Signal Analysis & Communication

ECE 355H1 F

ch 2.3:

Properties of LTI Systems (Contd.)

Lec 1, Wk 5

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Properties of LTI Sys. (Contd.)

⑦ LTI Stability:

Recap: BIBO stability: if $|x(t)| < \infty$ for all t, then $|y(t)| < \infty$ for all 't'.

For LTI, Suppose $|x(t)| < B$, $\forall t$, for some constant B, then

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right|$$

$$= \left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right| \quad \text{Convolution's Commutative property}$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)| |x(t-\tau)| d\tau \quad \text{Schwartz Inequality.}$$

- $|x(t-\tau)| < B$ for all values of 't' & 'τ'.

$$y(t) \leq B \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

- If $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$, then LTI sys. is stable

Conversely: Suppose $\int_{-\infty}^{\infty} h(\tau) d\tau = \infty$

Let $x(t) = \begin{cases} 0 & \text{if } h(-t) = 0 \\ \frac{h(-t)}{|h(-t)|} & \text{if } h(-t) \neq 0 \end{cases}$ — ①

basically sign of $h(-t)$

Eqn ① $\Rightarrow$ $|x(t)| \leq 1$, $\forall t$

Let's evaluate $y(0)$ [for simplicity - to prove]

$$y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(0-\tau)\, d\tau$$

Convolution's Commutative property

$$= \int_{-\infty}^{\infty} h(\tau) \frac{h(\tau)}{|h(\tau)|}\, d\tau$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \infty$$

Not stable although $|x(t)| < \infty$

Theorem: CT LTI Sys. $h(t)$ is BIBO stable iff $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$. ← "absolutely integratable"

Example 1

$$h(t) = \delta(t - t_0)$$

check →

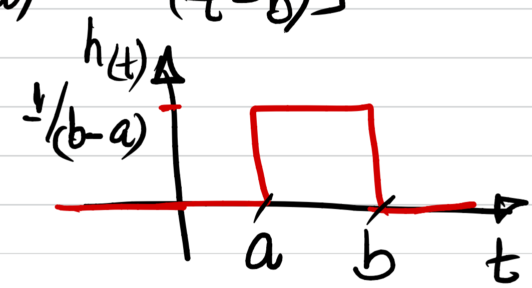
$$\int_{-\infty}^{\infty} |\delta(t - t_0)|\, dt = 1 \quad \text{area of an impulse}$$

$< \infty$

∴ BIBO stable

Example 2

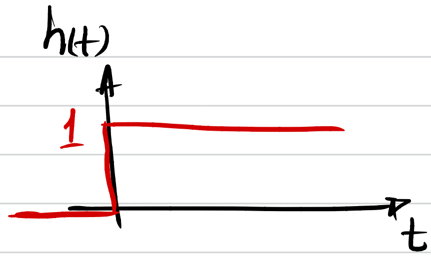
$$h(t) = \frac{1}{(b-a)} [u(t-a) - u(t-b)]$$



check → $\int_{a}^{b} \frac{1}{(b-a)}\, dt = 1 < \infty$

∴ BIBO stable

Example 3 $h(t) = u(t)$

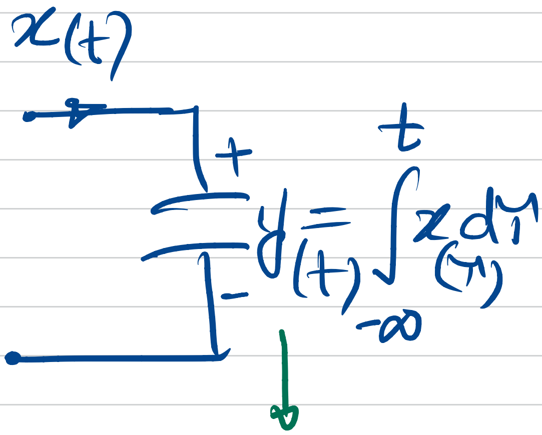


$$\int_0^{\infty} |1| \, dt = \infty$$

NOT STABLE

Practical Example

integrator $\int_{-\infty}^{t} x(\tau)\, d\tau$



impulse response

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

NOT STABLE!

as seen in example 3, when $h(t) = u(t) \rightarrow$ NOT STABLE

Similarly,

Theorem for DT LTI Sys.

For the DT LTI Sys. to be BIBO stable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

"absolutely summable"

(8) Unit Step Response of an LTI Sys.:

CT:

$$u(t) \rightarrow \boxed{h(t)} \rightarrow s(t)$$

unit step response

$$S(t) = u(t) * h(t)$$
$$= \int_{-\infty}^{t} h(\tau)\, d\tau$$

Convolution's Commutative property.

$\Rightarrow$ impulse response $h(t) = \frac{d}{dt} S(t)$, IR is diff. of unitstep response.

Similarly,

DT :

$$S[n] = \sum_{k=-\infty}^{n} h[k]$$

$$S[n] = \sum_{k=-\infty}^{\infty} u[k]\, h[n-k]$$
$$= \sum_{k=0}^{\infty} h[n-k]$$

$$h[n] = S[n] - S[n-1]$$

NOTE: 1 If $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$$x'(t) * h(t) = y'(t) \quad \text{where} \quad x'(t) = \frac{d}{dt} x(t)$$

LHS $= \int_{-\infty}^{\infty} h(\tau) \frac{d}{dt} x(t-\tau)\, d\tau$ Conv. Comm. prop.

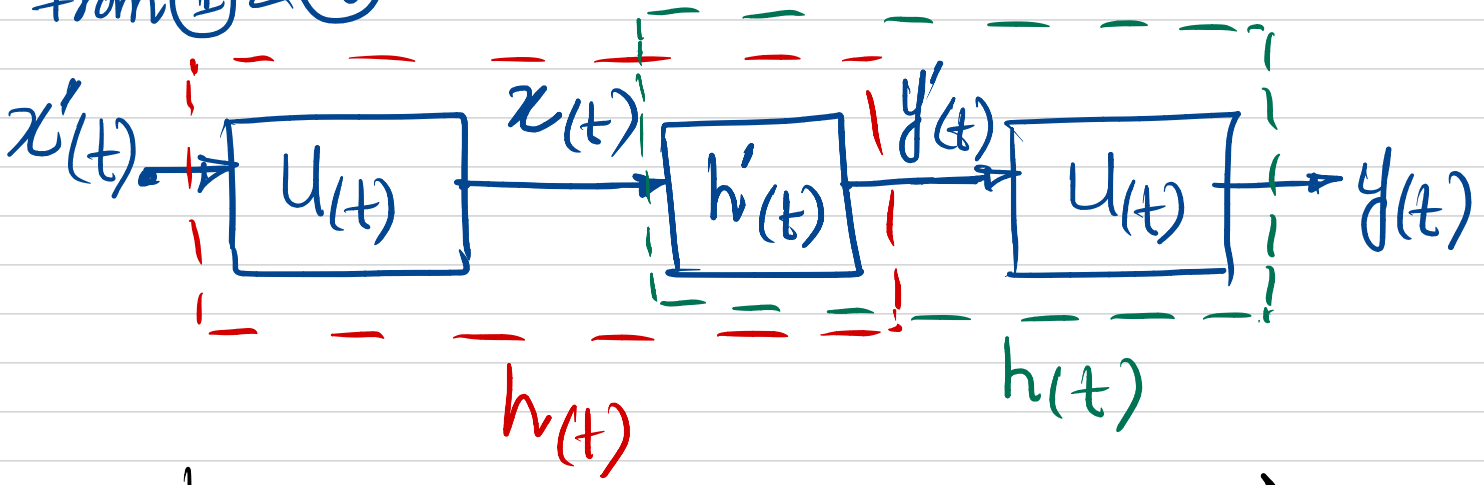
$$= \frac{d}{dt} \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \frac{d}{dt} y(t) = y'(t)$$

2. $u(t) * h'(t) = h'(t) * u(t)$ Conv. Comm. prop.

$$= \int_{-\infty}^{t} \frac{d}{d\tau} h(t-\tau)\, d\tau = h(t)$$

From 1 & 2



$$h(t) * u(t) = u(t) * h(t) = s(t)$$

Conclusions :

1. $x'(t) \longrightarrow \boxed{h(t)} \longrightarrow y'(t)$

2. $x(t) \longrightarrow \boxed{h'(t)} \longrightarrow y'(t)$

3. $\int_{-\infty}^{t} x(\tau)\,d\tau \longrightarrow \boxed{h'(t)} \longrightarrow y(t)$

4. $\int_{-\infty}^{t} x(\tau)\,d\tau \longrightarrow \boxed{h(t)} \longrightarrow \int_{-\infty}^{t} y(\tau)\,d\tau$

5. $x(t) \longrightarrow \boxed{\int_{-\infty}^{t} h(\tau)\,d\tau} \longrightarrow \int_{-\infty}^{t} y(\tau)\,d\tau$

Example: Given $x(t) = 2e^{-3t}u(t-1) \xrightarrow{h(t)} y(t)$

$$y'(t) = -3y(t) + e^{-2t}u(t) \quad — (1)$$

Find $h(t)$

$$x'(t) = -6e^{-3t}u(t-1) + 2e^{-3t}\delta(t-1)$$

$$= -3\left[2e^{-3t}u(t-1)\right] + 2e^{-3t}\delta(t-1)$$

$$= -3x(t) + 2e^{-3t}\delta(t-1) \quad — (2)$$

We know that

$$x'(t) \longrightarrow \boxed{h(t)} \longrightarrow y'(t)$$

$\therefore$ (2) $\Rightarrow$

$$y'(t) = \left[-3x(t) + 2e^{-3t}\delta(t-1)\right] * h(t)$$

$$= -3y(t) + \int_{-\infty}^{\infty} 2e^{-3\tau}\underbrace{\delta(\tau-1)}_{\text{exists at } \tau=1} h(t-\tau)\,d\tau$$

$$= -3y(t) + 2e^{-3}h(t-1)\cancelto{1}{\int_{-\infty}^{\infty}\delta(\tau-1)\,d\tau}$$

$$= -3y(t) + 2e^{-3}h(t-1) \quad — (3)$$

Comparing (1) & (3)

$$2e^{-3}h(t-1) = e^{-2t}u(t)$$

$$h(t-1) = \frac{1}{2}e^{-2t+3}u(t)$$

$$\rightarrow h(t) = \frac{1}{2}e^{-2(t+1)+3}u(t+1)$$

$$= \frac{1}{2}e^{-2t+1}u(t+1)$$