Purpose:

Analysis of the electric field due to a dipole in a sphere assumed to be the human skull. Importance of the problem includes prediction of EEG voltages on surface of the scalp, modelling of seizures, and other quantitative results. The analytical solution is given. A finite difference model will be developed to verify our the results. I will be using spherical coordinates to analyze the system.

Introduction:

When a dipole is placed in the middle of a sphere (the human skull) it gives rise to an electric field (see Figure 1 – field due to a single charge). The electric field passes through the brain matter and in-turn gives rise to voltage according to the equation

$$E = -\nabla V \quad (1.0)$$

If we assume the dipole gives a potential

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right) \qquad (1.1)$$

At a close distance, then we can calculate the potential at an inner ring which will be used as a boundary condition (See **Dirichlet Boundary)** for solving the Laplace equation (zero charge in space)

$$\nabla^2 V = 0 \ (1.2)$$

Where ∇^2 is the Laplacian.

Since the skull can be represented by a sphere – spherical coordinates are convenient (see Figure 2).

Figure 1. Electric Charge and its Field

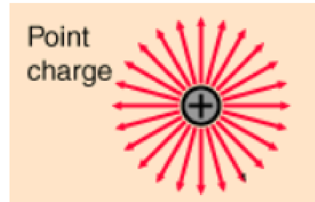
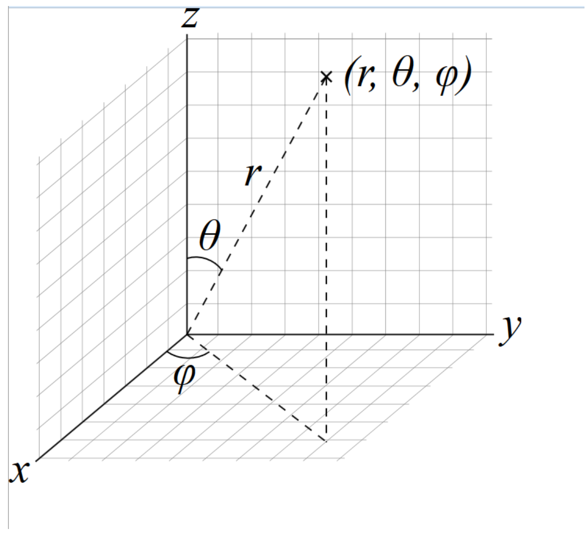


Figure 2. Spherical Coordinate System



Analytical:

Solving (1.2) analytically is not trivial. A solution was provided in class:

$$V(r,\theta) = \frac{P_z}{4\pi\sigma a^2}\left[\frac{2r}{a} + \left(\frac{a}{r}\right)^2\right] cos(\theta) \ (2.0)$$

According to (2.0) we can see that at θ=90˚ the potential is 0. Therefore on the XY-plane the voltage is 0 – our system is symmetric about z=0. As well we can observe, in spherical coordinate system, that the voltage does not change with φ. This is due to the fact the relative length r, from both charges (I call these r_1 and r_2) does not change with the change in φ. This observation is also backed up by (2.0) since φ does not appear in the equation and therefore the voltage is independent of it.

Numerical Analysis:

In order to solve the problem numerically we must first setup our finite difference equations. To solve (1.2), we derive the equation in spherical coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{cot(\theta)}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0 \qquad (3.0)$$

As we mentioned in the previous section the last term will **be zero** due to no change of voltage with respect to φ.

$$0 = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{cot(\theta)}{r^2}\frac{\partial V}{\partial \theta} \qquad (3.1)$$

According to (3.1) we must now solve the partial derivatives involved and discretize our mesh. Using central difference we can derive.

$$\frac{\partial V}{\partial r} = \frac{V(r_0 + \Delta r, \theta_0) - V(r_0 - \Delta r, \theta_0)}{2\Delta r} \qquad (3.2)$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{V(r_0 + \Delta r, \theta_0) - 2V(r_0, \theta_0) + V(r_0 - \Delta r, \theta_0)}{\Delta r^2} \qquad (3.3)$$

$$\frac{\partial V}{\partial \theta} = \frac{V(r_0, \theta_0 + \Delta\theta) - V(r_0, \theta_0 - \Delta\theta)}{2\Delta\theta} \qquad (3.4)$$

$$\frac{\partial^2 V}{\partial \theta^2} = \frac{V(r_0, \theta_0 + \Delta\theta) - 2V(r_0, \theta_0) + V(r_0, \theta_0 - \Delta\theta)}{\Delta\theta^2} \qquad (3.5)$$

Plugging (3.2) – (3.5) into (3.1)

$$\begin{aligned} 0 = \Bigg[&\frac{V(r_0 + \Delta r, \theta_0) - 2V(r_0, \theta_0) + V(r_0 - \Delta r, \theta_0)}{\Delta r^2} + \frac{2}{r_0}\frac{V(r_0 + \Delta r, \theta_0) - V(r_0 - \Delta r, \theta_0)}{2\Delta r} \\ &+ \frac{1}{{r_0}^2}\frac{V(r_0, \theta_0 + \Delta\theta) - 2V(r_0, \theta_0) + V(r_0, \theta_0 + \Delta\theta)}{\Delta\theta^2} \\ &+ \frac{cot(\theta_0)}{r_0^2}\frac{V(r_0, \theta_0 + \Delta\theta) - V(r_0, \theta_0 - \Delta\theta)}{2\Delta\theta} \Bigg] \qquad (3.6) \end{aligned}$$

We can then common factor the expression in terms of V terms

$$0 = \boldsymbol{V(r_0 + \Delta r, \theta_0)}\left(\frac{1}{\Delta r^2} + \frac{1}{r_0 \Delta r}\right) + \boldsymbol{V(r_0, \theta_0)}\left(\frac{-2}{\Delta r^2} - \frac{2}{{r_0}^2 \Delta\theta^2}\right) + \boldsymbol{V(r_0 - \Delta r, \theta_0)}\left(\frac{-1}{r_0 \Delta r} + \frac{1}{\Delta r^2}\right) + \boldsymbol{V(r_0, \theta_0 + \Delta\theta)}\left(\frac{1}{{r_0}^2 \Delta\theta^2} + \frac{cot(\theta_0)}{2{r_0}^2 \Delta\theta}\right) + \boldsymbol{V(r_0, \theta_0 - \Delta\theta)}\left(\frac{1}{{r_0}^2 \Delta\theta^2} - \frac{cot(\theta_0)}{2{r_0}^2 \Delta\theta}\right) \quad (3.7)$$

This can now be used for a finite difference approximation to set up our matrix. Before we do that lets talk about our boundary conditions.

Dirichlet Boundary:

Of the form V(1,1) = 5. Since the in matrix (X), from (3.8), holding our V(1,1) – V(r_N, θ_N) points contains our initial boundary conditions, we can eliminate the top rows (θ_N rows) of our matrix (A – the invertible matrix). Since our second row depends on coefficients in the first; they can be replaced by the numbers from the boundary conditions and put into b (our resulting matrix). Here is how this looks:

$$A * X = b \quad (3.8)$$

Lets call the coefficients in (3.7) A, B, C, D and E, respectively. Then the equation for V(2,2) will be

$$0 = \boldsymbol{V(3,2)}(A) + \boldsymbol{V(2,2)}(B) + \boldsymbol{V(1,2)}(C) + \boldsymbol{V(2,3)}(D) + \boldsymbol{V(2,1)}(E) \quad (3.9)$$

But since V(2,1) is on the boundary, **we know it's value** and therefore $V(2{,}1)(E)$ can go on the other side of the equation: into b of (3.8). Doing these for all points V(2,θ_N) we can then take the first row out of our matrices A, X and b.

Neumann Boundary:

For this type of boundary condition (BC) we have:

$$\frac{\partial V}{\partial r}|_{r=N} = 0 \quad (3.10)$$

Since the current leaving the skull is zero. We can now do a backward difference at the boundary to solve for the value at the boundary; according to this

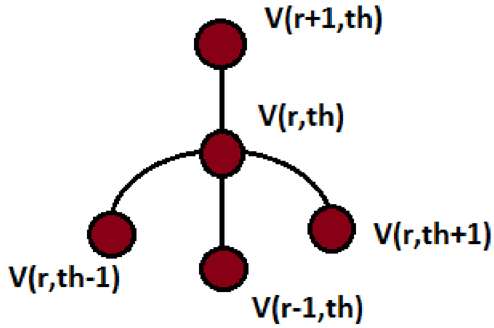
$$\frac{\partial V}{\partial r}|_{r=N} = \frac{V(r_N, \theta) - V(r_{N-1}, \theta)}{\Delta r} \quad (3.11)$$

This, due to the derivative being equal to zero can simplify to

$$V(r_N, \theta) = V(r_{N-1}, \theta) \quad (3.12)$$

Therefore our last points on the boundary are equal to the ones just before the boundary. See my stencil in Figure 3.

Figure 3. Stencil used for sampling

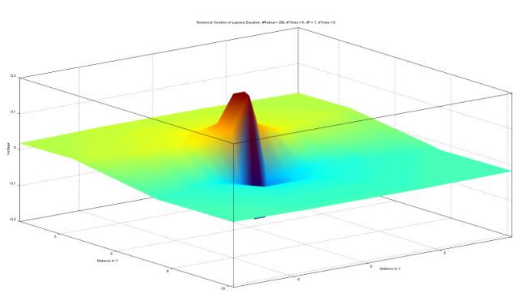


With this result we can substitute the **second last row**'s coefficients with entries from the **last row**. If we follow (3.9) and solve for a point near the boundary, $V(r_{N-1},\theta)$ it would result in

Results:

The solution of the voltages calculated are first mapped to the corresponding Cartesian set of points – **pair of XY** coordinates per **voltage**. After a surface must be generated to fit the points and display a result. The Octave function "Trisuft" is used together with Delaunay triangulation to draw surface of the voltage over the plane. See the resulting figures 4,5,6 and 7.

Figure 4. Perspective view. Voltage on XY. N=200, Theta=8



$$\begin{aligned} 0 = {} & V(r_n,2)(A) + V(r_{n-1},2)(B) \\ & + V(r_{n-2},2)(C) \\ & + V(r_{n-1},3)(D) \\ & + V(r_{n-1},1)(E) \quad (3.13) \end{aligned}$$

Using (3.12) we can replace $V(r_n,2)$ = $V(r_{n-1},2)$ and we would get

$$\begin{aligned} 0 = {} & V(r_{n-1},2)(B+A) + V(r_{n-2},2)(C) \\ & + V(r_{n-1},2)(D) \\ & + V(r_{n-1},1)(E) \quad (3.13) \end{aligned}$$

We can do this to all our values on radius r_n and then our matrix size would be: A = (N-2,N-2), a decrease of 2 rows and 2 cols.

Figure 5. View of X plane. N=200, Theta=8.

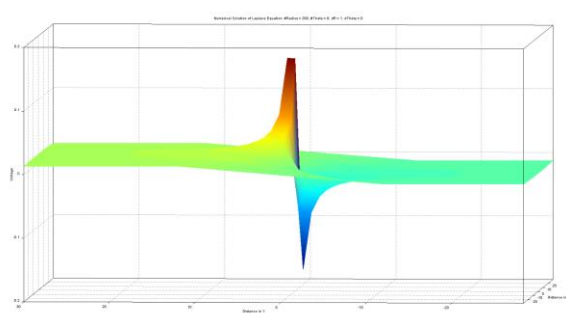


Figure 6. Top view of XY plane. N=200, Theta=8

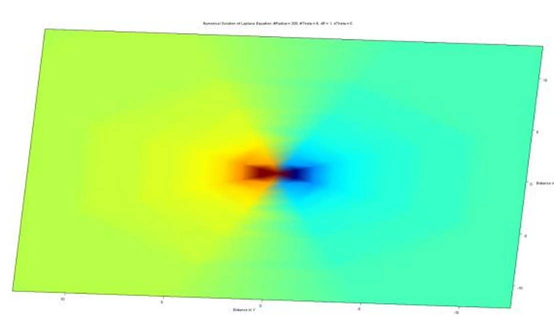
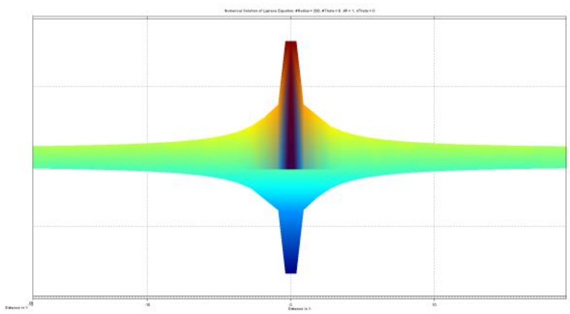


Figure 7. View of Y plane. N=200, Theta=8



Theta is calculated on eight instances, and caution is used to avoid the infinite values arising from the cotan in equation (3.7).

$$\theta_i = \frac{\pi}{4} i + \frac{\pi}{8} \qquad (4.0)$$

The dipole gives rise to a voltage which decays around the plane of the outer surface (the final radius - assumed to be the skull).

A dipole that changes with time (rectangular pulse of length T) would result in a proportional change in voltage around the plane, and on the edge of the skull. It would be in phase with the input signal. In reality this change in voltage would occur after a small time delay.

Modelling using finite difference extents to various geometries where we don’t have analytical solutions. It provides results with some error which can be further minimized, for example by increasing the order of approximation of the finite difference equations, in our case we used the first order approximation. It is also optimized by Octave’s quick matrix operation routines, as a next step: finite difference can also be implemented in custom hardware – like an FPGA for greater boost of computational speed where each clock cycle can be optimized.