Enter the first letter of **your last name** in this box:

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UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DIVISION OF ENGINEERING SCIENCE

ECE 355 – Signal Analysis and Communications

Final Examination, 18:30 - 21:00, December 13, 2016

Examination Type: D
Examiner: Ben Liang

Instructions

- You are allowed one 8.5×11 sheet of handwritten notes (double-sided) and a non-programmable calculator.
- Remember to enter your name on the first page. If there are loose pages, please put your name on every page that you turn in.
- Use the space provided to enter your answers and clearly label them. If you use the back of a page to enter your answers, make sure you clearly indicate that on the font of the page.
- Show intermediate steps for partial credits. Answers without justification will not be accepted.
- Unless otherwise specified, you may use without proof any formula in the Fourier transform tables on Page 2.
- Unless otherwise requested, all final answers must be expressed in simplified form.

MARKS

Question	1	2	3	4	5	6	7	Total
Value	5	5	10	10	15	15	15	75
Mark								

CTFT Pairs and Properties

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Transform pairs:

$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^n}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases} \leftrightarrow \frac{2\sin(\omega T)}{\omega}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Properties:

$$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$$

$$e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))$$

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

$$x(t) * y(t) \leftrightarrow X(j\omega)Y(j\omega)$$

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega-\theta))d\theta$$

If $x(t) \leftrightarrow X(j\omega) = G(\omega)$, then $G(t) \leftrightarrow 2\pi x(-\omega)$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

DTFT Pairs and Properties

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Transform pairs:

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}, \text{ for } |a| < 1$$

$$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^r}, \text{ for } |a| < 1$$

$$\begin{cases} 1, & |n| \le N \\ 0, & |n| > N \end{cases} \leftrightarrow \frac{\sin(\omega(N+1/2))}{\sin(\omega/2)}$$

$$\frac{\sin(Wn)}{\pi n} \leftrightarrow \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}, \text{ for } 0 < W < \pi$$

$$\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Properties:

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

$$nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

$$x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$$

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})Y(e^{j(\omega-\theta)})d\theta$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

1 (5 marks)
A moving-average filter is defined by the following relation between input $x(t)$ and output $y(t)$:

$$y(t) = \int_{t-T}^{t+T} x(\tau)d\tau \ .$$

Determine the frequency response of this filter.

$$h(t) = \int_{t-T}^{t+T} \delta(\tau)\, d\tau = \begin{cases} 1, & |t|<T \\ 0, & |t|>T \end{cases}$$

$$H(j\omega) = \frac{2\sin(\omega T)}{\omega}$$

2 (5 marks)
Within interval $-\pi < \omega < \pi$, the Fourier transform of a discrete-time signal $x[n]$ is given by

$$X(e^{j\omega}) = e^{-j\frac{\omega}{2}} \cdot$$

Determine $x[n]$.

$$X[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\frac{\omega}{2}} e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\omega(-j\frac{1}{2} + jn)} \, d\omega$$

$$= \frac{1}{2\pi} \frac{1}{jn - j\frac{1}{2}} \left(e^{\pi j(-\frac{1}{2} + n)} - e^{-\pi j(-\frac{1}{2} + n)} \right)$$

$$= \frac{1}{\pi(n - \frac{1}{2})} \sin\left(\pi\left(-\frac{1}{2} + n\right)\right)$$

$$= \frac{(-1)^{n+1}}{\pi(n - \frac{1}{2})}$$

3 (5 + 5 = 10 marks)

A periodic signal $x(t)$ with fundamental period T has Fourier series representation $\sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$.

(a) Show that the harmonically related complex exponentials are orthonormal, i.e.,

$$\frac{1}{T}\int_0^T e^{jm\frac{2\pi}{T}t}\left(e^{jn\frac{2\pi}{T}t}\right)^* dt = \delta[m-n], \text{ for all } m, n \in \mathbb{Z}.$$

intergrating over 1.

$$\frac{1}{T}\int_0^T e^{j\frac{2\pi}{T}t(m-n)}\, dt = \begin{cases} 1, & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases} = \delta[m-n]$$

since the intergrand is perodic with T and we are intergrately over 1 period.

(b) Use Part (a) to show Parseval's relation: $\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$.

$$\frac{1}{T}\int_0^T x(t)x^*(t)\,dt = \frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)\left(\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\right) dt$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n a_m^* \underbrace{\frac{1}{T}\int_0^T e^{jn\omega_0 t} e^{-jm\omega_0 t}\, dt}_{\substack{0 \text{ if } m\neq n \\ 1 \text{ if } m=n}}$$

$$= \sum_{k=-\infty}^{\infty} a_k a_k^*$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

4 (5 + 5 = 10 marks)
The Hilbert transform is given by the following frequency response:

$$H(j\omega) = \begin{cases} -j, & \omega > 0, \\ j, & \omega < 0. \end{cases}$$

(a) Find the corresponding impulse response $h(t)$. (*Hint: compare $H(j\omega)$ with the unit step function and use the duality property.*)

$$H_1(j\omega) = \begin{cases} -1, & \omega > 0 \\ 1, & \omega < 0 \end{cases} = u(-\omega) - u(\omega) \xleftrightarrow{FT} \frac{1}{2\pi}\left(\frac{1}{jt} + \pi\delta(t)\right) - \frac{1}{2\pi}\left(\frac{1}{-jt} + \pi\delta(t)\right)$$

$$\frac{j}{2\pi}\left(\frac{2}{jt}\right) = \boxed{\frac{1}{\pi t}}$$

(b) Find the output of the Hilbert transform when the input is $\cos(\omega_0 t)$, where $\omega_0 \neq 0$.

$$\cos(\omega_0 t) \xleftrightarrow{FT} \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

$$Y(j\omega) = \begin{cases} -j\pi\delta(\omega - \omega_0), & \omega > 0 \\ j\pi\delta(\omega + \omega_0), & \omega < 0 \end{cases} = j\pi\left(-\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

$$= \frac{\pi}{j}\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$

$$y(t) = -\sin(\omega_0 t) \xleftrightarrow{FT} = -\frac{\pi}{j}\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right)$$

5 (5 + 5 + 5 = 15 marks)
The design objective of a certain digitally controlled stabilization system is to achieve a causal LTI relation between the continuous-time input $x_c(t)$ and continuous-time output $y_c(t)$, expressed by the following differential equation:

$$y_c''(t) + 5y_c'(t) + 6y(t) = x_c'(t) + 4x_c(t) \; .$$

(a) Find the frequency response, $H_c(j\omega)$, of this system. Plot $|H_c(j\omega)|$ and specify its value at $\omega = 0, \pm 2, \pm 4$.

(b) Find the impulse response, $h_c(t)$, of this system.

(c) Suppose the system is implement by first sampling $x_c(t)$ at frequency $\omega_s = 25\pi$ to obtain discrete-time signal $x[n]$, then filtering $x[n]$ by discrete-time LTI system $h_d[n]$ to obtain $y[n]$, and finally using ideal interpolation on $y[n]$ at frequency $\omega_s = 25\pi$ to recover $y(t)$. Find the required Fourier transform of $h_d[n]$.

$H_d(e^{j\Omega}) = H_c(j\omega/T)$, repeat with period of 2π.

$\underbrace{= H_c(\frac{25}{2}j\omega)}_{\text{for } |\omega| < \pi}$

$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{25\pi} = \frac{2}{25}$

6 (5 + 5 + 5 = 15 marks)
A discrete-time recording of some sensor is denoted by $x[n]$. When the sensor transmits the recorded signal to its data collector, it can choose to send either the original $x[n]$ or a down-sampled version $y[n] = x[3n]$. (In practice, this choice often depends on the available transmission energy, data rate, and fidelity requirement.)

(a) Let $p[n]$ be the $\frac{1}{3}$ sampling pulse train, i.e.,

$$p[n] = \begin{cases} 1, & \text{if } n \text{ is a multiple of } 3 \\ 0, & \text{otherwise.} \end{cases}$$

Show that its Fourier transform is $P(e^{j\omega}) = \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{3})$.

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-3k] \xleftrightarrow{FT} P(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\tfrac{2\pi}{3}), \text{ since } p[n] \text{ is periodic.}$$

$\frac{1}{3}$

$$a_k = \frac{1}{3} \sum_{n=0}^{2} x[n] e^{-jk\omega_0 n} = \frac{1}{3}$$

$x[2] = x[1] = 0$

$x[0] = 1$

(b) Use Part (a) to find the Fourier transform of $y[n]$.

let $x[n] \xleftrightarrow{FT} X(e^{j\omega})$

$$x_p[n] = x[n]p[n] \xleftrightarrow{FT} X_p(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X(e^{j(\omega - k\frac{2\pi}{3})})$$

$$Y(e^{j\omega}) = X_p(e^{j\omega/3}) = \frac{1}{3} \sum_{k=0}^{2} X(e^{j(\omega/3 - k\frac{2\pi}{3})})$$

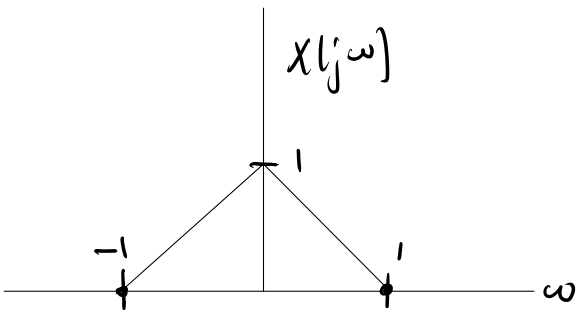
(c) Suppose $x[n]$ is bandlimited by some $\omega_M < \pi$, i.e., $X(e^{j\omega}) = 0$ for $\omega_M < |\omega| < \pi$. In order for the data collector to exactly recover $x[n]$ using $y[n]$, what ω_M values are allowed?

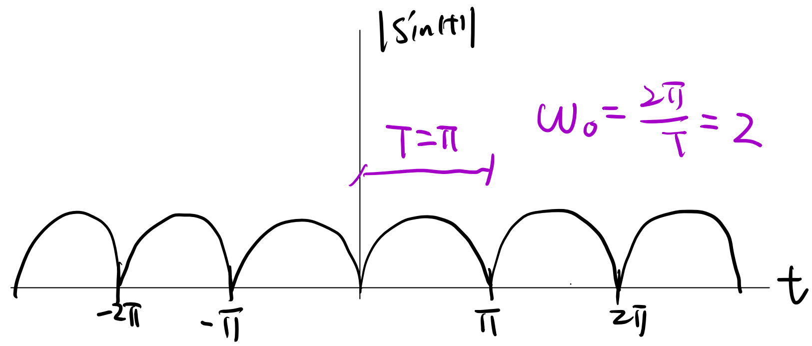
$\frac{2\pi}{3} = \omega_s > 2\omega_m \Rightarrow \frac{\pi}{3} > \omega_m$

7 (10 + 5 = 15 marks)
Consider amplitude modulation using carrier signal $c(t) = |\sin(t)|$, where $|\cdot|$ is the absolute-value operator. Let $x(t)$ be the modulating signal with spectrum

$$X(j\omega) = \begin{cases} 1 - |\omega|, & \text{for } |\omega| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find and **plot** the spectrum of the modulated signal $y(t) = x(t)c(t)$.





$$C(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2k)$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} \sin(t) e^{-jk2t}\,dt = \frac{1}{2j\pi}\int_0^{\pi} e^{jt}e^{-jk2t} - e^{-jt}e^{-jk2t}\,dt$$

$$= \frac{1}{2j\pi}\int_0^{\pi} e^{t(j-jk2)} - e^{t(-j-jk2)}\,dt$$

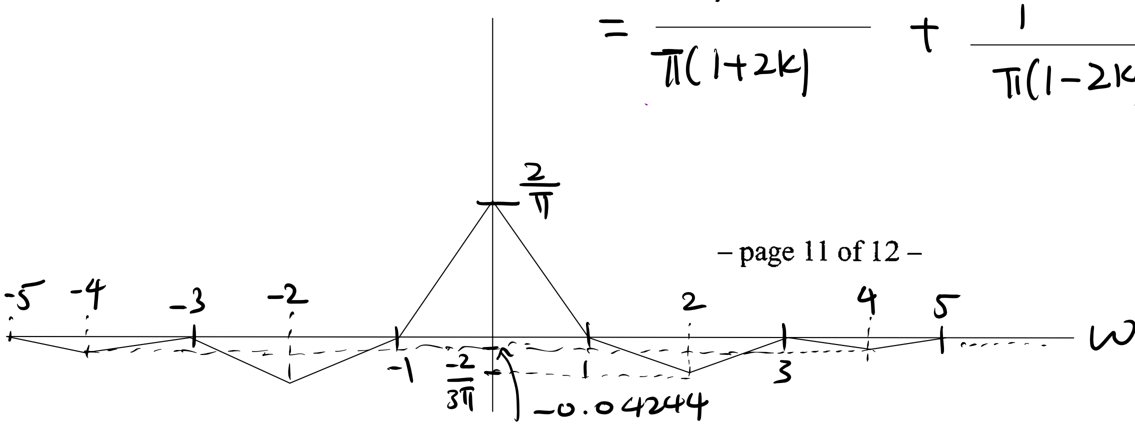
$$= \frac{1}{j2\pi}\frac{1}{j-jk2}\Big(\underbrace{e^{\pi(j-jk2)}}_{-1} - 1\Big) - \frac{1}{j2\pi}\frac{1}{-j-jk2}\Big(\underbrace{e^{\pi(-j-jk2)}}_{-1} - 1\Big)$$

$$= \frac{-1}{j\pi(j-jk2)} + \frac{1}{j\pi(j-jk2)}$$

$$= \frac{1}{\pi(1+2k)} + \frac{1}{\pi(1-2k)}$$

$$Y(j\omega) = \frac{1}{2\pi} C(j\omega) * X(j\omega)$$
$$= \sum_{k=-\infty}^{\infty} a_k X(j(\omega - 2k))$$

$$a_0 = \frac{2}{\pi}$$
$$a_1 = \frac{1}{3\pi} - \frac{1}{\pi} = \frac{-2}{3\pi}$$
$$a_2 = \frac{1}{\quad}$$



(b) Find the LTI filter $h(t)$ such that $x(t)$ can be recovered exactly by $y(t) * h(t)$.

$$H(j\omega) = \begin{cases} \frac{\pi}{2}, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$$

$$\frac{\cancel{\pi}}{2} \frac{1}{\cancel{2}\cancel{\pi}} \frac{\cancel{2}\sin(t)}{t} = \frac{\sin(t)}{2t} = h(t)$$