

$$+ r^2 \frac{Z}{W/r} \quad Z \sim \text{Normal}(0, 1) \quad W \sim \chi^2(r)$$

STA260 Summer 2025 - Tutorial 3

July 2025

$$F(r_1, r_2) = \frac{W_1/r_1}{W_2/r_2} \quad W_1 \sim \mathcal{N}(0, 1) \quad W_2 \sim \chi^2(r_2)$$

## Relevant Review from Lecture

$t_v = \frac{Z}{\sqrt{W/v}}$  where  $Z \sim \text{Normal}(0, 1)$ ,  $W \sim \chi^2_{(v)}$ , and  $Z, W$  are independent

$F_{v1, v2} = \frac{W_1/v_1}{W_2/v_2}$  where  $W_1 \sim \chi^2_{(v_1)}$ ,  $W_2 \sim \chi^2_{(v_2)}$  and  $W_1, W_2$  are independent

$$F_{n_1-1, n_2-1} \sim \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

## Question 1

If the random variable  $F$  has an F-distribution with  $r_1 = 5$ ,  $r_2 = 10$ , the degree of freedom of numerator and denominator respectively. Find  $a, b$  such that  $P(F \leq a) = 0.05$ ,  $P(a < F < b) = 0.9$ .

$$F \sim F(r_1=5, r_2=10)$$

$$\frac{(n-1)S^2}{\sigma^2} = \sum \frac{(y_i - \bar{y})^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$P(F \leq a) = 0.05$$

$$\Rightarrow P(a \leq F) = 0.95$$

$$\frac{1}{F} = \frac{W_2/r_2}{W_1/r_1} \sim F(r_2=10, r_1=5)$$

$$\Rightarrow$$

$$P(F \leq a) = 0.05 \Rightarrow P\left(\frac{1}{F} \geq \frac{1}{a}\right) = 0.05$$

$$\Rightarrow P\left(F_2 > \frac{1}{a}\right) = 0.05$$

$$\text{Let } F_2 = \frac{1}{F} \sim F(n_1, n_2)$$

$$\Rightarrow \frac{1}{a} = 4.74 \Rightarrow a = \frac{1}{4.74} \approx 0.210$$

↓ Masoud likes this question

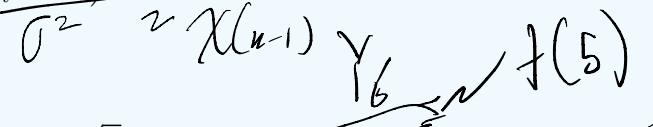
## Question 2

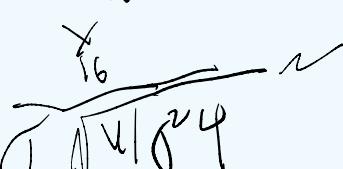
Let  $Y_1, Y_2, \dots, Y_5$  be a random sample of size 5 from a normal population with mean 0 and variance 1.

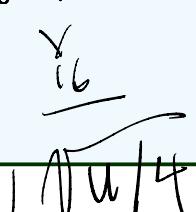
Let  $\bar{Y} = \frac{1}{5} \sum_{i=1}^5 Y_i$ .

Let  $Y_6$  be another independent observation from the same population.

Let  $W = \sum_{i=1}^5 Y_i^2$ ,  $U = \sum_{i=1}^5 (Y_i - \bar{Y})^2$ . What is the distribution of

(a)  $\frac{(n-1)s^2}{\sigma^2} = \frac{\sqrt{5}Y_6}{\sqrt{W}}$  

(b)  $\frac{2Y_6}{\sqrt{U}}$  

(c)  $\frac{2(5\bar{Y}^2 + Y_6^2)}{U} = \frac{Y_6}{\sqrt{U/4}}$  

Let  $Y_1, Y_2, \dots, Y_5 \sim N(0, 1)^2 = \chi^2(1)$

$$\bar{Y} = \frac{1}{5} \sum_{i=1}^5 Y_i \sim N\left(0, \frac{1}{5}\right)$$

$$\text{Let } W = \sum_{i=1}^5 Y_i^2, U = \sum_{i=1}^5 (Y_i - \bar{Y})^2 = \frac{(n-1)S^2}{\sigma^2}$$

$\sim \chi^2(4)$

a)  $\frac{\sqrt{5} Y_6}{\sqrt{W}} \sim \chi^2(5)$

$$= \frac{Y_6}{\sqrt{\frac{W}{5}}} = \frac{N(0, 1)}{\sqrt{\frac{\chi^2(5)}{5}}} \sim t(5)$$

$$W = \sum_{i=1}^5 Y_i^2 = \sum_{i=1}^5 (N(0, 1))^2 = \sum_{i=1}^5 \chi^2(1) = \chi^2(5)$$

$$b) \frac{2Y_6}{V} \quad V = \sum_{i=1}^n (y_i - \bar{y})^2 = \frac{(n-1)s^2}{\sigma^2} = \sum_{i=1}^n \left( \frac{(y_i - \bar{y})^2}{(\bar{y})^2} \right)$$

$$\therefore V \propto \chi^2(n-1) = \chi^2(n)$$

$$= \frac{Y_6}{\sqrt{\frac{V}{4}}} = \frac{Z}{\sqrt{\frac{\chi^2(n)}{4}}} \text{ n.f.(4)}$$

Another method to get

$$U: V = \sum_{i=1}^n (y_i - \bar{y})^2$$

$$= (n-1) \sum_{i=1}^n \frac{(y_i - \bar{y})^2}{n-1}$$

$$= \frac{(n-1)s^2}{(1)^2} = \frac{(n-1)s^2}{\sigma^2}$$

( )

$$\sqrt{U}$$

(c)

$$\frac{2(5\bar{Y}^2 + Y_6^2)}{U}$$

$$\frac{2(5\bar{Y}^2 + Y_6^2)}{U}$$

$$\bar{Y} \sim N(0, \frac{1}{5})$$

$$U \sim \chi^2(4)$$

$$= \frac{(5\bar{Y}^2 + Y_6^2) / 2}{U / 4}$$

$$5\bar{Y}^2 = (\sqrt{5}\bar{Y})^2$$

$$\sqrt{5}\bar{Y} \sim N(0, 1)$$

$$= \frac{\chi^2(2) / 2}{\chi^2(4) / 4} \sim F(2, 4) \quad \begin{aligned} &\because 5\bar{Y}^2 \sim \chi^2(1) \\ &\Rightarrow 5\bar{Y}^2 + Y_6^2 \sim \chi^2(2) \\ &\sim \chi^2(6) \end{aligned}$$