University of Toronto Mississauga
STA260 Term Test 1
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Fall 2023

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Question	**1**	**2**	**3**	**TOTAL**
Value	10	20	20	**50**
Points				

INSTRUCTIONS and POLICIES:

- Please **DO NOT** rip off any page from this booklet.
- For all questions, complete solutions are required. Show all your work to earn full marks.
- You are allowed to use a non-programmable calculator.
- Simplify final answers and round to **3 decimal** places where appropriate.

Question 1 (10 points). A recent study shows 20% of residents in Mississauga prefer a white telephone over other colours.

What is the probability that among the next 1000 telephones installed in Mississauga at least 185 will be white?

Hint: *Use normal approximation to the binomial distribution.*

Let A: = r.v. be event of white telephones installed for 1000 telephones

$X \sim \text{Binomial}(n=1000, p=0.20)$

$E[X] = np = 1000 \times 0.20 = 200$

$Var[X] = np(1-p) = (200)(0.8) = 160$

$$P(X \geq 185) \simeq P(\underset{\sim N(200,\,160)}{Y} \geq 184.5)$$

$$= P\left(\frac{Y-200}{\sqrt{160}} \geq \frac{184.5-200}{\sqrt{160}}\right)$$

$$= P(Z \geq -1.23)$$

$$= 1 - P(Z \geq 1.23)$$

$$= 1 - 0.1093 = 0.8907$$

Question 1 (Continued).

$\sim \mu, \sigma^2$

Question 2 (20 points = 10+5+5). Let $Y_1, Y_2, \ldots, Y_n$ be n independent observations (not necessarily identically distributed), each with mean μ and finite variance σ^2. Let us define the sample variance as follows:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2. \rightarrow$$ going to be a chi-square distribution somehow

a) Show that

$$\mathbb{E}[S^2] = \sigma^2.$$

b) Show that

$$\mathbb{E}[S] \leq \sigma.$$

c) If we further know that $Y_1, Y_2, \ldots, Y_n$ is a random sample from a normal distribution with mean μ and finite variance σ^2, find the constant a such that

$$\mathbb{P}\left(\frac{3S^2}{\sigma^2} \geq a\right) = 0.90. \qquad \frac{(n-1)S^2}{\sigma^2} = W$$

a)

$$E[S^2] = \frac{1}{n-1} E\left[\sum_{i=1}^{n} (Y_i - \overline{Y})^2\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2\right]$$

$$= \frac{1}{n-1}\left(nE[Y_i^2] - nE[\overline{Y}^2]\right)$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

$$P\left(\frac{3S^2}{\sigma^2} \geq a\right)$$

Let $n=4$, then $P\left(\frac{(n-1)S^2}{\sigma^2} \geq a\right) = P\left(\frac{3S^2}{\sigma^2} \geq a\right) = 0.90$

$$\Rightarrow P\left(\chi^2_{(3)} \geq a\right) = 0.90 \Rightarrow a = 0.584$$

Question 2 (Continued).

$$\mathrm{Var}(y_i) = E[y_i^2] - E[y]^2$$

$$\mathrm{Var}(y_i) + E[y_i]^2 = E[y_i^2]$$

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}$$

$$E[y_i^2] = \sigma^2 + (\mu)^2$$

$$\mathrm{Var}(\bar{y}) = E[\bar{y}^2] - E[\bar{y}]^2$$

$$\mathrm{Var}(\bar{y}) + E[\bar{y}]^2 = E[\bar{y}^2]$$

$$\Rightarrow E[\bar{y}^2] = \frac{\sigma^2}{n} + (\mu)^2$$

$$\Rightarrow E[s^2] = \left(\frac{1}{n-1}\right)\left[\cancel{n}\left[\sigma^2 + \cancel{\mu^2}\right] - \cancel{n}\left(\frac{\sigma^2}{n} + \cancel{\mu^2}\right)\right]$$

$$= \frac{1}{n-1}\left(n\sigma^2 - \sigma^2\right)$$

$$= \frac{\sigma^2(n-1)}{n-1} = \sigma^2$$

$$g(E[X]) \leq E[g(X)]$$

Question 2 (Continued).

b) $E[S] \leq \sigma$

we know that $g(x) = -\sqrt{x}$ is convex for $x > 0$, so by Jensen's inequality:

$$-\sqrt{E[S^2]} \leq E\left[-\sqrt{S^2}\right]$$

$$\Rightarrow -\sqrt{E[S^2]} \leq -E\left[\sqrt{S^2}\right]$$

$$\Rightarrow -\sqrt{\sigma^2} \leq -E[S]$$

$$\Rightarrow E[S] \leq \sqrt{\sigma^2} \Rightarrow E[S] \leq \sigma$$

$\because \sigma^2 \geq 0,$

$S_{S^2} \geq 0$

Question 3 (20 points = 10+10). Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a population with a uniform distribution on the interval $(0, \theta)$, where parameter $\theta > 0$.

$Y_1, Y_2, \ldots, Y_n \sim U(0, \theta)$, $\theta > 0$

a) Let $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$. Show that

$$\mathbb{E}[Y_{(n)}] = \left(\frac{n}{n+1}\right)\theta.$$

b) Let $Z = Y_{(n)}/\theta$. Show that Z has distribution function

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ z^n, & 0 \le z \le 1 \\ 1, & z > 1 \end{cases}$$

$$F_y(y) = \int_0^y \frac{1}{\theta}\,dt = \frac{y}{\theta} - \frac{1}{\theta}(0) = \frac{y}{\theta}$$

$$f_{Y_{(n)}}(y) = n f_Y(y) F_Y(y)^{n-1} = n\left(\frac{1}{\theta}\right)\left(\frac{y}{\theta}\right)^{n-1} = \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1}$$

$$E[Y_{(n)}] = \int_0^\theta y\left(\frac{n}{\theta}\right)\left(\frac{y}{\theta}\right)^{n-1} dy$$

$$= \left(\frac{\theta}{n}\right)(y)\left(\frac{y}{\theta}\right)^n \Big|_0^\theta - \frac{\theta}{n}\int_0^\theta \left(\frac{y}{\theta}\right)^n dy$$

Let $u = y$

$dv = \left(\frac{y}{\theta}\right)^{n-1} dy$

$du = dy$

$v = \frac{\theta}{n}\left(\frac{y}{\theta}\right)^n$

$$= \frac{\theta^2}{n} - \frac{\theta^2}{n(n+1)}\left(\frac{y}{\theta}\right)^{n+1}\Big|_0^\theta$$

$$= \frac{\theta^2}{n} - \frac{\theta^2}{n(n+1)} = \frac{(n+1)\theta^2 - \theta^2}{n(n+1)}$$

$$= \left(\frac{\theta^{\cancel{2}}}{n+1}\right)\left(\frac{n}{\cancel{\theta}}\right)$$

$$= \frac{n\theta}{n+1} = \frac{n}{n+1}\theta$$

b) Let $Z = Y_{(n)}/\theta$. Show that Z has distribution function

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ z^n, & 0 \le z \le 1 \\ 1, & z > 1 \end{cases}$$

$Z = Y_{(n)} / \theta$

CDF method:

$$P(Z \le z) = P\left(\frac{Y_{(n)}}{\theta} \le z\right) = P(Y_{(n)} \le z\theta)$$

$$= F_{Y_{(n)}}(z\theta)$$

$$F_{Y_{(n)}}(y) = \int_0^y \left(\frac{n}{\theta}\right)\left(\frac{t}{\theta}\right)^{n-1} dt = \left(\frac{\cancel{n}}{\cancel{\theta}}\right)\left(\frac{\cancel{\theta}}{\cancel{n}}\right)\left(\frac{t}{\theta}\right)^n \Bigg|_0^y$$

$$= \left(\frac{y}{\theta}\right)^n$$

$0 \le y \le \theta$

$= z^n$ for $0 \le z \le 1 \implies 0 \le z \le 1$

So, given definition of CDF,

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ z^n, & 0 \le z \le 1 \\ 1, & z > 1 \end{cases}$$

Question 3 (Continued).

Question 3 (Continued).

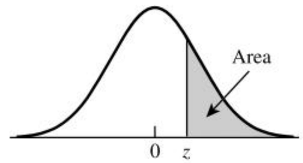
END OF EXAM*

* This page will not be graded unless you clearly indicate in the exam that something here is to be marked.

Table of Probability Distribution Functions and Their Properties

Distribution	**pdf/pmf**	**Mean**	**Variance**	**mgf**
Bernoulli	$p^y(1-p)^{1-y}$ $y = 0, 1, ..., n$	p	$p(1-p)$	$pe^t + (1-p)$
Binomial	$\binom{n}{y} p^y(1-p)^{n-y}$ $y = 0, 1, ..., n$	np	$np(1-p)$	$\left[pe^t + (1-p)\right]^n$
Geometric	$p(1-p)^{y-1}$ $y = 1, 2, ...$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$	$\dfrac{pe^t}{1-(1-p)e^t}$
Poisson	$\dfrac{\lambda^y e^{-\lambda}}{y!}$ $y = 0, 1, 2, ...$	λ	λ	$\exp\left[\lambda\left(e^t - 1\right)\right]$
Uniform	$\dfrac{1}{\theta_2 - \theta_1}$ $\theta_1 \leq y \leq \theta_2$	$\dfrac{\theta_1 + \theta_2}{2}$	$\dfrac{(\theta_2 - \theta_1)^2}{12}$	$\dfrac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$\dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(y-\mu)^2}{2\sigma^2}\right]$ $-\infty < y < \infty$	μ	σ^2	$\exp\left(\mu t + \dfrac{t^2\sigma^2}{2}\right)$
Exponential	$\dfrac{1}{\beta} e^{-y/\beta}$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$\left[\dfrac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta}$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$\dfrac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)}$ $0 < y < \infty$	v	$2v$	$(1 - 2t)^{-v/2}$
Beta	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$ $0 < y < 1$	$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	N/A

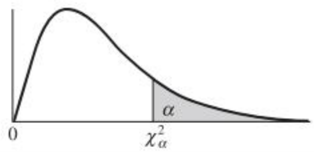
Table 4 Normal Curve Areas
Standard normal probability in right-hand tail
(for negative values of z, areas are found by symmetry)



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).

Table 6 Percentage Points of the χ^2 Distributions



df	$\chi^2_{0.995}$	$\chi^2_{0.990}$	$\chi^2_{0.975}$	$\chi^2_{0.950}$	$\chi^2_{0.900}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720
3	0.0717212	0.114832	0.215795	0.351846	0.584375
4	0.206990	0.297110	0.484419	0.710721	1.063623
5	0.411740	0.554300	0.831211	1.145476	1.61031
6	0.675727	0.872085	1.237347	1.63539	2.20413
7	0.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100	67.3276	70.0648	74.2219	77.9295	82.3581

Table 6 (*Continued*)

$\chi^2_{0.100}$	$\chi^2_{0.050}$	$\chi^2_{0.025}$	$\chi^2_{0.010}$	$\chi^2_{0.005}$	df
2.70554	3.84146	5.02389	6.63490	7.87944	1
4.60517	5.99147	7.37776	9.21034	10.5966	2
6.25139	7.81473	9.34840	11.3449	12.8381	3
7.77944	9.48773	11.1433	13.2767	14.8602	4
9.23635	11.0705	12.8325	15.0863	16.7496	5
10.6446	12.5916	14.4494	16.8119	18.5476	6
12.0170	14.0671	16.0128	18.4753	20.2777	7
13.3616	15.5073	17.5346	20.0902	21.9550	8
14.6837	16.9190	19.0228	21.6660	23.5893	9
15.9871	18.3070	20.4831	23.2093	25.1882	10
17.2750	19.6751	21.9200	24.7250	26.7569	11
18.5494	21.0261	23.3367	26.2170	28.2995	12
19.8119	22.3621	24.7356	27.6883	29.8194	13
21.0642	23.6848	26.1190	29.1413	31.3193	14
22.3072	24.9958	27.4884	30.5779	32.8013	15
23.5418	26.2962	28.8454	31.9999	34.2672	16
24.7690	27.5871	30.1910	33.4087	35.7185	17
25.9894	28.8693	31.5264	34.8053	37.1564	18
27.2036	30.1435	32.8523	36.1908	38.5822	19
28.4120	31.4104	34.1696	37.5662	39.9968	20
29.6151	32.6705	35.4789	38.9321	41.4010	21
30.8133	33.9244	36.7807	40.2894	42.7956	22
32.0069	35.1725	38.0757	41.6384	44.1813	23
33.1963	36.4151	39.3641	42.9798	45.5585	24
34.3816	37.6525	40.6465	44.3141	46.9278	25
35.5631	38.8852	41.9232	45.6417	48.2899	26
36.7412	40.1133	43.1944	46.9630	49.6449	27
37.9159	41.3372	44.4607	48.2782	50.9933	28
39.0875	42.5569	45.7222	49.5879	52.3356	29
40.2560	43.7729	46.9792	50.8922	53.6720	30
51.8050	55.7585	59.3417	63.6907	66.7659	40
63.1671	67.5048	71.4202	76.1539	79.4900	50
74.3970	79.0819	83.2976	88.3794	91.9517	60
85.5271	90.5312	95.0231	100.425	104.215	70
96.5782	101.879	106.629	112.329	116.321	80
107.565	113.145	118.136	124.116	128.299	90
118.498	124.342	129.561	135.807	140.169	100

From "Tables of the Percentage Points of the χ^2-Distribution." *Biometrika*, Vol. 32 (1941), pp. 188–189, by Catherine M. Thompson.