

Event-triggered stabilization of linear systems under bounded *bit* rates

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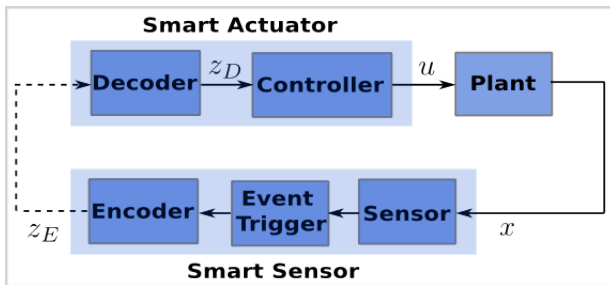
Jacobs School of Engineering

Department of Mechanical and Aerospace Engineering

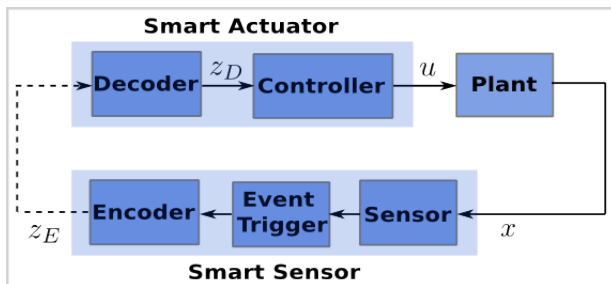
Conference on Decision and Control, 15 December 2014

Acknowledgements: National Science Foundation (Grant CNS-1329619)

Networked control systems



Networked control systems

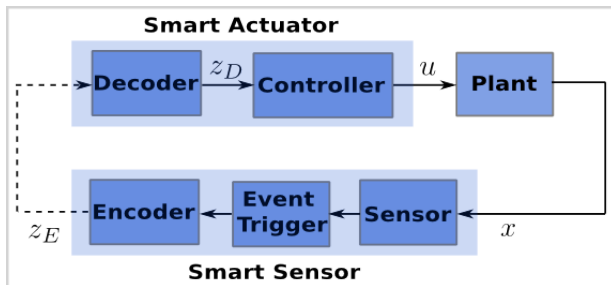


- **When to transmit:**

Event-triggered strategies

- A trigger function encodes the control goal
- Transmissions occur only when necessary
- Better use of resources than time-triggered

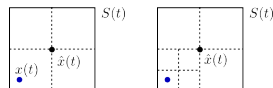
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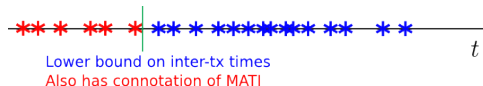
- **What to transmit:**

Information-theory based data rate theorems

- Quite successful in the discrete-time setting
- Tight necessary and sufficient data rates are available



Unanswered questions



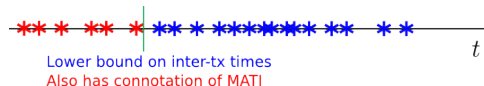
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Event-triggered control:

- What is the average inter-tx time?

Unanswered questions



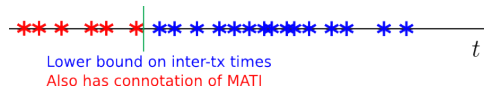
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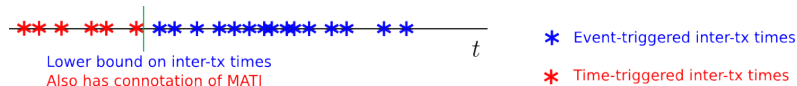
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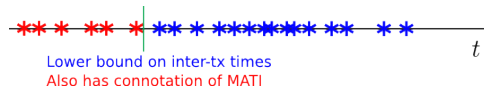
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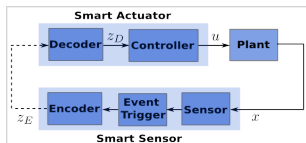
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The two themes have complementary strengths

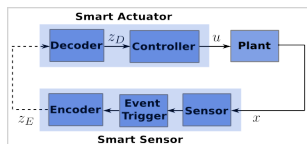
System description



Plant dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t), \quad u(t) = K\hat{x}(t)$$
$$\|v(t)\|_2 \leq \nu, \quad \forall t \in [0, \infty)$$

System description



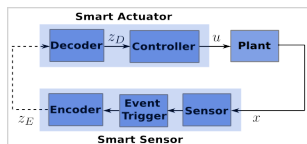
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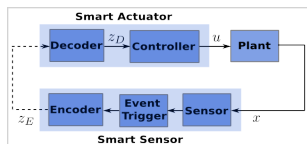
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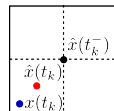
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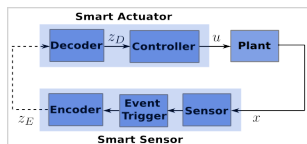
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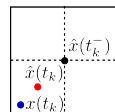
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Closed loop flow, for $t \in [r_k, r_{k+1})$

$$\dot{x}(t) = \bar{A}x(t) - BKx_e(t) + v(t), \quad \bar{A} \triangleq A + BK$$

$$\dot{x}_e(t) = Ax_e(t) + v(t), \quad x_e \triangleq x - \hat{x} \text{ (encoding error)}$$

Quantization and coding (instant communication)

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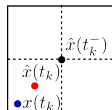
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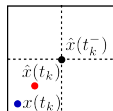
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Non-instant communication: more involved

Control objective

Suppose $\bar{A} = A + BK$ is Hurwitz $\iff P\bar{A} + \bar{A}^T P = -Q$

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Desired performance function: $V_d(t) = (V_d(t_0) - V_0)e^{-\beta(t-t_0)} + V_0$

Performance objective: ensure $b(t) \triangleq \frac{V(x(t))}{V_d(t)} \leq 1$, for all $t \geq t_0$

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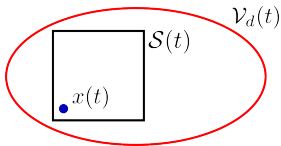
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Design objective:

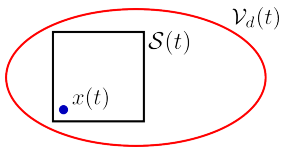
- Design event-triggered communication policy that recursively determines $\{t_k\}$ and np_k
- Ensure a uniform positive lower bound for $\{t_k - t_{k-1}\}_{k \in \mathbb{N}}$
- Ensure np_k is upper bounded by the given “channel capacity”
- Quantify the average data rate

Necessary data rate (non-state-triggered transmissions)



Set $\mathcal{S}(t)$ must lie within the set
 $\mathcal{V}_d(t) \triangleq \{\xi \in \mathbb{R}^n : V(\xi) \leq V_d(t)\}$ at all times.

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Number of bits necessary to be transmitted between t_0 and t to meet the control goal:

$$\mathcal{B}(t, t_0) \geq \left(\text{tr}(A) + \frac{n\beta}{2} \right) \log_2(e)(t - t_0) + \log_2 \left(\frac{\text{vol}(\mathcal{S}(t_0))}{c_P(V_d(t_0))^{\frac{n}{2}}} \right)$$

$$R_{\text{as}} \triangleq \lim_{t \rightarrow \infty} \frac{\mathcal{B}(t, t_0)}{t - t_0} \geq \left(\text{tr}(A) + \frac{n\beta}{2} \right) \log_2(e)$$

Assuming all eigenvalues of A have real parts greater than $-\beta$.

Control with arbitrary finite communication rate

Theorem

Assuming control goal is met with continuous and unquantized feedback, let

$$t_{k+1} = \min \left\{ t \geq t_k : b(t) \geq 1, \dot{b}(t) \geq 0 \right\}, \quad b(t) = \frac{V(x(t))}{V_d(t)}$$

$$np_k \geq \underline{np_k} \triangleq n \left\lceil \log_2 \left(\frac{d_e(t_k^-)}{c\sqrt{V_d(t_k)}} \right) \right\rceil, \quad np_k : \# \text{ bits sent at } t_k$$

Then

- *Inter-transmission times have a uniform positive lower bound,*
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No uniform bound on $\underline{p_k}$: for special initial conditions $\underline{p_k}$ can be arbitrarily large

Upper bound on the sufficient data rate

Corollary

If no disturbances, then for any $k \in \mathbb{N}$,

$$n(\underline{p}_k + \sum_{i=1}^{k-1} p_i) \leq n\left(\|A\|_\infty + \frac{\beta}{2}\right) \log_2(e)(t_k - t_0) + n \log_2\left(\frac{d_e(t_0)}{c\sqrt{V_d(t_0)}}\right) + n.$$

- Linear dependence on $t_k - t_0$
- Similar to the necessary data rate (e.g. $\text{tr}(A) \rightarrow n\|A\|_\infty$)
- If more bits than sufficient are transmitted in the past, ($p_i > \underline{p}_i$ for some $i < k$), then fewer bits are sufficient at t_k
- For any $k \in \mathbb{N}$, if $t_k - t_{k-1}$ is bounded, then so is \underline{p}_k
- Data rate is bounded even though “communication rate” (\underline{p}_k) is not uniformly bounded

Control under bounded channel capacity

Channel-trigger function:

$$h_{\text{ch}}(t) \triangleq \frac{d_e(t)}{c\sqrt{V_d(t)}\rho_T(b(t))}, \quad \rho_T(b) \triangleq \frac{(w+\theta)(1-b)}{W(e^{(w+\theta)T}-1)} + 1,$$

$T > 0$ is a fixed design parameter.

Interpretation: $n \log_2(h_{\text{ch}}(t))$ is a sufficient number of bits that, if transmitted at time t , ensures $b = \frac{V(x(t))}{V_d(t)} \leq 1$ for the next $TT = \min\{\Gamma_1(1, 1), T\}$ units of time.

Control under bounded channel capacity

Theorem

Suppose all previous assumptions hold and that $h_{\text{ch}}(t_0) \leq 2^{\bar{p}}$, where $n\bar{p}$ is the upper bound on the number of bits that can be sent per transmission. Let

$$t_{k+1} = \min\{t \geq t_k : b(t) \geq 1, \dot{b}(t) \geq 0 \text{ OR } \frac{h_{\text{ch}}(t)}{2^{\bar{p}}} \geq 1\}$$

$$np_k \geq \underline{np_k} \triangleq n \lceil \log_2 (h_{\text{ch}}(t_k^-)) \rceil, \quad np_k : \# \text{ bits sent at } t_k$$

Then

- $\underline{p_1} \leq \bar{p}$. Further for each $k \in \mathbb{N}$, if $p_k \in \mathbb{N} \cap [\underline{p_k}, \bar{p}]$, then $\underline{p_{k+1}} \leq \bar{p}$.
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- Inter-transmission times have a uniform positive lower bound,
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Non-instant communication: given an upper bound on the maximum communication time, T_M , main idea is to anticipate the threshold crossing of $b(t)$ and $\frac{h_{\text{ch}}(t)}{2^{\bar{p}}}$ well ahead.

Upper bound on the sufficient data rate

Corollary (Non-instant communication, disturbance)

Let $\bar{\theta} = \|A\|_\infty + \frac{\beta}{2}$. For any $k \in \mathbb{N}$,

$$\underline{p}_k \leq \log_2 \left(\frac{e^{\bar{\theta} T_M}}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)) - \alpha(T_M))} \right) + 1 + \log_2 \left(\frac{e^{\bar{\theta}(t_k - t_0)}}{\prod_{j=1}^{k-1} 2^{p_j}} \epsilon(t_0) + \sum_{i=0}^{k-1} \prod_{j=i+1}^{k-1} \frac{e^{\bar{\theta} T_j}}{2^{p_j}} \alpha(T_i) \right).$$

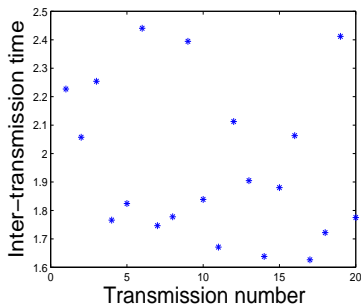
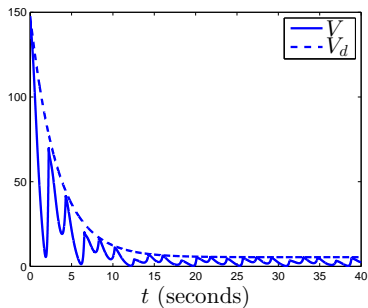
Corollary (Non-instant communication, no disturbance)

Let $\bar{\theta} = \|A\|_\infty + \frac{\beta}{2}$. For any $k \in \mathbb{N}$,

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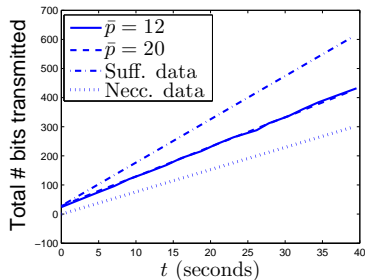
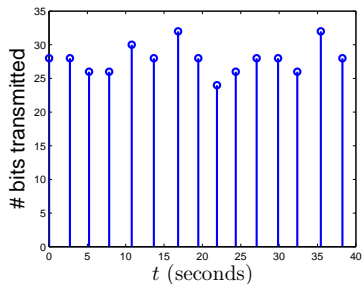
- In the general case, only an implicit characterization
- Effect of non-instant communication (through T_M) has only a “transient” effect on sufficient data rate
- If no disturbance and instant communication ($T_M = 0$), then we recover the data rate of the basic implementation

Simulation results: 2D linear system



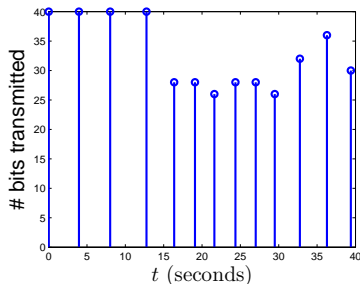
Non-instantaneous communication, with disturbance, $\bar{p} = 20$.

Simulation results: 2D linear system

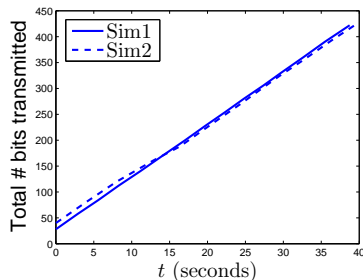


Instant communication and no disturbance

Simulation results: 2D linear system



(a)



(b)

Non-instantaneous communication without disturbance and $\bar{p} = 20$, (a) shows the number of bits on each transmission for “Sim2” (b) shows a comparison of the interpolated total number of bits transmitted in “Sim1,2”.

Contribution:

- Fusion of complementary strengths of event-triggered control and information-theoretic control
- Stabilization with prescribed convergence rate
- Control under bounded and specified channel capacity
- Instantaneous and non-instantaneous transmissions
- Analysis of average data rate

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Future work:

- Overcoming the assumption on synchronized encoder and decoder in non-instant communication
- Efficient quantization and coding schemes
- Stochastic time varying channels

Thank You

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