MitM TIP 2023 — Monday

NUMBERS AND INVESTIGATION

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1 *Know your sequences!*. You already know about odds and evens, but you need to have, at the very least, passive familiarity with as many other sequences as possible. Here are a few.

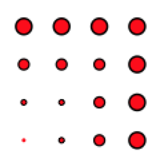
- The **triangular numbers** are the sums of consecutive integers, starting with 1. The first few are $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, \ldots$.
- The **squares** are $1, 4, 9, 16, 25, 26, 49, 64, 81, 100, 121, 144, 169, 196, 225 \ldots$.
- The **powers of two** are the numbers of the form 2^k for non-negative integers k. The first few terms are $1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \ldots$, since $2^0 = 1$.
- The **Fibonacci numbers** f_n are defined by $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. For example, $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8$. The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots.$$

2 *Investigating these sequences*. Try to ask questions that involve one or more sequence, and then investigate them. Here are a few suggestions.

- Is there a relationship between odd numbers and squares?

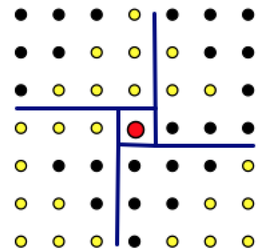
  Solution: Indeed, this picture shows that the nth square is the sum of the first n odd numbers, explicitly demonstrating that $1 + 3 + 5 + 7 = 4^2$.



- Are square numbers ever triangular numbers as well?

  Solution: We didn't really look into this, other than observe that $t_1 = 1$ and $t_8 = 36$. A computer or calculator will allow you to discover far more triangular numbers that are also squares; it is a fascinating exploration that leads to interesting patterns that amazingly, involve $\sqrt{2}$.
- Make a list of triangular numbers; then look at what happens when you multiply each number by 8 and then add 1.

  Solution: Proof by picture shows that if $T = 1 + 2 + 3$, then 8 copies of T can be arranged in a "pinwheel" pattern that makes a perfect square minus 1 (the red dot).



3 *Adding and multiplying*. Here's a fun and simple activity: Write a multiplication table, say from 1×1 up to 10×10, and figure out how to add up the entries of the table. Without doing a lot of work!

Solution: A little experimenting with smaller tables gets us the nice answer of t_n^2 for an $n\times n$ table. To see why, look at the multiplication table tor $n=3$.

	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

The sum of the entries is

$$(1+2+3)+2(1+2+3)+3(1+2+3)=(1+2+3)(1+2+3).$$

4 A number is called **trapezoidal** if it can be expressed as a sum of two or more consecutive positive integers. For example, $7=3+4$ and $10=1+2+3+4$ and $12=3+4+5$ are all trapezoidal. Investigate, generate questions, come up with conjectures.

Solution: The main thing that we determined is that a number is trapezoidal if and only if it is NOT a power of 2. In other words, if it does not have an odd factor. Thus we need to show two things:

1. If a number is trapezoidal, it must have an odd factor (besides 1, of course).
2. If a number has an odd factor (besides 1) it is trapezoidal.

To prove #1, we observe that if a number is trapezoidal with an odd number of addends, then it automatically has an odd factor. For example, the 5-addend sum $10+11+12+13+14+15=5\cdot 12$. This is because of the "balance-beam" principle, that the numbers all balance around the center point of 12.

But what if there are an even number of addends? Then we use the fact that the first and last numbers are of different parity, so their sum is odd, and we can collect several copies of this odd number. For example, consider the 8-addend sum

$$16+17+18+19+20+21+22+23.$$

There is no central balance point, but instead we just observe that the entire sum is equal to

$$(16+23)+(17+22)+(18+21)+(19+20)=4\cdot 39,$$

a multiple of the odd number 39. Clearly this method works for any even-addend trapezoidal number.

Thus, we know that trapezoidal numbers have odd factors, so they cannot be powers of two. But we haven't shown that ALL numbers with odd factors can be written "trapezoidally," which is what #2 asserts So let's prove that.

Suppose we have a number which has an odd factor, say $21=3\cdot 7$. We can write

$$21=7+7+7=6+7+8,$$

where we used the "balance-beam" principle.

But this method doesn't always work smoothly. Suppose we have $22=2\cdot 11$. We write

$$22=2+2+2+2+2+2+2+2+2+2+2,$$

and the balance-beam method yields

$$22=(-3)+(-2)+(-1)+0+1+2+3+4+5+6+7,$$

which is a sum of 11 consecutive integers, but of course they are not all positive. But so what! Just let the negatives cancel out their positive sisters, and we get the sum

$$22=4+5+6+7,$$

so it is trapezoidal.

5 *Fibonacci investigations*. Here are just a few suggestions.

- Investigate parity (odd or even), divisibility by 3, divisibility by 5, perfect squares, etc. for the Fibonacci numbers.
- Try adding the Fibonacci numbers.
- Try adding squares of Fibonacci numbers.

6 *Added during the session*. Given an $n\times n$ grid, count the number of squares possible. Call this number S_n. Also count the number of rectangles possible. Call this R_n. For example, you should be able to verify that $S_1=R_1=1$, and $S_2=5, R_2=9$, and $S_3=14, R_3=36$. Can you find formulas for S_n and R_n?

Solution: Let's try to compute R_3 by drawing a 3×3 grid, where we label each grid square with the number of rectangles possible that have that grid square as its upper-left corner.

9	6	3
6	4	2
3	2	1

Notice that for each square we label, we are merely counting the number of squares that there are to the right and below our square (including our square itself), since any of the other squares to the right and below can serve as the lower-right corner of the rectangle. We are exactly doing the multiplication table problem (problem 3), so the answer is t_3, and in general we have deduced that

$$R_n = t_n^2.$$

Let's now do the same analysis for S_3. We get the following grid.

3	2	1
2	2	1
1	1	1

for a total of 14. Likewise, the grid for S_4 is

4	3	2	1
3	3	2	1
2	2	2	1
1	1	1	1

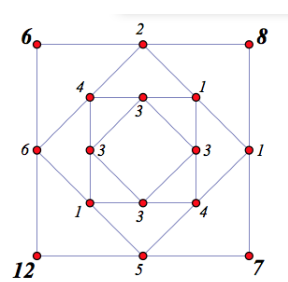
Let's think recursively. How do we get from S_3 to S_4? We merely increase every value of the S_3 grid by 1, and then add a boundary layer of 1s on the right and bottom. Since we are interested in the *sum* of the grid values, we are just adding a 4×4 grid consisting of all 1s to the S_3 grid sum. In other words,

$$S_4 = S_3 + 4^2.$$

Since $S_1 = 1 = 1^2$, we have deduced that in general,

$$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

7 *The difference game*. Start by labeling the vertices of a square with numbers. Then write the difference of the values at two adjacent vertices on the midpoint of the line joining them; this produces four new values at the vertices of a smaller square. Keep repeating this process, generating smaller and smaller squares until the process ends. In the example below, we started with the values $6, 8, 7, 12$ (shown in larger font) which generated the values $2, 1, 5, 6$, then $4, 1, 4, 1$. The final square shown has all vertices equal to 3; clearly the next square (and all subsequent squares) will have only zeros at each vertex.



Investigate, generate questions, come up with conjectures.

Solution: There are two big questions (at least). The first is whether you always end up with zeros eventually, and the second is whether you can hold out for an arbitrary length of time.

The answers are YES to both questions (provided you start with positive integers). To see why you eventually get only zeros, we will verify two simple observations.

- **The maximum of the four values NEVER goes up; it either stays the same or decreases.** This is pretty obvious. However, this is not enough to force the values to all become zero, since perhaps there could be some "oscillation." We need the next observation.
- **Eventually all the values will be even.** This is not obvious, but there are only 6 different parity cases to try, using 0 for even and 1 for odd: 0000,1000,1100,1010,1110,1111. With each case, we end up with all evens (all 0s) in at most 4 turns.

Thus, no matter what numbers you get, eventually, you will end up with a square whose values are all even. Call the values $2a, 2b, 2c, 2d$. Now, when you continue the process, everything is multiplied by this factor of 2, and you can visualize it as two identical squares playing the difference game. At some point, both of *those* squares will have all even values. But it was really just one square, so putting the factor of 2 back, we now have all values being *multiples of 4*.

This process will continue indefinitely, which forces the values to eventually be zeros, since no positive integer can have an arbitrarily high power of two dividing it!

Now that we know we will eventually get all zeros, we need to find a way to hold off that fate as long as possible. We can design a starting square that will take at least N turns to zero out, for any N. The secret is "tribonacci numbers," the sequence $1,1,1,3,5,9,17,\ldots$, where the first three terms are 1 and each subsequent term is the sum of the *three* preceding terms.

To see how this works, imagine that we put the values $t_{13}, t_{12}, t_{11}, t_{10}$ on the vertices of a square (clockwise, in that order), where t_n is the $n$the tribonacci number. After one turn, the vertices of the new square are

$$t_{13} - t_{12}, \quad t_{12} - t_{11}, \quad t_{11} - t_{10}, \quad t_{13} - t_{10}.$$

Using the definition of tribonacci numbers, we see that

$$t_{13} - t_{12} = (t_{12} + t_{11} + t_{10}) - t_{12} = t_{11} + t_{10}.$$

Likewise, $t_{12} - t_{11} = t_{10} + t_9$ and $t_{11} - t_{10} = t_9 + t_8$. The last difference is a slightly different pattern (since the indices differ by 3 instead of 1), but the magic of tribonacci yields

$$t_{13} - t_{10} = (t_{12} + t_{11} + t_{10}) - t_{10} = t_{12} + t_{11}.$$

It doesn't matter which vertex is "first," as long as we go in order. So observe that the values of this new square, going clockwise, are

$$t_{12} + t_{11}, \quad t_{11} + t_{10}, \quad t_{10} + t_9, \quad t_9 + t_8.$$

Now if we use the ”two squares” idea, we see that this new square can really be thought of as one square with vertices of $t_{12}, t_{11}, t_{10}, t_9$, and the other, with corresponding vertices of t_{11}, t_{10}, t_9, t_8. Well, we know what happens when we take differences; these are just squares with consecutive tribonacci values (only shifted backwards a bit).

You can see the pattern. If we let S_n denote the square whose values are the consecutive tribonacci numbers $t_n, t_{n-1}, t_{n-2}, t_{n-3}$, we see that after one turn, S_n becomes ”$S_{n-1} + S_{n-2}$,” and after k turns, we will have a ”sum” of 2^k tribonacci squares. The values of these tribonacci squares stay non-zero and non-constant until you hit S_4; after four turns that zeros out.

Notice that this ”multiple squares” idea requires that the values of the squares be monotonic in the same direction; this allows us to couple or uncouple squares without interference. So at the very least, we can be assured of getting at least $n-4$ turns if we start with S_n.

8 *Pizza slicing*. Imagine a giant pizza. For each n, what is the maximum number of pieces you can get if you slice this pizza with straight line cuts? The lines are *infinite*; they are not line segments.

That’s the warm up. Now investigate a slight modification that makes the problem even more interesting: Remove the words “the maximum” above, and replace it with “are the possible.” For example, if $n = 3$, the *maximum* number of pieces will be 7 (verify!) but it is possible to get fewer. If all three lines coincide, you will get just 2 pieces. If all three lines are parallel, you will get 4 pieces. If two are parallel, and one is not, you get 6.

Solution: The answer to the warm up is a classic problem; for n lines it is just 1 more than the nth triangular number. I have no idea about the more general problem. It is fun and rich and there are plenty of partial solutions possible.

MitM TIP 2023 — Tues AM

Combinatorial Games

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For all but the last of these, two players alternate turns, with the same rule for making legal turns. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players. Try to prove that your strategy works. And, always, try to generalize!

1 *Breaking the Bar.* Start with a rectangular chocolate bar which is 6×8 squares in size. A legal move is breaking a piece of chocolate along a single straight line bounded by the squares. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. What about the general case (the starting bar is $m \times n$)?

2 *Basic Takeaway.* A set of 16 pennies is placed on a table. Two players take turns removing pennies. At each turn, a player must remove between 1 and 4 pennies (inclusive).

Solution: Call the starting value s; there is nothing special about $s = 17$. Examining smaller values, it is easy to see that player A wins on her first move if $1 \le s \le 4$. If $s = 5$, no matter what A does, B will be presented with a value between 1 and 4, inclusive, and will then win in a single move. Indeed, B is now the first player "up," and hence has the same fate that A had; i.e., winning.

In other words, presenting one's opponent with the value 5 guarantees that you (not your opponent) will win. So if $6 \le s \le 9$, player A will take away enough pennies to present B with 5. Continuing this analysis further, it is clear that as long as you present your opponent with a multiple of 5, you will win. Why?

- If you present your opponent with a multiple of 5, she *must* present you with a non-multiple of 5.
- Conversely, if you present your opponent with a non-multiple of 5, she *can choose to* present you with a multiple of 5.

Consequently, if s is a non-multiple of 5, then A can guarantee a win by presenting her opponent with successively smaller multiples of 5, culminating in 0.

Using coloring. We will systematically color the game positions with green for winning and red for losing, by applying two simple rules. Consider the number line starting at 0. We start by coloring 0 green, of course, since if we presented our opponent with 0, then we have won!

Next, moving to right on the number line, we apply the following coloring rules:

1. *Red rule.* Color a value red if, starting from this position, we can move to a green position *in one move*.
2. *Green rule.* Color a value green if there are no smaller uncolored values.

Verify that this procedure will color the multiples of 5 green, and every other value red.

The coloring method is a simple way to analyze a game without thinking too hard. It is a good way to get your hands dirty in an initial investigation.

3 *Don't Be Greedy*. Start with some pennies. The first player can take any positive number away as long as he or she doesn't take all the pennies. After that, you must take a positive number of pennies, but you may not take more pennies than your opponent just took. For example, if you start with 20 pennies, and player A takes 5, leaving 15, then player B can take away $1,2,3,4$, or 5, but not 6 or more.

Solution: The crux idea is to express the values in binary (base-2). The winning strategy is for the first player to look at the number of pennies and find the rightmost 1, as long as it is not the first digit, in its base-2 representation. Thus the number of pennies will end with a 1 and some zeros, and this will not be the total number. (In other words, the number of pennies cannot be a power of two.) Player A will the take away the value equal to this rightmost 1 followed by the zeros. For example, suppose (we will only use binary), that the number of pennies is 10010100. Then player A will take away 100.

This accomplishes the following: the new number of pennies now has a rightmost 1 that is *strictly to the left* of the original rightmost 1. But since player B cannot take more than player A just did, after she makes her move, the rightmost 1 will again be *at least as far to the right* as it was when player A made her move.

Continuing the same example, after player A's first move, the number of pennies is now 10010000. Player B can take away any number between 1 and 100. If she takes away 100, the new number of pennies will be 10001100, and once again the rightmost 1 is where it started. If instead she took away 10, then the number of pennies would be 10001110, etc.

Why is this a winning strategy for player A? Because she controls how far to the right the rightmost 1 is, and when it is her turn, this rightmost 1 will always be at the same position that she started with, or it will be strictly further to the right. And whenever she makes a move, it is of the form 2^a, where the number of pennies was (before her move) 2^aQ, where Q is odd. So when A makes a move, she is taking away a power of two from a value that is an odd number times this power of two. If B were to respond with this same power of two, eventually B would lose. If B instead takes away a smaller number, then A will once again be faced with a number of pennies that is an odd number times a smaller power of two (possibly 2^0).

If the starting number of pennies is a power of two, then A cannot win as long as B plays optimally.

4 *Don't Be Doubly Greedy*. This is just like Don't Be Greedy, only now the rule is that you cannot take more than *twice* what your opponent just took. So if player A takes 5, player B can take any number between 1 and 10, inclusive.

Solution: We did not discuss this problem, but the idea is to use the techniques of Don't Be Greedy, bu instead of base-2, we use base-F, the so-called "base Fibonacci" or Zeckendorf representation, writing the number of pennies as a sum of fibonacci numbers with the stipulation that two consecutive fibonacci numbers are not allowed. For example, $10 = 8 + 2$, so in base-F, we get 10010. For a challenge, STOP READING NOW, AND SEE IF YOU CAN ADAPT THIS METHOD TO SOLVE THE PROBLEM!

Now the strategy works the same way as before: As long as the starting value is not a Fibonacci number, player A begins by "removing the rightmost 1," and this will be a winning strategy. The key idea behind this strategy is the fact that F_{k+2} is always more than twice F_k, since $F_{k+2} = F_{k+1} + F_k$, and $F_{k+1} \geq F_k$.

For example, suppose we are starting with 20 pennies, which is $13+5+2$ or 101010_F. Player A takes away a value equal to the rightmost 1 followed by any zeros that may occur to the right of it. We will call this a "rightmost 1" move. In this case, player A takes away 10_F or 2 pennies, leaving the value $18 = 101000_F$. Since there can be no consecutive 1s in base-F, the rightmost 1 in this new number is always *at least* two places to the left of where it was before A moved.

Now player B can take between 1 and 4 pennies, inclusive. Notice that 4 is strictly less than 5, which two fibonacci numbers after 2. In other words, player B will end up taking away a value that may have a 1 on the spot to the left of what A took away, but not two spots to the left. Consequently, B *cannot* remove the rightmost 1.

For example, suppose that B takes away $3 = 100_F$. Now the number of pennies will be $15 = 13+2 = 100010_F$. Following the same strategy, A takes away $10_F = 2$ pennies, leaving $13 = 100000_F$. Once again, B can take between 1 and 4. Suppose now that B takes $4 = 101_F$. Now there are $9 = 10001_F$ pennies left, and A's response will be to take away just 1, leaving $8 = 10000_F$.

So we see that as long as the starting value is not a Fibonacci number, player A can *always* perform a rightmost 1 move, but B can *never* respond with a rightmost 1 move. Among other things, B will never be able to take the last pennies, because B is always stuck with removing a value whose starting digit is at most one to the left of the starting digit of what A removed, yet the number of pennies has a starting digit that is at least two spots to the left of what A removed. Eventually, A will have a Fibonacci number of pennies, and will be able to perform a rightmost 1 move on this, taking away the entire number and winning.

One full example, to see the entire process: Start with $36 = 34+2 = 10000010_F$. Also, keep in mind that one never takes more than one-third of the number of pennies, since the next player could then take at most two-thirds and win.

	take	left
A	$2=10$	$34=10000000$
B	$3=100$	$31=1010010$
A	$2=10$	$29=1010000$
B	$4=101$	$25=1000101$
A	$1=1$	$24=1000100$
B	$2=10$	$22=1000001$
A	$1=1$	$21=1000000$
B	$1=1$	$20=101010$
A	$2=10$	$18=101000$
B	$3=100$	$15=100010$
A	$2=10$	$13=100000$
B	$3=100$	$10=10010$
A	$2=10$	$8=10000$
B	$2=10$	$6=1001$
A	$1=1$	$5=1000$
B	$1=1$	$4=101$
A	$1=1$	$3=100$
B	$1=1$	$2=10$
A	$2=10$	WIN!

5 *Divide and Conquer*. Start with 100 pennies. Each player can remove a divisor of the number of pennies remaining as long as the divisor is strictly less than the number of pennies remaining. For example, at the start, A could remove 1, 2, 4, 10, 20, 25, or 50 pennies, but not 100 pennies. This game ends when exactly 1 penny is left, since the only divisor of 1 is 1, which is not less than 1.

Solution: The first thing you will do is color the value 1 green. Then the red rule colors 2 red, and the green rule makes 3 green. Next, the red rule colors both 4 and 6 red, and the green rule makes 5 green, etc.

A few minutes of industrious coloring leads to the conjecture that the winning (green) positions are precisely the odd numbers, and the losing (red) positions are even. But why? In hindsight, it is easy to see that if you are presented with an even number, you can always present your opponent with an odd number (if necessary, by subtracting 1), whereas if you are presented with an odd number, all divisors are odd, so you must hand your opponent an even number. In sum: the winning strategy is to present your opponent with an odd number, which forces the play until you finally present her with the odd number 1, and win.

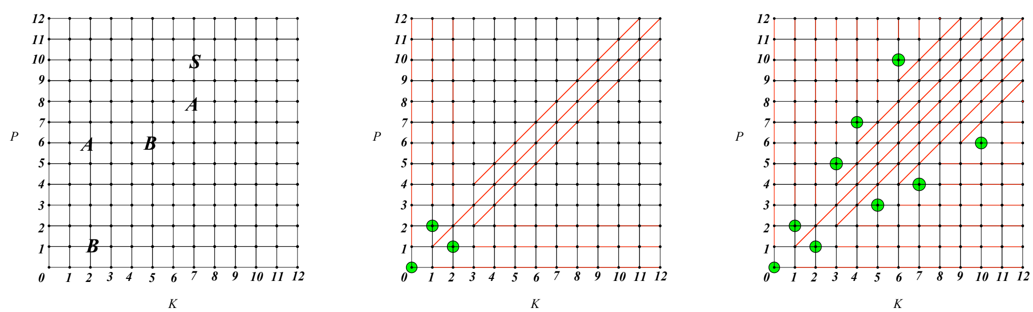
6 *Putdown*. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny that is already on the table. The table starts out completely bare.

7 *Puppies and Kittens*. We start with a pile of 7 kittens and 10 puppies. Two players take turns; a legal move is removing any number of puppies or any number of kittens or an equal number of both puppies and kittens.

Solution: the game states are two-dimensional. The game can be visualized geometrically using a horizontal kitten and vertical puppy axis. The game starts at a lattice point in the plane, and legal moves are due west, south, or southwest, with the objective of reaching the origin first. The leftmost grid in the figure below illustrates a sample game. Starting with 10 puppies and 7 kittens, player A adopted 2 puppies, then B adopted 2 of each, then A took home 3 kittens, and B took 5 puppies, leaving A with the game state $(2,1)$. It should be clear that this forces a win for B.

In other words, $(2,1)$—and by symmetry, $(1,2)$—are green points, along with $(0,0)$. A moment's thought shows that the entire axes and diagonal lattice points northeast of the origin are colored red, as depicted in the middle grid (the red points are shown by lines; the green points are circles). This is what forces $(2,1)$ and $(1,2)$ to be green: they are the "first" points that *cannot* reach $(0,0)$ in one move. Instead, a player whose position is, say, $(1,2)$ has no choice but to move to a red point.

Continuing in this way, we see that there are infinitely many red points that are one move away from $(1,2)$ and $(2,1)$; namely all points north, east, and due northeast of them. A careful perusal of the graph now shows that the new green points are $(3,5)$ and $(5,3)$. Starting from either of these, any south, west, or southwest move lands us on a red point, and from there, one can chose a move to a green point, etc. Continuing this process, we can easily work out several more green points, as shown in the rightmost grid. Listing just the ones where $k \leq p$, our list so far is $(0,0),(1,2),(3,5),(4,7),(6,10)$.



The graphical method, while appealing, is not necessary. For example, let's find the next green point after $(6,10)$. Since all points to the north, east, and northeast of the current green set are colored red, we have eliminated all points with coordinates used so far (i.e., all points with coordinates equal to $0,1,2,3,4,5,6,7,10$) as well as all points where the difference of the coordinates is $0,1,2,3,4$. The next green point will have a coordinate difference of 5, and cannot use any of the forbidden values. Thus it must be $(8,13)$. Using this method, we can easily compute more green points. Here is a table of the first values, where d denotes the difference of coordinates, and we just choose the points where $k \leq p$ (so if, say, $(8,13)$ is green, then so is $(13,8)$, etc.).

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13
k	0	1	3	4	6	8	9	11	12	14	16	17	19	21
p	0	2	5	7	10	13	15	18	20	23	26	28	31	34

A crucial question is whether there is a closed-form formula for the green coordinates in terms of d. Fibonacci numbers seem to lurk in this table. Clearly something interesting is going on involving them.

Advanced Digression: The magical formula! The problems below will help you to discover the stunning fact that the golden ratio is intimately involved with the Puppies and Kittens game. Let (x_n, y_n) be the Puppies and Kittens green point (winning position) satisfying $y_n - x_n = n$. For example,

$$x_1 = 1, y_1 = 2, \quad x_2 = 3, y_2 = 5, \quad x_3 = 4, y_3 = 7.$$

Our goal is to show, for all $n = 1, 2, 3, \ldots$, that $x_n = \lfloor n\tau \rfloor$, and thus $y_n = \lfloor n(\tau + 1) \rfloor$, where τ is the famous, ubiquitous *Golden Ratio*:

$$\tau = \frac{1+\sqrt{5}}{2}.$$

Notice that τ is the reciprocal of the number ζ used in Conway's Checker Problem (p. **??**). It is easy to see that $\tau^2 = \tau + 1$ and, thus,

$$\frac{1}{\tau} + \frac{1}{\tau+1} = 1.$$

(a) Two disjoint sets whose union is the natural numbers $\mathbf{N} = \{1, 2, 3, \ldots\}$ are said to *partition* $\mathbf{N}$. In other words, if A and B partition $\mathbf{N}$, then every natural number is a member of exactly one of the sets A, B. No overlaps, and no omissions.

Let (x_n, y_n) be the Puppies and Kittens green point satisfying $y_n - x_n = n$. For example,

$$x_1 = 1, y_1 = 2, x_2 = 3, y_2 = 5.$$

Verify that the two sets

$$A = \{x_1, x_2, x_3, \ldots\} \text{ and } B = \{y_1, y_2, y_3, \ldots\}$$

partition the natural numbers.

(b) Let α be a positive real number. Define the set of *multiples* of α to be the positive integers

$$\{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}.$$

For example, if $\alpha = 2$, then the multiples of α are just the even positive integers. Notice that α need not be an integer, or even rational.

Does there exist α such that the multiples of α are the *odd* positive integers?

(c) Suppose that there are two numbers α, β whose sets of multiples partition the natural numbers. In other words, every natural number is equal to the floor of an integer times *exactly one* of α or β, and there are no overlaps.

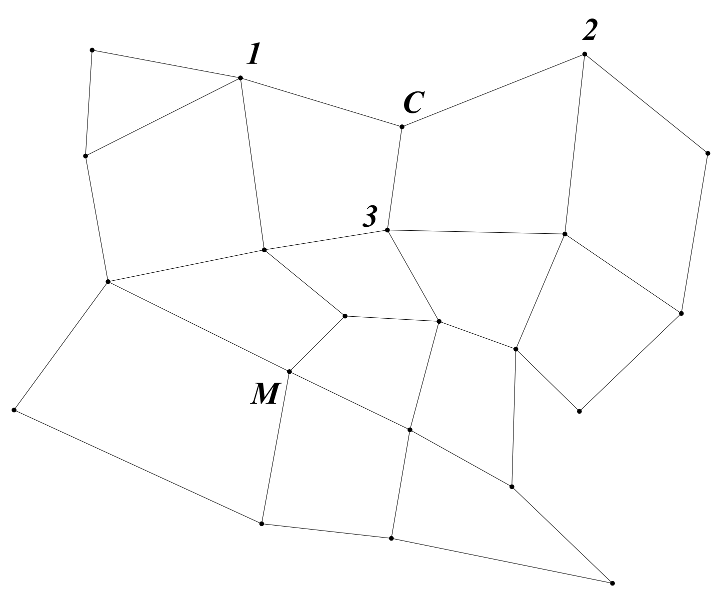
1. Prove that both α and β are greater than 1.
2. Suppose that $1 < \alpha < 1.1$. Show that $\beta \geq 10$.
3. Prove that
$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$
4. Prove that α and β must be irrational.

(d) Prove the converse of the above problem; i.e., if α, β are positive irrational numbers satisfying
$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$
then the multiples of α and the multiples of β partition the natural numbers.

(e) Why does this now show that $x_n = \lfloor n\tau \rfloor$, and thus $y_n = \lfloor n(\tau + 1) \rfloor$?

8 *Color the Grids*. You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.

9 *Cat and Mouse*. A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win?

MitM TIP 2023 — Tues PM

MATH MAGIC FOR MUGGLES

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Here are several easy-to-perform feats that suggest supernatural powers such as telepathy, "seeing fingers," predicting the future, photographic memory, etc. Each trick uses simple mathematical ideas that allow information to flow effortlessly and sneakily, among them

- parity and other invariants
- symmetry
- probability

One can approach these activities in many ways. At first, you may want to figure out HOW to do a trick. Then, you want to know WHY it works. Finally, you should strive to understand REALLY WHY it works: is there a simple theme or principle behind your possibly complex explanation? Look for simple and general guiding principles.

Several of these tricks were researched, perfected, and classroom-tested in 2012 at the San Francisco Math Circle by SFSU grad students Jessica Delgado and Kelly Walker. I am indebted to them. In turn, they (and I) are also indebted to the lovely book *Magical Mathematics*, by Persi Diaconis and Ron Graham (Princeton University Press, 2012).

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1 *Warm-up: Fingers That Can See*. The Magician (M) deals cards on a table (not in a pile), placing them face up or face down on the command of P (P), and stops dealing when P says so.

Then M is blindfolded. M proceeds to put the cards into two piles, using his magical seeing fingers, so that, miraculously, each pile has exactly the same number of face-up cards!

2 *An allegory for the previous problem*. Bottle A contains a quart of milk and bottle B contains a quart of black coffee. Pour a small amount from B into A, mix well, and then pour back from A into B until both bottles again each contain a quart of liquid. What is the relationship between the fraction of coffee in A and the fraction of milk in B?

3 *Zvonkin's Magic Table*. This trick is adapted from A. Zvonkin's book *Math From 3 to 7*, which I helped to translate and edit. Zvonkin ran a math circle for small kids in Moscow and entertained them by having them cover any four consecutive numbers in the table below (vertical or horizontal), and then he would instantly determine the sum! Was it a feat of memory? Telepathy?

5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0
7	1	7	2	3	7	1	7	2	3	7	1
2	7	6	1	4	2	7	6	1	4	2	7
5	6	1	6	2	5	6	1	6	2	5	6
5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0
7	1	7	2	3	7	1	7	2	3	7	1
2	7	6	1	4	2	7	6	1	4	2	7
5	6	1	6	2	5	6	1	6	2	5	6
5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0

4 *The Kruskal Count*. This telepathy trick can be done with cards or numbers. With cards, M deals out an entire deck face up on a table, and asks the participant to mentally pick one of the first dozen or so cards and then use that card to tell him or her where to go next. If the card is an Ace, move one spot to the next card. If it's 2 through 9, go that many places. If it's a face card, move the number of letter of the card (i.e., Jack or King means move four, Queen means move five). Keep doing this until you can go no further. For example, if you start with the Jack of Hearts, you then move 4 cards down and perhaps that is an Ace of clubs. Then you move to the next card, the 7 of spades, and move 7 down, etc.

When the participant gets to the final card (the one where you cannot go further, because you'd go past the last card in the deck), he or she thinks hard about it. And M manages to deduce the card.

The trick can also use a random list of numbers, or a semi-random one, such as the digits of π below.

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4
6	2	6	4	3	3	8	3	2	7	9	5	0	2	8	8	4	1	9	7
1	6	9	3	9	9	3	7	5	1	0	5	8	2	0	9	7	4	9	4
4	5	9	2	3	0	7	8	1	6	4	0	6	2	8	6	2	0	8	9
9	8	6	2	8	0	3	4	8	2	5	3	4	2	1	1	7	0	6	7

With a number table, the rule is simpler: Pick any starting point in the row, and move that many places, unless you hit 0, in which case you move one place. For example, if you start with the second digit (1), you move one place, to 4, then 4 more places, to 2, then 2 places, to 5, etc. Once again, P mentally chooses a starting point, concentrates on the ending number, and M magically guesses it!

5 *Hummer Shuffle Tricks*. The three tricks below all employ the "Hummer Shuffle," which consists of picking up the first two cards of a deck, turning the two cards over, and replacing them on the top of the deck (i.e., card #1 becomes card #2 and card #2 becomes card #1, and both get turned over), followed by cutting the deck (you take the top n cards, where n is up to you, and lift them off the deck, then place them at the bottom, without turning the

n cards over, so that now the top card is the previous $(n+1)$st and the bottom card is the previous nth card, etc. After doing a bunch of Hummer Shuffles, the cards in a deck are hopelessly messed up, since not only is the order permuted, but some of the cards will be face up and some will be face down. However, this shuffle is surprisingly orderly, as you will see.

(a) *Baby Hummer*. This trick only uses four cards. P takes four cards, all facing the same way, and sneaks a peek at the bottom card. Then P does the following:

1. Take the top card and place it on the bottom
2. Turn the current top card face up
3. Perform several Hummer Shuffles
4. Turn over the top card and put it on bottom
5. Put the current top card on the bottom without turning it over
6. Turn the top card over and leave it on top

Now spread the cards out and three cards will be facing one way and your original bottom card with be facing the other!

(b) *Nearly Perfect Mind Reading?* M gives P ten cards from A to 10, in order. P then performs several Hummer Shuffles, thoroughly messing up the cards. M is blindfolded. Then, P starts reading off the cards in order, from the top of the disordered pile, telling M what card it is. M is able to guess whether the card is face up or face down, with nearly flawless accuracy (much better than 5 correct—the expected number due to random guessing)!

(c) *What is the name of this trick?*. M takes about half a deck and shows the cards in it to P, who is invited to shuffle them. The magician then apparently messes the cards up further in a random way with respect to orientation (face-up vs. face-down). Then M invites P to continue messing up the cards with some Hummer-type shuffles. Then M deals the cards into two piles, puts them together, and spreads them out. Several of these cards are face-down. When turned over, the audience goes crazy!

6 *Random Numbers*. M asks P to choose a random number n between 1 and 20, and share this number with the audience without letting M know. P then removes the top n cards from the deck.

Next, M deals 20 cards from the top of the diminished deck (which is missing n cards), and he asks the audience to notice the nth card dealt (without giving it away with body language!).

Next, an audience member is asked to estimate half the size of the now very diminished deck (it is missing $20+n$ cards). We call this number h. M then deals h cards from the top, face-down. Then he places the stack of 20 cards on top of this, and puts the rest of the diminished deck on top of that (so the n cards removed at the start are still missing).

Finally, M deals cards off the top, but at some miraculous point, stops, and it is the one that the audience noted!

The Mysteries, Revealed!

1 *Fingers That Can See*. M watches and keeps track of the total number of face up cards. Call this number u. Then while blindfolded, M merely collects any u cards into a pile (making sure to keep their original orientation) and then flips this entire pile upside-down. Then this pile and the remaining cards have the same number of face-up cards. The reason: suppose that, among the u cards collected, that f of them are face up. Then $u-f$ are face down. However, in the pile of non-collected cards, $u-f$ must be face up (since the total number of face up cards is u). So flipping the chosen cards does what we want!

2 *The milk and coffee allegory*. The idea—absolutely equivalent to the above—is to think about conservation of mass. Imagine that the milk and coffee is discrete, say, one bucket of 100 white ping pong balls and another bucket 100 black ping pong balls. Then if we remove 10 black ping pong balls from the second bucket, mix them into the first, and then remove 10 balls from this bucket and put back into the first bucket, it is OBVIOUS that the amount of black "pollution" in the first bucket is equal to the amount of white "pollution" in the second!

3 *Zvonkin's Magic Table*. The table is a repeating grid of 5×5 numbers arranged so that each row and each column sums to 20. Such grids are easy to make—try it yourself!—and now the trick is obvious: just look at the number adjacent to the covered area, and subtract this from 20.

4 *The Kruskal Count*. This trick works for the same reason that putting a hotel on Park Place is almost always a winning Monopoly strategy: eventually, someone will land at Park Place!

Pick the very first card (or digit) and plot out the evolution of this pick. Imagine, say, that each card (digit) that gets visited is colored green. Now consider a different starting point. This will engender a new sequence of visited locations. But observe that as soon as we reach a green location, we are locked into all the rest of the green locations.

So now, think of the green locations as "mines" or "Monopoly hotels that belong to our opponent." We start at some random point, and then our course is preordained (by the actual values of the cards or digits) but is also, in some sense, random. With digits, each step will be have length from 1 to 9, with each choice approximately equal (1 is more likely, since landing on 0 leads to a step size of 1). With cards, step sizes of 4 and 5 are somewhat more likely than the others, but otherwise, it's a random choice between 1 and 9.

In other words, each random sequence of digits (or shuffled deck of cards) plus a starting point yields a random sequence of step lengths, with approximately equal probabilities for each step length.

How do you avoid green locations? At each step, look for the nearest green location, and make sure not to step that distance. Just like Monopoly: if you are 8 steps from Park Place, you toss your dice, hoping not to get an 8. Since there are 9 possible step lengths, and only one bad one, at each turn, you have an $8/9$ probability of missing the next green location. Consequently, if you do this 15 times, the probability of missing *all* of the green locations is $(8/9)^{15}$, which is about 17%. Hence there is an 83% probability that you will hit a green spot and then get locked into the sequence that began with the very first location.

So that's how the Magician does the trick, by starting from the first spot and knowing that, with high probability, the Participant and the Magician will end up in the same place.

5 *Hummer Shuffle Tricks.* Consider a pile of cards, where some possibly are face up. Each card has a position (from #1, the top card, down to the last card), a value (where $A = 1$ and J, Q, K respectively equal 11, 12, 13), and an orientation (either face-up or face-down). All of these tricks depend on using an *even* number of cards and use one or both of the following lemmas.

> **Lemma 1**: Start with a pile of $2n$ cards, *all face-down*. After any number of Hummer Shuffles is performed, the number of odd-position cards that are face-up will equal the number of even-position cards that are face-up.

> **Lemma 2**: Start with a pile of $2n$ cards, *all face-down*, and *arranged in numerical order* (for example, $5,6,7,8,9,10,J,Q$). then do any number of Hummer Shuffles. For each card, the sum of its position, value, and orientation (where we assign 1 to "face-up" and 0 to "face-down") will have the same parity.

For example, suppose the cards start with $5,6,7,8$ from top to bottom, all face-down, and we turn over the first two and cut by taking the top card and putting on the bottom and then turn over the top two and cut by taking the top two cards and putting them on the bottom. Then we get, in order (using a bar to indicate "face-up"), from the starting position:

$$5,6,7,8 \to \bar{6},\bar{5},7,8 \to \bar{5},7,8,\bar{6} \to \bar{7},5,8,\bar{6} \to 8,\bar{6},\bar{7},5.$$

Now let's compute the sum of position plus value plus orientation for each card. The first card's sum is $1+8+0=9$. Card #2's sum is $2+6+1=9$. Card #3's is $3+7+1=11$, and the final sum is $4+5+0=9$. All of these are odd.

I leave it to the reader to prove these lemmas, but this should not be difficult. The harder part is thinking of the lemmas in the first place! We also leave it to the reader to use these lemmas (or other similar ideas) to explain (a).

Lemma 2 is used for (b), the Nearly Perfect Mind Reader trick. The Magician merely guesses the first answer, but of course the Participant will tell the Magician if he or she is correct or not. This establishes the parity of the sum, and the rest is (fairly) easy, but requires paying attention.

For (c), the Magician makes sure that there are an even number of cards in the pile, and that a royal flush is included among them. Then M cleverly arranges the orientation of the cards by examining successive pairs and flipping over the odd-positioned card ONLY if it belongs to the royal flush, and flipping over the even-positioned card ONLY if it doesn't belong to the royal flush. I am right handed, so I start looking at the cards from the right, so I use the mnemonic aid "**R**oyal flush cards get flipped if they are the **R**rightmost one in the pair."

At this point, some cards are face-up and some are face-down, but the following regularity has been imposed:

The odd-positioned royal cards have the same orientation as the even-positioned ordinary cards. Likewise, the even-positioned royal cards have the same orientation as the odd-positioned ordinary cards.

Notice (VERIFY!) that Hummer Shuffling will not change this situation! So after a bunch of Hummer Shuffles (even ones where you flip over the top 4 cards, or any even number of cards), the Magician finally deals the cards out into two piles, alternating cards. M observes which pile has face-up royal cards, and takes this pile and surreptitiously turns it over and places it on the other pile. Now the only cards that are face down will be the royal cards!

6 *Random Numbers*. The crux idea behind this trick is that $n+(-n)=0$. Keep it simple for a moment, and suppose that $h=0$. Then P takes n cards off the top of the deck, and M draws out 20 from the $(52-n)$-card deck, with the audience noting the nth one. Since $h=0$, M just puts the $(32-n)$-card deck on top of the 20-card deck. However, the audience's card is the nth from the top of the 20-card deck. Adding, we get $32-n+n=32$; thus M merely counts down to the 32nd card and this will be the target.

In the more general case, there will be h cards at the bottom, 20 cards in the middle (with the target card at the nth position from the top) and $32-n-h$ cards on top. So now M counts to the $32-h$th card. Easy!

Scissors Congruence

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The **Wallace-Bolyai-Gerwein** theorem states that any two polygons in the plane with equal area are "scissors congruent;" i.e., you can cut one polygon into pieces which can be perfectly fit together (no holes, no overlaps) to form the other. I am indebted to Lalit Jain, a high school teacher at the time, who taught me about this at a workshop in Berkeley.

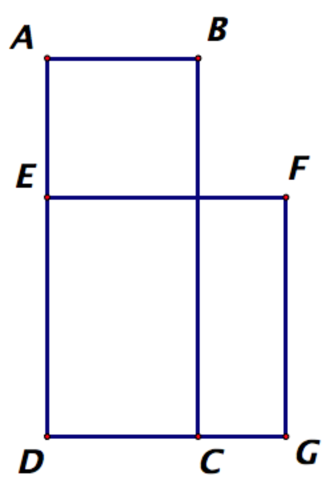
1 Prove the formula for the area of a triangle in as many ways as possible, including using paper and scissors. Does your proof work for any triangle? Why does it work?

2 Why does the area of a parallelogram only depend on its height and base? Explain this in more than one way.

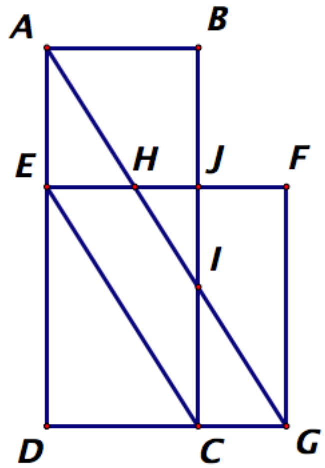
3 Can any polygon be dissected into triangles? Why? Have you examined both the convex and non-convex cases?

4 Show that two rectangles of equal area are scissors congruent.

Solution: Consider rectangles $ABCD$ and $DEFG$ below, both with the same area. I drew this in Geometer's Sketchpad so that the first rectangle was 5×2 while the second was $\sqrt{10} \times \sqrt{10}$.

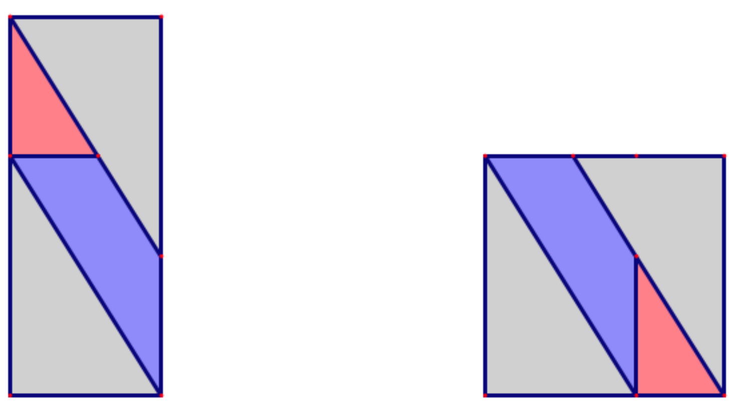


The key idea is to find some cut that "marries" both rectangles. Since the areas are equal, we have $DC \cdot AD = DG \cdot ED$, so $AD/DG = ED/DC$. This suggests *similar triangles*, and practically *demands* that you draw in the line joining A and G and the line joining E and C!



Because $EDC \sim ADG$, lines AG and EC are parallel, and there are many more similar triangles: $EJC \sim HFG \sim ABI$, and in fact, these three triangles are *congruent* because $AB = DC$ and $JC = FG$. Likewise, since $EHGC$ is a parallelogram, $EH = CG$, and triangles AEH and ICG are not just similar, but congruent.

If you think about it, this is enough to suggest a very simple dissection, requiring just two straight-line cuts!



Remark: Notice also that this method allows you to take any rectangle and dissect it to become another rectangle using any specified base. For example, you can take the 2×5 rectangle, and draw line DG to have length, say, 2π. Then we can draw line AG as before, find the intersection point I, and cut along AG and slide triangle ABI down so that I coincides with G. Then the new location of point B is the upper-right corner of the rectangle with base 2π and height $5/\pi$, preserving the area of 10. The point of this: *we can take any two rectangles and dissect them to form a bigger, single rectangle.*

One caveat: what if the the dimensions are really wacky? For example, suppose one rectangle is 1×1000000 and the other is 1000×1000. Then (verify!) you may need to modify the above method and do more translations (similar to what you may need to do with parallelograms).

5 Do the above problems allow you to prove the WBG theorem?

Solution: Yes, since any polygon can be triangulated, and each of these triangles can be turned into rectangles. And you can take any two rectangles and build a bigger rectangle (longer or taller) using the remark above. Eventually, you can dissect any polygon and turn it into a single rectangle. And likewise you can turn the other polygon into a single rectangle. You can dissect one of these giant rectangles into the other and then play the film backwards. It's not pretty, but—in theory—it can be done using only the triangle-to-parallelogram, parallelogram-to-rectangle, and rectangle-to-rectangle dissections!

Remark: The WBG theorem is false in 3 dimensions (due to Dehn), and other 2-dimensional shapes are poorly understood.

Other Dissection Problems

7 Do you know any proofs of the Pythagorean Theorem that use dissections? What does this have to do with the WBG theorem?

8 *Infinite dissections*. (Thanks to Sam Vandervelde.) Can you dissect a square into infinitely many line segments? Of course you can. (A line segment, by the way, is straight and has two endpoints and infinitely many points in-between; in other words, it has positive, non-zero length. A single point is *not* a line segment. And "dissecting into line segments" means decomposing into disjoint line segments (no two line segments have any point in common). So using this definition of line segment, here are a few harder questions. Which of the following can you dissect into infinitely many line segments?

(a) A rectangle.

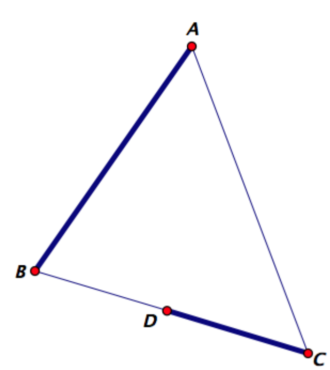
Solution: This is pretty obvious. Just draw parallel line segments.

(b) A trapezoid.

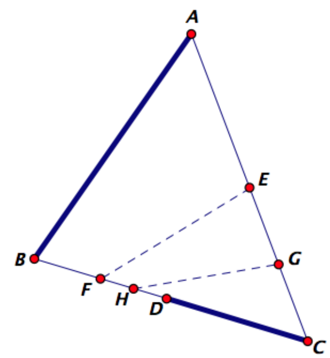
Solution: So is this.

(c) A triangle.

Solution: This is a nice example of using wishful thinking. If only a triangle were a trapezoid, say. *Make it so!*

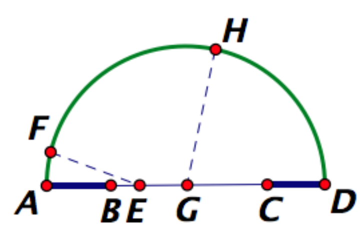


Who said that any of the sides of the trapezoid need to be parallel? Now we proceed as if we had a trapezoid. The two shaded lines AB and CD are the initial two line segments of the dissection and then we can draw line segments that include the rest of the triangle as follows: Along line BD, for example, the midpoint is F. We join F with the midpoint of AC. Likewise, point H is $3/4$ of the way from B to D, so we join it to G, which is also $3/4$ of the way between A and C. In this way, we can pick any point on BD and find a corresponding point on AC to join it to. The picture below illustrates this. (Note that in this picture, the line segments are indicated by thick lines or dashed lines; the thin lines are not one of the dissection line segments, but merely guidelines.)



(d) A semicircle.

Solution: We use the same idea as with the triangle, only twice. Start with segments AB and CD. Then we imagine that these two segments are the sides of a "trapezoid" whose top is the arc of the semicircle and whose bottom is BC. For example, EF and GH are two of the segments in the dissection. (Note that in this picture, the line segments are indicated by thick lines or dashed lines; the thin line is not one of the dissection line segments, but merely a guideline.)



If you are not satisfied with this picture, and want a formula, you can assign to each point on the interval between B and C an angle between 180 and 0 degrees, and then join each point on the BC to the corresponding point on the arc with that angle. For example, B joins with A (angle 180), and C joins with D (angle is 0) and E joins with 170 degrees, G joins with 77 degrees, etc.

(e) A circle.

Solution: Here's a picture to illustrate the idea. Essentially, we take two "semicircles" and put them together, but leave space for a "belt" of parallel lines. Again, the thick and dashed lines are line segments used in the dissection, but the thin lines are merely guidelines.