The value of timing information in event-triggered control: The scalar case

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Abstract— The problem of event-triggered control with rate-limited communication is considered. For continuous-time scalar systems without disturbances, a *phase transition* behavior of the transmission rate required for stabilization as a function of the communication delay is revealed. It is shown that for low values of the delay the timing information carried by the triggering events is large and the system can be stabilized with any positive rate. On the other hand, when the delay exceeds a certain threshold that depends on the given triggering strategy, the timing information alone is not enough to achieve stabilization and the rate must begin to grow, eventually becoming larger than what required by the classic *data-rate theorem*. The critical point where the transmission rate equals the one imposed by the data-rate theorem occurs when the delay equals the inverse of the *entropy rate* of the plant, representing the intrinsic rate at which the system generates information. At this critical point, the timing information supplied by event triggering is completely balanced by the information loss due to the communication delay. Exponential convergence guarantees are also discussed, and an explicit construction providing a sufficient condition for stabilization is given.

I. INTRODUCTION

Cyber-Physical Systems (CPS) [1] are next-generation engineering systems that integrate computing, communication, and control. They arise in diverse areas such as robotics, energy, civil infrastructure, manufacturing, and transportation. Due to their integration of different components, their modeling, analysis, and design present new and challenging problems to the control engineer.

One key aspect is the presence of finite-rate, digital communication channels in the feedback loop. To quantify their effect on the ability to stabilize the system, *data rate theorems* have been developed. These results essentially state that, in order to achieve stabilization, the communication rate available in the feedback loop should be at least as large as the *intrinsic entropy rate* of the system, corresponding to the sum of the logarithms of the unstable modes. In this way, the controller can compensate for the expansion of the state occurring during the communication process. Early formulations of data rate theorems include [2]–[4]. Later key contributions are [5], [6]. More recent extensions include stochastic, time-varying, and Markovian feedback communication channels [7]–[9], as well as formulations for nonlinear sytems [10]–[12]. Connections with information theory are highlighted in [12]–[16]. Extended surveys of the literature appear in [17] and in the book [18].

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Another important aspect of CPS is the need to use distributed resources efficiently. In this context, event-triggering control techniques [19], [20] have emerged. These are based on the idea of sending information in an opportunistic manner between the controller and the plant. In this way, communication occurs only when needed, and the primary focus is on minimizing the number of transmissions while guaranteeing the control objectives. Some recent results about event-triggered implementations in the presence of data rate constraints appear in [21], [22].

A key point raised in [22] is that if the channel does not introduce any delay, then an event-triggering strategy can achieve stabilization for any positive rate of transmission, thus apparently contradicting classic data-rate theorem formulations that require a periodic and sufficiently high data exchange between the plant and the controller. This apparent contradiction is resolved by realizing that the timing of the triggering events itself carries information, revealing the state of the system. When communication occurs without delay, the state can be tracked with arbitrary precision, and transmitting a single bit at every triggering event is enough to compute the appropriate control action. Hence, from the point of view of information exchange, event triggering with zero delay is analogous to conveying an infinite amount of information to the controller.

The main contribution of this paper is to extend the above observation to the whole spectrum of possible delay values. We consider the point of view of the sensor that sends a quantized observation to the controller at each triggering event, and distinguish between the *information access rate*, that is the rate at which the controller needs to receive data, regulated by the classic data-rate theorem; and the *information transmission rate*, that is the rate at which the sensor needs to send data, regulated by a given triggering control strategy. We show that for sufficiently low values of the delay the timing information carried by the triggering events is large enough and the system can be stabilized with any positive information transmission rate. We also show the existence of a critical value of the delay at which the required information transmission rate begins to grow, and the existence of a second critical value of the delay at which the required information transmission rate becomes larger than the information access rate imposed by the data-rate theorem. Finally, we provide necessary conditions on the access rate for asymptotic stabilizability and observability with exponential convergence guarantees; necessary conditions on the transmission rate for asymptotic observability with exponential convergence guarantees; as well as a sufficient condition with the same asymptotic behavior.

All results presented here are limited to scalar systems without disturbances. Extensions to the vector case and to systems subject to disturbances are planned in future work. Other possible extensions regard different families of triggering functions.

Organization: In Section II, we illustrate the system model and formulate the problem. In Section III we present necessary conditions on the information access rate for asymptotic observability and stabilizability with exponential convergence guarantees. In Section IV we determine the necessary transmission rate for an event-triggered implementation. In section V we explain the phase transition behavior of the transmission rate. Section VI present a corresponding sufficient condition. Section VII concludes the paper and mentions some open problems.

Notation: Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of positive integers. We let $m(.)$ denote the Lebesgue measure on $\mathbb{R}$, and we indicate a ball of radius ϵ by $\mathcal{B}(\epsilon)$. We let $\log(.)$ and $\ln(.)$ denote the logarithm with the base 2 and the natural logarithm, respectively. For a function $f : \mathbb{R} \to \mathbb{R}^n$ and any $t \in \mathbb{R}$, we let $f(t^+)$ denote the limit from the right, namely $\lim_{s \downarrow t} f(s)$. We let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. We denote the modulo function by $\mathrm{mod}(x, y)$, which is the remainder left after dividing x by y.

II. PROBLEM FORMULATION

We now describe the problem setup, including the system evolution, the model of the communication channel, and the role of event triggering in the controller design for observability and stabilizability.

A. System model

We consider a networked control system composed by the plant-sensor-channel-controller tuple indicated in Figure 1. The plant dynamics are described by a scalar, continuous-
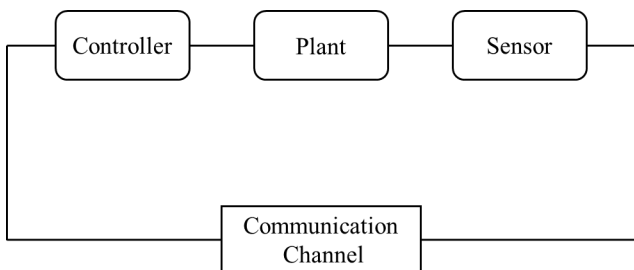

Fig. 1.   System model.

time, linear time-invariant (LTI) system

$$\dot{x} = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$ for $t \in [0, \infty)$ are the plant state and control input, respectively. Here, A is a positive real number, $B \in \mathbb{R}$, and

$$|x(0)| \le L$$

for some positive real number L.

The sensor can measure the state of the system exactly, and the controller can apply the control input to the plant with infinite precision and without delay. However, sensor and controller communicate through a channel that can support only a finite data rate.

B. Information access rate

Letting $b_c(t)$ be the number of bits encoding the state of the system at time t, and that are also available to the controller at the same time t, we define the *information access rate*

$$R_c = \lim_{t \to \infty} \frac{b_c(t)}{t}. \tag{2}$$

In this setting, data-rate theorems describe the trade-off between the information access rate and the ability to stabilize the system. There is a large literature on them, see [17] for a review. They are generally stated for discrete-time systems, albeit similar arguments hold in continuous time as well, see e.g. [21]. They are based on the fundamental observation that there is an inherent entropy rate

$$h = \frac{A}{\ln 2} \tag{3}$$

at which the plant generates information. It follows that to guarantee stability it is necessary for the controller to have access to state information at a rate

$$R_c > h. \tag{4}$$

This result indicates what is required by the controller, and it does not depend on the feedback structure - including aspects such as communication delays, information pattern at the sensor and the controller, and whether the times at which transmissions occur are state dependent, as in event-triggered control, or not, as in time-triggered control.

C. Information transmission rate

We now make two key observations regarding data-rate theorems of the form illustrated above. First, $b_c(t)$ in definition (2) represents the quantized state at time t that is also accessible by the controller at the same time t. In the presence of communication delays the state estimate received by the controller might be slightly out of date, so that the sensor might need to send data at a *higher rate* than what indicated in (4) to make-up for such discrepancy.

A second observation is that in the case of event-triggered transmissions, the timing of the triggering events in itself carries some information. In this case, if the communication channel does not introduce any delay, then a triggering event may reveal the state of the system very precisely, and effectively carry an unbounded amount of information. The controller may then be able to stabilize the system even if the sensor uses the channel very sparingly, transmitting at a *smaller rate* than what indicated in (4).

Motivated by these observations, we now consider the point of view of the sensor rather then the controller. We let $b_s(t)$ be the number of bits transmitted by the sensor up

to time t, and define the *information transmission rate*

$$R_s = \lim_{t \to \infty} \frac{b_s(t)}{t}. \tag{5}$$

The main objective is to quantify the transmission rate R_s required to stabilize the system in the presence of communication delays and using an event-triggering strategy, and to explore its relationship with the corresponding requirement for the information access rate R_c.

D. Event triggering and delay

We denote by $\{t_s^k\}_{k \in \mathbb{N}}$ the sequences of triggering times at which the sensor performs a transmission of a packet composed of p bits. We let $\{t_c^k\}_{k \in \mathbb{N}}$ be the sequence of times at which the controller receives the complete packet of data and decodes it. We assume that the communication delays are uniformly upper-bounded by γ, a finite non-negative real number, namely

$$\Delta_k = t_c^k - t_s^k \le \gamma, \tag{6}$$

where Δ_k is the kth *communication delay*. For all $k \ge 1$ we also define the kth *triggering interval*

$$\Delta_k' = t_s^{k+1} - t_s^k. \tag{7}$$

When referring to a generic triggering time or reception time, we shall skip the super-script k in t_r^k and t_c^k.

Since at every triggering interval, the sensor sends p bits, we have

$$R_s = \lim_{N \to \infty} \frac{Np}{\sum_{k=1}^{N} \Delta_k'}. \tag{8}$$

The objective is now to precisely quantify the value of R_s required for stabilization when γ is in the interval $[0, \infty)$.

E. Controller dynamics

We let $\hat{x}$ be the state estimate available at the controller, which evolves according to

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \tag{9}$$

with $\hat{x}(0) = 0$. We define the *state estimation error*

$$z(t) = x(t) - \hat{x}(t), \tag{10}$$

where $z(0) = x(0)$. A triggering event occurs when

$$|z(t)| = v(t), \tag{11}$$

where $v(t)$ is the *event-triggering function*

$$v(t) = De^{-\sigma t}, \tag{12}$$

and D and σ are positive real numbers.

If the controller knows the triggering time t_s, then it also knows that $x(t_s) = \pm v(t_s) + \hat{x}(t_s)$. It follows that in this case it may compute the exact value of $x(t_s)$ by just transmitting one single bit at every triggering time. In general, however, the controller does not have knowledge of t_s, but only knows the bound in (6). Let $\bar{z}(t_c)$ be an estimate of $z(t_c)$ constructed by the controller knowing that $|z(t_s)| = v(t_s)$ and using (6) and the decoded packet received through the communication channel. We define the following updating procedure, called *jump strategy*

$$\hat{x}(t_c^+) = \bar{z}(t_c) + \hat{x}(t_c). \tag{13}$$

Note that with this jump strategy, we have

$$z(t_c^+) = x(t_c) - \hat{x}(t_c^+) = z(t_c) - \bar{z}(t_c).$$

We conclude this section by stating the definitions of asymptotic observability and asymptotic stabilizability of our continuous-time system.

Definition 1 (Asymptotic observability): The system (1) is asymptotically observable if there exists an encoder and decoder such that for every trajectory of control $u(t)$, for $t \in [0, \infty)$ we have

- $\forall \epsilon > 0, \exists \delta > 0$ such that $|x(0)| \le \delta$ implies $|z(t)| \le \epsilon$ for all $t \in [0, \infty)$.
- $\forall \epsilon > 0$ and $\forall \delta > 0, \exists T$ such that $|x(0)| \le \delta$ implies $|z(t)| \le \epsilon$, for all $t \ge T$.

Definition 2 (Asymptotic stabilizability): The system (1) is asymptotically stabilizable if there exists an encoder, decoder, and a controller such that

- $\forall \epsilon > 0, \exists \delta > 0$ such that $|x(0)| \le \delta$ implies $|x(t)| \le \epsilon$ for all $t \in [0, \infty)$.
- $\forall \epsilon > 0$ and $\forall \delta > 0, \exists T$ such that $|x(0)| \le \delta$ implies $|x(t)| \le \epsilon$, for all $t \ge T$.

III. NECESSARY CONDITION ON THE ACCESS RATE

We begin illustrating our results by showing a data-rate theorem for the information access rate that is required for exponential convergence of the estimation error and the plant state to zero.

Theorem 1: The following necessary conditions hold:

- If the state estimation error satisfies

$$|z(t)| \le |z(0)| \, e^{-\sigma t},$$

  then

$$b_c(t) \ge t \, \frac{A + \sigma}{\ln 2} + \log \frac{L}{|z(0)|}. \tag{14}$$

- If the system in (1) is stabilizable and

$$|x(t)| \le |x(0)| \, e^{-\sigma t},$$

  then

$$b_c(t) \ge t \, \frac{A + \sigma}{\ln 2} + \log \frac{L}{|x(0)|}. \tag{15}$$

- In both of the above cases the information access rate is

$$R_c > \frac{A + \sigma}{\ln 2}. \tag{16}$$

Proof: The solution to (1) is given by

$$x(t) = e^{At}x(0) + \alpha(t), \quad \alpha(t) = e^{At} \int_0^t e^{-A\tau} Bu(\tau)d\tau.$$

We then define,

$$\Gamma_t = \{x(t) : x(t) = e^{At}x(0) + \alpha(t) \; ; \; |x(0)| \le L\},$$

that is a set which represents the uncertainty at time t given the bound L on the norm of the initial condition $x(0)$ and $\alpha(t)$. The state of the system can be any point in this uncertainty set. We can find a lower bound on $b_c(t)$ by counting the number of balls of radius $\epsilon(t)$, that cover Γ_t, where $\epsilon(t) = |z(0)| e^{-\sigma t}$. Therefore $b_c(t)$, the number of bits of information that the controller must have access to by time t, should satisfy

$$b_c(t) \geq \log \frac{m(\Gamma_t)}{m(\mathcal{B}(\epsilon(t)))} = \log \frac{e^{At} m(|x(0)| \leq L)}{2|z(0)| e^{-\sigma t}}$$
$$= t \log e^{A+\sigma} + \log \frac{L}{|z(0)|}.$$

With access to $b_c(t)$ bits of information, the controller can at best be able to identify $x(t)$ up to a ball of radius $\epsilon(t)$. If at any time t, if $\hat{x}(t)$ does not belong to the identified ball, then $\hat{x}(t)$ could be re-initialized to an arbitrary point in the ball. Consequently, the result on exponential observability follows.

Recall that $|x(0)| \leq L$. For any given control trajectory $\{u(\tau)\}_{\tau=0}^{\tau=t}$ define

$$\Pi_{\{u(\tau)\}_{\tau=0}^{\tau=t}} = \{x(0), |x(t)| < \epsilon(t)\},$$

where $\epsilon(t) = |x(0)| e^{-\sigma t}$. These are the sets of all initial conditions for which by choosing the control trajectory $\{u(\tau)\}_{\tau=0}^{\tau=t}$, the plant state at time t, $x(t)$, will be in a ball of radius $\epsilon(t)$. As discussed above, we have

$$x(t) = e^{At} x(0) + e^{At} \int_0^t e^{-A\tau} Bu(\tau) d\tau.$$

Thus, $x(t)$ depends linearly on $\{u(\tau)\}_{\tau=0}^{\tau=t}$. As a consequence, all of the sets $\Pi_{\{u(\tau)\}_{\tau=0}^{\tau=t}}$, are linear transformation of each other. So, all of them have the same measure which is equal to $m(\Pi) = 2|x(0)|e^{-At}e^{-\sigma t}$. We can then determine a lower bound for $b_c(t)$ by counting the number of Π sets (for different control trajectories $\{u(\tau)\}_{\tau=0}^{\tau=t}$) which takes to cover the ball $|x(0)| \leq L$. Thus, the controller must have access to at least $b_c(t)$ bits by time t, where

$$b_c(t) \geq \log \frac{m(|x(0)| \leq L)}{m(\Pi)} = \log \frac{2L}{2|x(0)|e^{-(A+\sigma)t}}$$
$$= t \frac{A+\sigma}{\ln 2} + \log \frac{L}{|x(0)|},$$

and this proves (15).

The proof of (16) now follows immediately by dividing (14) and (15) by t and taking the limit for $t \to \infty$. ∎

Remark 1: The proof of above theorem follows the same argument of [5] for discrete time systems. A similar result for continuous systems appears in [21]. The result in [21], however, is restricted to linear feedback controllers. The classic condition of the data-rate theorem (4) obtained in [5] is a special case of our theorem with $\sigma \to 0$. •

IV. NECESSARY CONDITION ON THE TRANSMISSION RATE

We now quantify the necessary transmission rate R_s for exponential convergence of the estimation error in the pres-

ence of communication delays, using our event-triggering and jump strategy for control.

Theorem 2: For the system in (1) when $|z(0)| \leq D$, using the event triggering strategy (11), triggering function (12) and the jump strategy (13), if the state estimation error satisfies

$$|z(t)| \leq e^{(A+\sigma)\gamma} v(t),$$

then

$$R_s > \frac{A+\sigma}{\ln 2} \max \left\{ 0, 1 + \frac{\log(e^{A\gamma} - 1)}{-\log(\rho_0 e^{-\sigma\gamma})} \right\}, \quad (17)$$

where ρ_0 is a constant in the interval $(0, 1)$.

Proof: We define

$$\Omega_{t_c} = \{y : y = z(t_s)e^{A(t_c - t_s)}, t_c \in [t_s, t_s + \gamma]\} \quad (18)$$

that is the set of uncertainty for the sensor at time t_s without knowing t_c. We wish to cover Ω_{t_c} with balls $\mathcal{B}(\rho(t_s))$, where $\rho(t) = \rho_0 e^{-\sigma\gamma} v(t)$, and $0 < \rho_0 < 1$ is a constant design parameter. We show that this choice of radius guarantees $|z(t_c^+)| < \rho_0 v(t_c)$, and

$$\frac{v(t_c)}{v(t_s)} = \frac{De^{-\sigma t_c}}{De^{-\sigma t_s}} = e^{-\sigma(t_c - t_s)}.$$

Since $t_c - t_s \leq \gamma$; we have that $v(t_c) \geq v(t_s)e^{-\sigma\gamma}$. Hence, by choosing

$$\rho(t) = \rho_o e^{-\sigma\gamma} v(t_s) \leq \rho_0 v(t_c),$$

we can guarantee $|z(t_c^+)| \leq \rho_0 v(t_c)$ as follows. Let us define,

$$H_{\rho(t_s)} = \log \frac{m(\Omega_{t_c})}{m(\mathcal{B}(\rho(t_s)))} = \log \frac{m(\Omega_{t_c})}{2\rho(t_s)}, \quad (19)$$

the second equality is true specifically in the scalar system case. Thus, we see that $H_{\rho(t_s)}$ quantifies the number of bits required to bring the uncertainty from Ω_{t_c} to a ball of radius $\rho(t_s)$. Now, denote the number of bits that we need to send at t_s by $g(t_s)$. Then, in order to ensure that $|z(t_c^+)| < \rho_0 v(t_c)$, we must have

$$g(t_s) \geq \max \left\{ 0, H_{\rho(t_s)} \right\}.$$

The first step for calculating this lower bound is looking at the differential equation which governs $z(t)$;

$$\dot{z}(t) = A(x(t) - \hat{x}(t)) = Az(t).$$

Consequently,

$$z(t) = \pm v(t_s)e^{A(t - t_s)},$$

where t can be any number in $[t_s, t_c]$, with $t_c \leq t_s + \gamma$. Hence,

$$g(t_s) \geq \max \left\{ 0, \log \frac{2 \max_{s \in (t_s, t_s + \gamma)} v(s) m(\chi_\gamma)}{2\rho_0 e^{-\sigma\gamma} v(t_s)} \right\},$$

where $\chi_\gamma = \{e^{At} : t \in [0, \gamma]\}$. Therefore,

$$g(t_s) \geq \max \left\{ 0, \log \frac{2v(t_s)(e^{A\gamma} - 1)}{2\rho_0 e^{-\sigma\gamma} v(t_s)} \right\}$$

or,

$$g(t_s) \geq \max\left\{0, \log(e^{A\gamma} - 1) - \log \rho_0 e^{-\sigma\gamma}\right\}. \qquad (20)$$

At this point, we have covered the uncertainty space with balls of radius $\rho(t_s)$. As a result, by this scheme the controller knows that $z(t_c)$ is in one of these balls; therefore, the controller can choose an arbitrary point in this specific ball, namely, $\bar{z}(t_c)$. Since $|\bar{z}(t_c) - z(t_c)| \leq \rho(t_s)$, by using the information in $\bar{z}(t_c)$, the controller updates $\hat{x}(t_c^+)$ according to the jump strategy $\hat{x}(t_c^+) = \hat{x}(t_c) + \bar{z}(t_c)$. Hence, we have

$$|z(t_c^+)| = |x(t_c) - \hat{x}(t_c^+)| = |z(t_c) - \bar{z}(t_c)| \leq \rho_0 e^{-\sigma\gamma} v(t_s).$$

Let t_s^k and t_s^{k+1} be two successive triggering times. We have that $|z(t_c^{k+})| \leq \rho_0 e^{-\sigma\gamma} v(t_s)$, and therefore we have an upper bound for the difference between two consecutive triggering times which is independent of k

$$|z(t_c^{k+}) e^{A(t_s^{k+1} - t_c^k)}| = |v(t_s^{k+1})|.$$

Since $|z(t_c^{k+})| \leq \rho_0 e^{-\sigma\gamma} v(t_s)$, we have

$$\rho_0 e^{-\sigma\gamma} v(t_s^k) e^{A(t_s^{k+1} - t_c^k)} \geq v(t_s^{k+1}).$$

Since we also have $t_s^k \leq t_c^k$, it follows that

$$\rho_0 e^{-\sigma\gamma} D e^{-\sigma t_s^k} e^{A(t_s^{k+1} - t_s^k)} \geq D e^{-\sigma t_s^{k+1}},$$

$$e^{A(t_s^{k+1} - t_s^k)} \geq \frac{e^{-\sigma(t_s^{k+1} - t_s^k)}}{\rho_0 e^{-\sigma\gamma}},$$

$$A(t_s^{k+1} - t_s^k) \geq -\ln(\rho_0 e^{-\sigma\gamma}) - \sigma(t_s^{k+1} - t_s^k).$$

Thus, the triggering intervals (7) have a uniform lower bound

$$\Delta_k' = t_s^{k+1} - t_s^k \geq \frac{-\ln(\rho_0 e^{-\sigma\gamma})}{A + \sigma}. \qquad (21)$$

Consequently, we have the upper bound on triggering rate, namely

$$R_t = \frac{1}{t_s^{k+1} - t_s^k} \leq \frac{A + \sigma}{-\ln(\rho_0 e^{-\sigma\gamma})}. \qquad (22)$$

We want our communication rate to be larger than this upper bound, and in every triggering time we transfer $g(t_s)$ bits. As a result, the required bit rate should satisfy this inequality

$$R_s > \max\left\{0, \frac{A + \sigma}{-\ln(\rho_0 e^{-\sigma\gamma})} \log\left(\frac{e^{A\gamma} - 1}{\rho_0 e^{-\sigma\gamma}}\right)\right\}.$$

The last statement can be seen directly from the definition of R_s in (8) too. Since we proved that the number of bits that the encoder needs to send at time t_s^k which will be received after a delay Δ_k is uniformly upper bounded as (20), and Δ_k' is upper bounded uniformly as (21).

Since

$$|z(t_c)| \leq v(t_s) e^{A\gamma} = D e^{-\sigma t_s} e^{A\gamma}$$
$$\leq D e^{-\sigma(t_c - \gamma)} e^{A\gamma} = D e^{-\sigma t_c} e^{(A+\sigma)\gamma},$$

the result follows. ∎

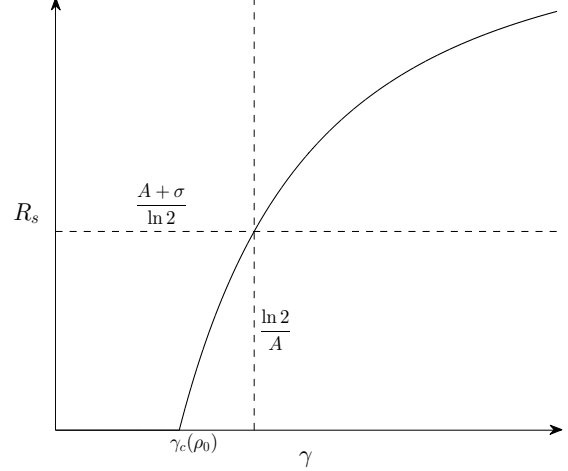*Remark 2:* While the result in Theorem 1 is independent

Fig. 2. Illustration of the necessary transmission rate for asymptotic observability versus the upper bound of delay in the communication channel. Here, $A = 5$, $\sigma = 3$, and $\rho_0 = 0.7$.

of the feedback structure, the result in Theorem 2 depends on the delay, event triggering and jump strategy used for control. This is to be expected for a result on the transmission rate of a specific event-triggering implementation. •

Remark 3: It is easy to check that the condition (17) with $\sigma \to 0$ also gives a necessary condition for asymptotic stability (provided the system in (1) is stabilizable), although it does not provide any exponential convergence guarantee of the state to zero. •

V. PHASE TRANSITION BEHAVIOR

Figure 2 illustrates the result of Theorem 2. For small values of γ, the amount of timing information carried by the triggering events is higher than what is needed to stabilize the system and the value of R_s is arbitrarily close to zero. This means that if the delay is sufficiently small then only a positive transmission rate is required to track the state of the system with an exponential convergence guarantee on the estimation error, and the controller can successfully stabilize the system by receiving a single bit of information at every triggering event. This situation persists until a critical value $\gamma = \gamma_c(\rho_0)$, that is solution of the equation $e^{A\gamma} - \rho_0 e^{-\sigma\gamma} = 1$, is reached. For this level of delay, which depends on the given jump strategy, the timing information of the triggering events becomes so much out of date that the transmission rate must begin to increase.

When γ reaches a second critical value $(\ln 2)/A$, that equals the inverse of the intrinsic entropy rate of the system indicated in (3), the timing information carried by the triggering events compensates exactly the loss of information due to the delay introduced by the communication channel. This situation is analogous to having no delay, but also no timing information. It follows that in this case the required transmission rate matches the access rate in Theorem 1, and we have $R_s = (A + \sigma)/\ln 2$. For $\sigma \to 0$, there is no
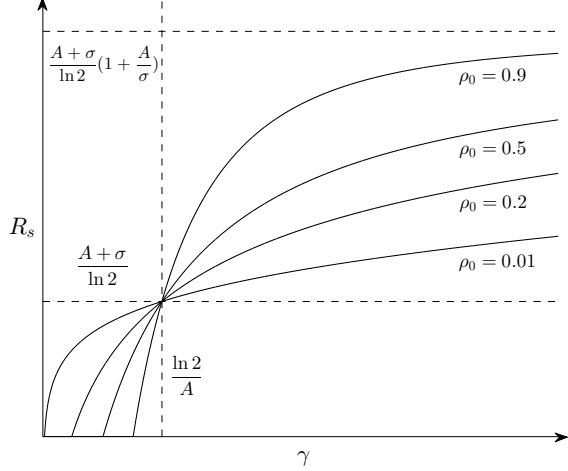
Fig. 3. Illustration of the necessary transmission rate for asymptotic observability versus the upper bound of delay for different values of ρ_0. Here $A = 1$, $\sigma = 0.5$, and $\rho_0 \in \{0.01, 0.2, 0.5, 0.9\}$.

exponential convergence guarantee and the classic data-rate theorem is recovered with the critical rate $R_s = A/\ln 2$.

If γ is increased even further, then the timing information carried by event triggering is excessively out of date and cannot fully compensate for the channel's delay. The required transmission rate then exceeds the access rate imposed by the data-rate theorem. In practice, a more precise estimate of the state must be sent at every triggering time to compensate for the larger delay. Another interpretation of this behavior follows by considering the definition $H_{\rho(t_s)}$ in (19). The value $\gamma = (\ln 2)/A$ marks a transition point for $H_{\rho(t_s)}$ from negative to positive values. For $\gamma > (\ln 2)/A$ event triggering does not supply enough information and $H_{\rho(t_s)}$ presents a positive information balance in terms of number of bits required to cover the uncertainty space. On the other hand, for $\gamma < (\ln 2)/A$, event triggering supplies more than enough information, and $H_{\rho(t_s)}$ presents a negative information balance. We can then think of event triggering as a "source" supplying information, the controller as a "sink" consuming information, and $H_{\rho(t_s)}$ as measuring the balance between the two, indicating whether additional information is needed in terms of quantized observations sent through the channel.

Figure 3 illustrates the result of Theorem 2 for different values of ρ_0. For $\gamma < (\ln 2/A)$, the timing information carried by the triggering events is useful for stabilization. Since smaller values of ρ_0 imply fewer triggering events, it follows that curves associated to smaller values of ρ_0 must have larger transmission rates to compensate for the lack of timing information. On the other hand, for $\gamma > (\ln 2/A)$ the situation is reversed. The timing information carried by the triggering events is now completely exhausted by the delay and the controller relies only on the additional information due to the quantized packets sent through the channel. Since smaller values of ρ_0 now imply larger packets sent through the channel, and the information in the larger

packets becomes out of date at a slower rate than that in the smaller packets, it follows that in this case curves associated to smaller values of ρ_0 correspond to smaller transmission rates. Finally, we observe that all curves have the same asymptotic behavior for large values of γ, which is independent of ρ_0. This occurs because as γ increases, more information needs to be sent through the channel and also the triggering rate decreases. When both of these effects are taken into account the asymptotic value of the transmission rate $\frac{A+\sigma}{\ln 2}(1 + \frac{A}{\sigma})$ is obtained.

VI. SUFFICIENT CONDITION ON THE TRANSMISSION RATE

We now consider a sufficient condition for asymptotic observability and asymptotic stabilizability of our system.

Theorem 3: For the system in (1) when $|z(0)| \leq D$, using the event triggering strategy (11), triggering function (12) and the jump strategy (13), a sufficient transmission rate for

$$|z(t)| \leq e^{(A+\sigma)\gamma} v(t),$$

is

$$R_s \geq \frac{A+\sigma}{-\ln(\rho_0 e^{-\sigma\gamma})} \max\left\{0, 1 + \log \frac{b\gamma(A+\sigma)}{\ln(1 + \rho_0 e^{-(\sigma+A)\gamma})}\right\},$$
(23)

where ρ_0 is a constant in the interval $(0,1)$, and $b > 1$.

Proof: Similar to the proof of Theorem 2, we want to cover the uncertainty space with balls of radius $\rho(t) = \rho_o e^{-\sigma\gamma} v(t_s)$. However, here we need to explicitly construct the covering. Recall that $\bar{z}(t_c)$ denotes the estimate of $z(t_c)$ at time t_c constructed by the controller. We want to ensure

$$|z(t_c) - \bar{z}(t_c)| \leq \rho_0 e^{-\sigma\gamma} v(t_s) \leq \rho_0 v(t_c). \quad (24)$$

Assume the sensor sends a quantized version of t_s to the controller, which we denote by $q(t_s)$. After receiving $q(t_s)$, the controller approximates $z(t_c)$ with

$$\bar{z}(t_c) = v(q(t_s))e^{A(t_c - q(t_s))}.$$

It follows that

$$|z(t_c) - \bar{z}(t_c)| = v(t_s)e^{A(t_c - t_s)}\left|1 - \frac{v(q(t_s))}{v(t_s)}e^{A(t_s - q(t_s))}\right|,$$

and to satisfy (24) we need

$$\left|1 - \frac{v(q(t_s))}{v(t_s)}e^{A(t_s - q(t_s))}\right| \leq \rho_0 e^{-A(t_c - t_s)}\frac{v(t_c)}{v(t_s)}.$$

The smallest possible values for $v(t_c)/v(t_s)$ and $e^{-A(t_c-t_s)}$ are $e^{-\sigma\gamma}$ and $e^{-A\gamma}$, respectively. Hence, we need

$$\left|1 - \frac{v(q(t_s))}{v(t_s)}e^{A(t_s - q(t_s))}\right| \leq \rho_0 e^{-(\sigma+A)\gamma}.$$

This condition can be rewritten as

$$\left|1 - \frac{De^{-\sigma q(t_s)}}{De^{-\sigma t_s}}e^{A(t_s - q(t_s))}\right| \leq \rho_0 e^{-(\sigma+A)\gamma},$$

$$\left|1 - e^{(A+\sigma)(t_s - q(t_s))}\right| \leq \rho_0 e^{-(\sigma+A)\gamma},$$

$$-\rho_0 e^{-(\sigma+A)\gamma} \le 1 - e^{(A+\sigma)(t_s - q(t_s))} \le \rho_0 e^{-(\sigma+A)\gamma},$$

$$1 - \rho_0 e^{-(\sigma+A)\gamma} \le e^{(A+\sigma)(t_s - q(t_s))} \le 1 + \rho_0 e^{-(\sigma+A)\gamma}.$$

Taking logarithms and dividing by $(A + \sigma)$, we obtain

$$\frac{1}{A+\sigma} \ln(1 - x) \le t_s - q(t_s) \le \frac{1}{A+\sigma} \ln(1 + x),$$

where $x = \rho_0 e^{-(\sigma+A)\gamma}$. It follows that to satisfy (24) it is enough that

$$|t_s - q(t_s)| \le \min\{|\frac{1}{A+\sigma} \ln(1 - x)|, |\frac{1}{A+\sigma} \ln(1 + x)|\},$$

or by the properties of the logarithmic function that

$$|t_s - q(t_s)| \le \frac{1}{A+\sigma} \ln(1 + \rho_0 e^{-(\sigma+A)\gamma}). \qquad (25)$$

We now design a quantizer to construct a packet $p(t_s)$ of length $g(t_s)$ that the encoder sends to the decoder. Using the packet $p(t_s)$ the decoder reconstructs $q(t_s)$ that satisfies (25). First bit of $p(t_s)$ is used to determine the sign of $z(t_s)$, i.e., to determine whether $z(t_s) = +v(t_s)$ or $z(t_s) = -v(t_s)$. The second bit of $p(t_s)$ is mod $\left(\lfloor \frac{t_s}{b\gamma} \rfloor, 2\right)$. This second bit informs the decoder that $t_s \in [jb\gamma, (j+1)b\gamma]$ for some fixed j. As a part of quantization process the encoder divides this interval uniformly into $2^{g(t_s)-2}$ sub-interval. Thus, t_s belongs to at least one of these sub-intervals. After receiving the packet, the decoder chooses $q(t_s)$ as the middle point of the sub-interval that t_s belongs to. The packet length $g(t_s)$ is large enough that

$$\frac{b\gamma}{2^{g(t_s)-1}} \le \frac{1}{A+\sigma} \ln(1 + \rho_0 e^{-(\sigma+A)\gamma}). \qquad (26)$$

With this strategy, since

$$|t_s - q(t_s)| \le \frac{b\gamma}{2^{g(t_s)-1}},$$

it follows that (25) is satisfied. From (26), we have

$$g(t_s) \ge \max\left\{0, 1 + \log \frac{b\gamma(A+\sigma)}{\ln(1 + \rho_0 e^{-(\sigma+A)\gamma})}\right\},$$

and this choice also guarantees (24).

Finally, the controller updates $x(t_c^+)$ according to the jump strategy, so that $|z(t_c^+)| \le \rho_0 e^{-\sigma\gamma} v(t_s)$. We can then use the upper bound on the triggering rate (22) and since we want a communication rate larger than this upper bound it follows that a sufficient transmission rate should also satisfy

$$R_s \ge \frac{A+\sigma}{-\ln(\rho_0 e^{-\sigma\gamma})} \max\left\{0, 1 + \log \frac{b\gamma(A+\sigma)}{\ln(1 + \rho_0 e^{-(\sigma+A)\gamma})}\right\},$$

and the proof is complete. ∎

Corollary 1: If (A, B) is a stabilizable pair, and $u(t) = -K\hat{x}(t)$, then inequality (23) is a sufficient condition for asymptotic stabilizability.

Proof: Since (1) can be rewritten as

$$\dot{x}(t) = (A - KB)x(t) + Bz(t),$$

we have

$$x(t) = e^{(A-KB)t}x(0) + e^{(A-KB)t}\int_0^t e^{-(A-KB)\tau} Bz(\tau)d\tau.$$

We proved that (23) is sufficient for $\lim_{t\to\infty} z(t) = 0$. Since (A, B) is a stabilizable pair, and $A + BK$ is Hurwitz it follows that, criterion (23) is sufficient for $\lim_{t\to\infty} x(t) = 0$ too. On top of that, $\forall \epsilon > 0$ there exist an L small enough such that $|x(t)| \le \epsilon$ for all $t \in [0, \infty)$. ∎

Figure 4-(a) shows that for small values of γ, both necessary and sufficient conditions reduce to $R_s > 0$. Figure 4-(b) shows that for large values of γ, both necessary and sufficient conditions converge to same asymptote. As we discussed in the previous section, the asymptotic value of the necessary transmission rate is $\frac{A+\sigma}{\ln 2}(1 + \frac{A}{\sigma})$. For sufficient transmission rate we have

$$\lim_{\gamma\to\infty} \max\left\{0, 1 + \log \frac{b\gamma(A+\sigma)}{\ln(1 + \rho_0 e^{-(\sigma+A)\gamma})}\right\}$$

$$= 1 + \log \frac{b\gamma(A+\sigma)}{\rho_0 e^{-(\sigma+A)\gamma}}$$

$$= 1 + \log b\gamma(A+\sigma) + \frac{1}{\ln 2}(-\ln\rho_0 + A\gamma + \sigma\gamma).$$

Hence, the limit of the lower bound in (23) as $\gamma \to \infty$ is equal to

$$\lim_{\gamma\to\infty} \frac{A+\sigma}{\ln 2}(1 + \frac{\ln 2(1 + \log b\gamma(A+\sigma)) + A\gamma}{-\ln\rho_0 + \sigma\gamma})$$

$$= \frac{A+\sigma}{\ln 2}(1 + \frac{A}{\sigma}).$$

It remains to be seen if the gap between the sufficient and necessary conditions could be closed completely.

VII. CONCLUSIONS

In this paper we introduced an event-triggered control strategy for stablization in the presence of delay in the communication channel between the plant and the controller. We characterized the value of the timing information carried by each triggering event and derived data-rate theorems from the point of view of the sensor sending quantized state estimates to the controller. We determined the necessary and sufficient transmission rate for stabilizability and observability of the system. The central result revealed critical transition points for the transmission rate as a function of the communication delay. Important open problems for future research include extensions to vector systems, including the effect of system disturbances, tightening necessary and sufficient conditions, and obtain exponential convergence guarantees for the stabilizability of the system.

VIII. ACKNOWLEDGMENT

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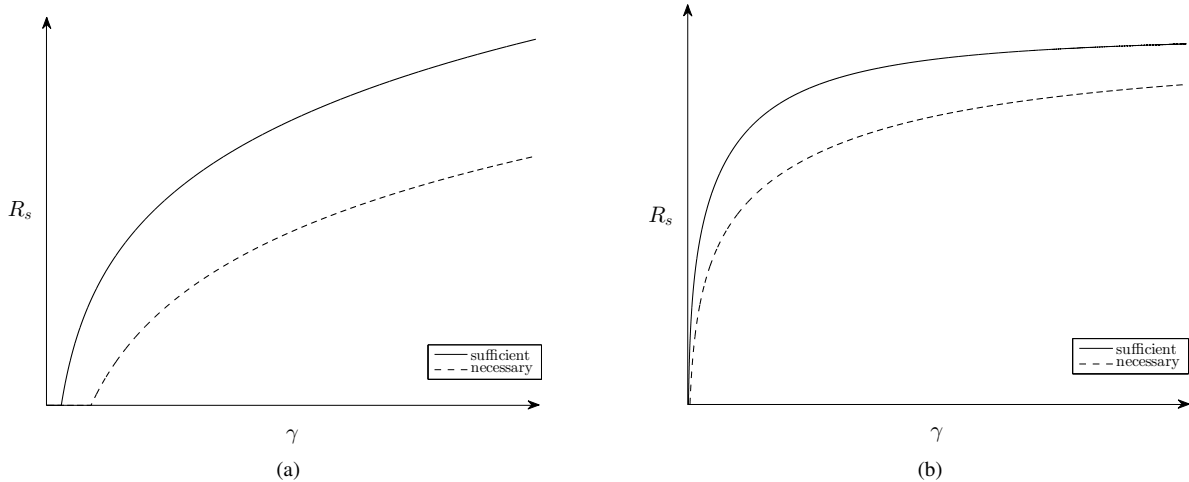
Fig. 4. Illustration of the sufficient and necessary transmission rate for asymptotic observability for small (a) and large (b) values of delay. Here, $A = 1$, $\sigma = 0.5$, $b = 1.0001$, and $\rho_0 = 0.1$.

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