

## ON THE EQUIVALENCE OF FLOATING- AND FIXED-STRIKE ASIAN OPTIONS

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### Abstract

There are two types of Asian options in the financial markets which differ according to the role of the average price. We give a symmetry result between the floating- and fixed-strike Asian options. The proof involves a change of numéraire and time reversal of Brownian motion. Symmetries are very useful in option valuation, and in this case the result allows the use of more established fixed-strike pricing methods to price floating-strike Asian options.

*Keywords:* Asian option; floating-strike Asian option; put call symmetry; change of numéraire; time reversal; Brownian motion

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### 1. Introduction

The purpose of this paper is to establish a useful symmetry result between floating- and fixed-strike Asian options. A change of probability measure or numéraire and a time-reversal argument are used to prove the result for models where the underlying asset follows exponential Brownian motion.

There are many known symmetry results in financial option pricing dating back to Kruizenga [11]. Bartels [2] derives relationships between European puts and calls with different strikes, under general assumptions on the stock price process. Vecer [13] discusses some more recent work on symmetries. Such results are useful for transferring knowledge about one type of option to another, and may be used to simplify coding of one type of option when the other is already coded. For example, Henderson and Hobson [10] derive an equivalence between a passport option and a fixed-strike lookback option. However, care must be taken as the transformed option may not exist in the market or have a sensible economic interpretation.

These tricks become very useful for exotic options, when perhaps no closed-form solution exists but an equivalence relation holds, together with an accurate computational procedure for the related class. This is of particular interest for the Asian option since much is known about the fixed-strike case, but comparatively little work has been done for the floating-strike option.

In contrast to the above symmetries, our results for Asian options relate two *different* types of arithmetic Asians, a floating-strike call (or put) and a fixed-strike put (or call) option. This is interesting, since from the parity shown by Alziary *et al.* [1] involving all four Asians and a

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European put and call, it does not seem likely that there would be such a result. It is simple, however, to obtain a put–call parity between two Asians of the same type; see [1].

Asian options have a payoff which depends on the average price of the underlying asset during some part of its life. The average is usually arithmetic, and if the asset price is assumed to follow exponential Brownian motion, an explicit option price is not available as the arithmetic average of a set of log-normal distributions is not known explicitly. There are two types of Asian options: the fixed-strike option, where the average relates to the underlying asset and the strike is fixed; and the floating-strike option, where the average relates to the strike price.

Pricing of the fixed-strike Asian has been the subject of much research over the last ten years and academic interest in these options has experienced a recent revival; see [3], [5], which continue the earlier work of [9]. The reader is referred to the recent papers [7] and [6] for an overview of the literature on pricing fixed-strike Asian options.

The floating-strike Asian option has received much less attention in the literature, perhaps because the problem is more difficult in that the joint law of  $\{S_t, A_t\}$  is needed. Chung *et al.* [4] generalise earlier efforts which derive approximations using joint log-normality. A PDE approach can also be taken; see [12], and [13] for an excellent new method. Even so, pricing methods for floating-strike options are underdeveloped compared with the more established methods for the fixed-strike option.

## 2. The model

We consider the standard Black–Scholes economy with a risky asset (stock) and a money market account. We take as given a complete probability space  $(\Omega, \mathcal{F}, P)$  with a filtration  $(\mathcal{F})_{0 \leq t \leq \infty}$  which is right-continuous and such that  $\mathcal{F}_0$  contains all the  $P$ -null sets of  $\mathcal{F}$ . We also assume the existence of a risk-neutral probability measure  $Q$  (equivalent to  $P$ ) under which discounted asset prices are martingales, implying no arbitrage. We denote expectation under measure  $Q$  by  $E$ , and under  $Q$  the stock price follows

$$\frac{dS_t}{S_t} = (r - \delta) dt + \sigma dW_t, \quad (1)$$

where  $r$  is the constant continuously compounded interest rate,  $\delta$  is a continuous dividend yield,  $\sigma$  is the instantaneous volatility of asset return and  $W$  is a  $Q$ -Brownian motion.

An Asian option contract is written at  $t = 0$  and expires at  $T > 0$ . The arithmetic average  $A$  can be calculated at  $T$  given the price history from time  $t_0 < T$ . Let  $D > 0$  denote the duration of averaging  $D = T - t_0$ . If  $T = D$ , the average is computed over the whole life of the option, which is termed the ‘plain vanilla’ option. If  $T > D$ , the option is ‘forward starting’, and if  $T < D$ , the Asian option is ‘in progress’. The arithmetic average is defined by

$$A_t = \frac{1}{D} \int_{t_0}^t S_u du,$$

where  $t \geq 0$ .

The fixed-strike Asian call option is defined to be the security which pays  $(A_T - K)^+$  at time  $T$ ; the floating-strike Asian call option pays  $(\lambda S_T - A_T)^+$ . By arbitrage arguments, the price of a *fixed-strike call*  $c_x$  at time  $t = 0$  is given by:

$$c_x(K, S_0, r, \delta, 0, T) = c_x = e^{-rT} E(A_T - K)^+,$$

whilst the price of a *floating-strike call*  $c_f$  at time  $t = 0$  is given by

$$c_f(S_0, \lambda, r, \delta, 0, T) = c_f = e^{-rT} E(\lambda S_T - A_T)^+,$$

with  $\lambda = 1$  being the important case in financial option pricing. Asian put options are defined analogously with prices denoted by  $p_x(K, S_0, r, \delta, 0, T)$  and  $p_f(S_0, \lambda, r, \delta, 0, T)$  for fixed- and floating-strike calls respectively.

### 3. A symmetry between floating-strike and fixed-strike Asian options

**Theorem 1.** *Under the assumption that  $S$  follows exponential Brownian motion in (1), the following symmetry results hold:*

- (i)  $c_f(S_0, \lambda, r, \delta, 0, T) = p_x(\lambda S_0, S_0, \delta, r, 0, T)$  and
- (ii)  $c_x(K, S_0, r, \delta, 0, T) = p_f(S_0, K/S_0, \delta, r, 0, T)$ .

*Proof.* We prove (i) first. Taking  $t_0 = 0$ , the floating-strike Asian call price expressed in units of stock as numéraire is

$$c_f^* \equiv \frac{c_f}{S_0} = \frac{e^{-rT}}{S_0} E[(\lambda S_T - A_T)^+] = E\left[\frac{S_T e^{-rT}}{S_0} \frac{(\lambda S_T - A_T)^+}{S_T}\right].$$

By changing numéraire to  $S$  via

$$\frac{S_T e^{-rT}}{S_0 e^{-\delta T}} = e^{-(\sigma^2/2)T + \sigma W_T} = \frac{dQ^*}{dQ},$$

we define the measure  $Q^*$  (see [8]). Under  $Q^*$ ,  $W_t^* = W_t - \sigma t$  is a Brownian motion, using the Girsanov theorem. Now the price becomes

$$c_f^* = e^{-\delta T} E^*[(\lambda - A_T^*)^+],$$

with

$$A_T^* \equiv \frac{A_T}{S_T} = \frac{1}{T} \int_0^T \frac{S_u}{S_T} du = \frac{1}{T} \int_0^T S_u^*(T) du,$$

where for  $u \leq T$  we define the  $\mathcal{F}_T$ -measurable random variable

$$S_u^*(T) \equiv \frac{S_u}{S_T} = \exp\{(r - \delta + \frac{1}{2}\sigma^2)(u - T) + \sigma(W_u^* - W_T^*)\},$$

using  $W_t^*$ , a  $Q^*$  Brownian motion. Now note that if, for all  $t$ ,  $\hat{W}_t \equiv -W_t^*$  is a  $Q^*$ -Brownian motion starting at zero, then  $W_u^* - W_T^* \stackrel{\text{law}}{=} \hat{W}_{T-u}$  and

$$A_T^* \stackrel{\text{law}}{=} \hat{A}_T \equiv \frac{1}{T} \int_0^T e^{\sigma \hat{W}_{T-u} + (r - \delta + (1/2)\sigma^2)(u - T)} du.$$

Reversing time via the variable change  $s = T - u$  gives

$$\hat{A}_T = \frac{1}{T} \int_0^T e^{\sigma \hat{W}_s - (r - \delta + (1/2)\sigma^2)s} ds.$$

Thus  $c_f^* = e^{-\delta T} E^*(\lambda - A_T^*)^+ = e^{-\delta T} E^*(\lambda - \hat{A}_T)^+$  and so the result (i) is proved. The result (ii) follows directly from (i) using put-call parity.

**Remark 1.** The above results extend trivially to forward-start options where the averaging period begins at  $t_0 > 0$ , and the option is priced at times up to  $t_0$ . They do not extend to ‘in progress’ Asians due to the extra term created by the average to date  $A_t$ , at time  $t$ .

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