Signal Analysis & Communication
ECE355H1F

ch. 3.5

Properties of Fourier Series

Lec 2, Wk 7
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Properties of Fourier Series (ch. 3.5)

Notation: $x(t)$ periodic
fundamental period T
fundamental freq. $\omega_0 = 2\pi/T$

$$x(t) \xleftrightarrow{FS} a_k$$

① Linearity:

If $y(t)$ has fundamental period T &

$$y(t) \xleftrightarrow{FS} b_k$$

then

$$Ax(t) + By(t) \longleftrightarrow Aa_k + Bb_k$$

Proof:

$$LHS = A\left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right) + B\left(\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}\right)$$

$$= \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k)\, e^{jk\omega_0 t}$$

② Time Shifting:

$$x(t-t_0) \xleftrightarrow{FS} a_k e^{-jk\omega_0 t_0}$$

No change to the magnitude of the FS coefficient

Proof:

Finding FS coefficient for $x(t-t_0)$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

$\tau = t - t_0$

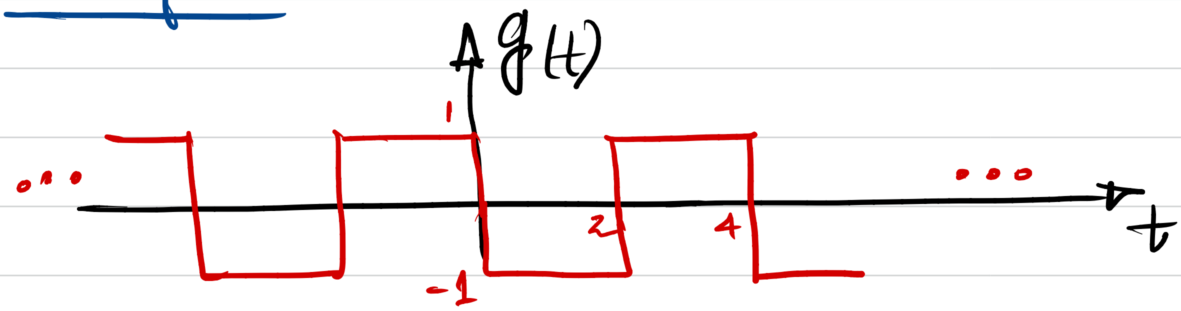
$$= \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau+t_0)}\, d\tau$$

unchanged as it is over any period!

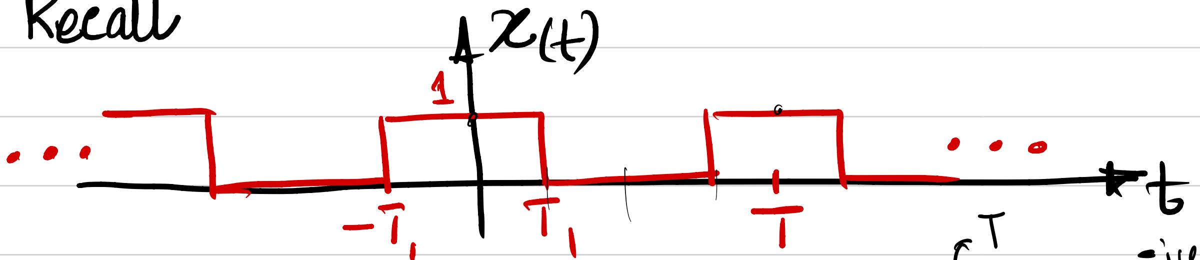
$$= e^{-jk\omega_0 t_0} \underbrace{\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau}_{a_k}$$

$$= a_k\, e^{-jk\omega_0 t_0}$$

Example:



Recall



$T_1 = T/4$

$$a_k = \int_0^T x(t)\, e^{-j\omega_0 k t} dt$$

$$= \int_0^{T/4} e^{-j\omega_0 k t} dt + \int_{3T/4}^{T} e^{-j\omega_0 k t} dt$$

$\updownarrow$ FS

$$a_k = \begin{cases} \dfrac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}, & k \neq 0 \\ 1/2, & k = 0 \end{cases}$$

Relating $g(t)$ to $x(t)$: Let $T = 4 \Rightarrow \omega_0 = \pi/2$

$$g(t) = \underbrace{2x(t-1)}_{\mathrm{I}} \underbrace{-1}_{\mathrm{II}} \xleftrightarrow{FS} c_k \quad \text{—} \quad (1)$$

I. $\quad 2x(t-1) \xleftrightarrow{FS} 2e^{-jk\pi/2} a_k$ (using time shift property)

II. $\quad -1 \xleftrightarrow{FS} \begin{cases} -1 , & k=0 \\ 0 , & k\neq 0 \end{cases}$ (using FS formula $\underline{a_k}$)

I & II $\to$ ①

$$\Rightarrow \begin{cases} c_0 = 2a_0 - 1 = 0 \; ; & k=0 \\ c_k = 2e^{-jk\pi/2} \dfrac{\sin k\pi/2}{k\pi} \; ; & k\neq 0 \end{cases}$$

③ Time Scaling:

For $\alpha > 0$, $x(\alpha t) \xleftrightarrow{FS} a_k$, with fundamental freq. $\alpha\omega_0$

Proof:

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(\alpha t)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

Fundamental freq. now is $\alpha\omega_0$

④ Time Reversal:

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

Proof:

$$x(-t) = \sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0(-t)}$$

$$x(-t) \overset{m=-k}{=} \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t}$$

Special Cases:

I. $x(t)$ – even

$\Rightarrow \quad x(t) = x(-t)$

$\Rightarrow \quad a_k = a_{-k}$

$\Rightarrow \quad$ even a_k

II. $x(t)$ – odd

$\Rightarrow \quad x(t) = -x(-t)$

$\Rightarrow \quad a_k = -a_{-k}$

$\Rightarrow \quad$ odd a_k

⑤ Conjugation:

$$x^*(t) \overset{FS}{\longleftrightarrow} a^*_{-k}$$

Proof:

$$x^*(t) = \left(\sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t} \right)^*$$

$$= \sum_{m=-\infty}^{\infty} a^*_m e^{-jm\omega_0 t}$$

$$\overset{k=-m}{=} \sum_{k=-\infty}^{\infty} \underline{a^*_{-k}} e^{jk\omega_0 t}$$

Special Cases:

I. $x(t)$ - real

$$x(t) = x^*(t)$$

$$\Rightarrow a_k = a^*_{-k}$$

$$\Rightarrow \boxed{a_{-k} = a_k^*}$$ "Conjugate Symmetric" (Hermitian)

FS for this case:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + \underbrace{a_{-k} e^{-jk\omega_0 t}}_{=(a_k e^{jk\omega_0 t})^*}\right)$$

↳ imaginary part cancels.

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\{a_k e^{jk\omega_0 t}\} \quad \text{—(I)}$$

Suppose $a_k = A_k e^{j\theta_k}$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

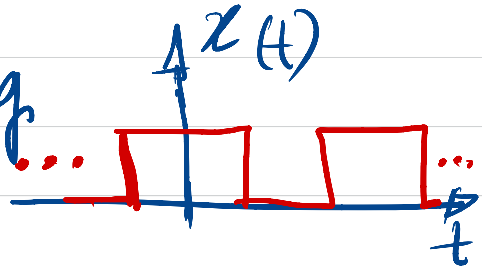
$$\underbrace{= a_0 + \sum_{k=1}^{\infty} 2A_k \cos(k\omega_0 t + \theta_k)}_{\text{use when } x(t) \text{ is real valued func.}}$$

II. $x(t)$ - real & even

$$\Rightarrow a_k = a_{-k} = a_k^*$$

$$\boxed{\text{real \& even } \{a_k\}}$$

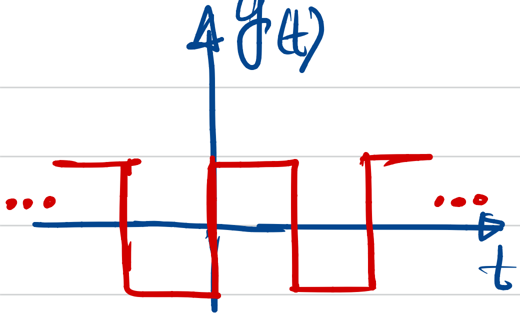
eg.



III. $x(t)$ – real & odd

$$\Rightarrow \quad a_k = -a_{-k} = -a_k^*$$

$$\boxed{\text{purely imaginary \& odd } \{a_k\}}$$



* Encourage you to check using (I)

⑥ Multiplication:

If $\quad x(t) \xleftrightarrow{FS} a_k \quad \& \quad y(t) \xleftrightarrow{FS} b_k$

$$x(t)\,y(t) \xleftrightarrow{FS} c_k$$

where $\displaystyle c_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$

Proof:

$$x(t)\,y(t) = \left(\sum_{l=-\infty}^{\infty} a_l\, e^{jl\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} a_m\, e^{jm\omega_0 t}\right)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_l\, b_m\, e^{j(l+m)\omega_0 t}$$

$$\overset{k=l+m}{=} \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty} a_l\, b_{k-l}\right) e^{jk\omega_0 t}$$

⑦ Parseval's Relation:

If $x(t) \xleftrightarrow{FS} a_k$

then $$\boxed{\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$

Parseval's Power Theorem

Proof:

From Conjugation property ⑤

$$x^*(t) \longleftrightarrow a^*_{-k}$$

From multiplication property ⑥

$$\underbrace{x(t)\,x^*(t)}_{=|x(t)|^2} \longleftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$= \sum_{l=-\infty}^{\infty} a_l a^*_{l-k}$$

$$\text{Average of } |x(t)|^2 = c_0 = \sum_{l=-\infty}^{\infty} a_l a_l^*$$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{l=-\infty}^{\infty} |a_l|^2$$

NOTE: Average Power of the kth component

$$= \frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt$$

$$= |a_k|^2$$

⑨ Other properties: Table 3.1 (textbook)