Signal Analysis & Communication

ECE355H1 F

ch 2.4

Linear Constant Coefficient Differential/
Difference Equations (LCCDE)

Lec 2, WK 5
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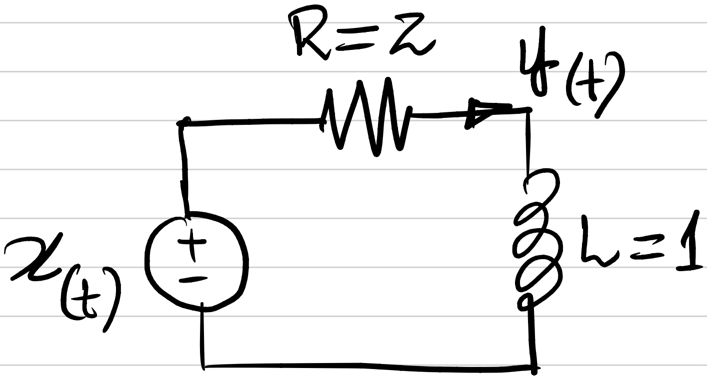
Linear Constant Coefficient Differential/ Difference Equations LCCDE (ch. 2.4)

- I/P & O/P are related through LCCDE. ← Implicit specification
- LCCDE describe wide variety of systems & physical phenomenon.
- Eg. RC, RLC systems, mech. systems, chemical reactions etc.
- Can solve LCCDE to find explicit expression between I/P & O/P

<u>Example</u>



$$L\frac{dy(t)}{dt} + Ry(t) = x(t)$$

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

} Implicit

To find O/P explicitly solving this LCCDE for given conditions.

Eg. Given $x(t) = e^{3t}u(t)$ and $y(t)=0$ for $t<0$, find $y(t)$ for $t>0$.

From calculus course, we know:

$$y(t) = y_h(t) + y_p(t) \quad \text{— (A)}$$

$y_{h(t)}$: homogeneous solution, a solution of

$$\frac{d}{dt}y_{(t)} + 2y_{(t)} = 0 \quad — (1)$$ "Natural Response"

"Unforced Response"

$y_{p(t)}$: particular solution, a special solution due to the I/p

"Forced Response"

→ To find $y_{h(t)}$, guess the solution of the form:

$$y_{h(t)} = Ae^{st}, \quad \text{for all 't'}$$

$$(1) \Rightarrow \quad Ase^{st} + 2Ae^{st} = 0$$

$$Ae^{st}(s+2) = 0$$

$$s+2 = 0$$

$$s = -2$$

$$\therefore y_{h(t)} = Ae^{-2t} \text{ (for any A)} \quad — (2)$$

→ To find $y_{p(t)}$, guess

$$y_{p(t)} = Be^{3t} \quad \text{for } t>0$$

a sig. of the same form as the I/p

$$3Be^{3t} + 2Be^{3t} = e^{3t}$$

$$3B + 2B = 1$$

$$B = 1/5$$

$$\therefore y_{p(t)} = \frac{1}{5}e^{3t} \quad \text{for } t>0 \quad — (3)$$

Substituting (2) & (3) into (A)

$$y_{(t)} = Ae^{-2t} + \frac{1}{5}e^{3t} \quad — ④$$

To determine A, we use the initial condition $y_{(0)}=0$

$$④ \Rightarrow \quad 0 = A + \frac{1}{5}$$

$$A = -1/5$$

Hence, $$y_{(t)} = -\frac{1}{5}e^{-2t} + \frac{1}{5}e^{3t} \qquad t>0$$

<u>NOTE:</u> Suppose the initial condition was $y_{(0)}=1$, then

$$1 = A + \frac{1}{5}$$

$$A = \frac{4}{5}$$

$$④ \Rightarrow \quad y_{(t)} = \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t}, \; t>0 \quad — ⑤$$

But in this case the <u>initial rest condition</u> is not satisfied, which is, $y_{(t)}=0$ for $t<t_0$ if $x_{(t)}=0$ for $t<t_0$, $\forall t, t_0$.

For <u>Causal LTI Systems</u>, we consider in this course, initial rest condition should be satisfied!

Sys. in ⑤ is either not causal or not an LTI.

★ Causality also ensures time invariancy of LTI!
That is, with zero initial values, the response at different times will be the same (with the same I/p) [assuming R and C donot change over time).

General Form of LCCDE (CT)

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

* The considered Causal LTI Systems provides close approximation to most LTI Systems. [initial rest condition]

Standard Solution:

$$y(t) = y_{h}(t) + y_{p}(t)$$

$y_h(t)$: Use roots of characteristic Eqn.

$y_p(t)$: Guess →

$x(t)$	$y_p(t)$
e^{st}	Be^{st}
$\cos(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$
⋮	⋮

Auxilliary conditions: (Initial conditions-Rest)

$$y(t_0) = \frac{d}{dt} y(t_0) = \cdots = \frac{d^{N-1}}{dt^{N-1}} y(t_0) = 0$$

(assume continuity of $y(t)$ & its derivatives)

t_0: time when I/Ps become active.

* We will see later - solving LCCDE is more straight forward in freq. domain.

Property of system
↳ BIBO stable?
↳ Causal is $h(t) = 0$ for $t < 0$?
↳

DT LCCDE:

$$\frac{d}{dt} \rightarrow d^1 y[n] \triangleq y[n] - y[n-1]$$

$$\frac{d^2}{dt^2} \rightarrow d^2 y[n] \triangleq \underbrace{(y[n] - y[n-1])}_{dy[n]} - \underbrace{(y[n-1] - y[n-2])}_{dy[n-1]}$$

$$= y[n] - 2y[n-1] + y[n-2] \quad \text{— (B)}$$

$$\vdots$$

$$d^k y[n] = d^k y[n] - d^{k-1} y[n-1]$$

DT analogous of CT LCCDE:

$$\hat{a}_0 y[n] + \sum_{k=1}^{N} \hat{a}_k d^k y[n] = \hat{b}_0 x[n] + \sum_{k=1}^{M} \hat{b}_k d^k x[n]$$

Equivalenty, (as (B) is written)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The differential order for input is not nessarily the same as the output

Solution Options:

(1) similar to CT case

$$y[n] = y_h[n] + y_p[n]$$

Assume initial rest conditions for Causal LTI i.e., if $x[n] = 0$ for all $n < n_0$, then $y[n] = 0$ for $n < n_0$, $\forall n, n_0$.

② $a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right]$$

Compute recursively!

Ex. 2.15 $y[n] - \frac{1}{2} y[n-1] = x[n]$

$$y[n] = x[n] + \frac{1}{2} y[n-1]$$

y[n] depends on y for past values

$$x[n] = K\delta[n]$$

since $x[n]=0$ for $n \le -1$

$\Rightarrow y[n]=0$ for $n \le -1$

↳ initial rest condition assumption.

$$y[0] = x[0] + \frac{1}{2}\underbrace{y[-1]}_{=0} = K$$

$$y[1] = x[1] + \frac{1}{2} y[0] = \frac{1}{2}K$$

$$y[2] = x[2] + \frac{1}{2} y[1] = \frac{1}{4}K$$

$\vdots \qquad \vdots \qquad \vdots$

$$y[n] =$$

Check that!

$$\frac{dx(t)}{dt} \rightsquigarrow x[n] - x[n-1]$$

$$\frac{dx(t-1)}{dt} \rightsquigarrow x[n-1] - x[n-2]$$

focus on LTI system.

prepared formula sheet for T2
self formula sheet for FE