Signal Analysis & Communication

ECE355H1 F

ch. 3.3 (contd.)

Fourier Series for CT Periodic Signals

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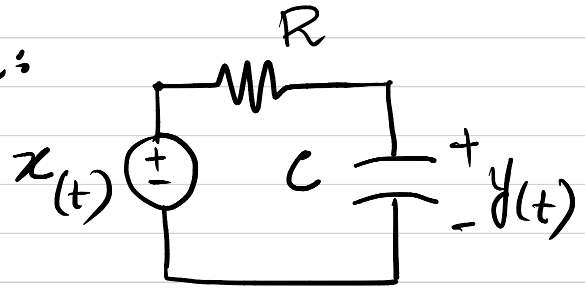
Examples of Fourier Series with LTI:

FS: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

LTI: $x(t) \rightarrow \boxed{h(t)} \rightarrow \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$

Example:



$H(s) = ?$

We assume initial rest condition

$$RC \frac{d}{dt} y(t) + y(t) = x(t) \quad — \ 1$$

Way 1:

I. Solving LCCDE with $x(t) = \delta(t)$

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

↑ starting with $x(t) = u(t)$

II. $$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt = \dots$$

Way 2:

I. Use $x(t) = e^{st}$

$$\Rightarrow \quad y(t) = H(s) e^{st}$$

II. $$1 \Rightarrow RC\, H(s)\, s\, e^{st} + H(s) e^{st} = e^{st}$$

$$H(s) = \frac{1}{1+RCs}$$

Similarly,

$$H(j\omega) = \frac{1}{1+jRC\omega}$$

$$H(jk\omega_0) = \frac{1}{1+jkRC\omega_0}$$

Suppose $RC=1$, & $x(t) = 1+\cos(2\pi t)+\cos(4\pi t)+\cos(6\pi t)$

Find $y(t)$

$$\omega_0 = 2\pi$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k t}$$

where $a_0 = 1$

$$a_1 = a_{-1} = a_2 = a_{-2} = a_3 = a_{-3} = 1/2$$

& $a_k = 0$ for $|k| > 3$

$$H(jk\omega_0) = \frac{1}{1+j2\pi k} = \frac{1}{\sqrt{1+4\pi^2k^2}} e^{-j\tan^{-1}(2\pi k)}$$

$*$ $a+jb = \sqrt{a^2+b^2}\, e^{j\tan^{-1}(b/a)}$ $*$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k \frac{1}{\sqrt{1+4\pi^2k^2}} e^{j(2\pi k t - \tan^{-1}(2\pi k))}$$

$$y(t) = a_0 + \sum_{k=1}^{\infty} 2a_k \frac{1}{\sqrt{1+4\pi^2k^2}} \cos\left(2\pi k t - \tan^{-1}(2\pi k)\right)$$

$*$ Remember: $e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$\therefore$ 'cos' components doubles $*$

'sin' components cancels out!

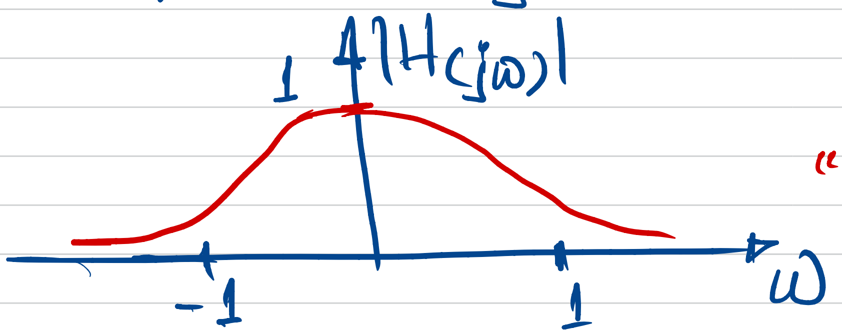
$$y(t) = 1 + \frac{1}{\sqrt{1+4\pi^2}} \cos\left(2\pi t - \tan^{-1}(2\pi)\right) + \frac{1}{\sqrt{1+16\pi^2}} \cos\left(4\pi t - \tan^{-1}(4\pi)\right) + \frac{1}{\sqrt{1+36\pi^2}} \cos\left(6\pi t - \tan^{-1}(6\pi)\right)$$

Freq. dependent attenuation

Freq. dependent phase shift

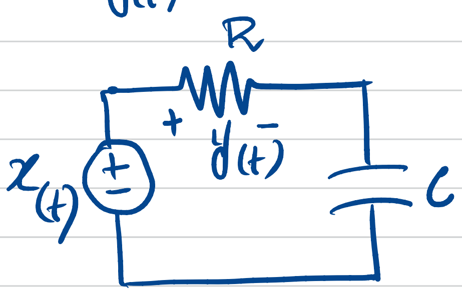
NOTE 1

$$|H(j\omega)| \downarrow = \left| \frac{1}{1+j\omega} \right| = \frac{1}{\sqrt{1+\omega^2}} \uparrow$$



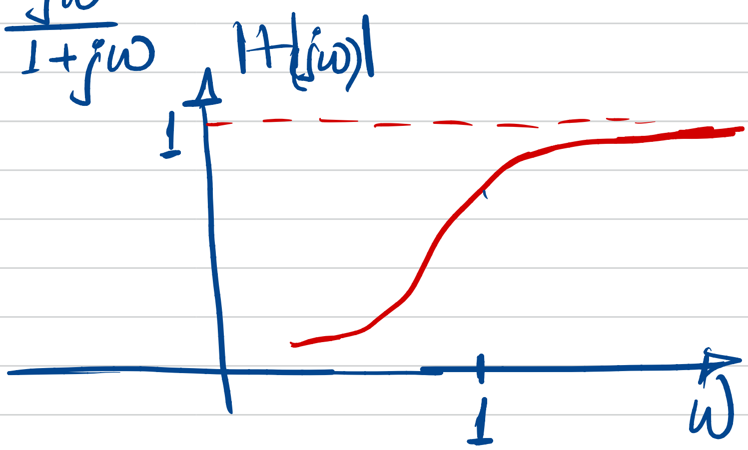
"Low Pass Filter"

NOTE 2 If o/p $y(t)$ is across in RC Circuit



$$H(j\omega) = 1 - \frac{1}{1+j\omega} = \frac{j\omega}{1+j\omega}$$

$$|H(j\omega)| = \frac{\omega}{\sqrt{1+\omega^2}}$$



Calculation of CT Fourier Series

Theorem:

If $x(t)$ has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{—} \; (A)$$

then

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt \quad \text{—} \; (B)$$

integration over any period T

eg. $\int_0^T \ldots$

Proof:

$$\int_T x(t) e^{-jk\omega_0 t} dt$$

$$= \int_T \sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t} \cdot e^{-jk\omega_0 t} dt$$

$$= \sum_{m=-\infty}^{\infty} a_m \underbrace{\int_T e^{j(m-k)\omega_0 t} dt}_{= \begin{cases} 0 & m \neq k \\ T & m = k \end{cases} = T\delta[m-k]}$$

$\omega_0 = \frac{2\pi}{T}$

$$= a_k T$$

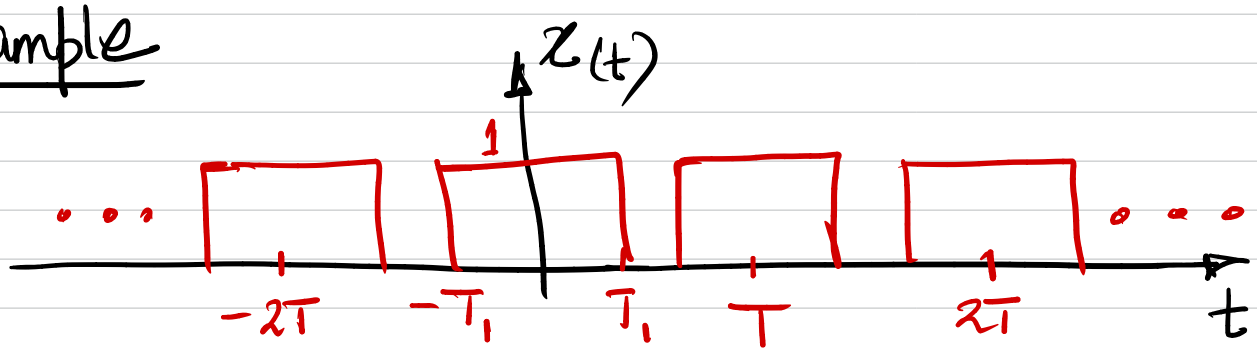
NOTE:

1. Eqn (A) : Synthesis Equation
   Eqn (B) : Analysis Equation

2. $\{a_k\}$: Fourier Series Coefficients

$$a_0 = \frac{1}{T}\int_T x(t)\,dt \leftarrow \text{average value of } x(t)$$

Example



$$T > 2T_1$$

$$x(t) = \sum_{k=-\infty}^{\infty} p_{T_1}(t+kT)$$

$$\text{where } p_{T_1} = \begin{cases} 1 & |t| < T_1 \\ 0 & o/w \end{cases}$$

Find Fourier Series of $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ — (A)

I. Fundamental freq. $= \omega_0 = 2\pi/T$

II. $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}$

$\underline{\text{iii}}$. $$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{-jk\omega_0 T}\, e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi}$$

<u>Eg.</u> Let $T = 4T_1 \Rightarrow \omega_0 = \frac{2\pi}{4T_1} = \frac{\pi}{2T_1}$

$$a_k = \begin{cases} \dfrac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}, & k \neq 0 \\ 1/2, & k = 0 \end{cases}$$

$a_k = 0$ for k: even

$\left[\sin(\text{odd integer} \times \pi) = 0\right]$

For $k=0 \xrightarrow[\text{Rule}]{\text{L'Hpital}} \frac{\sin k\omega_0 T_1}{k\pi} = \frac{\cos k\omega_0 T_1 \times \omega_0 T_1}{\pi}$

$$= \frac{1 \times \pi/2T_1 \times T_1}{\pi} = \frac{1}{2}$$

A $\Rightarrow$

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\cos\left(\frac{2\pi}{T}t\right) - \frac{2}{3\pi}\cos\left(\frac{6\pi}{T}t\right) + \frac{2}{5\pi}\cos\left(\frac{10\pi}{T}t\right) - \dots$$

* <u>again</u>

'cos' components doubles.

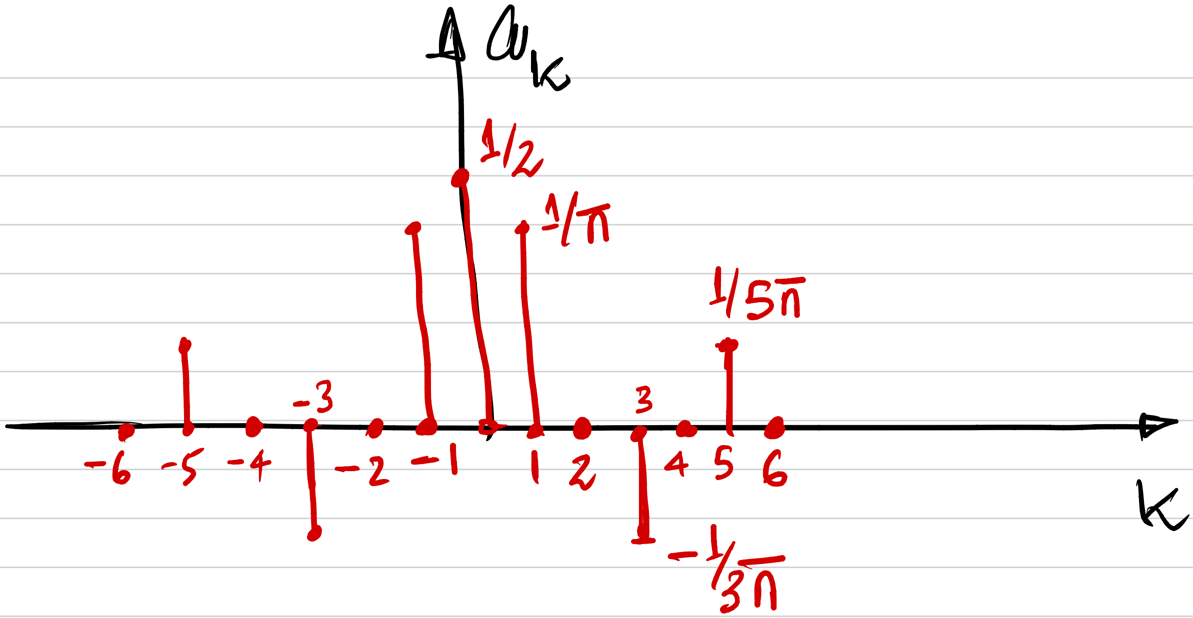
'sin' components cancels out!

$[e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta]$

*

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2a_k \cos(k\omega_0 t)$$



a_k
$1/2$
$1/\pi$
$1/5\pi$
-3
3
-6 -5 -4 -2 -1 1 2 4 5 6
k
$-1/3\pi$