STA260 Summer 2025 - Tutorial 3

July 2025

$t_r = \frac{Z}{\sqrt{W/r}}$, $Z \sim \text{Normal}(0,1)$, $W \sim \chi^2(r)$

$F(r_1, r_2) = \frac{W_1/r_1}{W_2/r_2}$, $W_1 \sim \chi^2(r_1)$, $W_2 \sim \chi^2(r_2)$

Relevant Review from Lecture

$$t_v = \frac{Z}{\sqrt{W/v}} \text{ where } Z \sim \text{Normal}(0,1), W \sim \chi^2_{(v)}, \text{ and } Z, W \text{ are independent}$$

$$F_{v1,v2} = \frac{W_1/v_1}{W_2/v_2} \text{ where } W_1 \sim \chi^2_{(v_1)}, W_2 \sim \chi^2_{(v_2)} \text{ and } W_1, W_2 \text{ are independent}$$

$$F_{n_1-1,n_2-1} \sim \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

Question 1

If the random variable F has an F-distribution with $r_1 = 5$, $r_2 = 10$, the degree of freedom of numerator and denominator respectively. Find a, b such that $P(F \leq a) = 0.05$, $P(a < F < b) = 0.9$.

$F \sim F(r_1 = 5, r_2 = 10)$

$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum (Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2(n-1)$

$P(F \leq a) = 0.05$

$\Rightarrow P(a \leq F) = 0.95$

$\frac{1}{F} = \frac{W_2/r_2}{W_1/r_1} \sim F(r_2 = 10, r_1 = 5)$

$\Rightarrow$

$$P(F \leq a) = 0.05 \Rightarrow P\left(\frac{1}{F} \geq \frac{1}{a}\right) = 0.05$$

$$\Rightarrow P\left(F_2 \geq \frac{1}{a}\right) = 0.05$$

Let $F_2 = \frac{1}{F} \sim F(r_2, r_1)$

$$\Rightarrow \frac{1}{a} = 4.74 \Rightarrow a = \frac{1}{4.74} \simeq 0.2110$$

↓ Masoud likes this question

Question 2

Let $Y_1, Y_2, ..., Y_5$ be a random sample of size 5 from a normal population with mean 0 and variance 1.
Let $\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$.
Let Y_6 be another independent observation from the same population.
Let $W = \sum_{i=1}^{5} Y_i^2, U = \sum_{i=1}^{5}\frac{(Y_i - \bar{Y})^2}{\sigma^2} \sim \chi(n-1)$. What ~~is~~ the distribution of

(a) $\frac{(n-1)s^2}{\sigma^2} =$ $$\frac{\sqrt{5}Y_6}{\sqrt{W}}$$ $\frac{Y_6}{\sqrt{W/5}} \sim t(5)$

(b) $$\frac{2Y_6}{\sqrt{U}}$$ $\frac{Y_6}{\sigma\sqrt{U/\sigma^2 4}} \sim$

(c) $$\frac{2(5\bar{Y}^2 + Y_6^2)}{U}$$ $= \frac{Y_6}{\sqrt{U/4}}$

Let $Y_1, Y_2, \ldots Y_5 \sim N(0,1)^2 = \chi^2(1)$

$$\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i \sim N\left(0, \frac{1}{5}\right)$$

Let $W = \sum_{i=1}^{5} Y_i^2$, $U = \sum_{i=1}^{5} (Y_i - \bar{Y})^2 = \frac{(n-1)S^2}{\sigma^2}$

$\sim \chi^2(4)$

a) $\frac{\sqrt{5}\,Y_6}{\sqrt{W}}$ $\quad W \sim \chi^2(5)$

$$= \frac{Y_6}{\sqrt{W/5}} = \frac{N(0,1)}{\sqrt{\frac{\chi^2(5)}{5}}} \sim t(5)$$

$$W = \sum_{i=1}^{5} Y_i^2 = \sum_{i=1}^{5} (N(0,1))^2 = \sum_{i=1}^{5} \chi^2(1) = \chi^2(5)$$

b) $\dfrac{2Y_6}{\sqrt{U}}$ $\quad U = \sum_{i=1}^{n} (y_i - \bar{y})^2 = \dfrac{(n-1)S^2}{\sigma^2 = 1} = \sum_{i=1}^{n} \left(\dfrac{(y_i - \bar{y})}{(1)^2}\right)^2$

$\therefore U \sim \chi^2(n-1) = \chi^2(4)$

$$= \frac{Y_6}{\sqrt{\frac{U}{4}}} = \frac{Z}{\sqrt{\frac{\chi(4)}{4}}} \sim t(4)$$

Another method to get U: $U = \sum_{i=1}^{n} (y_i - \bar{y})^2$

$$= (n-1) \sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{n-1}$$

$$= \frac{(n-1)S^2}{(1)^2} = \frac{(n-1)S^2}{\sigma^2}$$

c)

$$\sqrt{U}$$

(c)

$$\frac{2(5\bar{Y}^2 + Y_6^2)}{U}$$

$$\frac{2(5\bar{Y}^2 + Y_6^2)}{U} = \frac{(5\bar{Y}^2 + Y_6^2)/2}{U/4} = \frac{\chi^2(2)/2}{\chi^2(4)/4} \sim F(2,4)$$

$\bar{Y} \sim N(0, \frac{1}{5})$

$U \sim \chi^2(4)$

$5\bar{Y}^2 = (\sqrt{5}\bar{Y})^2$

$\sqrt{5}\bar{Y} = Z \sim N(0,1)$

$\therefore 5\bar{Y}^2 \sim \chi^2(1)$

$\Rightarrow 5\bar{Y}^2 + Y_6^2 \sim \chi^2(2)$

$\sim \chi^2(1)$