ECE 606, Fall 2019, Assignment 10

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1. *Proof.* Suppose $s \leq_k t$ and let m_1 be the polynomial time reduction function such that $x \in s$ if and only if $m_1(s) \in t$. Similarly, suppose $t \leq_k u$ and let m_2 be the polynomial time reduction function such that $x \in t$ if and only if $m_2(x) \in u$. Then we can compute $m_2 \circ m_1$ in polynomial time, and $x\ ins$ if and only if $m_2(m_1(x)) \in u$.

   Therefore, $s \leq_k u$, which means $\leq_k$ is transitive.

2. *Proof.* By construction. We propose the following mapping m. Given an instance $< G =< V_G, E_G >, H =< V_H, E_H >>$ of ISO, we first check if $|V_G| = |V_H|$ and $|E_G| = |E_H|$, then m directly maps $< G, H >$ as the instance of SUBISO, else, removed a vertex from G as G', then m maps $< G, H >$ to $< G, G' >$.

   We first observe that m is computable in time polynomial in the size of G and H.

   For the "only if" direction, suppose G and H is isomorphic, then, G and H must have the same size of vertexes and edges, $< G, H >$ must be a true instance of SUBISO.

   For the "if" direction, suppose G and H is not isomorphic, then, we do a case analysis. 1) If G and H has the same size, for the purpose of contradiction, H has a subgraph which is isomorphic to G, then G is also isomorphic to H because they have the same vertexes and edges numbers, which is a contradiction. 2) If G and H don't have the same size, for the purpose of contradiction, G' has a subgraph which is isomorphic to G, then the size of G' must not less than G, which is a contradiction.

   Therefore, ISO $\leq_k$ SUBISO.

3. (a) Since HAMPATH $\leq_k$ HAMPATHSTARTEND, if we can prove HAMPATHSTARTEND $\leq_k$ HAMCYCLE, then, HAMPATH $\leq_k$ HAMCYCLE.

   Proof. By construction. We propose the following mapping m. Given an instance $< G =< V, E >, a, b >$ of HAMPATHSTARTEND, introduce two new vertexes $x, y \notin V$. Let $V' = V \cup \{x, y\}$, then introduce three new edges $< x, a >, < b, y >, < x, y >$. Let $G' =< V, E' >$. The function m maps $< G, a, b >$ to $< G' >$.

   We first observe that m is computable in time polynomial in the size of G.

   For the "only if" direction, suppose G has a Hamiltonian path $a \rightsquigarrow b$, then G' also have a Hamiltonian cycle $x \to a \rightsquigarrow b \to y \to x$.

   For the "if" direction, suppose G has no Hamiltonian path from a to b, and for the purpose of contradiction, G' has a Hamiltonian cycle. Then, such a cycle must be $x \to a \rightsquigarrow b \to y \to x$, thus, G has a Hamiltonian path $a \rightsquigarrow b$, which is a contradiction.

   Therefore, HAMPATHSTARTEND $\leq_k$ HAMCYCLE, HAMPATH $\leq_k$ HAMCYCLE.

   (b) Since HAMPATHSTARTEND $\leq_k$ HAMPATH, if we can prove HAMCYCLE $\leq_k$ HAMPATHSTARTEND, then, HAMCYCLE $\leq_k$ HAMPATH.

   Proof. By construction. We propose the following mapping m. Given an instance $< G =< V, E >>$ of HAMCYCLE, introduce two new vertexes $x, y \notin V$. Let $V' = V \cup \{x, y\}$, then introduce two new edges $< x, s >, < y, r >$, where s, r is two connected vertex in V. Let $G' =< V, E' >$. The function m maps $< G >$ to $< G', x, y >$.

   We first observe that m is computable in time polynomial in the size of G.

   For the "only if" direction, suppose G has a Hamiltonian cycle $s \rightsquigarrow r \to s$, then G' also have a Hamiltonian path $x \to s \rightsquigarrow r \to y$.

   For the "if" direction, suppose G has no Hamiltonian cycle, and for the purpose of contradiction, G' has a Hamiltonian path from x to y. Then, such a path must be $x \to s \rightsquigarrow r \to y$, thus, G has a Hamiltonian cycle $s \rightsquigarrow s$ because s and r are connected, which is a contradiction.

Therefore, HAMCYCLE $\leq_k$ HAMPATHSTARTEND, HAMCYCLE $\leq_k$ HAMPATH.

4. *Proof.* First, we define a decision problem UNSAT: given a boolean formula in propositional logic, is it unsatisfiable?

   Then, for every input f of UNSAT, we can simply maps it to $\neg f$, therefore, SAT $\leq_c$ UNSAT because SAT can be polynomial-time if UNSAT is polynomial-time.

   Hence, a formula f is unsatisfiable if and only if $\neg f$ is tautology, therefore, UNSAT can be polynomial-time if TAUT is polynomial-time, UNSAT $\leq_c$ TAUT.

   Finally, SAT $\leq_c$ UNSAT $\leq_c$ TAUT, then, SAT $\leq_c$ TAUT.