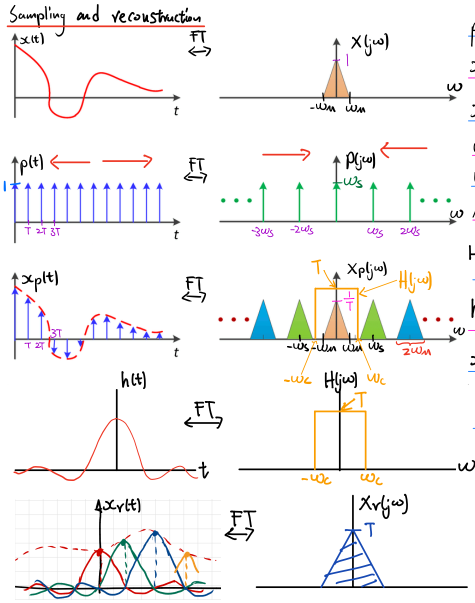
Sampling and reconstruction



$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$

$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$

$\omega_M < \omega_c < \omega_s - \omega_M$

$\omega_c = \frac{\omega_s}{2} = \frac{\pi}{T}$ usually

Nyquist rate: $2\omega_M$

$H(j\omega) = \begin{cases} T & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$

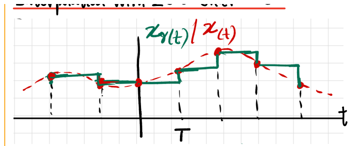
$h(t) = \frac{T\sin(\omega_c t)}{\pi t}$

$x_r(t) = x_p(t) * h(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t-nT)$

$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\omega_c T}{\pi}\frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}$ — cut off frequency $\omega_c = \frac{\omega_s}{2}$

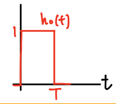
band limited interpolation

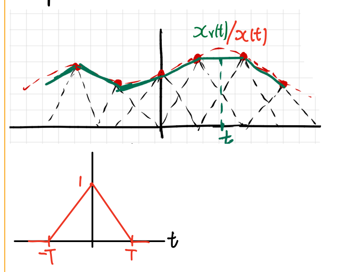
Interpolation with zero order hold



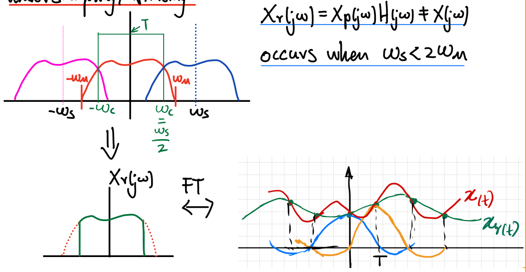
$h_0(t) = \begin{cases} 1, & 0<t<T \\ 0, & \text{otherwise} \end{cases}$

interpolation with first order hold





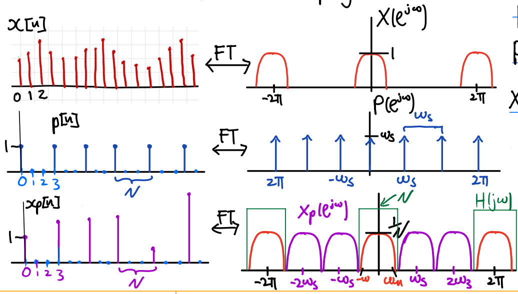
under sampling / Aliasing



$X_r(j\omega) = X_p(j\omega)H(j\omega) \neq X(j\omega)$

occurs when $\omega_s < 2\omega_M$

Discrete Sampling

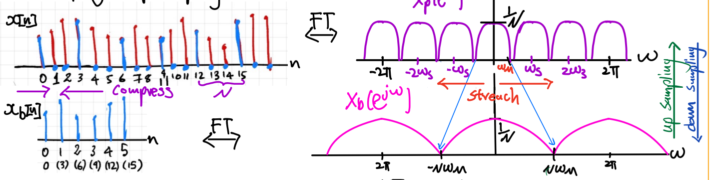


$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$

$P(e^{j\omega}) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$

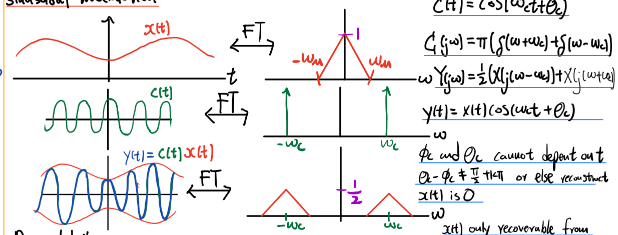
$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})$

Down sampling / upsampling



downsampling: $x_b[n] = x_p[nN] = x[nN] \xleftrightarrow{FT} X_b(e^{j\omega}) = X_p(e^{j\omega/N})$

Sinusoidal modulation



$c(t) = \cos(\omega_c t + \theta_c)$

$C(j\omega) = \pi(\delta(\omega+\omega_c) + \delta(\omega-\omega_c))$

$Y(j\omega) = \frac{1}{2}(X(j(\omega-\omega_c)) + X(j(\omega+\omega_c)))$

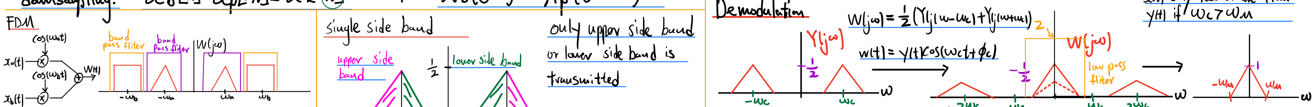
$y(t) = x(t)\cos(\omega_c t + \theta_c)$

ϕ_c and θ_c cannot depart out

$\theta_c - \phi_c \neq \frac{\pi}{2} + k\pi$ or else reconstruct $x(t)$ is 0

$x(t)$ only recoverable from $y(t)$ if $\omega_c > \omega_M$

FDM



Single side band

only upper side band or lower side band is transmitted

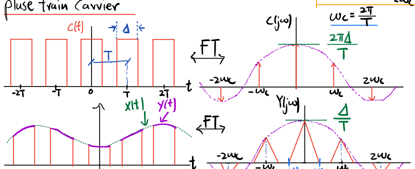
Demodulation

$W(j\omega) = \frac{1}{2}(Y(j(\omega-\omega_c)) + Y(j(\omega+\omega_c)))$

$w(t) = y(t)\cos(\omega_c t + \phi_c)$

low pass filter

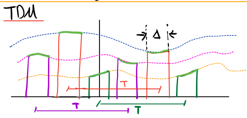
plse train carrier



$\omega_c = \frac{2\pi}{T}$, $\frac{2\pi\Delta}{T}$, $\frac{\Delta}{T}$

$x(t)$ only recoverable if $\omega_c > 2\omega_M$

TDM



$\uparrow T \downarrow \Delta \Rightarrow$ more signal can be transmitted

$\omega_c = \frac{2\pi}{T} > 2\max(\omega_{M_1}, \omega_{M_2}, \dots)$

$\frac{T}{\Delta}$ = max # of signal to fit in a one period

Angle modulation

Carrier signal: $c(t) = A_c\cos(\omega_c t)$

modulated signal: $y(t) = A_c\cos(\theta(t))$

phase modulation: $\theta(t) = \omega_c t + k_p x(t)$

freq. modulation: $\theta(t) = \omega_c t + k_f\int x(t)dt$

$|y(t)|^2$ transmission power is independent of $x(t)$

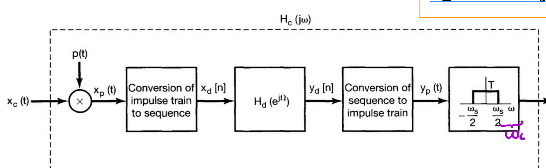
$\frac{1}{T_0}\int_{T_0} |y(t)|^2 dt \approx \frac{1}{2}A_c^2$ ← Average power (any period)

Integration by parts

$\int_a^b t e^{ct} dt = \left(\frac{t}{c}e^{ct} - \frac{1}{c^2}e^{ct}\right)\Big|_a^b$

geometric series

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ $\quad \sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}$



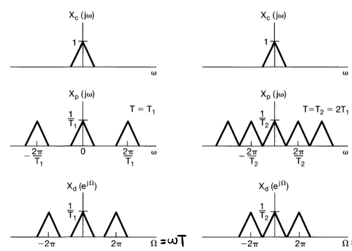
$H_d(e^{j\Omega}) = H_c(j\omega/T)$ ← repeated with 2π

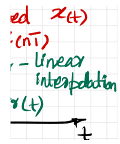
expansion on x axis (multiply by T)

$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T)$

$x_d[n] = X_c(nT) \xleftrightarrow{FT} X_d(e^{j\Omega}) = X_p(j\omega/T)$

compression on the x axis $n = \frac{t}{T}$ $\quad$ expansion on the x axis $\Omega = \omega T$





Even odd decomposion

$Ev[x(t)] = \frac{1}{2}[x(t) + x(-t)]$

$od[x(t)] = \frac{1}{2}[x(t) - x(-t)]$

$x(t) = Ev[x(t)] + od[x(t)]$

odd signal must be zero at $t=0$

Periodic signal

$x(t)$ is periodic $\Leftrightarrow x(t) = x(t+T)$, $T = \frac{2\pi}{\omega_0}$ $T, \omega \in \mathbb{R}$

$x[n]$ is periodic $\Leftrightarrow x[n] = x[n+N]$, $N = \frac{2\pi}{\omega_0}M$ $N, M \in \mathbb{Z}$

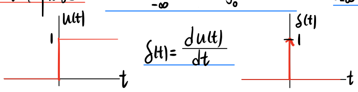
- for continuous ω_0 can be anything
- for discrete $0 \le \omega_0 < 2\pi$

Continuous unit step and unit impulse

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{0}^{\infty}\delta(t-\tau)d\tau \qquad \int_{-\infty}^{\infty}\delta(t)dt = 1$$

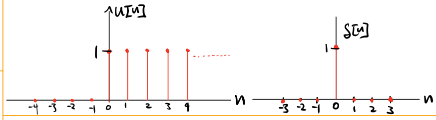
$$\delta(t) = \frac{du(t)}{dt}$$



unit step and unit impulse

$\delta[n] = u[n] - u[n-1]$

$$u[n] = \sum_{m=-\infty}^{n}\delta[m] = \sum_{k=0}^{\infty}\delta[n-k]$$



Sampling property of $\delta[n]$

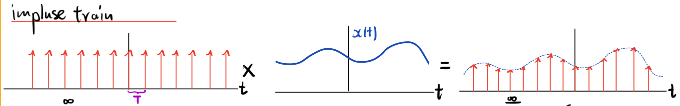
$x[n]\delta[n] = x[0]\delta[n]$

$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$

Sampling property of $\delta(t)$

$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

impulse train



$$x_\delta(t) = \sum_{n=-\infty}^{\infty}\delta(t-Tn) \qquad x(t) \qquad x(t)x_\delta(t) = \sum_{n=-\infty}^{\infty}x(nT)\delta(t-Tn)$$

System properties

① memoryless

if the output of a system at t_0 only depends on the input at t_0

② with memory

presence of a mechanism in the sys. that retains or stores info. about I/P values at non-current time (future or past)

③ invertibility

If distinct inputs leads to distinct outputs all the time

④ Causality

A system is causal if the output at any time depends only on values of the input at present time and/or in the past.

⑤ stability

A sys. is BIBO stable if bounded I/P signals $|x(t)|<\infty$, $\forall t$, leads to a bounded o/p signal $|y(t)|<\infty, \forall t$

⑥ time invariance

if $x(t) \xrightarrow{S} y(t)$ then $x(t-t_0) \xrightarrow{S} y(t-t_0)$

$S\{x(t-t_0)\} = S\{x(t)\}(t-t_0) = y(t-t_0)$

⑦ linearity

if $S\{x_1(t)\} = y_1(t)$ and $S\{x_2(t)\} = y_2(t)$ then

$S\{ax_1(t) + x_2(t)\} = aS\{x_1(t)\} + S\{x_2\} = ay_1(t) + y_2(t)$

Convolution

for any LTI system, the output $y[n]/y(t)$ can be written as:

$$y[n] = \sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

impulse responds $h(t) = S\{\delta(t)\}$

$$y(t) = \int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

Properties of LTI sys./Convolution

① Convolution is commutative

$x(t) * h(t) = h(t) * x(t)$

② Convolution is Associative

③ Convolution is distributive

$x(t)*[h_1(t) + h_2(t)] = x(t)*h_1(t) + x(t)h_2(t)$

④ LTI memory

LTI sys. is memoryless $\Leftrightarrow$ $h(t) = k\delta(t)$ and $y(t) = kx(t)$

⑤ LTI invertibility

if an LTI system is invertible then its inverse system is also LTI

$h(t) * h^{I}(t) = \delta(t)$

⑥ LTI Causality

A CT LTI sys. is causal $\Leftrightarrow h(t) = 0$ for $t<0$

① LTI Stability

$$\int_{-\infty}^{\infty}|h(t)|dt \le \int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty \Leftrightarrow \text{LTI sys. is stable}$$

$$\left|\sum_{n=-\infty}^{\infty}h[n]\right| \le \sum_{n=-\infty}^{\infty}|h[n]| < \infty \Leftrightarrow \text{LTI sys. is stable}$$

if $x(t) \to \boxed{h(t)} \to y(t)$ then

① $x'(t) \to \boxed{h(t)} \to y'(t)$

② $x(t) \to \boxed{h'(t)} \to y'(t)$

③ $\int_{-\infty}^{t} x(\tau)d\tau \to \boxed{h'(t)} \to y(t)$

④ $\int_{-\infty}^{t} x(\tau)d\tau \to \boxed{h(t)} \to \int_{-\infty}^{t} y(\tau)d\tau$

⑤ $x(t) \to \boxed{\int_{-\infty}^{t} h(\tau)d\tau} \to \int_{-\infty}^{t} y(\tau)d\tau$

6. multiplication

$$x(t)y(t) \xleftrightarrow{FS} c_k = \sum_{l=-\infty}^{\infty}a_l b_{k-l}$$

7. Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|a_k|^2$$

Fourier series for perdic func.

$$x(t) = \sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_0 t}dt$$

integrate over any period

FT of Periodic Signal (DT and CT)

$$X(j\omega) = \sum_{k=-\infty}^{\infty}2\pi a_k \delta(\omega - k\omega_0)$$

Find a_k and ω_0

Discrete fourier Series

Sythesis

$$x[n] = \sum_{k=\langle N\rangle}a_k e^{jk\omega_0 n}$$

Analysis

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle}x[n]e^{-jk\omega_0 n}$$

Solving LCCDE (find y(t)) using FT (Continuous) for a given x(t)

① Find impulse responds in frequency domain ($H(j\omega)$)

set $x(t) = \delta(t)$ and $y(t) = h(t)$

take the fourier transform of both sides

Solve for $H(j\omega)$

② use partial fraction to find $h(t) = F^{-1}\{H(j\omega)\}$

③ find $X(j\omega)$ and find $H(j\omega)X(j\omega)$

④ $y(t) = h(t) * x(t) = \underbrace{F^{-1}\{H(j\omega)X(j\omega)\}}_{\text{use partial fraction}}$

Partial fraction

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
	• where $x^2 + bx + c$ cannot be factorised further	

time	CTFT	CTFS
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty}\frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k

$$a + jb = \sqrt{a^2+b^2}\, e^{\tan^{-1}\left(\frac{b}{a}\right)}$$

Properties of F.S.

1. linearity $Ax(t) + By(t) \xleftrightarrow{FS} Aa_k + Bb_k$
2. time shifting $x(t-t_0) \xleftrightarrow{FS} a_k e^{-jk\omega_0 t_0}$
3. time scaling For $\alpha>0$, $x(\alpha t) \xleftrightarrow{FS} a_k$, with fun. fre. $\alpha\omega_0$
4. time Reversal $x(-t) \xleftrightarrow{FS} a_{-k}$
   - I. $x(t)$ is even $\Leftrightarrow a_k = a_{-k}$ even a_k
   - II. $x(t)$ is odd $\Leftrightarrow a_k = -a_{-k}$ odd a_k
5. Conjugation $x^*(t) \xleftrightarrow{FS} a^*_{-k}$
   - I. $x(t)$ is real $\Leftrightarrow a_{-k} = a_k^*$
   - II. $x(t)$ is real and even $\Leftrightarrow a_{-k} = a_k = a_k^*$ real & even a_k
   - III. $x(t)$ is real and odd $\Leftrightarrow a_k = -a_{-k} = -a_k^*$ purely Im & odd a_k

DTFT properties

Differencing

$x[n] - x[n-k] \xleftrightarrow{F} (1 - e^{-j\omega k})X(e^{j\omega})$

Conjugation

$x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})$

$X(e^{j\omega}) = X^*(e^{-j\omega}) \Leftrightarrow x[n]$ is real

$x[n]$ is real and even $\Leftrightarrow X(e^{j\omega})$ is purely real

$x[n]$ is real & odd $\Leftrightarrow X(e^{j\omega})$ is purely Im

duality between DTFT and CTFS

$x[n] \xleftrightarrow{FT} X(e^{j\omega}) \xleftrightarrow{FS} x[-n]$

$\operatorname{sinc}(\theta) = \frac{\sin\pi\theta}{\pi\theta}$

sinc function is even

$\operatorname{sinc}(0) = 1$

$f(x) = \frac{\sin a\pi x}{\pi x} \Rightarrow f(0) = a$

Properties of CTFT

Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ purely real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even " Im " "	$X(j\omega)$ real and even " Im " "
Symmetry for Real and Odd Signals	$x(t)$ real and odd " Im " "	$X(j\omega)$ purely imaginary and odd " " Re " "
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

$x(t)$ is not even and odd $\Leftrightarrow X(j\omega)$ is not purely Re and purely Im

time	CTFT	CTFS
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise

LTI and Complex exponentials

$x(t) \to \boxed{h(t)} \to y(t)$

Continuous

if $x(t) = e^{st}$ & $H(s)$ exists then $y(t) = H(s)e^{st}$ (eigenvalue, eigen. func.)

$$H(s) = \int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau$$

if $x(t) = \sum_k a_k e^{s_k t}$ then $y(t) = \sum_k a_k H(s_k)e^{s_k t}$

Discrete

if $x[n] = e^{sn}$ & $H(s)$ exists then $y[n] = H(s)e^{sn}$

$$H(s) = \sum_{k=-\infty}^{\infty}h[k]e^{sk}$$