OPTIMIZATION

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FWC22011 IITH Future Wireless Communication (FWC) Assignment

Contents

1 Problem

show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is 1/3 h.

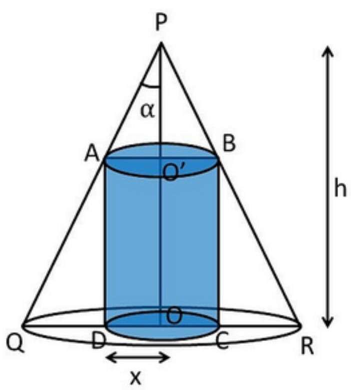

Figure.1

2 Solution

Given, Height of cone is 'h'.

$$\implies h = \|\mathbf{P} - \mathbf{O}\| \tag{1}$$

Let ,radius of cylinder be 'x'

$$\implies x = \|\mathbf{A} - \mathbf{O'}\| = \|\mathbf{O'} - \mathbf{B}\| \tag{2}$$

From figure,

$$\tan\alpha = \frac{\|\mathbf{A} - \mathbf{O'}\|}{\|\mathbf{P} - \mathbf{O'}\|} \tag{3}$$

$$\implies \|\mathbf{P} - \mathbf{O'}\| = x\cot\alpha \tag{4}$$

Height of cylinder say 'H' can be written as,

$$H = \|\mathbf{O'} - \mathbf{O}\| = \|\mathbf{P} - \mathbf{O}\| - \|\mathbf{P} - \mathbf{O'}\| \tag{5}$$

$$\implies H = h - x\cot\alpha \tag{6}$$

Volume of cylinder = (Area of base)x(Height of cylinder)

$$V(x) = (\pi x^2)(h - x\cot\alpha) \tag{7}$$

$$\tag{8}$$

objective function:

$$V = \max_{\mathbf{x}} \pi x^2(h - x\cot\alpha) \tag{9}$$

$$\tag{10}$$

constraints:

x>0

h>0

0< α <90

h, α are constants.

$$\frac{dV(x)}{dx} = (2\pi x h) - (3\pi x^2\cot\alpha) \tag{11}$$

$$\tag{12}$$

Equating $\frac{dV(x)}{dx}$ to 0,

$$\implies x = \frac{2h}{3\cot\alpha} \tag{13}$$

Derivating equation (8) results,

$$\frac{d^2V(x)}{dx^2} = 2\pi h - 6\pi x\cot\alpha \tag{14}$$

putting (9) in (10) results,

$$\frac{d^2V(x)}{dx^2} = -2\pi h < 0 \tag{15}$$

$\therefore$ V(x) is maximum at $x = \frac{2h}{3\cot\alpha}$.
Equation (6) $\implies$ Height of cylinder= $h - x\cot\alpha$.

$$= h - \frac{2h}{3\cot\alpha}\cot\alpha \tag{16}$$

$$\implies H = \frac{h}{3} \tag{17}$$

$\therefore$ The height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{h}{3}$.
Volume of cylinder:-$V(x) = \pi x^2(h - x\cot\alpha)$

substitute equation (9) in above gives,

$$V(x) = \pi\left(\frac{2h}{3\cot\alpha}\right)^2\left(h - \frac{2h}{3\cot\alpha}.\cot\alpha\right) \tag{18}$$

$$\implies V(x) = \frac{4}{27}\pi h^3\tan^2\alpha. \tag{19}$$

Let $\alpha = 45°$ and 'h'=9.

$$\implies V(x) = \pi x^2(9-x) \tag{20}$$
$$\frac{dV(x)}{dx} = 18\pi x - 3\pi x^2 \tag{21}$$
$$\frac{dV(x)}{dx} = 0 \implies x = 6. \tag{22}$$
$$\frac{d^2V(x)}{dx^2} = 18\pi - 6\pi x \tag{23}$$
$$\tag{24}$$

putting $x = 6$ in above equation gives,

$$\frac{d^2V(x)}{dx^2} = 18\pi - 36\pi = -18\pi < 0. \tag{25}$$

$\therefore$ V(x) is maximum at $x = 6$.
Height of the cylinder (H)=$h - x\cot\alpha$

$$\implies H = 9 - 6\cot 45° = 3 = \frac{1}{3} \text{ (Height of cone)}. \tag{26}$$

Geometric programming:-

$$f(x) = \sum_{j=1}^{N} C_j \prod_{i=1}^{n} x_i^{a_{ij}} \tag{27}$$
$$C_j > 0, x_i \geq 0, a_{ij} \text{is real.} \tag{28}$$

Any non linear function which obeys the conditions in (28) can be solvable using geometric programming.

But , the function $V(x)$ have negative coefficient for x^3.

so ,we approximate the function $V(x)$ using the equations below.

$$f(w) \approx f(x) \prod_{i=1}^{n} \left(\frac{w_i}{x_i}\right)^{a_i} \tag{29}$$

where

$$a_i = \frac{x_i}{f(x)} \frac{\partial f}{\partial x_i} \tag{30}$$

By keeping the constraints like

$$x \geq \frac{x_k}{1 + error} \tag{31}$$
$$x \leq (1 + error)(x_k) \tag{32}$$

where $x_k = x - 10\delta$,

The above formulation can be iterated till the problem converges at a local maximum. Taking n = 1000 ,δ= 0.001 and error=0.005 the optimal solution obtained using cvxpy is

$$\boxed{V_{max} = 339.3032173204817}$$
$$\boxed{x = 6.022349936219689}$$

For Maxima :
Using gradient ascent method,

$$x_{n+1} = x_n + \alpha \nabla V(x_n) \tag{33}$$
$$\implies x_{n+1} = x_n + \alpha(3\pi x(6-x)) \tag{34}$$

In equation (24), α is a variable parameter known as step size. x_{n+1} is the next position.The positive sign refers to the maximization part of gradient ascent. By following the above method, we keep doing iterations until $x_{n+1} - x_n$ becomes less than the value of precision.

Taking $\alpha = 0.001$,$x_0 = 2$ and precision=0.00000001, values obtained using python are :

$$\boxed{\text{Maxima} = 339.29200658769685 \approx 339}$$
$$\boxed{\text{Maxima Point} = 5.999999834068126 \approx 6}$$

$\therefore$ Maxima value of $V(x)$ at $x = 6$ is 339.

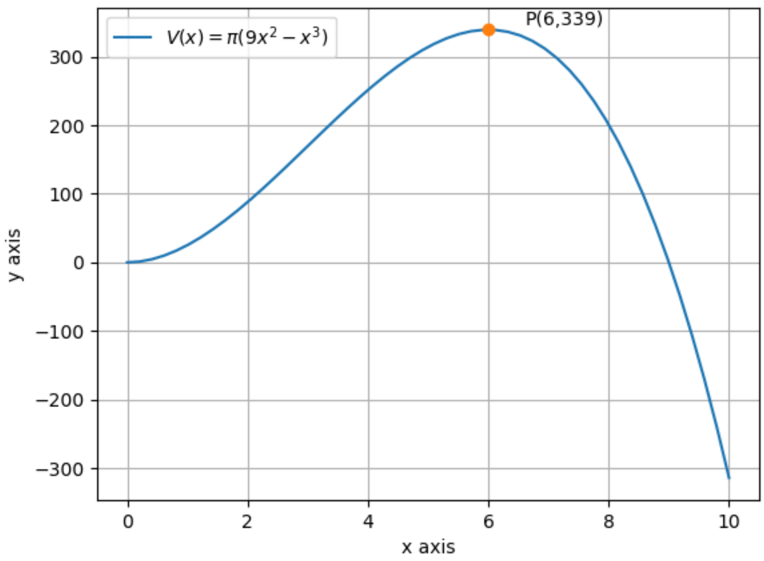

Figure.2:$V(x) = \pi x^2(9-x)$

Below link shows python code to verify maxima of the function $V(x)$.

https://github.com/FWC_module1/blob/main/optimization/advanced.py