Signal Analysis & Communication
ECE355H1F

ch. 4.1

Continuous-Time Fourier Transform
(CTFT)

Lec 3, Wk 7
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(CTFT)

Introduction:

Periodic signals $\xrightarrow{\text{Represented as}}$ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

This representation can be used in describing the effect of LTI systems on signals.

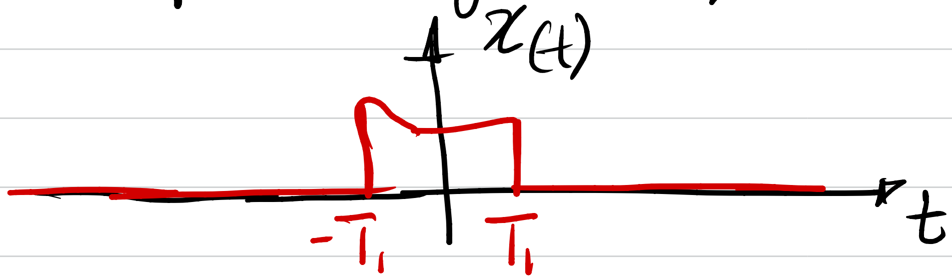
(using) $e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$

- Let's extend these concepts to apply to signals that are aperiodic.

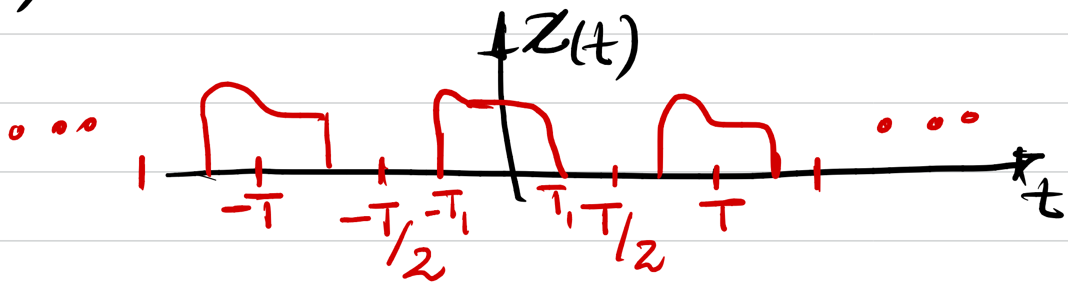
- Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesimally close in freq., & the representation in terms of linear combination takes the form of an integral rather than a sum.

- The resulting spectrum of coefficients in this representation is called the "Fourier Transform"

- Consider aperiodic signal $x(t)$



- From this aperiodic sig., we can construct a periodic sig. $\tilde{x}(t)$ for which $x(t)$ is one period.



$$\tilde{x}(t) = x(t) \quad \text{for } -\frac{T}{2} \leq t \leq \frac{T}{2}$$

- Fourier series representation of this sig. is:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{--- (1)}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$ & $x(t) = 0$ o/w

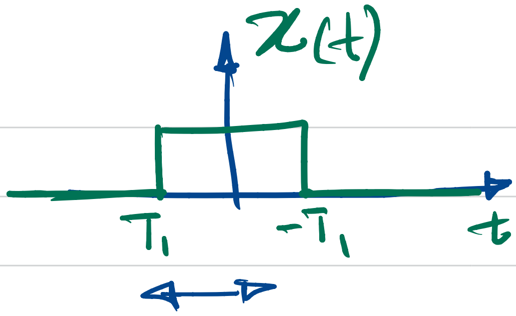
$$Ta_k = \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt = X(jk\omega_0) \text{ --- (2)}$$

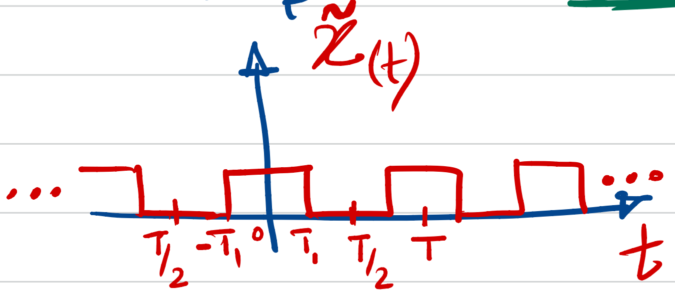
- Define envelope of Ta_k as $X(j\omega)$

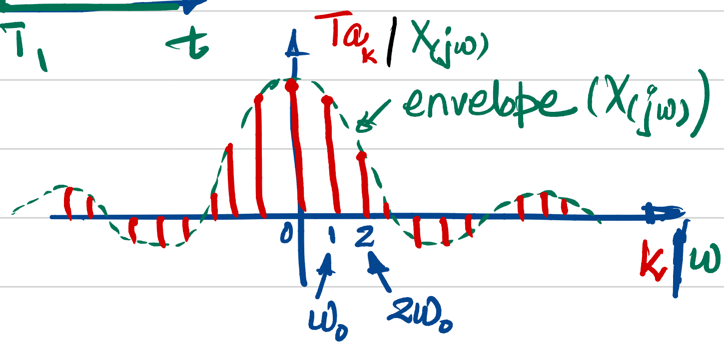
$k\omega_0 = \omega$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{--- (3)}$$

For example:







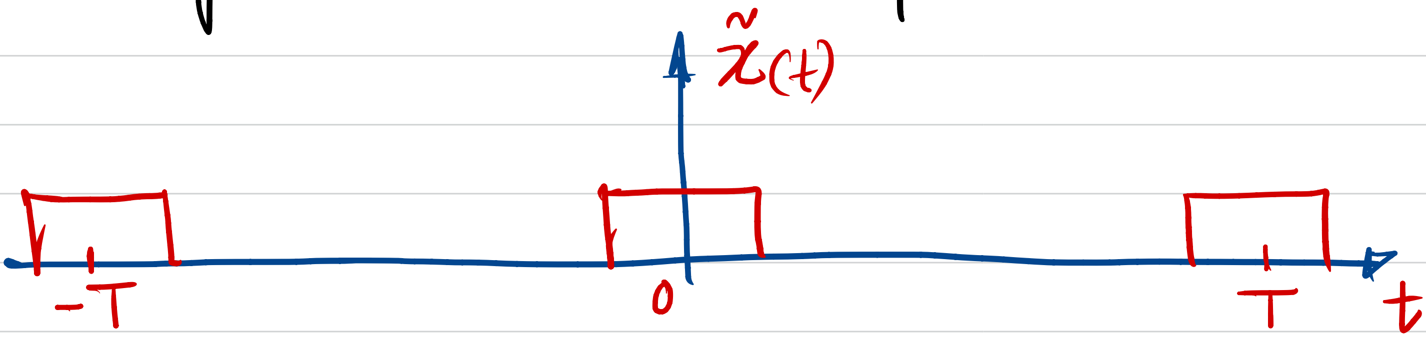
- Combining ① & ②, we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t}$$

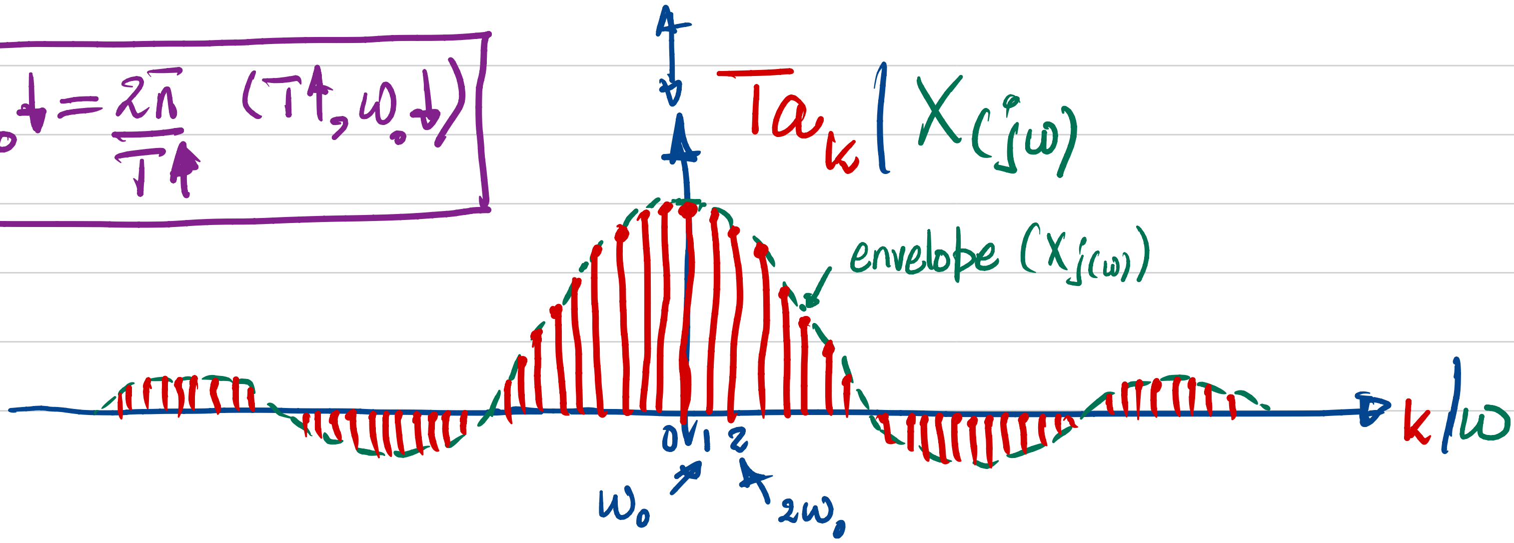
Since $\omega_0 = 2\pi/T \Rightarrow T = 2\pi/\omega_0$

$$\therefore \tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0 \quad - ④$$

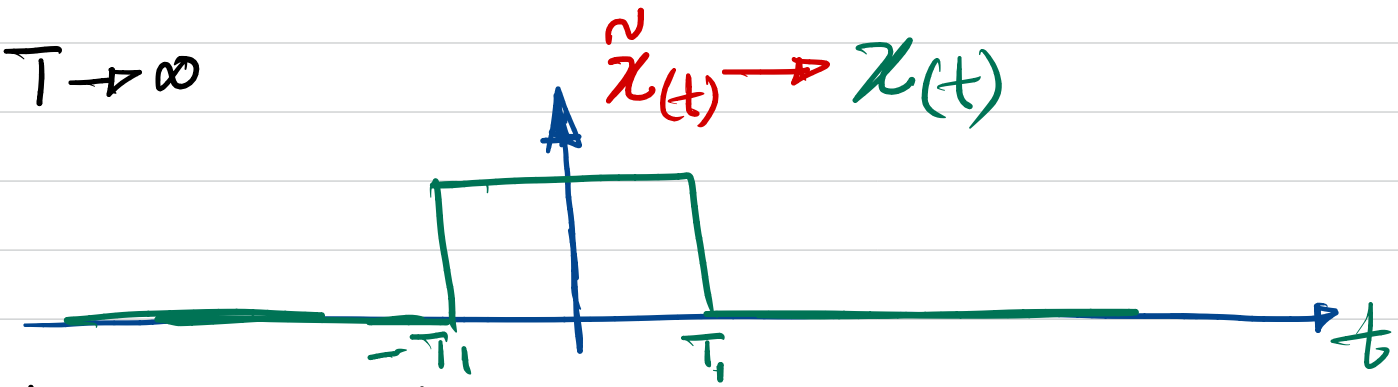
- Larger 'T' for above example will result:



$$\boxed{\omega_0\downarrow = \frac{2\pi}{T\uparrow} \quad (T\uparrow, \omega_0\downarrow)}$$



- As $T \to \infty$

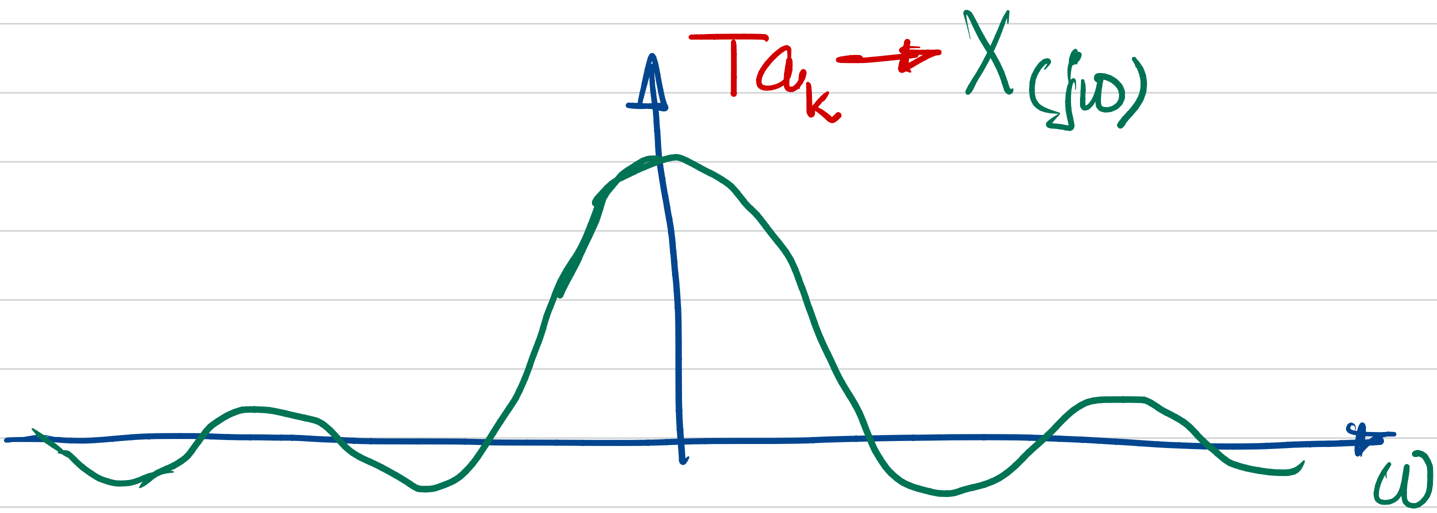


- And consequently, in the limit ④ becomes a representation of $x(t)$.

- Moreover, $\omega_0 \to 0$ as $T \to \infty$, & the RHS of ④ passes to an integral.

④ $\Rightarrow$ $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

& ③ $\Rightarrow$ $$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$



CONCLUSION:

- In general, a sig. $x(t)$ can be represented as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$ Synthesis (Inverse FT)

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$ Analysis (FT of $x(t)$)

- The transform $X(j\omega)$ of an aperiodic sig. $x(t)$ is commonly referred to as the spectrum of $x(t)$.
- It provides us with the information needed for describing $x(t)$ as a linear combination (specifically, an integral) of sinusoidal signals at different frequencies.

Notation: Fourier Transform pair

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$

NOTE: Convergence Condition — similar to FS.

Example 1:

$$x(t) = e^{-at} u(t), \quad a > 0$$ aperiodic

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{(-a-j\omega)} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

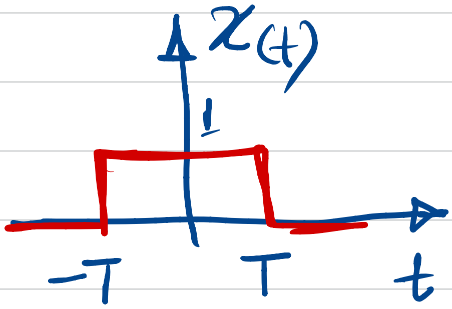
$$= \frac{1}{a+j\omega}$$

i.e., $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a+j\omega} e^{j\omega t} d\omega$$

Example 2:

Rectangular Pulse Signal:

$$x(t) = \begin{cases} 1, & |t| < T \\ 0 & |t| > T \end{cases}$$



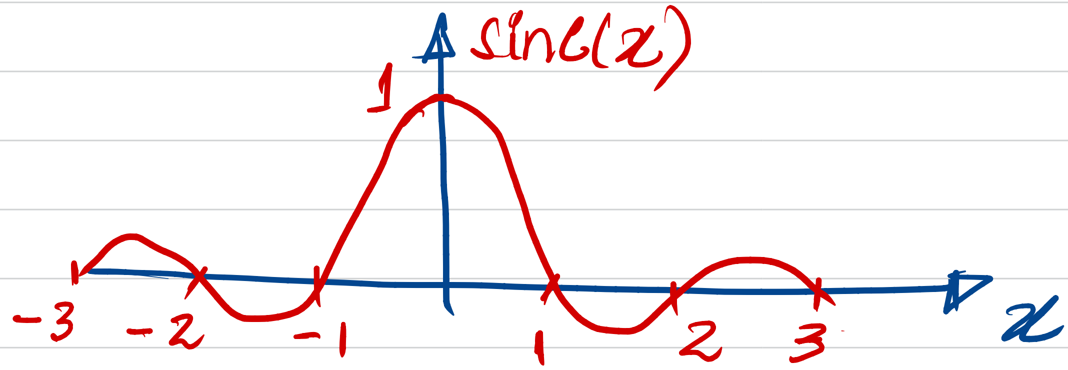
$$X(j\omega) = \int_{-T}^{T} e^{-j\omega t} dt$$

$$= \frac{1}{j\omega} e^{-j\omega t} \Big|_{-T}^{T} = 2 \frac{\sin \omega T}{\omega}$$

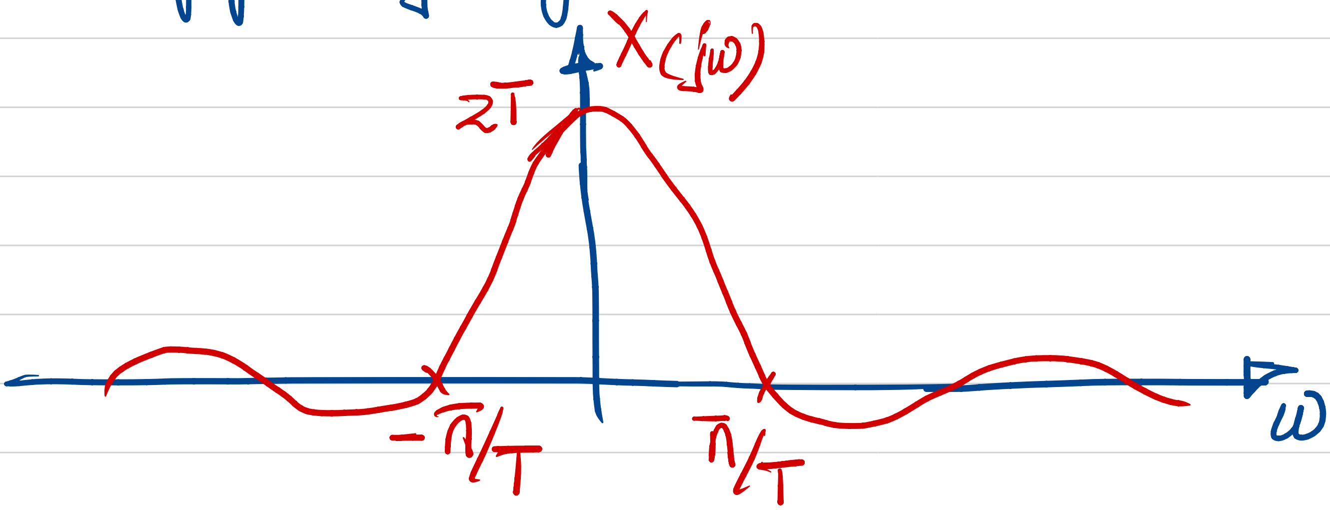
$$X(j\omega) = 2T\,\frac{\sin \omega T}{\omega T} = 2T\,\frac{\sin\left(\pi\left(\omega T/\pi\right)\right)}{\pi \times \omega T/\pi}$$

$$= 2T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right) \quad \text{— (A)}$$

Definition: $\boxed{\mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}}$



- Accordingly, $X(j\omega)$ given in (A) is drawn as:



larger T, narrower spectrum band!