ECE 606, Fall 2019, Assignment 1

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1. (a) Suppose n is a natural number whose digits, in order of most- to least-significant, are $n_{k-1}n_{k-2}\ldots,n_0$, where each n_i is one of $0,...,9$. If n is divisible by 3, then the sum of the digits of n, $S_n = \sum_{i=0}^{k-1} n_i$, is divisible by 3.

   (b) *Proof.* Base case: $k = 1$. Then $n = n_0 = S_n$, i.e., n has only one digit. Then for n to be divisible by 3, n must be one of 3, 6 or 9. In each case $n = S_n$, so the S_n is also divisible by 3.

   Step: We assume that given any n that has $k = 1, \ldots, i-1$ digits, for some $i \geq 2$, if n is divisible by 3, then so is S_n. And now we need to prove that given some n of i digits, if n is divisible by 3, then so is S_n. Here we use the notation $()_{10}$ to indicate when we write a number in base-10, i.e., its digits from most- to least-significant.
   We have $n = (n_{i-1}n_{i-2}\ldots n_0)_{10}$. Therefore, $n = 10^{i-1}n_{i-1} + 10^{i-2}n_{i-2} + \cdots + 10^0 n_0 = 10^{i-1}n_{i-1} + (n_{i-2}\ldots n_0)_{10}$. Here we appeal often to the **Claim 1**: given three natural numbers x, y, z such that $x + y = z$ and any two are divisible by 3, then so is the third.

   - Suppose $(n_{i-2}\ldots n_0)_{10}$ is divisible by 3. Therefore, for n is divisible by 3, $10^{i-1}n_{i-1}$ must be divisible by 3 by Claim 1, then n_{i-1} must be divisible by 3, too. That is, $n_{i-1} = 3a$ for some natural number a. Then, $S_n = n_{i-1} + \sum_{j=0}^{i-2} n_j$. As $(n_{i-2}\ldots n_0)_{10}$ is divisible by 3, by the induction assumption that $\sum_{j=0}^{i-2} n_j$ is divisible by 3. Therefore, by the Claim 1, $S_n = n_{i-1} + \sum_{j=0}^{i-2} n_j$ is divisible by 3.
   - Suppose $(n_{i-2}\ldots n_0)_{10}$ is not divisible by 3. Then, $(n_{i-2}\ldots n_0)_{10} = 3a + b$, for some natural numbers a, and for b either 1 or 2. Here we do a case analysis of those two cases for b.
     - if $b = 1$, then $n_{i-1} = 3a' + 2$ for some natural numbers a', to ensure that n is divisible by 3. Therefore, we have:

       $$S_n = 3a' + 2 + \sum_{j=0}^{i-2} n_j$$

       * if $n_0 \leq 7$, then, we have $(n_{i-2}\ldots(n_0+2))_{10}$. Therefore, $2 + \sum_{j=0}^{i-2} n_j$ is divisible by 3 by the induction assumption because $(n_{i-2}\ldots(n_0+2))_{10} = 3a + 1 + 2 = 3(a+1)$. Now, $3a'$ and $(2 + \sum_{j=0}^{i-2} n_j)$ are both divisible by 3, S_n is divisible by 3 by Claim 1.
       * if $n_0 > 7$, then, we have $(n_{i-2}\ldots(n_1+1)(n_0-8))_{10}$. Therefore, $2 + \sum_{j=0}^{i-2} n_j$ is divisible by 3 by the induction assumption because $(n_{i-2}\ldots(n_1+1)(n_0-8))_{10} = 3a+1+10-8 = 3(a+1)$. Now, $3a'$ and $(2 + \sum_{j=0}^{i-2} n_j)$ are both divisible by 3, S_n is divisible by 3 by Claim 1.
     - if $b = 2$, then $n_{i-1} = 3a' + 1$ for some natural numbers a', to ensure that n is divisible by 3. Therefore, we have:

$$S_n = 3a' + 1 + \sum_{j=0}^{i-2} n_j$$

* if $n_0 \leq 8$, then, we have $(n_{i-2} \ldots (n_0 + 1))_{10}$. Therefore, $1 + \sum_{j=0}^{i-2} n_j$ is divisible by 3 by the induction assumption because $(n_{i-2} \ldots (n_0 + 1))_{10} = 3a + 2 + 1 = 3(a + 1)$. Now, $3a'$ and $(1 + \sum_{j=0}^{i-2} n_j)$ are both divisible by 3, S_n is divisible by 3 by Claim 1.
* if $n_0 > 8$, then, we have $(n_{i-2} \ldots (n_1 + 1)(n_0 - 9))_{10}$. Therefore, $1 + \sum_{j=0}^{i-2} n_j$ is divisible by 3 by the induction assumption because $(n_{i-2} \ldots (n_1+1)(n_0-9))_{10} = 3a+2+10-9 = 3(a+1)$. Now, $3a'$ and $(1 + \sum_{j=0}^{i-2} n_j)$ are both divisible by 3, S_n is divisible by 3 by Claim 1.

2. It's TRUE about the statement: suppose $n, m \in \mathbb{N}$, and n pigeons fly in to m pigeon holes, there is a hole with at least $\lceil n/m \rceil$ pigeons.

   Proof. Assume, for the purpose of contradiction, that every hole has a number of pigeons less than $\lceil n/m \rceil$. Then, let the number of pigeons in each hole denotes to $l_1, \ldots l_m$. Therefore, $\sum_{i=1}^{m} l_i < m\lceil n/m \rceil$. Then, $\sum_{i=1}^{m} l_i < m(n/m) = n$. This contradicts our assumption because $\sum_{i=1}^{m} l_i = n$.

3. (a) *Proof.* Suppose $|\mathcal{U}| = n$. $\mathcal{F}$ is the set of all possible functions with domain $\mathcal{U}$ and codomain $\{0, 1, \ldots, m-1\}$, so $|\mathcal{F}| = m^n$. Then, for each $i \in \mathcal{U}$, we have $m^{(n-1)}$ possible $f(i) = j, j \in \{0, 1, \ldots, m-1\}$. So, when we pick the function f from $\mathcal{F}$ uniformly, $Pr\{f(i) = j\} = 1/m$. Therefore, f is a random function.

   (b) a1p1.py