$$\begin{pmatrix} 0 & 0 & B_R \\ 0 & K_{CC} & K_{CR} \\ B_R^T & K_{RC} & K_{RR} \end{pmatrix} \begin{pmatrix} \lambda \\ u_C \\ u_R \end{pmatrix} = \begin{pmatrix} d \\ f_C \\ f_R \end{pmatrix}$$

C: corners.

$$u_C = \begin{pmatrix} u_P \\ u_D \end{pmatrix} \leftarrow \text{corner dofs with Dirichlet conditions.}$$

$$K_{CC} = \begin{pmatrix} K_{PP} & K_{PD} \\ K_{DP} & K_{DD} \end{pmatrix}, \quad f_C = \begin{pmatrix} f_P \\ f_D \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & B_R \\ 0 & K_{PP} & K_{PD} & K_{PR} \\ 0 & K_{DP} & K_{DD} & K_{DR} \\ B_R^T & K_{RP} & K_{RD} & K_{RR} \end{pmatrix} \begin{pmatrix} \lambda \\ u_P \\ u_D \\ u_R \end{pmatrix} = \begin{pmatrix} d \\ f_P \\ f_D \\ f_R \end{pmatrix}$$

We remove u_D from the unknowns.

$$\begin{pmatrix} 0 & 0 & B_R \\ 0 & K_{PP} & K_{PR} \\ B_R^T & K_{RP} & K_{RR} \end{pmatrix} \begin{pmatrix} \lambda \\ u_P \\ u_R \end{pmatrix} = \begin{pmatrix} d \\ \tilde{f}_P \\ \tilde{f}_R \end{pmatrix}$$

$$\tilde{f}_P = f_P - K_{PD} u_D$$

$$\tilde{f}_R = f_R - K_{RD} u_D$$

(This is the same system as before, but with $B_P = 0$, and $f_P \to \tilde{f}_P$ and $f_R \to \tilde{f}_R$.)

Then we split the system above as:

$$B_R u_R = d \qquad (1)$$

$$K_{PP} u_P + K_{PR} u_R = \tilde{f}_P \qquad (2)$$

$$B_R^T \lambda + K_{RP} u_P + K_{RR} u_R = \tilde{f}_R \qquad (3)$$

From (2): $K_{PP} u_P = \tilde{f}_P - K_{PR} u_R$

From (3): $u_R = K_{RR}^{-1} \left(\tilde{f}_R - K_{RP} u_P - B_R^T \lambda \right)$

Plugging (3) into (2):

$$K_{PP} u_P = \tilde{f}_P - K_{PR} K_{RR}^{-1} \tilde{f}_R + K_{PR} K_{RR}^{-1} K_{RP} u_P + K_{PR} K_{RR}^{-1} B_R^T \lambda$$

$$u_P = S_{PP}^{-1} \left(\tilde{f}_P - K_{PR} K_{RR}^{-1} \tilde{f}_R + K_{PR} K_{RR}^{-1} B_R^T \lambda \right) \quad (4)$$

with $S_{PP} = K_{PP} - K_{PR} K_{RR}^{-1} K_{RP}$ (5)

We plug (4) into (3):

$$u_R = K_{RR}^{-1} \tilde{f}_R - K_{RR}^{-1} K_{RP} S_{PP}^{-1} \tilde{f}_P + K_{RR}^{-1} K_{RP} S_{PP}^{-1} K_{PR} K_{RR}^{-1} \tilde{f}_R$$
$$+ K_{RR}^{-1} K_{RP} S_{PP}^{-1} K_{PR} K_{RR}^{-1} B_R^T \lambda + K_{RR}^{-1} B_R^T \lambda$$

$$u_R = \left[K_{RR}^{-1} \left(I + K_{RP} S_{PP}^{-1} K_{PR} K_{RR}^{-1} \right) \tilde{f}_R - K_{RR}^{-1} K_{RP} S_{PP}^{-1} \tilde{f}_P \right]$$
$$+ K_{RR}^{-1} \left(K_{RP} S_{PP}^{-1} K_{PR} K_{RR}^{-1} + I \right) B_R^T \lambda \quad (6)$$

Finally, we plug (6) in (1)

$$B_R K_{RR}^{-1} \left(\left(I + K_{RP} S_{PP}^{-1} K_{PR} \right) K_{RR}^{-1} \tilde{f}_R - K_{RP} S_{PP}^{-1} \tilde{f}_P \right)$$
$$+ B_R K_{RR}^{-1} \left(K_{RP} S_{PP}^{-1} K_{PR} K_{RR}^{-1} + I \right) B_R^T \lambda = d$$

That can be written as:

$$F \lambda = \bar{d}, \quad \text{where}$$

$$F = B_R K_{RR}^{-1} \left(K_{RP} S_{PP}^{-1} K_{PR} K_{RR}^{-1} + I \right) B_R^T$$
$$\bar{d} = d - B_R K_{RR}^{-1} \left(\left(I + K_{RP} S_{PP}^{-1} K_{PR} \right) K_{RR}^{-1} \tilde{f}_R - K_{RP} S_{PP}^{-1} \tilde{f}_P \right)$$