University of Toronto Mississauga
STA260 Term Test 1
Dr. Masoud Ataei
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Last Name / Surname (please print): Yun

First Name (please print): Paul

Student ID Number: 1010288617

Question	**1**	**2**	**3**	**4**	**TOTAL**
Value	10	10	20	20	**60**
Points					

INSTRUCTIONS and POLICIES:

- Please **DO NOT** rip off any page from this booklet.
- For all questions, complete solutions are required. Show all your work to earn full marks.
- You are allowed to use a non-programmable calculator.
- Simplify final answers and round to **3 decimal** places where appropriate.

★ Bring a calculator to the test

Question 1 (10 points). A factory produces 300 light bulbs per day, where each light bulb has a 5% chance of being defective, independent of other bulbs. What is the probability that more than 20 light bulbs are defective on a given day?

Hint: *Use normal approximation to the binomial distribution.*

Let $A :=$ r.v. denoting # of defective light bulbs.

$A \sim \text{Binomial}(n=300, p=0.05)$

$p(x) = \binom{n}{x}(p)^x(1-p)^{n-x}$

$E[A] = np = 300 \times 0.05 = 15$

$Var(A) = np(1-p) = 15(0.95) = 14.25$

$P(X > 20) = P(X \geq 21)$

$(Y \sim N(15, 14.25))$

$\simeq P(Y \geq 20.5) = P\left(\frac{Y - E[Y]}{\sqrt{Var(Y)}} \geq \frac{20.5-15}{3.774}\right)$

give this more room

$= P(Z \geq 1.46) = 0.0722$

Question 1 (Continued).

Question 2 (10 points). Let $Y_1, Y_2, \ldots, Y_5$ denote a random sample from a normal population $N(0,1)$, and let Y_6 be another independent observation from the same normal distirubiton. Consider two statistics defined as follows:

$$W = \sum_{i=1}^{5} Y_i^2 \quad \text{and} \quad U = \sum_{i=1}^{5} \left(Y_i - \bar{Y}\right)^2.$$

Find the distribution of the following statistics

$W \sim \chi^2(5)$, $U \sim \chi^2(4)$

a) $\dfrac{\sqrt{5}Y_6}{\sqrt{W}}$,

b) $\dfrac{2\left(5\bar{Y}^2 + Y_6^2\right)}{U}$.

a) $= \dfrac{Y_6}{\sqrt{W/5}} = \dfrac{Z}{\sqrt{\frac{\chi^2(5)}{5}}} \sim t(5)$

b) $\bar{Y} \sim N\left(0, \frac{1}{5}\right) \Rightarrow \sqrt{5}\,\bar{Y} \sim N(0,1)$

$\Rightarrow 5\bar{Y}^2 \sim \chi^2(1)$

$\Rightarrow 5\bar{Y}^2 + Y_6^2 \sim \chi^2(2)$

$$\frac{2(5\bar{Y}^2 + Y_6^2)}{U} = \frac{(5\bar{Y}^2 + Y_6^2)/2}{U/4} = \frac{\chi^2(2)/2}{\chi^2(4)/4} \sim F_{(r_1=2,\, r_2=4)}$$

Question 2 (Continued).

Question 3 (20 points = 5+10+5). Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n drawn from a uniform distribution on the interval $(\theta, \theta+1)$, where $\theta > 0$ is an unknown parameter. A combined estimator $\hat{\theta}$ for θ is defined as:

r.v r.v. r.v

$$\hat{\theta} = w\hat{\theta}_1 + (1-w)\hat{\theta}_2,$$

where $0 < w < 1$ is a constant weight, and the individual estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are given by:

$$\hat{\theta}_1 = \bar{Y} - \frac{1}{2} \quad \text{and} \quad \hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}.$$

$Y_1 \quad Y_2 \quad \ldots \quad Y_n$

a) Prove that the *probability density function* (PDF) of $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ is given by

$$f_{Y_{(n)}}(y) = n(y-\theta)^{n-1}, \qquad \theta < y < \theta + 1.$$

b) Prove that the combined estimator $\hat{\theta}$ is an unbiased estimator of θ.

c) Let the variances of $\hat{\theta}_1$ and $\hat{\theta}_2$ be denoted by σ_1^2 and σ_2^2, respectively. Find the optimal value of the weight $w^\star$ that minimizes the variance of the combined estimator $\hat{\theta}$.

a) $f_{Y_{(n)}}(y) = n\left(f_Y(y)\right)\left(F_Y(y)\right)^{n-1} \Rightarrow$

(1) pdf ung Y_i: $f_{Y_i}(y_i) = \dfrac{1}{(\theta+1)-\theta} = 1$

cdf: $F_Y(y) = \displaystyle\int_\theta^y 1\, dt = y - \theta$

$f_{Y_{(n)}}(y) = n(1)(y-\theta)^{n-1} = n(y-\theta)^{n-1}, \ \theta < y < \theta+1$

Question 3 (Continued).

b) $E[\hat{\theta}] = E[w\hat{\theta}_1 + (1-w)\hat{\theta}_2]$

$= E\left[w\left(\bar{Y} - \frac{1}{2}\right) + (1-w)\left(Y_{(n)} - \frac{n}{n+1}\right)\right]$

$=^{①} E\left[w\left(\bar{Y} - \frac{1}{2}\right)\right] + {}^{②} E\left[(1-w)\left(Y_{(n)} - \frac{n}{n+1}\right)\right]$

① $= wE\left[\bar{Y} - \frac{1}{2}\right] = w\,E[\bar{Y}] - w\frac{1}{2} = w\left(\frac{\theta + (\theta+1)}{2}\right) - w\frac{1}{2}$

$= w\left(\frac{2\theta+1}{2}\right) - w\frac{1}{2}$

$= \frac{w(2\theta)}{2} = w\theta$

② $= (1-w)\,E\left[\left(Y_{(n)} - \frac{n}{n+1}\right)\right]$

$= (1-w)E[Y_{(n)}] - \frac{(1-w)n}{n+1}$

$= (1-w)\left(\theta + \frac{n}{n+1}\right) - \frac{(1-w)n}{n+1}$

$= (1-w)\theta$

$E[Y_{(n)}] = \int_{\theta}^{\theta+1} ny(y-\theta)^{n-1}\,dy$

by parts:

$= y(y-\theta)^n\Big|_{\theta}^{\theta+1} - \int_{\theta}^{\theta+1}(y-\theta)^n dy$

$\int v\,du = uv - \int u\,dv$

$= (\theta+1)(\theta+1-\theta)^n - \frac{1}{n+1}(y-\theta)^{n+1}\Big]_{\theta}^{\theta+1}$

$u = ny$

$du = n\,dy$

$dv = (y-\theta)^{n-1}dy$

$v = (y-\theta)^n$

$= (\theta+1) - \frac{1}{n+1} = \theta + \frac{n}{n+1}$

① + ② $=$

$(1-w)\theta + w\theta$

$= \theta$

Question 3 (Continued).

c) $\text{var}(\hat{\theta}) = \text{var}\left(w\hat{\theta}_1 + (1-w)\hat{\theta}_2\right)$

$$= w^2 \text{var}(\hat{\theta}_1) + (1-w)\,\text{var}(\hat{\theta}_2)$$

$$= w^2(\sigma_1^2) + (1-w)^2(\sigma_2^2)$$

$$= w^2\sigma_1^2 + (1-2w+w^2)\sigma_2^2$$

$$= w^2\sigma_1^2 + \sigma_2^2 - 2w\sigma_2^2 + w^2\sigma_2^2$$

$$g'(w) = 2\sigma_1^2 w - 2\sigma_2^2 + 2\sigma_2^2 w = 0$$

$$w(2\sigma_1^2 + 2\sigma_2^2) = 2\sigma_2^2$$

$$w = \frac{2\sigma_2^2}{(2\sigma_1^2 + 2\sigma_2^2)} = \frac{\sigma_2^3}{\sigma_1^2+\sigma_2^2}$$

$$g''(w) = 2\sigma_1^2 + 2\sigma_2^2 > 0$$

Question 3 (Continued).

Question 4 (20 points = 5+10+5). Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a normal population $N(\mu, \sigma^2)$, where parameters $-\infty < \mu < \infty$ and $\sigma > 0$. Consider two estimators for σ^2 given as follows:

$$S_1^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 \quad \text{and} \quad S_2^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2.$$

a) Derive the *bias* of each estimator S_1^2 and S_2^2.

b) Compute the *mean squared error* (MSE) for each estimator.

c) Compare the two estimators in terms of their bias and MSE, and determine which estimator is preferable under the *principles of unbiasedness and minimum variance*.

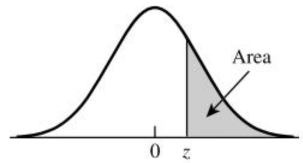
Question 4 (Continued).

Question 4 (Continued).

Table of Probability Distribution Functions and Their Properties

Distribution	pdf/pmf	Mean	Variance	mgf
Bernoulli	$p^y(1-p)^{1-y}$ $y = 0, 1, ..., n$	p	$p(1-p)$	$pe^t + (1-p)$
Binomial	$\binom{n}{y} p^y (1-p)^{n-y}$ $y = 0, 1, ..., n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(1-p)^{y-1}$ $y = 1, 2, ...$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Poisson	$\frac{\lambda^y e^{-\lambda}}{y!}$ $y = 0, 1, 2, ...$	λ	λ	$\exp\left[\lambda\left(e^t - 1\right)\right]$
Uniform	$\frac{1}{\theta_2 - \theta_1}$ $\theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$ $-\infty < y < \infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$\frac{1}{\beta} e^{-y/\beta}$ $0 < y < \infty$	β	β^2	$(1-\beta t)^{-1}$
Gamma	$\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta}$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1-\beta t)^{-\alpha}$
Chi-square	$\frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)}$ $0 < y < \infty$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	N/A

Table 4 Normal Curve Areas
Standard normal probability in right-hand tail
(for negative values of *z*, areas are found by symmetry)



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).