order statistics:

Let $X_1, X_2, \ldots, X_n$ be a random sample from cont. distribution

Theorem 4.4.1. : Let $Y_1 < Y_2 < \cdots < Y_n$

Then joint pdf is $g(y_1, y_2, \ldots, y_n) = \begin{cases} n! f(y_1) f(y_2) \ldots f(y_n) \\ 0 \text{ elsewhere} \end{cases}$

$1 \times 2 \times \cdots \times n = n!$

$X \quad X_2 \quad \cdots \quad X_n$

$5 \quad 6 \quad 9$

$f(y_1) \quad f(y_2) \quad f(y_n)$

$y_{(1)} \quad y_{(2)} \quad y_{(n)}$

$F(x) = \int_a^x f(w)\,dw \quad a < x < b$

$g(y_1, y_2, y_3) = \begin{cases} 6 f(y_1) f(y_2) f(y_3) \\ 0 \text{ elsewhere} \end{cases}$

$X_1, X_2, X_3 \quad 3!$

$y_1 \quad y_2 \quad y_3$

$f(y_1) \quad f(y_2) \quad f(y_3)$

$(y_1) \quad y_2 \quad y_3$

Heuristic definition of order probability

Functions of order statistics:

Sample range: given by $Y_n - Y_1$ → new distribution

Sample midrange: $(Y_1 + Y_n)/n$

Sample median given by: $Q_2 = \begin{cases} Y_{(n+1)/2} & \text{if } n \text{ is odd} \\ (Y_{n/2} + Y_{(n/2)+1})/2 & \text{if } n \text{ is even} \end{cases}$

Ex 4.4.3.

Review Practice Problems: Order Statistics

Use the formula sheet.

1. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$. **Answer:** $f_Y(y) = n[F_X(y)]^{n-1} f_X(y)$.

2. Let $X_1, \ldots, X_n$ be independent exponential random variables with parameter λ (i.e., $f_{X(x)} = \lambda e^{-\lambda y}$). Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$. **Answer:** $f_Y(y) = n[(1 - e^{-\lambda y})]^{n-1} \lambda e^{-\lambda y}$, $y > 0$.

3. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$. Find the density $f_Y(y)$. **Answer:** $f_Y(y) = n[1 - F_X(y)]^{n-1} f_X(y)$.

4. Let $X_1, \ldots, X_n$ be independent random variables with the exponential distribution with mean λ, where $\lambda > 0$. Let $Y = \min(X_1, \ldots, X_n)$.

   (a) Find the density $f_Y(y)$. **Answer:** $\frac{n}{\lambda} e^{-ny/\lambda}$.

   (b) State the name of the distribution of Y. Don't forget to specify the parameters. **Answers:** $Y \sim \text{Exponential}(n/\lambda)$. Here $E(Y) = \lambda/n$.

   (c) Find $Var(2Y + 1)$. **Answer:** $4\lambda^2/n^2$.

5. Let X_1 and X_2 be two independent random variables with the following common probability density function:
$$f_X(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}.$$

   Let $Y = \max(X_1, X_2)$. Find $E\left(\frac{1}{Y}\right)$. **Answer:** 6/5.

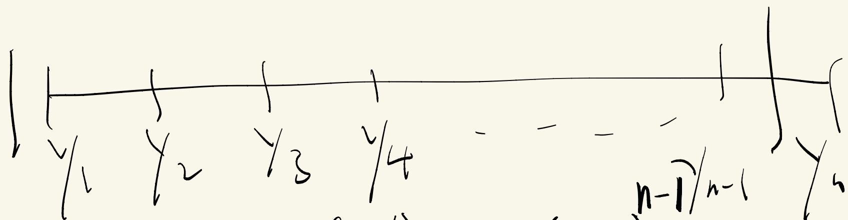
Use the formula sheet.

1. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$.
Answer: $f_Y(y) = n[F_X(y)]^{n-1} f_X(y)$.

Let $Y = X_{(n)}$

Find pdf of $X_{(n)}$:

Using heuristic method:



$$f_Y(Y) = \frac{n!}{(n-1)!\,1!} f_X(Y) F_X(Y)^{n-1} = n F_X(y)^{n-1} f_X(y)$$

from this random variable

different scope/context

$e^{\cdots})]^{\cdots}\lambda e^{\cdots}$, $y > 0$.

3. Let $X_1, \ldots, X_n$ be independent random variables with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$. Find the density $f_Y(y)$.
Answer: $f_Y(y) = n[1 - F_X(y)]^{n-1} f_X(y)$.

$Y = \min(X_1, \ldots, X_n)$ heuristic method:

$$f_{X_{(1)}}(y) = \frac{n!}{(n-1)!\,1!}(1 - F_X(y))^{n-1} f_X(y)$$

$$= n[1 - F_X(y)]^{n-1} f_X(y)$$

$f(x) = \lambda e^{-\lambda x}$

Answer: $f_Y(y) = n[F_X(y)]^{n-1} f_X(y)$.

2. Let $X_1, \ldots, X_n$ be independent exponential random variables with parameter λ (i.e., $f_{X(x)} = \lambda e^{-\lambda y}$). Let $Y = \max(X_1, \ldots, X_n)$. Find the density $f_Y(y)$. **Answer:** $f_Y(y) = n[(1 - e^{-\lambda y})]^{n-1} \lambda e^{-\lambda y}$, $y > 0$.

$X_1, X_2, \ldots, X_n \sim \mathrm{Exp}(\lambda)$ $\quad f(x) = \lambda e^{-\lambda x}$

$f_Y(y) = n f_X(y) F_X(y)^{n-1} = n \lambda e^{-\lambda y} (1 - e^{-\lambda y})^{n-1}$, $y > 0$

$f_X(y) = \lambda e^{-\lambda y}$

$F_X(y) = \int_0^y \lambda e^{-\lambda t} dt = \left[-e^{-\lambda t} \right]_0^y = 1 - e^{-\lambda y}$

Answer: $f_Y(y) = n[1 - F_X(y)]^{n-1} f_X(y)$.

4. Let $X_1, \ldots, X_n$ be independent random variables with the exponential distribution with mean λ, where $\lambda > 0$. Let $Y = \min(X_1, \ldots, X_n)$.

   (a) Find the density $f_Y(y)$. **Answer:** $\frac{n}{\lambda}e^{-ny/\lambda}$.

   (b) State the name of the distribution of Y. Don't forget to specify the parameters. **Answers:** $Y \sim$ Exponential(n/λ). Here $E(Y) = \lambda/n$.

   (c) Find $Var(2Y + 1)$. **Answer:** $4\lambda^2/n^2$.

Let $X_1, \ldots, X_n \sim Exp(1/\lambda)$ $\quad f(x) = \lambda e^{-\lambda x}$

$F(x) = 1 - \lambda e^{-\lambda x}$

(a) $f_Y(y) = n\left[1 - \left(1 - \frac{1}{\lambda}e^{-\frac{y}{\lambda}}\right)\right]^{n-1} \frac{1}{\lambda}e^{-\frac{y}{\lambda}}$

$$= n\left(\frac{1}{\lambda}e^{-\frac{y}{\lambda}}\right)^{n-1}\left(\frac{1}{\lambda}e^{-\frac{y}{\lambda}}\right)$$

$$= n\left(\frac{1}{\lambda}e^{-\frac{y}{\lambda}}\right)^{n} = \frac{n}{\lambda}e^{-\frac{ny}{\lambda}}$$

b) $Y \sim Exp\left(\frac{n}{\lambda}\right)$ with mean $\mu = \frac{\lambda}{n}$

$Var(2Y+1)$

Gamma function definition:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,dx$$

c) Find Var(2Y+1)

$$Var(2Y+1) = Var(2Y) = 4\,Var(Y)$$

$$E[Y^2] = \int_0^\infty Y^2 \frac{n}{\lambda} e^{-\frac{ny}{\lambda}} dy = \frac{n}{\lambda} \int_0^t Y^2 e^{-\frac{n}{\lambda}y} dy$$

$\lim t \to \infty$

Based on Gamma Function property

$$= \left(\frac{n}{\lambda}\right)^2$$

$$= 4 \frac{n^2}{\lambda^2} = \frac{4n^2}{\lambda^2}$$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

(c) Find $Var(2Y+1)$. **Answer:** $4\lambda^2/n^2$.

5. Let X_1 and X_2 be two independent random variables with the following common probability density function:

$$f_X(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}.$$

Let $Y = \max(X_1, X_2)$. Find $E\left(\frac{1}{Y}\right)$. **Answer:** $6/5$.

$Y = \max(X_1, X_2)$

$$E\left(\frac{1}{Y}\right) = \int_0^1 \frac{1}{Y} f_Y(y)\,dy = \int_0^1 6y^4\,dy$$

$$f_Y(y) = (2)(F_X(y))(f_X(y)) = \frac{6}{5}y^5 \Big|_0^1$$

$$= (2)(y^3)(3y^2) = 6y^5, \quad = \frac{6}{5}$$

$0 < y < 1$