Homework - 1

1.1)

$$A = \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$$

1) Characteristic polynomial of a matrix is calculated ~~given~~ by $|A - \cancel{\lambda} I| = 0$ λ - characteristic polynomial

$$\left| \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3/2 - \lambda & -1 \\ -1/2 & 1/2 - \lambda \end{vmatrix} = 0$$

$$(3/2 - \lambda)(1/2 - \lambda) - (-1/2)(-1) = 0$$

$$\frac{3}{4} - \frac{1}{2}\lambda - \frac{3}{2}\lambda + \lambda^2 - 1/2 = 0$$

$$\lambda^2 - 2\lambda - 1/4 = 0 \rightarrow 4\lambda^2 - 8\lambda - 1 = 0_{//}$$

$\therefore$ characteristic polynomial of A is $4\lambda^2 - 8\lambda - 1 = 0_{//}$

2) Eigen value - $4\lambda^2 - 8\lambda - 1 = 0$

Roots of quadratic equation - $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ -

$\therefore$ eigen values are $\lambda_1 = \dfrac{2 - \sqrt{3}}{2}$ $\lambda_2 = \dfrac{2 + \sqrt{3}}{2}$

3) Eigen vectors are given by $(A - \lambda I).V_1 = 0$

$(A - \lambda_2 I).V_2 = 0$

$$\left[\begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} \frac{2+\sqrt{3}}{2} & 0 \\ 0 & \frac{2+\sqrt{3}}{2} \end{bmatrix}\right] \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{+1 \cancel{+} \sqrt{3}}{2} & -1 \\ -1/2 & \frac{-1 \cancel{+} \sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0 \qquad \begin{bmatrix} \frac{1 - \sqrt{3}}{2} & -1 \\ -1/2 & \frac{-1 - \sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$$\left(\frac{1-\sqrt{3}}{2}\right)v_{11} - v_{12} = 0 \qquad -\frac{1}{2}v_{11} + \left(\frac{-1-\sqrt{3}}{2}\right)v_{12} = 0$$

$$v_{12} = \left(\frac{1-\sqrt{3}}{2}\right)v_{11} \qquad (-1-\sqrt{3})\,v_{12} = v_{11}$$

//

$$A = \begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$$

For another set of values using λ_2, $(A - \lambda_2 I).v = 0$

$$\left[\begin{bmatrix} 3/2 & -1 \\ -1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} \frac{2-\sqrt{3}}{2} & 0 \\ 0 & \frac{2-\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}\right] = 0$$

$$\begin{bmatrix} \frac{1+\sqrt{3}}{2} & -1 \\ -1/2 & \frac{\sqrt{3}-1}{2} \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$\left(\frac{1+\sqrt{3}}{2}\right)v_{21} - v_{22} = 0$$

$$-\frac{v_{21}}{2} + \left(\frac{\sqrt{3}-1}{2}\right)v_{22} = 0$$

$\therefore$ $v_{22} = \left(\frac{1+\sqrt{3}}{2}\right)v_{21}$, $v_{21} = (\sqrt{3}-1)\,v_{22}$

$\therefore$ Eigen vectors are $\begin{bmatrix} -(1+\sqrt{3})\,v_{12} \\ v_{12} \end{bmatrix}$ and $\begin{bmatrix} (\sqrt{3}-1)\,v_{22} \\ v_{22} \end{bmatrix}$

Solving these equations, we get

$$v_1 = \begin{bmatrix} 0.5906 \\ 0.8069 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0.9391 \\ -0.34372 \end{bmatrix}$$

or $v_1 = k_1 \begin{bmatrix} 1 \\ 1.366 \end{bmatrix}$ $\qquad v_2 = k_2 \begin{bmatrix} 1 \\ -0.366 \end{bmatrix}$

or $v_1 = k_1 \begin{bmatrix} 0.732 \\ 1 \end{bmatrix}$ $\qquad v_2 = k_2 \begin{bmatrix} -2.732 \\ 1 \end{bmatrix}$ //

1.2) $Z_1 \; N(2,1)$ $\quad Z_2 \; N(1,5)$

Z_1 & Z_2 are correlated with covariance 2

$$Z_1 + Z_2 = N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho(1,2)\sigma_1\sigma_2)$$

where ρ is coefficient of correlation.

Probability distribution is given by,

$$f_{xy}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[x^2 - 2\rho xy + y^2\right]\right\}$$

$$\rho(x,y) = \rho(1,2) = \frac{\text{Covariance}(1,2)}{\sigma_1 \sigma_2} = \frac{2}{\sqrt{1} \times \sqrt{5}}$$

$$\rho(1,2) = 0.8944$$

$$Z_1 + Z_2 = N(2+1, 1+5+2\times 0.8944 \times \sqrt{1} \times \sqrt{5})$$

$$Z_1 + Z_2 = N(3, 10)$$ → which implies it is a Normal distribution

$$f_{xy}(x,y) = \frac{1}{2\pi\sqrt{1-(0.8944)^2}} \exp\left\{ \frac{-1}{2(1-0.8944^2)} \left[x^2 - 2\times 0.8944 + y^2\right]\right\}$$

∴ Probability distribution function is,

$$f_{xy}(x,y) \Rightarrow 0.355 \times \exp\left\{-2.5\left[x^2 - 1.788 + y^2\right]\right\}$$

1.3) $N > 300 \rightarrow$ divided to group of 'K' and control groups of size $N-K$

i) Null hypothesis:

Let μ_1 be the mean effectiveness of placebo and μ_2 be the mean effectiveness of new drug

$H_0 : \mu_2 - \mu_1 = \Delta_0$ (typically 0)

$\Rightarrow H_0 : P_1 - P_2 = \Delta_0$

$\therefore$ Alternate hypothesis is $H_a : \mu_2 - \mu_1 \neq \Delta_0$

~~$H_a : P_2 - P_1 \neq \Delta_0$~~

$H_a : P_2 - P_1 \neq \Delta_0$

2) Statistical test: 'Z' statistical test for normal distribution

In our case, test statistic: $Z = \dfrac{\bar{X} - \bar{Y} - (\mu_2 - \mu_1)}{\sqrt{\dfrac{\sigma_2^2}{K_1} + \dfrac{\sigma_1^2}{(N-K)}}}$

assuming mean is given

$$Z = \frac{\bar{X} - \bar{Y} - (\Delta_0)}{\sqrt{\dfrac{S_2^2}{K} + \dfrac{S_1^2}{(N-K)}}} = \frac{\bar{X} - \bar{Y}}{\sqrt{\dfrac{S_2^2}{K} + \dfrac{S_1^2}{(N-K)}}} \quad (\because \Delta_0 = 0 \text{ usually})$$

Based on population proportion,

$$Z = \frac{\hat{P}_2 - \hat{P}_1}{\sqrt{\dfrac{\hat{P}_2(1-\hat{P}_2)}{K} + \dfrac{\hat{P}_1(1-\hat{P}_1)}{(N-K)}}}$$

3) For optimal value of $K \left(= N/2\right)$

$$K = \begin{cases} N/2 & \text{if } N \text{ is even} \\ N+1/2 & \text{if } N \text{ is odd} \end{cases}$$

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_2 - \mu_1)}{\sqrt{\dfrac{\sigma_2^2}{K} + \dfrac{\sigma_1^2}{(N-K)}}}$$

To reject Null hypothesis, value must be >0.
Considering a 95% confidence ~~[illegible]~~ and a_0

$z = 1.96$

$$\bar{x} - \bar{y} - (\mu_2 - \mu_1) > 0 \quad \text{(for rejection)}$$

$\sigma_1 = \sigma_2$ (since the S.D. of sample space is same)

$$z = \frac{\bar{x} - \bar{y} - (\bar{\mu}_2 - \bar{\mu}_1)}{\sigma\sqrt{\frac{1}{K} + \frac{1}{N-K}}} \qquad \cancel{\frac{\bar{\mu}_2 + \bar{\mu}_1}{\sigma\sqrt{\frac{1}{K} + \frac{1}{N-K}}}}$$

~~Dummy~~

$$z > \frac{a_0}{\sigma\sqrt{\frac{1}{K} + \frac{1}{N-K}}} \Rightarrow z \times \sigma\sqrt{\frac{1}{K} + \frac{1}{N-K}} > a_0$$

($\because$ $\bar{x} - \bar{y} - (\bar{\mu}_2 - \bar{\mu}_1) > 0$ for rejection

$$\therefore 2 \times \sigma \times \sqrt{\frac{N}{K(N-K)}} > a_0 \qquad (\because z = 2)$$

$$4 \times \sigma^2 \times \frac{N}{K(N-K)} > a_0^2$$

$4N > a_0^2 \times (KN - K^2)$ (assuming variance $= 1$)

$$a_0^2 K^2 - a_0^2 KN + 4N > 0$$

Finding the roots for the quadratic equation,

$$(a_0^2 N)^2 - 4(a_0^2)(4N) \geq 0 \qquad (\because b^2 - 4ac \geq 0)$$

$$a_0^4 N^2 - 16 a_0^2 N \geq 0 \quad \therefore N \geq \frac{16}{a_0^2}$$

roots are $\rightarrow$ $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Assuming $b^2 - 4ac = 0$, $K = \frac{-b}{2a} = \frac{a_0^2 N}{2a_0^2}$

$$\boxed{K = N/2}$$

1.47) i) The mean is expected value in a geometric distribution. $E(x) = \sum_{k=1}^{\infty} k \cdot f_x(k)$

$$E(x) = \sum_{k=1}^{\infty} f_x(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1} = p(1 + 2(1-p) + 3(1-p)^2 \ldots\ldots)$$

$$= p\left(\sum_{k=1}^{\infty}(1-p)^{k-1} + \sum_{k=2}^{\infty}(1-p)^{k-1} + \sum_{k=3}^{\infty}(1-p)^{k-1} + \ldots\ldots\right)$$

we can notice that since p lies between 0 & 1, 1-p also lies between 0 & 1. $\therefore \sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}$

$S_\infty = \frac{a}{1-r}$ (using GP summation)

$$\rightarrow p(1 + (1-p)\left[(2 + 3(1-p) + 4(1-p)^2 + \ldots\ldots\right]$$

$$= p\left(\sum_{k=1}^{\infty}(1-p)^{k-1} + \sum_{k=2}^{\infty}(1-p)^{k-1} + \sum_{k=3}^{\infty}(1-p)^{k-1} + \ldots\ldots\right)$$

$$= p \cdot x \underbrace{\left(\frac{1}{1-(1-p)} + \frac{1-p}{1-(1-p)} + \frac{(1-p)^2}{1-(1-p)} + \ldots\ldots\right)}$$

$1, (1-p), (1-p)^2, \ldots\ldots$ another GP $\Rightarrow \frac{1}{1-(1-p)}$

$$E(x) = \frac{1}{1-(1-p)} = \frac{1}{p}$$ //

27) Variance $\rightarrow E(x^2) \Rightarrow \sum_{k=1}^{\infty} k^2 \times p(k)$

$$= \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} \times p = p \sum_{k=1}^{\infty} k^2 \times (1-p)^{k-1}$$

Rewriting $k^2 (1-p)^{k-1}$ ~~as~~ as $-\frac{d}{dp} k(1-p)^k$

$$= p \sum_{k=1}^{\infty} -\frac{d}{dp} (k(1-p)^k) = p \times -\frac{d}{dp} \sum_{k=1}^{\infty} k \times (1-p)^{k-1} \times p \left(\frac{1-p}{p}\right)$$

$$= -p \times \frac{d}{dp} \sum_{1} k \times (1-p)^{k-1} \times p \left(\frac{1-p}{p}\right)$$

$$= -p \times \frac{d}{dp}\left(\frac{1-p}{p}\right) \underbrace{\sum_{k=1}^{\infty} k \times (1-p)^{k-1} \times p}_{E(x) = 1/p.}$$

$$= -p \times \frac{d}{dp}\left(\frac{1-p}{p}\right) \times \frac{1}{p} = -p \times \frac{d}{dp}\left(\frac{1}{p^2} - \frac{1}{p}\right)$$

$$= -p \frac{d}{dp}(p^{-2} - p^{-1}) = -p\left(-2p^{-3} + p^{-2}\right)$$

$$\therefore E(x^2) = \frac{2}{p^2} - \frac{1}{p} = \frac{2-p}{p^2}$$

Variance $= E(x^2) - (E(x))^2 \rightarrow \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$ //