Signal Analysis & Communication

ECE355H1F

ch. 2.3

Properties of LTI Systems

Lec 2, Wk 4

28-09-2022

Properties of LTI Systems (ch. 2.3)

① Convolution is Commutative

CT: $x(t) * h(t) = h(t) * x(t)$

Proof:

LHS

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

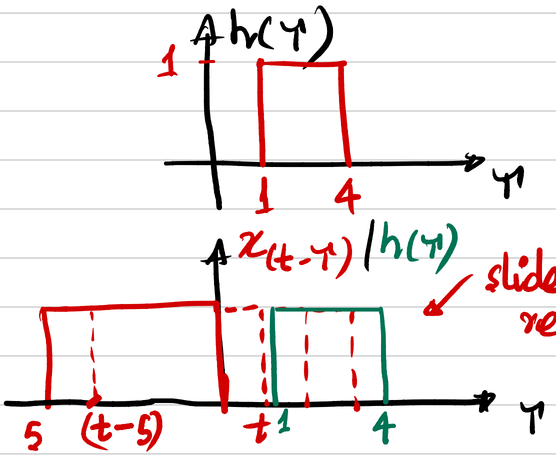
let $s = t - \tau$; $ds = -d\tau$; limits: $\infty \to -\infty$ $\Rightarrow$ ds Limits: $-\infty \to +\infty$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(t-s)\, h(s)\, ds$$

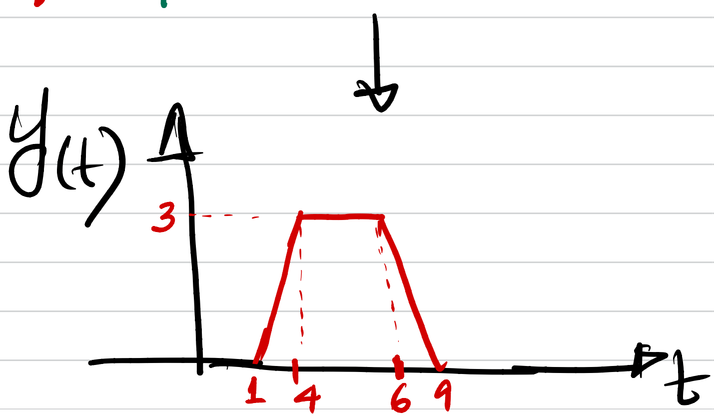
$$= \int_{-\infty}^{\infty} h(s)\, x(t-s)\, ds = \text{RHS}$$

DT: Similar

Example 1:
(previous Ex. 2)



slide 't' for different regions of overlap.



Example 2: $x(t) \rightarrow \boxed{\delta(t)} \rightarrow x(t)$

$$\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

identity system

By property,

$\delta(t) \rightarrow \boxed{x(t)} \rightarrow x(t)$

$$\int_{-\infty}^{+\infty} \delta(\tau)\, x(t-\tau)\, d\tau$$

exist at $\tau = 0$

$\Downarrow = x(t)$

② Convolution is Associative

CT: $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$

Proof LHS

$$= \Big[\underbrace{\int_{-\infty}^{+\infty} x(\tau)\, h_1(t-\tau)\, d\tau}_{x_1(t)}\Big] * h_2(t)$$

$$= \int_{-\infty}^{+\infty} \Big[\int_{-\infty}^{+\infty} x(\tau)\, h_1(s-\tau)\, d\tau\Big]\, h_2(t-s)\, ds$$

$$= \int_{-\infty}^{+\infty} x(\tau) \Big[\int_{-\infty}^{+\infty} h_1(s-\tau)\, h_2(t-s)\, ds\Big] d\tau \quad \text{— ①}$$

RHS

$$= \int_{-\infty}^{+\infty} x(\tau)\, \underbrace{[h_1 * h_2](t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau) \Big[\int_{-\infty}^{+\infty} h_1(\tau')\, h_2(t-\tau-\tau')\, d\tau'\Big] d\tau$$

let $\tau' = s - \tau$, $d\tau' = ds$, limits: same

$$= \int_{-\infty}^{+\infty} x(\tau) \Big[\int_{-\infty}^{+\infty} h_1(s-\tau)\, h_2(t-s)\, ds\Big] d\tau \quad \text{— ②}$$

NOTE:

$$x(t) * h(t) = (x * h)(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$(x * h)(t-1) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-1-\tau)\, d\tau$$

$$\neq x(t-1) * h(t-1)$$

① = ②

systems in series

$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t) \quad \longleftrightarrow \quad x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$

NOTE: 1. We can simply write $x(t) * h_1(t) * h_2(t)$

2. Commutative + Associative

$$x(t) * h_1(t) * h_2(t) = x(t) * h_2(t) * h_1(t)$$

3. Extend to any number of sys. in series.

$$x(t) * h_1(t) * h_2(t) * \cdots\cdots * h_n(t) = y(t)$$

DT: Similar

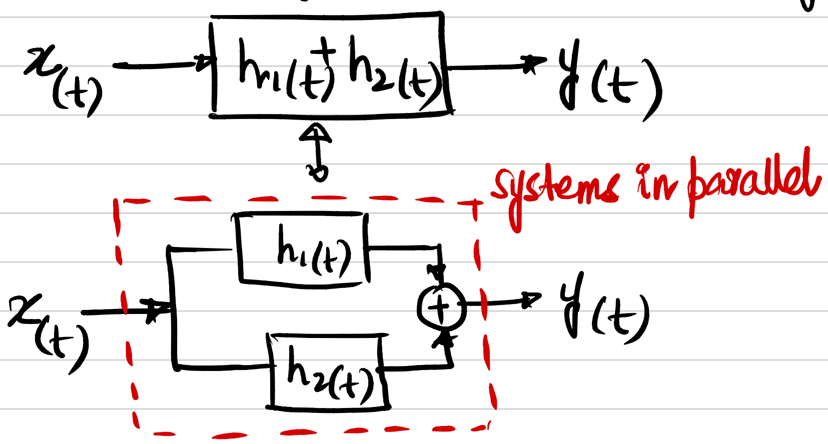
③ Convolution is Distributive:

CT: $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

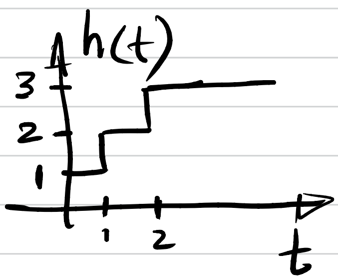
Proof

I. Using eqn. LHS = RHS, $\int_{-\infty}^{+\infty} x(\tau) \cdots d\tau = \cdots$

II. View $x(t)$ as sig. & $[h_1(t) + h_2(t)]$ as sys.



Example $h(t) = u(t) + u(t-1) + u(t-2)$



$x(t) = e^{-at} u(t)$

Find $y(t) = x(t) * h(t)$

$x(t) * u(t) = \frac{1}{a}(1 - e^{-at}) u(t)$ (previous Ex.)

$x(t) * u(t-1) = \frac{1}{a}(1 - e^{-a(t-1)}) u(t-1)$

$x(t) * u(t-2) = \frac{1}{a}(1 - e^{-a(t-2)}) u(t-2)$

So,

$$x(t)*h(t) = \frac{1}{a}\left[(1-e^{-at})u(t) + (1-e^{-a(t-1)})u(t-1) \cdots\right]$$