, \ldots, \mathrm{z}\}$), etc.
- Mathematically: a dataset $D = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{B}$ from a sample space $\mathcal{X}^n$ is mapped to $\{0, 1\}^m$ by $C\colon \mathcal{X}^n \mapsto \{0, 1\}^m$.
  - In the first examples above, we could have $\mathcal{X}^n = \{x_1\} = \{'\mathrm{Hello, world}'\}$ and $C(x_1) = 0$.
- We demand the mapping C to be *uniquely* decodable.
  - No multiple interpretations allowed.

From Data to Code

- Suppose we would like to encode a binary data sequence of length 2.
- We are not sure what outcome we observe.
- Let $X_i \in \{0, 1\}$ be the random variable at position $i = 1, 2$.
  - The complete data sequence becomes $X_1X_2 \in \{00, 10, 01, 11\}$.
- Every sequence in $\{00, 10, 01, 11\}$ is assigned a code.
- In general, for the binary alphabet, for n positions there are 2^n possible data sequences.
  - For example, when $n = 3$ we have the data sequences $\{000, 100, 010, \ldots, 111\}$.
  - Without loss of generality, every non-binary alphabet can be mapped to the binary alphabet $\{0, 1\}$.

From Code to Code Length

- Given a data sequence x_i and its corresponding code $C(x_i)$, then we are interested in the code length $L(x_i)$ with $L\colon \mathcal{X}^n \mapsto \mathbb{R}_+$.
- For example, to map an integer from $\{1, \ldots, n\}$ in a uniform way, we need $\lceil \log_2(n) \rceil$ bits.
  - Note that n has to be known in advance.
  - For example, take $n = 64$. Then we have 64 binary data sequences of length $\lceil \log_2(64) \rceil = 6$.
  - Or take $n = 10$. Then we need $\lceil \log_2(10) \rceil = 4$ bits.
    - Note that $2^4 = 16$ data sequences are possible while we only use 10 of them.
- What coding scheme results in the smallest number of *expected* bits?

From Code to Code Length

- Recall the data sequences $\{00, 10, 01, 11\}$.
- Uniform method: the data sequence is the code.
  - $C(00) = 00$, $C(10) = 10$, $C(01) = 01$, and $C(11) = 11$.
  - Expected number of bits: 2.
- In general we can do better!
  - That is, $\mathbb{P}[X_i = 1] \neq \frac{1}{2}$ for at least one $i \in \{1, \ldots, n\}$.

From Code Length to Probabilities

- Suppose $\mathbb{P}[X_i = 1] = \frac{1}{4}$.
- We reserve code 0 for $X_1X_2 = 00$ so $C(00) = 0$, $C(10) = 100$, $C(01) = 110$, and $C(11) = 1110$.
- Then the expected number of bits is

$$1 \cdot \frac{3}{4}^2 + 3 \cdot \frac{3}{4} \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} \cdot \frac{3}{4} + 4 \cdot \frac{1}{4}^2 = 1.9375 < 2$$

- This the Shannon-Fano coding scheme.
  - More general, reserve $\lceil -\log_2(p_i) \rceil$ bits for probability p_i corresponding to data sequence i.
  - The Huffman code is optimal and improves on the Shannon-Fano code.
- Main message:

$$\boxed{\text{higher probability} = \text{smaller code length} = \text{less bits required}}$$

From Probabilities to Model Selection

- To describe any dataset we need $L(D, H)$ bits.
  - Both the description method H and the data D require bits.
- The MDL principle employed for model selection is to minimise the sum of
  - the number of bits to encode the description mechanism $L(H)$ and
  - the number of bits to encode, with the description mechanism H, the data observed $L(D|H)$.
- Information 'length' is this sum $L(H) + L(D|H)$.
  - $L(H)$ is the *model complexity*, $L(D|H)$ is the *fit of the data*.
- The MDL principle is to choose the model as to minimise this sum.
- Main message:

$$\boxed{\text{smaller description length} = \text{better model selection}}$$

MDL and LASSO

- More mathematically, given a set of candidate models $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$, select the optimal model $\mathcal{H}^\star \in \mathcal{H}$ as

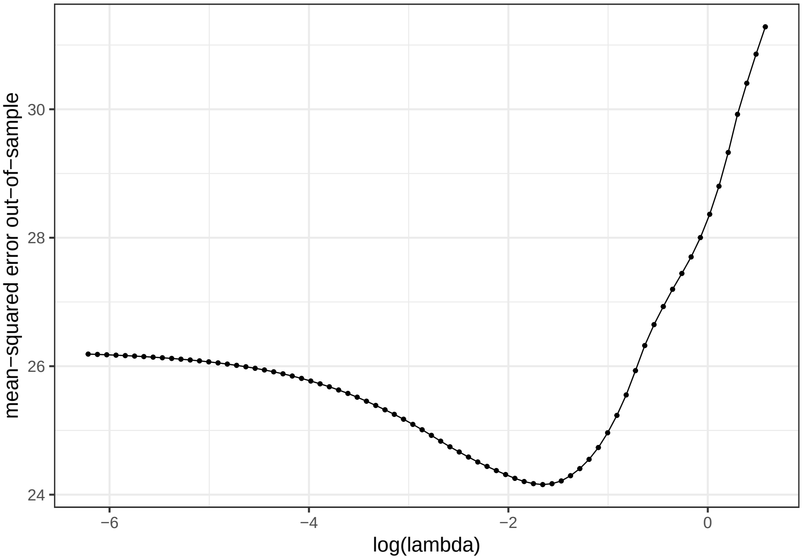
$$\mathcal{H}^\star := \arg\min_{H \in \mathcal{H}} L(D, H) = \arg\min_{H \in \mathcal{H}} \{L(H) + L(D|H)\}$$

- Note the resemblance with penalised model fitting, such as LASSO:
  - with LASSO we apply cross-validation to minimise

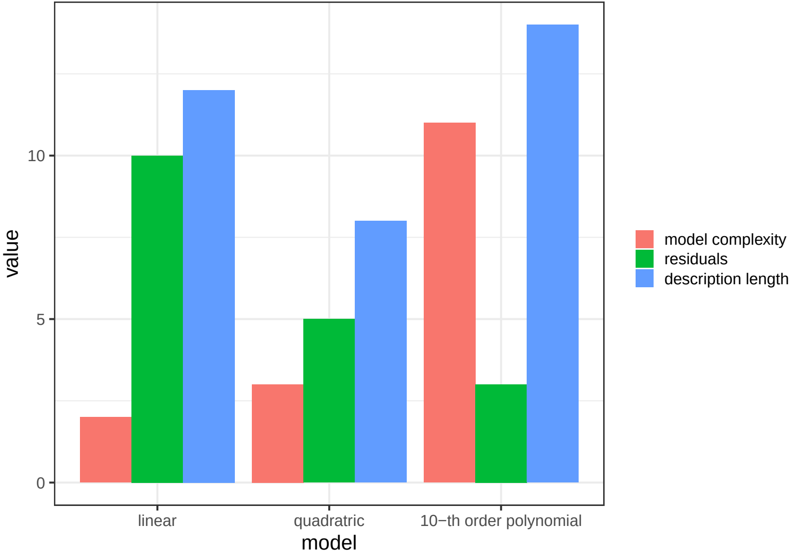
$$\arg\min_{\boldsymbol{\beta}} \left\{ \frac{1}{n} \|\boldsymbol{y} - \boldsymbol{X}\beta\|_2^2 + \lambda \|\boldsymbol{\beta}\|_1 \right\}$$

with the data part $\frac{1}{n}\|\boldsymbol{y} - \boldsymbol{X}\beta\|_2^2$ ($L(D|H)$ in MDL) and $\lambda\|\beta\|_1$ the model complexity part ($L(H)$ in MDL).

Example of LASSO



Example of dealing with overfitting (continued)



Questions to be answered

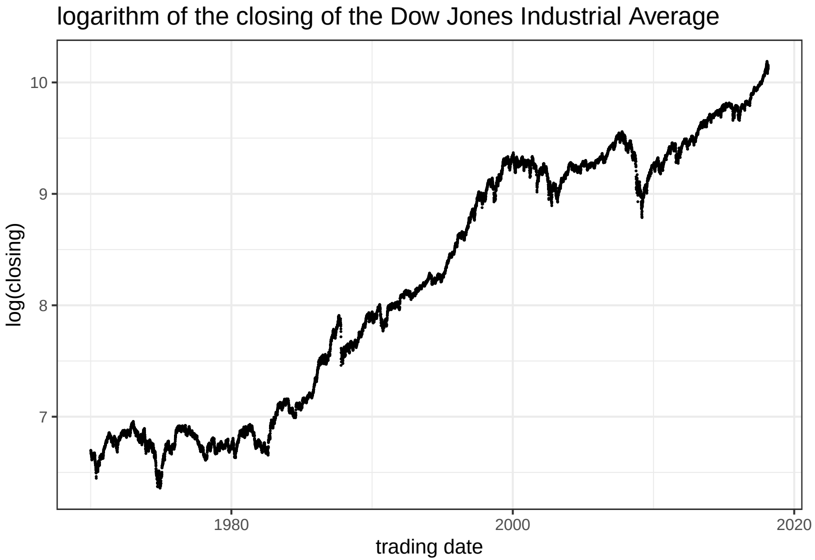
- How does MDL work in practice?
- How do the following approaches compare: frequentist statistics, Bayesian statistics, machine learning, and MDL?
- The coding system to put down the description mechanism should not matter. How can we choose a universal programming language?
  - For example, whether we use `Matlab`, `R` or whatever programming language should not matter for the number of bits at use.
- What are good encoding systems?
- What patterns are generalisable and what not?
- How to incorporate prior knowledge?

MDL in Practice

Dow Jones Industrial Average

- How can we employ MDL to compress a dataset?
  - Example taken from (Hansen and Yu 2001).
- Take the log daily return R_t and the volatility V_t of the Dow Jones Industrial Average (DJIA).
  - t runs from $t_0 =$ July 1962 until $T =$ June 1988, i.e. 6,430 trading days.
  - $R_t = P_t - P_{t-1}$ with P_t the logarithm of the DJIA at day t.
  - $V_t = 0.9V_{t-1} + 0.1R_t^2$ and V_0 the variance of the series P_t.
- R_t has a corresponding indicator: 1 (0) when $P_t > P_{t-1}$ ($P_t < P_{t-1}$).
  - Analogously for V_t.
  - Two binary strings of length 6,430 - 1 = 6,429.
  - R_t has 3,181 (49.49%) ups.
  - V_t has 2,023 (31.47%) ups.

Dow Jones Industrial Average



Dow Jones Industrial Average

- Without data compression we require 6,429 bits per series.
- Using MDL we save 10% on the volatility series V_t and 4% on R_t.
- n is known in advance so costs us $\lceil \log_2 n \rceil$ bits.
- Model the up or down indicator as a Bernoulli probability distribution with probability p on success.
  - Maximum likelihood yields $\hat{p} = k/n$ with k the number of successes.
- Ignoring rounding errors we have $L(H) = \log_2(n)$ and

$$L(D|H) = k\left[-\log_2\left(\frac{k}{n}\right)\right] + (n-k)\left[-\log_2\left(1-\frac{k}{n}\right)\right]$$

Dow Jones Industrial Average

- With $n = 6,429$ and
  - for R_t we have $k = 3181$ and $\hat{p} = \frac{k}{n} = \frac{3181}{6429} = 0.49$ we need

    $$\log_2(6429) + 3181[-\log_2 0.49] + 3248[-\log_2 0.51] = 6442 \text{ bits}$$

  - for V_t we have $k = 2023$ and $\hat{p} = \frac{k}{n} = \frac{2023}{6429} = 0.31$ we need

    $$\log_2(6429) + 2023[-\log_2 0.31] + 4406[-\log_2 0.69] = 5789 \text{ bits}$$

- In conclusion, for R_t we have gained nothing, even lost some bits, but the improvement for V_t is substantial.
  - The savings in storage increases as well as n increases or k/n approaches 0 or 1.
- Main message:

$$\boxed{\text{good data compression} = \text{good model performance}}$$

Comparing Models

- The model that requires the smallest number of bits, is the best.
  - Automatically prevents overfitting while aiming for the best model performance out-of-sample.
- For example, in the DJIA example (Hansen and Yu 2001) achieve a 4% improvement (a savings of 253 bits) on $R_t - R_{t-1}$ by applying a first-order Markov model.
  - This is because $R_t - R_{t-1}$ has a relatively high first-order autocorrelation of -0.42.
- The reduction in $L(D, H)$ justifies a more complex model: for $R_t - R_{t-1}$ a first order Markov-chain is preferred over a simple Bernoulli model.

Approaches compared

How do the approaches compare: Frequentist

- Also known as orthodox or non-Bayesian.
- Main goal: models should be consistent.
  - At least asymptotically: as $n \to \infty$ the estimated model coefficients should converge to the 'true' model parameters.
  - In the example above, one would expect $\hat{\beta} = (5\,0\,2\,1)$ for $n \to \infty$.
- When the model assumptions are fulfilled, this approach yields the best possible predictions.
- Pros: widely known and applicable, easy calculation, models are good with infinite amount of data.
- Cons: assumptions are not realistic (e.g. almost no residual follows a normal probability distribution).

How do the approaches compare: Bayesian

- Main goal: incorporate prior knowledge by assigning a prior probability distribution.
    - Once the prior probability distribution is assigned, the posterior probability distribution can be derived / calculated.
    - Frequentist models consider the data random and the parameters fixed.
    - Bayesian models consider the data and the parameters random.
    - In practice, asymptotically frequentist and Bayesian models converge.
- Pros: mathematically elegant and precise, encompasses the frequentist approach.
- Cons: hard to come up with a useful prior probability distribution.

How do the approaches compare: Machine Learning

- Main goal: find and fit a model that is able to generalise from train to test data.
  - No assumptions about the data generating process.
- Pros: no assumptions required, good model performance.
- Cons: hard to explain predictions, hard to study the data generating process.

How do the approaches compare: MDL

- Main goal: infer useful information from the data and use that to achieve good data compression.
  - Good data compression implies good learning.
  - No assumptions about the data generating process required but are allowed.
- Pros: no assumptions required, has best of all worlds (Bayesian and Machine Learning): employ prior knowledge when viable, good statistical properties, and decent performance.
- Cons: relatively new in literature; intersection of state-of-the-art knowledge of information theory, measure theory and statistics; not standard available in mainstream programming languages, can be computationally challenging.

Discussion

Application to Expected Credit Loss

- Suppose we apply the MDL principle on the calculation of the Expected Credit Loss.
  - No coding scheme discussed here, just the principle.
- Discussion:

  take a moving average of the realised credit losses to calculate the Expected Credit Loss

- Pros:
  - Occam's Razor: simpler is better.
  - Saves us a complete IFRS team to perform tedious calculations.
- Cons:
  - No new developments (e.g. Corona) taken into consideration. Can we with our current models?
  - No intermediate analyses available (e.g. use PD to rank customers in need).

Conclusion

Summary

- MDL is a guidance for model selection, an instrument against overfitting and exploratory data analysis.
- Sound statistics and decent model performance could make MDL a challenger for other statistical approaches.
- Much more to be explored.

What's next?

- A workshop to take a deepdive into the applications of MDL, backed up by statistical theory.
- Take a look at the open questions raised in this presentation.
  - The coding system to put down the description mechanism should not matter. How can we choose a universal programming language?
  - What are good encoding systems?
  - What patterns are generalisable and what not?
  - How to incorporate prior knowledge?
- Dedicated application to IFRS and Regulatory Capital calculations.
- Please let Eelko (eelko.ubels@devolksbank.nl) or me (paul.vanleeuwen@devolksbank.nl) know whether you would like to join!

Questions

?

Appendix

Kolmogorov complexity

- Any application of MDL should not be affected by the programming language of choice.
- Andrey Kolmogorov postulated the invariance theorem:

given any description of an object in a description language A, said description may be used in the optimal description language B with a constant overhead

- The description follows two steps:
  1. describe optimal language B in language A;
  2. describe object in optimal language B via language A.
- Comparable with a user-friendly programming language A (e.g. Matlab, R, Python) and whose code is compiled in an efficient programming language B (e.g. C++, C, FORTRAN).
- Part 1. is overhead independent of the object to be described, hence as the object size grows the overhead becomes small.

Kolmogorov complexity (example)

- Suppose we would like to describe the binary data sequence `1 0 0 0` in C via Matlab. How many bits does it take?
- First translate any code from C to Matlab needed to describe `1 0 0 0`.
  - E.g. C has a certain mapping procedure from the data type bit to its memory.
- The corresponding number of bits of the Matlab-code is $L_{\mathrm{C} \mapsto \mathrm{M}}$, e.g. 10k bits.
- To describe $D =$ `1 0 0 0` in C via Matlab takes $L_{\mathrm{C} \mapsto \mathrm{M}} + L_{\mathrm{M}}(D)$ bits.
- But $L_{\mathrm{C} \mapsto \mathrm{M}}$ is independent of the object to be described.
  - Whether we put in the works of Shakespeare or `Hello, World`, $L_{\mathrm{C} \mapsto \mathrm{M}}$ remains the same.

Bibliography

Grünwald, Peter D. 2007. *The Minimum Description Length Principle*. MIT press.

Hansen, Mark H, and Bin Yu. 2001. "Model Selection and the Principle of Minimum Description Length." *Journal of the American Statistical Association* 96 (454): 746–74.