Signal Analysis & Communication

ECE355H1F

Ch. 2.3

Properties of LTI Systems (Contd.)

Lec 3, Wk 4

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Properties of LTI Systems (Contd.)

④ LTI Memory

Recap: A sys. is memoryless iff for all t_0, o/P $y(t_0)$ only depends on $x(t_0)$ (at the same time).

$$y(t_0) = \int_{-\infty}^{+\infty} x(\tau)\, h(t_0-\tau)\, d\tau$$

memoryless $\Leftrightarrow$ $h(t_0-\tau) = 0$ for all $\tau \neq t_0$

OR $h(t) = 0$ for $t \neq 0$

$$\Rightarrow \quad y(t_0) = \int_{-\infty}^{+\infty} x(t_0)\, h(t_0-\tau)\, d\tau$$

$$= x(t_0) \underbrace{\int_{-\infty}^{+\infty} h(t_0-\tau)\, d\tau}$$

$\rightarrow K = \int_{-\infty}^{+\infty} h(\tau)\, d\tau$

$$= K\, x(t_0)$$

Impulse Response: $x(t) = \delta(t)$

$$y(t) = K\,\delta(t)$$

$$h(t) = K\,\delta(t)$$ Response when I/P is $\delta(t)$

Theorem: (CT) LTI sys. is memoryless iff its impulse response, $h(t) = K\delta(t)$ for some constant K.

(In that case, $y(t) = Kx(t)$ for all $x(t)$)

DT: $h[n] = K\,x[n]$

⑤ LTI Invertibility

Recap: A sys. is invertible iff every output $y(t)$ corresponds to a unique input $x(t)$.

OR A sys. whose inverse sys. exists.

$$x(t) \longrightarrow \boxed{S} \xrightarrow{\;y(t)\;} \boxed{S^{-1}} \longrightarrow x(t)$$

Theorem: If an LTI sys. is invertible, then its inverse sys. is also LTI.

Proof: Suppose $x(t) \longrightarrow \boxed{h(t)} \xrightarrow{\;y(t)\;} \boxed{S^{-1}} \longrightarrow x(t)$

Linear:

If $x_1(t) * h(t) = y_1(t)$ (and $y_1(t) \xrightarrow{S^{-1}} x_1(t)$)

$x_2(t) * h(t) = y_2(t)$ (and $y_2(t) \xrightarrow{S^{-1}} x_2(t)$)

then $[a_1 x_1(t) + a_2 x_2(t)] * h(t) = a_1 y_1(t) + a_2 y_2(t)$

hence $a_1 y_1(t) + a_2 y_2(t) \xrightarrow{S^{-1}} a_1 x_1(t) + a_2 x_2(t)$

Time invariance:

Since $x(t-t_0) * h(t) = y(t-t_0)$

we have $y(t-t_0) \xrightarrow{S^{-1}} x(t-t_0)$

NOTE: $x(t) \longrightarrow \underbrace{\boxed{h(t)} \longrightarrow \boxed{h_1(t)}}_{\text{inverse}} \longrightarrow x(t)$

$\Rightarrow h(t) * h_1(t) = \delta(t)$ as $\displaystyle\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$

Example: $h(t) = u(t)$

$$x(t) \longrightarrow \boxed{u(t)} \xrightarrow{\int_{-\infty}^{t} x(\tau)\,d\tau} \boxed{\frac{d(\cdot)}{dt}} \longrightarrow x(t)$$

since integral of $x(t)$ is i/p. for inverse it has to be differential!

has to be LTI

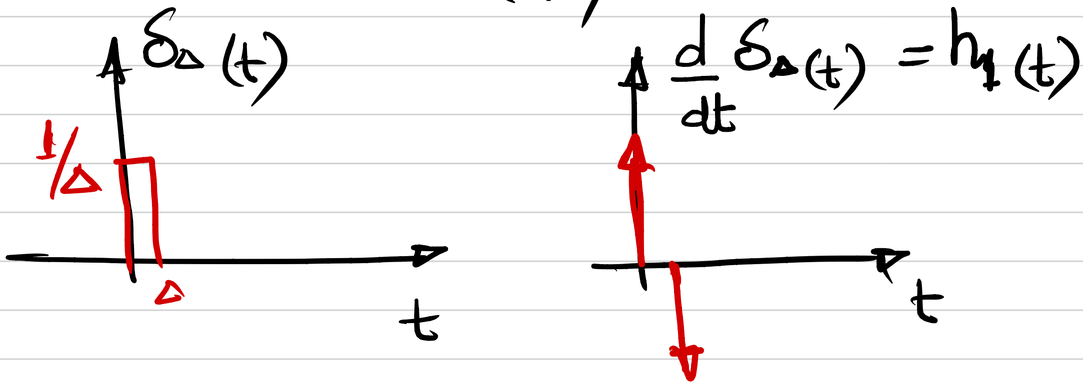
$$h_1(t) = \frac{d}{dt}\delta(t)$$

$$h(t) * h_1(t) = h_1(t) * h(t) \qquad \text{(Commutative property)}$$

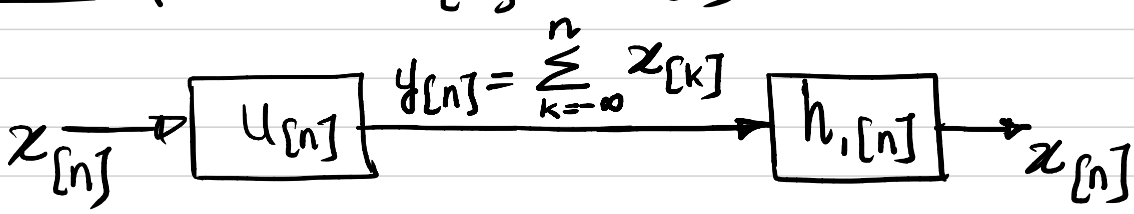
$$= \int_{-\infty}^{\infty} \frac{d}{dt}\delta(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} \frac{d}{dt}\delta(\tau)\, d\tau$$

$$= \delta(\tau)$$



<u>DT Example:</u> $h[n] = u[n]$



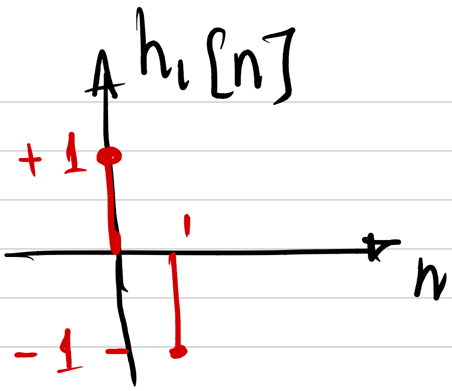
$$x[n] = y[n] - y[n-1]$$

$$h_1[n] = \delta[n] - \delta[n-1] \qquad \text{identity sys.}$$

$$h[n] * h_1[n] = u[n] * (\delta[n] - \delta[n-1])$$

$$= \sum_{k=-\infty}^{n} \delta[k] - \sum_{k=-\infty}^{n-1} \delta[k]$$

$$= u[n] - u[n-1] = \delta[n]$$



⑥ LTI Causality

<u>Recap</u>: A sys. is causal iff O/P $y(t)$ for $t<t_0$ depends only on values of $x(t)$ for $t<t_0$.

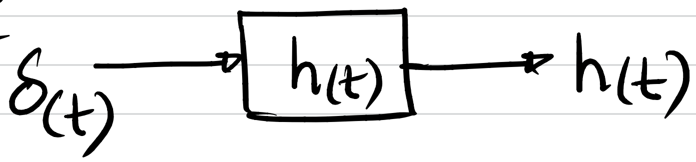
$$y(t) = \int_{-\infty}^{+\infty} \underline{x(\tau)}\, h(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

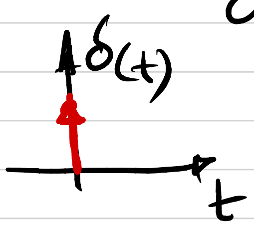
causality $\Rightarrow$ $h(t-\tau) = 0$ for $\tau > t$ $h(t-\tau)=0$

$\Leftrightarrow$ $h(t) = 0$ for $t<0$

<u>Theorem</u>: A CT LTI sys. is causal iff $h(t)=0$, $\forall\ t<0$.

<u>Intuition</u>





Input has action only at $t=0$

Causality $\Rightarrow$ output $h(t)$ must be '0' before $t=0$

<u>Example</u> $h[n] = u[n]$

$h(t) = u(t)$

NOTE:

1. Causality for LTI is equivalent to the initial rest condition (Prob. 1.44)

   If $x_{(t)} = 0$ for $t < t_0$, then $y_{(t)} = 0$ for $t < t_0$

2. If $h_{(t)}$ is causal then

$$y_{(t)} = \int_{-\infty}^{+\infty} x_{(\tau)} h_{(t-\tau)} d\tau$$

$$= \int_{-\infty}^{t} x_{(\tau)} h_{(t-\tau)} d\tau \quad * \text{ only exists } \tau < t$$

$$= \int_{\infty}^{0} h_{(s)} x_{(t-s)} (-ds)$$

$s = t - \tau$
$ds = -d\tau$
Limits:
$\infty \rightarrow 0$

$$= \int_{0}^{\infty} h_{(s)} x_{(t-s)} ds$$

3. Sometimes a signal $x_{(t)}$ is called "causal" if $x_{(t)} = 0$ for $t < 0$.