Enter the first letter of **your last name** in this box:

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UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DIVISION OF ENGINEERING SCIENCE

ECE 355 – Signal Analysis and Communications

Final Examination, 14:00 - 16:30, December 12, 2017

Examination Type: D
Examiner: Ben Liang

Instructions

- You are allowed one 8.5×11 sheet of handwritten notes (double-sided) and a non-programmable calculator.
- Remember to enter your name on the first page. If there are loose pages, please put your name on every page that you turn in.
- Use the space provided to enter your answers and clearly label them. If you use the back of a page to enter your answers, make sure you clearly indicate that on the font of the page.
- Show intermediate steps for partial credits. Answers without justification will not be accepted.
- Unless otherwise specified, you may use without proof any formula in the Fourier transform tables on Page 2.
- Unless otherwise requested, all final answers must be expressed in simplified form.

MARKS

Question	1	2	3	4	5	6	7	8	Total
Value	5	5	10	15	10	10	15	10	80
Mark									

CTFT Pairs and Properties

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Transform pairs:

$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^n}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases} \leftrightarrow \frac{2\sin(\omega T)}{\omega}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Properties:

$$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$$

$$e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))$$

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

$$x(t) * y(t) \leftrightarrow X(j\omega)Y(j\omega)$$

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega-\theta))d\theta$$

If $x(t) \leftrightarrow X(j\omega) = G(\omega)$, then $G(t) \leftrightarrow 2\pi x(-\omega)$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

DTFT Pairs and Properties

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Transform pairs:

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}, \text{ for } |a| < 1$$

$$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^r}, \text{ for } |a| < 1$$

$$\begin{cases} 1, & |n| \le N \\ 0, & |n| > N \end{cases} \leftrightarrow \frac{\sin(\omega(N+1/2))}{\sin(\omega/2)}$$

$$\frac{\sin(Wn)}{\pi n} \leftrightarrow \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}, \text{ for } 0 < W < \pi$$

$$\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Properties:

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

$$nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

$$x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$$

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})Y(e^{j(\omega-\theta)})d\theta$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

1 (5 marks)
A continuous-time system has the following relation between its input $x(t)$ and output $y(t)$:

$$y(t) = x(t)\sin(t^2).$$

Is this system linear? Explain why or why not.

let $y_1(t) = x_1(t)\sin(t^2)$, $y_2(t) = x_2(t)\sin(t^2)$

$$x_1(t) + a x_2(t) \longrightarrow \boxed{S} \quad (x_1(t) + a x_2(t))\sin t^2 = x_1(t)\sin(t^2) + a x_2(t)\sin(t^2)$$

$$= y_1(t) + a y_2(t)$$

$\Rightarrow$ the system is linear.

2 (5 marks)
Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefficients of $\mathfrak{Re}\{x(t)\}$ (i.e., the real part of $x(t)$).

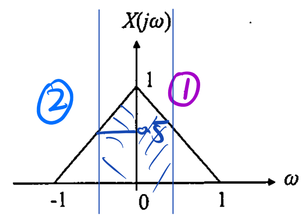
$$\mathrm{Re}\{x(t)\} = \frac{1}{2}(x(t) + x^*(t)) \xleftrightarrow{FS} \frac{1}{2}a_k + \frac{1}{2}a^*_{-k}$$

Since $x^*(t) \xrightarrow{FS} a^*_{-k}$ and fourier series is linear.

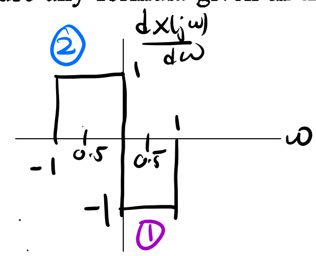
$$\mathrm{Im}\{x(t)\} = \frac{1}{2}(-x^*(t) + x(t)) \xleftrightarrow{FS} \frac{1}{2}a_k - \frac{1}{2}a^*_{-k}$$

3 (5 + 5 = 10 marks)

The Fourier transform of continuous-time signal $x(t)$ is given in the figure below.



(a) Find $x(t)$. You must show all derivation details, but you may use any formula given in the tables on Page 2.



① $$-\frac{1}{2\pi}\frac{2\sin(\frac{1}{2}t)}{t}e^{j\frac{1}{2}t} = \frac{tx(t)}{j} \xleftrightarrow{FT} \frac{dX(j\omega)}{d\omega}, \quad \omega > 0$$

② $$\frac{1}{2\pi}\frac{2\sin(\frac{1}{2}t)}{t}e^{-j\frac{1}{2}t} = \frac{tx(t)}{j} \xleftrightarrow{FT} \frac{dX(j\omega)}{d\omega}, \quad \omega < 0$$

$$\frac{-j}{\pi}\frac{\sin(\frac{1}{2}t)}{t^2}e^{j\frac{1}{2}t} + \frac{j}{\pi}\frac{\sin(\frac{1}{2}t)}{t^2}e^{-j\frac{1}{2}t} = \frac{\sin(\frac{1}{2}t)}{\pi t^2}\, j\left(-e^{j\frac{1}{2}t} + e^{-j\frac{1}{2}t}\right)$$

$$= \frac{2\sin(\frac{1}{2}t)}{\pi t^2}\,\frac{1}{2j}\left(e^{j\frac{1}{2}t} - e^{-j\frac{1}{2}t}\right)$$

$$= \frac{2\sin^2(\frac{1}{2}t)}{\pi t^2}$$

(b) Suppose $x(t)$ is processed with the following ideal low-pass filter:

$$H(j\omega) = \begin{cases} 1, & |\omega| < 0.5\,, \\ 0, & |\omega| > 0.5\,. \end{cases}$$

Determine the amount of energy of $x(t)$ that is **removed** by such filtering.

$$\frac{1}{2\pi}\left(\int_{-1}^{-0.5}(\omega+1)^2\,d\omega + \int_{0.5}^{1}(-\omega+1)^2\,d\omega\right) = 0.013263$$

4 (5 + 5 + 5 = 15 marks)

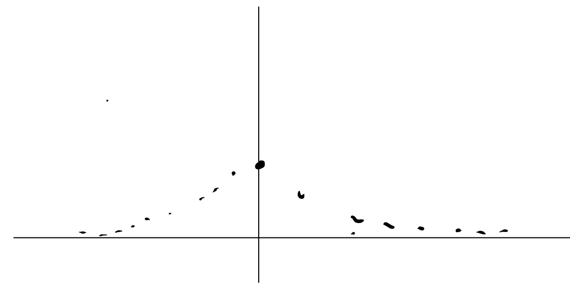
Consider the discrete-time LTI system with impulse response $h[n] = \left(\frac{1}{2}\right)^{|n|}$.

(a) Is this system causal or BIBO stable? Explain why or why not.

BIBO stable

since $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

not causal since $h[n] \neq 0$ for $n<0$



(b) Derive the frequency response $H(e^{j\omega})$.

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ \left(\frac{1}{2}\right)^{-n}, & n < 0 \end{cases} = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^{-n} u[-n] - \delta[n]$$

$$\Big\updownarrow \text{FT}$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{1-\frac{1}{2}e^{j\omega}} - 1$$

$$= 2\,\mathrm{Re}\left\{\frac{1}{1-\frac{1}{2}e^{j\omega}}\right\} - 1 \quad \text{with fund. fre. } 2\pi.$$

c) $a_K = h[-k] = \left(\frac{1}{2}\right)^{|k|}$, using duality.

$$H(e^{jt}) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} e^{jk\underset{\uparrow\, w_0}{(2\pi)} t}$$

(5)

a)

$$s^2 H + 4sH + 3H = 1$$

let $s = j\omega$

$$H(s) = \frac{1}{s^2 + 4s + 3} = \frac{1}{(s+1)(s+3)}$$

$$A(s+3) + B(s+1) = 1$$

$s = -3$ $\quad$ $s = -1$

$B = -\frac{1}{2}$ $\quad$ $A = \frac{1}{2}$

$$= \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}$$

$$h(t) = \left(e^{-t} - e^{-3t}\right)\frac{1}{2}u(t)$$

b)

$$X(s) = \frac{1}{2+s}$$

$$Y(s) = X(s)H(s) = \frac{1}{(s+1)(s+3)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+3)} + \frac{C}{(s+2)}$$

$$A(s+3)(s+2) + B(s+1)(s+2) + C(s+1)(s+3) = 1$$

$s = -3$ $\quad$ $s = -2$ $\quad$ $s = -1$

$B(-2)(-1) = 1$ $\quad$ $C(-1)(1) = 1$ $\quad$ $A(2)(1) = 1$

$B = \frac{1}{2}$ $\quad$ $C = -1$ $\quad$ $A = \frac{1}{2}$

$$y(t) = \left(\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} - e^{-2t}\right)u(t)$$

⑥ $\omega_n = 2000\pi$ $T = 0.0005$

a)

$$X_d(e^{j\omega}) = X_c(e^{j\omega/T}), \text{ repeat with } 2\pi.$$

$$\underbrace{X_d(e^{j\omega T})}_{} = X_c(e^{j\omega})$$

for $-\pi < \omega < \pi$

$X_c(e^{j\omega}) = 0$ for $1500\pi = \frac{3\pi}{4}\frac{1}{T} \le |\omega| \le \frac{\pi}{T} = 2000\pi$

$\boxed{\text{tiny tiny bit less than } 15000\pi}$

b)

$$\frac{2\pi}{T} = \omega_s > 2\omega_n = 3000\pi$$

minimum sampling frequency is 3000π.

$\omega_s = 2\pi$ $T = 1$

⑦ a) $V_r(t) = \sum_{n=-\infty}^{\infty} 5 \frac{\sin(\pi(t-n))}{\pi(t-n)}$

let $\omega_c = \frac{\omega_s}{2} = \frac{2\pi}{2} = \pi$

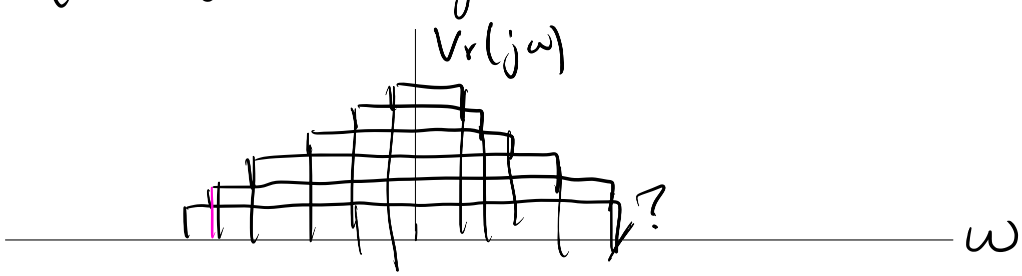
b) $\omega_n < \omega_c < \omega_s - \omega_n$ let $\omega_c = \frac{\omega_s}{2} = \pi$

$\omega_n < \pi$

the maximum angular frequency is smaller than π.

each sinc is a low pass filter in freq. domain.

then we add them together!



$$\frac{1}{2\pi}\frac{2\sin(tT)}{t} \overset{FT}{\longleftrightarrow} \begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases}$$

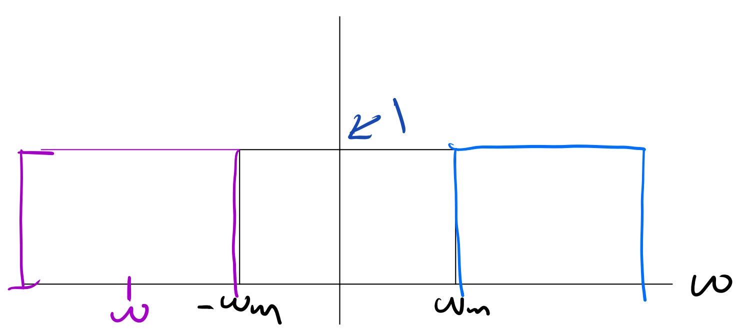
all sinc has $T=\pi$ so maximum angular frequency is $\underline{\pi}$

c) there is no aliasing so there is only one possible reconstruction from the 5 les that is determined in a).

$$V_r(t) = \sum_{n=-\infty}^{\infty} 5 \frac{\sin(\pi(t-n))}{\pi(t-n)} = \underline{5 = V(t)}$$ w/ max angular frequency of π.

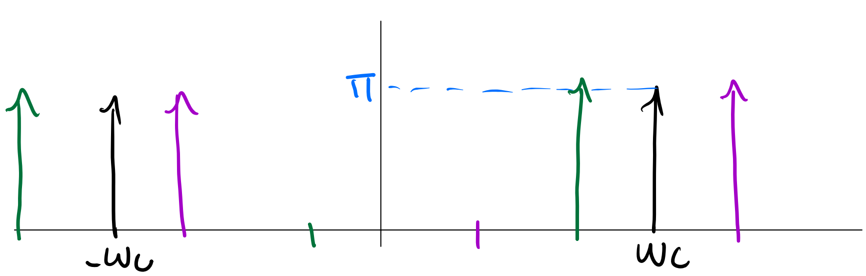
⑧ $y(t) = x^2(t) + 2x(t)\cos(\omega_c t) + \cos^2(\omega_c t)$

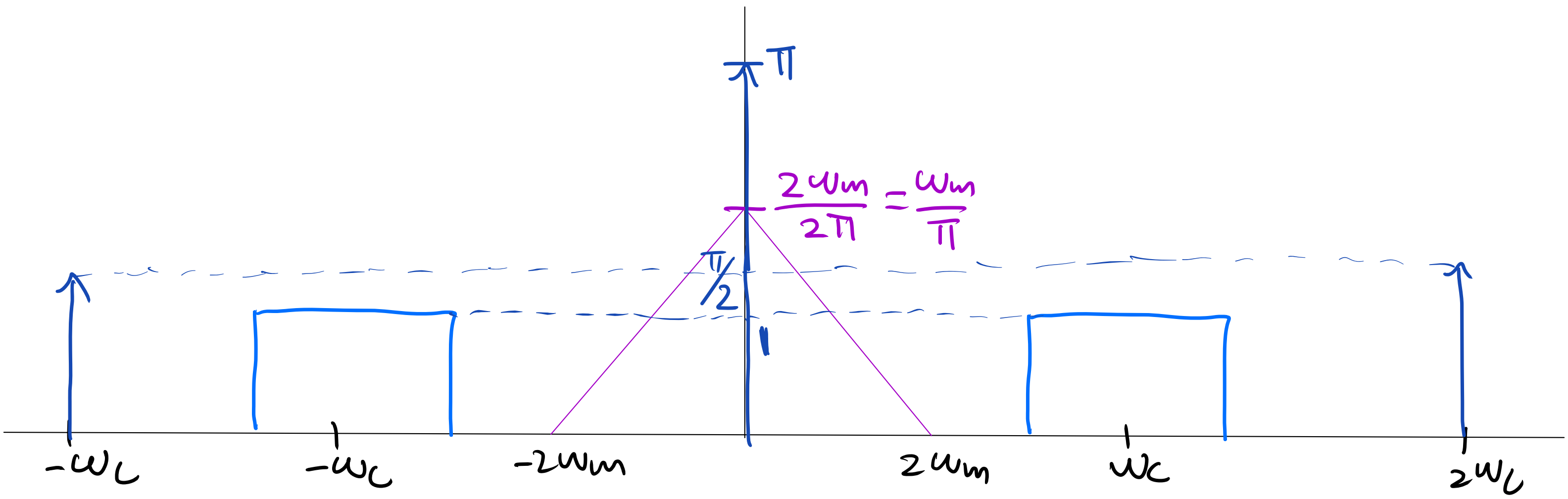
$$F\{x^2(t)\} = \frac{1}{2\pi} X(j\omega) * X(j\omega) = \begin{cases} 0, & 2\omega_m < |\omega| \\ -\omega + 2\omega_m, & 0 < \omega < 2\omega_m \\ \omega + 2\omega_m & -2\omega_m < \omega < 0 \end{cases}$$



$F\{\cos(\omega_c t)\} = \pi(\delta(\omega - \omega_c) + \delta(\omega + \omega_c))$

$$F\{\cos^2(\omega_c t)\} = \frac{1}{2\pi}\cos(\omega_c t) * \cos(\omega_c t) = \underbrace{\frac{1}{2\pi}(2\pi^2)}_{\pi}\delta(\omega) + \frac{1}{2\pi}(\pi^2)\delta(\omega - 2\omega_c) + \frac{1}{2\pi}(\pi^2)\delta(\omega + 2\omega_c)$$





π
$\frac{2\omega_m}{2\pi} = \frac{\omega_m}{\pi}$
π/2
1
$-\omega_L$
$-\omega_c$
$-2\omega_m$
$2\omega_m$
ω_c
$2\omega_L$