Fourier series for perdic func.

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

integrate over any period

Fourier transform of a perdioc func.

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt$$

Convergence of fourier series

Perdic func $x(t)$ can be written as FS except at isolate point of discontinuity if:

① $\int_{-\infty}^{\infty}|x(t)|\,dt<\infty$

② there is a finite # of maxium and minima during any single period of $x(t)$

③ In any finite interval of time, there are only a finite # of discontinuites. discontinuities is finite

$$\sin(t)=\frac{1}{2j}\left(e^{jt}-e^{-jt}\right)$$

$$\cos(t)=\frac{1}{2}\left(e^{jt}+e^{-jt}\right)$$

$$\mathrm{sinc}(\theta)=\frac{\sin\pi\theta}{\pi\theta}$$

sinc function is even

sinc(0) = π

FT of Periodic Signal

$$X(j\omega)=\sum_{k=-\infty}^{\infty}2\pi a_k\,\delta(\omega-k\omega_0)$$

Find a_k and ω_0

$$a+jb=\sqrt{a^2+b^2}\,e^{\tan^{-1}\left(\frac{b}{a}\right)}$$

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
$x(at+b)=x(a(t+d))$ scale by a then shift by d		
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ purely real	$X(j\omega)=X^*(-j\omega)$; $\Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\}$; $\Im m\{X(j\omega)\}=-\Im m\{X(-j\omega)\}$; $\|X(j\omega)\|=\|X(-j\omega)\|$; $\sphericalangle X(j\omega)=-\sphericalangle X(-j\omega)$
Symmetry for Real and Even Signals	$x(t)$ real and even; " Im " "	$X(j\omega)$ real and even; " Im " "
Symmetry for Real and Odd Signals	$x(t)$ real and odd; " Im " "	$X(j\omega)$ purely imaginary and odd; " Re " "
Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real]; $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$; $j\Im m\{X(j\omega)\}$

$x(t)$ is not even and odd ⇔ $X(j\omega)$ is not purely Re and purely Im

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}$	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta(\omega-k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$a_1=1$; $a_k=0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$a_1=a_{-1}=\frac{1}{2}$; $a_k=0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$a_1=-a_{-1}=\frac{1}{2j}$; $a_k=0$, otherwise
$x(t)=1$	$2\pi\delta(\omega)$	$a_0=1$, $a_k=0$, $k\neq 0$ (this is the Fourier series representation for any choice of $T>0$)
Periodic square wave $x(t)=\begin{cases}1, & \|t\|<T_1\\ 0, & T_1<\|t\|\le\frac{T}{2}\end{cases}$ and $x(t+T)=x(t)$	$\sum_{k=-\infty}^{+\infty}\frac{2\sin k\omega_0 T_1}{k}\delta(\omega-k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)=\frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$	$a_k=\frac{1}{T}$ for all k
$x(t)\begin{cases}1, & \|t\|<T_1\\ 0, & \|t\|>T_1\end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega)=\begin{cases}1, & \|\omega\|<W\\ 0, & \|\omega\|>W\end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	—
$\delta(t-t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\Re e\{a\}>0$	$\frac{1}{a+j\omega}$	—
$te^{-at}u(t)$, $\Re e\{a\}>0$	$\frac{1}{(a+j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\}>0$	$\frac{1}{(a+j\omega)^n}$	—

Properties of F.S.

$x(t)\xleftrightarrow{FS}a_k$ $\quad y(t)\xleftrightarrow{FS}b_k$ $\quad x(t)$ has fun. per. T, $y(t)$ " " " fre. $\omega_0=\frac{2\pi}{T}$

1. linearity $Ax(t)+By(t)\xleftrightarrow{FS}Aa_k+Bb_k$
2. time shifting $x(t-t_0)\xleftrightarrow{FS}a_k e^{-jk\omega_0 t_0}$
3. time scaling For $\alpha>0$, $x(\alpha t)\xleftrightarrow{FS}a_k$, with fun. fre. $\alpha\omega_0$
4. time Reversal $x(-t)\xleftrightarrow{FS}a_{-k}$
   - I. $x(t)$ is even ⇔ $a_k=a_{-k}$ even a_k
   - II. $x(t)$ is odd ⇔ $a_k=-a_{-k}$ odd a_k
5. Conjugation $x^*(t)\xleftrightarrow{FS}a^*_{-k}$
   - I. $x(t)$ is real ⇔ $a_{-k}=a_k^*$
   - II. $x(t)$ is real and even ⇔ $a_{-k}=a_k=a_k^*$ real & even a_k
   - III. $x(t)$ is real and odd ⇔ $a_k=-a_{-k}=-a_k^*$ purely Im & odd a_k
6. multiplication $x(t)y(t)\xleftrightarrow{FS}c_k=\sum_{l=-\infty}^{\infty}a_l b_{k-l}$
7. Parseval's Relation $\frac{1}{T}\int_T|x(t)|^2dt=\sum_{k=-\infty}^{\infty}|a_k|^2$

LTI and complex exponentials

$x(t)\rightarrow\boxed{h(t)}\rightarrow y(t)$

Continuous

if $x(t)=e^{st}$ & $H(s)$ exists then $y(t)=\underbrace{H(s)}_{\text{eigenvalue}}\underbrace{e^{st}}_{\text{eigen func.}}$

$$H(s)=\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau$$

if $x(t)=\sum_k a_k e^{s_k t}$ then $y(t)=\sum_k a_k H(s_k)e^{s_k t}$

Discrete

if $x[n]=e^{sn}$ & $H(s)$ exists then $y[n]=H(s)e^{sn}$

$$H(s)=\sum_{k=-\infty}^{\infty}h[k]e^{-sk}$$

Discrete fourier Seriers

Sythesis

$$x[n]=\sum_{k=\langle N\rangle}a_k e^{jk\omega_0 n}$$

Analysis

$$a_k=\frac{1}{N}\sum_{k=\langle N\rangle}x[n]e^{-jk\omega_0 n}$$

$\langle N\rangle = N$ successive values

$\langle N\rangle=0,1,\ldots,N-1$ $\langle N\rangle=-1,0,\ldots,N$

$N=\frac{2\pi}{\omega_0}M$ $\quad N,M\in\mathbb{Z}$

N,M has no common factors

For $x[n]$ with fundamental N

Discret time fourier transform

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

Analysis

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

Synthesis

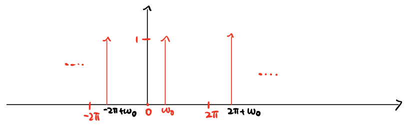
The synthesis equation converge if

$$\sum_{n=-\infty}^{\infty}|x[n]|<\infty$$

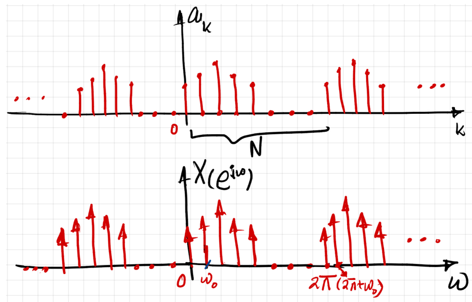
$X(e^{j\omega})$'s period is always 2π

DTFT for Periodic Signals

For $x[n] = e^{j\omega_0 n}$, $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi)$



For periodic $x[n]$, $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta(\omega-k\omega_0)$



Duality for CTFT

if $x(t) \xleftrightarrow{F} X(j\omega) = g(\omega)$

then $g(t) \xleftrightarrow{F} 2\pi\, x(-\omega)$

Properties of discrete FT

given $x[n] \xleftrightarrow{F} X(e^{j\omega})$

① Periodic

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

② linearity

$$a_1x_1[n] + a_2x_2[n] \xleftrightarrow{F} a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$$

③ time & frequency shifting

$$x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0}X(e^{j\omega})$$

$$e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

④ Conjugation

$$x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})$$

if $x[n]$ is real

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

if $x[n]$ is real and even

$$x[n] \xleftrightarrow{F} \mathrm{Re}\{X(e^{j\omega})\}$$

if $x[n]$ is real & odd

$$x[n] \xleftrightarrow{F} \mathrm{Im}\{X(e^{j\omega})\}$$

⑤ Differencing

$$x[n] - x[n-k] \xleftrightarrow{F} (1-e^{-j\omega k})X(e^{j\omega})$$

⑥ Accumulation:

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

⑦ time Reversal

$$x[-n] \xleftrightarrow{F} X(e^{-j\omega})$$

⑧ differentiation in frequency

$$-jn\,x[n] \longleftrightarrow \frac{d}{d\omega}X(e^{j\omega})$$

⑨ Parseval's Relation

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2\,d\omega$$

⑩ Convolution

$$x[n] * h[n] \xleftrightarrow{F} X(e^{j\omega})H(e^{j\omega})$$

⑪ multiplication

$$x_1[n]\,x_2[n] \longleftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,d\theta$$

⑫ duality between DTFT and CTFS

$$x[n] \xleftrightarrow{F} X(e^{j\omega}) \xleftrightarrow{FS} x[-n]$$

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
	• where x^2+bx+c cannot be factorised further	

Solving LCCDE (find $y(t)$) using FT (Continuous) for a given $x(t)$

① Find impulse responds in frequency domain ($H(j\omega)$)

set $x(t)=\delta(t)$ and $y(t)=h(t)$

take the fourier transform of both sides

Solve for $H(j\omega)$

② use partial fraction to find $h(t) = F^{-1}\{H(j\omega)\}$

③ find $X(j\omega)$ and find $H(j\omega)X(j\omega)$

④ $y(t) = h(t) * x(t) = F^{-1}\{H(j\omega)X(j\omega)\}$

use partial fraction

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases}1, & k=m, m\pm N, m\pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos\omega_0 n$	$\pi\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases}\frac{1}{2}, & k=\pm m, \pm m\pm N, \pm m\pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin\omega_0 n$	$\frac{\pi}{j}\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0-2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$, $a_k = \begin{cases}\frac{1}{2j}, & k=r, r\pm N, r\pm 2N,\dots\\ -\frac{1}{2j}, & k=-r, -r\pm N, -r\pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$	$a_k = \begin{cases}1, & k=0, \pm N, \pm 2N,\dots\\ 0, & \text{otherwise}\end{cases}$
Periodic square wave $x[n] = \begin{cases}1, & \lvert n\rvert \le N_1\\ 0, & N_1 < \lvert n\rvert \le N/2\end{cases}$ and $x[n+N] = x[n]$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin[2\pi k/2N]}$, $k \neq 0, \pm N, \pm 2N,\dots$; $a_k = \frac{2N_1+1}{N}$, $k = 0, \pm N, \pm 2N,\dots$
$\sum_{k=-\infty}^{+\infty}\delta[n-kN]$	$\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$	—
$x[n]\begin{cases}1, & \lvert n\rvert \le N_1\\ 0, & \lvert n\rvert > N_1\end{cases}$	$\frac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wn}{\pi}\right)$, $0 < W < \pi$	$X(\omega) = \begin{cases}1, & 0 \le \lvert\omega\rvert \le W\\ 0, & W < \lvert\omega\rvert \le \pi\end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$	—
$\delta[n-n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^r}$	—

Enter the first letter of **your last name** in this box:

FIRST NAME: ________________ LAST NAME: ________________ ID#: ________________

UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DIVISION OF ENGINEERING SCIENCE

ECE 355 – Signal Analysis and Communications

Final Examination, 18:30 - 21:00, December 7, 2018

Examination Type: D
Examiner: Ben Liang

Instructions

- You are allowed one 8.5×11 sheet of ***handwritten*** notes (double-sided) and a non-programmable calculator.
- Remember to enter your name on the first page. If there are loose pages, please put your name on every page that you turn in.
- Use the space provided to enter your answers and clearly label them. If you use the back of a page to enter your answers, make sure you clearly indicate that on the font of the page.
- Show intermediate steps for partial credits. Answers without justification will not be accepted.
- Unless otherwise specified, you may use without proof any formula in the Fourier transform tables on Page 2.
- Unless otherwise requested, all final answers must be expressed in simplified form.

MARKS

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Total**
Value	10	5	5	15	10	10	10	15	80
Mark									

CTFT Pairs and Properties

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Transform pairs:

$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^n}, \text{ for } \Re\mathfrak{e}\{a\} > 0$$

$$\begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases} \leftrightarrow \frac{2\sin(\omega T)}{\omega}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Properties:

$$x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$$

$$e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))$$

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

$$x(t) * y(t) \leftrightarrow X(j\omega)Y(j\omega)$$

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega-\theta))d\theta$$

If $x(t) \leftrightarrow X(j\omega) = G(\omega)$, then $G(t) \leftrightarrow 2\pi x(-\omega)$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

DTFT Pairs and Properties

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Transform pairs:

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}, \text{ for } |a| < 1$$

$$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\omega})^r}, \text{ for } |a| < 1$$

$$\begin{cases} 1, & |n| \le N \\ 0, & |n| > N \end{cases} \leftrightarrow \frac{\sin(\omega(N+1/2))}{\sin(\omega/2)}$$

$$\frac{\sin(Wn)}{\pi n} \leftrightarrow \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}, \text{ for } 0 < W < \pi$$

$$\sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Properties:

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

$$nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

$$x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$$

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})Y(e^{j(\omega-\theta)})d\theta$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

1 (5 + 5 = 10 marks)
Consider an LTI system with impulse response

$$h(t) = \frac{1}{(2 + jt)^2} .$$

(a) Find its frequency response $H(j\omega)$.

(b) Is this system BIBO stable? (*Hint:* $|h(t)| = \left|\frac{1}{2+jt}\right|^2$.)

2 (5 marks)
Starting from the definition of the Fourier transform, show that if $x(t) \leftrightarrow X(j\omega)$ then for all $a \neq 0$ we have

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

3 (5 marks)
Periodic discrete-time signal $x[n]$ has period 8 and Fourier series coefficients

$$a_k = \cos(\frac{k\pi}{4}) + \sin(\frac{3k\pi}{4}) \, .$$

Find $x[n]$.

4 (5 + 5 + 5 = 15 marks)
Consider an LTI system with impulse response

$$h(t) = \frac{d^2}{dt^2}\delta(t) \, .$$

(a) Find its frequency response $H(j\omega)$.

(b) Is this system invertible? If not, explain why not. If yes, find the impulse response of the inverse LTI system.

(c) Suppose we wish to implement LTI system $h(t)$ in discrete-time (e.g., using a microprocessor), with sampling period T and cutoff frequency ω_c. Assume that sampling and reconstruction are ideal, and there is no aliasing. Find and **plot** the required discrete-time frequency response $H_d(e^{j\Omega})$.

5 (5 + 5 = 10 marks)
Consider the periodic signal

$$x(t) = \frac{\sin(\frac{5t}{2})}{\sin(\frac{t}{2})}.$$

(a) Express $x(t)$ in Fourier series. (*Hint: use the tables on Page 2. Don't forget to specify the fundamental frequency.*)

(b) Find the Fourier series of $y(t) = x(2t)$.

6 (5 + 5 = 10 marks)
A causal LTI system with input $x(t)$ and output $y(t)$ is described by the following differential equation:

$$y''(t) + 5y'(t) + 6y(t) = x(t) \; .$$

(a) Find the impulse response, $h(t)$, of this system.

(b) If $x(t) = e^{-3t}u(t)$, what is the output $y(t)$?

7 (5 + 5 = 10 marks)
We sample a continuous-time voltage signal $v(t)$ using impulse train

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - \frac{k}{3}) \,.$$

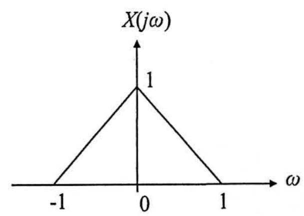
Assume that the sampling rate is above the Nyquist rate with respect to $v(t)$. Suppose the outcome of this sampling is the following (somewhat strange-looking) discrete-time voltage signal:

$$v[n] = \delta[n] \,.$$

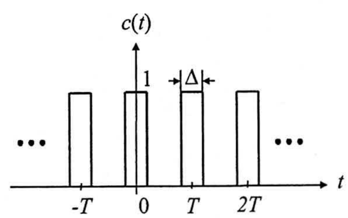
(a) Find and **plot** the Fourier transform of $v_p(t) = v(t)p(t)$.

(b) Find and **plot** $v(t)$.

8 (5 + 5 + 5 = 15 marks)
The spectrum of continuous-time signal $x(t)$ is given in the figure below.



The signal is transmitted by amplitude modulation with carrier $c(t)$ as shown below, where $T = 2$ and $\Delta = 0.5$.



(a) Find and **plot** the spectrum of $c(t)$.

(b) Find and **plot** the spectrum of the transmitted signal $y(t) = x(t)c(t)$.

(c) Suppose time-division multiplexing is used, with an additional carrier $c_1(t) = c(t - \frac{T}{2})$ modulated by signal $x_1(t)$. Suppose $x_1(t) = x(t - \frac{T}{2})$. Find the spectrum of the overall transmitted signal $y(t) = x(t)c(t) + x_1(t)c_1(t)$. You do **not** need to plot it.