Signal Analysis & Communication

ECE355H1F

Ch. 2.2

CT LTI Systems

Lec 1, Wk4

26-09-2022

CT LTI Systems (ch. 2.2)

- Characterization of CT LTI Sys. in terms of its (unit) impulse response - analogous to DT LTI Sys.

<u>Def^n</u>: The unit impulse response $h(t)$ of a CT LTI Sys. is its output when $\delta(t)$ is the input.

$$\delta(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow h(t)$$

For linear & time invariant systems:

integral of weighted shifted impulse responses

any sig. can be represented as integral of weighted shifted impulses

$$\underbrace{\int_{\tau} x(\tau)\,\delta(t-\tau)\,d\tau}_{x(t)} \xrightarrow{\text{LTI}} \underbrace{\int_{\tau} x(\tau)\,h(t-\tau)\,d\tau}_{y(t)}$$

analogous to $\sum_k x[k]\,\delta[n-k] \rightarrow \sum_k x[k]\,h[n-k]$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\underbrace{\delta(t-\tau)}_{\text{exists at } \tau=t,\ \text{'0' o/w}}\,d\tau$$

$$= \int_{-\infty}^{+\infty} x(t)\,\delta(t-\tau)\,d\tau \quad \text{(sampling property of } \delta(t))$$

$$= x(t)\underbrace{\int_{-\infty}^{+\infty} \delta(t-\tau)\,d\tau}_{\text{area is '1'}} = x(t)\ \checkmark$$

<u>Theorem</u>: For a CT LTI sys. with impulse response $h(t)$, when $x(t)$ is the I/P, the O/P is given as

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau \triangleq x(t) * h(t)$$

Example 1:

$$x(t) = e^{-at}\, u(t) \quad (a > 0)$$

$$h(t) = u(t)$$

$y(t) = ?$

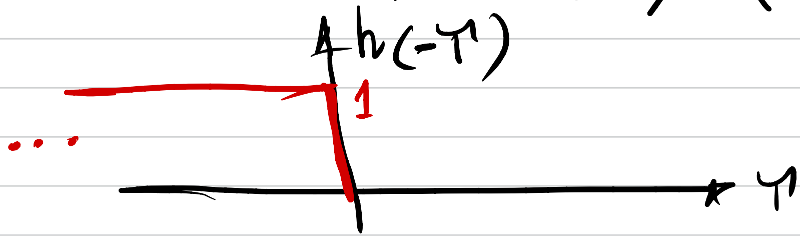
$$y(t) = \int_{-\infty}^{+\infty} e^{-a\tau}\, u(\tau)\, u(t-\tau)\, d\tau$$

Step I

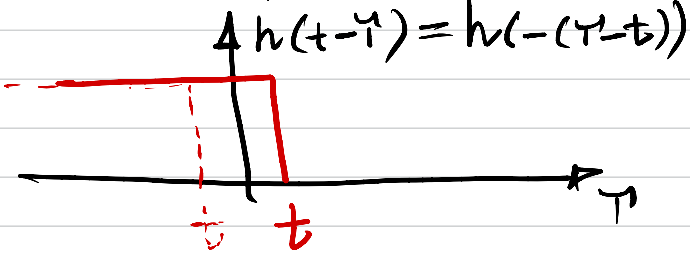
$$x(\tau) = e^{-a\tau}\, u(\tau)$$

Step II

$h(-\tau) = u(-\tau)$ ($u(\tau)$ flipped)



Step III $h(t-\tau)$ (slide $h(-\tau)$ to the right by 't')



(Check with different values of 't' → different regions of overlap of $x(\tau)$ & $h(t-\tau)$, as well as non-overlaps)

for example, here we can see two regions:

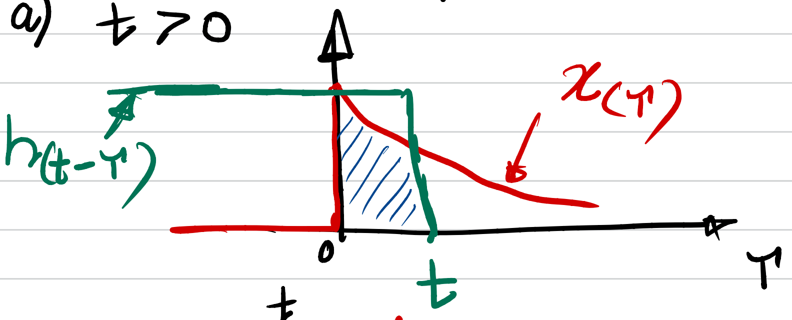
$t < 0$ No overlap

$t > 0$ Overlap.

Step IV As per overlapping regions in step III, multiply $x(\tau)$ & $h(t-\tau)$ and integrate.

For the non-overlapping regions, the product of $x(\tau)$ & $h(t-\tau)$ would be zero!

a) $t > 0$

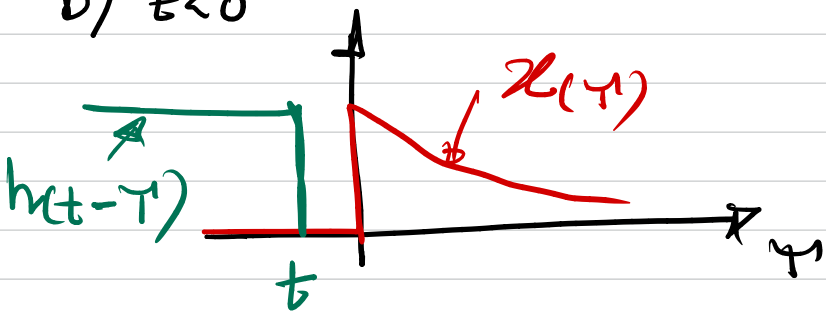


"$t>0$"

$$y(t) = \int_0^t e^{-a\tau}(1)\,d\tau$$

$$= \frac{e^{-a\tau}}{-a}\Big|_0^t = \frac{1}{a}(1-e^{-at}) \quad \text{—— (1)}$$

b) $t < 0$



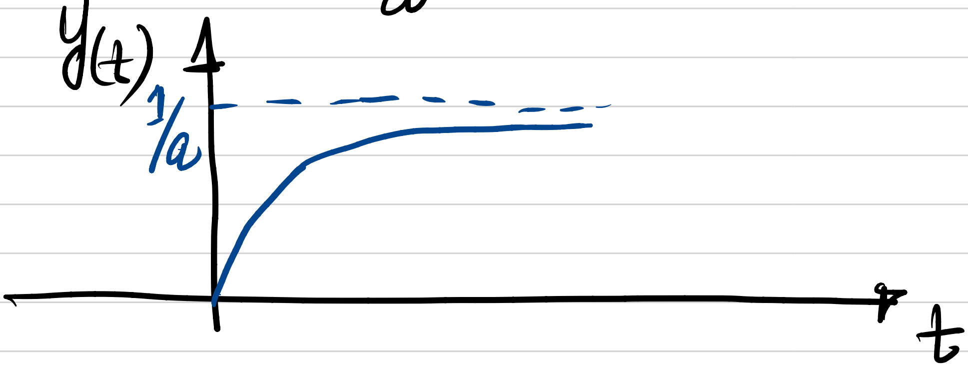
"$t<0$"

$$y(t) = 0 \quad \text{—— (2)}$$

Step V Eqn (1) & (2) $\Rightarrow$

$$y(t) = \begin{cases} 0 & t<0 \\ \frac{1}{a}(1-e^{-at}) & t>0 \end{cases}$$

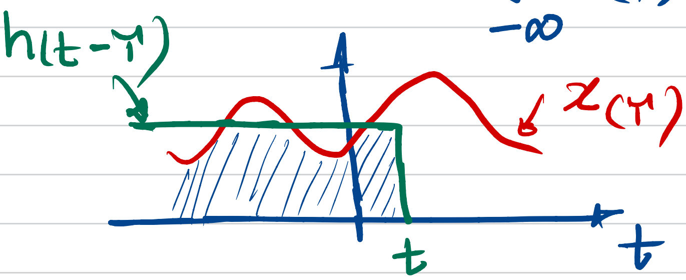
$$= \frac{1}{a}(1-e^{-at})\,u(t)$$



NOTE: For any $x_{(t)}$ ranging from $-\infty$ to $+\infty$, if $h_{(t)} = u_{(t)}$,

$$y_{(t)} = x_{(t)} * h_{(t)} = \int_{-\infty}^{+\infty} x_{(\tau)} h_{(t-\tau)} \, d\tau$$

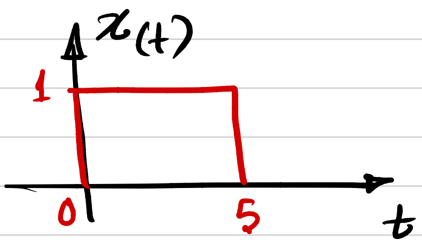
$$= \int_{-\infty}^{t} x_{(\tau)} \, d\tau \rightarrow \text{Cumulative Integration}$$



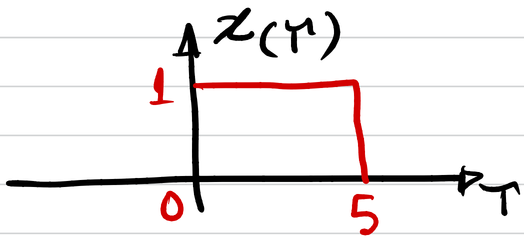
For example: if $x_{(t)} = \delta_{(t)}$ & $h_{(t)} = u_{(t)}$

$$y_{(t)} = \int_{-\infty}^{t} \delta_{(\tau)} \, d\tau = u_{(t)}$$ verifies (IR is $u_{(t)}$)

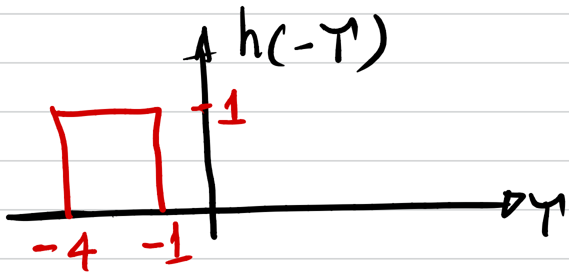
Example 2:



Step I



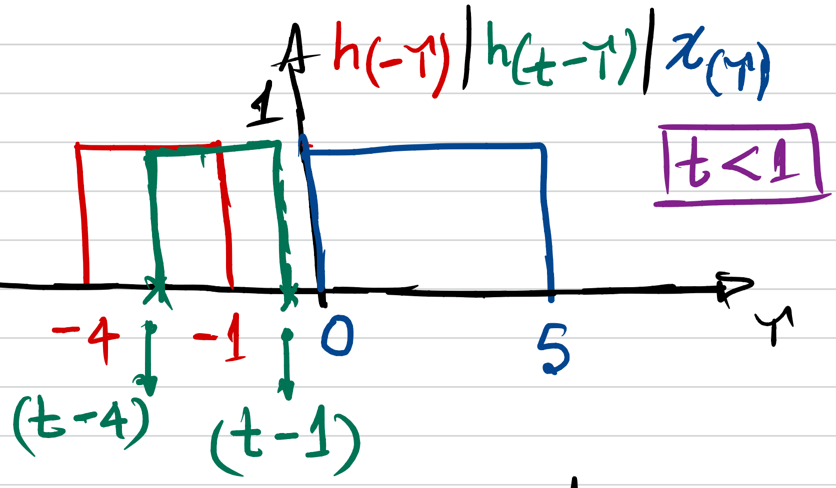
Step II



Step III

Step IV

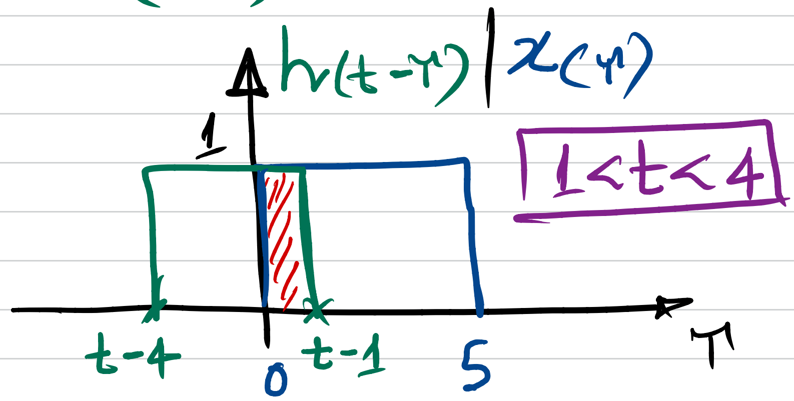
a)



No overlap

$$y(t) = 0 \quad -(1)$$

b)

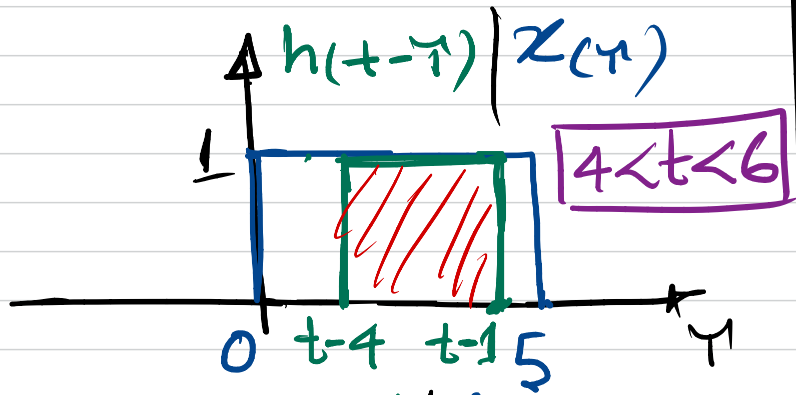


Overlap (0 → t-1)

$$y(t) = \int_0^{t-1} (1)(1)\, d\tau$$

$$= t-1 \quad -(2)$$

c)

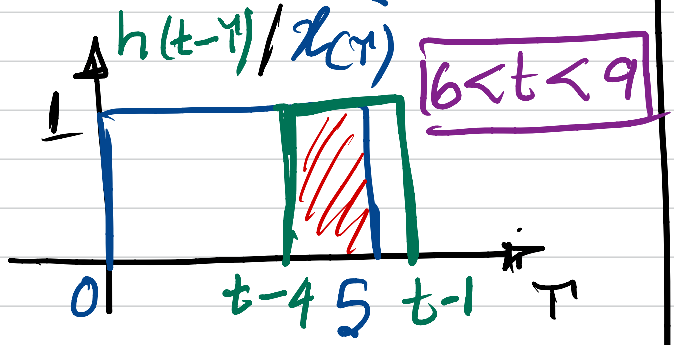


Overlap ((t-4) → (t-1))

$$y(t) = \int_{t-4}^{t-1} (1)(1)\, d\tau$$

$$= 3 \quad -(3)$$

d)

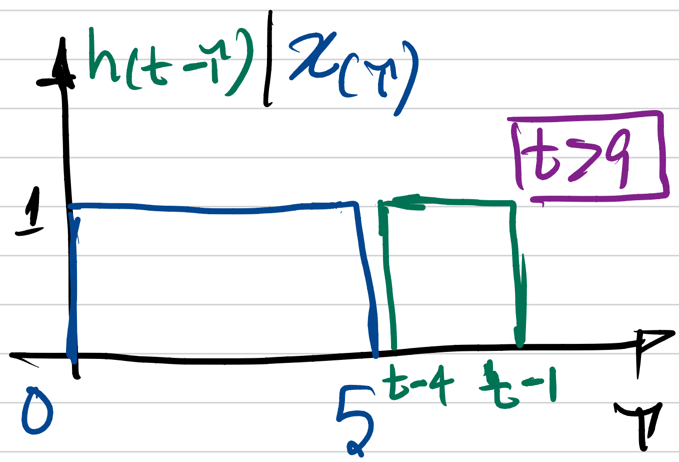


Overlap ((t-4) → 5)

$$y(t) = \int_{t-4}^{5} (1)(1)\, d\tau$$

$$= -t+9 \quad -(4)$$

e)

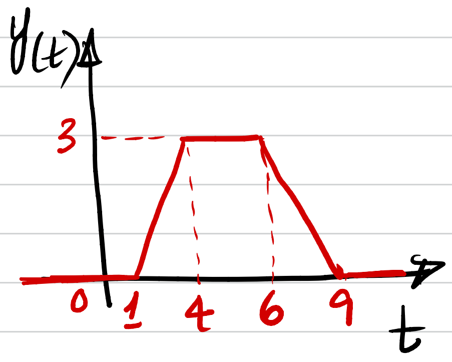


No overlap

$$y(t) = 0 \quad -(5)$$

Step V Eqn ①, ②, ③, ④, & ⑤ $\Rightarrow$

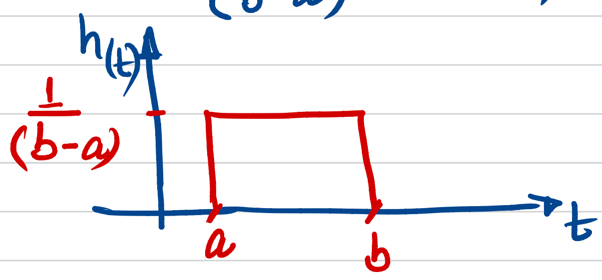
$$y(t) = \begin{cases} 0 & t<1 \\ t-1 & 1<t<4 \\ 3 & 4<t<6 \\ -t+9 & 6<t<9 \\ 0 & t>9 \end{cases}$$



Remember: while performing convolution; you should look for all the possible regions of overlap.

NOTE: If $h(t)$ is a rectangular pulse given as:

$$h(t) = \frac{1}{(b-a)}\left[u(t-a) - u(t-b)\right] \quad (a<b)$$



Then for any $x(t)$

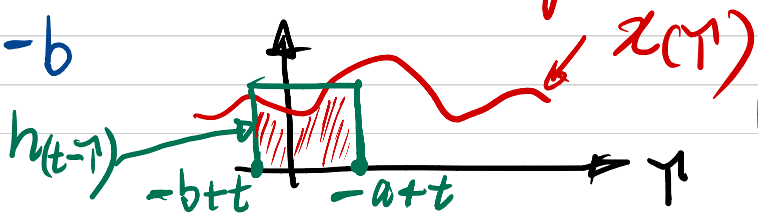
$$y(t) = x(t) * h(t) = \frac{1}{(b-a)} \int_{-\infty}^{+\infty} x(\tau) \underbrace{\left[u(t-a-\tau) - u(t-b-\tau)\right]}_{\text{For the pulse to exist, i.e., '=1'}} d\tau$$

For the pulse to exist, i.e., '=1'

$t-a-\tau > 0 \Rightarrow \tau < t-a$

$t-b-\tau < 0 \Rightarrow \tau > t-b$

$$y(t) = \frac{1}{(b-a)} \int_{t-b}^{t-a} x(\tau)\, d\tau \quad \text{(average)}$$



moving average