Enter the first letter of **your last name** in this box:

FIRST NAME: ________________ LAST NAME: ________________ ID#: ________________

UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DIVISION OF ENGINEERING SCIENCE

ECE 355 – Signal Analysis and Communications

Final Examination, 18:30 - 21:00, December 7, 2018

Examination Type: D
Examiner: Ben Liang

Instructions

- You are allowed one 8.5×11 sheet of ***handwritten*** notes (double-sided) and a non-programmable calculator.
- Remember to enter your name on the first page. If there are loose pages, please put your name on every page that you turn in.
- Use the space provided to enter your answers and clearly label them. If you use the back of a page to enter your answers, make sure you clearly indicate that on the font of the page.
- Show intermediate steps for partial credits. Answers without justification will not be accepted.
- Unless otherwise specified, you may use without proof any formula in the Fourier transform tables on Page 2.
- Unless otherwise requested, all final answers must be expressed in simplified form.

MARKS

Question	1	2	3	4	5	6	7	8	Total
Value	10	5	5	15	10	10	10	15	80
Mark									

CTFT Pairs and Properties

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Transform pairs:

$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^n}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases} \leftrightarrow \frac{2\sin(\omega T)}{\omega}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Properties:

$$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$$

$$e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))$$

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

$$x(t) * y(t) \leftrightarrow X(j\omega)Y(j\omega)$$

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega-\theta))d\theta$$

If $x(t) \leftrightarrow X(j\omega) = G(\omega)$, then $G(t) \leftrightarrow 2\pi x(-\omega)$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

DTFT Pairs and Properties

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Transform pairs:

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}, \text{ for } |a| < 1$$

$$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^r}, \text{ for } |a| < 1$$

$$\begin{cases} 1, & |n| \le N \\ 0, & |n| > N \end{cases} \leftrightarrow \frac{\sin(\omega(N+1/2))}{\sin(\omega/2)}$$

$$\frac{\sin(Wn)}{\pi n} \leftrightarrow \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}, \text{ for } 0 < W < \pi$$

$$\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Properties:

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

$$nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

$$x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$$

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})Y(e^{j(\omega-\theta)})d\theta$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

1 (5 + 5 = 10 marks)
Consider an LTI system with impulse response

$$h(t) = \frac{1}{(2+jt)^2}\,.$$

(a) Find its frequency response $H(j\omega)$.

$$2\pi \frac{-\omega}{(2-1)!} e^{2\omega} u(-\omega) = -2\pi \omega e^{2\omega} u(-\omega)$$

(b) Is this system BIBO stable? (*Hint:* $|h(t)| = \left|\frac{1}{2+jt}\right|^2$.)

$$\left| \int_{-\infty}^{\infty} h(\tau)\, d\tau \right| = \left| \int_{-\infty}^{\infty} \frac{1}{(2+j\tau)^2} d\tau \right| \leq \int_{-\infty}^{\infty} \left| \frac{1}{2+j\tau} \right|^2 d\tau < \infty$$

$\Rightarrow$ it is Bibo stable

2 (5 marks)
Starting from the definition of the Fourier transform, show that if $x(t) \leftrightarrow X(j\omega)$ then for all $a \neq 0$ we have

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

$$X_1(j\omega) = \int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt$$

$u = at \quad \frac{1}{a} du = dt$

$\frac{u}{a} = t$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(u) e^{-j\omega \frac{u}{a}}\, du$$

$$= \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(u) e^{-j\omega \frac{u}{a}}\, du, & a > 0 \\ \frac{-1}{a} \int_{-\infty}^{\infty} x(u) e^{-j\omega \frac{u}{a}}\, du, & a < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{a} X\left(\frac{j\omega}{a}\right), & a > 0 \\ \frac{-1}{a} X\left(\frac{j\omega}{a}\right), & a < 0 \end{cases}$$

$$= \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

3 (5 marks)
Periodic discrete-time signal $x[n]$ has period 8 and Fourier series coefficients

$$a_k = \cos(\frac{k\pi}{4}) + \sin(\frac{3k\pi}{4})\,.$$

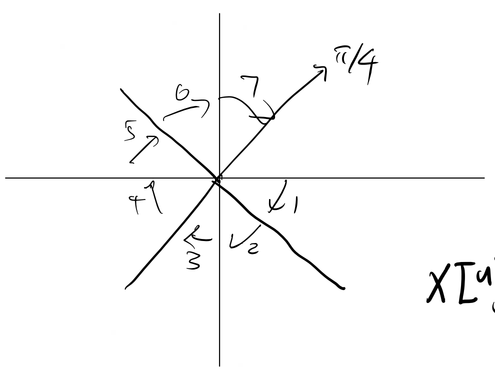
$\omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

Find $x[n]$.

$$a_k = \frac{1}{2}e^{jk\frac{\pi}{4}} + \frac{1}{2}e^{-jk\frac{\pi}{4}} + \frac{1}{2j}e^{3jk\frac{\pi}{4}} - \frac{1}{2j}e^{-3jk\frac{\pi}{4}}$$

$$= \frac{1}{2}e^{-j7k\omega_0} + \frac{1}{2}e^{-jk\omega_0} + \frac{1}{2j}e^{-j5k\omega_0} - \frac{1}{2j}e^{-j3k\omega_0}$$

$$= \frac{1}{8}x[7]e^{-j7k\omega_0} + \frac{1}{8}x[1]e^{-j1k\omega_0} + \frac{1}{8}x[5]e^{-j5k\omega_0} + \frac{1}{8}x[3]e^{-j3k\omega_0}$$



$x[7] = 4 \quad x[1] = 4 \quad x[5] = \frac{4}{j}$

$x[3] = -\frac{4}{j}$ for $\langle N \rangle = 0, 1, \ldots, 7$

$$x[n] = 4\delta[n-7] + 4\delta[n-1] + \frac{4}{j}\delta[n-5] - \frac{4}{j}\delta[n-3]$$

and repeat with $N=8$

4 (5 + 5 + 5 = 15 marks)
Consider an LTI system with impulse response

$$h(t) = \frac{d^2}{dt^2}\delta(t) \, .$$

(a) Find its frequency response $H(j\omega)$.

$H(j\omega) = -\omega^2$

(b) Is this system invertible? If not, explain why not. If yes, find the impulse response of the inverse LTI system.

No, differentiating takes out the constant.

$x_1(t) = t^2 + 5 \longrightarrow \boxed{h(t)} \rightarrow 2$

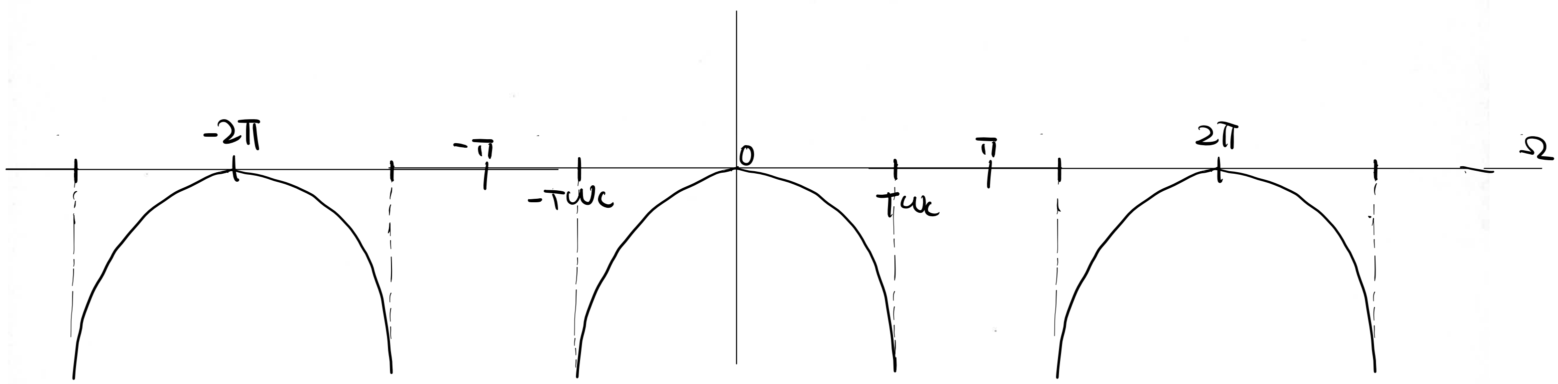
$x_2(t) = t^2 + 8 \longrightarrow \boxed{h(t)} \rightarrow 2$

(c) Suppose we wish to implement LTI system $h(t)$ in discrete-time (e.g., using a microprocessor), with sampling period T and cutoff frequency ω_c. Assume that sampling and reconstruction are ideal, and there is no aliasing. Find and **plot** the required discrete-time frequency response $H_d(e^{j\Omega})$.

$H_d(e^{j\Omega}) = H(j\omega/T)$, repeated with 2π cutoff at ω_c

$H(j\omega) = -\omega^2$

$H_d(e^{j\Omega}) = -\left(\frac{\Omega}{T}\right)^2$, repeated with 2π cutoff at ω_c



$\frac{2\pi}{T} > 2\omega_c$ $\frac{\pi}{T} > \omega_c$ $\pi > \omega_c T$

5 (5 + 5 = 10 marks)
Consider the periodic signal

$$x(t) = \frac{\sin(\frac{5t}{2})}{\sin(\frac{t}{2})}.$$

(a) Express $x(t)$ in Fourier series. (*Hint: use the tables on Page 2. Don't forget to specify the fundamental frequency.*)

$$y[n] \overset{DTFT}{\longleftrightarrow} x(\omega) \overset{FS}{\longleftrightarrow} y[-n]$$

$$\frac{5}{2} = N + \frac{1}{2} \qquad N = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$$

$$a_n = y[-n] = y[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & |n| > 2 \end{cases}$$

T_0 for $x(t)$ is 2π since it's the result of DTFT

(b) Find the Fourier series of $y(t) = x(2t)$.

if $x(t) \overset{FS}{\longleftrightarrow} a_n$ then

$y(t) = x(2t) \overset{FS}{\longleftrightarrow} a_n$ with fund. fre. of $2\omega_0 = 4\pi$

same as part a

6 (5 + 5 = 10 marks)
A causal LTI system with input $x(t)$ and output $y(t)$ is described by the following differential equation:

$$y''(t) + 5y'(t) + 6y(t) = x(t) \; .$$

(a) Find the impulse response, $h(t)$, of this system.

let $s = j\omega$

$s^2 H + 5sH + 6H = 1$

$H(s) = \frac{1}{s^2 + 5s + 6}$

$= \frac{1}{(s+3)(s+2)}$

$A(s+2) + B(s+3) = 1$

$s = -2$ $\quad$ $s = -3$

$B = 1$ $\quad$ $A = -1$

$= \frac{-1}{s+3} + \frac{1}{s+2}$

$h(t) = -e^{-3t} u(t) + e^{2t} u(t)$

(b) If $x(t) = e^{-3t}u(t)$, what is the output $y(t)$?

let $s = j\omega$

$X(s) = \frac{1}{3+s}$ $\qquad$ $X(s)H(s) = \frac{1}{(s+3)^2(s+2)}$

$\frac{A}{(s+3)^2} + \frac{B}{(s+3)} + \frac{C}{s+2}$

$A(s+2) + B(s+2)(s+3) + C(s+3)^2 = 1$ $\qquad$ $y(t) = (-te^{-3t} - e^{-3t} + e^{-2t})u(t)$

$s = -2$ $\quad$ $s = -3$ $\quad$ $s = 0$

$C = 1$ $\quad$ $A = -1$ $\quad$ $-2 + B6 + 9 = 1$

$B = \frac{1-9+2}{6}$

$= -1$

7 (5 + 5 = 10 marks)
We sample a continuous-time voltage signal $v(t)$ using impulse train

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - \frac{k}{3}) .$$

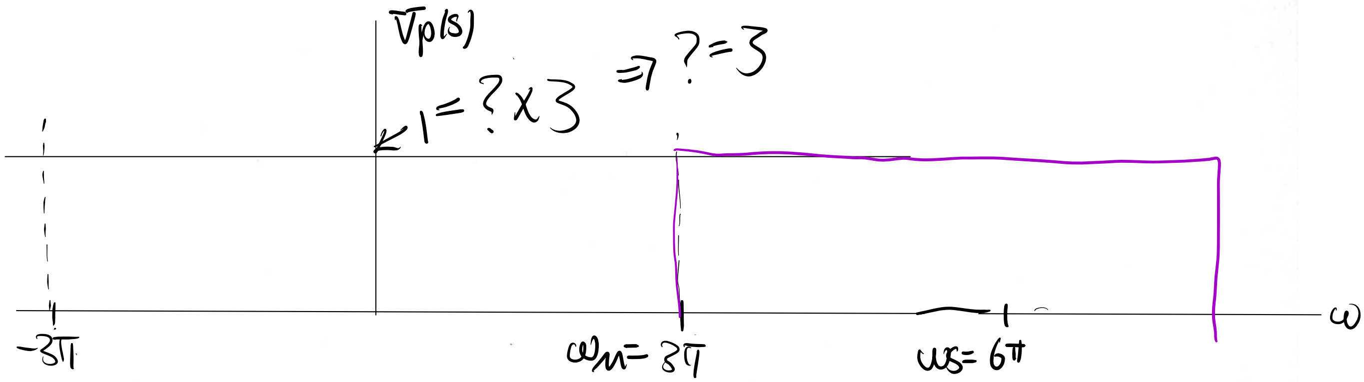
Assume that the sampling rate is above the Nyquist rate with respect to $v(t)$. Suppose the outcome of this sampling is the following (somewhat strange-looking) discrete-time voltage signal:

$$v[n] = \delta[n] .$$

(a) Find and **plot** the Fourier transform of $v_p(t) = v(t)p(t)$.

$v(t)_p = \delta(t) \xleftrightarrow{FT} V_p(s) = 1$

$1 = ? \times 3 \Rightarrow ? = 3$

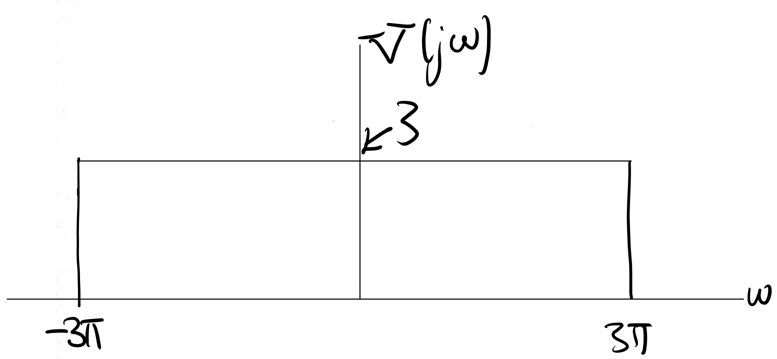


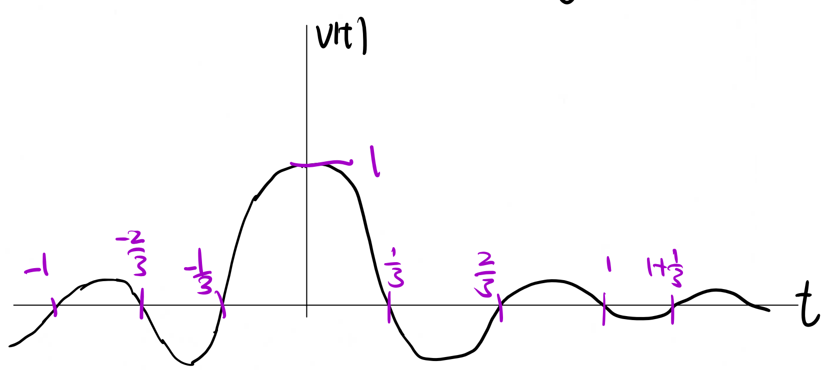
(b) Find and **plot** $v(t)$.

$V_p(0) = 1$

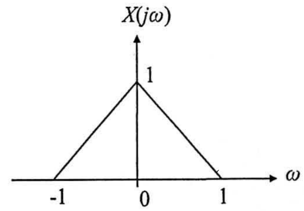
$v_p(t) = \sum_{n=-\infty}^{\infty} v(n\tfrac{1}{3})\,\delta(t - n\tfrac{1}{3}) = \delta(t) \quad v(n\tfrac{1}{3}) = 0 \text{ for } n \in \mathbb{Z}$

$T = \frac{1}{3} \quad \omega_s = \frac{2\pi}{T} = 6\pi \qquad v(t) = \frac{6 \sin(t3\pi)}{2\pi \; t} = \frac{\sin(3\pi t)}{t}$

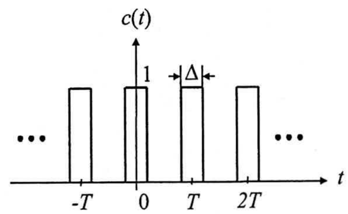




8 (5 + 5 + 5 = 15 marks)
The spectrum of continuous-time signal $x(t)$ is given in the figure below.



The signal is transmitted by amplitude modulation with carrier $c(t)$ as shown below, where $T = 2$ and $\Delta = 0.5$.



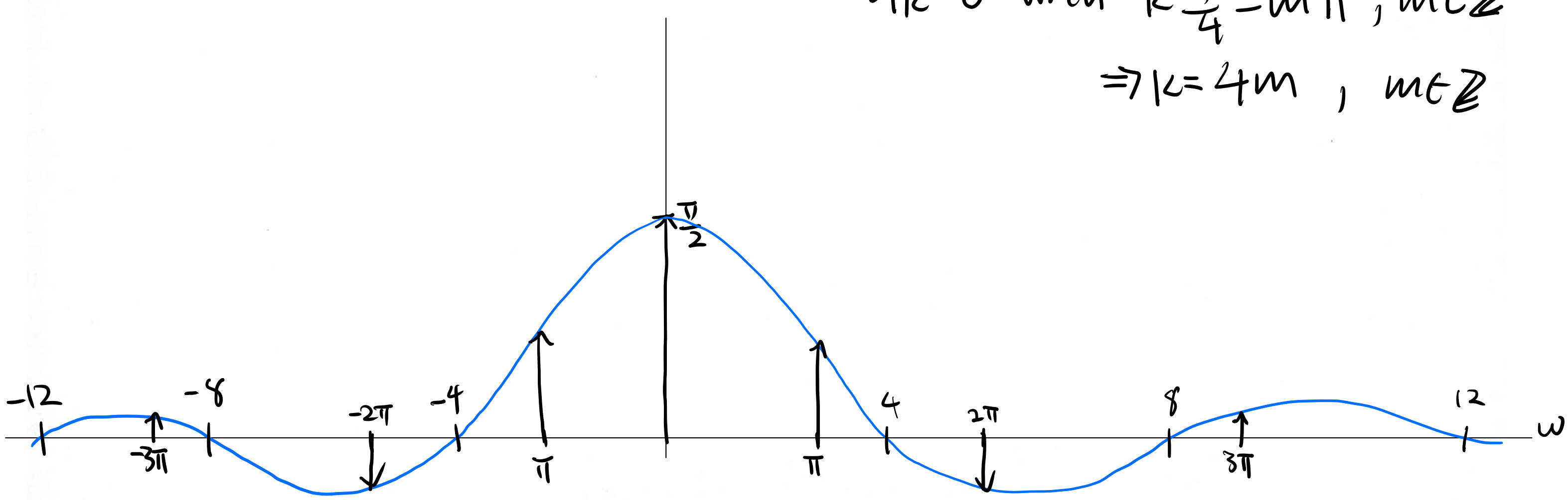
(a) Find and **plot** the spectrum of $c(t)$.

$$C(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\pi) = \sum_{k=-\infty}^{\infty} \overbrace{\frac{2\sin(k\frac{\pi}{4})}{k}}^{2\pi a_k} \delta(\omega - k\pi)$$

$$a_k = \frac{\sin(k\frac{2\pi}{T}\Delta)}{k\pi} = \frac{\sin(k\frac{\pi}{4})}{k\pi} \qquad a_0 = \frac{1}{4} \qquad \omega_0 = \pi \quad T = 2$$

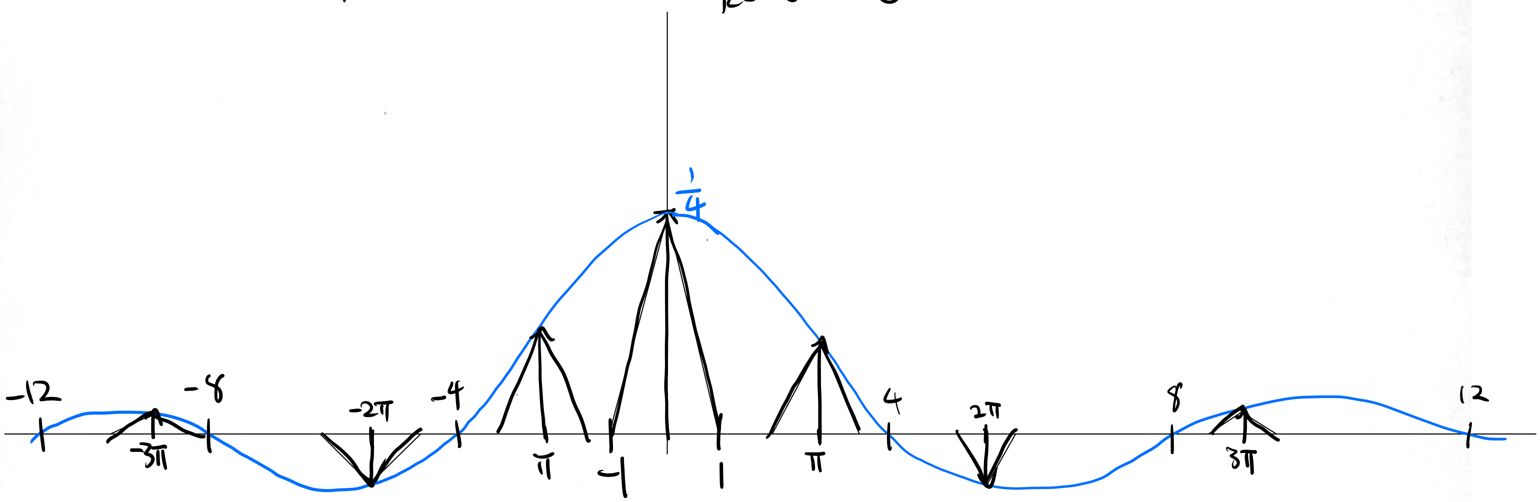
$a_k = 0$ when $k\frac{\pi}{4} = m\pi$, $m \in \mathbb{Z}$

$\Rightarrow k = 4m$, $m \in \mathbb{Z}$



(b) Find and **plot** the spectrum of the transmitted signal $y(t) = x(t)c(t)$.

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * C(j\omega) = \sum_{k=-\infty}^{\infty} a_k X(j(\omega - k\pi))$$



(c) Suppose time-division multiplexing is used, with an additional carrier $c_1(t) = c(t - \frac{T}{2})$ modulated by signal $x_1(t)$. Suppose $x_1(t) = x(t - \frac{T}{2})$. Find the spectrum of the overall transmitted signal $y(t) = \underbrace{x(t)c(t)}_{\updownarrow FT\; Y_1(j\omega)} + \underbrace{x_1(t)c_1(t)}_{\updownarrow FT\; Y_2(j\omega)}$. You do **not** need to plot it.

$$y_2(t) = y_1(t - \tfrac{T}{2}) = y_1(t-1)$$

$$Y_2(j\omega) = e^{-j\omega} Y_1(j\omega)$$

$$Y(j\omega) = (1 + e^{-j\omega}) \sum_{k=-\infty}^{\infty} a_k X(j(\omega - k\pi))$$

Same as part a