Signal Analysis & Communication

ECE355H1F

Lec. 2, Wk 2

14-09-2022

Exponential & Sinusoidal Signals (ch 1.3)

- Occur frequently
- Basic Building block for constructing many other signals

A. CT Complex Exponential & Sinusoidal Signals:

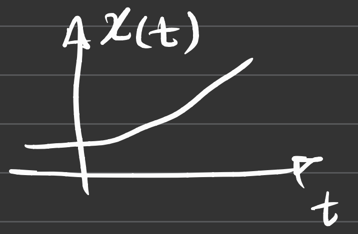
$$x(t) = Ce^{at}$$

C and a are complex numbers

(a) Real Exponential

If C and a are real

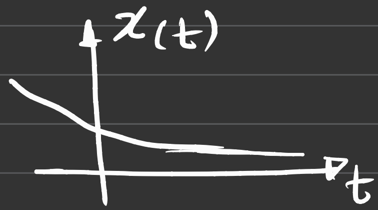
'a' +ve



Examples

- ✓ Chain reaction in atomic explosions
- ✓ Complex chemical reactions

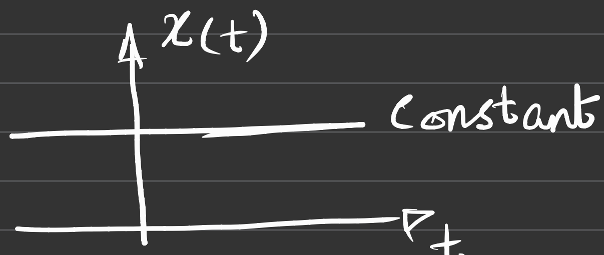
'a' -ve



Examples

- ✓ RC Circuit
- ✓ Radioactive Decay
- ✓ Damped mech. circuit

'a' 0



(b) Periodic Complex Exponential & Sinusoidal

Case I

'a' imaginary, 'C' real (=1 for simplicity)

$$x(t) = e^{j\omega_0 t}$$

- Important property - Periodic

$$\Rightarrow \quad e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}$$

To be periodic: $e^{j\omega_0 T} = 1$

two cases: $\omega_0 = 0 \Rightarrow x_{(t)} = 1$ constant periodic for any period

$\omega_0 \neq 0$ In this case, to be periodic

$$T_0 = \frac{2\pi}{|\omega_0|}$$

as $e^{j2\pi} = 1$

Also, $e^{j\omega_0 T}$ & $e^{-j\omega_0 T}$ both have same period.

Example

$$x_{(t)} = \cos\omega_0 t$$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$= \mathrm{Re}\left[e^{j\omega_0 t}\right]$$

ω: rad/s

$\omega_0 = 2\pi f_0$

f_0: cycles/sec

Similarly,

$$x_{(t)} = \sin\omega_0 t$$

$$= \mathrm{Im}\left[e^{j\omega_0 t}\right]$$

Case II

'a' is imaginary, c is complex

$$c = |c| e^{j\phi}$$

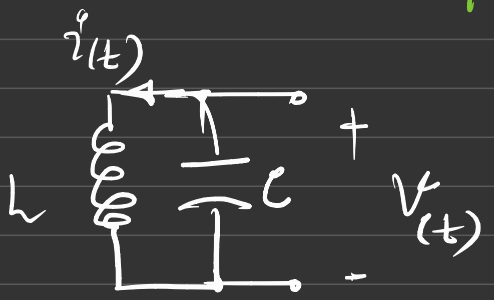
$$x_{(t)} = c e^{j\omega_0 t}$$

$$= |c| e^{j(\omega_0 t + \phi)}$$

$$= |c|\cos(\omega_0 t + \phi) + j|c|\sin(\omega_0 t + \phi)$$

Periodic with period $T_0 = \frac{2\pi}{\omega_0}$

Examples: - physical system in which energy is conserved (LC Circuit)
- Mech System
- Accoustic pressure to single music tone.



$$C\frac{dV}{dt} = -i(t)$$

$$L\frac{di}{dt} = V(t)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

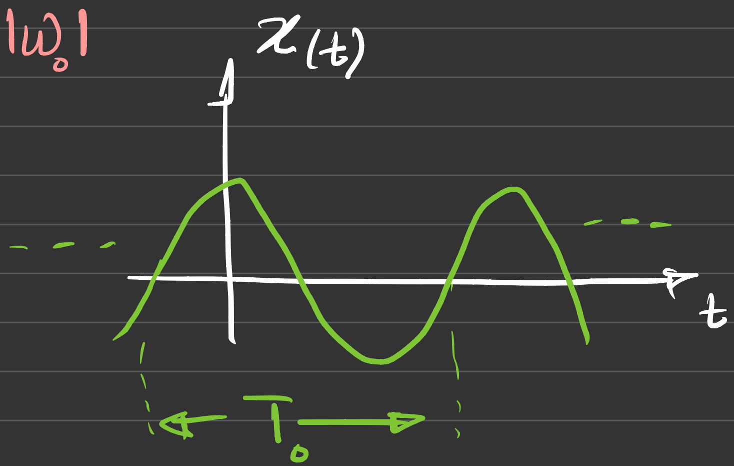
If $L = C$

$$\frac{1}{L} = \frac{1}{C} = \omega_0$$

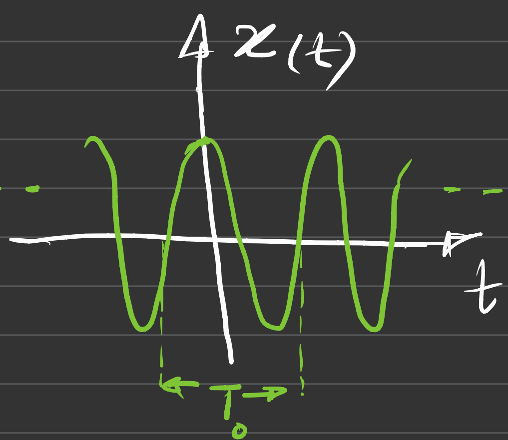
with initial conditions
$V(0) = V_0$, $i(0) = 0$

$$T_0 \propto \frac{1}{|\omega_0|}$$

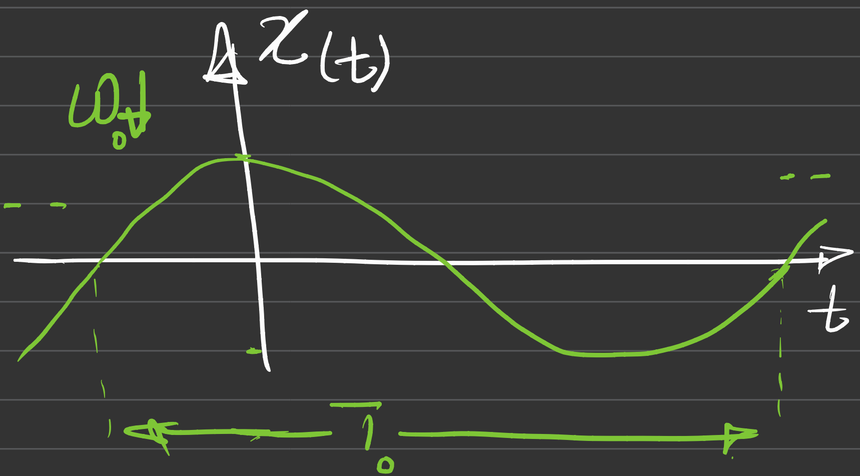
ω_0



$\omega_0 \uparrow$



$\omega_0 \downarrow$



for $\omega_0 = 0 \rightarrow$ Constant Signal

Energy & Power of Periodic Complex Exp. Sig.

$$E_{period} = \int_0^{T_0} |e^{j\omega_0 t}|^2 dt$$ Case I

$$= \int_0^{T_0} 1 \times dt = T_0$$ (finite)

$$E_\infty = \infty$$ (infinite)

$$P_{period} = \frac{E_{period}}{T_0} = 1$$ (finite)

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 dt = 1$$ (finite)

(c) General Complex Exponential Signals

$$C e^{at}$$

$$C = |C| e^{j\theta}$$ polar form

$$a = r + j\omega_0$$ cartesian form

$$x_{(t)} = |C| e^{j\theta} e^{(r + j\omega_0)t}$$

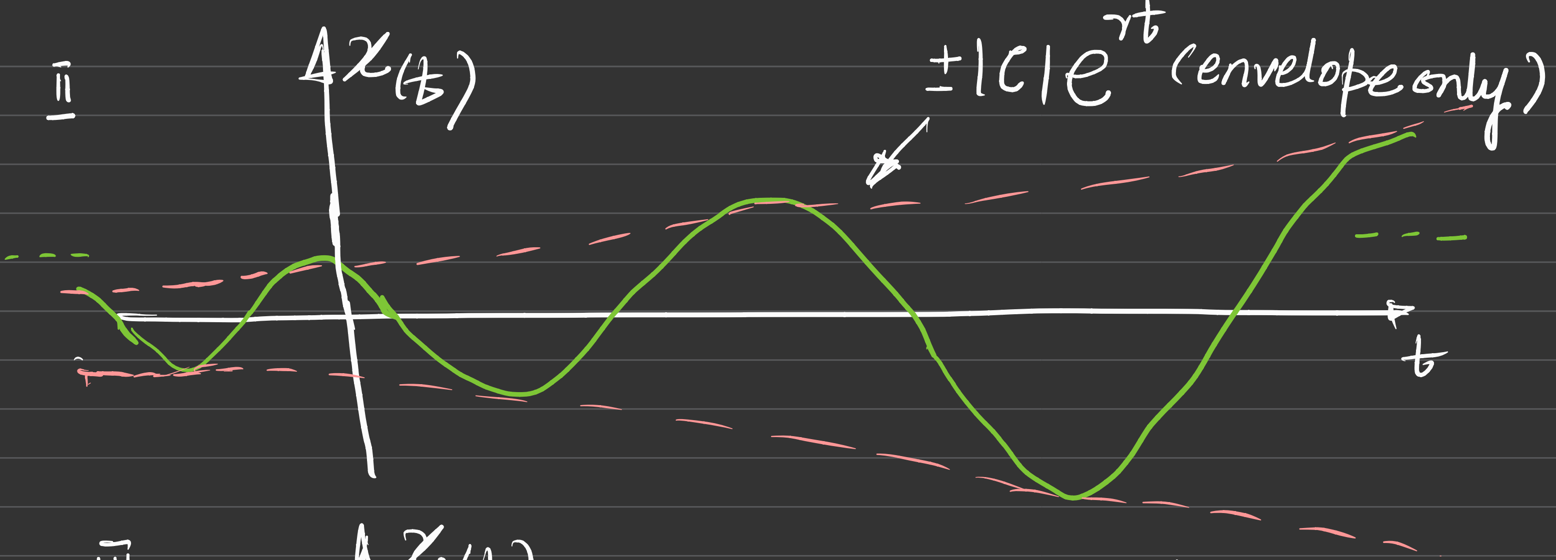
$$= |C| e^{rt} e^{j(\omega_0 t + \theta)}$$

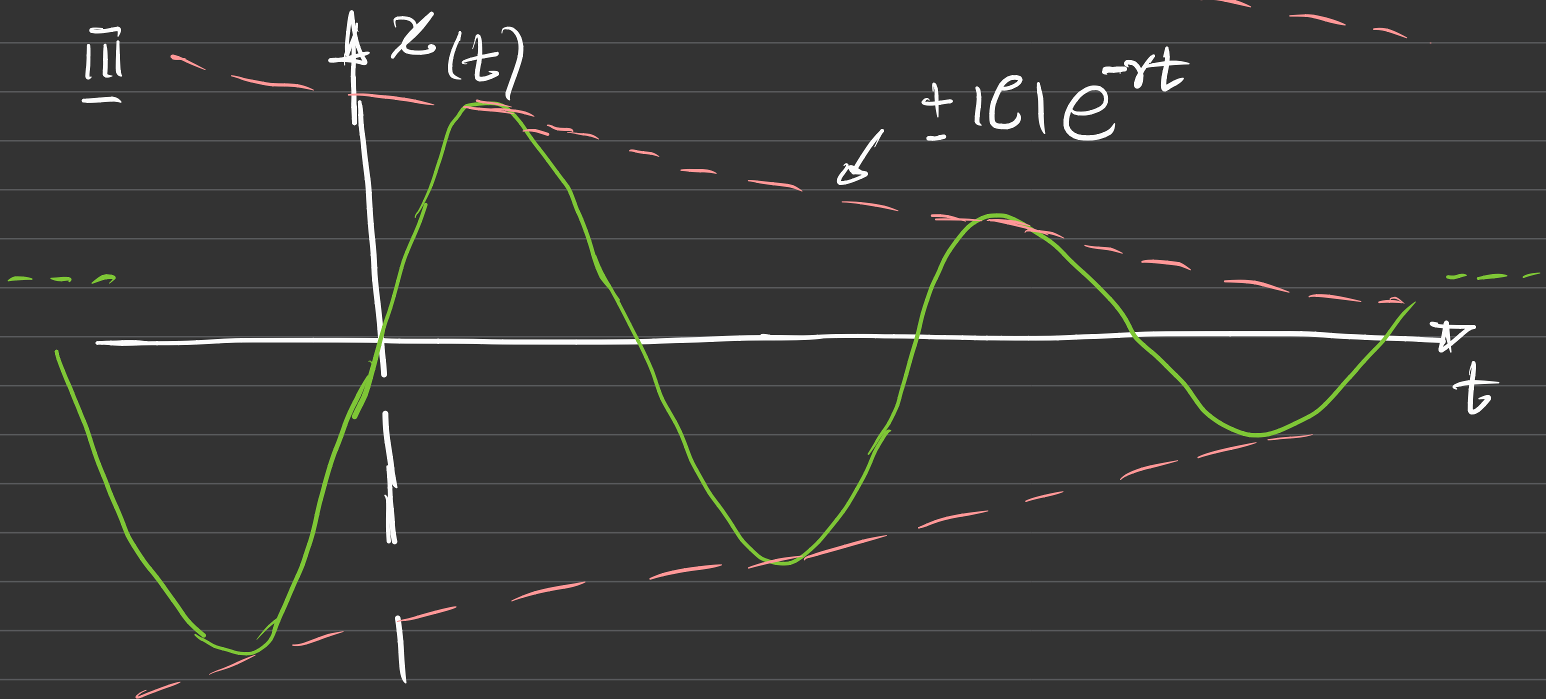
$$Re\{x_{(t)}\} = |C| e^{rt} \cos(\omega_0 t + \theta)$$

Case I : $r = 0$ the real & imaginary parts - sinusoidal

Case II : $r > 0$ Sinusoidal Sig. multiplied by a growing exp.

Case III : $r < 0$ Sinusoidal Sig. multiplied by a decaying exp.





Examples: - Damped Sinusoid
- RLC Circuit (R dissipates energy)
- mech. Sys. (friction dissipates energy)
- Pushing a swing.

B. DT Complex Exponential & Sinusoidal Signals

$$x[n] = C\alpha^n$$ (Convenient representation for DT)

C & α : Complex numbers

$$x[n] = Ce^{\beta n}$$ (analogous to CT sig)

where, $\alpha = e^{\beta}$

(a) Real Exp. Sig.

C & α real

Cases:

- $\rightarrow |\alpha| > 1$ growing exp.
- $\rightarrow |\alpha| < 1$ decaying exp.
- $\rightarrow \alpha$ +ve $\quad C\alpha^n$ - same sign for all 'n'
- $\rightarrow \alpha$ -ve $\quad C\alpha^n$ - sign alternates
- $\rightarrow \alpha = 1 \quad C$
- $\rightarrow \alpha = -1$ alternates b/w $+C$ & $-C$

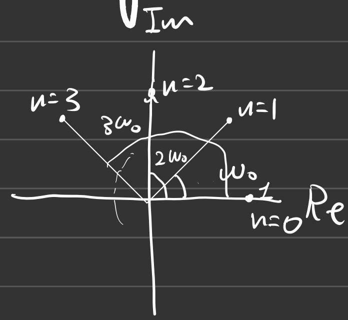
Example: Total return on investment as a function of month/yr.

(b) Sinusoidal Sig.

$$x[n] = Ce^{\beta n} = C\alpha^n$$

Case I $\quad \beta$: imaginary, C: real

$$x[n] = e^{j\omega_0 n} \quad \text{(assuming } C = 1)$$



Example: $x[n] = \cos\omega_0 n$

Remember, $e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$.

Also, $\cos\omega_0 n = Re\{e^{j\omega_0 n}\}$

$\sin\omega_0 n = Im\{e^{j\omega_0 n}\}$

Case II $\quad a$: imaginary, C: complex

$$x[n] = |C|e^{j\phi}e^{j\omega_0 n}$$

$$= |C|e^{j(\omega_0 n + \phi)}$$

$$= |C|\cos(\omega_0 n + \phi) + j|C|\sin(\omega_0 n + \phi)$$

ⓒ General Complex Exponential Sig.

$$x[n] = C\alpha^n$$

Both C & α are complex

$$C = |C|e^{j\theta} \quad \text{(polar)}$$

$$\alpha = |\alpha|e^{j\omega_0} \quad \text{(polar)}$$

$$x[n] = |C||\alpha|^n e^{j\theta} e^{j\omega_0 n}$$

magnitude as a function of n.

$$= |C||\alpha|^n e^{j(\omega_0 n+\theta)}$$

$$= |C||\alpha|^n [\cos(\omega_0 n+\theta) + j\sin(\omega_0 n+\theta)]$$

Case I: $|\alpha| = 1$ Sinusoidal

Case II: $|\alpha| > 1$ Sinusoidal sequence - growing exp.

Case III: $|\alpha| < 1$ Sinusoidal sequence - decaying exp.

* Periodicity properties of DT Complex Exp. Sig.

- Different than that of CT Complex Exp. Sig.
- CT: Complex Exp. Sig. $e^{j\omega_0 t}$ compression

① $\omega_0 \uparrow$, $T_0 \downarrow$ & vice versa (Distinct)

② periodic for any value of ω_0

with $T_0 = 2\pi/|\omega_0|$ ← any value.

- DT: Complex Exp. Sig $e^{j\omega_0 n}$ n is integers.

$$① \; \underbrace{e^{j(\omega_0+2\pi)n} = e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}}_{\text{Not Distinct}}$$

The sig. with freq. ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$ are all identical.
Therefore, we only consider freq. interval of length 2π to choose ω_0:

- Either $0 \leq \omega_0 < 2\pi$
- OR $-\pi \leq \omega_0 < \pi$

- As $\omega_0 \uparrow$ from zero — rate of oscillation $\uparrow$
- until we reach π
- as $\omega_0 \uparrow$ further — rate of oscillation $\downarrow$
- until we reach 2π
- At $\omega_0 = 2\pi$ — same constant sequence as at $\omega_0 = 0$
- Near $\omega_0 = 0, 2\pi$, even multiples of π — low freq. sig.
- Near $\omega_0 = \pm\pi$, odd multiples of π — high freq. sig.

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n \text{ — oscillate rapidly!}$$

② In order for DT sig. to be periodic:

$$e^{j\omega_0 (n+N)} = e^{j\omega_0 n} \qquad N\text{—period} \quad N>0$$

$$\Rightarrow \quad e^{j\omega_0 N} = 1$$

Suggests '$\omega_0 N$' should be an integral multiple of 2π
i.e., $\omega_0 N = 2\pi m$

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

$\therefore$ $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number & is not periodic otherwise.

The same is true for DT Sinusoids.

If $\omega_0 \neq 0$ & if m & N have no factors in common, then the fundamental period of $e^{j\omega_0 n}$

$$N = m\left(\frac{2\pi}{\omega_0}\right)$$

& the fundamental freq. is:

$$\frac{\omega_0}{m} = \frac{2\pi}{N}$$