Signal Analysis & Communication
ECE355H1 F

Ch 3.2:

Response of LTI systems to Complex Exponentials.

Ch 3.3:

Fourier Series Representation of CT Periodic Signals.

Lec 3, Wk 5

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Response of LTI to Complex Exponentials (ch. 3.2)

Recap:

LTI — Important special case of LCCDE

Solution of LCCDE: e.g., with $x(t) = e^{3t}u(t)$

guess $y_p(t) = Be^{3t}u(t)$

CT: $e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ?$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

Conv. Comm. property

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

Define $$\boxed{H(s) \triangleq \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}$$

"Laplace transform of $h(\cdot)$"

"Transfer function" of LTI system

Theorem: If $x(t) = e^{st}$ & $H(s)$ exists,

$$y(t) = H(s)e^{st}$$

Two interpretations: ① Same complex exponential as I/P e^{st}

② Scaled by constant $H(s)$

Eigen value

Eigen function of LTI sys. $h(t)$

apply e^{st} to LTS system get a scaled version of e^{st}

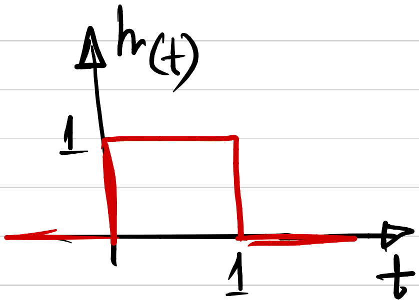
Corollary: If the I/p to an LTI sys. with impulse response $h(t)$ is $x(t) = \sum_k a_k e^{s_k t}$, then its o/p is given by

any periodic period can by written as this.

$$y(t) = \sum_k a_k H(s_k) e^{s_k t} \quad \text{—} \; (A)$$

Eg.

$$h(t) = u(t) - u(t-1)$$



$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$= \int_0^1 e^{-s\tau} d\tau$$

$$= \begin{cases} \frac{1}{s}(1-e^{-s}), & s \neq 0 \\ 1, & s = 0 \end{cases}$$

If $x(t) = \cos(4t)$

$$= \underbrace{\frac{1}{2}}_{a_1, a_2} \left[e^{\underbrace{j4}_{s_1} t} + e^{\underbrace{j(-4)}_{s_2} t} \right]$$

using (A)

$$y(t) = \frac{1}{2} \left[\frac{1}{j4}(1-e^{-j4}) e^{j4t} + \frac{1}{-j4}(1-e^{j4}) e^{-j4t} \right]$$

$$= \frac{1}{4} \left[\sin(4t) - \sin(4t-4) \right]$$

DT: $\quad e^{sn} = z^n \quad$ where $\quad z \triangleq e^{s}$

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$ Conv. Comm. property

$$= z^n \sum h[k]\, z^{-k}$$

Define $$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$ "z-transform" "transfer function"

If $H(z)$ exists, then

$$y[n] = \underbrace{H(z)}_{\text{Eigen value}}\ \underbrace{z^n}_{\text{Eigen function}}$$

<u>NOTE:</u> $(e^x)^y \neq e^{xy}$ for general $x, y \in \mathbb{C}$

But $(e^x)^n = e^{xn}$ for $n \in \mathbb{Z}$

<u>NOTE:</u> In Fourier Series/transforms, we focus on sinusoidal complex exponentials of the form $e^{j\omega t}$ & $e^{j\omega n}$

$e^{j\omega t}$: periodic, fundamental period $\frac{2\pi}{|\omega|}$

$e^{j\omega n}$: Periodic iff $\omega = \frac{2\pi k}{N}$ for some $k, N \in \mathbb{Z}$.

CT Fourier Series (ch. 3.3)

Recall: ✓ Periodic signal $x(t)$: $x(t) = x(t+T), \forall t$

✓ Fundamental period: minimum positive value of 'T'

✓ Fundamental (angular) frequency, $\omega_0 = \frac{2\pi}{T}$

FACT: "Almost all" periodic signals can be expressed as a sum of harmonically related complex exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k t}{T}}$$

$a_k e^{jk\omega_0 t}$: $|k|$th harmonic component, for $k \neq 0$

$k = \pm 1$: first harmonic components "fundamental components"

a_0 : constant component "DC"

Circuit Analysis

Eg. $x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$

Fundamental period $T = 1$

frequency $\omega_0 = 2\pi$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt}$$

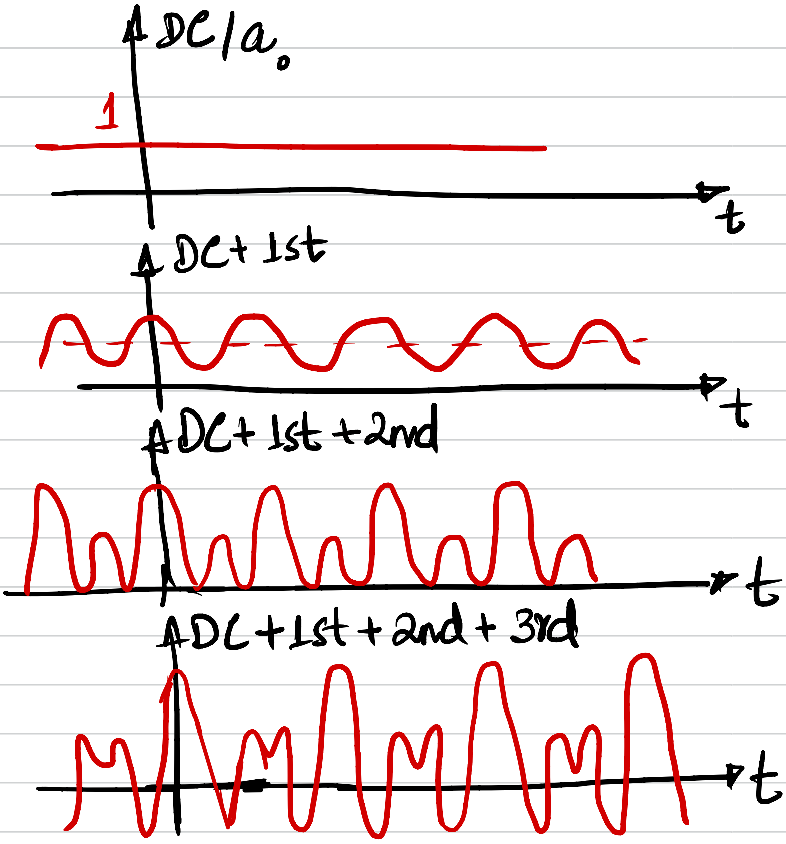
when $a_0 = 1$

$a_1 = a_{-1} = 1/4$

$a_2 = a_{-2} = 1/2$

$a_3 = a_{-3} = 1/3$

$a_k = 0$, o/w



periodic

NOTE:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$x(t) = \sum_k a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_k a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$H(j\omega)$: Frequency Response of LTI System $h(t)$